Killing vector fields for some meromorphic affine connections.

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Abstract

We give a geometric assumption on a meromorphic affine connection for its Killing vector fields to be univaluated. For, we prove a general result on the infinitesimal automorphisms for a subcategory of meromorphic Cartan geometries, and uses the equivalence with the former geometric structures. This result is applied to the classification on complex compact manifolds with algebraic dimension zero, extending partially the main result of [3].

1 Introduction

Many geometric structures over complex or real manifolds are infinitesimal versions of some *homogeneous* geometric structure on a model space, meaning that the global automorphisms of the structure act transitively on this space. Thus, the *infinitesimal isometries*, i.e differentiable or holomorphic local vector fields whose local flows preserve the geometric structure, are of special interset when we try to classify them. When these vector fields span the tangent space at any point, we say that the geometric structure is *locally homogeneous* (resp. *quasi-homogeneous* if an open dense subset is locally homogeneous). The classification of simply connected manifolds (complex or real) admitting a *locally homogeneous* geometric structure is a classical problem in differential geometry, and started with the riemanniann case. In particular, Riemann, Killing and Hopf proved that any simply connected and locally homogeneous riemanniann manifold is in fact *homogeneous* (see [18]).

The quasi-homogeneity of the holomophic geometric structure on a compact complex manifolds is sometimes obtained from complex-geometrical assumptions on the base manifold. For example, any *rigid* holomorphic geometric structures (see [8]) on a compact complex manifold whose meromorphic functions are constants is quasi-homogeneous, by a result proved by S. Dumitrescu in [11]. It was extended to *holomorphic Cartan geometries* and their *infinitesimal automorphisms* (or *Killing fields*). These are objects related with holomorphic geometric structures in the sense that any holomorphic Cartan geometry on a base complex manifold M, with model a complex Klein geometry (G, P), *induces* a holomorphic geometric structure whose model only depends on (G, P). For example, when (G, P) is the affine Klein geometry, the holomorphic geometric structure is a holomorphic affine connection (see Corollary 5.1).

The facts mentionned above were used in [3] to prove a classification result : any simply connected complex compact manifold M, whose meromorphic functions are constants, cannot bear a holomorphic (G, P)-Cartan geometry. Two important steps in the proof are as follow. First, it is proved that any germ of infinitesimal automorphism of the Cartan geometry is the germ of a global infinitesimal automorphism : this is a generalization of a result by Nomizu ([17]) for analytic riemanniann metrics, and follows from the fact that the former objects form a local system on M. Next, it is proved that there is a family of global infinitesimal automorphisms, whose existence is ensured by the previous fact, which integrates into an action by automorphisms of a complex abelian Lie group L with an open dense orbit in M. The conclusion follow from detailed study of the geometry of such manifolds M, which implies that the Cartan geometry is flat.

In the meromorphic category, the two steps may fail : infinitesimal automorphisms could be multivaluated, and a meromorphic univaluated infinitesimal automorphism may not have a well defined flow at some point of the pole. In this paper, we give a sufficient condition on some meromorphic Cartan geometries to recover the first step, and apply it to the classification of meromorphic affine connections.

2 Preliminaries and notations

2.1 Locally free modules and meromorphic connections

Let (M, D) be a **pair**, i.e a complex manifold M equipped with a divisor D. Let $\mathcal{L}$ be a coherent $\mathcal{O}_M$-module. The **order** $ord_D^{\mathcal{L}}(s)$ at D of a section s of $\mathcal{L}[\star D]$ defined on an open subset $U \subset M$ is the greatest integer $d \in \mathbb{Z}$ such that s is also a section of $\mathcal{L}(-dD)|_U$ on U.

Definition 2.1. *A meromorphic connection on (M, D) is a couple $(\mathcal{V}, \nabla)$ where $\mathcal{V}$ is a locally free $\mathcal{O}_M$-module of finite rank, and ∇ is a morphism of $\mathbb{C}$-sheaves from $\mathcal{V}[\star D]$ to $\Omega_M^1 \otimes \mathcal{V}[\star D]$ satisfying the Leibniz identity $\nabla(fs) = d(f)s + f\nabla(s)$ for any $s \in \mathcal{V}(U)$ and $f \in \mathcal{O}_M[\star D](U)$ (U is an open subset of M).*

If $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla)$ are two meromorphic connections related by $\mathcal{L} = \bigoplus_{i=1}^{r} \mathcal{O}_M s_i$, $\mathcal{L}' = \bigoplus_{i=1}^{r} \mathcal{O}_M t_i$ and $t_i = \sum_{j=1}^{r} q_{ji} s_j$ for a meromorphic matrix Q on M, then the matrices A and A' respectively associated to the basis $(s_i)_{i=1,\ldots,r}$ and $(t_i)_{i=1,\ldots,r}$ are related by the gauge-transformation formula :

$$A' = Q^{-1} dQ + Q^{-1} A Q \tag{1}$$

where d stands for the deRham derivative.

A **meromorphic affine connection** on (M, D) is a meromorphic connection ∇ on TM with poles supported at D. The **torsion** of a meromorphic affine connection ∇ on (M, D) is the meromorphic section T_∇ of $\Omega_M^1 \otimes End(TM)$ defined by :

$$T_\nabla(X)(Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y]_{TM} \tag{2}$$

Let (M, D) be a pair and $r \geq 1$ be an integer. Recall that mapping a holomorphic rank r vector bundle V over M to the locally free rank r $\mathcal{O}_M$-module $\mathcal{E}$ of local holomorphic sections of V is an equivalence of categories. It maps a isomorphism $\hat{\Psi} : V_1 \longrightarrow V_2$ of vector bundles with associated sheaves of sections $\mathcal{E}_1, \mathcal{E}_2$, covering an isomorphism $\varphi : M_1 \longrightarrow M_2$ of complex manifolds, to the isomorphism Φ of $\mathcal{O}_M$-modules between $\mathcal{E}_1$ and $\varphi^* \mathcal{E}_2$ defined by :

$$\begin{array}{cccc} \Phi & : & \mathcal{E}_1(U) & \longrightarrow & \varphi^*\mathcal{E}_2(U) \\ & & s & \longrightarrow & \hat{\Psi} \circ s \circ \varphi^{-1} \end{array} \tag{3}$$

where U is an open subset of M.

Definition 2.2. *A couple (φ, Φ) as above will be called a isomorphism of vector bundles between $\mathcal{E}_1$ and $\mathcal{E}_2$. More generally, we define a isomorphism of meromorphic bundles by replacing the sheaves of holomorphic sections by the corresponding of meromorphic sections with poles at D_1 and D_2.*

Let ∇_1, ∇_2 be two meromorphic connections on $\mathcal{E}_1$,$\mathcal{E}_2$ with poles supported at D_1,D_2. We say that an isomorphism of meromorphic vector bundles (φ, Φ) **preserves the connections** iff :

$$(\varphi, \Phi)^\star \nabla_2 = \nabla_1 \tag{4}$$

where $(\varphi, \Phi)^\star \nabla_2$ is the **pullback** of ∇_2 through (φ, Φ) defined by the commutative diagram :

$$\begin{tikzcd} \mathcal{E}_1 \arrow[r, "(\varphi,\phi)^\star \nabla_2"] \arrow[d, "\phi"'] & \Omega^1_{M_1} \otimes \mathcal{E}_1[\star D_1] \\ \varphi^*(\mathcal{E}_2[\star D_2]) \arrow[r, "\varphi^\star \nabla_2"'] & \varphi^* \Omega^1_{M_2} \otimes \varphi^*(\mathcal{E}_2[\star D_2]) \arrow[u, "(d\varphi)^* \otimes \phi^{-1}"'] \end{tikzcd}$$

where we denoted φ^* the pullback in the sheaf theoretic sense and $d\varphi$ the differential in the sheaf theoretic sense.

A **flat** meromorphic connection ∇ on $\mathcal{E}$ with poles at D is a meromorphic connection such that the subsheaf of **horizontal sections** $ker(\nabla)$ on $M \setminus D$ defined by :

$$\forall U \subset M \setminus D,\, ker(\nabla)(U) = \{s \in \mathcal{E}(U) \,|\, \nabla(s) = 0\} \tag{5}$$

is a **local system** (see [9]).

We recall that there is an equivalence of categories between the category of local systems of rank r on $M \setminus D$ with arrows being the isomorphisms, and the category of representations $\rho : \pi_1(M \setminus D, x) \longrightarrow K$ (for any $x \in M \setminus D$, and K is a $\mathbb{C}$-vector space of dimension r) with arrows being the isomorphisms of representations. Once a point $x \in M \setminus D$ is choosen, this equivalence is obtained by associating to the local system $\mathcal{K}$, the **monodromy map** $Mon^x(\mathcal{K}) : \pi_1(M \setminus D, x) \longrightarrow Aut(\mathcal{K}_x)$ (see [9]).

2.2 Atiyah sequence of the frame bundle

The **frame bundle** of a locally free $\mathcal{O}_M$-module $\mathcal{E}$ of rank r is the holomorphic $GL_r(\mathbb{C})$-principal bundle $E \xrightarrow{p} M$ whose fiber at $x \in M$ is the set of isomorphisms $\mathbb{C}^r \simeq \mathcal{E}(x)$. Here $\mathcal{E}(x) = \mathcal{E}_x/\mathfrak{m}_x$ stands for the fiber of $\mathcal{E}$ at x.

We recall that for any complex Lie group P and a holomorphic P-principal bundle $E \xrightarrow{p} M$, there is a notion of P-**linearization** for a $\mathcal{O}_E$-module $\mathcal{V}$: this is a family $(\phi_b)_{b\in P}$ of isomorphisms $\phi_b : \mathcal{V} \simeq r_b^*\mathcal{V}$ (where r_b is the right action of P) with nice properties (see [14]). A $\mathcal{O}_E$-module equipped with a P-linearization is said to be P-**equivariant**. In this context, there is an equivalence between the P-equivariant locally free $\mathcal{O}_E$-modules and the locally free $\mathcal{O}_M$-modules, and between the P-equivariant morphisms and the morphisms between the corresponding $\mathcal{O}_M$-modules (see []). For any representation $\rho : P \longrightarrow GL(\mathbb{V})$, and any holomorphic P-principal bundle $E \xrightarrow{p} M$, we denote by $E(\mathbb{V})$ the $\mathcal{O}_M$-module associated with the $\mathcal{O}_E$-module $\mathcal{O}_E \otimes \mathbb{V}$, where the P-linearization $(\phi_b)_{b\in P}$ is given by $\phi_b = r_b^* \otimes \rho(b^{-1})$. We call it the **representation module** associated with E and $\mathbb{V}$. For any isomorphism $\Psi : E_1 \longrightarrow E_2$ of holomorphic P-principal bundles covering $\varphi : M_1 \longrightarrow M_2$, the **representation isomorphism of associated vector bundles** corresponding to Ψ is the isomorphism

$$\Psi(\mathbb{V}) : E_1(\mathbb{V}) \longrightarrow \varphi^* E_2(\mathbb{V}) \tag{6}$$

associated to the P-equivariant isomorphism $\Psi^* \otimes Id_{\mathbb{V}}$ of trivial $\mathcal{O}_E$-modules.

Definition 2.3. *Let $\mathbb{V}$ be a representation of a complex Lie group P. Let $E_1 \xrightarrow{p_1} M_1$ and $E_2 \xrightarrow{p_1} M_2$ be two holomorphic P-principal bundles, and D_1, D_2 be respectively two divisors of M_1 and M_2. An isomorphism $\Psi : E_1|_{M_1 \setminus D_1} \longrightarrow E_2|_{M_2 \setminus D_2}$ of holomorphic P-principal bundles is $\mathbb{V}$-meromorphic between (M_1, D_1) and (M_2, D_2) iff the representation isomorphism $\Phi = \Psi(\mathbb{V})$ restricts to an isomorphism $\Phi : E_1(\mathbb{V})[\star D_1] \longrightarrow \varphi^* E_2(\mathbb{V})[\star D_2]$ (see 6).*

In particular, mapping holomorphic $GL_r(\mathbb{C})$-principal bundles E over M to the associated representation modules $E(\mathbb{C}^r)$gives an equivalence of categories. A pseudo-inverse is given by mapping a locally free $\mathcal{O}_M$-module $\mathcal{E}$ of rank r to its frame bundle E.

Consider $\mathfrak{p} = Lie(P)$ which is the adjoint representation of P. Let $At(E)$ be the $\mathcal{O}_M$-module associated with the P-equivariant locally free $\mathcal{O}_E$-module TE equipped with the P-linearization induced by the infinitesimal action of P on E : it is called the **Atiyah bundle** of E, and fits into the short exact sequence :

$$0 \longrightarrow E(\mathfrak{p}) \xrightarrow{\iota} At(E) \xrightarrow{q} TM \longrightarrow 0 \tag{7}$$

where ι is the morphism associated with the P-equivariant mophism which to any $A \in \mathcal{O}_E \otimes \mathfrak{p}$ associates the corresponding fundamental vector field on E, and q is the one associated with the P-equivariant morphism $dp : TE \longrightarrow p^*TM$.

