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Cost Allocation in Natural Gas Distribution Networks

David Lowing^a

^a*Univ Lyon, UJM Saint-Etienne, GATE UMR 5824, F-42023 Saint-Etienne, France.*

Abstract

A natural gas distribution network connects consumers to a source of gas. It is managed by a network operator, whose task incurs various costs, some of which may not be attributable to a particular consumer. Assuming that the operator wishes to recover these costs by charging for its services, the problem is then to determine how much each consumer should pay. In other words, how should these costs be allocated among consumers. In this paper, we address this problem and propose cost allocation rules that depend on the network and the demands of the consumers. To that end, we adopt a normative approach and oppose two principles: (i) the connection principle and (ii) the uniformity principle. The Connection rule is proposed in accordance with (i), while the Uniform rule is developed in line with (ii). It appears that (i) and (ii) are incompatible. To make a trade-off between these two principles, we propose the family of Mixed rules, which compromise between the Connection rule and the Uniform rule by mean of convex combinations. For each cost allocation rule, an axiomatic characterization is provided. Then, we show that the Connection rule coincides with the multi-choice Shapley value of a specific multi-choice game derived from the network and the demands of the consumers. Moreover, the Connection rule is in the Core of this specific multi-choice game. Similarly, we show that the Uniform rule coincides with the multi-choice Equal division value and the Mixed rules coincide with the multi-choice Egalitarian Shapley values.

Keywords: Natural gas distribution network; Cost allocation rules; Axiomatization; Multi-choice games

JEL codes: C71; D61

1. Introduction

Natural gas distribution is carried out through a network that takes gas from a source (e.g. a transmission network entry point or a gas storage facility) and delivers it to consumers. The network is under the management of an operator who is responsible for its safety and maintenance. In order to carry out its task properly, the network operator is confronted with various **operation costs**. Some of these costs cannot be directly attributed to a specific consumer. For example, there are costs related to gas storage (necessary to overcome seasonal fluctuations in gas consumption), costs related to the maintenance of the pipelines and costs related to public safety (monitoring the network by drone, conducting awareness campaigns, etc.). Under the assumption that the network operator recovers these operation costs by charging the consumers, our problem is then to determine how to allocate these costs among them.

Email address: david.lowing@outlook.com (David Lowing)

1.1. The gas distribution problem

To conduct this study, we formally define a problem adapted to the gas distribution setting. Fix a finite set $N = \{a, b, \dots, n\}$ of **consumers**, directly or indirectly, connected to a source of gas by means of a fixed **network**, which is represented by a rooted tree graph P . The nodes of the graph represent the consumers plus the source of gas \mathcal{S} , while the arcs of the graph represent the pipelines of the network. In a tree graph, each node has at most one direct predecessor. Therefore, an element in N will refer to both a consumer and the pipeline having this consumer at its tail.

Each pipeline has a certain **capacity**, which is represented by an integer. There is an industrial limit to the capacity of the pipelines given by $K \in \mathbb{K}$. Each consumer $i \in N$ has an **effective demand** in capacity, which is represented by an integer $q_i \leq K$. It corresponds to the capacity required to meet i 's highest gas consumption over a certain period of time. For instance, the effective demand of a regular household is often determined by its highest daily consumption during winter. The profile of all effective demands is given by $q = (q_a, \dots, q_n)$.

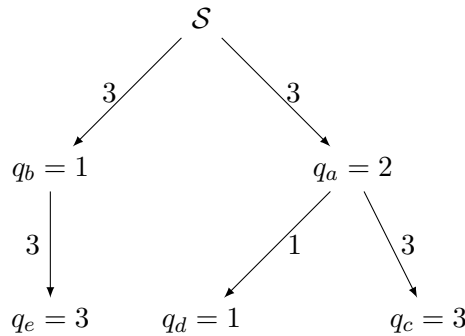


Figure 1: Network design

The network operator is responsible for fulfilling any effective demand at all times, and must therefore ensure that its network is **designed** accordingly. This means that each pipeline must have the capacity to meet the effective demands of its downstream consumers. In this paper, we assume that each pipeline has the capacity to meet the highest effective demand of the consumers located downstream of it, which we refer to as the pipeline's **effective capacity**. For instance, Figure 1 illustrates a situation in which consumer a , c and d have an effective demand of 2, 3 and 1, respectively. In this case, both pipelines a and c have an effective capacity of 3. For the same reason, pipeline d has an effective capacity of 1. There are alternatives to this approach to network design. Each approach, including this one, has its advantages and its drawbacks. We will discuss some of these approaches at the end of the paper.

A **cost function** $C : N \times \{1, \dots, K\} \rightarrow \mathbb{R}_+$ is a map that computes the cost $C(i, j)$ of operating any pipeline $i \in N$ of any capacity $j \leq K$. We assume that the map $C(i, \cdot) : \{1, \dots, K\} \rightarrow \mathbb{R}_+$ is non-decreasing, since a larger capacity leads to higher costs, and zero normalized. The **total cost** of operating the network is computed as the sum of the costs of all pipelines, where each pipeline meets its effective capacity.

The problem is then to determine appropriate rules to allocate this total cost among the consumers. This problem is called the **gas distribution problem** and is denoted by (N, q, C, P) . We propose three cost allocation rules to solve this problem. These rules are based on principles that may be considered desirable for a network operator.

1.2. Principles for cost allocation rules

To properly define cost allocation rules, let us endow each consumer $i \in N$ with the discrete set of **demand units** $\{1, \dots, q_i\}$. A **cost allocation rule**, or a rule for short, is a map f that allocates a real number $f_{ij}(N, q, C, P)$ to each demand unit $j \in \{1, \dots, q_i\}$ of each consumer $i \in N$. Each rule studied in this paper follows the two following principles.

- (i) **Budget balanced principle**: a cost allocation rule must recover the total cost of operating the network. This means that the network operator does not make any losses or profits from its activity.
- (ii) **Independence of higher demands principle**: the amount allocated to a consumer for a certain demand unit should be independent from any other greater demand unit. This principle makes it possible to avoid situations where the presence of a consumer with a high demand implies additional and unjustified costs to consumers with lower demands. The independence of higher demands principle was formally discussed by Moulin and Shenker (1992) in the context of cost sharing problems.

The quantity $f_{ij}(N, q, C, P)$ can be interpreted as the additional amount allocated to consumer i if he increases his demand from $j - 1$ to j . In other words, a rule describes how the amount allocated to a consumer varies according to his demand. This can be useful, for example, if one wishes to derive a pricing schedule from a rule. Obviously, it suffices to sum each $f_{ij}(N, q, C, P)$, $j \in \{1, \dots, q_i\}$, to obtain the **total allocation** of i . In addition to (i) and (ii), we discuss two other principles.

- (iii) **Connection principle**: a consumer should only be charged for the costs associated with the specific pipelines that connect him to the source. This ensures that costs are allocated fairly and that consumers are not burdened with the costs of infrastructure that they do not benefit from.
- (iv) **Uniformity principle**: two consumers with the same demands should be charged with the same amount regardless of their geographical location. For example, there should be no difference in treatment between rural and urban consumers, although the underlying distribution costs are different. In fact, (public) network operators in France highly favor rules that respect this principle (see Fleurbaey and Trannoy (1998)). The aim is to reduce inequalities and ensure that all residents have access to a similar level of public services.

Contrary to principles (i) and (ii), principles (iii) and (iv) are incompatible and cannot be satisfied by the same rule. Indeed, (iii) states that consumers should only pay for the portion of the network they use. Therefore, two consumers with the same demands can be charged with different amount depending on their position on the network, which contradicts (iv).

In this paper, we propose three rules. First, the **Connection rule**, follows principles (i), (ii) and (iii). It ensures that each consumer pays a fair share of the portion of the network that connect him to the source. Second, the **Uniform rule**, follows principles (i), (ii) and (iv). It ensures that two consumers are allocated the same amount for the same demand unit, regardless of any other parameters of the gas distribution problem. Since the principles (iii) and (iv) are incompatible, we propose a family of rules to compromise between them: the **Mixed rules**. We detail how these rules are computed in the next section.

We formalize (iii) and (iv) into axioms for rules. Additionally, we introduce other axioms that do not necessarily fit into the idea of principles (iii) and (iv) but can still be viewed as desirable properties in the context of gas distribution. Combining these axioms, we provide an axiomatic characterization for each rule introduced in this paper. Observe that principles (i) and (ii) are not formalized into axioms. Instead, we directly incorporate them into the formal definition of a rule (see Definition 2).

1.3. Computing the cost allocation rules

To clearly present our rules, assume that a pipeline $i \in N$ is built, step by step, by increasing its capacity from 0 to its effective capacity, which we denote by \bar{q}_i . At each step $j \in \{1, \dots, \bar{q}_i\}$, pipeline i undergoes an upgrade, called the **j-th upgrade of i**. Due to this upgrade, an **incremental cost** $A_{ij}^C = C(i, j) - C(i, j - 1)$ is generated. This incremental cost represents the additional operation costs incurred by the upgrade. This procedure allows us to explain how each upgrade impacts the cost allocations.

The **Connection rule** is computed as follows. Pick any pipeline $i \in N$. **Step 1:** let us assign a capacity of 1 to pipeline i . This corresponds to its 1-st upgrade. The Connection rule allocates the incremental cost A_{i1}^C equally among the first demand unit of all downstream consumers of i . **Step 2:** upgrade the capacity of pipeline i by one unit. This 2-nd upgrade generates a positive incremental cost A_{i2}^C . The Connection rule allocates this incremental cost equally among the second demand units of all downstream consumers of i whose effective demand exceeds or equals 2. This procedure continues until pipeline i reaches its effective capacity, i.e., until **Step \bar{q}_i** . This procedure is applied to each pipeline and defines the Connection rule (see Algorithm 1).

The **Uniform rule** is computed in a similar manner. Pick any pipeline $i \in N$. **Step 1:** the Uniform rule allocates A_{i1}^C equally among the first demand units of all consumers regardless of their position on the network. **Step 2:** the Uniform rule allocates A_{i2}^C equally among the second demand units of all consumers whose effective demand exceeds or equals 2. This procedure continues until **Step \bar{q}_i** . This procedure is applied to each pipeline and defines the Uniform rule (see Algorithm 2).

To reach a compromise between principles (iii) and (iv), we propose the family of **Mixed rules**. Each rule in this family achieves a compromise by means of convex combinations between the Connection rule and the Uniform rule. Pick any pipeline $i \in N$. **Step 1:** the Mixed rule allocates A_{i1}^C to the first demand unit of each consumer. This allocation is computed as a convex combination between the two allocations prescribed by the Connection rule and the Uniform rule. This combination depends on a parameter $\alpha^1 \in [0, 1]$. For instance, if $\alpha^1 = 0.8$, the final allocation consists of 80% of the allocation prescribed by the Connection rule, plus 20% of the allocation prescribed by the Uniform rule. **Step 2:** the Mixed rule allocates A_{i2}^C among the second demand units of all consumers whose effective demand exceeds or equals 2. This allocation is computed as a convex combination between the Connection rule and the Uniform rule. This second combination depends on a parameter $\alpha^2 \in [0, 1]$, which may be different from α^1 . This procedure continues until **Step \bar{q}_i** . This procedure is applied to each pipeline and defines the Mixed rules (see Algorithm 3).

