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Cost allocation in energy distribution networks

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Abstract

This paper presents a cost allocation problem arising from energy distribution and proposes cost allocation rules that depend on the distribution network and consumer demands. To determine relevant rules, we adopt a normative approach and compare two principles: (i) the connection principle and (ii) the uniformity principle. The Connection rule is proposed in accordance with (i), while the Uniform rule is developed in line with (ii). However, (i) and (ii) are incompatible. To make a trade-off between them, we propose a family of Mixed rules. Each rule is axiomatically characterized. Then, we demonstrate that the Connection rule coincides with the multi-choice Shapley value of a specific multi-choice game derived from the original problem. Moreover, the Connection rule is in the Core of this game. Similarly, we show that the Uniform rule and the Mixed rules coincide with other solution concepts from multi-choice games.

Keywords: Energy distribution network; Cost allocation rules; Axiomatization; Multi-choice games

JEL codes: C71; D61

1. Introduction

This paper considers situations in which some form of energy (e.g., electricity, gas, heating, etc.) is distributed through a network that channels the energy from a source to end consumers. The network is managed by an operator who is responsible for ensuring its safety and maintenance. To carry out its responsibilities, the network operator faces various **operation costs**. Some of these costs cannot be directly assigned to a specific part of the network. For example, there are costs associated with storage, which is necessary to address seasonal fluctuations in consumption. There are also costs associated with maintaining the network and ensuring public safety. Assuming that the network operator recovers these operation costs by charging consumers, the problem is to determine how to allocate these costs among them.

1.1. The energy distribution problem and optimistic design

To conduct this study, we define a model adapted to the energy distribution setting. Fix a finite set $N = \{a, b, \dots, n\}$ of **consumers**, directly or indirectly, connected to a unique source of energy \mathcal{S} by means of a fixed **network**, which is represented by a (directed) tree graph P . Figure 1 is an example of tree graph, but a formal definition can be found in Section 2. The

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nodes of the graph represent the consumers plus the source, while the links of the graph represent the infrastructure of the network. An element in N will refer to both a consumer and the link having this consumer at its head. Link b has consumer b at its head in Figure 1 for instance.

Each link has a varying **capacity**, which is represented by an integer. This capacity may represent the diameter of a gas pipeline or the ampacity of a power line for instance. There is an industrial limit to the capacity of the links represented by $K \in \mathbb{N}$. In addition, each consumer $i \in N$ has a **peak demand** in capacity, which is represented by an integer $q_i \leq K$. It corresponds to the capacity required to handle i 's highest energy usage over a certain period of time. For instance, the peak gas/electricity demand of a regular household is often determined by its consumption during winter. The profile of all peak demands is denoted by $q = (q_a, \dots, q_n)$.

The network operator is responsible for satisfying the peak demand of the consumers, and must ensure that its network is **designed** accordingly. This means that each link must have the capacity to handle the peak demands of its downstream consumers. For instance, link b must have the capacity to handle the peak demands of both b and e in Figure 1. In this paper, we assume that each link must have the capacity to handle the highest peak demand of the consumers located downstream of it, which we refer to as the link's **peak capacity**. For example, in Figure 1, links a and c have a peak capacity of 3, while link d has a peak capacity of 1, as consumers a , c , and d have peak demands of 2, 3, and 1, respectively. We call this methodology the **optimistic approach** to network design.

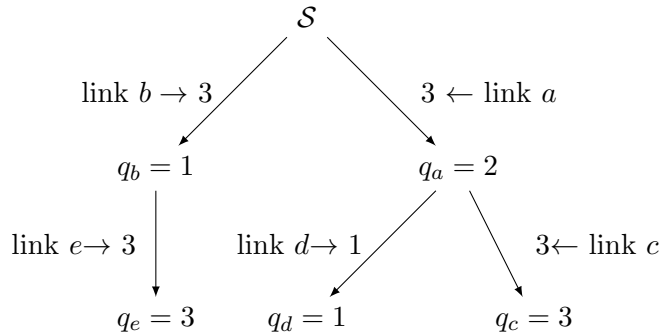


Figure 1: Optimistic approach to network design

The main drawback of the optimistic approach to network design is that a link can only satisfy a few peak demands at a time. In particular, the highest downstream peak demand of a link already saturates its capacity. However, this approach has its advantages since it does not expand the network too much and it limits the associated operation costs. A real-life example is an electricity distribution network serving sports stadiums or event venues. These venues often have high power requirements during events, with a significant spike in electricity demand during peak times when the place is filled with spectators, lights are at maximum intensity, and various electrical equipment is in use. To prevent network overload, consumers may coordinate and schedule their electricity usage to not consume at the same time. In this context, it makes sense to adopt an optimistic approach to network design.

An alternative is the **pessimistic approach** to network design: the network operator must be able to satisfy all peak demands at any time. Therefore each link should have the capacity to handle the sum of all the peak demands of its downstream consumers. Unlike the previous approach, this approach ensures a flawless service to consumers. The main drawback of this

approach are the important operation costs associated with it. This approach has recently been covered in Techer (2023) and Béal et al. (2023). The author applies the pessimistic approach to model a waste transportation network with a formalism close to ours. In this study, we primarily focus on the optimistic approach.

In this paper, we make the assumption that the methodology used to calculate the cost of operating a link may differ from one link to another. The idea is that the cost of operating a link may vary due to factors outside of the distribution network. Operating a gas pipeline in mountainous terrain poses greater challenges compared to operating one in an urban environment, for example. To model this idea, we define a **cost function** $C : N \times \{1, \dots, K\} \rightarrow \mathbb{R}_+$, which computes the cost $C(i, j)$ of operating any link $i \in N$ of any capacity $1 \leq j \leq K$. Each map $C(i, \cdot) : \{1, \dots, K\} \rightarrow \mathbb{R}_+$, $i \in N$, is non-decreasing since a larger capacity leads to higher costs, and is zero normalized. Two maps $C(i, \cdot) : \{1, \dots, K\} \rightarrow \mathbb{R}_+$ and $C(i', \cdot) : \{1, \dots, K\} \rightarrow \mathbb{R}_+$, $i, i' \in N$, may behave very differently if the two links i and i' differ according to some outside factors. The **total cost** of operating the network is computed as the sum of the costs of all the links, where each link meets its peak capacity. The problem is then to determine appropriate rules to allocate this total cost among the consumers. This problem is called the **distribution problem**. We propose three cost allocation rules to solve it. These rules are based on principles that can be viewed as desirable for a network operator.

1.2. Principles for cost allocation rules

To properly define cost allocation rules, let us endow each consumer $i \in N$ with the discrete set of **demand units** $\{1, \dots, q_i\}$. A **cost allocation rule**, or a rule for short, is a map f that allocates a real number f_{ij} to each demand unit $j \in \{1, \dots, q_i\}$ of each consumer $i \in N$. The real f_{ij} captures the **incremental allocation** assigned to consumer i for an increase in demand from $j - 1$ to j . In essence, an incremental allocation describes the marginal change in the amount allocated to a consumer per unit change in their demand. This can be particularly helpful when attempting to generate a pricing schedule. Obviously, it suffices to sum each $(f_{ij})_{j \leq q_i}$ to obtain the **total allocation** of consumer i . Each rule studied in this paper follows the following two principles.

- (i) **Budget balance principle:** a cost allocation rule must recover the total cost of operating the network. This means that the network operator neither incurs losses nor earns profits from its activity.
- (ii) **Independence to higher demands principle:** the amount allocated to a consumer for a certain demand unit should be independent of any other greater demand unit. This principle makes it possible to avoid situations where the presence of a consumer with a high demand implies additional and unjustified costs to consumers with lower demands. The Independence to higher demands principle was formally discussed by Moulin and Shenker (1992) in the context of cost-sharing problems.

In addition to (i) and (ii), we discuss two other principles.

- (iii) **Connection principle:** a consumer should only be charged for the costs associated with the specific links that connect him to the source. This ensures that costs are allocated fairly and that consumers are not burdened with the costs of infrastructure that they do not benefit from.

- (iv) **Uniformity principle**: two consumers with the same demands should be charged the same amount regardless of their geographical location. For example, there should be no difference in treatment between rural and urban consumers, although the underlying operation costs are different. In fact, (public) network operators in France highly favor rules that respect this principle (see Fleurbaey and Trannoy, 1998). The aim is to reduce inequalities and ensure that all residents have access to a similar level of public services.

Contrary to principles (i) and (ii), principles (iii) and (iv) are incompatible and cannot be satisfied by the same rule. Indeed, (iii) states that consumers should only pay for the portion of the network they use. Therefore, two consumers with the same demands can be charged with different amounts depending on their position on the network, which contradicts (iv).

In this paper, we propose three rules. First, the **Connection rule**, follows principles (i), (ii) and (iii). It ensures that each consumer pays a fair share of the portion of the network that connect him to the source. Second, the **Uniform rule**, follows principles (i), (ii) and (iv). It ensures that two consumers are allocated the same amount for the same demand unit, regardless of any other parameter of the distribution problem. Since the principles (iii) and (iv) are incompatible, we propose a family of rules to compromise between them: the **Mixed rules**. We detail how these rules are computed in the next sub-section.

We formalize (iii) and (iv) into axioms for rules. Additionally, we introduce other axioms that do not necessarily fit into the idea of principles (iii) and (iv) but can still be viewed as desirable properties in the context of distribution. Combining these axioms, we provide an axiomatic characterization for each rule introduced in this paper. Observe that principles (i) and (ii) are not formalized into axioms. Instead, we directly incorporate them into the formal definition of a rule (see Definition 2).

1.3. Computing the cost allocation rules

To clearly present our rules, assume that a link $i \in N$ is built, step by step, by increasing its capacity from 0 to its peak capacity, which we denote by \bar{q}_i . At each step $j \in \{1, \dots, \bar{q}_i\}$, link i undergoes an upgrade, called the **j-th upgrade of i**. Due to this upgrade, an **incremental cost** $A_{ij}^C = C(i, j) - C(i, j-1)$ is generated. This incremental cost represents the additional operation costs incurred by the upgrade. This procedure allows us to explain how each incremental cost impacts the cost allocations.

The **Connection rule** is computed as follows. Pick any link $i \in N$. *Step 1*: let us assign a capacity of 1 to link i . This corresponds to its 1-st upgrade. The Connection rule allocates the incremental cost A_{i1}^C equally among the first demand unit of all downstream consumers of i . Next comes *Step 2*: let us upgrade the capacity of link i by one unit. This 2-nd upgrade generates the incremental cost A_{i2}^C . The Connection rule allocates this incremental cost equally among the second demand units of all downstream consumers of i whose peak demand exceeds or equals 2. This procedure continues until link i reaches its peak capacity, i.e., until *Step* \bar{q}_i , and is applied to each link.

The **Uniform rule** is computed in a similar manner. Pick any link $i \in N$. *Step 1*: the Uniform rule allocates A_{i1}^C equally among the first demand units of all consumers regardless of their position on the network. *Step 2*: the Uniform rule allocates A_{i2}^C equally among the second demand units of all consumers whose peak demand exceeds or equals 2. This procedure continues until *Step* \bar{q}_i , and is applied to each link.

Principle (iii) is incompatible with principle (iv). To reach a compromise between them, we propose the family of **Mixed rules**. Each rule in this family achieves several compromises by

means of convex combinations between the Connection rule and the Uniform rule. Pick any link $i \in N$. *Step 1*: the Mixed rule allocates A_{i1}^C to the first demand unit of each consumer. This allocation is computed as a convex combination between the two allocations prescribed by the Connection rule and the Uniform rule. This combination depends on a parameter $\alpha^1 \in [0, 1]$. For instance, if $\alpha^1 = 0.8$, the allocation to the first demand unit of the consumers consists of 80% of the allocation prescribed by the Connection rule, plus 20% of the allocation prescribed by the Uniform rule. In that case, the compromise favors the connection principle specifically for the first demand unit of the consumers. *Step 2*: the Mixed rule allocates A_{i2}^C among the second demand units of all consumers whose peak demand exceeds or equals 2. This allocation is computed as a convex combination between the Connection rule and the Uniform rule. This second combination depends on a parameter $\alpha^2 \in [0, 1]$, which may be different from α^1 . For instance, if $\alpha^2 = 0.1$, the allocation consists of 10% of the allocation prescribed by the Connection rule, plus 90% of the allocation prescribed by the Uniform rule. In that case, the compromise favors the uniformity principle specifically for the second demand unit of the consumers. This procedure continues until *Step* \bar{q}_i , and is applied to each link.