The previous equivalence implies that P-equviariant meromorphic one forms on E, with poles at $\tilde{D} = p^{-1}(D)$, and values in $\mathbb{V}$ are in bijection with morphisms $\beta : At(E)[\star D] \longrightarrow E(\mathbb{V})[\star D]$, or equivalently with sections of $At(E) \otimes E(\mathbb{V})[\star D]$. This correspondance restricts to a bijective correspondance between :

- The set of morphisms β as above vanishing on the image of ι in (7), equivalently sections of $\Omega^1_M[\star D] \otimes E(\mathbb{V})$
- The set of meromorphic one forms $\tilde{\omega}$ on $(E, \tilde{D})$ with values in $\mathbb{V}$ vanishing on $ker(dp)$

3 Holomorphic branched Cartan geometries and the Killing connection

In this section, we fix a pair (M, D) where M is of complex dimension n. We define meromorphic Cartan geometries, and the subcategory of branched holomorphic Cartan geometries. We describe their infinitesimal automorphisms as sections for a meromorphic connection either on a trivial module over the principal bundle of the geometry, or on the corresponding module over the base manifold. We introduce the subcategory of *totally geodesic* meromorphic Cartan geometries : in the next section, we will see that their infinitesimal automorphisms are univaluated, in a sense that will be defined.

3.1 Meromorphic and holomorphic branched Cartan geometries

First, we have to define the models for Cartan geometries :

Definition 3.1. *A complex Klein geometry of dimension $n \geq 1$ is a couple (G, P) where G is a complex Lie group, and P is a complex Lie subgroup with $dim(G) - dim(P) = n$.*

Let (G, P) be as in Definition 3.1 and let $P' = ker(\underline{ad})$ where $\underline{ad} : P \longrightarrow GL(\mathfrak{g}/\mathfrak{p})$ is the representation induced by the ajdoint representation. Then any choice of a basis for $\mathfrak{g}/\mathfrak{p}$ identifies $Q = P/P'$ with a linear complex subgroup, and TG/P with the module $G(\mathfrak{g}/\mathfrak{p})$ associated to the P-principal bundle E and the representation $\mathfrak{g}/\mathfrak{p}$. Thus, the complex manifold G/P comes equiped with a holomorphic reduction $G \underset{P}{\times} Q$ of its holomorphic frame bundle $R^1(G/P)$, i.e a holomorphic Q-structure : namely G/P'.

This is in fact only due to the presence of a holomorphic 1-$\mathfrak{g}$-form with special properties on the total space of the holomorpihc P-principal bundle $G \longrightarrow G/P$, namely the *Maurer-Cartan form* ω_G of G. We can consider curved versions of theses objects, for which the previous fact still holds. Authorizing the one form to have poles on the P-principal bundle, we obtain their meromorphic analogues :

Definition 3.2. *Let (G, P) be a complex Klein geometry with $dim(G/P) = n$ and (M, D) be a pair. A meromorphic (G, P)-Cartan geometry is a couple (E, ω_0) where $E \overset{p}{\to} M$ is a holomorphic P-principal bundle, and ω_0 is a meromorphic 1-$\mathfrak{g}$-form on E, with poles on $\tilde{D} = p^{-1}(D)$, such that :*

(i) For any $x \in M \smallsetminus D$, $\iota_x^\star \omega_0$ coïncides with the Maurer-Cartan form $\omega_{E,x}$ (see above).

(ii) ω_0 is P-equivariant.

(iii) For any $e \in E \smallsetminus \tilde{D}$, $\omega_0(e)$ is an isomorphism between T_eE and $\mathfrak{g}$.

Namely, if (E, ω_0) is a meromorphic (G, P)-Cartan geometry on (M, D), then there is an isomorphism of $\mathcal{O}_{M\smallsetminus D}$-modules between $TM \smallsetminus D$ and the module $E(\mathfrak{g}/\mathfrak{p})|_{M\smallsetminus D}$ associated with E. We obtain as above a holomorphic reduction of $R^1(M \smallsetminus D)$ to Q, which prolongates to a holomorphic Q-reduction $\tilde{E}$ of the bundle of frames of $\mathcal{E} = \tilde{E}(\mathfrak{g}/\mathfrak{p})$ where $\tilde{E} = E \underset{P}{\times} Q$. The isomorphism of $\mathcal{O}_{M\smallsetminus D}$-modules is the restriction of an isomorphism of $\mathcal{O}_M[\star D]$-modules between $TM[\star D]$ and $\mathcal{E}[\star D]$.

They form a category :

Definition 3.3. *Let (G, P) be a complex Klein geometry and (M, D),(M', D') be two pairs with $dim(M) = dim(M') = dim(G/P)$. Let (E, ω_0) and (E', ω'_0) be respectively two meromorphic (G, P)-Cartan geometries on (M, D) and (M', D'). An isomorphism between (E, ω_0) and (E', ω'_0) is an isomorphism of holomorphic P-principal fiber bundles $\Psi : E \smallsetminus \tilde{D} \overset{\sim}{\to} E' \smallsetminus \tilde{D}'$ such that $\Psi^\star \omega'_0 = \omega_0$.*

The following object is central in the study of Cartan geometries :

Definition 3.4. *Let (E, ω_0) be a meromorphic (G,P)-Cartan geometry on (M,D). Its curvature function is the meromorphic function k_{ω_0} on E with values in $\mathbb{W} = \mathfrak{g}^* \wedge \mathfrak{g}^* \otimes \mathfrak{g}$ and defined by :*

$$k_{\omega_0} = d\omega_0 \circ (\omega_0^{-1} \wedge \omega_0^{-1}) + [,]_{\mathfrak{g}} \tag{8}$$

where $[,]_{\mathfrak{g}}$ is the Lie-bracket of $\mathfrak{g}$ identified with an element of $\mathbb{W}$.

Fix a Klein geometry (G,P) and choose a basis $(\mathfrak{e}_i)_{i=1,\dots,N}$ of $\mathfrak{g}$, with $(\mathfrak{e}_i)_{i=1,\dots,n}$ spanning a supplementary $\mathfrak{g}_-$ of $\mathfrak{p}$. Denote by $(\mathfrak{e}_i^*)_{i=1,\dots,N}$ the dual basis of $\mathfrak{g}^*$.

Definition 3.5. *Let (E, ω_0) be a meromorphic (G,P)-Cartan geometry on (M,D). The meromorphic functions :*

$$\gamma_{i,j}^k = \mathfrak{e}_k^* \circ k_{\omega_0}(\mathfrak{e}_i, \mathfrak{e}_j) \tag{9}$$

are called the structure coefficients of (E,ω_0).

A natural subcategory of the meromorphic (G,P)-Cartan geometries on pairs is the following :

Definition 3.6. *A branched holomorphic (G,P)-Cartan geometry on a pair (M,D) is a meromorphic (G,P)-Cartan geometry (E,ω_0) on (M,D) such that ω_0 extends as a holomorphic one form on E.*

An important feature of these objects for the classification is the existence of a holomorphic connection on the adjoint vector bundle. Indeed, let (E,ω_0) be a branched holomorphic (G,P)-Cartan geometry on (M,D), and $E_G = E \times_P G$ the extension of the holomorphic P-principal bundle E to the group G. By definition, E_G is the quotient of the product $E \times G$ by the action of G given by $(e,g)\cdot h = (e\cdot h, h^{-1}g)$. Consider the G-equivariant holomorphic one form $\overline{\omega}$ on $E\times G$ with values in $\mathfrak{g}$ given by :

$$\overline{\omega} = ad(\pi_2) \circ \pi_1^\star \omega_0 + \pi_2^\star \omega_G \tag{10}$$

where π_1, π_2 are the projections on each factor and ω_G is the Maurer-Cartan form of G. It is straightforward to verify that for any $A \in \mathfrak{g}_0$, $\overline{\omega}(\frac{d}{dt}|_{t=0}(e,h)\cdot exp_G(tA)) = 0$, i.e the vectors tangent to the fibers of

$$\pi_G : E \times G \longrightarrow E_G \tag{11}$$

are in the kernel of $\overline{\omega}$. Thus, $\overline{\omega}$ induces a holomorphic one form on E_G.

Definition 3.7. *The holomorphic G-principal connection $\tilde{\omega}$ on E_G induced by $\overline{\omega}$ is the tractor-connection of (E,ω_0). We denote by ∇^{ω_0} the corresponding holomorphic connection on $E_G(\mathfrak{g}) = E(\mathfrak{g})$ (see Proposition 5.0.1).*

The pullback $p^* E(\mathfrak{g})$ is the trivial module $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$.

Lemma 3.1. *The pullback $p^\star\nabla^{\omega_0}$ is $d - Ad(\omega_0)$ where d is the deRham diffential on the trivial module $\mathcal{V}$, $Ad(\omega_0)$ is the section of $\Omega_E^1 \otimes End(\mathfrak{g}) = \Omega_E^1 \otimes End(\mathcal{V})$ defined by :*

$$X \neg Ad(\omega_0)(s) = [\omega_0(X), s]_{\mathfrak{g}}$$

for any holomorphic vector field X of E and section s of $\mathcal{V}$. In particular, its curvature is $R_{p^\star\nabla^{\omega_0}} = Ad(d\omega_0) + Ad(\omega_0 \wedge \omega_0)$.

Proof. Since the ω_0-constant vector fields on E span T_eE at any $e \in E \smallsetminus \tilde{D}$, we can choose $\tilde{A} = \omega_0^{-1}(A)$ for $A \in \mathfrak{g}$ as a holomorphic vector field on $E \smallsetminus \tilde{D}$. Let s be any section of $\mathcal{V}(U)$, $U \subset E \smallsetminus \tilde{D}$ an open subset. By definition of ∇^{ω_0} and the remarks preceding Definition 3.7, we have :

$$\tilde{A} \neg p^\star \nabla^{\omega_0}(s) = (\overline{A} - \hat{A}) \neg d(\tilde{s})$$

where $\tilde{s}$ is the unique G-equivariant section of $\mathcal{O}_{E\times G} \otimes \mathfrak{g}$ which coïncides with s in restriction to $E \subset E \times G$, $\overline{A}$ is the unique G-invariant meromorphic vector field whose restriction to E coïncides with $\tilde{A}$, and $\hat{A}$ is the holomorphic vector field tangent to the fibers of $E \times G \xrightarrow{\pi_1} E$ such that $\pi_2^\star \omega_G(\hat{A}) = A$. Indeed, $\overline{A} - \hat{A}$ is the unique vector field which belongs to $ker(\overline{\omega})$ and projects to $\tilde{A}$ via $\pi_1 : E \times G \longrightarrow E$. Now, $\overline{A} \neg d(\tilde{s})$ coïncides with $\tilde{A} \neg d(s)$ in restriction to E, while $\hat{A} \neg d(\tilde{s}) = [A, s]_{\mathfrak{g}}$ because $\tilde{s}$ is G-equivariant. The first formula follows. For the curvature, it corresponds to the classical computation of the curvature in a trivialisation of a vector bundle. □

3.2 Meromorphic extension of the tangent sheaf

We now describe an object induced by any meromorphic Cartan geometry.