Observe that the Mixed rules allow to reach different compromises depending on the level of upgrade one is considering. For instance, it is possible to reach a compromise in favor of (iv) for low-capacity upgrades. This ensures that consumers with low effective demand are treated relatively equally, regardless of their geographical location. Meanwhile, it is possible to reach a compromise in favor of (iii) for high-capacity upgrades. In this way, consumers with high effective demand are allocated an amount (for their highest demand units) that reflects their true impact on the cost of the network. Finally, observe that each rule can be computed in polynomial time. This is obvious from the Algorithms 1, 2 and 3.

1.4. Comparison with multi-choice games

An appropriate game theoretic tool for modeling gas distribution problems are multi-choice (cooperative) games. Multi-choice games, introduced by Hsiao and Raghavan (1992) and van den Nouweland (1993), are a natural extension of TU-games in which each player is endowed with a certain number of activity levels. A (multi-choice) coalition is a profile describing each player's activity level within

this coalition. The worth of each coalition is measured by a characteristic function. Given a gas distribution problem, we derive a special multi-choice game associated with this problem. This game is called the gas distribution game. The player set represents the set of consumers, and the activity levels represent the demands of the consumers. The worth of a coalition corresponds to the total cost of the network in which each pipeline meets its effective capacity. The Connection rule applied to a gas distribution problem corresponds to the multi-choice Shapley value, introduced by Lowing and Techer (2022), of the corresponding gas distribution game. Similarly, the Uniform rule corresponds to the multi-choice Equal division value and the Mixed rules to the multi-choice Egalitarian Shapley values. Both values are introduced by Lowing and Techer (2022). Moreover, for each gas distribution problem, the multi-choice Shapley value of the corresponding gas distribution game is in the Core of this gas distribution game.

1.5. Related literature

The delivery of natural gas from production sites to consumers can be separated into two stages: transmission and distribution. While some papers address cost or resource sharing issues in the context of gas transmission, few studies have addressed similar problems in gas distribution. For instance, Junqueira et al. (2007) investigate cost allocation in energy transmission networks, and Bergantiños et al. (2017) examine the allocation of energy losses in transmission networks. On the other hand, some papers focus on cooperation aspects of gas-related issues, such as Massol and Tchong-Ming (2010), who explore cooperation among liquefied gas suppliers, and Csercsik et al. (2019) who model transfer profits as externalities using partition function form games.

The gas distribution problem shares some technical similarities with the polluted river problem studied by Ni and Wang (2007), Dong et al. (2012), van den Brink et al. (2018), and Li et al. (2022), to cite a few. In the polluted river problem, agents living along a river must bear the cost of cleaning the water, which carries pollutants. To clean the water, some costs are incurred and must be allocated among the agents. The main difference between the two models is that the gas distribution problem takes into account the demands of consumers (or agents), while the polluted river problem does not. The connection principle invoked in the gas distribution problem is analogous to the upstream responsibility principle used in the unlimited territorial integrity theory for polluted river problems, which states that agents located along the polluted river should pay for cleaning the portion of the river that connects them to the source of the river (see Dong et al. (2012) for details). Furthermore, the Connection rule is conceptually close to the Downstream equal sharing solution, introduced by Dong et al. (2012) for polluted river problems.

Similarly to how we make an analogy between gas distribution problems and multi-choice games, van den Brink et al. (2018) makes an analogy between polluted river problem and TU-games with a permission structure (see Gilles et al. (1992) for details on games with a permission structure). In particular, the authors show that the permission value (see van den Brink and Gilles (1996)) applied to a special TU-game constructed from the polluted river problem is equivalent to the Downstream equal sharing solution.

1.6. Overview of the paper

The rest of the paper is organized as follows. We define the gas distribution problem in Section 2. Section 3 presents the cost allocation rules and provide their axiomatic characterizations. Section 4 is dedicated to multi-choice games. Section 5 concludes and provides leads for future research. Section 6 is an appendix containing all the proofs of the results.

2. The model

Fix $N = \{a, b, \dots, n\}$, $|N| \geq 3$, a finite set of **consumers**. These consumers are connected to a source \mathcal{S} through **pipelines**. The consumers and the pipelines form a fixed **network**, which is modeled by a rooted tree graph. The rooted tree graph is a couple $(N \cup \mathcal{S}, P)$ where P is a map $P : N \cup \mathcal{S} \rightarrow 2^{N \cup \mathcal{S}}$ representing the links (pipelines) between the nodes. If no confusion arises, simply denote the network by P . The relationship $i' \in P(i)$ means that $i' \in N$ is supplied right after consumer i . Equivalently, denote $i \in P^{-1}(i')$ if and only if $i \in N$ is supplied right before consumer i' . In this case, consumer i is called the **predecessor** of i' while i' is the **successor** of i . Put differently, i is the **head** of the pipeline and i' is the **tail** of the pipeline. In a rooted tree graph, each node has at most one predecessor and only one node (the source) has no predecessor, i.e., $P^{-1}(\mathcal{S}) = \emptyset$. The transitive closure of a rooted graph P is a rooted graph \hat{P} such that, for each $i \in N$, $i' \in \hat{P}(i)$ if and only if there exists a path $i = h_1, h_2, \dots, h_k = i'$ such that $h_k \in P(h_{k-1}), \dots, h_2 \in P(h_1)$. The consumers in $\hat{P}(i)$ are called the **subordinates** of $i \in N$ in P , and the consumers in $\hat{P}^{-1}(i) := \{i' \in N : i \in \hat{P}(i')\}$ are called the **superiors** of $i \in N$ in P . Similarly, the set $\hat{P}(E)$ represents the subordinates of the consumers in $E \subseteq N$ and $\hat{P}^{-1}(E)$ represents the superiors of the consumers in $E \subseteq N$. For each $i \in N$, we call **pipeline i** the pipeline having consumer i at its tail. Consumer $i \in N$ and his subordinates are called the **downstream consumers** of pipeline i .

Each pipeline has a certain **capacity**, which is represented by an integer. Fix $K \in \mathbb{N}$ an upper bound for the capacity of the pipelines. Each consumer $i \in N$ has an **effective demand** in capacity given by $q_i \leq K$. The **profile of effective demands** is denoted by $q = (q_a, \dots, q_n)$. Without loss of generality, assume that $q_n \geq q_i$, for each $i \in N$. The set of all consumers whose effective demand exceeds or equals j is denoted by $Q(j) = \{i \in N : q_i \geq j\}$. We assume that $Q(1) = N$. A **cost function** is a map

$$C : N \times \{0, \dots, K\} \rightarrow \mathbb{R}_+$$

that measures the cost of any pipeline of any capacity. Formally, for each $i \in N$ and $j \in \{1, \dots, K\}$, $C(i, j)$ represents the cost of operating the pipeline i when it is designed have a capacity of j . We use the convention $C(i, 0) = 0$ for each $i \in N$. We assume that each map

$$\forall i \in N, \quad C(i, \cdot) : \{1, \dots, K\} \rightarrow \mathbb{R}_+$$

is non-decreasing: a larger pipeline's capacity leads to higher costs. On the other hand, for any $j \in \{1, \dots, K\}$ and any two $i, i' \in N$, we do not necessarily have $C(i, j) = C(i', j)$. Indeed, the cost of operating a pipeline may differ depending on the pipeline. These differences are due to exogenous features of the gas distribution network such as the length or the geographical location of the pipelines.

Equivalently, a cost function can be expressed as a matrix of incremental costs. We denote by A_{ij}^C the **incremental cost** generated by the j -th upgrade of pipeline i , i.e., when pipeline i is upgraded by one unit from capacity $j - 1$ to capacity j . Formally, it is defined as

$$\forall i \in N, \forall j \in \{1, \dots, K\}, \quad A_{ij}^C = C(i, j) - C(i, j - 1).$$

Obviously, $A_{ij}^C \geq 0$ for each $i \in N$ and each $j \in \{1, \dots, K\}$, since $C(i, \cdot)$ is a non-decreasing map. The (n, K) **matrix of incremental costs** collects all the incremental costs, and is denoted by A^C . Pick any $i \in N$ and any $j \leq K$. The **unit cost matrix** I^{ij} is the (n, K) matrix defined as

$$\forall k \in N, l \leq K, \quad I_{kl}^{ij} = \begin{cases} 1 & \text{if } k = i, l = j, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

In such matrix, only the j -th upgrade of pipeline i generates non-null incremental costs. Unit cost matrices will be used to focus on the specific upgrade of a specific pipeline. A matrix of incremental costs A^C can be expressed as the linear combination of $n \times K$ unit cost matrices:

$$A^C = \sum_{i \in N} \sum_{j \leq K} A_{ij}^C I^{ij}. \quad (2)$$

We propose an approach to compute the **total cost** of operating a gas distribution network from N , q , P and C . Alternative approaches are discussed at the end of this paper. Assume that each pipeline is always designed at its **effective capacity**, i.e., its capacity equals the highest effective demand of its downstream consumers. Then, for each $i \in N$, the cost of operating pipeline i is given by

$$\forall i \in N, \quad C(i, \bar{q}_i), \quad \text{where } \bar{q}_i = \max_{k \in \hat{P}(i) \cup i} q_k. \quad (3)$$

Equivalently, observe that the cost of operating a pipeline can be expressed using incremental costs.

$$\forall i \in N, \quad C(i, \bar{q}_i) = \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C. \quad (4)$$

Expression (4) highlights how the cost of operating a given pipeline varies according to its capacity. The **total cost** of operating the network is computed as the sum of the costs of all the pipelines designed at their effective capacity (3), which is given by

$$\sum_{i \in N} C(i, \bar{q}_i), \quad \text{or equivalently,} \quad \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C. \quad (5)$$

Definition 1 (The gas distribution problem). A gas distribution problem is denoted by (N, q, C, P) , or by (q, A^C) for short, since N and P are fixed and C and A^C are equivalent. The problem is to determine a way to allocate the total cost (5) among consumers based on their demands and their location on the distribution network. The set of gas distribution problems is denoted by GDP .

3. Rules and characterizations

We address gas distribution problems by defining **cost allocation rules** (rules for short). This section is devoted to the study of three rules: the Connection rule, the Uniform rule and the Mixed rules. Our rules describe how the allocation of a consumer varies according to his demand. An axiomatic characterization for each one of these rules is provided.

To properly define rules, we endow each consumer $i \in N$ with the discrete set of **demand units** $\{1, \dots, q_i\}$. A (cost) allocation $x \in \mathbb{R}_+^{n \times K}$ on GDP assigns a non-negative real number x_{ij} to each demand unit $j \in \{1, \dots, q_i\}$ of each consumer $i \in N$. We use the convention that for each $j \notin \{1, \dots, q_i\}$, $x_{ij} = 0$. We give a formal description of the Budget balanced principle and the Independence of higher demands principle for allocations.