1.4. Comparison with multi-choice games

An appropriate game theoretic tool for modeling distribution problems are multi-choice (cooperative) games. Multi-choice games, introduced by Hsiao and Raghavan (1992) and van den Nouweland (1993), are a natural extension of TU-games in which each player is endowed with a certain number of participation levels. A (multi-choice) coalition is a profile describing each player's participation level within this coalition, and each coalition's worth is measurable. Given a distribution problem, we derive a special multi-choice game associated with this problem. This game is called the distribution game. The player set represents the set of consumers, and the participation levels represent the demands of the consumers. The worth of a coalition corresponds to the total cost of the network in which each link meets its peak capacity. The Connection rule applied to a distribution problem corresponds to the multi-choice Shapley value, introduced by Lowing and Techer (2022), of the corresponding distribution game. Similarly, the Uniform rule corresponds to the multi-choice Equal division value and the Mixed rules to the multi-choice Egalitarian Shapley values. Both values are introduced by Lowing and Techer (2022). Moreover, for each distribution problem, the multi-choice Shapley value of the corresponding distribution game is in its Core (in the sense of Grabisch and Xie, 2007).

1.5. Related literature

The distribution problem shares some technical similarities with the polluted river problem studied by Ni and Wang (2007), Dong et al. (2012), van den Brink et al. (2018), and Li et al. (2023), to cite a few. In the polluted river problem, agents living along a river must bear the cost of cleaning the water, which carries pollutants. To clean the water, some costs are incurred and must be allocated among the agents. The main difference between the two models is that distribution problems discriminate consumers (or agents) by incorporating demands, while polluted river problems do not. The connection principle invoked in the distribution problem is analogous to the upstream responsibility principle used in the unlimited territorial integrity theory for polluted river problems, which states that agents located along the polluted river should pay for cleaning the portion of the river that connects them to the source of the river (see Dong et al., 2012). Furthermore, the Connection rule is conceptually close to the Downstream equal sharing solution, introduced by Dong et al. (2012) for polluted river problems. Similarly to

how we make an analogy between distribution problems and multi-choice games, van den Brink et al. (2018) makes an analogy between polluted river problem and TU-games with a permission structure (see Gilles et al., 1992). In particular, the authors show that the permission value (see van den Brink and Gilles, 1996) applied to a special TU-game constructed from the polluted river problem is equivalent to the Downstream equal sharing solution.

1.6. Overview of the paper

The rest of the paper is organized as follows. We define the distribution problem in Section 2. Section 3 presents the cost allocation rules and provide their axiomatic characterizations. Section 4 is dedicated to multi-choice games. Section 5 concludes and proposes leads for future research. Section 6 is an appendix containing all the proofs of the results.

2. The model

We denote by $|A|$ the number of elements in a finite set $A \subset \mathbb{N}$. For each non-empty $B \subseteq A$, we denote by $e_B \in \mathbb{R}^{|A|}$ the vector such that $(e_B)_i = 1$ if $i \in B$ and $(e_B)_i = 0$ otherwise. The power set of A is denoted by 2^A . Pick any $s \in \mathbb{R}^{|A|}$. The vector $(s_{-i}, j) \in \mathbb{R}^{|A|}$ is defined as $(s_{-i}, j)_i = j$ and $(s_{-i}, j)_{i'} = s_{i'}$ for each $i' \neq i$.

We start by introducing some definitions from graph theory. A **directed graph** is a pair (N, P) where $N \subset \mathbb{N}$ is a set of nodes and $P : N \rightarrow 2^N$ is a map describing the links between the nodes. We assume that $i \notin P(i)$ for any $i \in N$. In the following, we denote a directed graph (N, P) just by P . Pick any $i \in N$. The nodes in $P(i)$ are called the successors of i . The nodes in $P^{-1}(i) = \{i' \in N : i \in P(i')\}$ are called the predecessors of i . A (directed) path from $i \in N$ to $i' \in N$ is a sequence of nodes $i = h_1, h_2, \dots, h_k = i'$ such that $h_k \in P(h_{k-1}), \dots, h_2 \in P(h_1)$. The transitive closure of a directed graph P is a directed graph \hat{P} , such that, for each $i \in N$, $i' \in \hat{P}(i)$ if and only if there exists a path from $i \in N$ to $i' \in N$. The nodes in $\hat{P}(i)$ are called the subordinates of i in P , and the nodes in $\hat{P}^{-1}(i)$ are called the superiors of i in P . Similarly, the set $\hat{P}(E)$ represents the subordinates of the nodes in $E \subseteq N$ and the set $\hat{P}^{-1}(E)$ represents the superiors of the nodes in $E \subseteq N$. A directed graph is a (directed) **rooted tree** if and only if there is a unique root $i_0 \in N$ such that (i) the root has no predecessor, i.e., $P^{-1}(i_0) = \emptyset$, (ii) the subordinates of the root are all the other nodes, i.e., $\hat{P}(i_0) = N \setminus \{i_0\}$, and (iii), each node aside from the root has exactly one predecessor, i.e., $|P^{-1}(i)| = 1$ for each $i \in N \setminus \{i_0\}$.

Consider a finite set of **consumers** $N = \{a, b, \dots, n\}$ who are connected to a source \mathcal{S} via a fixed (energy distribution) network. This network is represented by a rooted tree $(N \cup \mathcal{S}, P)$, where the source plays the role of the root, the consumers are the other nodes, and $P : N \cup \mathcal{S} \rightarrow 2^{N \cup \mathcal{S}}$ models the links between the consumers and the source. If no confusion arises, simply denote the network by P . Pick any two $i, i' \in N \cup \mathcal{S}$ such that $i \in P(i')$. In that case, there is a link that goes from consumer i' to consumer i in the network. The consumers i' and i are respectively called the tail and the head of the link. We call **link i** the link with consumer i at its head, and the **downstream consumers** of link i are consumer i and its subordinates in P .

Each link has a certain **capacity** represented by an integer, which cannot exceed $K \in \mathbb{N}$ for operational reasons. Each consumer $i \in N$ has an **peak demand** in capacity $q_i \leq K$. This peak demand corresponds to the capacity required to handle i 's highest consumption over a certain period of time. For instance, the highest gas/electricity consumption of a regular household often occurs during the coldest days of winter. The profile of peak demands is denoted by $q = (q_a, \dots, q_n)$. Without loss of generality, assume that $q_n \geq q_i$, for each $i \in N$. The set of

all consumers whose peak demand exceeds or equals j is denoted by $Q(j) = \{i \in N : q_i \geq j\}$. We assume that $Q(1) = N$. For any two $a, b \in \{0, \dots, K\}^{|N|}$, $a \vee b$ and $a \wedge b$ denote their least upper bound and their greatest lower bound on $\{0, \dots, K\}^{|N|}$, respectively.

A **cost function** is a map $C : N \times \{0, \dots, K\} \rightarrow \mathbb{R}_+$ that measures the cost of any link of any capacity. In other words, for each $i \in N$ and each $j \in \{0, \dots, K\}$, $C(i, j)$ represents the cost of operating the link i when it is designed to have a capacity of j . We use the convention $C(i, 0) = 0$ for each $i \in N$. We assume that for each $i \in N$, the map $C(i, \cdot) : \{0, \dots, K\} \rightarrow \mathbb{R}_+$ is non-decreasing: a larger link's capacity leads to higher costs. On the other hand, for any $j \in \{0, \dots, K\}$ and any two $i, i' \in N$, we do not necessarily have $C(i, j) = C(i', j)$. The interpretation is that the cost of operating a link may vary due to factors that are not captured by N , q or P , such as the length or geographic location of the link.

We have the material to compute the total cost of operating the distribution network. To that end, we employ the **optimistic approach** to network design, as described in the introduction. Assume that each link is optimistically designed to meet its **peak capacity** \bar{q}_i , i.e., its capacity equals the highest peak demand of its downstream consumers. Then, for each $i \in N$, the cost of operating link i is given by

$$C(i, \bar{q}_i), \quad \text{where } \bar{q}_i = \max_{k \in \bar{P}(i) \cup i} q_k.$$

Following the optimistic approach, we define (optimistic) incremental costs. Assume that each link $i \in N$ is built, step by step, by increasing its capacity one unit at a time, each upgrade is referred to as the **j -th upgrade of i** . The incremental cost generated by the j -th upgrade of $i \in N$, $1 \leq j \leq K$, is denoted by A_{ij}^C and is defined as

$$A_{ij}^C = C(i, j) - C(i, j-1).$$

Obviously, $A_{ij}^C \geq 0$ for each $i \in N$ and each $j \in \{1, \dots, K\}$, since $C(i, \cdot)$ is a non-decreasing map. The $|N| \times K$ **matrix of incremental costs** collects all the incremental costs, and is denoted by $A^C \in \mathbb{R}_+^{|N| \times K}$. Pick any $i \in N$ and any $1 \leq j \leq K$. The **unit cost matrix** I^{ij} is the $|N| \times K$ matrix defined as

$$\forall k \in N, l \leq K, \quad I_{kl}^{ij} = \begin{cases} 1 & \text{if } k = i, l = j, \\ 0 & \text{otherwise.} \end{cases}$$

In such a matrix, only the j -th upgrade of link i generates a non-null incremental cost. Unit cost matrices will be used to focus on the specific upgrade of a specific link in the axiomatic discussions. Moreover, the matrix of incremental costs A^C can be expressed as the linear combination of unit cost matrices as follows

$$A^C = \sum_{i \in N} \sum_{1 \leq j \leq K} A_{ij}^C I^{ij}.$$

The **total cost** of operating the network is computed as the sum of the costs of all the optimistically designed links

$$\sum_{i \in N} C(i, \bar{q}_i), \quad \text{or equivalently, } \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C. \quad (1)$$

As mentioned in the introduction, the optimistic approach can model real-life situations such as an electricity distribution network serving sports stadiums or event venues (the consumers). Obviously, these consumers have very important needs in power. To prevent network overload, these consumers may coordinate and schedule their electricity usage to not consume at the same time. In this context, it makes sense to adopt an optimistic approach to network design to limit the total cost of operating the network. This leads us to the following problem: how to allocate this total cost among the consumers? Next definition formalizes this problem.

Definition 1 (Distribution problems). A distribution problem is denoted by (N, q, A^C, P) , or by (q, A^C) for short, since N and P are fixed. The problem is to determine a way to allocate the total cost (1) among the consumers based on their demands and their location on the distribution network. The set of distribution problems is denoted by DP .

3. Rules and characterizations

We address distribution problems by defining **cost allocation rules** (rules for short). In this section, we analyze and compare three different rules: the Connection rule, the Uniform rule, and the Mixed rules. Our rules describe the marginal change in the amount allocated to a consumer per unit change in their demand. These rules can be particularly helpful when attempting to determine a pricing schedule for instance. We also provide an axiomatic characterization for each rule. To properly define the rules, we endow each consumer $i \in N$ with the discrete set of **demand units** $\{1, \dots, q_i\}$.

Definition 2 (Cost allocation rule). A (cost allocation) rule $f : DP \rightarrow \mathbb{R}_+^{|N| \times K}$ for distribution problems is a map that assigns a $|N| \times K$ matrix $f(q, A^C)$ to each $(q, A^C) \in DP$. Each coordinate $f_{ij}(q, A^C) \in \mathbb{R}_+$ specifies a positive allocation to each demand unit $j \in \{1, \dots, q_i\}$ of each consumer $i \in N$. It captures the **incremental allocation** assigned to consumer i for an increase in demand from $j-1$ to j . We set the convention $f_{ij}(q, A^C) = 0$ for each $j \notin \{1, \dots, q_i\}$. Moreover, a rule satisfies (i) the Budget balance principle and (ii) the Independence to higher demands principle, for which we give a formal description.