Definition 3.8. *Let (M,D) be a pair.*

1. *A meromorphic extension of (M,D) is a couple $(\phi_0,\mathcal{E})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_M$-module and $\phi_0 : TM[\star D] \longrightarrow \mathcal{E}[\star D]$ is an isomorphism of $\mathcal{O}_M$-modules.*
2. *A holomorphic extension of (M,D) is a meromorphic extension $(\phi_0,\mathcal{E})$ such that $\phi_0(TM) \subset \mathcal{E}$.*
3. *The category $\mathcal{F}$ (resp. $\mathcal{F}^0$) of meromorphic extensions (resp. holomorphic extensions) over pairs is defined as follow. An arrow between two meromorphic extensions $(\phi_0,\mathcal{E})$ and $(\phi_0',\mathcal{E}')$ over (M_1,D_1) and (M_2,D_2) is a an isomorphism (φ,Φ) of meromorphic bundles (resp. of vector bundles, see Definition 2.2) between $\mathcal{E}$ and $\mathcal{E}'$ such that the following diagram commutes :*

$$\begin{tikzcd} TM_1 \arrow[r, "\phi_0"] \arrow[d, "d\varphi"'] & \mathcal{E}[\star D_1] \arrow[d, "\Phi"] \\ \varphi^* TM_2 \arrow[r, "\varphi^*\phi_0'"'] & \varphi^*\mathcal{E}'[\star D_1] \end{tikzcd} \tag{12}$$

4. *The category obtained by restricting to meromorphic extensions of (M,D) and to isomorphisms of meromorphic bundles of the form (Id_M,Φ) is denoted by $\mathcal{F}_{M,D}$ (resp. $\mathcal{F}^0_{M,D}$).*

Definition 3.9. *Let (M,D) be a pair.*

1. *Let $E \xrightarrow{p} M$ be a holomorphic P-principal bundle and $\tilde{D} = p^{-1}(D)$. A meromorphic solderform on $(E,\tilde{D})$ is a P-equivariant meromorphic 1-$\mathbb{C}^n$-form θ_0 on E, with poles supported at $\tilde{D}$, vanishing on $ker(dp)$, and such that $\theta_0(e)$ is surjective for any $e \in E \smallsetminus \tilde{D}$. A couple (E,θ_0) is called a meromorphic solder form over (M,D)*
2. *An arrow between two meromorphic solderforms (E,θ_0) and (E',θ_0') over (M_1,D_1) and (M_2,D_2) is an isomorphism of holomorphic P-principal bundles $\tilde{\Psi} : E \longrightarrow E'$ such that $\theta_0 = \tilde{\Psi}^\star\theta_0'$. This defines the category $\mathcal{D}$ of meromorphic solderforms over pairs.*

Proposition 3.0.1. *The map which to any meromorphic solder form (E,θ_0) over (M,D) (Definition 3.9) associates the meromorphic extension $(\phi_0,\mathcal{E})$ where $\phi_0 : TM[\star D] \xrightarrow{\sim} \mathcal{E}[\star D]$ is the isomorphism which corresponds to θ_0 (see remarks preceding Definition 5.1), extends to an equivalence of categories $m : \mathcal{D} \longrightarrow \mathcal{E}$.*

Proof. If $\tilde{\Psi} : E \longrightarrow E'$ is an arrow between two objects (E,θ_0) and (E',θ_0') of the category of solderforms over (M,D), we define $m(\tilde{\Psi}) = \Phi$ as the image of $\tilde{\Psi}$ through the equivalence of **??**. Consider the images $(\phi_0,\mathcal{E})$ and $(\phi_0',\mathcal{E})$ of (E,θ_0) and (E',θ_0'). Since $\theta_0' = \tilde{\Psi}^\star\theta_0$, by definition, $\Phi \circ \phi_0 = \phi_0'$ so m is a essentially surjective functor. Since it is the restriction of the equivalence of categories described in subsection 2.2, it is an equivalence of categories. □

Now let (E,ω_0) be any meromorphic (G,P)-Cartan geometry on (M,D). Then the meromorphic one form $\pi_{\mathfrak{g}/\mathfrak{p}} \circ \omega_0$ obtained by projecting ω_0 on $\mathfrak{g}/\mathfrak{p}$ is P-equivariant for the quotient adjoint action on $\mathfrak{g}/\mathfrak{p}$, and pointwise surjective on $E \smallsetminus p^{-1}(D)$. Moreover, its kernel contains $ker(dp)$. In view of the Atiyah sequence with poles at D (see Equation 7), it thus corresponds to a morphism of $\mathcal{O}_M$-modules :

$$\phi_0 \quad : \quad TM[\star D] \quad \longrightarrow \quad \mathcal{E}[\star D] \tag{13}$$

where we set $\mathcal{E} = E(\mathfrak{g}/\mathfrak{p})$. By construction, ϕ_0 is an isomorphism of meromorphic bundles and $(\mathcal{E},\phi_0)$ is thus a meromorphic extension on (M,D).

Definition 3.10. *The meromorphic extension $(\mathcal{E},\phi_0)$ obtained as above is the meromorphic extension induced by (E,ω_0). We denote by f the map from the set of meromorphic (G,P)-Cartan geometries on pairs to the set of meromorphic extensions which maps (E,ω_0) to its induced meromorphic extension $(\mathcal{E},\phi_0)$. This extends as a functor f between the corresponding categories.*

3.3 Infinitesimal automorphisms as horizontal sections

Important objects in the study of meromorphic Cartan geometries are the following :

Definition 3.11. *Let (E,ω_0) be a meromorphic (G,P)-Cartan geometry on (M,D). An infinitesimal automorphism of (E,ω_0) is a holomorphic vector field $\overline{X}$ on an open subset $U \subset M \setminus D$, lifting to a vector field X on $p^{-1}(U)$ such that $\phi_X^{t\star}\omega_0 = \omega_0$. We write $\mathfrak{kill}^{loc}_{M,\omega_0}$ for the subsheaf of $TM \setminus D$ whose sections are the local infinitesimal automorphisms, and $\mathfrak{kill}^{loc}_{E,\omega_0}$ for the subsheaf of $TE \setminus \tilde{D}$ whose sections are the lifts of sections of $\mathfrak{kill}^{loc}_{M,\omega_0}$.*

In order to study the sections of $\mathfrak{kill}^{loc}_{M,\omega_0}$, it is convenient to identify them with horizontal sections for a meromorphic connection on a trivial module over E. This is a classical approch for general meromorphic parallelisms (see for example [4]). Indeed, let's denote by T the torsion of the flat meromorphic connection ∇^0 whose horizontal sections are the ω_0-constant vector fields on E. Then :

Proposition 3.0.2. *Let (E,ω_0) be a meromorphic (G,P)-Cartan geometry on (M,D).*

1. *The sheaf $\mathfrak{kill}^{loc}_{E,\omega_0}$ coïncides with the sheaf $ker(\nabla^{rec}_{\omega_0})$ of horizontal sections for the reciprocal connection ∇^{rec} defined by :*
$$\nabla^{rec}_X = \nabla^0_X + T(X,\cdot) \tag{14}$$
for any local vector field X.

2. *The connection $\nabla^{rec}_{\omega_0}$ is invariant by the P-linearization $(dr_b)_{b\in P}$ corresponding to the action of principal P-bundle.*

Proof. 1. See Lemma 3.2 in [4].

2. This straightforwardly follows from the fact that the torsion of ∇_0 is P-invariant by definition.

□

Definition 3.12. *The Killing connection of a meromorphic (G,P)-Cartan geometry (E,ω_0) on (M,D) is the meromorphic connection $(\mathcal{V},\nabla^{\omega_0})$ where $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$ and*
$$\nabla^{\omega_0} = \Phi_{\omega_0}^{-1}\nabla^{rec}$$
where Φ_{ω_0} is the isomorphism of $\mathcal{O}_E[\star\tilde{D}]$-modules between $TE[\star\tilde{D}]$ and $\mathcal{V}[\star\tilde{D}]$.

Hence, $\mathfrak{kill}_{E,\omega_0}$ and $\mathfrak{kill}_{M,\omega_0}$ are respectively local systems on $E \setminus \tilde{D}$ and $M \setminus D$.

As explained in the introduction, our goal is to classify quasi-homogeneous meromorphic Cartan geometries (E,ω_0). This hypothesis is satisfied whenever the base manifold M has only constant meromorphic functions (see [10]). In this case, there exists a point $x_0 \in M$ and n independent germs of Killing vector fields for (E,ω_0). We want to see for a sufficient condition for these germs to come from global Killing vector fields, i.e for the following property to be satisfied :

Definition 3.13. *Let (E,ω_0) be a meromorphic (G,P)-Cartan geometry on a pair (M,D). It satisfies the extension property of infinitesimal automorphisms if and only the local system $\mathfrak{kill}_{M,\omega_0}$ on $M \setminus D$ extends as a local system $\mathfrak{k} \subset TM$ on M.*

3.4 Distinguished foliations and totally geodesic meromorphic Cartan geometries

We will restrict our attention on the following subcategory of meromorphic Cartan geometries. Let (E,ω_0) be a meromorphic (G,P)-Cartan geometry on (M,D), and $A \in \mathfrak{g} \setminus \{0\}$. Since ω_0 induces an isomorphism of meromorphic bundles between $TE[\star\tilde{D}]$ and $\mathcal{O}_E[\star\tilde{D}]\otimes\mathfrak{g}$, there exists a unique distribution of rank one (thus integrable) $\mathcal{T}_A \subset TE$ with the following property :
$$\omega_0(\mathcal{T}_A) \subset \mathcal{O}_E[\star\tilde{D}]A \tag{15}$$

We will call it the A-**distinguished foliation** of (E,ω_0), and a leaf Σ will be called a A-**distinguished curve** for (E,ω_0).

Definition 3.14. *A meromorphic (G,P)-Cartan geometry (E,ω_0) on a pair (M,D) is holomorphically totally geodesic iff for a dense subset $W \subset \tilde{D}$, and any $e_0 \in W$, there exists $A \in \mathfrak{g} \setminus \mathfrak{p}$ and a smooth leaf Σ of $\mathcal{T}_A$ (Equation 15) with $\Sigma \cap \tilde{D} = \{e_0\}$.*

The following lemma show that it suffices to check the existence of one distinguished curve Σ as in Definition 3.14 for each irreducible component $\tilde{D}_\alpha$ of $\tilde{D}$:

Lemma 3.2. *Let (E,ω_0) be a meromorphic (G,P)-Cartan geometry on a pair (M,D), with D a smooth irreducible divisor. Let $e_0 \in \tilde{D} = p^{-1}(D)$, and suppose there exists $A \in \mathfrak{g} \setminus \mathfrak{p}$ and a leaf Σ of $\mathcal{T}_A$ through e_0 s.t. $\Sigma \cap \tilde{D} = \{e_0\}$. Then there exists a holomorphic foliation of an open neighboorhood U of $x_0 = p(e_0)$ whose leaves $(\overline{\Sigma}_x)_{x \in U}$ are holomorphic A-geodesics of (E,ω_0) satisfying $\overline{\Sigma}_x \cap D = \{x_0(x)\}$. Moreover, there exists a dense subset $W \subset D \cap U$ such that for any $x_0 \in W$, $\overline{\Sigma}_{x_0}$ is a smooth submanifold of U.*

Proof. Let's prove the first statement. Since there is a leaf Σ of $\mathcal{T}_A$ through e_0, there exists an open neighboorhood $\tilde{U}$ of e_0 s.t. $\mathcal{T}_A|_{\tilde{U}}$ is a regular foliation, and defined by a non vanishing holomorphic vector field Z_A on $\tilde{U}$.

The condition that $\Sigma \cap \tilde{D}$ is a discrete set of points in $\tilde{D}$ is equivalent to $Z_A(e_0) \notin T_{e_0}\tilde{D}$. Denoting by z_1 an equation of $\tilde{U} \cap \tilde{D}$, this means (see [5]) :

$$dim_{\mathbb{C}}\, \mathcal{O}_{E,e_0}/\langle \mathcal{L}_{Z_A}(z_1), z_1\rangle_{e_0} < \infty \tag{16}$$

Up to restriction of $\tilde{U}$, for any $e \in \tilde{U}$ there is an isomorphism of vector spaces ρ_e between $\mathcal{O}_E(\tilde{U})$ and $\mathcal{O}_{E,e}$ associating to any holomorphic function its germ at e. Evidently, ρ_e restricts as an isomorphism between the subspaces $\langle \mathcal{L}_{Z_A}(z_1), z_1\rangle(\tilde{U})$ and $\langle \mathcal{L}_{Z_A}(z_1), z_1\rangle_e$. Thus, the function $e \mapsto dim_{\mathbb{C}}\, \mathcal{O}_{E,e}/\langle \mathcal{L}_{Z_A}(z_1), z_1\rangle_e$ is constant on $\tilde{U}$. In particular, by the above remark, any leaf Σ' of $\mathcal{T}_A|_{\tilde{U}}$ satisfies $\Sigma' \cap \tilde{D} = \{e_1, \ldots, e_k\}$. Up to restriction of $\tilde{U}$, we can assume $k = 1$. In particular, the leaf Σ_e of $\mathcal{T}_A|_{\tilde{U}}$ through $e \in \tilde{U}$ intersects $\tilde{D}$ in an unique point $e'_0(e)$.

Now, let $U = p(\tilde{U}$ and pick any holomorphic section σ of E over U with values in $\tilde{U}$. Then the family $(\overline{\Sigma}_{x'_0})_{x'_0 \in D \cap U}$ of holomorphic curves defined by :

$$\overline{\Sigma}_{x'_0} = p(\Sigma_{x'_0}) \tag{17}$$

defines a holomorphic foliation on U, whose leaves are holomorphic geodesics for (E,ω_0). By construction, they satisfy the required property.