- (i) **Budget balanced principle:** an allocation should recover the total cost of operating the network, i.e.,

$$\sum_{i \in N} \sum_{j \in \{1, \dots, q_i\}} x_{ij}(q, A^C) = \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C.$$

- (ii) **Independence of higher demands principle:** the amount allocated to a consumer for a certain demand unit is independent from any other greater demand unit, i.e., for each $(q, A^C) \in GDP$ and each $l \in \{0, \dots, K\}$,

$$\forall i \in Q(l), \forall j \leq l, \quad x_{ij}(q, A^C) = x_{ij}((q_k \wedge l)_{k \in N}, A^C).$$

Definition 2 (Cost allocation rule). A cost allocation rule on GDP is a map

$$f : GDP \rightarrow \mathbb{R}_+^{n \times K}$$

that associates an allocation $f(q, A^C) \in \mathbb{R}_+^{n \times K}$, satisfying (i) and (ii), to each gas distribution problem $(q, A^C) \in GDP$. The quantity $f_{ij}(q, A^C)$ can be interpreted as the additional amount allocated to i if he increases his demand from $j - 1$ to j , all other parameters being equal.

3.1. The Connection rule

We define the **Connection rule** on GDP . This rule ensures that each consumer pays a fair share of the portion of the network that connect them to the source of gas. Formally, for each pipeline k and each capacity j such that some downstream consumers of pipeline k are in $Q(j)$, the Connection rule allocates the incremental cost A_{kj}^C equally among the downstream consumers of pipeline k that are in $Q(j)$.

Definition 3 (Connection rule). The Connection rule Ψ is defined, for each $(q, A^C) \in GDP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Psi_{ij}(q, A^C) = \sum_{k \in \hat{P}^{-1}(i) \cup i} \frac{A_{kj}^C}{(|\hat{P}(k) \cup k| \cap Q(j))}. \quad (6)$$

From (6), it is clear that the Connection rule follows the Budget balanced principle. Moreover, the cost allocated to a consumer i for a demand unit j is computed using only the incremental costs generated by j -th upgrade of the pipelines. Therefore, the Connection rule also follows the Independence of higher demands principle. Algorithm 1 formalizes the procedure depicted in the Introduction to compute the Connection rule.

Algorithm 1: Connection rule

Data: Consumers N , upper bound K , matrix A^C , profile $q \in \mathbb{N}^n$ and graph P .
 $\Psi \leftarrow 0$
 \\\ This loops iterates over all pipelines
for $k \in N$ **do**
 | \\\ This loops iterates over each integer j below the pipeline's effective capacity. For each j , it
 | defines W as the set of downstream consumers of k whose effective demand exceeds or equals j
 | **for** $j \leq \bar{q}_k$ **do**
 | | $Q(j) \leftarrow \{i \in N : q_k \geq j\}$
 | | $D(k) \leftarrow \hat{P}(k) \cup \{k\}$
 | | $W \leftarrow D(k) \cap Q(j)$
 | | \\\ Last loop allocates an equal share of A_{kj}^C to the j -th demand unit of each consumers in W
 | | **for** $i \in W$ **do**
 | | | $\Psi_{ij} \leftarrow \Psi_{ij} + A_{kj}^C / |W|$
 | | **end**
 | **end**
end
Result: Ψ

An axiomatic characterization of the Connection rule is provided. It invokes two axioms. We say that a pipeline is **irrelevant** to a consumer if it is not a pipeline that helps to connect this consumer to the source. The first axiom is in line with the Connection principle as it ensures that a consumer is not charged for the costs generated by irrelevant pipelines. Formally, the axiom states that the cost charged to a demand unit of a consumer should not be impacted by an increase in incremental cost of an irrelevant pipeline.

Independence to Irrelevant Cost (IIC): Pick any $(q, A^C) \in GDP$. For each $j \leq K$, each $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon \in \mathbb{R}$,

$$\forall h \in Q(j), h \notin (\hat{P}(i) \cup \{i\}), \quad f_{hj}(q, A^C) = f_{hj}(q, A^C + \varepsilon I^{ij}).$$

The costs generated by a pipeline are relevant to any of its downstream consumers. The next axiom is an equal loss requirement, arguing that the rule must treat each downstream consumer equally regarding a given upstream cost. Formally, the axiom states that the amount allocated to the demand unit j of two downstream consumers should be equally impacted by the increase of an incremental cost generated by the j -th upgrade of an upstream pipeline.

Equal Loss for Downstream Consumers (EDC): Pick any $(q, A^C) \in GDP$. For each $j \leq K$, each $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon \in \mathbb{R}$,

$$\forall h, h' \in (\hat{P}(i) \cup \{i\}) \cap Q(j), \quad f_{hj}(q, A^C + \varepsilon I^{ij}) - f_{hj}(q, A^C) = f_{h'j}(q, A^C + \varepsilon I^{ij}) - f_{h'j}(q, A^C).$$

Observe that (EDC) does not necessarily fit into the idea of the Connection principle or the Uniformity principle. In fact, we will see that this axiom is also satisfied by the other rules presented in this paper. We now have the necessary material to characterize the Connection rule.

Theorem 1. *A rule f on GDP satisfies (IIC) and (EDC) if and only if $f = \Psi$.*

Proof. See Appendix 6.1 □

Remark 1. The axioms invoked in this characterization are conceptually close to those used in Dong et al. (2012) to characterize the Downstream equal sharing rule for polluted river problems. In short, their characterization uses axioms equivalent to (IIC), (EDC), the Budget balanced principle and an Additivity axiom. However, contrary to Dong et al. (2012), we do not use an Additivity axiom in our characterization. Moreover, the Budget balanced principle is already incorporated within Definition 2, thus we do not invoke it in the characterization.

3.2. The Uniform rule

This section defines the Uniform rule on GDP . This rule ensures that two consumers should always be allocated the same amount for the same demand unit, regardless of any other parameter of the gas distribution problem. Formally, the Uniform rule allocates each incremental cost A_{ij}^C equally among each consumer in $Q(j)$ (assuming that pipeline i 's effective capacity exceeds or equals j).

Definition 4 (Uniform rule). The Uniform rule Υ is defined, for each $(q, A^C) \in GDP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Upsilon_{ij}(q, A^C) = \frac{1}{|Q(j)|} \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^C. \quad (7)$$

From (7), it is clear that the Uniform rule follows the Budget balanced principle and the Independence of higher demands principle. Observe that the Uniform rule satisfies (EDC) but does not satisfy (IIC). Algorithm 2 formalizes the procedure depicted in the Introduction to compute the Uniform rule.

Algorithm 2: Uniform rule

Data: Consumers N , upper bound K , matrix A^C , profile $q \in \mathbb{N}^n$ and graph P .
 $\Psi \leftarrow 0$
 \\\This loops iterates over all pipelines
for $k \in N$ **do**
 \\\This loops iterates over each integer j below the pipeline's effective capacity. For each j , it defines $Q(j)$ as the set of consumers whose effective demand exceeds or equals j
for $j \leq \bar{q}_k$ **do**
 $Q(j) \leftarrow \{i \in N : q_k \geq j\}$
 \\\Last loop allocates an equal share of A_{kj}^C to the j -th demand unit each of consumers in $Q(j)$
for $i \in Q(j)$ **do**
 $\Upsilon_{ij} \leftarrow \Upsilon_{ij} + A_{kj}^C / |Q(j)|$
end
end
end
Result: Υ

To characterize the Uniform rule, we must introduce an additional axiom. Let f be a rule on GDP . Assume that one or several pipelines generate additional costs due to exogenous reasons (incident, natural disaster, etc), which leads to an increase of the incremental costs. Such event should not increase the inequalities between the allocations of the consumers. Formally, pick any capacity j and assume that all the incremental costs generated by the j -th upgrades of the pipelines increase. Compare the difference between the highest allocation and the lowest allocation among the consumers in $Q(j)$ for their demand unit j . This difference should not be increased when the incremental costs increase.

Non-Increasing Inequalities (NII): For each $(q, A^C), (q, A^{C'}) \in GDP$ such that $A_{ij}^{C'} \geq A_{ij}^C$, for each $i \in N$ and $j \leq q_n$,

$$\forall j \in \{1, \dots, q_n\}, \quad \max_{i \in Q(j)} f_{ij}(q, A^{C'}) - \min_{i \in Q(j)} f_{ij}(q, A^{C'}) \leq \max_{i \in Q(j)} f_{ij}(q, A^C) - \min_{i \in Q(j)} f_{ij}(q, A^C).$$

This axiom is in line with the uniformity principle and is sufficient to characterize the Uniform rule. Obviously, the Connection rule does not satisfy (NII).

Theorem 2. *A rule f on GDP satisfies (NII) if and only if $f = \Upsilon$.*

Proof. See Appendix 6.2 □

3.3. The Mixed rule

The Uniform rule does not satisfy (IIC), and the Connection rule does not satisfy (NII). This testifies that the uniformity principle and the connection principle are clearly incompatible. However, both principles can be highly desirable in the context of gas distribution problems. For this reason, a trade-off between the two principles is proposed by introducing a family of rules that compromises between the uniformity principle and the connection principle. The rules in this family are called

Mixed rules are computed by using convex combinations between the Connection rule and the Uniform rule. Moreover, Mixed rules allow for different compromises between the uniformity principle and the connection principle, depending on the level of demand.

Definition 5 (Mixed rules). Let $\alpha = \{\alpha^j\}_{1 \leq j \leq K}$ be a parameter system such that $\alpha^j \in [0, 1]$, for each $1 \leq j \leq K$. The α -Mixed rule μ^α is defined, for each $(q, A^C) \in GDP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \mu_{ij}^\alpha(q, A^C) = \alpha^j \Psi_{ij}(q, A^C) + (1 - \alpha^j) \Upsilon_{ij}(q, A^C).$$

By definition, it is clear that any α -Mixed rule follows the Budget balanced principle and the Independence of higher demands principle. Algorithm 3 formalizes the procedure depicted in the Introduction to compute the Mixed rules.

Algorithm 3: Mixed rules

Data: Consumers N , upper bound K , matrix A^C , profile $q \in \mathbb{N}^n$, a graph P , and a system $\alpha \in [0, 1]^K$.

$\mu \leftarrow 0$

*//*This loops iterates over all pipelines

for $k \in N$ **do**

*//*This loops iterates over each integer j below the pipeline's effective capacity.

for $j \leq \bar{q}_h$ **do**

$Q(j) \leftarrow \{i \in N : q_k \geq j\}$

$D(k) \leftarrow \hat{P}(k) \cup \{k\}$

$W \leftarrow D(k) \cap Q(j)$

*//*This loops iterates over all consumers in $Q(j)$. First, it allocates an equal share of $(1 - \alpha^j) \times A_{kj}^C$ to the j -th demand unit of each consumers in $Q(j)$. Then, it allocates an equal share of $\alpha^j \times A_{kj}^C$ to the j -th demand unit of consumers in $D(k)$.

for $i \in Q(j)$ **do**

$\mu_{ij} \leftarrow \mu_{ij} + (1 - \alpha^j) \times A_{kj}^C / |Q(j)|$

if $i \in D(k)$ **then**

$\mu_{ij} \leftarrow \mu_{ij} + \alpha^j \times A_{kj}^C / |W|$

end

end

end

Result: μ

Remark 2. A α -Mixed rule operates convex combinations between the Connection rule and the Uniform rule. For each demand $j \in \{0, \dots, q_n\}$, a consumer $i \in Q(j)$ receives an allocation lying between the Connection rule $\Psi_{ij}(q, A^C)$ and the Uniform rule $\Upsilon_{ij}(q, A^C)$. If α^j is closer to 1, then this allocation is closer to $\Psi_{ij}(q, A^C)$. On the other hand, if α^j is closer to 0, then this allocation is closer to $\Upsilon_{ij}(q, A^C)$.