(i) **Budget balance principle:** a rule recovers the total cost of operating the network, i.e.,

$$\sum_{i \in N} \sum_{j \in \{1, \dots, q_i\}} f_{ij}(q, A^C) = \sum_{i \in N} \sum_{j \in \{1, \dots, q_i\}} A_{ij}^C.$$

(ii) **Independence to higher demands principle:** the amount allocated to a demand unit of a consumer is independent from any other greater demand unit, i.e., for each $i \in N$ and $j \in \{1, \dots, q_i\}$,

$$\forall l \geq j, \quad f_{ij}(q, A^C) = f_{ij}(q \wedge l e_N, A^C).$$

3.1. The Connection rule

We define the **Connection rule** on DP as a means to ensure a fair allocation to each consumer based on their position on the network. For each link k and each capacity j , such that some downstream consumers of k belong to $Q(j)$, the Connection rule divides the incremental cost of A_{kj}^C equally among the downstream consumers of k in $Q(j)$. Specifically, each j -th demand unit of k 's downstream consumers in $Q(j)$ bears an equal portion of A_{kj}^C .

Definition 3 (Connection rule). The Connection rule Ψ is defined, for each $(q, A^C) \in DP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Psi_{ij}(q, A^C) = \sum_{k \in \hat{P}^{-1}(i) \cup i} \frac{A_{kj}^C}{|(\hat{P}(k) \cup k) \cap Q(j)|}. \quad (2)$$

From (2), it is clear that the Connection rule follows the Budget balance principle. Moreover, the cost allocated to a consumer i for a demand unit j is computed using only the incremental costs generated by j -th upgrade of the links. Therefore, the Connection rule also follows the Independence to higher demands principle. Moreover, the Connection rule can be computed in polynomial time.

Obviously, the Connection rule aligns with the connection principle: according to this rule, a consumer should only pay for the incremental costs associated with the links that connect them to the source. In other words, consumers are not burdened with the costs of infrastructure from which they do not benefit. It is worth noting that the allocations of a consumer increase as their distance from the source increases. Consequently, a consumer will always pay less than their subordinates for any demand unit they have in common. This disparity arises due to the non-negative nature of incremental costs and the fact that a consumer must contribute a portion of each incremental cost generated by the links that connect them to the source. Next, we provide an axiomatic foundation of the Connection rule. To that end, we introduce two axioms that define properties for a rule f on DP .

First, we say that a link is **irrelevant** to a consumer if it does not connect him to the source. The first axiom aligns with the Connection principle by ensuring that a consumer is not burdened with the costs of irrelevant links. Specifically, the axiom dictates that the cost allocated to a demand unit of a consumer should remain unaffected by any increase in the incremental cost of an irrelevant link.

Independence to Irrelevant Cost (IIC): Pick any $(q, A^C) \in DP$. For each $j \leq K$, each $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon \in \mathbb{R}$,

$$\forall h \in Q(j), h \notin (\hat{P}(i) \cup \{i\}), \quad f_{hj}(q, A^C) = f_{hj}(q, A^C + \varepsilon I^{ij}).$$

The next axiom is an equal loss requirement. It emphasizes the importance of treating each downstream consumer equally with respect to the costs generated by an upstream link. The idea is that the distance of a consumer to a certain (upstream) link should not matter when deciding how much they should pay for it. All downstream consumers of that link have the same usage of it, and no discrimination should occur. Formally, this axiom states that an increase of an incremental cost generated by the j -th upgrade of an upstream link should impact the allocation of the j -th demand unit of any two downstream consumers equally.

Equal Loss for Downstream Consumers (EDC): Pick any $(q, A^C) \in DP$. For each $j \leq K$, each $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon \in \mathbb{R}$,

$$\forall h, h' \in (\hat{P}(i) \cup \{i\}) \cap Q(j), \quad f_{hj}(q, A^C + \varepsilon I^{ij}) - f_{hj}(q, A^C) = f_{h'j}(q, A^C + \varepsilon I^{ij}) - f_{h'j}(q, A^C).$$

Observe that (EDC) does not necessarily fit into the idea of the Connection principle or the Uniformity principle. In fact, we will see that this axiom is also satisfied by the other rules presented in this paper. We now have the necessary material to characterize the Connection rule.

Theorem 1. *A rule f on DP satisfies (IIC) and (EDC) if and only if $f = \Psi$.*

Proof. See Appendix 6.1 □

Remark 1. The axioms invoked in this characterization are conceptually close to those used in Dong et al. (2012) to characterize the Downstream equal sharing rule for polluted river problems. In short, their characterization uses axioms equivalent to (IIC), (EDC), the Budget balance principle and an Additivity axiom. However, contrary to Dong et al. (2012), we do not use an Additivity axiom in our characterization. Moreover, the Budget balance principle is already incorporated within Definition 2, thus we do not invoke it in Theorem 1.

3.2. The Uniform rule

This section defines and characterizes the Uniform rule on DP . This rule ensures that two consumers should always be allocated the same amount for the same demand unit, regardless of any other parameter of the distribution problem. Formally, the Uniform rule allocates each incremental cost A_{kj}^C equally among consumers in $Q(j)$ (assuming that link k 's peak capacity exceeds or equals j).

Definition 4 (Uniform rule). The Uniform rule Υ is defined, for each $(q, A^C) \in DP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Upsilon_{ij}(q, A^C) = \frac{1}{|Q(j)|} \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^C. \quad (3)$$

From (3), it is clear that the Uniform rule follows the Budget balance principle and the Independence to higher demands principle. Observe that the Uniform rule satisfies (EDC) but does not satisfy (IIC). Obviously, the Uniform rule can be computed in polynomial time.

The Uniform rule aligns with the uniformity principle: according to this rule, two consumers with the same effective demand will have to pay the same amount regardless of their position on the network. While it does not consider the structure of the network, this rule allows to avoid significant disparities in the consumers' allocations. It reduces the likelihood of disputes, conflicts, or resentment among them. Moreover, a consumer with low financial capabilities would be able to locate itself far from the source, which may not be the case with the Connection rule. Next, we provide an axiomatic foundation of the Uniform rule. To that end, we introduce additional axioms.

The first axiom is called Solidarity. It aligns with the uniformity principle as it guarantees that every consumer contributes a certain amount when the overall cost of the network increases. Formally, consider two distribution problems. If the sum of the incremental costs generated by the j -th upgrade of the links used in the computation of the total cost (see (1)) are higher in one of the two problems, then, the axiom requires that the amount allocated to the j -th demand unit of the consumers should also be higher in that particular problem.

Solidarity (S) Pick any two $(q, A^C), (q, A^{C'}) \in DP$. If

$$\sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^C \geq \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^{C'}$$

for some $j \leq K$, then $f_{ij}(q, A^C) \geq f_{ij}(q, A^{C'})$ for each $i \in Q(j)$.

Observe that the Connection rule fails (S). The next axiom, Linearity, is a classical axiom in economic theory. An interpretation is provided in the context of energy distribution. The costs associated with operating a network can be categorized in various ways, such as maintenance costs and costs for accessing storage. The axiom of Linearity guarantees that there is no difference in considering these costs separately or jointly. Furthermore, if the costs are denominated in one currency, say USD, and we want the final allocation to be expressed in another currency, say EUR, Linearity guarantees that the conversion can be done either before or after applying the rule without affecting the outcome.

Linearity (L): For each $(q, A^C), (q, A^{C'}) \in DP$ and each $\beta \in \mathbb{R}_+$,

$$\forall i \in N, j \in \{1, \dots, q_i\}, \quad f_{ij}(q, A^C + \beta A^{C'}) = f_{ij}(q, A^C) + \beta f_{ij}(q, A^{C'}).$$

Obviously, both the Connection rule and the Uniform rule satisfy (L). The final axiom asserts that the downstream consumers of a link should not be allocated less than other consumers for this link. This is because the downstream consumers benefit the most from the link, and therefore should bear the largest share of the costs. Formally, consider any problem $(q, I^{ij}) \in DP$. The axiom states that any consumer located downstream of the cost generating link i should not be allocated less than other consumers.

Fairness (F): For each $(q, I^{ij}) \in DP$,

$$\forall k \in (\hat{P}(i) \cup \{i\}) \cap Q(j), \forall h \in Q(j), \quad f_{kj}(q, I^{ij}) \geq f_{hj}(q, I^{ij}).$$

Observe that both the Connection rule and the Uniform rule satisfy (F).

Theorem 2. *A rule f on DP satisfies (S), (L) and (F) if and only if $f = \Upsilon$.*

Proof. See Appendix 6.2 □

3.3. The Mixed rule

The Uniform rule does not satisfy (IIC) and the Connection rule does not satisfy (S). This testifies that the uniformity principle and the connection principle are clearly incompatible. However, both principles can be highly desirable in the context of distribution problems. To reconcile these principles, a compromise is proposed in the form of a family of rules called Mixed rules. These rules are obtained by using convex combinations of the Connection rule and the Uniform rule. Moreover, they allow for different types of compromise between the two principles depending on the demand unit.

Definition 5 (Mixed rules). Let $\alpha = \{\alpha^j\}_{1 \leq j \leq K}$ be a parameter system such that $\alpha^j \in [0, 1]$, for each $1 \leq j \leq K$. The α -Mixed rule μ^α is defined, for each $(q, A^C) \in DP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \mu_{ij}^\alpha(q, A^C) = \alpha^j \Psi_{ij}(q, A^C) + (1 - \alpha^j) \Upsilon_{ij}(q, A^C). \quad (4)$$

By (4), it is clear that any α -Mixed rule follows the Budget balance principle and the Independence to higher demands principle. Obviously, any Mixed rule can be computed in polynomial time.

Remark 2. A α -Mixed rule operates convex combinations between the Connection rule and the Uniform rule. For each demand $j \in \{0, \dots, K\}$, a consumer $i \in Q(j)$ receives an allocation lying between the Connection rule $\Psi_{ij}(q, A^C)$ and the Uniform rule $\Upsilon_{ij}(q, A^C)$. If α^j is closer to 1, then this allocation is closer to $\Psi_{ij}(q, A^C)$. On the other hand, if α^j is closer to 0, then this allocation is closer to $\Upsilon_{ij}(q, A^C)$.

We introduce two axioms to characterize the Mixed rules. The first axiom describes how the allocations of two distinct consumers are impacted by an irrelevant cost, and the second axiom describes how the allocation of a consumer is impacted by two distinct irrelevant costs. The first axiom states that any two distinct consumers are equally impacted by irrelevant costs, without specifying the extent of this impact. It relaxes the prescription of (IIC) as it allows consumers to be impacted by irrelevant costs. Formally, pick any $(q, I^{ij}) \in DP$, a unit cost matrix that possesses a unique non-null incremental cost. The axiom states that any two distinct consumers for whom this incremental cost is irrelevant should be allocated the same amount in (q, I^{ij}) .

Equal impact of irrelevant costs (EIC): For each $(q, I^{ij}) \in DP$,

$$\forall h, h' \in Q(j), h, h' \notin (\hat{P}(i) \cup \{i\}), \quad f_{hj}(q, I^{ij}) = f_{h'j}(q, I^{ij}).$$

Observe that both the Connection rule and the Uniform rule satisfy (EIC). Moreover, (IIC) implies (EIC). The converse is not true. The next axiom states that no matter where an irrelevant cost is located, the impact of that cost on a consumer, for whom that cost is irrelevant, remains the same. Formally, pick any two distinct problems $(q, I^{ij}), (q, I^{i'j}) \in DP$. Assume that each of the two unit cost matrices features a unique non-null incremental cost that can be considered irrelevant for a given consumer. The axiom states that the consumer for whom these incremental costs are not relevant should be allocated the same amount in both problems.

Location independence of irrelevant costs (LIC): For each $(q, I^{ij}), (q, I^{i'j}) \in DP$,

$$\forall h \in Q(j), h \notin [\hat{P}(i) \cup \{i\}] \cup [\hat{P}(i') \cup \{i'\}], \quad f_{hj}(q, I^{ij}) = f_{hj}(q, I^{i'j}).$$

Observe that both the Connection rule and the Uniform rule satisfy (LIC). Moreover, (IIC) implies (LIC). The converse is not true. We now have the material to characterize the Mixed rules.

