



HAL
open science

Sharing the Cost of a Gas Distribution Network

David Lowing

► **To cite this version:**

| David Lowing. Sharing the Cost of a Gas Distribution Network. 2022. hal-03680156v1

HAL Id: hal-03680156

<https://hal.science/hal-03680156v1>

Preprint submitted on 27 May 2022 (v1), last revised 13 Oct 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sharing the Cost of a Gas Distribution Network.

David Lowing^{a,b}

^a*Univ Lyon, UJM Saint-Etienne, GATE UMR 5824, F-42023 Saint-Etienne, France.*

^b*Gaz Réseau Distribution France, Direction Economie et Régulation.*

Abstract

A gas distribution network connects consumers to a source in gas. It is managed by a network operator, whose task incurs various costs, some of which may not be attributable to a particular consumer. Assuming that the operator wishes to recover these costs by charging for its services, the problem is then to determine how much each consumer should pay. In other words, how should these costs be shared among consumers. In this paper, we address this problem and propose cost sharing rules that depend on the network and the demands of the consumers. To that end, we adopt a normative approach and resort to three principles: (i) the independence of higher demands principle, (ii) the connection principle and (iii) the uniformity principle. Applying (i) and (ii), we derive the Connection rule and applying (i) and (iii), we derive the Uniform rule. It appears that (ii) and (iii) are incompatible. In order to make a trade-off between these two principles, we propose the Mixed rules, which compromise between the Connection rule and the Uniform rule. For each cost sharing rule, an axiomatic characterization is provided. Then, we show that the Connection rule coincides with the multi-choice Shapley value of a specific multi-choice game derived from the network and the demands of the consumers. Moreover, the Connection rule is in the Core of this specific multi-choice game. Similarly, we show that the Uniform rule coincides with the multi-choice Equal division value and the Mixed rules coincide with the multi-choice Egalitarian Shapley values.

Keywords: Gas distribution network; Cost sharing rules; Axiomatization; Multi-choice games

JEL codes: C71; D61

1. Introduction

1.1. The gas distribution problem

Gas distribution is carried out via a network that draws gas from a source (e.g. a transmission network arrival point or a gas storage) to deliver it to the consumer. The network is managed by an operator responsible for connecting each consumer to the source while ensuring the safety and maintenance of the network. To convey out its task properly, the network operator is confronted with various **operation costs**, some of which are not directly assignable to a given consumer. For instance, there are costs related to access the transmission network, costs related to access the gas storage (necessary to overcome seasonal fluctuations in consumer demands) and costs related to the maintenance of the pipelines making up the network. In this paper, we assume that a network operator recovers these operation costs by charging the consumers. The problem is then to determine how to

Email address: david.lowing@outlook.com (David Lowing)

share these costs among the consumers. We address this problem and propose cost sharing rules that depend on the network and the demands of the consumers.

To that end, we define a cost sharing problem adapted to the gas distribution setting. Fix a finite set $N = \{a, b, \dots, n\}$ of **consumers**, directly or indirectly, connected to a source by means of a fixed network, which is represented by a **rooted tree graph** P . In a tree graph, each node has at most one direct predecessor. The nodes of the graph represent the consumers plus the gas arrival point \mathcal{S} , while the arcs of the graph represent the pipelines of the network. An integer in N refers to both a consumer and the pipeline having this consumer at its tail.

Assume that there is a limit to the daily volume in gas that the network operator can provide. This limit is represented by the integer $K \in \mathbb{K}$. Each consumer $i \in N$ has an **effective demand**, which is represented by an integer $q_i \leq K$. It corresponds to the highest daily volume in gas that this consumer expects to demand in a year. For instance, the effective demand of a regular household is often determined by its consumption during winter. The effective demand is communicated to the operator in advance in order to design the network accordingly. The profile of all effective demands is given by $q = (q_a, \dots, q_n)$. Without loss of generality, assume that $q_n \geq q_i$, for each $i \in N$. Moreover, each consumer $i \in N$ is endowed with a discrete set $\{0, 1, \dots, q_i\}$ that describes its effective demand q_i and each of its **demand units** smaller than q_i . It can be viewed as the set of all daily volumes in gas that i is allowed to demand throughout the year. For instance, the gas consumption of an household may vary throughout a year from an almost null demand in summer to its effective demand in winter.

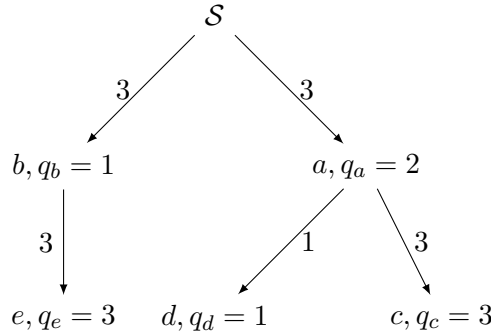


Figure 1: Network design

The network operator must be able to satisfy any effective demand at any time, and therefore, he must design its network accordingly. This means that each pipeline must have the capacity to meet the demands of its downstream consumers. In this paper, we assume that each pipeline is designed to meet the highest effective demand of the consumers located at the tail and downstream of that pipeline (the **highest downstream demand** of that pipeline for short). For instance, Figure 1 illustrates a situation in which consumer a , c and d have an effective demand of 2, 3 and 1, respectively. In this case, both pipelines a and c are designed to meet the demand 3, since it is their highest downstream demand. For the same reason, pipeline d is designed to meet the demand 1. There are alternatives to this approach to network design. Each approach, including this one, has its advantages and its drawbacks. We will discuss some of these approaches at the end of the paper.