To characterize the Mixed rules, we introduce four new axioms. The first axiom describes how an irrelevant cost impacts the allocation of two distinct consumers. The second axiom describes how the allocation of a consumer is impacted by two distinct irrelevant costs. The third axiom is a linearity requirement. Finally, the last axiom describes the impact of a given cost on two consumers. This cost being irrelevant for one of them, and relevant for the other.

Let f be a rule on GDP . The first axiom states that any two distinct consumers are equally impacted by irrelevant costs, without specifying the extent of this impact. Pick any $(q, I^{ij}) \in GDP$. Recall that a unit cost matrix I^{ij} possesses a unique non-null incremental cost. The next axiom states

that any two distinct consumers for whom this incremental cost is irrelevant should be allocated the same amount in (q, I^{ij}) .

Equal impact of irrelevant costs (EIC): For each $(q, I^{ij}) \in GDP$,

$$\forall h, h' \in Q(j), h, h' \notin (\hat{P}(i) \cup \{i\}), \quad f_{hj}(q, I^{ij}) = f_{h'j}(q, I^{ij}).$$

Observe that both the Connection rule and the Uniform rule satisfy (EIC). Moreover, (IIC) implies (EIC). The converse is not true. The next axiom states that no matter where an irrelevant cost is located, the impact of that cost on a consumer, for whom that cost is irrelevant, remains the same. Formally, pick any two distinct problems $(q, I^{ij}), (q, I^{i'j}) \in GDP$. Assume that each of the two unit cost matrices features a unique non-null incremental cost that can be considered irrelevant for a given consumer. The axiom states that the consumer for whom these incremental costs are not relevant should be allocated the same amount in both problems.

Location independence of irrelevant costs (LIC): For each $(q, I^{ij}), (q, I^{i'j}) \in GDP$,

$$\forall h \in Q(j), h \notin [\hat{P}(i) \cup \{i\}] \cup [\hat{P}(i') \cup \{i'\}], \quad f_{hj}(q, I^{ij}) = f_{hj}(q, I^{i'j}).$$

Observe that both the Connection rule and the Uniform rule satisfy (LIC). Moreover, (IIC) implies (LIC). The converse is not true. The next axiom, Linearity, is a classical axiom in economic theory. An interpretation is provided in the context of natural gas distribution. The costs of operating a network can be divided into several categories. For instance, one could separate the maintenance costs and the costs to access the gas storage. Linearity ensures that there is no difference by considering the two costs separately or together. Moreover, if the costs are expressed in USD and we want the rule to be expressed in EUR, then Linearity ensures that there is no difference between converting before or after applying of the rule.

Linearity (L): For each $(q, A^C), (q, A^{C'}) \in G$ and each $\beta \in \mathbb{R}_+$,

$$\forall (i, j) \in M^+, \quad f_{ij}(q, A^C + \beta A^{C'}) = f_{ij}(q, A^C) + \beta f_{ij}(q, A^{C'}).$$

Obviously, both the Connection rule and the Uniform rule satisfy (L). The last axiom advocates that the costs incurred by a pipeline should mostly be allocated to its downstream consumers. Formally, consider any problem $(q, I^{ij}) \in GDP$. The axiom states that any consumer located downstream of the cost generating pipeline i should not be allocated less than other consumers.

Fairness (F): For each $(q, I^{ij}) \in GDP$,

$$\forall k \in (\hat{P}(i) \cup \{i\}) \cap Q(j), \forall h \in Q(j), \quad f_{kj}(q, I^{ij}) \geq f_{hj}(q, I^{ij}).$$

Observe that both the Connection rule and the Uniform rule satisfy (F). We now have the material to characterize the Mixed rules.

Theorem 3. *A rule f on GDP satisfies (EIC), (LIC), (L) and (F) if and only if $f = \mu^\alpha$, for some parameter system α .*

Proof. See Appendix 6.3 □

3.4. Illustrative example

Consider the set of consumers $N = \{a, b, c, d, e\}$ and a source \mathcal{S} in gas. In this example, we consider the distribution network P and the profile of effective demands $q = (2, 1, 3, 1, 3)$, which are illustrated by Figure 2b. For convenience, we fix $K = 3$. We introduce the lengths of the pipelines, which is one of the possible exogenous features of a network. The profile $L = (1, 2, 1, 2, 1)$ collects these lengths. Consider the cost function given by

$$C : \{a, b, c, d, e, f\} \times \{1, 2, 3\} \rightarrow \mathbb{R}_+$$

$$(i, j) \mapsto L_i + j.$$

This cost function computes the cost of a pipeline as the sum of its length plus its capacity. This cost function is obviously overly simplified for the clarity of the example. But the cost of a pipeline is increasing with respect to both parameters, which is consistent with reality. The corresponding matrix of incremental costs is given by Figure 2a.

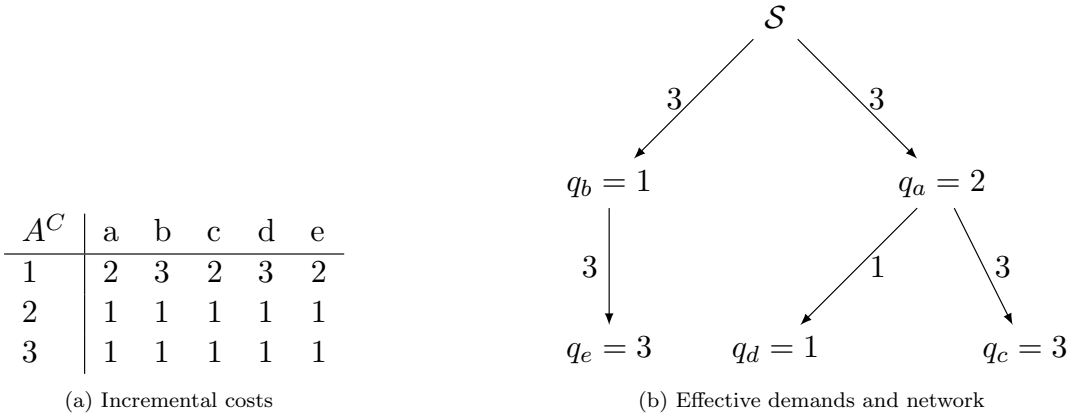


Figure 2: Gas distribution problem

Consider pipeline a . The effective capacity of pipeline a is $q_c = 3$. Therefore, this pipeline costs $C(a, q_c) = L_a + q_c = 4$. The total cost of operating the gas distribution network is given by

$$\sum_{i \in N} C(i, \bar{q}_i) = C(a, q_c) + C(b, q_e) + C(c, q_c) + C(d, q_d) + C(e, q_e)$$

$$= 20.$$

The allocations of consumer c obtained by applying the Connection rule Ψ and the Uniform rule Υ are given by Figure 3. In addition, we plot these allocations into Figure 4 to illustrate how c 's total allocation varies according to his demand, all other parameters being equal. Pick $\alpha = (0, 0.5, 1)$. The associated Mixed rule μ^α is given by Figure 3.

c	1	2	3	Total
Ψ	2.67	1.5	2	6.17
Υ	2.4	1.33	2	5.73
μ^α	2.4	1.415	2	5.815

Figure 3: Allocations of consumer c -table

The Mixed rule μ^α favors the uniformity principle for the demand units 2, i.e., $\alpha_1 = 0$. It compromises between the uniformity principle and the connection principle for the demand unit 2, i.e., $\alpha_2 = 0.5$. Finally, it favors the connection principle for the demand unit 3, i.e., $\alpha_3 = 1$. In Figure 4, any point located between a square and the corresponding star may represents the cost share obtained by using a Mixed rule.

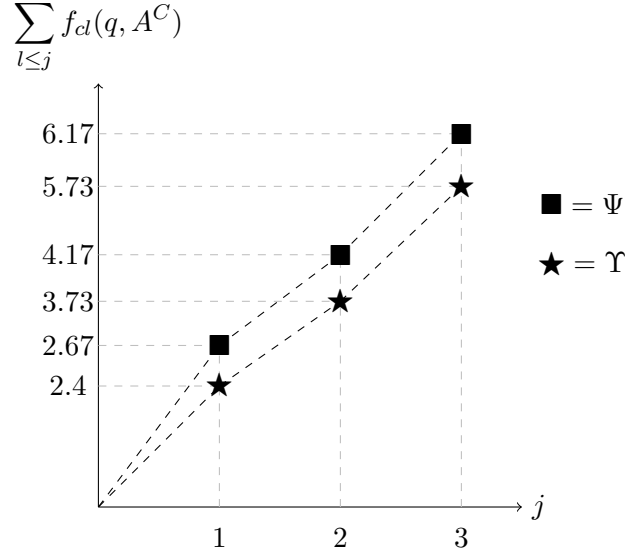


Figure 4: Allocations of consumer c -plot

4. Rules and Multi-choice games

This section presents multi-choice games as introduced by Hsiao and Raghavan (1992) and van den Nouweland (1993). For each gas distribution problem, we derive a specific a specific multi-choice game called the gas distribution game. Using this gas distribution game, we study how our rules relate to some solution concepts from multi-choice games. In particular, we show that the Connection rule is a **stable rule**, in the sense that consumers have an interest in becoming customers of the operator.

4.1. The gas distribution game

In order to remain consistent with gas distribution problems, some notation coincide with Section 2. Let $N = \{a, b, \dots, n\}$ be a fixed set of players and $K \in \mathbb{N}$. Each player $i \in N$ has a finite set of pairwise distinct activity levels $M_i := \{0, \dots, q_i\}$, where $q_i \leq K$. Define the set of players capable of playing at least activity level j as $Q(j) = \{i \in N : q_i \geq j\}$. Denote by \mathcal{M} the Cartesian product $\prod_{i \in N} M_i$. Each element $s = (s_1, \dots, s_n) \in \mathcal{M}$ specifies a participation profile for players and is referred to as a (multi-choice) coalition. So, a coalition indicates each player's activity level. Then, $q = (q_1, \dots, q_n) \in \mathcal{M}$ is the players' maximal participation profile that plays the role of the grand coalition, whereas $\Theta = (0, \dots, 0)$ plays the role of the empty coalition. We use the notations $M_i^+ := M_i \setminus \{0\}$ for each $i \in N$ and $M^+ := \bigcup_{i \in N} (\{i\} \times M_i^+)$. A pair $(i, j) \in M^+$ represents a player and one of his activity levels. A (cooperative) **multi-choice game** on N is a couple (q, v) where $v : \mathcal{M} \rightarrow \mathbb{R}$ is a characteristic function, with $v(\Theta) = 0$, that specifies a worth, $v(s)$, for each $s \in \mathcal{M}$.