Theorem 3. *A rule f on DP satisfies (EIC), (LIC), (L) and (F) if and only if $f = \mu^\alpha$, for some parameter system α .*

Proof. See Appendix 6.3 □

A direct consequence of Theorem 3 is that the Connection rule can be characterized by replacing (EDC) in the statement of Theorem 1 by (L) and (F), or, equivalently, by replacing (EIC) and (LIC) in the statement of Theorem 3 by (IIC).

Corollary 1. *A rule f on DP satisfies (IIC), (L) and (F) if and only if $f = \Psi$.*

Proof. Direct from the proof of Theorem 3.

3.4. Illustrative example

Consider the set of consumers $N = \{a, b, c, d, e\}$ and a source \mathcal{S} . In this example, we consider the distribution network P and the profile of peak demands $q = (2, 1, 3, 1, 3)$, which are illustrated by Figure 2b. For convenience, we fix $K = 3$. We introduce the lengths of the links, which is one of the possible exogenous features of a network. The profile $L = (1, 2, 1, 2, 1)$ collects these lengths. Consider the cost function given by

$$C : \{a, b, c, d, e\} \times \{1, 2, 3\} \rightarrow \mathbb{R}_+$$

$$(i, j) \mapsto L_i + j.$$

The cost function used in this example adds the length and capacity of the link to compute the overall cost. While this function is an oversimplification for the sake of clarity, it aligns with the real-world scenario where operation costs increase with both of these parameters. The corresponding matrix of incremental costs is given by Figure 2a.

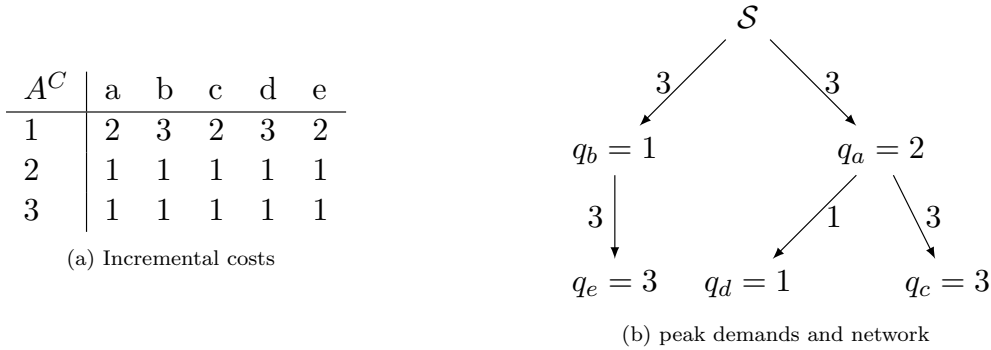


Figure 2: distribution problem

Consider link a . The peak capacity of link a is $q_c = 3$. Therefore, this link costs $C(a, q_c) = L_a + q_c = 4$. Similarly, it is possible to determine the cost of each remaining link. The costs associated with each link are summed to obtain the total cost of operating the distribution network given by

$$\sum_{i \in N} C(i, \bar{q}_i) = C(a, q_c) + C(b, q_e) + C(c, q_c) + C(d, q_d) + C(e, q_e)$$

$$= 20.$$

We want to allocate the total cost among consumers. Let us consider consumer c . Figure 3a shows consumer c 's allocation according to the Connection rule Ψ and the Uniform rule Υ . Moreover, we also show consumer a 's allocation for comparison. Consumer c 's allocations are also plotted in Figure 3b to highlight the marginal change in his allocation with respect to his demand. To derive a Mixed rule μ^α , we select $\alpha = (0, 0.5, 1)$, as depicted in Figure 3a.

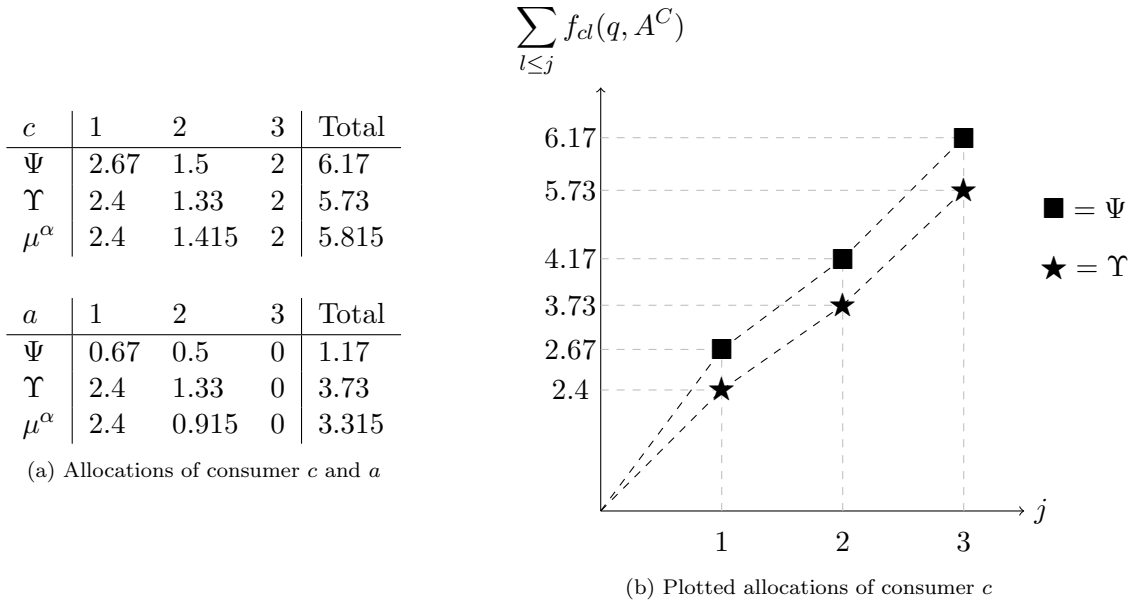


Figure 3: Cost allocations of consumers

The Connection and Uniform rules exemplify the connection and uniformity principles, respectively. This is evident in the allocations made by these rules for consumers a and c . The former utilizes a smaller portion of the distribution network and thus receives a smaller allocation than the latter under the Connection rule. On the other hand, the Uniform rule assigns the same allocation to both consumers for their first two demand units, regardless of their location on the network.

The Mixed rule μ^α prioritizes the uniformity principle for demand unit 1, as indicated by $\alpha_1 = 0$. It reaches a compromise between the uniformity and connection principles for demand unit 2 by setting $\alpha_2 = 0.5$. For demand unit 3, the connection principle is favored with $\alpha_3 = 1$. In Figure 3b, any allocation per demand unit resulting from applying a Mixed rule can be represented by a point between the corresponding square and star.

4. Rules and multi-choice games

In this section, we introduce multi-choice games, as defined by Hsiao and Raghavan (1992) and van den Nouweland (1993), and apply them to distribution problems. Specifically, we derive a distribution (multi-choice) game for each problem and analyze how our rules relate to solution concepts from multi-choice games. We demonstrate that the Connection rule is a **stable rule**, meaning that consumers have an incentive to become customers of the operator.

4.1. The distribution game

In order to remain consistent with distribution problems, some notations coincide with Section 2. Let $N = \{a, b, \dots, n\}$ be a fixed set of players and $K \in \mathbb{N}$. Each player $i \in N$ has a finite set of pairwise distinct participation levels $M_i := \{0, \dots, q_i\}$, where $q_i \leq K$. Define the set of players capable of participating at least at level j as $Q(j) = \{i \in N : q_i \geq j\}$. Denote by \mathcal{M} the Cartesian product $\prod_{i \in N} M_i$. Each element $s = (s_1, \dots, s_n) \in \mathcal{M}$ specifies a participation

profile for players and is referred to as a (multi-choice) coalition. Then, $q = (q_1, \dots, q_n) \in \mathcal{M}$ is the players' maximal participation profile that plays the role of the grand coalition, whereas $\Theta = (0, \dots, 0)$ plays the role of the empty coalition. We use the notations $M_i^+ := M_i \setminus \{0\}$ for each $i \in N$ and $M^+ := \bigcup_{i \in N} (\{i\} \times M_i^+)$. A pair $(i, j) \in M^+$ represents a player and one of his participation levels. A (cooperative) **multi-choice game** on N is a couple (q, v) where $v : \mathcal{M} \rightarrow \mathbb{R}$ is a characteristic function, with $v(\Theta) = 0$, that specifies a worth, $v(s)$, for each $s \in \mathcal{M}$. The full class of multi-choice games is denoted by \mathcal{G} . A multi-choice game is **sub-modular** if, for each $s, t \in \mathcal{M}$,

$$v(s \vee t) + v(s \wedge t) \leq v(s) + v(t).$$

Consider $(q, v) \in \mathcal{G}$. A payoff matrix $x \in \mathbb{R}^{|N| \times K}$ for the game (q, v) assigns a payoff $x_{ij} \in \mathbb{R}$ to each (i, j) , $i \in N$ and $1 \leq j \leq K$. If $(i, j) \notin M^+$, $x_{ij} = 0$. A set-valued solution on \mathcal{G} is a map F that assigns a collection of payoff matrices $F(q, v)$ to each $(q, v) \in \mathcal{G}$. A value f is a single-valued solution on \mathcal{G} , that assigns a unique payoff matrix $f(q, v)$ to each $(q, v) \in \mathcal{G}$. We define distribution games as specific multi-choice games based on distribution problems. A distribution game, derived from a distribution problem (q, A^C) , measures the total cost of each distribution problem (s, A^C) in which each consumer $i \in N$ has a peak demand $s_i \leq q_i$. Put differently, a distribution game is a collection of all the total costs that could be generated by (optimistically designed) networks smaller than the one they are derived from.

Definition 6 (The distribution game). For each $(q, A^C) \in DP$, its associated distribution (multi-choice) game $(q, v^{C,P})$ is defined as

$$\forall s \leq q, \quad v^{C,P}(s) = \sum_{i \in N} C(i, \bar{s}_i) = \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{s}_i\}} A_{ij}^C$$

where $\forall i \in N, \quad \bar{s}_i = \max_{k \in \hat{P}(i) \cup i} s_k.$

In a distribution game, the players represent the consumers and the participation levels represent the demand units of the consumers. The worth $v^{C,P}(s)$ represents the cost of operating a (optimistically designed) network (s, A^C) in which each consumer $i \in N$ has a peak demand of s_i . Obviously, $v^{C,P}(q)$ coincides with the total cost of operating the network, and $v^{C,P}(\Theta) = 0$ since $C(i, 0) = 0$ for each $i \in N$.

Remark 3. Pick any $(q, A^C) \in DP$. Since $C(i, \cdot)$ is non-decreasing for each $i \in N$, it follows that the associated distribution game is **monotonic**, meaning that for each $s \geq t$, it holds that $v^{C,P}(s) \geq v^{C,P}(t)$.

4.2. Rules and solution concepts for multi-choice games

This section presents several solution concepts for multi-choice games and compare them with the rules of Section 3. First, we present the multi-choice Shapley value as introduced by Lowing and Techer (2022) for multi-choice games. To that end, we must present a certain coalition formation process. We introduce **restricted orders** over the set of pairs M^+ as defined by Grabisch and Xie (2007). Consider a distribution game $(q, v^{C,P})$. Restricted orders are such that no pair $(i, j) \in M^+$ is ordered before a pair $(i', j') \in M^+$ with a strictly lower $j' < j$. This means that no consumer can demand $j + 1$ unless each consumer $i \in Q(j)$ demands at least

j . Formally, a restricted order over the set of pairs is a bijection $\sigma : M^+ \rightarrow \{1, \dots, \sum_{i \in N} q_i\}$ defined as

$$\forall (i, j), (i', j') \in M^+, \quad [j < j'] \implies [\sigma(i, j) < \sigma(i', j')].$$

Denote by \bar{O} the set of all restricted orders over the set of pairs. Let $\sigma \in \bar{O}$ be a restricted order and $h \in \{1, \dots, \sum_{i \in N} q_i\}$. Denote by $s^{\sigma, h}$ the coalition formed after h steps. We use the convention $s^{\sigma, 0} = \Theta$. Formally, it is defined as

$$\forall i \in N, \quad s_i^{\sigma, h} = \max \{j \in \{1, \dots, q_i\} : \sigma(i, j) \leq h\} \cup \{0\}.$$

For each $\sigma \in \bar{O}$, the marginal vector $\eta^\sigma(q, v^{C,P})$ is defined as

$$\forall (i, j) \in M^+, \quad \eta_{ij}^\sigma(q, v^{C,P}) = v^{C,P}(s^{\sigma, \sigma(i, j)}) - v^{C,P}(s^{\sigma, \sigma(i, j) - 1}).$$

Each $\eta_{ij}^\sigma(q, v^{C,P})$ is called the marginal contribution of the pair (i, j) to the coalition $s^{\sigma, \sigma(i, j) - 1}$, which is formed after $\sigma(i, j) - 1$ steps with respect to the restricted order σ . The marginal contribution of the pair (i, j) can be interpreted as the additional costs generated when consumer i increases his demand from $j - 1$ to j . We have the material to define the multi-choice Shapley value for distribution games. This value assigns to each pair $(i, j) \in M^+$ its expected marginal contribution assuming that each restricted orders over the set of pairs occurs with the same probability.