The second statement follow straightforwardly from the fact that, for any $x_0 \in D \cap U$, the tangent sheaf $T\overline{\Sigma}_{x_0}$ is the restriction $\mathcal{T}_A|_{\overline{\Sigma}_{x_0}}$. Since $\mathcal{T}_A$ is a locally free submodule of TU when restricted to te complement of a codimension 2 open subset, the statement follows. □

Moreover, locally, the isomorphism class of totally geodesic meromorphic Cartan geometries and the one of holomorphic totally geodesic Cartan geometries on a pair (M,D) coïncide :

Lemma 3.3. *Let (M,D) be a pair and (G,P) be a complex Klein geometry with $dim(G/P) = dim(M)$. Let (E,ω_0) be a totally geodesic meromorphic (G,P)-Cartan geometry on (M,D), and denote by $W \subset D$ the corresponding dense subset. Then for any $x_0 \in W$, there exists a neighboorhood U of x_0 in M, a holomorphically totally geodesic meromorphic (G,P)-Cartan geometry (E',ω_0) on (U,D) and an isomorphism $\Psi : E|_{U\setminus D} \longrightarrow E'|_{U\setminus D}$ between $(E|_U,\omega_0)$ and (E',ω'_0).*

Proof. Pick $x_0 \in W$. By definition, there exists $A \in \mathfrak{g} \setminus \{0\}$, and a meromorphic A-distinguished curve $\tilde{\gamma} : D(0,\epsilon) \longrightarrow E$ with $p \circ \tilde{\gamma}(0) = x_0$ and $p \circ \tilde{\gamma}(t) \notin D$ for $t \neq 0$. Denote by $\overline{\Sigma}$ the image of $\gamma = p \circ \tilde{\gamma}$, which is a smooth one dimensional submanifold of M by definition of W. Thus, there are local coordinates $(\overline{u}_1, \ldots, \overline{u}_n)$ on an open neighboorhood U of x_0 such that $\overline{\Sigma}$ is the common zero loci of $\overline{u}_2, \ldots, \overline{u}_n$. This enables to consider a section σ of E over $U \setminus D$, with the property $\sigma \circ \gamma = \tilde{\gamma}$, by assuming σ to be constant (with respect to any holomorphic section $\sigma_0 : U \longrightarrow E$) on level sets of $(\overline{u}_2, \dots, \overline{u}_n)$. A such section is $\mathfrak{g}$-meromorphic on (U, D), in the sense that the induced trivialisation $\psi_\sigma(\mathfrak{g}) : E(\mathfrak{g})|_{U \smallsetminus D} \longrightarrow \mathcal{O}_{U \smallsetminus D} \otimes \mathfrak{g}$ maps the $\mathcal{O}_U$-submodule $j^* E(\mathfrak{g})[\star D]$ (where $j : U \smallsetminus D \hookrightarrow U$ is the inclusion) to the $\mathcal{O}_U$-submodule $j^* \mathcal{O}_U[\star D] \otimes \mathfrak{g}$. Moreover, if X is a meromorphic vector field on U with pole at $D \cap U$, then the unique P-invariant holomorphic vector field $\hat{X}$ on $p^{-1}(U \smallsetminus D)$ satisfying $\hat{X} \circ \sigma = T\sigma(X)$ extends as a meromorphic vector field on $p^{-1}(U)$. Indeed, it suffices to check it for $X = \frac{\partial}{\partial \overline{u}_1}$, for which it follows from $\hat{X}|_\Sigma = h \circ p\omega_0^{-1}(A)$ with h a holomorphic function on U, and the equivariancy of ω_0.

We then define the trivial holomorphic P-principal bundle $E' = U \times P$, with canonical holomorphic section σ_0' and the isomorphism of holomorphic P-principal bundles $\Psi : E|_{U \smallsetminus D} \longrightarrow E'|_{U \smallsetminus D}$ defined by

$$\Psi \circ \sigma = \sigma_0'$$

Then the unique holomorphic (G, P)-Cartan connection on $E'|_{U \smallsetminus D}$ satisfying $\Psi^\star \omega_0' = \omega_0$ extends as a meromorphic Cartan connection on E'. Indeed, $(\sigma_0')^\star \omega_0' = \sigma^\star \omega_0$ is a meromorphic one form on U by the above remarks. Moreover, the intersection of te smooth submanifold $\Sigma' = \sigma_0'(\overline{\Sigma})$ with $E'|_{U \smallsetminus D}$ coïncide with $\Psi(\Sigma)$ by construction, thus Σ' is a A-distinguished curve. This means that $\overline{\Sigma}$ is a holomorphic geodesic for (E', ω_0'), which ends the proof. □

Lemma 3.4. *Let (E, ω_0) be a regular meromorphic parabolic geometry on (M, D), with model (G, P). Suppose (E, ω_0) is holomorphically totally geodesic and let W be the corresponding dense subset of D. Then for any $x_0 \in D$, there exists $e_0 \in p^{-1}(x_0)$, a vector $A \in \mathfrak{g}_{-1} \smallsetminus \{0\}$ and a leaf Σ of $\mathcal{T}_A$ through e_0 satisfies $\Sigma \cap \tilde{D} = \{e_0\}$.*

Proof. □

4 Infinitesimal automorphisms of meromorphic parabolic geometries

A classical result in riemaniann geometry states that any Killing vector field X for a riemannian metric g is a Jacobi field : for any geodesic γ, its scalar product $g(X(\gamma(t)), \gamma'(t)))$ with the velocity of γ is constant. There is a natural generalization of riemannian metrics to the holomorphic category, and the corresponding objects are equivalent to torsionfree holomorphic affine connections preserving a holomorphic reduction to the orthogonal group. The holomorphic version of the previous result can be seen as a result on some torsionfree holomorphic affine Cartan geometries (see Corollary 5.1). In this section, we will see a general result for meromorphic Cartan geometries. In particular, this will imply that the local system of infinitesimal automorphisms for any totally geodesic *regular meromorphic parabolic geometry* prolongates as a local system on the whole base manifold.

4.1 Regular meromorphic parabolic geometries

A **complex parabolic Klein geometry** is a complex Klein geometry (G, P) where G is a complex semi-simple Lie group, and P a parabolic subgroup. A **meromorphic parabolic geometry** is a meromorphic (G, P)-Cartan geometry for some complex parabolic Klein geometry. We refer the reader to [7] for a detailed introduction.

With the subgroup P is associated a grading $(\mathfrak{g}_i)_{i \in \mathbb{Z}}$ of the Lie algebra $\mathfrak{g} = Lie(G)$, meaning $[\mathfrak{g}_{i_1}, \mathfrak{g}_{i_2}]_\mathfrak{g} \subset \mathfrak{g}_{i+j}$ for any indices $i_1, i_2 \in \mathbb{Z}$. We call it the **parabolic graduation** associated with P. It induces a grading of any representation of G, in particular $\mathbb{W} = (\bigwedge^2 \mathfrak{g}_-^*) \otimes \mathfrak{g}$ is graded by **homogeneous degrees** $\mathbb{W}_l$, and we denote by π_l the corresponding projections.

The **parabolic degree** of (G, P) is the smallest positive integer $k \geq 1$ such that $\mathfrak{g}_i = \{0\}$ for any $|i| > k$. The subspaces $\mathfrak{p} = Lie(P)$ and the subspace

$$\mathfrak{g}_- = \bigoplus_{i=-k}^{1} \mathfrak{g}_i \tag{18}$$

are clearly subalgebras of $\mathfrak{g}$. For any $i \in \{-k, \dots, k\}$, we will denote $\mathfrak{g}^i = \bigoplus_{i' \geq i} \mathfrak{g}_{i'}$, inducing a filtration $(\mathfrak{g}^i)_{i=-k,\dots,k}$ of $\mathfrak{g}$.

By a result of C. Chevalley, we can always pick a basis $(\mathfrak{e}^i_j)_{\substack{i=-k,\dots,k \\ j=1,\dots,n_i}}$ of $\mathfrak{g}$, such that $(\mathfrak{e}^i_j)_{j=1,\dots,n_i}$ is a basis for $\mathfrak{g}_i$ for any $i \in \{-k, \dots, k\}$, and $[\mathfrak{e}^{i_1}_{j_1}, \mathfrak{e}^{i_2}_{j_2}]_{\mathfrak{g}}$ is either 0 or $\mathfrak{e}^{i_1+i_2}_j$ for some $j \in \{1, \dots, n_{i_1+i_2}\}$. We will refer to it as a **graded basis** of $\mathfrak{g}$ for (G, P).

The homogeneous space G/P associated with a complex parabolic Klein geometry (G, P) bears the following holomorphic geometric structures. Its tangent bundle is filtered by subbundles $(T^{-i}G/P)_{i=1,\dots,k}$ where $T^{-i}G/P$ is the projection of $\omega_G^{-1}(\mathfrak{g}^i)$ through the map $Tp_{G/P}$, where $p_{G/P} : G \longrightarrow G/P$ is the natural projection. The Lie bracket of holomorphic vector fields on G/P induces a Lie bracket of holomorphic vector bundle on the corresponding graded bundle $gr(TG/P)$. The Lie algebra bundle thus obtained is locally isomorphic to $(U \times \mathfrak{g}_-, [,]_{\mathfrak{g}_-})$.

The **regular meromorphic parabolic geometries** are the infinitesimal versions of this model. More precisely, these are meromorphic (G, P)-Cartan geometries (E, ω_0) on (M, D) for which the homogeneous component $\pi_l(k_{\omega_0})$ of degree l of the Cartan curvature vanishes identically whenever $l \leq 0$ (see above). This amounts to the following property. Let $T^{-i}M[\star D]$ be the image of $\omega_0^{-1}(\mathfrak{g}^{-i})[\star \tilde{D}]$ through Tp. This gives a filtration of $TM[\star \tilde{D}]$, and (E, ω_0) is regular if and only if the Lie bracket of vector fields on M induces a structure of Lie algebras bundle on the graded $gr(TM \setminus D)$, locally isomorphic to $(U \times \mathfrak{g}_-, [,]_{\mathfrak{g}_-})$.

4.2 Bott connections and infinitesimal automorphisms of Cartan geometries

Now, we come back to a general complex Klein geometry (G, P). Let (M, D) be a complex pair of dimension $n \geq 1$, and (E, ω_0) be a meromorphic (G, P)-Cartan geometry on it. Fix $A \in \mathfrak{g} \setminus \{0\}$ and consider the holomorphic foliation $\mathcal{T}_A$ from Equation 15. To any such holomorphic foliation is associated a $\mathcal{T}_A$-partial holomorphic connection $\nabla^{\mathcal{T}_A}$ on $TE/\mathcal{T}_A$, the **Bott-connection** of $\mathcal{T}_A$, defined as follow. Let X be a holomorphic vector field on $U \subset E$, $[X]$ its class in $TE/\mathcal{T}_A(U)$, and $Z \in \mathcal{T}_A(U)$. Then :

$$Z \neg \nabla^{\mathcal{T}_A}([X]) = [[Z, X]_{TE}] \tag{19}$$

Let $t \in \mathbb{C}$ and $V \subset U$ such that the flow $\phi = \phi^t_Z$ is well defined on V. Then clearly $d\phi(\mathcal{T}_A) \subset \phi^* \mathcal{T}_A$, so ϕ induces a morphism $[d\phi]$ of $\mathcal{O}_V$-modules defined by the commutative diagram :

$$\begin{tikzcd} TV \arrow[r, "d\phi"] \arrow[d, "q"'] & \phi^* T\phi(V) \arrow[d, "\phi^* q"] \\ TV/\mathcal{T}_A \arrow[r, "{[d\phi]}"'] & \phi^* T\phi(V)/\mathcal{T}_A \end{tikzcd} \tag{20}$$

By the formula (19), the horizontal sections for $\nabla^{\mathcal{T}_A}$ are the $[X]$ which are invariant by the isomorphisms of holomorphic vector bundle $(\phi, [d\phi])$ defined as before.

It will be more convenient to work with the images of meromorphic vector fields on E through the isomorphism Φ_{ω_0} between $TE[\star \tilde{D}]$ and $\mathcal{V}[\star \tilde{D}]$, where $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$. We will write :

$$\mathcal{K} = \Phi_{\omega_0}(\mathfrak{kill}_{E,\omega_0}) \tag{21}$$

for the corresponding local system on $E \setminus \tilde{D}$. Clearly, the image of $\mathcal{T}_A[\star \tilde{D}]$ is $\mathcal{V}_A = \mathcal{O}_E[\star \tilde{D}]A$. The class of a section s of $\mathcal{V}[\star \tilde{D}](U)$ (where $U \subset E$ is an open subset) in $\mathcal{V}/\mathcal{V}_A[\star \tilde{D}]$ will be denoted by $[s]_{\mathcal{V}/\mathcal{V}_A}$. Since Φ_{ω_0} induces an isomorphism of $\mathcal{O}_E$-modules between $TE/\mathcal{T}_A[\star \tilde{D}]$ and $\mathcal{V}/\mathcal{V}_A[\star \tilde{D}]$, for any $Z \in \mathcal{T}_A(U)$, the morphism $[d\phi]$ defined by (20) corresponds to an isomorphism

$$\overline{d\phi} : \mathcal{V}/\mathcal{V}_A[\star \tilde{D}]|_V \longrightarrow \phi^* \mathcal{V}/\mathcal{V}_A[\star \tilde{D}]|_{\phi(V)} \tag{22}$$

and thus an isomorphism $(\phi, \overline{d\phi})$ of meromorphic bundles.