The full class of multi-choice games is denoted by \mathcal{G} . A multi-choice game is **sub-modular** if, for each $s, t \in \mathcal{M}$,

$$v(s \vee t) + v(s \wedge t) \leq v(s) + v(t).$$

Consider $(q, v) \in \mathcal{G}$. A payoff vector $x \in \mathbb{R}^{M^+}$ for the game (q, v) assigns a payoff $x_{ij} \in \mathbb{R}$ to each pair $(i, j) \in M^+$. A set-valued solution on \mathcal{G} is a map F that assigns a collection of payoff vectors $F(q, v)$ to each $(q, v) \in \mathcal{G}$. A value f is a single-valued solution on \mathcal{G} , that assigns a unique payoff vector $f(q, v)$ to each $(q, v) \in \mathcal{G}$. We define gas distribution games, which are specific multi-choice games based on gas distribution problems. A gas distribution game, derived from a gas distribution problem (q, A^C) , measures the total cost of each gas distribution problem (s, A^C) in which each consumer $i \in N$ has an effective demand $s_i \leq q_i$. Put differently, a gas distribution game is a collection of all the total costs that could be generated by networks smaller than the one they are derived from.

Definition 6 (The gas distribution game). For each $(q, A^C) \in GDP$, its associated gas distribution (multi-choice) game $(q, v^{C,P})$ is defined as

$$\forall s \leq q, \quad v^{C,P}(s) = \sum_{i \in N} C(i, \bar{s}_i) = \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{s}_i\}} A_{ij}^C$$

where $\forall i \in N, \quad \bar{s}_i = \max_{k \in \dot{P}(i) \cup i} s_k.$

In a gas distribution game, the players represent the consumers and the activity levels represent the demand units of the consumers. The worth $v^{C,P}(s)$ represents the cost of operating a network (s, A^C) in which each consumer $i \in N$ has an effective demand of s_i . Obviously, $v^{C,P}(q)$ coincides with the total cost of operating the network, and $v^{C,P}(0, \dots, 0) = 0$ since $C(i, 0) = 0$ for each $i \in N$.

Remark 3. Pick any $(q, A^C) \in GDP$. Since $C(i, \cdot)$ is non-decreasing for each $i \in N$, it follows that the associated gas distribution game is **monotonic**, meaning that for each $s \geq t$, it holds that $v^{C,P}(s) \geq v^{C,P}(t)$.

4.2. Rules and solution concepts for multi-choice games

This section presents several solution concepts for multi-choice games and compare them with the rules of Section 3. First, we present the multi-choice Shapley value as introduced by Lowing and Techer (2022) for multi-choice games. To that end, we introduce **restricted orders** over the set of pairs M^+ as defined by Grabisch and Xie (2007). Consider a gas distribution game $(q, v^{C,P})$. Restricted orders are such that no pair $(i, j) \in M^+$ is ordered before a pair $(i', j') \in M^+$ with a strictly lower activity level $j' < j$. This means that no consumer can demand $j + 1$ unless each consumer $i \in Q(j)$ demands at least j . Formally, a restricted order over the set of pairs is a bijection $\sigma : M^+ \rightarrow \{1, \dots, \sum_{i \in N} q_i\}$ defined as

$$\forall (i, j), (i', j') \in M^+, \quad [j < j'] \implies [\sigma(i, j) < \sigma(i', j')].$$

Denote by \bar{O} the set of all restricted orders over the set of pairs. Let $\sigma \in \bar{O}$ be a restricted order and $h \in \{1, \dots, \sum_{i \in N} q_i\}$. Denote by $s^{\sigma, h}$ the coalition formed after h steps. We use the convention $s^{\sigma, 0} = \emptyset$. Formally, it is defined as

$$\forall i \in N, \quad s_i^{\sigma, h} = \max \{j \in \{1, \dots, q_i\} : \sigma(i, j) \leq h\} \cup \{0\}.$$

For each $\sigma \in \overline{O}$, the marginal vector $\eta^\sigma(q, v^{C,P})$ is defined as

$$\forall (i, j) \in M^+, \quad \eta_{ij}^\sigma(q, v^{C,P}) = v^{C,P}(s^{\sigma, \sigma(i,j)}) - v^{C,P}(s^{\sigma, \sigma(i,j)-1}).$$

Each $\eta_{ij}^\sigma(q, v^{C,P})$ is called the marginal contribution of the pair (i, j) to the coalition $s^{\sigma, \sigma(i,j)-1}$, which is formed after $\sigma(i, j) - 1$ steps with respect to the restricted order σ . The marginal contribution of the pair (i, j) can be interpreted as the additional costs generated when consumer i increases his demand from $j - 1$ to j , according to the restricted order σ . We have the material to define the multi-choice Shapley value for gas distribution games. This value assigns to each pair $(i, j) \in M^+$ its expected marginal contribution assuming that each restricted orders over the set of pairs occurs with the same probability.

Definition 7 (The multi-choice Shapley value). For each gas distribution game $(q, v^{C,P})$, the multi-choice Shapley value is defined as

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(q, v^{C,P}) = \frac{1}{\prod_{j \in \{1, \dots, q_n\}} |Q(j)|} \sum_{\sigma \in \overline{O}} \eta_{ij}^\sigma(q, v^{C,P}).$$

The next result states that, for each gas distribution problem, the Connection rule applied to the problem coincides with the multi-choice Shapley value applied to the gas distribution game associated with the problem.

Theorem 4. For each gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$,

$$\varphi(q, v^{C,P}) = \Psi(q, A^C).$$

Proof. See Appendix 6.4 □

Next, we present the multi-choice Equal division value as introduced by Lowing and Techer (2022) for multi-choice games. Assume that all consumers agree on forming a coalition in which everyone has the same demand, let us say j . Each consumer $i \in N$ with an effective demand $q_i \leq j$ demands q_i . We call such coalition a j -synchronized coalition. For each activity level $j \in \{1, \dots, q_n\}$, the multi-choice Equal division value shares the difference in worth between the j -synchronized coalition and the $j - 1$ -synchronized coalition equally among the pairs containing the activity level j .

Definition 8 (The multi-choice Equal division value). For each gas distribution game $(q, v^{C,P})$, the multi-choice Equal division value ξ is defined as

$$\forall (i, j) \in M^+, \quad \xi_{ij}(q, v^{C,P}) = \frac{1}{|Q(j)|} \left[v^{C,P}((j \wedge q_k)_{k \in N}) - v^{C,P}(((j-1) \wedge q_k)_{k \in N}) \right].$$

The next result states that, for each gas distribution problem, the Uniform rule applied to the gas distribution problem coincides with the multi-choice Equal division value applied to the gas distribution game associated with the problem.

Theorem 5. For each gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$,

$$\xi(q, v^{C,P}) = \Upsilon(q, A^C).$$

Proof. The proof follows directly from Definition (6) and Definition (8). \square

Next, we present the multi-choice Egalitarian Shapley values as defined by Lowing and Techer (2022) for multi-choice games. These values operate a trade-off between the multi-choice Shapley value and the multi-choice Equal division value by mean of convex combinations between the two values. In particular, a convex combination is done for each demand, meaning that the trade-off between the two values can differ depending on the demand.

Definition 9 (The multi-choice Egalitarian Shapley values). Let $\alpha = \{\alpha^j\}_{1 \leq j \leq K}$ be a parameter system such that $\alpha^j \in [0, 1]$ for each $1 \leq j \leq K$. For each gas distribution game $(q, v^{C,P})$, the multi-choice Egalitarian Shapley value χ^α is defined as

$$\forall (i, j) \in M^+, \quad \chi_{ij}^\alpha(q, v^{C,P}) = \alpha^j \varphi_{ij}(q, v^{C,P}) + (1 - \alpha^j) \xi_{ij}(q, v^{C,P}).$$

The next result states that, for each gas distribution problem, the set of Mixed rules applied to the problem coincides with the set of multi-choice Egalitarian Shapley values applied to the gas distribution game associated with the problem.

Theorem 6. For each gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$, and for each parameter system α ,

$$\chi^\alpha(q, v^{C,P}) = \mu^\alpha(q, A^C).$$

Proof. The proof follows directly from Theorem 4 and Theorem 5. \square

We introduce one last solution concept: the Core of a gas distribution game, inspired from the Core for multi-choice games introduced by Grabisch and Xie (2007). The Core is defined by two principles. The first principle (8) states that no coalition can achieve, by itself, a better outcome than the one prescribed by the payoff vectors in the Core. The second principle (9) states that any l -synchronized coalition achieves the same outcome than the one prescribed by the payoff vectors in the Core.

Definition 10 (The Core). The Core of a gas distribution game $(q, v^{C,P})$, denoted by $\mathcal{C}(q, v^{C,P})$, is the set of payoff vectors $x \in \mathbb{R}^{\sum_{i \in N} q_i}$ defined as

$$x \in \mathcal{C}(q, v^{C,P}) \iff \begin{cases} \forall s \leq q, & \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \leq v^{C,P}(s), & (8) \\ \forall l \leq q_n, & \sum_{i \in N} \sum_{j=1}^{l \wedge q_i} x_{ij} = v^{C,P}((l \wedge q_i)_{i \in N}). & (9) \end{cases}$$

If we re-interpret the Core principles in terms of rule for gas distribution problems, principle (8) ensures that consumers always pay less than the cost of supplying themselves in gas and (9) states that if a group of consumers, which synchronize their demands in the sense of a synchronized coalition, decide to supply themselves without resorting to the network operator, then they should pay the same amount as they would have been charged by the operator. Thus, the Core can be viewed as the set of **stable rules**, in the sense that consumers have an interest in becoming customers of the operator. We point out that principle (8) is the opposite from the original definition of the Core provided by Grabisch and Xie (2007). For each gas distribution problem, the multi-choice Shapley value applied to a gas distribution game is always in the Core. Therefore, the Connection rule is a stable rule.

Theorem 7. For each gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$,

$$\varphi(q, v^{C,P}) \in \mathcal{C}(q, v^{C,P}).$$

Proof. See Appendix 6.5 □

5. Conclusion

In this paper, we define gas distribution problems and determine three cost allocation rules for these problems on the basis of principles. Applying the connection principle, we propose the Connection rule. Applying the uniformity principle, we propose the Uniform rule. To compromise between the connection principle and the uniformity principle, we propose the Mixed rules. We provide an axiomatic characterization for each of these rules. In addition, these rules coincide with solution concepts from multi-choice games. In particular, the Connection rule applied to a gas distribution problem belongs to the Core of a specific multi-choice game derived from this problem.

Throughout this study, we assume that each pipeline is designed to meet its highest downstream demand. Let us call this approach the **optimistic approach** to pipeline design. The main drawback of this approach is that a pipeline can only satisfy a few effective demands at a time. In particular, the highest downstream demand of a pipeline already saturates its capacity. However, this approach has its advantages since it can be implemented at a low cost while ensuring a minimal service to consumers.

An alternative is the **pessimistic approach** to pipeline design: the network operator must be able to satisfy all effective demand at any time. Therefore each pipeline should have the capacity to meet the sum of all the effective demands of its downstream consumers. Unlike the previous approach, this approach ensures a flawless service to consumers. The main drawback of this approach is the resulting total cost of operating the network. Indeed, a network designed this way would probably be too expensive to operate.

The two approaches have their advantages and their drawbacks. Therefore, a compromise between these two approaches may be interesting to investigate. A hybrid approach would allow the network to handle multiple effective demands simultaneously while limiting overall operation costs. Exploring such an approach could be an interesting path for future research.

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6. Appendix

To properly conduct the proofs of our results, we introduce a Lemma. By the Independence of higher demands principle, the total amount allocated to consumers in $Q(j)$ for their demand unit j is equal to the sum of all incremental costs, generated by the j -th upgrades of the pipelines that connect the consumers in $Q(j)$ to the source.