Definition 7 (The multi-choice Shapley value). For each distribution game $(q, v^{C,P})$, the multi-choice Shapley value is defined as

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(q, v^{C,P}) = \frac{1}{\prod_{j \in \{1, \dots, q_n\}} |Q(j)|!} \sum_{\sigma \in \bar{O}} \eta_{ij}^\sigma(q, v^{C,P}).$$

The next result states that, for each distribution problem, the Connection rule applied to the problem coincides with the multi-choice Shapley value applied to the distribution game associated with the problem.

Theorem 4. For each distribution problem $(q, A^C) \in DP$ and its associated distribution game $(q, v^{C,P})$,

$$\varphi(q, v^{C,P}) = \Psi(q, A^C).$$

Proof. See Appendix 6.4 □

Next, we present the multi-choice Equal division value as introduced by Lowing and Techer (2022) for multi-choice games. Assume that all consumers agree on forming a coalition in which everyone has the same peak demand, let us say j . Each consumer $i \notin Q(j)$ demands q_i instead. We call such coalition a **j-synchronized** coalition. For each demand unit $j \in \{1, \dots, q_n\}$, the multi-choice Equal division value allocates the difference in worth between the j -synchronized coalition and the $j - 1$ -synchronized coalition equally among the pairs featuring the demand unit j .

Definition 8 (The multi-choice Equal division value). For each distribution game $(q, v^{C,P})$, the multi-choice Equal division value ξ is defined as

$$\forall (i, j) \in M^+, \quad \xi_{ij}(q, v^{C,P}) = \frac{1}{|Q(j)|} \left[v^{C,P}(q \wedge j e_N) - v^{C,P}(q \wedge (j-1) e_N)_{k \in N} \right].$$

The next result states that, for each distribution problem, the Uniform rule applied to the distribution problem coincides with the multi-choice Equal division value applied to the distribution game associated with the problem.

Theorem 5. For each distribution problem $(q, A^C) \in DP$ and its associated distribution game $(q, v^{C,P})$,

$$\xi(q, v^{C,P}) = \Upsilon(q, A^C).$$

Proof. The proof follows directly from Definition (6) and Definition (8). □

In the following, we introduce the multi-choice Egalitarian Shapley values for multi-choice games, as defined by Lowing and Techer (2022). These values reach a compromise between the multi-choice Shapley value and the multi-choice Equal division value by using convex combinations of the two values. Specifically, for each demand unit, a convex combination is computed, resulting in a compromise that may vary depending on the demand unit.

Definition 9 (The multi-choice Egalitarian Shapley values). Let $\alpha = \{\alpha^j\}_{1 \leq j \leq K}$ be a parameter system such that $\alpha^j \in [0, 1]$ for each $1 \leq j \leq K$. For each distribution game $(q, v^{C,P})$, the multi-choice Egalitarian Shapley value χ^α is defined as

$$\forall (i, j) \in M^+, \quad \chi_{ij}^\alpha(q, v^{C,P}) = \alpha^j \varphi_{ij}(q, v^{C,P}) + (1 - \alpha^j) \xi_{ij}(q, v^{C,P}).$$

The next result states that, for each distribution problem, the set of Mixed rules applied to the problem coincides with the set of multi-choice Egalitarian Shapley values applied to the distribution game associated with the problem.

Theorem 6. For each distribution problem $(q, A^C) \in DP$ and its associated distribution game $(q, v^{C,P})$, and for each parameter system α ,

$$\chi^\alpha(q, v^{C,P}) = \mu^\alpha(q, A^C).$$

Proof. The proof follows directly from Theorem 4 and Theorem 5. □

We introduce one last solution concept: the **Core of a distribution game**, derived from the Core for multi-choice games introduced by Grabisch and Xie (2007). The Core is defined by two principles. The first principle (5) states that no coalition can achieve, by itself, a better outcome than the one prescribed by the payoff matrices in the Core. The second principle (6) states that any l -synchronized coalition achieves the same outcome than the one prescribed by the payoff matrices in the Core.

Definition 10 (The Core). The Core of a distribution game $(q, v^{C,P})$, denoted by $\mathcal{C}(q, v^{C,P})$, is the set of payoff matrices $x \in \mathbb{R}^{\sum_{i \in N} q_i}$ defined as

$$x \in \mathcal{C}(q, v^{C,P}) \iff \begin{cases} \forall s \leq q, & \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \leq v^{C,P}(s), & (5) \\ \forall l \leq q_n, & \sum_{i \in N} \sum_{j=1}^{\min\{q_i, l\}} x_{ij} = v^{C,P}(q \wedge l e_N). & (6) \end{cases}$$

If we re-interpret the Core principles in terms of rule for distribution problems, principle (5) ensures that consumers always pay less than the cost of supplying themselves and (6) states that if a group of consumers, which synchronize their peak demands, decide to supply themselves without resorting to the network operator, then they should pay the same amount as they would have been charged by the operator. Thus, the Core can be viewed as the set of **stable rules**, in the sense that consumers have an interest in becoming customers of the operator.

Remark 4. We emphasize that principle (5) is the opposite of the original definition of the Core provided by Grabisch and Xie (2007). Furthermore, principle (6) is implied by both the Budget balance principle and the Independence to Higher Demands principle. To see this, we refer the reader to Lemma 1 and its proof, both contained in the Appendix.

Alternative definitions of the Core for multi-choice games can be found in van den Nouweland (1993) and Hwang and Liao (2010). However, these definitions of the Core do not relate to our solution concepts and will not be covered. For each distribution problem, the multi-choice Shapley value applied to a distribution game is always in the Core. Therefore, the Connection rule is a stable rule.

Theorem 7. For each distribution problem $(q, A^C) \in DP$ and its associated distribution game $(q, v^{C,P})$,

$$\varphi(q, v^{C,P}) \in \mathcal{C}(q, v^{C,P}).$$

Proof. See Appendix 6.5 □

Remark 5. Observe that the Uniform rule and the Mixed rules may not be stable. To see this, consider Example 3.4. According to Figure 3a, consumer a is allocated 2.4 for its demand unit 1 by both the Uniform rule and the α -Mixed rule. However, observe that $v^{C,P}(1, 0, 0, 0, 0) = A_{a1}^C = 2$. Then, we have $\Upsilon_{a1}(q, A^C) > v^{C,P}(1, 0, 0, 0, 0)$ and $\mu_{a1}^\alpha(q, A^C) > v^{C,P}(1, 0, 0, 0, 0)$, which contradicts the Core principle (5).

Remark 6. Assume that we follow the pessimistic approach to network design as defined by Techer (2023) and Béal et al. (2023). In that case, each link has the capacity to handle the sum of all the peak demands of its downstream consumers. For each $i \in N$, the pessimistic cost of operating link i is given by $C(i, \sum_{k \in \hat{P}(i) \cup \{i\}} q_k)$. Following Techer (2023), pessimistic incremental costs are defined, for each $i \in N$ and each $j \in \{1, \dots, K\}$, as

$$\bar{A}_{ij}^C = C(i, \sum_{k \in \hat{P}(i) \cup \{i\}} \min\{j, q_k\}) - C(i, \sum_{k \in \hat{P}(i) \cup \{i\}} \min\{j-1, q_k\}).$$

In this context, a (pessimistic) incremental cost \bar{A}_{ij}^C represents the increase in the operation cost of link i when all of its downstream consumers increase their demand by one unit, transitioning from demand level $j - 1$ to demand level j (assuming they do not exceed their peak demand). A distribution problem with a pessimistically designed network is denoted by (q, \bar{A}^C) . The class of such problems is denoted by \overline{DP} .

By modifying the form of the incremental costs in their expression, one can adapt the Connection rule, the Uniform rule, and their convex combinations to the framework of distribution problems with a pessimistically designed network. Most results can also be adapted in a similar manner. However, we cannot adapt Theorem 7 to this alternative framework. To see this, let us adapt the Connection rule to the framework distribution problems with a pessimistically designed network. We obtain the allocation rule defined, for each $(q, \bar{A}^C) \in \overline{DP}$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Psi_{ij}(q, \bar{A}^C) = \sum_{k \in \hat{P}^{-1}(i) \cup i} \frac{\bar{A}_{kj}^C}{|(\hat{P}(k) \cup k) \cap Q(j)|}.$$

In addition, it is possible to define a distribution game associated with any distribution problem with a pessimistically designed network. For each $(q, \bar{A}^C) \in \overline{DP}$, its associated distribution game $(q, \bar{v}^{C,P})$ is defined as

$$\forall s \leq q, \quad \bar{v}^{C,P}(s) = \sum_{i \in N} C(i, \bar{s}_i)$$

where $\forall i \in N, \quad \bar{s}_i = \sum_{k \in \hat{P}(i) \cup i} s_k.$

This game measures the cost of operating any hypothetical distribution network with a pessimistically designed network. By adapting Theorem 4, we obtain $\Psi(q, \bar{A}^C) = \varphi(q, \bar{v}^{C,P})$ for each $(q, \bar{A}^C) \in \overline{DP}$. It remains to show that there exists a $(q, \bar{A}^C) \in \overline{DP}$ such that $\varphi(q, \bar{v}^{C,P}) \notin \mathcal{C}(q, \bar{v}^{C,P})$. Consider the example introduced in section 3.4. We replace C with a new cost function defined as

$$C' : \{a, b, c, d, e\} \times \{1, 2, 3\} \rightarrow \mathbb{R}_+$$

$$(i, j) \mapsto j^2.$$

Consider the resulting new problem $(q, \bar{A}^{C'})$ and its associated game $(q, \bar{v}^{C',P})$. Observe that $\bar{A}_{b1}^{C'} = C'(b, 2) = 4$. Consequently,

$$\varphi_{b1}(q, \bar{v}^{C',P}) = \frac{\bar{A}_{b1}^{C'}}{2}$$

$$= 2.$$

Moreover, observe that $\bar{v}^{C',P}(\Theta_{-b}, 1) = C'(b, 1) = 1$. Since $\varphi_{b1}(q, \bar{v}^{C',P}) > \bar{v}^{C',P}(\Theta_{-b}, 1)$ the Core condition (5) does not hold. Therefore, it holds that $\varphi(q, \bar{v}^{C',P}) \notin \mathcal{C}(q, \bar{v}^{C',P})$. This shows that we cannot adapt Theorem 7 to the class of distribution problems with a pessimistically designed network.

5. Conclusion

In this paper, we define distribution problems and propose three cost allocation rules for these problems on the basis of principles. The Connection rule is proposed in accordance with the connection principle, while the Uniform rule is developed in line with the uniformity principle. To compromise between the connection principle and the uniformity principle, we propose the Mixed rules. For each cost allocation rule, we provide an axiomatic characterization. In addition, these rules coincide with solution concepts from multi-choice games. In particular, the Connection rule applied to a distribution problem belongs to the Core of a specific multi-choice game derived from this problem.

Throughout this study, we assume that each link is optimistically designed to handle its highest downstream demand. Alternatively, the pessimistic approach to network design states that the network operator must be able to satisfy all peak demands at any time. Therefore, each link should have the capacity to handle the sum of all of its peak downstream demands. The two approaches both have their advantages and their drawbacks. Therefore, a compromise between these two approaches may be interesting to investigate. A **hybrid approach** would allow the network to handle multiple peak demands simultaneously while limiting overall operation costs. Exploring network design in this direction could be an interesting path for future research.