The isomorphism of meromorphic bundles Φ_{ω_0} (see above) maps $\mathcal{T}_A[\star \tilde{D}]$ to $\mathcal{V}_A[\star \tilde{D}]$, and we denote by $\overline{\Phi}_{\omega_0} : TE/\mathcal{T}_A[\star \tilde{D}] \longrightarrow \mathcal{V}/\mathcal{V}_A[\star \tilde{D}]$ the isomorphism induced by Φ_{ω_0}. Then :

Lemma 4.1. *Let s be a section of $\mathcal{K}$ on an open subset $U \subset E \setminus \tilde{D}$. Then its class $[s]_{\mathcal{V}/\mathcal{V}_A}$ is invariant by any isomorphism of meromorphic bundles $(\phi, \overline{d\phi})$ constructed as above.*

Proof. Let X be any holomorphic vector field on $U \subset E$, and $[X]$ its class in $TE/\mathcal{T}_A$. By definition, for any $Z_A = h\tilde{A}$ (where h is a meromorphic function on U and $\tilde{A} = \omega_0^{-1}(A)$) we have :

$$\begin{aligned} 0 &= [\tilde{A}, X]_{TE} \\ &= \frac{1}{h}[Z_A, X]_{TE} \qquad \mathrm{mod} \quad \mathcal{T}_A[\star \tilde{D}](U) \end{aligned} \tag{23}$$

In other words, the classes of $d\phi(X)$ and $\phi^* X$ in $TE/\mathcal{T}_A[\star\tilde{D}]$, well defined on $U \cap \phi(U)$, coïncides i.e s is invariant by $(\phi, \overline{d\phi})$. □

Now, we suppose M to be simply connected. We wish to prove the extension property for (E, ω_0) (Definition 3.13). We will use the following general fact on meromorphic Cartan geometries :

Proposition 4.0.1. *Let (G,P) be a complex Klein geometry, and (M,D) be a pair with $dim(M) = dim(G/P)$. Let (E,ω_0) be a meromorphic (G,P)-Cartan geometry on (M,D). Let $x_0 \in D$ belonging to an unique irreducible component D_α, and suppose that there exists $A \in \mathfrak{g} \setminus \mathfrak{p}$ and a A-geodesic $\overline{\Sigma}$ for (E,ω_0) such that $\overline{\Sigma} \cap D_\alpha = \{x_0\}$. Then there exists a neighborhood U of x_0 with the following properties:*

1. *Let s be a section of $\mathcal{K}$ on $V \subset p^{-1}(U)$. Then the image $\mu \in GL(\mathcal{V}_e)$ of any loop at $e \in V$ in $p^{-1}(U \setminus \tilde{D})$ by the local monodromy of $\mathcal{K}$ (at e) satisfies : $[\mu(s_e)]_{\mathcal{V}/\mathcal{V}_A} = [s_e]_{\mathcal{V}/\mathcal{V}_A}$.*
2. *For any section s of $\mathcal{K}$ on $p^{-1}(U \setminus D)$, $[s]_{\mathcal{V}/\mathcal{V}_A}$ extends as a section of $\mathcal{V}/\mathcal{V}_A[\star\tilde{D}](p^{-1}(U))$. In other words $j_*\pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K}) \subset \mathcal{V}/\mathcal{V}_A$ where j is the inclusion of $E \setminus \tilde{D}$ in E and $\pi_{\mathcal{V}/\mathcal{V}_A}$ the morphism which map s to its class $[s]_{\mathcal{V}/\mathcal{V}_A}$.*

Proof.

1. It is a classical result that the image $\mathcal{K}_A = \pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K})$ of the local system $\mathcal{K}$ on $E \setminus \tilde{D}$ is a local system on $E \setminus \tilde{D}$. The image $\mu_A \in GL((\mathcal{K}_A)_e)$ of any loop of $E \setminus \tilde{D}$ at e is given by :

   $$\pi_{\mathcal{V}/\mathcal{V}_A} \circ \mu = \mu_A \circ \pi_{\mathcal{V}/\mathcal{V}_A} \tag{24}$$

   By the correspondance between local systems and their monodromy, it is thus sufficient to prove that $\mathcal{K}_A$ is a constant sheaf on a open subset $\tilde{U} \setminus \tilde{D}$ where $\tilde{U}$ is a neighboorhood of $e_0 \in p^{-1}(x_0)$.

   By Lemma 3.3, we can suppose, without loss of generality, that $\overline{\Sigma}$ is a holomorphic geodesic for (E,ω_0). By Lemma 3.2, there exists an open neighborhood $\tilde{U}_0$ of some point $e_0 \in p^{-1}(x_0)$ such that any leaf of $\mathcal{T}_A|_{\tilde{U}_0}$ intersects $\tilde{D}$ in exactly one point. Equivalently, there exists a holomorphic nonvanishing vector field Z defining $\mathcal{T}_A|_{\tilde{U}_0}$ such that $\phi_Z^t(e_0') \notin \tilde{D}$ for any $e_0' \in \tilde{D} \cap \tilde{U}_0$ and $t \neq 0$ with $|t|$ small enough. Pick a point $e = \phi_Z^{-t}(e_0)$ of the leaf Σ through e_0, which does'nt belong to $\tilde{D}$. Thus, there is a simply connected neighborhood V of e such that $\mathcal{K}_A$ is a constant sheaf, and thus admits a basis $[s_1]_{\mathcal{V}/\mathcal{V}_A}, \ldots, [s_r]_{\mathcal{V}/\mathcal{V}_A}$.

   By the Lemma 4.1, the family $\overline{d\phi}([s_1]_{\mathcal{V}/\mathcal{V}_A}), \ldots, \overline{d\phi}([s_r]_{\mathcal{V}/\mathcal{V}_A})$, where $\overline{d\phi}$ is the morphism (22), is a basis of $\phi^*\mathcal{K}_A(\phi(V) \setminus \tilde{D})$. Then $\mathcal{K}_A|_{\phi(V)}$ is a constant sheaf. The open neighborhood $U = p(\phi(V))$ of x_0 thus satisfies the required property.

2. Since(22) is an isomorphism of meromorphic bundles, we have proved in 1. that the local system $\mathcal{K}_A = \pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K})$ extends as a constant sheaf, included in $\mathcal{V}/\mathcal{V}_A[\star\tilde{D}]|_{p^{-1}(U)}$ since $(\phi, \overline{d\phi})$ is an automorphism of meromorphic bundles for $\mathcal{V}/\mathcal{V}_A$.

□

4.3 Affine and degree one parabolic models

We know apply Proposition 4.0.1 to prove the extension property for infinitesimal automorphisms of some totally geodesic meromorphic (G,P)-Cartan geometries. More precisely, we let (G,P) be a complex parabolic Klein geometry of dimension $n \geq 2$ and degree $k = 1$ (see subsection 4.1), or the complex affine Klein geometry of dimension $n \geq 2$. For the first model, we denote by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the parabolic graduation associated with P (see subsection 4.1). For the second one, $\mathfrak{g}_{-1}$ will stand for the abelian subalgebra of infinitesimal generators for the translations in $\mathbb{C}^n$.

These two kind of models are of special interest beacause P acts transitively on $\mathbb{P}(\mathfrak{g}/\mathfrak{p})$ through the projectivized quotient adjoint action. Indeed, if (G,P) is a degree one parabolic model, for any $A \in \mathfrak{g}_{-1} \smallsetminus \{0\}$, $Ad(\mathfrak{g}_0)[A]$ must span $\mathfrak{g}_{-1}$, since $\mathfrak{g}$ is semi-simple and $\mathfrak{g}_{-1}$ is an abelian subalgebra. If (G,P) is the affine model, the isotropy group $P = GL_n(\mathbb{C})$ clearly acts transitively on $\mathfrak{g}_{-1}$.

By the above remarks, for any $A \in \mathfrak{g}_{-1} \smallsetminus \{0\}$, there exists a basis $(\mathfrak{e}_i)_{i=1,\dots,n}$ of the abelian subalgebra $\mathfrak{g}_{-1} \subset \mathfrak{g} = Lie(G)$, and n elements $b_1, \dots, b_n \in P$ such that

$$ad(b_i^{-1})[A] \in \mathbb{C}\mathfrak{e}_i \tag{25}$$

for $1 \leq i \leq n$.

Lemma 4.2. *Let (E,ω_0) be a meromorphic (G,P)-Cartan geometry on (M,D), and let $x_0 \in D$ belonging to a unique irreducible component D_α. Suppose there exists a geodesic $\overline{\Sigma}$ for (E,ω_0) with $\overline{\Sigma} \cap D = \{x_0\}$. Then, for any $1 \leq i \leq n$, there exists a $\mathfrak{e}_i$-geodesic $\overline{\Sigma}_i$ for (E,ω_0) with $\overline{\Sigma}_i \cap D = \{x_0\}$.*

Proof. By Lemma 3.3, we can suppose, without loss of generality, that $\overline{\Sigma}$ is a A-holomorphic geodesic for $A \in \mathfrak{g}_- \smallsetminus \{0\}$. Thus, there exists $e_0 \in p^{-1}(x_0)$ and a A-distinguished smooth complex curve Σ with $p(\Sigma) = \overline{\Sigma}$ and $\Sigma \cap \tilde{D} = \{e_0\}$. Let $b_1, \dots, b_n \in P$ as in (25). By equivariancy of ω_0, for any $1 \leq i \leq n$, the b_i-translated Σ_i of Σ is a $\mathfrak{e}_i$-distinguished smooth complex curve, with $\Sigma_i \cap \tilde{D} = \{e_0 \cdot b_i\}$. By Lemma 3.2, there is an open neighboorhood U_i of $e_0 \cdot b_i$ in E such that for any $e \in U_i$, the leaf Σ_i of $\mathcal{T}_{\mathfrak{e}_i}|_{U_i}$ through e intersects $\tilde{D}$ in an unique point e_0'. In fact, $U_i \cap p^{-1}(x_0)$ is a Zariski-dense subset of the fiber $p^{-1}(x_0)$, so that $\tilde{V}_{x_0} = \bigcap_{i=1}^{n} U_i \cap p^{-1}(x_0)$ is a Zariski dense subset of $p^{-1}(x_0)$. Pick any $e_0' \in \tilde{V}$. Then by construction, for any $1 \leq i \leq n$ there is a $\mathfrak{e}_i$-distinguished smooth curve Σ_i' with $\Sigma_i' \cap \tilde{D} = \{e_0'\}$. Their projections $\overline{\Sigma}_i'$ throuhg p are $\mathfrak{e}_i$-geodesics for (E,ω_0), and the proof is thus achieved. ☐

Corollary 4.1. *Let (E,ω_0) be a totally geodesic meromorphic (G,P)-Cartan connection on (M,D), with M simply connected. Then :*

1. *(E,ω_0) satisfies the extension property for the infinitesimal automorphisms.*
2. *Suppose moreover that (E,ω_0) is branched holomorphic on (M,D). Then any section s of $ker(\nabla^\kappa_{\omega_0})(U)$, where $U \subset E$ is an open subset, is a section of $\mathcal{V}(U)$.*

Proof. 1. Since the complement of a codimension 2 subset of M has the same fundamental group as M, and in vertue of the equivalence between local systems and representations of the fundamental group, it suffices to find a codimension 1 subset W of D, pick a point x_0 on an unique irreducible component D_α of $D \cap W$, and show the existence of a neighboorhood U of x_0 in M such that the restriction of $ker(\nabla^\kappa_{\omega_0})$ to $p^{-1}(U \smallsetminus D)$ extends as a local system on $p^{-1}(U)$, included in $TE[\star D]|_U$. The property on $\mathfrak{kill}_{M,\omega_0}$ is clearly invariant under isomorphisms of meromorphic Cartan geometries, so we assume, without loss of generality, that (E,ω_0) is holomorphically totally geodesic (Lemma 3.3).

Pick $x_0 \in W \cap D_\alpha$, where W is the dense subset in D of Definition 3.14. Thus, there exists a holomorphic geodesic $\overline{\Sigma}$ of (E,ω_0) with $\overline{\Sigma} \cap D = \{x_0\}$. We now apply Lemma 4.2 to obtain, for any $1 \leq i \leq n$, a holomorphic $\mathfrak{e}_i$-geodesic $\overline{\Sigma}_i$ with $\overline{\Sigma}_i \cap D = \{x_0\}$. More precisely, the proof of the lemma implies the existence of $e_0 \in p^{-1}(x_0)$ such that the $\mathfrak{e}_i$-distinguished curve Σ_i projecting onto $\overline{\Sigma}_i$ satisfies $\Sigma_i \cap \tilde{D} = \{e_0\}$. Using the Proposition 4.0.1 for each geodesic, we obtain neighboorhoods U_i of x_0 such that the restriction of the local system $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_i}}(ker(\nabla^\kappa_{\omega_0}))$ to $p^{-1}(U_i)$ is a constant sheaf.