Lemma 1. *Pick a rule f on GDP. For each $(q, A^C) \in GDP$,*

$$\forall l \leq q_n, \quad \sum_{i \in Q(l)} f_{il}(q, A^C) = \sum_{i \in \hat{P}^{-1}(Q(l)) \cup Q(l)} A_{il}^C.$$

Proof. Pick a rule f on GDP. By the Budget balanced principle, for each $(q, A^C) \in GDP$,

$$\begin{aligned} \sum_{i \in N} \sum_{j \in \{1, \dots, q_i\}} f_{ij}(q, A^C) &= \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C \\ \Leftrightarrow \sum_{j \in \{1, \dots, q_n\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &= \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C \end{aligned} \quad (10)$$

Recall that, for each $i \in N$,

$$\bar{q}_i = \max_{h \in \hat{P}(i) \cup i} q_h.$$

Observe that, for each $l \leq q_n$, and each $i \in N$,

$$\begin{aligned} \overline{((l \wedge q)_{k \in N})}_i &= \max_{h \in \hat{P}(i) \cup i} ((l \wedge q)_{k \in N})_h \\ &= \max_{h \in \hat{P}(i) \cup i} (l \wedge q_h) \\ &= \begin{cases} l & \text{if } \max_{h \in \hat{P}(i) \cup i} q_h \geq l \\ \max_{h \in \hat{P}(i) \cup i} q_h & \text{otherwise.} \end{cases} \\ &= l \wedge \bar{q}_i. \end{aligned} \quad (11)$$

For each $l \leq q_n$,

$$\begin{aligned} \sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &= \sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}(\overline{((l \wedge q)_{k \in N})}_i, A^C) \\ &= \sum_{i \in N} \sum_{j \in \{1, \dots, \overline{((l \wedge q)_{k \in N})}_i\}} A_{ij}^C \\ &= \sum_{i \in N} \sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C \end{aligned}$$

where the first equality follows from the Independence of higher demands principle, the second from (10), and the third from (11). Similarly,

$$\begin{aligned} \sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &= \sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(\overline{((l-1) \wedge q)_{k \in N}}_i, A^C) \\ &= \sum_{i \in N} \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C. \end{aligned}$$

Therefore, for each $l \leq q_n$,

$$\begin{aligned}
\sum_{i \in Q(l)} f_{il}(q, A^C) &= \sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) - \sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) \\
&= \sum_{i \in N} \sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C - \sum_{i \in N} \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C \\
&= \sum_{i \in N} \left[\sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C - \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C \right].
\end{aligned}$$

To conclude this proof, observe that, for each $i \in N$ such that $l \wedge \bar{q}_i = (l-1) \wedge \bar{q}_i$ (i.e., $\bar{q}_i \leq l-1$),

$$\sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C - \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C = 0.$$

To the contrary, for each $i \in N$ such that $l \wedge \bar{q}_i > (l-1) \wedge \bar{q}_i$ (i.e., $\bar{q}_i \geq l$),

$$\sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C - \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C = A_{il}^C.$$

Observe that $\bar{q}_i \geq l$ if and only if $i \in \hat{P}^{-1}(Q(l)) \cup Q(j)$. Therefore, for each $l \leq q_n$, we obtain the desired result

$$\sum_{i \in Q(l)} f_{il}(q, A^C) = \sum_{i \in \hat{P}^{-1}(Q(l))} A_{il}^C.$$

□

6.1. Proof of Theorem 1

Existence: We show that Ψ satisfies the axioms of the statement of Theorem 1.

(IIC): The proof follows directly from the definition of the Connection rule. The allocated to a consumer i for a demand unit j is computed using only the incremental costs generated by the pipelines that connect i to the source. Therefore, the incremental costs generated by the pipelines irrelevant to i are not taken into account by the Connection rule when computing the allocation of i for his demand unit j .

(EDC): Pick any $(q, A^C) \in GDP$, any $i \in N$, any $j \leq q_n$, and any $\varepsilon \in \mathbb{R}$. Assume that there exists two distinct consumers $h, h' \in (\hat{P}(i) \cup \{i\}) \cap Q(j)$. For h , it holds that

$$\begin{aligned}
\Psi_{hj}(q, A^C + \varepsilon I^{ij}) - \Psi_{hj}(q, A^C) &= \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{A_{kj}^C + \varepsilon I_{kj}^{ij}}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} - \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{A_{kj}^C}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} \\
&= \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{\varepsilon I_{kj}^{ij}}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} \\
&= \frac{\varepsilon}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|}.
\end{aligned}$$

Similarly, for h' , it holds that

$$\Psi_{h'j}(q, A^C + \varepsilon I^{ij}) - \Psi_{h'j}(q, A^C) = \frac{\varepsilon}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|},$$

which shows that Ψ satisfies the axiom.

Uniqueness: We show that the Connection rule is the only rule on GDP that satisfies all the axioms of the statement of Theorem 1. Pick any $(q, A^C) \in GDP$. Let f be a rule that satisfies all the axioms of the statement of Theorem 1 on GDP . Let us show that $f(q, A^C)$ coincides with the Connection rule. By definition of a rule, $f_{ij}(q, A^C) = \Psi_{ij}(q, A^C) = 0$ whenever $j > q_i$, for each $i \in N$. Pick any $i \in N$ and any $j \leq q_i$, we show that

$$f_{ij}(q, A^C) = \Psi_{ij}(q, A^C).$$

Consider the problem $(q, A^j) \in GDP$, in which A^j is defined as

$$\forall k \in N, \forall l \leq K, \quad A_{kl}^j = \begin{cases} 0 & \text{if } l = j \text{ and } k \in \hat{P}^{-1}(Q(j)) \cup Q(j) \\ A_{kl}^C & \text{otherwise.} \end{cases}$$

By Lemma 1,

$$\begin{aligned} \sum_{k \in Q(j)} f_{kj}(q, A^j) &= \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^j \\ &= 0 \end{aligned}$$

By definition, a rule assigns non-negative allocations. Therefore,

$$\forall k \in Q(j), \quad f_{kj}(q, A^j) = 0.$$

In (q, A^j) , the incremental cost A_{kj}^j of each pipeline $k \in \hat{P}^{-1}(Q(j)) \cup Q(j)$ has been nullified. We have shown that $f_{kj}(q, A^j) = 0$ for each $k \in Q(j)$. In particular, $f_{ij}(q, A^j) = 0$. In this part of the proof we consider, one by one, the pipelines that have been nullified in $\hat{P}^{-1}(Q(j)) \cup Q(j)$. For each pipeline $k \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, we will un-nullified the incremental cost A_{kj}^j into A_{kj} and discuss the impact on the allocation of i for his demand unit j . This way, we reconstruct the problem (q, A) to show the desired result.

Pick any $a \in \hat{P}^{-1}(Q(j)) \cup Q(j)$. If $a \notin \hat{P}^{-1}(i) \cup \{i\}$, then by (IIC), it holds that

$$\begin{aligned} f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= f_{ij}(q, A^j) \\ &= 0. \end{aligned}$$

If $a \in \hat{P}^{-1}(i) \cup \{i\}$, then by (EDC),

$$\begin{aligned} \forall h \in (\hat{P}(a) \cup \{a\}) \cap Q(j), \quad f_{ij}(q, A^j + A_{aj}^C I^{aj}) - f_{ij}(q, A^j) &= f_{hj}(q, A^j + A_{aj}^C I^{aj}) - f_{hj}(q, A^j) \\ \iff f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= f_{hj}(q, A^j + A_{aj}^C I^{aj}). \end{aligned} \tag{12}$$

By (IIC),

$$\begin{aligned} \forall h \in Q(j), h \notin \hat{P}(a) \cup \{a\}, \quad f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= f_{hj}(q, A^j) \\ &= 0. \end{aligned} \tag{13}$$

It follows that

$$\begin{aligned} \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} (A^j + A_{aj}^C I^{aj})_{kj} \\ \iff \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= A_{aj}^C \\ \iff \sum_{h \in (\hat{P}(a) \cup \{a\}) \cap Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= A_{aj}^C \\ \implies f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|}, \end{aligned}$$

where the first equality follows from Lemma 1, the second equality follows from the definition of A^j , the third equality follows from (13), and the fourth equality follows from (12). To summarize,

$$f_{ij}(q, A^j + A_{aj}^C I^{aj}) = \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ 0 & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases}$$

Next, pick $b \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, $b \neq a$. If $b \in \hat{P}^{-1}(i) \cup \{i\}$, then by (EDC), it holds that

$$\begin{aligned} \forall h \in (\hat{P}(b) \cup \{b\}) \cap Q(j), \quad f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{ij}(q, A^j + A_{aj}^C I^{aj}) \\ = f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{hj}(q, A^j + A_{aj}^C I^{aj}) \\ = \lambda, \end{aligned} \tag{14}$$

for some $\lambda \in \mathbb{R}$. By (IIC),

$$\forall h \in Q(j), h \in \hat{P}(b) \cup \{b\}, \quad f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = f_{hj}(q, A^j + A_{aj}^C I^{aj}). \tag{15}$$

It follows that

$$\begin{aligned} \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) &= A_{aj}^C + A_{bj}^C \\ \iff \sum_{h \in (\hat{P}(b) \cup \{b\}) \cap Q(j)} (f_{hj}(q, A^j + A_{aj}^C I^{aj}) + \lambda) \\ &+ \sum_{h \notin (\hat{P}(b) \cup \{b\}) \cap Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) = A_{aj}^C + A_{bj}^C \\ \iff \sum_{h \in (\hat{P}(b) \cup \{b\}) \cap Q(j)} \lambda + \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= A_{aj}^C + A_{bj}^C \\ \iff |(\hat{P}(b) \cup \{b\}) \cap Q(j)| \lambda + A_{aj}^C &= A_{aj}^C + A_{bj}^C \\ \iff |(\hat{P}(b) \cup \{b\}) \cap Q(j)| \lambda &= A_{bj}^C \\ \iff \lambda = \frac{A_{bj}^C}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|}, \end{aligned}$$

where the first equality follows from Lemma 1 and the definition of A^j , the second equality follows from (14), and the fourth equality follows from (15) and Lemma 1. We obtain

$$f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = f_{ij}(q, A^j + A_{aj}^C I^{aj}) + \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|}$$

$$= \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} + \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|} & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases}$$

If $b \notin \hat{P}^{-1}(i) \cup \{i\}$, then

$$f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = f_{ij}(q, A^j + A_{aj}^C I^{aj})$$

$$= \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ 0 & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases}$$

We have considered the pipelines a and b in $\hat{P}^{-1}(Q(j)) \cup Q(j)$. Continue this reasoning until all the pipelines in $\hat{P}^{-1}(Q(j)) \cup Q(j)$ have been considered once. Then, we obtain

$$f_{ij}(q, A^C) = f_{ij}(q, \sum_{h \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{hj}^C I^{hj})$$

$$= \sum_{k \in \hat{P}^{-1}(i) \cup \{i\}} \frac{A_{kj}^C}{(|\hat{P}(k) \cup \{k\} \cap Q(j)|)}$$

$$= \Psi_{ij}(q, A^C).$$

We have shown that $f_{ij}(q, A^C) = \Psi_{ij}(q, A^C)$ for any $i \in N$ and any $j \leq q_i$. This concludes the proof of Theorem 1. \square

The axioms of the statement of Theorem 1 are logically independent, as shown by the following alternative solutions.