In addition, a strategic approach to distribution problems could be interesting. For instance, Lowing and Techer (2022) demonstrate that the multi-choice Shapley value is related to the Serial cost sharing mechanism proposed by Moulin and Shenker (1992) for cost sharing problems. Consequently, the Connection rule is also related to this mechanism. It appears that Moulin and Shenker (1992) conduct a **strategic analysis** of the Serial cost sharing mechanism and provide a characterization. Therefore, it could be interesting to adapt their strategic analysis to the framework of distribution problems, thus providing an alternative perspective on the problem.

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6. Appendix

To properly conduct the proofs of our results, we introduce a Lemma. By the Budget balance principle and the Independence to higher demands principle, the total amount allocated to consumers in $Q(j)$ for their demand unit j is equal to the sum of all incremental costs, generated by the j -th upgrades of the links that connect the consumers in $Q(j)$ to the source.

Lemma 1. *Pick a rule f on DP . For each $(q, A^C) \in DP$,*

$$\forall l \leq q_n, \quad \sum_{i \in Q(l)} f_{il}(q, A^C) = \sum_{i \in \hat{P}^{-1}(Q(l)) \cup Q(l)} A_{il}^C.$$

Proof. Pick a rule f on DP . By the Budget balance principle, for each $(q, A^C) \in DP$,

$$\begin{aligned} \sum_{i \in N} \sum_{j \in \{1, \dots, q_i\}} f_{ij}(q, A^C) &= \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C \\ \Leftrightarrow \sum_{j \in \{1, \dots, q_n\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &= \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C \end{aligned} \quad (7)$$

Recall that, for each $i \in N$,

$$\bar{q}_i = \max_{h \in \hat{P}(i) \cup i} q_h.$$

Observe that, for each $l \leq q_n$, and each $i \in N$,

$$\begin{aligned} \overline{(q \wedge l e_N)}_i &= \max_{h \in \hat{P}(i) \cup i} (q \wedge l e_N)_h \\ &= \max_{h \in \hat{P}(i) \cup i} \min\{q_h, l\} \\ &= \begin{cases} l & \text{if } \max_{h \in \hat{P}(i) \cup i} q_h \geq l \\ \max_{h \in \hat{P}(i) \cup i} q_h & \text{otherwise.} \end{cases} \\ &= \min\{\bar{q}_i, l\}. \end{aligned} \quad (8)$$

For each $l \leq q_n$,

$$\begin{aligned} \sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &= \sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}(q \wedge l e_N, A^C) \\ &= \sum_{i \in N} \sum_{j \in \{1, \dots, \overline{(q \wedge l e_N)}_i\}} A_{ij}^C \\ &= \sum_{i \in N} \sum_{j \in \{1, \dots, \min\{\bar{q}_i, l\}\}} A_{ij}^C \end{aligned}$$

where the first equality follows from the Independence to higher demands principle, the second from (7), and the third from (8). Similarly,

$$\begin{aligned} \sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &= \sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(q \wedge (l-1)e_N, A^C) \\ &= \sum_{i \in N} \sum_{j \in \{1, \dots, \min\{\bar{q}_i, l-1\}\}} A_{ij}^C. \end{aligned}$$

Therefore, for each $l \leq q_n$,

$$\begin{aligned}
\sum_{i \in Q(l)} f_{il}(q, A^C) &= \sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) - \sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) \\
&= \sum_{i \in N} \sum_{j \in \{1, \dots, \min\{\bar{q}_i, l\}\}} A_{ij}^C - \sum_{i \in N} \sum_{j \in \{1, \dots, \min\{\bar{q}_i, l-1\}\}} A_{ij}^C \\
&= \sum_{i \in N} \left[\sum_{j \in \{1, \dots, \min\{\bar{q}_i, l\}\}} A_{ij}^C - \sum_{j \in \{1, \dots, \min\{\bar{q}_i, l-1\}\}} A_{ij}^C \right].
\end{aligned}$$

To conclude this proof, observe that, for each $i \in N$ such that $\min\{\bar{q}_i, l\} = \min\{\bar{q}_i, l-1\}$ (i.e., $\bar{q}_i \leq l-1$),

$$\sum_{j \in \{1, \dots, \min\{\bar{q}_i, l\}\}} A_{ij}^C - \sum_{j \in \{1, \dots, \min\{\bar{q}_i, l-1\}\}} A_{ij}^C = 0.$$

On the contrary, for each $i \in N$ such that $\min\{\bar{q}_i, l\} > \min\{\bar{q}_i, l-1\}$ (i.e., $\bar{q}_i \geq l$),

$$\sum_{j \in \{1, \dots, \min\{\bar{q}_i, l\}\}} A_{ij}^C - \sum_{j \in \{1, \dots, \min\{\bar{q}_i, l-1\}\}} A_{ij}^C = A_{il}^C.$$

Observe that $\bar{q}_i \geq l$ if and only if $i \in \hat{P}^{-1}(Q(l)) \cup Q(j)$. Therefore, for each $l \leq q_n$, we obtain the desired result

$$\sum_{i \in Q(l)} f_{il}(q, A^C) = \sum_{i \in \hat{P}^{-1}(Q(l))} A_{il}^C.$$

□

6.1. Proof of Theorem 1

Existence: We show that Ψ satisfies the axioms of the statement of Theorem 1.

(IIC): The proof follows directly from the definition of the Connection rule. The allocation of a consumer i for a demand unit j is computed using only the incremental costs generated by the links that connect i to the source. Therefore, the incremental costs generated by the links irrelevant to i are not taken into account by the Connection rule when computing the allocation of i for his demand unit j .

(EDC): Pick any $(q, A^C) \in DP$, any $i \in N$, any $j \leq q_n$, and any $\varepsilon \in \mathbb{R}$. Assume that there exist two distinct consumers $h, h' \in (\hat{P}(i) \cup \{i\}) \cap Q(j)$. For h , it holds that

$$\begin{aligned}
\Psi_{hj}(q, A^C + \varepsilon I^{ij}) - \Psi_{hj}(q, A^C) &= \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{A_{kj}^C + \varepsilon I_{kj}^{ij}}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} - \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{A_{kj}^C}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} \\
&= \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{\varepsilon I_{kj}^{ij}}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} \\
&= \frac{\varepsilon}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|}.
\end{aligned}$$

Similarly, for h' , it holds that

$$\Psi_{h'j}(q, A^C + \varepsilon I^{ij}) - \Psi_{h'j}(q, A^C) = \frac{\varepsilon}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|},$$

which shows that Ψ satisfies the axiom.

Uniqueness: We show that the Connection rule is the only rule on DP that satisfies all the axioms of the statement of Theorem 1. Pick any $(q, A^C) \in DP$. Let f be a rule that satisfies all the axioms of the statement of Theorem 1 on DP . Let us show that $f(q, A^C)$ coincides with the Connection rule. By definition of a rule, $f_{ij}(q, A^C) = \Psi_{ij}(q, A^C) = 0$ whenever $j > q_i$, for each $i \in N$. Pick any $i \in N$ and any $j \leq q_i$, we show that

$$f_{ij}(q, A^C) = \Psi_{ij}(q, A^C).$$

Consider the problem $(q, A^j) \in DP$, in which A^j is defined as

$$\forall k \in N, \forall l \leq K, \quad A_{kl}^j = \begin{cases} 0 & \text{if } l = j \text{ and } k \in \hat{P}^{-1}(Q(j)) \cup Q(j) \\ A_{kl}^C & \text{otherwise.} \end{cases}$$

By Lemma 1,

$$\begin{aligned} \sum_{k \in Q(j)} f_{kj}(q, A^j) &= \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^j \\ &= 0 \end{aligned}$$

By definition, a rule assigns non-negative allocations. Therefore,

$$\forall k \in Q(j), \quad f_{kj}(q, A^j) = 0.$$

In (q, A^j) , the incremental cost A_{kj}^j of each link $k \in \hat{P}^{-1}(Q(j)) \cup Q(j)$ has been nullified. We have shown that $f_{kj}(q, A^j) = 0$ for each $k \in Q(j)$. In particular, $f_{ij}(q, A^j) = 0$. In this part of the proof we consider, one by one, the links that have been nullified in $\hat{P}^{-1}(Q(j)) \cup Q(j)$. For each link $k \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, we will un-nullified the incremental cost A_{kj}^j into A_{kj} and discuss the impact on the allocation of i for his demand unit j . This way, we reconstruct the problem (q, A) to show the desired result.

Pick any $a \in \hat{P}^{-1}(Q(j)) \cup Q(j)$. If $a \notin \hat{P}^{-1}(i) \cup \{i\}$, then by (IIC), it holds that

$$\begin{aligned} f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= f_{ij}(q, A^j) \\ &= 0. \end{aligned}$$

If $a \in \hat{P}^{-1}(i) \cup \{i\}$, then by (EDC),

$$\begin{aligned} \forall h \in (\hat{P}(a) \cup \{a\}) \cap Q(j), \quad f_{ij}(q, A^j + A_{aj}^C I^{aj}) - f_{ij}(q, A^j) &= f_{hj}(q, A^j + A_{aj}^C I^{aj}) - f_{hj}(q, A^j) \\ \iff f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= f_{hj}(q, A^j + A_{aj}^C I^{aj}). \end{aligned} \quad (9)$$

By (IIC),

$$\begin{aligned} \forall h \in Q(j), h \notin \hat{P}(a) \cup \{a\}, \quad f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= f_{hj}(q, A^j) \\ &= 0. \end{aligned} \quad (10)$$

It follows that

$$\begin{aligned} \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} (A^j + A_{aj}^C I^{aj})_{kj} \\ \iff \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= A_{aj}^C \\ \iff \sum_{h \in (\hat{P}(a) \cup \{a\}) \cap Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= A_{aj}^C \\ \implies f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|}, \end{aligned}$$

where the first equality follows from Lemma 1, the second equality follows from the definition of A^j , the third equality follows from (10), and the fourth equality follows from (9). To summarize,

$$f_{ij}(q, A^j + A_{aj}^C I^{aj}) = \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ 0 & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases}$$

Next, pick $b \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, $b \neq a$. If $b \in \hat{P}^{-1}(i) \cup \{i\}$, then by (EDC), it holds that

$$\begin{aligned} \forall h \in (\hat{P}(b) \cup \{b\}) \cap Q(j), \quad f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{ij}(q, A^j + A_{aj}^C I^{aj}) \\ = f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{hj}(q, A^j + A_{aj}^C I^{aj}) \\ = \lambda, \end{aligned} \quad (11)$$

for some $\lambda \in \mathbb{R}$. By (IIC),

$$\forall h \in Q(j), h \in \hat{P}(b) \cup \{b\}, \quad f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = f_{hj}(q, A^j + A_{aj}^C I^{aj}). \quad (12)$$

It follows that

$$\begin{aligned} \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) &= A_{aj}^C + A_{bj}^C \\ \iff \sum_{h \in (\hat{P}(b) \cup \{b\}) \cap Q(j)} (f_{hj}(q, A^j + A_{aj}^C I^{aj}) + \lambda) \\ &+ \sum_{h \notin (\hat{P}(b) \cup \{b\}) \cap Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) = A_{aj}^C + A_{bj}^C \\ \iff \sum_{h \in (\hat{P}(b) \cup \{b\}) \cap Q(j)} \lambda + \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= A_{aj}^C + A_{bj}^C \\ \iff |(\hat{P}(b) \cup \{b\}) \cap Q(j)| \lambda + A_{aj}^C &= A_{aj}^C + A_{bj}^C \\ \iff |(\hat{P}(b) \cup \{b\}) \cap Q(j)| \lambda &= A_{bj}^C \\ \iff \lambda &= \frac{A_{bj}^C}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|}, \end{aligned}$$

where the first equality follows from Lemma 1 and the definition of A^j , the second equality follows from (11), and the fourth equality follows from (12) and Lemma 1. We obtain

$$f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = f_{ij}(q, A^j + A_{aj}^C I^{aj}) + \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|}$$

$$= \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} + \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|} & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases}$$

If $b \notin \hat{P}^{-1}(i) \cup \{i\}$, then

$$f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = f_{ij}(q, A^j + A_{aj}^C I^{aj})$$

$$= \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ 0 & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases}$$

We have considered the links a and b in $\hat{P}^{-1}(Q(j)) \cup Q(j)$. Continue this reasoning until all the links in $\hat{P}^{-1}(Q(j)) \cup Q(j)$ have been considered once. Then, we obtain

$$f_{ij}(q, A^C) = f_{ij}(q, \sum_{h \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{hj}^j I^{hj})$$

$$= \sum_{k \in \hat{P}^{-1}(i) \cup i} \frac{A_{kj}^C}{(|\hat{P}(k) \cup k| \cap Q(j))}$$

$$= \Psi_{ij}(q, A^C).$$

We have shown that $f_{ij}(q, A^C) = \Psi_{ij}(q, A^C)$ for any $i \in N$ and any $j \leq q_i$. This concludes the proof of Theorem 1. \square

The axioms of the statement of Theorem 1 are logically independent, as shown by the following alternative solutions.