Let $U = \bigcap_{i=1}^{n} U_i$. Since $\mathfrak{e}_1, \mathfrak{e}_2$ are independant vectors of $\mathfrak{g}$, the morphism of $\mathcal{O}_E$-modules :

$$\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}} \oplus \pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}} \quad : \quad \mathcal{V}[\star \tilde{D}] \quad \longrightarrow \quad \mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}[\star \tilde{D}] \oplus \mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}[\star \tilde{D}] \tag{26}$$

is an isomorphism onto its image. Thus, it restricts to $ker(\nabla^{\kappa}_{\omega_0})$ as an isomorphism of $\mathbb{C}$-sheaves onto its image, a subsheaf of the local system $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}}(ker(\nabla^{\kappa}_{\omega_0})) \oplus \pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}}(ker(\nabla^{\kappa}_{\omega_0}))$. By the above remark, this local system is a constant sheaf when restricted to $p^{-1}(U)$. Thus, the same is true for $ker(\nabla^{\kappa}_{\omega_0})$, i.e (E, ω_0) satisfies the extension property for the infinitesimal automorphisms.

2. Since (E, ω_0) is a branched holomorphic Cartan geometry, the isomorphism of meromorphic bundles (22) is in fact an isomorphism of holomorphic vector bundles from $\mathcal{V}/\mathcal{V}_A$ to itself. Applying this remark to $A = \mathfrak{e}_1$ and $A = \mathfrak{e}_2$, we obtain that the image of $ker(\nabla^{\kappa}_{\omega_0})$ through $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}}$ and $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}}$ respectively extends as subsheaves of $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}$ and $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}$ on E. Since the morphism (26) clearly restricts to a morphism between $\mathcal{V}$ and $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1} \oplus \mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}$, this proves the assertion.

□

4.4 Parabolic geometries of higher degree

Now, we let (G, P) be a complex parabolic Klein geometry of degree $k > 1$, and denote by $\mathfrak{g}_{-k} \oplus \ldots \mathfrak{g}_0 \oplus \ldots \mathfrak{g}_k$ the parabolic graduation. We refer the reader to [7] for the definitions and a complete introduction on this subject.

The group P no longer acts transitively on $\mathbb{P}(\mathfrak{g}/\mathfrak{p})$. Instead, we use the principal result of [6] which implies the following :

Lemma 4.3. *Let (E, ω_0) be a regular meromorphic (G, P)-Cartan geometry on a pair (M, D). Then there exists a morphism of $\mathbb{C}$-sheaves :*

$$\mathcal{L} \quad : \quad \mathcal{V}_{\mathfrak{g}_{-k}}[\star \tilde{D}] \quad \longrightarrow \quad \mathcal{V}[\star \tilde{D}] \tag{27}$$

with the following properties :

(i) Let $\pi_{-k} : \mathcal{V}[\star \tilde{D}] \longrightarrow \mathcal{V}_{\mathfrak{g}_{-k}}[\star \tilde{D}]$ be the projection on $\mathcal{V}_{\mathfrak{g}_{-k}}$ with respect to $\mathcal{V}_{\mathfrak{g}^{-k+1}}[\star \tilde{D}]$. Then $\pi_{-k} \circ \mathcal{L} = Id_{\mathcal{V}_{\mathfrak{g}_{-k}}}$.

(ii) The restriction of $\mathcal{L} \circ \pi_{-k}$ to $ker(\nabla^{\kappa}_{\omega_0})$ is the identity on $ker(\nabla^{\kappa}_{\omega_0})$.

Proof. The Theorem 4 in [6] is exactly the regular version of this lemma, i.e when D is empty. Its proof uses only differential operators constructed with the de Rham differential on trivial modules, and morphisms of modules obtained by tensorizing a linear map with the identity on holomorphic functions. Thus, it straightforwardly extends to the meromorphic category since such operators preserves the sheaves of meromorphic sections. □

Corollary 4.2. *Let (E, ω_0) be a regular meromorphic (G, P)-Cartan geometry on a pair (M, D). Suppose that for any irreducible component D_α of D, there exists $A \in \mathfrak{g}_- \setminus \mathfrak{g}_{-k}$ and a A-geodesic $\overline{\Sigma}$ of (E, ω_0) with $\overline{\Sigma} \cap D_\alpha = \{x_0\}$. Then (E, ω_0) satisfies the extension property for the infinitesimal automorphisms.*

Proof. The property of regularity is invariant under isomorphisms of meromorphic Cartan geometries, so we assume as in the proof of Corollary 4.1 that (E, ω_0) is holomorphically totally geodesic without losing any generality. We pick $x_0 \in W \cap D_\alpha$, where W is the dense subset of D in Definition 3.14 and D_α the unique irreducible component of D containing x_0, and prove that $ker(\nabla^{\kappa}_{\omega_0})$ extends as constant $\mathbb{C}$-subsheaf of $\mathcal{V}[\star \tilde{D}]$ when restricted to some neighboorhood $p^{-1}(U)$ of $e_0 \in p^{-1}(x_0)$.

By definition of W, there exists $A \in \mathfrak{g}_- \setminus \mathfrak{g}_{-k}$ and a A-holomorphic geodesic $\overline{\Sigma}$ of (E, ω_0) with $\overline{\Sigma} \cap D = \{x_0\}$, and thus a leaf Σ of $\mathcal{T}_A$ with $\Sigma \cap \tilde{D} = \{e_0\}$ for some $e_0 \in p^{-1}(x_0)$. We now apply the Proposition 4.0.1 to the A-holomorphic geodesic $\overline{\Sigma}$. Since $k > 1$, $\mathbb{C}A$ and $\mathfrak{g}_{-k}$ are independant subspaces in $\mathfrak{g}$. Thus, the projection $\pi_{-k}(ker(\nabla^{\kappa}_{\omega_0}))$ extends as a constant $\mathbb{C}$-subsheaf of $\mathcal{V}_{\mathfrak{g}_{-k}}[\star \tilde{D}]$ on a neighboorhood U of e_0. The image of a constant sheaf by a morphism of $\mathbb{C}$-sheaves is a constant sheaf, so by Lemma 4.3, $ker(\nabla^{\kappa}_{\omega_0})$ extends as a constant $\mathbb{C}$-subsheaf of $\mathcal{V}[\star \tilde{D}]$ on U. The proof is then achieved. □

5 Application to the classification of meromorphic affine connections

5.1 Equivalence between meromorphic principal connections and meromorphic connections

We now prove the equivalence between meromorphic connections on a locally free $\mathcal{O}_M$-module $\mathcal{E}$ and meromorphic principal connections on its frame bundle E. It straightforwardly restricts as an equivalence between meromorphic connections preserving a holomorphic reduction $E_1 \subset E$ to a subgroup $P \subset GL_r(\mathbb{C})$, and meromorphic P-principal connections on E_1. In the regular setting, this was first proved by C.Erhesmanh ([13]) using the formalism of horizontal lifts for paths, and reformulated in an equivariant way by M. Atiyah ([1]). We adopt the point of view of M. Atiyah in order to extend the result to the meromorphic category.

The starting point is that for $P = GL_r(\mathbb{C})$, there is a canonical isomorphism ([1], Proposition 9) :

$$E(\mathfrak{p}) = End(\mathcal{E}) \tag{28}$$

There is a bijection between the set of meromorphic connections ∇ on $\mathcal{E}$ and the one of $\mathcal{O}_M$-linear splittings $\delta : \mathcal{E}[\star D] \longrightarrow J^1(\mathcal{E})[\star D]$ of the exact sequence of $\mathbb{C}$-sheaves :

$$0 \longrightarrow \Omega^1_M[\star D] \otimes \mathcal{E} \longrightarrow J^1(\mathcal{E})[\star D] \longrightarrow \mathcal{E}[\star D] \longrightarrow 0 \tag{29}$$

Let $\sigma : U \longrightarrow E$ be a holomorphic section of the holomorphic frame bundle. This corresponds to a basis $(s_1, \dots, s_r)$ of $\mathcal{E}|_U$, and we denote in the following lines by d the pullback of the de Rham differential through the corresponding isomorphism $\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus r}$. The former equivalence is given by $\nabla = d - \delta$. Indeed, this clearly defines a meromorphic connection, and if ∇ is a meromorphic connection on $\mathcal{E}|_U$, then $\delta_1 = d - \nabla$ is a morphism of $\mathcal{O}_U$-modules from $\mathcal{E}|_U[\star D]$ to $\Omega^1_U[\star D] \otimes \mathcal{E}|_U$, and we obtain a splitting $\delta = (Id_{\mathcal{E}|_U}, \delta_1)$ of (29).

Definition 5.1. *A meromorphic principal connection on a holomorphic $GL_r(\mathbb{C})$-principal bundle $E \xrightarrow{p} M$ with poles at $\tilde{D} = p^{-1}(D)$ (shortly on $(E, \tilde{D})$) is a meromorphic one form $\tilde{\omega}$ on E with values in $\mathfrak{p}$, which is P-equivariant and such that $\tilde{\omega}$ coïncides with the Maurer-Cartan form of P when restricted to any fiber $p^{-1}(x) \subset E$.*

Using the correspondance for equivariant one forms as in subsection 2.2, a meromorphic P-principal connection on $(E, \tilde{D})$ is equivalent to a morphism $\beta : At(E)[\star D] \longrightarrow E(\mathfrak{p})[\star D]$ such that $\iota \circ \beta = Id_{At(E)}$, where ι is defined in (7). Its kernel defines a splitting

$$\tau : TM[\star D] \longrightarrow At(E)[\star D] \tag{30}$$

of (7), which uniquely determines β. The following lemma straightforwardly follows from the equivalence described before between equivariant morphisms of modules over principal bundles and morphisms between the corresponding modules over the base manifolds :

Lemma 5.1. *Let (M_1, D_1) and (M_2, D_2) be two pairs of same dimension. Let $\tilde{\Psi} : E_1 \longrightarrow E_2$ be an isomorphism of holomorphic P-principal bundles over M_1 and M_2 covering a morphism of pairs $\varphi : M_1 \longrightarrow M_2$ (i.e $\varphi(D_1) = D_2$). Let $\tilde{\omega}_2$ be a meromorphic principal connection on $(E_1, \tilde{D}_1)$ where $\tilde{D}_1$ is the preimage of D_1 (resp. $\tilde{\omega}_1 = \tilde{\Psi}^\star \tilde{\omega}_1$), and τ_1 (resp. τ_2) be the splitting as in* (30)*. Then the diagram below is commutative :*

$$\begin{tikzcd} TM_1 \arrow[r, "\tau_1"] \arrow[d, "d\varphi"'] & At(E_1)[\star D_1] \arrow[d, "p_{1\star} d\tilde{\Psi}"] \\ \varphi^\star TM_2 \arrow[r, "\varphi^\star \tau_2"'] & \varphi^\star At(E_2)[\star D_1] \end{tikzcd} \tag{31}$$

where p_1 is the footmap of E_1.

Denote by $\tilde{d}$ the usual de Rham differential on $\mathcal{O}_E[\tilde{D}]\otimes\mathbb{V}$. Since the P-linearization $(\phi_b^{\mathbb{V}})_{b\in P}$ preserves the subsheaf of constant functions with values in $\mathbb{V}$ on E, the pushforward $p_*\tilde{d}$ restricts to $p_*\tilde{d} : \mathcal{E} \longrightarrow p_*\Omega^1_E \otimes \mathcal{E}$. This defines a meromorphic connection ∇ on $\mathcal{E}$ by :

$$\nabla = \tau \neg p_* \tilde{d} \tag{32}$$

where $\tilde{d}$ is defined above and $\neg$ stands for the contraction by a vector field.