- The Uniform rule satisfies (EDC) but does not satisfy (IIC).
- Pick any $(q, A^C) \in GDP$ and fix any arbitrary integer $\beta_{i'} \in \{1, 2\}$ for each $i' \in N$. The rule f^β given by

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}^\beta(q, A^C) = \sum_{k \in \hat{P}^{-1}(i) \cup \{i\}} \frac{\beta_i}{\sum_{x \in Q(j)} \beta_x} \times A_{kj}^C,$$

satisfies (IIC) but does not satisfy (EDC).

6.2. Proof of Theorem 2

Existence: The proof is straightforward since each consumer is charged with the same amount for a given demand. Thus the Uniform rule satisfies (NII).

Uniqueness: We show that the Uniform rule is the only rule on GDP that satisfies all the axioms of the statement of Theorem 2. Consider $(q, A^C) \in GDP$ and f a rule that satisfies all the axioms of the statement of Theorem 2 on GDP . Let us show that $f(q, A^C)$ is uniquely determined. By Lemma 1, for each $j \leq q_n$,

$$\sum_{i \in Q(j)} f_{ij}(q, A^C) = \sum_{i \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{ij}^C. \quad (16)$$

Consider the matrix of incremental costs $A^{0,j}$ in which each incremental cost generated by a j -th upgrade is null, i.e.,

$$\forall k \in N, \forall l \leq K, \quad A_{kl}^{0,j} = \begin{cases} 0 & \text{if } l = j \text{ and } k \in Q(j) \\ A_{kl}^C & \text{otherwise.} \end{cases}$$

By definition of a rule, each allocation is positive. Thus, by (16),

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}(q, A^{0,j}) = 0.$$

The difference between any two allocations is always null. Therefore, by (NII) and the positivity of the allocations, we get

$$\forall j \leq q_n, \forall i, i' \in Q(j), \quad f_{ij}(q, A^C) = f_{i'j}(q, A^C). \quad (17)$$

Combining (16) and (17), $f(q, A^C)$ is uniquely determined, which concludes the proof. \square

6.3. Proof of Theorem 3

Pick any α parameter system and consider the Mixed rule μ^α . By definition, and the fact that μ^α is computed as a convex combination of Ψ and Υ , μ^α satisfies all the axioms of the statement of Theorem 3.

It remains to show that the Mixed rules are the only rules that satisfy all the axioms of the statement of Theorem 3. Consider $(q, A^C) \in GDP$ and f on GDP a rule that satisfies all the axioms of the statement of Theorem 3. First, pick any I^{ij} such that $j \leq q_n$ and $i \notin \hat{P}^{-1}(Q(j)) \cup Q(j)$. The Budget balanced principle implies that

$$\begin{aligned} \sum_{k \in N} \sum_{l \in \{1, \dots, q_k\}} f_{kl}(q, I^{ij}) &= \sum_{k \in N} \sum_{l \in \{1, \dots, \bar{q}_k\}} I_{kl}^{ij} \\ &= 0. \end{aligned}$$

Since the cost share charged to each demand of each consumer is non-negative, it follows that, for each $k \in N$ and each $l \in \{1, \dots, q_k\}$,

$$f_{kl}(q, I^{ij}) = 0.$$

Therefore, $f(q, I^{ij})$ is uniquely determined whenever $j \leq q_n$ and $i \notin \hat{P}^{-1}(Q(j)) \cup Q(j)$.

Now, Pick any I^{ij} such that $j \leq q_n$ and $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$. By (EIC), for each $h, h' \in Q(j)$ such that $h, h' \notin \hat{P}(i) \cup \{i\}$,

$$\begin{aligned} f_{hj}(q, I^{ij}) &= f_{h'j}(q, I^{ij}) \\ &= Y, \end{aligned} \tag{18}$$

for some $Y \geq 0$. By (F), for each $h \in (\hat{P}(i) \cup \{i\}) \cap Q(j)$,

$$\begin{aligned} f_{hj}(q, I^{ij}) &\geq Y \\ \iff f_{hj}(q, I^{ij}) &= Y + W, \end{aligned} \tag{19}$$

for some $W \geq 0$. By Lemma 1,

$$\sum_{h \in Q(j)} f_{hj}(q, I^{ij}) = 1. \tag{20}$$

Observe that $Y \leq \frac{1}{|Q(j)|}$. By 19 and (20), it follows that

$$\begin{aligned} &|(\hat{P}(i) \cup \{i\}) \cap Q(j)|(Y + W) + |Q(j) \setminus ((\hat{P}(i) \cup \{i\}) \cap Q(j))|Y = 1 \\ \iff &|(\hat{P}(i) \cup \{i\}) \cap Q(j)|W = 1 - |Q(j)|Y \\ \implies &0 \leq |(\hat{P}(i) \cup \{i\}) \cap Q(j)|W \leq 1 \\ \iff &0 \leq W \leq \frac{1}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|} \\ \iff &W = \frac{\alpha^j}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|}, \end{aligned} \tag{21}$$

for some $0 \leq \alpha^j \leq 1$. Combining (18), (19),(20) and (21), we obtain

$$\begin{aligned} &|Q(j)|Y + \alpha^j = 1 \\ \iff &Y = \frac{1 - \alpha^j}{|Q(j)|}. \end{aligned} \tag{22}$$

Combining (19) and (22), for each $h \in (\hat{P}(i) \cup \{i\}) \cap Q(j)$, we obtain

$$f_{hj}(q, I^{ij}) = \frac{1 - \alpha^j}{|Q(j)|} + \frac{\alpha^j}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|}. \tag{23}$$

By (20) and by the Budget balanced principle, for each $l \neq j$ and $h \in Q(l)$,

$$f_{hl}(q, I^{ij}) = 0. \tag{24}$$

Finally, combining (22),(23) and (24), we obtain

$$\forall l \leq q_n, h \in Q(l),$$

$$f_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ \frac{1 - \alpha^j}{|Q(j)|} & \text{if } l = j \text{ and } h \notin \hat{P}(i) \cup \{i\}, \\ \frac{1 - \alpha^j}{|Q(j)|} + \frac{\alpha^j}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|} & \text{if } l = j \text{ and } h \in \hat{P}(i) \cup \{i\}, \end{cases}$$

for some $0 \leq \alpha^j \leq 1$. However, observe that

$$\forall l \leq q_n, h \in Q(l),$$

$$\Psi_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ 0 & \text{if } l = j \text{ and } h \notin \hat{P}(i) \cup \{i\}, \\ \frac{1}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|} & \text{if } l = j \text{ and } h \in \hat{P}(i) \cup \{i\}. \end{cases}$$

and

$$\Upsilon_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ \frac{1}{|Q(j)|} & \text{if } l = j \text{ and } h \notin \hat{P}(i) \cup \{i\}, \\ \frac{1}{|Q(j)|} & \text{if } l = j \text{ and } h \in \hat{P}(i) \cup \{i\}. \end{cases}$$

Thus, for each $h \in N$ and $l \leq q_h$,

$$f_{hl}(q, I^{ij}) = \alpha^l \Psi_{hl}(q, I^{ij}) + (1 - \alpha^l) \Upsilon_{hl}(q, I^{ij}), \quad (25)$$

for some parameter system α such that α^j is the one determined above. Observe that, for any $I^{i'j}$ such that $i' \in \hat{P}^{-1}(Q(j)) \cup Q(j)$ and $i' \neq i$, (EIC) and (LIC) ensure that 19 still holds. It follows that (25) still holds for the same parameter α^j even when considering a different incremental cost $I^{i'j}$. By (L), we conclude that $f(q, A^C) = \mu^\alpha(q, A^C)$. The proof of Theorem 3 is completed. \square

The axioms of the statement of Theorem 3 are logically independent, as shown by the following alternative solutions. To properly explain the first two alternative solutions, we introduce the following claim. Pick any $(q, A^C) \in GDP$. Pick any sequence $(f^{ij})_{\substack{i \in N \\ j \leq K}}$ of $n \times K$ rules on GDP . Then, define the map f as

$$f(q, A^C) = \sum_{\substack{i \in N \\ j \leq K}} f^{ij}(q, I^{ij} A_{ij}^C). \quad (26)$$

This map also qualifies as a rule on GDP . Indeed, by the properties of the sum, f allocates a positive payoff to each demand unit of each consumer. Moreover, by (5), f satisfies the Budget balanced principle and the Independence of higher demands principle. In the next two alternative solutions, we define rules computed in a similar manner than (26).

- Pick any $(q, A^C) \in GDP$ and any $h \in N$. Let us consider the problem $(q, I^{k1} A_{k1}^C)$, $k \in N$. Recall that $Q(1) = N$. Define the allocation $\Psi_{i1}^{-h}(q, I^{k1} A_{k1}^C)$ of a consumer $i \in N$ for his demand unit 1 by

$$\Psi_{i1}^{-h}(q, I^{k1} A_{k1}^C) = \begin{cases} \frac{A_{k1}^C}{N} & \text{if } i \notin \hat{P}(k) \cup \{k\} \text{ and } i = h, \\ 0 & \text{if } i \notin \hat{P}(k) \cup \{k\} \text{ and } i \neq h, \\ A_{k1}^C \frac{1 - \frac{1}{N}}{|\hat{P}(i) \cup \{i\}|} & \text{if } i \in \hat{P}(k) \cup \{k\} \text{ and } h \notin \hat{P}(k) \cup \{k\}, \\ \Psi_{i1}(q, I^{k1} A_{k1}^C) & \text{otherwise.} \end{cases}$$

This allocation rule behaves just like the Connection rule, but it gives special treatment to h by allocating him a non-null payoff for irrelevant costs. Indeed, for each $i \in N$, the amount allocated by the Connection rule in $(q, I^{k1}A_{k1}^C)$ is given by

$$\Psi_{i1}(q, I^{k1}A_{k1}^C) = \begin{cases} 0 & \text{if } i \notin \hat{P}(k) \cup \{k\}, \\ \frac{A_{k1}^C}{\hat{P}(i) \cup \{i\}} & \text{if } i \in \hat{P}(k) \cup \{k\}. \end{cases}$$

The main difference between the two rules Ψ_{i1}^{-h} and Ψ_{i1} is that the former allocates $\frac{A_{k1}^C}{N}$ to h whenever A_{k1}^C is an irrelevant cost for h , whereas the later allocates a null share. In case A_{k1}^C is a relevant cost for h , then the two allocations coincide.

Using the rule Ψ^{-h} , let us define another rule denoted by f . For each demand level $j > 1$ and each consumer $i \in Q(j)$, $f_{ij}(q, A^C) = \Psi_{ij}(q, A^C)$. As for the demand level 1, f is defined by

$$\forall i \in N, \quad f_{i1}(q, A^C) = \sum_{k \in N} \Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C).$$

The rule f satisfies all the axioms except (EIC) since consumer h is getting a special treatment for his first level of demand.