- The Uniform rule satisfies (EDC) but does not satisfy (IIC).
- Fix any arbitrary integer $\beta_{i'} \in \{1, 2\}$ for each $i' \in N$. The rule f^β defined, for each $(q, A^C) \in DP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}^\beta(q, A^C) = \sum_{k \in \hat{P}^{-1}(i) \cup \{i\}} \frac{\beta_i}{\sum_{x \in Q(j) \cap (P(k) \cup \{k\})} \beta_x} \times A_{kj}^C,$$

satisfies (IIC) but does not satisfy (EDC).

6.2. Proof of Theorem 2

Existence: This part of the proof is straightforward by definition of the Uniform rule.

Uniqueness: We show that the Uniform rule is the only rule on DP that satisfies all the axioms of the statement of Theorem 2. Consider $(q, A^C) \in DP$ and f a rule that satisfies all the axioms of the statement of Theorem 2 on DP . Let us show that $f(q, A^C) = \Upsilon(q, A^C)$. First, pick any I^{ij} such that $j \leq q_n$ and $i \notin \hat{P}^{-1}(Q(j)) \cup Q(j)$. The Budget balance principle implies that

$$\begin{aligned} \sum_{k \in N} \sum_{l \in \{1, \dots, q_k\}} f_{kl}(q, I^{ij}) &= \sum_{k \in N} \sum_{l \in \{1, \dots, q_k\}} I_{kl}^{ij} \\ &= 0. \end{aligned}$$

Since an allocation is a positive real matrix, it follows that, for each $k \in N$ and each $l \in \{1, \dots, q_k\}$,

$$f_{kl}(q, I^{ij}) = 0.$$

Therefore, $f(q, I^{ij})$ is the null matrix whenever $j \leq q_n$ and $i \notin \hat{P}^{-1}(Q(j)) \cup Q(j)$. Now, Pick any two I^{ij} and $I^{i'j}$ such that $j \leq q_n$ and $i, i' \in \hat{P}^{-1}(Q(j)) \cup Q(j)$. By Lemma 1,

$$\begin{aligned} \sum_{k \in Q(j)} f_{kj}(q, I^{ij}) &= \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} I_{kj}^{ij} \\ &= 1, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \sum_{k \in Q(j)} f_{kj}(q, I^{i'j}) &= \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} I_{kj}^{i'j} \\ &= 1. \end{aligned} \tag{14}$$

By (S), (13) and (14), $f_{kj}(q, I^{ij}) = f_{kj}(q, I^{i'j})$ for each $k \in Q(j)$. In particular, if $k \in \hat{P}(i) \cup \{i\}$, then by (F), for any $k' \neq k$,

$$f_{kj}(q, I^{ij}) \geq f_{k'j}(q, I^{ij}).$$

However, if $k' \in P(i') \cup \{i'\}$, then

$$f_{k'j}(q, I^{i'j}) \geq f_{kj}(q, I^{i'j}).$$

Since $f_{k'j}(q, I^{ij}) = f_{k'j}(q, I^{i'j})$, we obtain

$$\begin{aligned} f_{kj}(q, I^{ij}) &= f_{kj}(q, I^{i'j}) \geq f_{k'j}(q, I^{ij}) = f_{k'j}(q, I^{i'j}) \\ \text{and } f_{k'j}(q, I^{i'j}) &= f_{k'j}(q, I^{ij}) \geq f_{kj}(q, I^{i'j}) = f_{kj}(q, I^{ij}). \end{aligned}$$

Therefore, $f_{kj}(q, I^{ij}) = f_{k'j}(q, I^{ij})$ and $f_{kj}(q, I^{i'j}) = f_{k'j}(q, I^{i'j})$. Since this reasoning holds for any two I^{ij} and $I^{i'j}$ such that $j \leq q_n$ and $i, i' \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, it follows that for each I^{ij} such that $j \leq q_n$ and $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$,

$$\forall k, k' \in Q(j), \quad f_{kj}(q, I^{ij}) = f_{k'j}(q, I^{ij}).$$

By the Budget balance condition and (13), for each I^{ij} such that $j \leq q_n$ and $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$,

$$\forall k \in Q(j), \quad f_{kj}(q, I^{ij}) = \frac{1}{|Q(j)|}.$$

By (L), we obtain the desired result.

$$\begin{aligned} \forall j \leq K, \forall k \in Q(j), \quad f_{kj}(q, A^C) &= \frac{1}{|Q(j)|} \sum_{i \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{ij}^C \\ &= \Upsilon_{kj}(q, A^C). \end{aligned}$$

This concludes the proof of Theorem 2. \square

The axioms of the statement of Theorem 2 are logically independent, as shown by the following alternative solutions.

- The Connection rule satisfies (L) and (F) but does not satisfy (S).
- The rule f defined, for each $(q, A^C) \in DP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}(q, A^C) = \begin{cases} \sum_{i \in N} C(i, \bar{q}_i) & \text{if } i = 1, j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies (L), (S) but does not satisfy (F).

- The rule f given, for each $(q, A^C) \in DP$, by

$$\forall j \leq q_n, \forall i \in Q(j),$$

$$f_{ij}(q, A^C) = \begin{cases} \Upsilon_{ij}(q, A^C) & \text{if } \sum_{i \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{ij}^C \leq 10, \\ \frac{i}{\sum_{i' \in Q(j)} i'} \left[\left(\sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^C \right) - 10 \right] + \frac{10}{Q(j)} & \text{otherwise.} \end{cases}$$

satisfies all the axioms except (L).

6.3. Proof of Theorem 3

Pick any α parameter system and consider the Mixed rule μ^α . By definition, and the fact that μ^α is computed as a convex combination of Ψ and Υ , μ^α satisfies all the axioms of the statement of Theorem 3.

It remains to show that the Mixed rules are the only rules that satisfy all the axioms of the statement of Theorem 3. Consider $(q, A^C) \in DP$ and f on DP a rule that satisfies all the axioms of the statement of Theorem 3. First, pick any I^{ij} such that $j \leq q_n$ and $i \notin \hat{P}^{-1}(Q(j)) \cup Q(j)$. The Budget balance principle implies that

$$\begin{aligned} \sum_{k \in N} \sum_{l \in \{1, \dots, q_k\}} f_{kl}(q, I^{ij}) &= \sum_{k \in N} \sum_{l \in \{1, \dots, \bar{q}_k\}} I_{kl}^{ij} \\ &= 0. \end{aligned}$$

Since an allocation is a positive real matrix, it follows that, for each $k \in N$ and each $l \in \{1, \dots, q_k\}$,

$$f_{kl}(q, I^{ij}) = 0.$$

Therefore, $f(q, I^{ij})$ is uniquely determined whenever $j \leq q_n$ and $i \notin \hat{P}^{-1}(Q(j)) \cup Q(j)$.

Now, Pick any I^{ij} such that $j \leq q_n$ and $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$. By (EIC), for each $h, h' \in Q(j)$ such that $h, h' \notin \hat{P}(i) \cup \{i\}$,

$$\begin{aligned} f_{hj}(q, I^{ij}) &= f_{h'j}(q, I^{ij}) \\ &= Y, \end{aligned} \tag{15}$$

for some $Y \geq 0$. By (F), for each $h \in (\hat{P}(i) \cup \{i\}) \cap Q(j)$,

$$\begin{aligned} f_{hj}(q, I^{ij}) &\geq Y \\ \iff f_{hj}(q, I^{ij}) &= Y + W, \end{aligned} \tag{16}$$

for some $W \geq 0$. By Lemma 1,

$$\sum_{h \in Q(j)} f_{hj}(q, I^{ij}) = 1. \tag{17}$$

Observe that $Y \leq \frac{1}{|Q(j)|}$. By (16) and (17), it follows that

$$\begin{aligned} &|(\hat{P}(i) \cup \{i\}) \cap Q(j)|(Y + W) + |Q(j) \setminus ((\hat{P}(i) \cup \{i\}) \cap Q(j))|Y = 1 \\ \iff &|(\hat{P}(i) \cup \{i\}) \cap Q(j)|W = 1 - |Q(j)|Y \\ \implies &0 \leq |(\hat{P}(i) \cup \{i\}) \cap Q(j)|W \leq 1 \\ \iff &0 \leq W \leq \frac{1}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|} \\ \iff &W = \frac{\alpha^j}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|}, \end{aligned} \tag{18}$$

for some $0 \leq \alpha^j \leq 1$. Combining (15), (16), (17) and (18), we obtain

$$\begin{aligned} &|Q(j)|Y + \alpha^j = 1 \\ \iff &Y = \frac{1 - \alpha^j}{|Q(j)|}. \end{aligned} \tag{19}$$

Combining (16) and (19), for each $h \in (\hat{P}(i) \cup \{i\}) \cap Q(j)$, we obtain

$$f_{hj}(q, I^{ij}) = \frac{1 - \alpha^j}{|Q(j)|} + \frac{\alpha^j}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|}. \tag{20}$$

By (17) and by the Budget balance principle, for each $l \neq j$ and $h \in Q(l)$,

$$f_{hl}(q, I^{ij}) = 0. \tag{21}$$

Finally, combining (19), (20) and (21), we obtain

$$\forall l \leq q_n, h \in Q(l),$$

$$f_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ \frac{1 - \alpha^j}{|Q(j)|} & \text{if } l = j \text{ and } h \notin \hat{P}(i) \cup \{i\}, \\ \frac{1 - \alpha^j}{|Q(j)|} + \frac{\alpha^j}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|} & \text{if } l = j \text{ and } h \in \hat{P}(i) \cup \{i\}, \end{cases}$$

for some $0 \leq \alpha^j \leq 1$. However, observe that

$$\forall l \leq q_n, h \in Q(l),$$

$$\Psi_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ 0 & \text{if } l = j \text{ and } h \notin \hat{P}(i) \cup \{i\}, \\ \frac{1}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|} & \text{if } l = j \text{ and } h \in \hat{P}(i) \cup \{i\}. \end{cases}$$

and

$$\Upsilon_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ \frac{1}{|Q(j)|} & \text{if } l = j \text{ and } h \notin \hat{P}(i) \cup \{i\}, \\ \frac{1}{|Q(j)|} & \text{if } l = j \text{ and } h \in \hat{P}(i) \cup \{i\}. \end{cases}$$

Thus, for each $h \in N$ and $l \leq q_h$,

$$f_{hl}(q, I^{ij}) = \alpha^l \Psi_{hl}(q, I^{ij}) + (1 - \alpha^l) \Upsilon_{hl}(q, I^{ij}), \quad (22)$$

for some parameter system α such that α^j is the one determined above. Observe that, for any $I^{i'j}$ such that $i' \in \hat{P}^{-1}(Q(j)) \cup Q(j)$ and $i' \neq i$, (EIC) and (LIC) ensure that (16) still holds. It follows that (22) still holds for the same parameter α^j even when considering a different incremental cost $I^{i'j}$. By (L), we conclude that $f(q, A^C) = \mu^\alpha(q, A^C)$. The proof of Theorem 3 is completed. \square

The axioms of the statement of Theorem 3 are logically independent, as shown by the following alternative solutions. To properly explain the first two alternative solutions, we introduce the following claim. Pick any $(q, A^C) \in DP$. Pick any sequence $(f^{ij})_{\substack{i \in N \\ j \leq K}}$ of $|N| \times K$ rules on DP . Then, define the map f as

$$f(q, A^C) = \sum_{\substack{i \in N \\ j \leq K}} f^{ij}(q, I^{ij} A_{ij}^C). \quad (23)$$

This map also qualifies as a rule on DP . Indeed, by the properties of the sum, f allocates a positive payoff to each demand unit of each consumer. Moreover, f satisfies the Budget balance principle and the Independence to higher demands principle. In the next two alternative solutions, we define rules computed in a similar manner than (23).