Proposition 5.0.1. *Mapping a meromorphic principal connection $(E,\tilde{\omega})$ over (M,D) to the meromorphic connection $(\mathcal{E},\nabla)$ on (M,D) defined by* (32) *induces an equivalence of categories between :*

- *The category of principal meromorphic (resp. holomorphic) connections over pairs, where the arrows are the $\mathbb{C}^r$-meromorphic isomorphisms (Definition 2.3 of principal bundles between pairs preserving the principal connections (resp. isomorphisms of holomorphic principal bundles preserving the principal connections)*

- *The category of meromorphic (resp. holomorphic) connections on (M,D) with isomorphisms of meromorphic bundle (resp. holomorphic vector bundles, see Definition 2.2) preserving connections (in the sense of* (4)*).*

Proof. Let's first prove that this maps induces a functor. Let $\tilde{\Psi} : E_1 \longrightarrow E_2$ be an isomorphism of meromorphic principal connections between $(E_1,\tilde{\omega}_1)$ and $(E_2,\tilde{\omega}_1)$ over (M,D) and $(\mathcal{E}_1,\nabla_1)$ and $(\mathcal{E}_2,\nabla_2)$ obtained as in (32). Let (φ,Φ) be the associated isomorphism of vector bundles (see (3)). Fix any open subset $U \subset M_1$ and a basis $(s_i)_{i=1,\ldots,r}$ of $\mathcal{E}_1|_U$ and denote by $(\varphi^* t_i)_{i=1,\ldots,r}$ its image through Φ. Denote by $(\tilde{s}_i)_{i=1,\ldots,r}$ and $(\tilde{t}_{i=1,\ldots,r})$ respectively the corresponding equivariant functions on $p_1^{-1}(U)$ and $p_2^{-1}(\varphi(U))$. Thus $\tilde{t}_i = \tilde{s}_i \circ \tilde{\Psi}$ by definition of Φ. By definition of $\Phi^{-1}\varphi^\star\nabla_2$, we can compute :

$$\Phi^{-1}\varphi^\star\nabla_2(s_i) \quad = \quad (Id_{\Omega^1_{M_1}} \otimes \Phi^{-1})[d\varphi\neg(\varphi^\star\nabla_2(\varphi^* t_i))] \tag{33}$$

Using the definition of ∇_1 and ∇_2, and Lemma 5.1, we get :

$$\begin{aligned} \Phi^{-1}\varphi^\star\nabla_2(s_i) &= (Id_{\Omega^1_{M_1}} \otimes \Phi^{-1})[(\varphi^*\tau_2 \circ d\varphi)\neg\varphi^* p_{2*}\tilde{d}_2(\tilde{t}_i))] \\ &= \tau_1\neg(p_{1*}\tilde{d}_1\tilde{t}_i \circ \tilde{\Psi}) \\ &= \nabla_1(s_i) \end{aligned} \tag{34}$$

where we denoted by $\tilde{d}_1$ and $\tilde{d}_2$ the usual deRham differentials on $\mathcal{O}_{E_1}\otimes\mathbb{C}^r$ and $\mathcal{O}_{E_2}\otimes\mathbb{C}^r$. Hence we can map $\tilde{\Psi}$ to the vector bundle isomorphism (φ,Φ) which preserves the linear meromorphic connections ∇_1 and ∇_2.

Now, we construct the pseudo-inverse. Let $(\mathcal{E},\nabla)$ be a meromorphic connection over a pair (M,D). Denote by E its frames bundle. Let $x \in M$ and U be a neighbhoorhood equipped with a holomorphic section $\sigma : U \longrightarrow E$. Denote by $(s_1,\ldots,s_r)$ the corresponding basis of $\mathcal{E}|_U$. The section σ induces a splitting $TE|_{p^{-1}(U)} = p^*TU \oplus ker(dp)$ which is P-equivariant, hence a splitting

$$At(E)|_U = TU \oplus E(\mathfrak{p})|_U \tag{35}$$

We denote by τ_0 the splitting of the exact sequence (7) restricted to U induced by (35), and by d the pullback of the deRham differential through the trivialization associated with $(s_i)_{i=1,\ldots,r}$. Let $\delta = \nabla - d$, which vanishes on the image of $E(\mathfrak{p})$ through ι (see (7)). Its kernel thus define a morphism $\Theta : TU \longrightarrow At(E)|_U[\star D]$, and we obtain a splitting

$$\tau = \tau_0 + \Theta \tag{36}$$

of (7) over U. From the remarks above, this is equivalent to a meromorphic principal connection $\tilde{\omega}_U$ on $p^{-1}(U)$ with poles at $\tilde{D} \cap p^{-1}(U)$.

Now, let U, U' be two open subset and $(s_i)_{i=1,\dots,r}$ and $(s'_i)_{i=1,\dots,r}$ be two basis of $\mathcal{E}|_U$ and $\mathcal{E}|_{U'}$ corresponding to holomorphic sections σ, σ' of E on U and U'. Let d and d' be the corresponding de Rham differentials, then :

$$d - d'(s'_i) = d(s'_i) = d(\sum_{j=1}^{r} b^{-1}_{ji} s_j) = \sum_{j=1}^{r} (b d_0(b^{-1}))_{ji} s'_j \tag{37}$$

where b is the meromorphic function on $U \cap U'$ with values in P such that $\sigma' = \sigma \cdot b$, and d_0 is the usual de Rham differential on $\mathfrak{p}$-valued functions. Denote by τ and τ' constructed as before. Thus :

$$\tau' - \tau = [(\sigma, b^\star \omega_P)] \tag{38}$$

Thus $\nabla' - \nabla = d' - d + \tau' - \tau = 0$ and the corresponding meromorphic principal connections $\tilde{\omega}$ and $\tilde{\omega}'$ coïncide over $p^{-1}(U \cap U')$. We obtain a global meromorphic principal connection $\tilde{\omega}$ on $(E, \tilde{D})$ inducing ∇ as in (32).

If (φ, Φ_0) is an isomorphism of vector bundles preserving the meromorphic connections ∇_1, ∇_2, then from subsection 2.2 it induces an isomorphism $\tilde{\Psi}$ of holomorphic principal bundles between E and E'. Since the action of P on $\mathbb{C}^r$ is free, by definition of ∇_1 and ∇_2 we get that $\varphi^\star \tau_2 = \tau_1 \circ d\varphi$. By Lemma 5.1 we obtain $\tilde{\Psi}^\star \tilde{\omega}_2 = \tilde{\omega}_1$.

□

5.2 Equivalence between meromorphic affine connections and meromorphic affine Cartan geometries

In this subsection, we consider the complex affine group G of dimension $n \geq 1$, and the complex linear group $P \subset G$. The restricted adjoint representation $ad : P \longrightarrow GL(\mathfrak{g})$ splits as the sum of two irreducible representations $\mathfrak{g}_-$, the subalgebra corresponding to the infinitesimal generators for the translations in $Aff(\mathbb{C}^n)$, and $\mathfrak{p} = Lie(P)$. Consequently, if $E \xrightarrow{p} M$ is a holomorphic P-principal bundle and ω_0 is a meromorphic (G,P)-Cartan connection on $(E, \tilde{D})$, then it splits as the sum :

$$\omega_0 = \theta_0 \oplus \tilde{\omega} \tag{39}$$

of a meromorphic solderform θ_0 on $(E, \tilde{D})$ (see Definition 3.9) and a meromorphic P-principal connection $\tilde{\omega}$ on $(E, \tilde{D})$.

Consider the category $\mathcal{F}^{conn}$ whose objects are triples $(\phi_0, \mathcal{E}, \overline{\nabla})$ formed by a meromorphic extension $(\phi_0, \mathcal{E})$ over a pair (M, D) and a meromorphic connection $(\mathcal{E}, \overline{\nabla})$ on (M, D), and the arrows are the isomorphisms of vector bunlde (see subsection 2.2) preserving the meromorphic connections (see (4)). Define the map f from the category $\mathcal{G}_{aff}$ of meromorphic (G,P)-Cartan geometries on (M, D) to $\mathcal{F}^{conn}$ as follows. If (E, ω_0) is an object of $\mathcal{G}_{aff}$, consider the meromorphic solderform (E, θ_0) (see Definition 3.9) defined by (39), and $\overline{\nabla}$ the meromorphic connection on $\mathcal{E} = E(\mathbb{C}^n)$ associated with $\tilde{\omega}$ (see Proposition 5.0.1).

Now, consider the subcategory $\mathcal{G}^0_{aff}$ of $\mathcal{G}_{aff}$ whose objects are holomorphic branched (G,P)-Cartan geometries, together with their isomorphisms. Consider a subcategory $\mathcal{F}^{conn,0}$ of $\mathcal{F}^{conn}$ obtained by intersecting with $\mathcal{F}^0$ (Definition 3.8).

Proposition 5.0.2. *Let (M, D) be a pair. The map f extends as an equivalence of categories between $\mathcal{G}_{aff}$ (resp. $\mathcal{G}^0_{aff}$) and $\mathcal{F}^{conn}$ (resp. $\mathcal{F}^{conn,0}$).*

Proof. Let $\Psi : E \longrightarrow E'$ is an arrow between two meromorphic (G,P)-Cartan geometries (E, ω_0) and (E', ω'_0) over (M_1, D_1) and (M_2, D_2) , and $(\phi_0, \mathcal{E}, \overline{\nabla})$ and $(\phi'_0, \mathcal{E}', \overline{\nabla}')$ their images through f. So Ψ is a morphism of holomorphic rincipal bundles between the frame bundles and we define $f(\Psi) = (\varphi, \Phi)$ as the image of Ψ through the equivalence described in subsection 2.2. By construction, since $\Psi^\star \omega'_0 = \omega_0$, we have $\Psi^\star \theta'_0 = \theta_0$ and $\Psi^\star \tilde{\omega}' = \tilde{\omega}$. The first condition implies that (φ, Φ) is an arrow of meromorphic extensions (Definition 3.8), while the second one implies that it preserves the meromorphic connections ∇ and ∇' (see Lemma 5.1). Hence, f is a functor. Since it is the restriction of the equivalence of categories from Proposition 3.0.1, we obtain an equivalence of categories. □

Let $(\phi_0, \mathcal{E}, \overline{\nabla})$ be an object of $\mathcal{F}^{conn}_{M,D}$. Then :

$$\nabla = \phi_0^{-1}\overline{\nabla} \tag{40}$$

defines a meromorphic affine connection on (M, D), we will call it the **meromorphic affine connection induced by** $(\phi_0, \mathcal{E}, \overline{\nabla})$. Thus, there is a functor

$$\mu : \mathcal{F}^{conn} \longrightarrow \mathcal{A} \tag{41}$$

to the category $\mathcal{A}$ of meromorphic affine connections on pairs.

We denote by T_∇ the torsion of ∇ (Equation 2). There is the analagous notion of $\mathfrak{g}_-$**-torsion** for an object (E, ω_0) of $\mathcal{G}_{aff}$ on (M, D). It is the P-equivariant meromorphic function τ_{ω_0} on E with values in $\mathbb{W}_{\mathfrak{g}_-} = \bigwedge^2(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-$ defined as the projection of the Cartan curvature k_{ω_0} of (E, ω_0) (see Definition 3.4) on $\mathbb{W}_{\mathfrak{g}_-}$ respective to $\bigwedge^2(\mathfrak{g}_-)^* \otimes \mathfrak{p}$.

Let (E, ω_0) be an object of $\mathcal{G}_{aff}$ on a pair (M, D) and $(\phi_0, \mathcal{E}, \overline{\nabla})$ its image through the equivalence in Proposition 5.0.2. Let ∇ be the meromorphic affine connection on (M, D) induced by this object (see above). Identify the $\mathfrak{g}_-$-torsion $\tau_{\omega_0} \circ (\omega_0 \wedge Id_{\mathfrak{g}_-})$ with a meromorphic one form on $(E, \tilde{D})$ with values in $\mathfrak{g}_-^* \otimes \mathfrak{g}_- = End(\mathbb{C}^n)$. Let $\overline{\tau}_{\omega_0}$ be the section of $\Omega^1_M[\star D] \otimes End(\mathcal{E})[\star D]$ corresponding to τ_{ω_0} through the equivalence preceding Definition 5.1. Identify it with a section of $\Omega^1_M \otimes End(TM)[\star D]$ through the isomorphism $\phi_0 : TM[\star D] \longrightarrow \mathcal{E}[\star D]$. Then :

Lemma 5.2. *The two sections $\overline{\tau}_{\omega_0}$ and T_∇ defined as above coïncides.*

Proof. It suffices to prove the statement for the restriction $(E|_{M\smallsetminus D}, \omega_0)$ and $\nabla|_{M\smallsetminus D}$ over $M \smallsetminus D$, in the holomorphic categories. We refer to [16], Theorem 5.1. ☐

The vanishing of the torsion is clearly preserved by the isomorphisms of $\mathcal{G}_{aff}$ and we define $\mathcal{G}^{norm}_{aff}$ as its subcategory whose objects are meromorphic (G, P)-Cartan geomtries whose torsion vanishes identically. We define $\mathcal{A}^{norm}$ as the subcategory of $\mathcal{A}$ whose objects are meromorphic affine connections on pairs whose torsions identically vanish.