- Pick any $(q, A^C) \in GDP$. For each $k \in N$, fix any parameter $\alpha^k \in [0, 1]$. The rule f^α is defined, for each $(q, A^C) \in GDP$, as

$$\begin{aligned} & \forall j \leq q_n, \forall i \in Q(j), \\ & f_{ij}^\alpha(q, A^C) = \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} \left[\alpha^k \Psi_{ij}(q, I^{kj}A_{kj}^C) + (1 - \alpha^k) \Upsilon_{ij}(q, I^{kj}A_{kj}^C) \right]. \end{aligned}$$

This rule operates different compromises between the connection principle and the uniformity principle for each pipeline of the network. For instance, pipeline $k \in N$ may be attributed a parameter $\alpha^k = 1$, in which case the rule f^α will share the incremental costs generated by k according to the connection principle. Alternatively, pipeline $k' \in N$ may be attributed a parameter $\alpha^{k'} = 0$, in which case the rule f^α will share the incremental costs generated by k' according to the uniformity principle. The rule f^α satisfies all the axioms except (LIC).

- The rule f given, for each $(q, A^C) \in GDP$, by

$$\begin{aligned} & \forall j \leq q_n, \forall i \in Q(j), \\ & f_{ij}(q, A^C) = \begin{cases} \Upsilon_{ij}(q, A^C) & \text{if } \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C \leq 10, \\ \Psi_{ij}(q, A^C) & \text{otherwise.} \end{cases} \end{aligned}$$

satisfies all the axioms except (L).

- The rule f is defined, for each $(q, A^C) \in GDP$, as

$$\begin{aligned} & \forall j \leq q_n, \forall i \in Q(j), \\ & f_{ij}(q, A^C) = \left[\left(\frac{1}{|Q(j)|^2} + \frac{1}{|Q(j)|} \right) \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^C \right] - \frac{\Psi_{ij}(q, A^C)}{|Q(j)|}. \end{aligned}$$

This rule satisfies all the axioms except Fairness.

6.4. Proof of Theorem 4

In order to prove Theorem 4, we recall some definitions on TU-games. Consider a finite set of players $N = \{a, b, \dots, n\}$. A TU-game is a couple (N, v) , where $v : 2^N \rightarrow \mathbb{R}$ is a characteristic function assigning to each coalition of players $S \in 2^N$ its worth $v(S)$. For each (N, v) , the Shapley value is defined as

$$\forall i \in N, \quad Sh_i(N, v) = \frac{1}{n!} \sum_{\sigma \in O} (v(E_i^\sigma \cup i) - v(E_i^\sigma)), \quad (27)$$

where O is the set of all orders over N and E_i^σ is the coalition formed by the players ordered before i according to the order σ .

Next, we introduce an intermediary result. To that end, consider a gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$. For each $j \leq q_n$, define the TU-game $(N, w_j^{(q, v^{C,P})})$ as

$$\forall E \subseteq N, \quad w_j^{(q, v^{C,P})}(E) = v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right] - v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right].$$

The worth $w_j^{(m, v^{C,P})}(E)$ can be interpreted as the surplus in cost generated in $v^{C,P}$ when a group of players E decide to increase their activity level from $j-1$ to j while all the other players play the activity level $j-1$ or their maximal feasible activity level if they are unable to play $j-1$. The next Lemma is already proved on the class of monotonic multi-choice games in Lowing and Techer (2022), and so is omitted.

Lemma 2. *For each gas distribution problem $(q, A^C) \in GDP$, its associated gas distribution game $(q, v^{C,P})$ verifies*

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(q, v) = Sh_i(N, w_j^{(q, v^{C,P})}). \quad (28)$$

We have the material to prove Theorem 4. Consider $(q, A^C) \in GDP$ and $j \leq q_n$. For each $k \in N$, define the TU-game $(Q(j), R^k)$ as

$$\forall E \subseteq Q(j), \quad R^k(E) = \begin{cases} 1 & \text{if } E \cap (\hat{P}(k) \cup k) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of the Shapley value,

$$\forall i \in N, \quad Sh_i(Q(j), R^k) = \begin{cases} \frac{1}{|(\hat{P}(k) \cup k) \cap Q(j)|} & \text{if } i \in \hat{P}(k) \cup k, \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

Consider $(q, v^{C,P}) \in \mathcal{G}$, the gas distribution game associated to (q, A^C) . Consider the TU-game

$(Q(j), w^{(q, v^{C,P})})$ and observe that, for each $E \subseteq Q(j)$,

$$\begin{aligned} w_j^{(q, v^{C,P})}(E) &= v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right] - v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right] \\ &\stackrel{(6)}{=} \sum_{i \in N} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right] \end{aligned} \quad (30)$$

$$\begin{aligned} &= \sum_{i \in \hat{P}^{-1}(E) \cup E} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right] \\ &+ \sum_{i \notin \hat{P}^{-1}(E) \cup E} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right]. \end{aligned} \quad (31)$$

For each $i \in \hat{P}^{-1}(E) \cup E$, there exists at least one $h \in \hat{P}(i) \cup i$ such that $h \in E \subseteq Q(j)$. Therefore, for each $i \in \hat{P}^{-1}(E) \cup E$,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h = j. \quad (32)$$

On the other hand,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h = j-1. \quad (33)$$

For each $i \notin \hat{P}^{-1}(E) \cup E$, there is no $h \in \hat{P}(i) \cup i$ such that $h \in E$. Therefore, for each $i \notin \hat{P}^{-1}(E) \cup E$,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h = \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h. \quad (34)$$

For the sake of clarity, for each $i \notin \hat{P}^{-1}(E) \cup E$, we denote

$$Z_i = \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h.$$

Therefore, by (32), (33) and (34), (30) becomes

$$\begin{aligned}
w_j^{(q, v^{C,P})}(E) &= \sum_{i \in \hat{P}^{-1}(E) \cup E} [C(i, j) - C(i, j-1)] + \sum_{i \notin \hat{P}^{-1}(E) \cup E} [C(i, Z_i) - C(i, Z_i)] \\
&= \sum_{i \in \hat{P}^{-1}(E) \cup E} A_{ij}^C + 0 \\
&= \sum_{i \in N} A_{ij}^C \times R^i(E).
\end{aligned}$$

By Lemma 2, for each $i \in N$ and $j \in \{1, \dots, q_i\}$,

$$\begin{aligned}
\varphi_{ij}(q, v^{C,P}) &= Sh_i \left(Q(j), \sum_{l \in N} A_{lj}^C \times R^l \right) \\
&= \sum_{l \in N} A_{lj}^C \times Sh_i(Q(j), R^l) \\
&= \sum_{l \in \hat{P}^{-1}(i) \cup i} \frac{A_{lj}^C}{|(\hat{P}(l) \cup l) \cap Q(j)|} \\
&= \Psi_{ij}(q, A^C).
\end{aligned}$$

The second equality follows from the Linearity of the Shapley value and the third equality follows from (29). This concludes the proof. \square

6.5. Proof of Theorem 7

Lowing and Techer (2022) show that the multi-choice Shapley value belongs to the Core of sub-modular (cost) games. Therefore, it suffice to show that gas distribution games are sub-modular games to prove Theorem 8. Consider a gas distribution game $(q, v^{C,P}) \in \mathcal{G}$ associated with a gas distribution problem $(q, A^C) \in GDP$. For each $s, t \in \mathcal{M}$,

$$\begin{aligned}
v^{C,P}(t) + v^{C,P}(s) &= \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} s_k) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} t_k), \\
\text{and } v^{C,P}(s \vee t) + v^{C,P}(s \wedge t) &= \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \vee t_k)) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} s_k \wedge t_k).
\end{aligned}$$

Take any $i \in N$. Without loss of generality, assume that

$$\max_{k \in \hat{P}(i) \cup i} s_k \geq \max_{k \in \hat{P}(i) \cup i} t_k.$$

Then, on the one hand,

$$\begin{aligned}
\max_{k \in \hat{P}(i) \cup i} (s_k \vee t_k) &= \max_{k \in \hat{P}(i) \cup i} s_k \\
\implies C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \vee t_k)) &= C(i, \max_{k \in \hat{P}(i) \cup i} s_k).
\end{aligned}$$

On the other hand, since C is non-decreasing, it holds that

$$\begin{aligned} & \forall k \in \hat{P}(i) \cup i, \quad s_k \wedge t_k \leq t_k \\ \iff & \max_{k \in \hat{P}(i) \cup i} (s_k \wedge t_k) \leq \max_{k \in \hat{P}(i) \cup i} t_k \\ \implies & C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \wedge t_k)) \leq C(i, \max_{k \in \hat{P}(i) \cup i} t_k). \end{aligned}$$

This shows that, for each $i \in N$,

$$\sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \vee t_k)) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \wedge t_k)) \leq \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} s_k) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} t_k),$$

and so, we obtain the desired result

$$v^{C,P}(s \vee t) + v^{C,P}(s \wedge t) \leq v^{C,P}(s) + v^{C,P}(t).$$

□

References

- Bergantiños, G., González-Díaz, J., González-Rueda, Á.M., de Córdoba, M.P.F., 2017. Loss allocation in energy transmission networks. *Games and Economic Behavior* 102, 69–97.
- van den Brink, R., Gilles, R.P., 1996. Axiomatizations of the conjunctive permission value for games with permission structures. *Games and Economic Behavior* 12, 113–126.
- van den Brink, R., He, S., Huang, J.P., 2018. Polluted river problems and games with a permission structure. *Games and Economic Behavior* 108, 182–205.
- Csercsik, D., Hubert, F., Sziklai, B.R., Kóczy, L.Á., 2019. Modeling transfer profits as externalities in a cooperative game-theoretic model of natural gas networks. *Energy Economics* 80, 355–365.
- Dong, B., Ni, D., Wang, Y., 2012. Sharing a polluted river network. *Environmental and Resource Economics* 53, 367–387.
- Fleurbaey, M., Trannoy, A., 1998. La péréquation territoriale en question. *FLUX Cahiers scientifiques internationaux Réseaux et Territoires* 14, 91–98.
- Gilles, R.P., Owen, G., van den Brink, R., 1992. Games with permission structures: the conjunctive approach. *International Journal of Game Theory* 20, 277–293.
- Grabisch, M., Xie, L., 2007. A new approach to the core and weber set of multichoice games. *Mathematical Methods of Operations Research* 66, 491–512.
- Hsiao, C.R., Raghavan, T., 1992. Monotonicity and dummy free property for multi-choice cooperative games. *International Journal of Game Theory* 21, 301–312.
- Junqueira, M., da Costa, L.C., Barroso, L.A., Oliveira, G.C., Thomé, L.M., Pereira, M.V., 2007. An Aumann-Shapley approach to allocate transmission service cost among network users in electricity markets. *IEEE Transactions on Power Systems* 22, 1532–1546.
- Li, W., Xu, G., van den Brink, R., 2022. Two new classes of methods to share the cost of cleaning up a polluted river. *Social Choice and Welfare*, 1–25.
- Lowing, D., Techer, K., 2022. Marginalism, egalitarianism and efficiency in multi-choice games. *Social Choice and Welfare*.
- Massol, O., Tchong-Ming, S., 2010. Cooperation among liquefied natural gas suppliers: Is rationalization the sole objective? *Energy Economics* 32, 933–947.
- Moulin, H., Shenker, S., 1992. Serial cost sharing. *Econometrica* 60, 1009–1037.
- Ni, D., Wang, Y., 2007. Sharing a polluted river. *Games and Economic Behavior* 60, 176–186.
- van den Nouweland, C.G.A., 1993. *Games and graphs in economic situations*. tilburg university.