- Pick any $(q, A^C) \in DP$ and any $h \in N$. Let us consider the problem $(q, I^{k1}A_{k1}^C)$, $k \in N$. Recall that $Q(1) = N$. Define the allocation $\Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C)$ of a consumer $i \in N$ for his demand unit 1 by

$$\Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C) = \begin{cases} \frac{A_{k1}^C}{N} & \text{if } i \notin \hat{P}(k) \cup \{k\} \text{ and } i = h, \\ 0 & \text{if } i \notin \hat{P}(k) \cup \{k\} \text{ and } i \neq h, \\ A_{k1}^C \frac{1 - \frac{1}{N}}{|\hat{P}(i) \cup \{i\}|} & \text{if } i \in \hat{P}(k) \cup \{k\} \text{ and } h \notin \hat{P}(k) \cup \{k\}, \\ \Psi_{i1}(q, I^{k1}A_{k1}^C) & \text{otherwise.} \end{cases}$$

This allocation rule behaves just like the Connection rule, but it gives special treatment to h by allocating him a non-null payoff for irrelevant costs. Indeed, for each $i \in N$, the amount allocated by the Connection rule in $(q, I^{k1}A_{k1}^C)$ is given by

$$\Psi_{i1}(q, I^{k1}A_{k1}^C) = \begin{cases} 0 & \text{if } i \notin \hat{P}(k) \cup \{k\}, \\ \frac{A_{k1}^C}{|\hat{P}(i) \cup \{i\}|} & \text{if } i \in \hat{P}(k) \cup \{k\}. \end{cases}$$

The main difference between the two rules Ψ_{i1}^{-h} and Ψ_{i1} is that the former allocates $\frac{A_{k1}^C}{N}$ to h whenever A_{k1}^C is an irrelevant cost for h , whereas the later allocates a null amount. In case A_{k1}^C is a relevant cost for h , then the two allocations coincide.

Using the rule Ψ^{-h} , let us define another rule denoted by f . For each demand unit $j > 1$ and each consumer $i \in Q(j)$, $f_{ij}(q, A^C) = \Psi_{ij}(q, A^C)$. As for the demand unit 1, f is defined by

$$\forall i \in N, \quad f_{i1}(q, A^C) = \sum_{k \in N} \Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C).$$

The rule f satisfies all the axioms except (EIC) since consumer h is getting a special treatment for his first level of demand.

- Pick any $(q, A^C) \in DP$. For each $k \in N$, fix a parameter $\alpha^k \in [0, 1]$. The rule f^α is defined, for each $(q, A^C) \in DP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \\ f_{ij}^\alpha(q, A^C) = \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} \left[\alpha^k \Psi_{ij}(q, I^{kj}A_{kj}^C) + (1 - \alpha^k) \Upsilon_{ij}(q, I^{kj}A_{kj}^C) \right].$$

This rule operates different compromises between the connection principle and the uniformity principle for each link constituting the network. For instance, link $k \in N$ may be attributed a parameter $\alpha^k = 1$, in which case the rule f^α will allocate the incremental costs generated by k according to the connection principle. Alternatively, link $k' \in N$ may be attributed a parameter $\alpha^{k'} = 0$, in which case the rule f^α will allocate the incremental costs generated by k' according to the uniformity principle. The rule f^α satisfies all the axioms except (LIC).

- The rule f given, for each $(q, A^C) \in DP$, by

$$\begin{aligned} & \forall j \leq q_n, \forall i \in Q(j), \\ & f_{ij}(q, A^C) = \begin{cases} \Upsilon_{ij}(q, A^C) & \text{if } \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C \leq 10, \\ \Psi_{ij}(q, A^C) & \text{otherwise.} \end{cases} \end{aligned}$$

satisfies all the axioms except (L).

- The rule f is defined, for each $(q, A^C) \in DP$, as

$$\begin{aligned} & \forall j \leq q_n, \forall i \in Q(j), \\ & f_{ij}(q, A^C) = \left[\left(\frac{1}{|Q(j)|^2} + \frac{1}{|Q(j)|} \right) \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^C \right] - \frac{\Psi_{ij}(q, A^C)}{|Q(j)|}. \end{aligned}$$

This rule satisfies all the axioms except Fairness.

6.4. Proof of Theorem 4

In order to prove Theorem 4, we recall some definitions on TU-games. Consider a finite set of players $N = \{a, b, \dots, n\}$. A TU-game is a couple (N, v) , where $v : 2^N \rightarrow \mathbb{R}$ is a characteristic function assigning to each coalition of players $E \in 2^N$ its worth $v(E)$. For each (N, v) , the Shapley value is defined as

$$\forall i \in N, \quad Sh_i(N, v) = \frac{1}{n!} \sum_{\sigma \in O} (v(E_i^\sigma \cup i) - v(E_i^\sigma)), \quad (24)$$

where O is the set of all orders over N and E_i^σ is the coalition formed by the players ordered before i according to the order σ .

Next, we introduce an intermediary result. To that end, consider a distribution problem $(q, A^C) \in DP$ and its associated distribution game $(q, v^{C,P})$. For each $j \leq q_n$, define the TU-game $(N, w_j^{(q, v^{C,P})})$ as

$$\forall E \subseteq N, \quad w_j^{(q, v^{C,P})}(E) = v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right] - v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right].$$

The worth $w_j^{(m, v^{C,P})}(E)$ can be interpreted as the surplus in cost generated in $v^{C,P}$ when a group of players E decide to increase their participation level from $j-1$ to j while all the other players play the participation level $j-1$ or their maximal feasible participation level if they are unable to play $j-1$. The next Lemma is already proved on the class of monotonic multi-choice games in Lowing and Techer (2022), and so is omitted.

Lemma 2. *For each distribution problem $(q, A^C) \in DP$, its associated distribution game $(q, v^{C,P})$ verifies*

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(q, v) = Sh_i(N, w_j^{(q, v^{C,P})}). \quad (25)$$

We have the material to prove Theorem 4. Consider $(q, A^C) \in DP$ and $j \leq q_n$. For each link $k \in N$, define the TU-game $(Q(j), R^k)$ as

$$\forall E \subseteq Q(j), \quad R^k(E) = \begin{cases} 1 & \text{if } E \cap (\hat{P}(k) \cup k) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of the Shapley value,

$$\forall i \in Q(j), \quad Sh_i(Q(j), R^k) = \begin{cases} \frac{1}{|(\hat{P}(k) \cup k) \cap Q(j)|} & \text{if } i \in \hat{P}(k) \cup k, \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

Consider $(q, v^{C,P}) \in \mathcal{G}$, the distribution game associated to (q, A^C) . Consider the TU-game $(Q(j), w^{(q, v^{C,P})})$ and observe that, for each $E \subseteq Q(j)$,

$$\begin{aligned} w_j^{(q, v^{C,P})}(E) &= v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right] - v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right] \\ &\stackrel{(6)}{=} \sum_{i \in N} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right] \end{aligned} \quad (27)$$

$$\begin{aligned} &= \sum_{i \in \hat{P}^{-1}(E) \cup E} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right] \\ &\quad + \sum_{i \notin \hat{P}^{-1}(E) \cup E} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right]. \end{aligned} \quad (28)$$

For each $i \in \hat{P}^{-1}(E) \cup E$, there exists at least one $h \in \hat{P}(i) \cup i$ such that $h \in E \subseteq Q(j)$. Therefore, for each $i \in \hat{P}^{-1}(E) \cup E$,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h = j. \quad (29)$$

On the other hand,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h = j-1. \quad (30)$$

For each $i \notin \hat{P}^{-1}(E) \cup E$, there is no $h \in \hat{P}(i) \cup i$ such that $h \in E$. Therefore, for each $i \notin \hat{P}^{-1}(E) \cup E$,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h = \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h. \quad (31)$$

For the sake of clarity, for each $i \notin \hat{P}^{-1}(E) \cup E$, we denote

$$Z_i = \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h.$$

Therefore, by (29), (30) and (31), (27) becomes

$$\begin{aligned} w_j^{(q, v^{C,P})}(E) &= \sum_{i \in \hat{P}^{-1}(E) \cup E} \left[C(i, j) - C(i, j-1) \right] + \sum_{i \notin \hat{P}^{-1}(E) \cup E} \left[C(i, Z_i) - C(i, Z_i) \right] \\ &= \sum_{i \in \hat{P}^{-1}(E) \cup E} A_{ij}^C + 0 \\ &= \sum_{i \in N} A_{ij}^C \times R^i(E). \end{aligned}$$

By Lemma 2, for each $j \in \{1, \dots, q_n\}$ and each $i \in Q(j)$,

$$\begin{aligned} \varphi_{ij}(q, v^{C,P}) &= Sh_i \left(Q(j), \sum_{k \in N} A_{kj}^C \times R^k \right) \\ &= \sum_{k \in N} A_{kj}^C \times Sh_i(Q(j), R^k) \\ &= \sum_{k \in \hat{P}^{-1}(i) \cup i} \frac{A_{kj}^C}{|\hat{P}(k) \cup k \cap Q(j)|} \\ &= \Psi_{ij}(q, A^C). \end{aligned}$$

The second equality follows from the Linearity of the Shapley value and the third equality follows from (26).

Therefore, $\varphi_{ij}(q, v^{C,P}) = \Psi_{ij}(q, A^C)$ for each $i \in N$ and $j \leq q_i$. In addition, by definition of a payoff matrix for multi-choice games, $\varphi_{ij}(q, v^{C,P}) = 0$ for each $i \in N$ and $j > q_i$. Since $\Psi_{ij}(q, A^C) = 0$ for each $i \in N$ and $j > q_i$, we obtain the desired result: $\varphi(q, v^{C,P}) = \Psi(q, A^C)$. This concludes the proof. \square

6.5. Proof of Theorem 7

Lowing and Techer (2022) show that the multi-choice Shapley value belongs to the Core of sub-modular (cost) games. Therefore, it suffices to show that distribution games are sub-modular games to prove Theorem 7. Consider a distribution game $(q, v^{C,P}) \in \mathcal{G}$ associated with a distribution problem $(q, A^C) \in DP$. For each $s, t \in \mathcal{M}$,

$$v^{C,P}(t) + v^{C,P}(s) = \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} s_k) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} t_k),$$

$$\text{and } v^{C,P}(s \vee t) + v^{C,P}(s \wedge t) = \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s \vee t)_k) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s \wedge t)_k).$$

Take any $i \in N$. Without loss of generality, assume that

$$\max_{k \in \hat{P}(i) \cup i} s_k \geq \max_{k \in \hat{P}(i) \cup i} t_k.$$

Then, on the one hand,

$$\begin{aligned} \max_{k \in \hat{P}(i) \cup i} (s \vee t)_k &= \max_{k \in \hat{P}(i) \cup i} s_k \\ \implies C(i, \max_{k \in \hat{P}(i) \cup i} (s \vee t)_k) &= C(i, \max_{k \in \hat{P}(i) \cup i} s_k). \end{aligned}$$

On the other hand, since $s \wedge t \leq t$ and C is non-decreasing, it holds that

$$\begin{aligned} \iff \max_{k \in \hat{P}(i) \cup i} (s \wedge t)_k &\leq \max_{k \in \hat{P}(i) \cup i} t_k \\ \implies C(i, \max_{k \in \hat{P}(i) \cup i} (s \wedge t)_k) &\leq C(i, \max_{k \in \hat{P}(i) \cup i} t_k). \end{aligned}$$

This shows that, for each $i \in N$,

$$\sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s \vee t)_k) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s \wedge t)_k) \leq \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} s_k) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} t_k),$$

and so, we obtain the desired result

$$v^{C,P}(s \vee t) + v^{C,P}(s \wedge t) \leq v^{C,P}(s) + v^{C,P}(t).$$

□

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