Finally, let's remark that for any object $(\mathcal{E}, \phi_0, \overline{\nabla})$ of $\mathcal{F}^{conn,0}$, the meromorphic affine connection (40) restricts as a holomorphic connection on the submodule $\mathcal{E}$. We then define :

Definition 5.2. *The category $\mathcal{A}^0$ is the subcategory of $\mathcal{A}$ whose objects are the meromorphic affine connections on (M, D) preserving a locally free $\mathcal{O}_M$-module $\mathcal{E}$ with $TM \subset \mathcal{E} \subset TM[\star D]$, in the above sense. Its objects are called holomorphic branched affine connections.*

Lemma 5.3. *Let ∇ be a holomorphic branched affine connection on (M, D). Then the submodule $\mathcal{E} \subset TM[\star D]$ from Definition 5.2 is unique.*

Proof. Let E be the bundle of holomorphic frames for $\mathcal{E}$, and $\tilde{\omega}$ be the meromorphic principal connection on $R^1(M)$ corresponding to ∇ (Proposition 5.0.1). Suppose there exists another rank n locally free submodule $\mathcal{E}'$ of $TM[\star D]$ such that ∇ restricts to a holomorphic connection on $\mathcal{E}'$, and let $\tilde{\omega}'$ be the corresponding holomorphic princpal connection on its bundle of holomorphic frames E'.

Pick a point $x \in M$, and a neighbhoorhood U of x in M with two basis $(\overline{s}_1, \dots, \overline{s}_n)$ of $\mathcal{E}|_U$ and $(\overline{t}_1, \dots, \overline{t}_n)$ of $\mathcal{E}'|_U$. Denote by σ, σ' the holomorphic sections of $R^1(M \smallsetminus D)$ on $U \smallsetminus D$ corresponding respectively to these basis, and b be the unique holomorphic function on $U \smallsetminus D$ with values in $GL_n(\mathbb{C})$ such that $\sigma' = \sigma \cdot b$. Thus, ${\sigma'}^\star \tilde{\omega} = \sigma^\star \tilde{\omega} = 0$, so that :

$$b^\star \omega_{GL_n(\mathbb{C})} = {\sigma'}^\star \tilde{\omega} - \sigma^\star \omega = 0 \tag{42}$$

Hence, b prolongates on U as a holomorphic funcion. This being true for any $x \in M$, we obtain that the two prolongations E and E' of $R^1(M \smallsetminus D)$ coïncide, i.e $\mathcal{E} = \mathcal{E}'$. ☐

Corollary 5.1. *The composition of the equivalence from Proposition 5.0.2 and the map given by* (41) *gives an equivalence of categories between* $\mathcal{G}_{aff}$ *(resp.* $\mathcal{G}^0_{aff}$*) and* $\mathcal{A}$ *(resp.* $\mathcal{A}^0$*). It restricts as an equivalence between* $\mathcal{G}^{norm}_{aff}$ *and* $\mathcal{A}^{norm}$*.*

Recall that, given any meromorphic affine connection ∇ on (M,D), a local holomorphic vector field $\overline{X}$ on an open subset $U \subset M$ is a *Killing field* for ∇ iff the pullback of ∇ by its flows is again ∇. We denote by $\mathfrak{kill}_\nabla$ the subsheaf of $TM \setminus D$ whose sections are the Killing field for ∇. By Corollary 5.1, we obtain :

Lemma 5.4. *If* (E,ω_0) *is an object of* $\mathcal{G}_{aff}$ *on a pair* (M,D)*, and* ∇ *the corresponding meromorphic affine connection on* (M,D)*, then* $\mathfrak{kill}_{M,\omega_0} = \mathfrak{kill}_\nabla$*.*

5.3 Totally geodesic branched affine connections in algebraic dimension zero

Now, we will give some application of the results of section 4 to the classification of affine meromorphic connections on some simply connected complex compact manifolds M. Most of them are adaptations of the arguments used in the proof of the principal theorem in [3], using the results of this article.

Theorem 5.1. *Let* (M,D) *be a pair, with* M *a simply connected complex compact manifold. If* (M,D) *bears a quasihomogeneous and totally geodesic meromorphic affine connection* ∇*, then it admits a meromorphic parallelism* $(\overline{X}_1,\dots,\overline{X}_n)$*, such that* $\overline{X}_i$ *is a Killing vector field for* ∇ *when restricted to* $M \setminus D$*.*

Proof. Let (E,ω_0) be the meromorphic affine Cartan geometry on M corresponding to ∇ (see Corollary 5.1). By the Corollary 4.1, it satisfies the extension property of infinitesimal automorphisms, i.e the local system $\mathfrak{kill}_\nabla$ on $M \setminus D$ extends as a local system $\mathcal{K}$ on M, with $\mathcal{K} \subset TM[\star D]$. Since M is simply connected, this is a constant sheaf on M. Since ∇ is assumed quasihomogeneous, we can pick $x \in M$ and a $\mathcal{O}_{M,x}$-basis $\overline{X}_{1,x},\dots,\overline{X}_{n,x}$ of $(TM)_x$ formed by germs of Killing fields for ∇. These germs are thus restrictions of global meromorphic vector fields $\overline{X}_1,\dots,\overline{X}_n$ whose restrictions to $M \setminus D$ are elements of $\mathfrak{kill}_\nabla(M \setminus D)$. Since their germs at x are independants, there exists a Zariski-dense open subset $M \setminus S$ such that the restrictions of $\overline{X}_1,\dots,\overline{X}_n$ to any subset $U \subset M \setminus S$ are independant elements of $TM(U)$, i.e $(\overline{X}_1,\dots,\overline{X}_n)$ is a meromorphic parallelism on M.

□

Corollary 5.2. *Let* M *be a compact complex manifold with finite fundamental group, and whose meromorphic functions are constants. Then* M *doesn't bear any totally geodesic branched holomorphic affine connection.*

Proof. Suppose that ∇ is a totally geodesic branched holomorphic affine connection on (M,D), denote by $\mathcal{E}$ the submodule of $TM[\star D]$ from Lemma 5.3. Complete the meromorphic parallelism $(\overline{X}_i)_{i=1,\dots,n}$ from Theorem 5.1 into a basis $(\overline{X}_j)_{j=1,\dots,r}$ for the global meromorphic Killing fields of ∇. A meromorphic parallelism is a rigid geometric structure (see [12]), so by the Theorem 2 of [11], the juxtaposition of $(\overline{X}_j)_{j=1,\dots,r}$ and ∇ is quasihomogeneous. Since ∇ satisfies the extension property for the Killing vector fields (Corollary 4.1) and M is simply connected, we obtain a meromorphic parallelism $(\overline{X}'_i)_{i=1,\dots,n}$ such that the restriction of each $\overline{X}'_i$ to $M \setminus D$ is a Killing field for ∇ and commutes with each $\overline{X}_j$. In particular, each $\overline{X}'_i$ is a $\mathbb{C}$-linear combination of the $(\overline{X}_j)_{j=1,\dots,r}$, so $(\overline{X}'_i)_{i=1,\dots,n}$ are commuting meromorphic vector fields.

Now, let pick any Gauduchon metric on M ([15]) and let's prove that the degree $deg(\mathcal{E})$ of $\mathcal{E}$ with respect to this metric is zero. Let (E,ω_0) be the branched holomorphic affine Cartan geometry on (M,D) corresponding to ∇ (Corollary 5.1). Then $\mathcal{E} = E(\mathfrak{g}/\mathfrak{p}) = E(\mathfrak{g})/E(\mathfrak{p})$ by definition of (E,ω_0), and since $P = GL_n(\mathbb{C})$, we have $deg(E(\mathfrak{p})) = 0$ (see [2], Corollary 4.2).

We must then prove that $deg(E(\mathfrak{g})) = 0$. For, it is sufficient to prove that $C_1(R_{\nabla^{\omega_0}})$ vanishes identically, where C_1 is the trace on $End(E(\mathfrak{g}))$ and ∇^{ω_0} is the tractor connection (see Definition 3.7). We will prove that the meromorphic one form $\eta_i = X'_i \neg C_1(R_{\nabla^{\omega_0}})$ vanishes identically on M for any

$1 \leq i \leq n$. By Lemma 3.1, we have :

$$p^\star \eta_i = \tilde{X}_i' \neg C_1(R_{p^\star \nabla^{\omega_0}}) = \underbrace{dC_1(Ad(s_i))}_{\tilde{\eta}_i^0} + \underbrace{\tilde{X}_i' \neg Ad(\omega_0 \wedge \omega_0)}_{\tilde{\eta}_i^1} \tag{43}$$

where $\tilde{X}_i'$ is the lifting of X_i' to E and $s_i = \omega_0(X_i')$.

The meromorphic one form $\tilde{\eta}_i^0$ is exact and P-equivariant. By a classical result on exact invariant forms on connected Lie groups, the restriction of $\tilde{\eta}_i^0$ to any fiber of $E \xrightarrow{p} M$ corresponds to a homomorphism $\chi : P \longrightarrow \mathbb{C}$. Because $P = GL_n(\mathbb{C})$, any such homomorphism is trivial, so that $\tilde{\eta}_i^0$ vanishes on the fibers of $E \xrightarrow{p} M$. Thus, it is the pullback of a meromorphic exact one form η_i^0 on M. Moreover, by Corollary 4.1, s_i is a holomorphic section of $\mathcal{V}$ on E, so that $\tilde{\eta}_i^0$ is a holomorphic one form. Thus, η_i^0 is an exact holomorphic one form on a simply connected compact complex manifold, i.e vanishes everywhere.

Now, let's prove that $\tilde{\eta}_i^1 = p^\star \eta_i$ vanishes everywhere. Consider $E_G = E \times G \xrightarrow{pi_G} E$ and the holomorphic tractor connection $\tilde{\omega}$ on it (Definition 3.7). Using the splitting $TE_G = ker(\tilde{\omega}) \oplus ker(d\pi_G)$, the pullback $\hat{\eta}_i^1 = \pi_G^\star \tilde{\eta}_i^1$ uniquely decomposes as a sum :

$$\hat{\eta}_i^1 = \hat{\eta}_i^H \oplus \hat{\eta}_i \tag{44}$$

with $\hat{\eta}_i^H$ a G-invariant meromorphic one form on E_G, vanishing on $ker(\tilde{\omega})$, and $\hat{\eta}_i^V$ vanishing on $ker(\tilde{\omega})$. In particular, $\hat{\eta}_i^H$ is the pullback of η_i through the composition $p_G = p \circ \pi_G$, so that $\hat{\eta}_i^V$ vanishes everywhere. Now, using Corollary 4.1, $\tilde{\eta}_i^1$ is a holomorphic one form, so that η_i is a holomorphic one form on M. Using the Lie-Cartan formula, we have :

$$d\eta_i(X_j', X_k') = \mathcal{L}_{X_j'} \eta_i(X_k') - \mathcal{L}_{X_k'} \eta_i(X_j') - \eta_i([X_j', X_k']_{TM}) \tag{45}$$

Since the only meromorphic functions on M are the constants, we obtain $\mathcal{L}_{X_j'} \eta_i(X_k') = \mathcal{L}_{X_k'} \eta_i(X_j') = 0$, and since the meromorphic vector fields $(X_i')_{i=1,\ldots,n}$ commute, η_i is a closed holomorphic one form. Since M is simply connected and compact, η_i vanishes everywhere. This proves that $C_1(R_{\nabla^{\omega_0}})$ vanishes everywhere, i.e $deg(E(\mathfrak{g})) = 0$.

Hence, $deg(\mathcal{E}) = 0$. Let $\overline{s}_1, \ldots, \overline{s}_n$ be the images of $X_1', \ldots, X_n'$ through the morphism ϕ_0, where $(\phi_0, \mathcal{E})$ is the holomorphic extension image of (E, ω_0) as in Proposition 5.0.2. Since $s_i = \omega_0(\tilde{X}_i')$ is a section of $\mathcal{V}(E)$ for any $1 \leq i \leq n$ from Corollary 4.1, each $\overline{s}_i$ is a section of $\mathcal{E}(M)$. Since they are independant, the holomorphic section $\bigwedge_{n}^{i=1} \overline{s}_i$ of $det(\mathcal{E})$ is not identically vanishing, thus $det(\mathcal{E})$ is trivial and $\bigwedge_{n}^{i=1} \overline{s}_i$ never vanishes. It therefore forms a basis of $\mathcal{E}$ on M, and the dual sections $\overline{s}_1^*, \ldots, \overline{s}_n^*$ are holomorphic sections of $\mathcal{E}^*$ on M. We obtain a branched holomorphic $(\mathbb{C}^n, \{0\})$-Cartan geometry (M, η) on (M, D), where :

$$\eta = \bigwedge_{i=1}^{n} (\overline{s}_i^* \circ \phi_0) \otimes \mathfrak{e}_i \tag{46}$$

where $(\mathfrak{e}_i)_{i=1,\ldots,n}$ is the canonical basis of $\mathbb{C}^n$. Because the η-constant vector fields $(X_i')_{i=1,\ldots,n}$ commute, it is a flat branched holomorphic Cartan geometry. Since M is simply connected and compact, there is a holomorphic submersion $dev : M \longrightarrow \mathbb{C}^n$. This is impossible by the maximum principle, so M cannot bear any totally geodesic branched holomorphic affine connection.

□

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