A **cost function** $C : N \times \{1, \dots, K\} \rightarrow \mathbb{R}_+$ is a non-decreasing map that computes the cost $C(i, j)$ of any pipeline $i \in N$ designed to meet any demand $j \leq K$. From this function, the **total cost** of operating the network is computed as the sum of the costs of all pipelines, where each pipeline is designed to meet its highest downstream demand. The problem is then to determine an appropriate

rule to share this total cost among the consumers. This problem is called the cost sharing problem of gas distribution (the **gas distribution problem** for short) and is denoted by (N, q, C, P) .

1.2. Normative approach

A **cost sharing rule**, or a rule for short, is a map f that divides the total cost among the positive demand units of the consumers. Precisely, it associates a cost share $f_{ij}(N, q, C, P)$ to each demand unit $j \in \{1, \dots, q_i\}$ of each consumer $i \in N$. Each cost share $f_{ij}(N, q, C, P)$ can be interpreted as the additional amount that consumer i must pay in order to increase its demand from $j - 1$ to j . The total cost share of a consumer i is given by $\sum_{j \leq q_i} f_{ij}(N, q, C, P)$. A rule highlights how the amount allocated to a consumer evolves according to her demand. Additionally, a rule must satisfy the budget balanced condition, meaning that it must recover the total cost of operating the network. We adopt a normative approach to identify appropriate rules. To that end, three desirable principles for rules are discussed.

1. The first principle is the **independence of higher demands principle**. It indicates that a consumer should not have to pay for costs generated by demands higher than its own. This principle makes it possible to avoid situations where the presence of a consumer with a high demand implies additional and unjustified costs to consumers with lower demands. For example, it seems unjustified to increase the bill of a household whenever a factory with a large consumption moves in next door. The independence of higher demands principle was originally discussed by Moulin and Shenker (1992) in the context of cost sharing problems.
2. The second principle is called the **connection principle**. In a gas distribution network, a consumer depends on a number of pipelines that connect it to the gas arrival point. The connection principle defends the idea that we should only charge this consumer for costs generated by these pipelines, i.e., the part of the network it uses.
3. The last principle is called the **uniformity principle**. This principle simply states that two consumers with the same demands should be charged with the same amount regardless of their geographical location. For example, there should be no difference in treatment between rural and urban consumers, although the underlying distribution costs are different. Obviously, the uniformity principle is an egalitarian principle. In fact, network operators in France highly favor rules that respect this principle as much as possible (see Fleurbaey and Trannoy (1998)).

Observe that a rule cannot satisfy both the connection principle and the uniformity principle on the full class of gas distribution problems. Indeed, the connection principle states that consumers should only pay for the portion of the network they use. Therefore, two consumers with the same demands can be charged with different amount depending on their position on the network, which contradicts the uniformity principle.

1.3. Cost sharing rules

We define three rules on the basis of these principles. Recall that a rule charges a cost share to each demand unit of each consumer. To clearly present our rules, assume that a pipeline $i \in N$ is built, step by step, by increasing its capacity from 0 to its highest downstream demand denoted by \bar{q}_i . At each step $j \in \{1, \dots, \bar{q}_i\}$, pipeline i undergoes an upgrade, called the **j -th upgrade of i** . Due to this upgrade, an incremental cost $C(i, j) - C(i, j - 1)$ is generated and is going to be charged to

the consumers. This procedure allows us to see how each upgrade impacts the cost charged to the demand units of the consumers.

The first rule is called the **Connection rule** and is computed as follows. Pick any pipeline. Assume that each consumer located downstream of this pipeline has an effective demand of at most 1. Then, this pipeline is designed to meet a demand of 1, this corresponds to its 1st upgrade. The Connection rule shares the cost of this pipeline equally among the first demand units of all downstream consumers. Next, assume that each consumer located downstream of this pipeline now has an effective demand of at most 2. Then, upgrade this pipeline in order to meet a demand of 2. This 2nd upgrade generates a positive incremental cost. The Connection rule shares this incremental cost equally among the second demand units of all downstream consumers whose demand is at least 2. This procedure continues until the demand for which this pipeline is designed, i.e., its highest downstream demand is reached. The procedure is applied to each pipeline. This way, Connection rule recovers the total cost of the network and respects the connection principle. It should be pointed out that it also respects the independence of higher demands principle. Indeed, a consumer will never have to pay for upgrades with the purpose of meeting demands higher than its own effective demand.

The second rule is called the **Uniform rule** and is computed in a similar manner the the Connection rule. Pick any pipeline. Assume that this pipeline is designed to meet a demand of 1. The Uniform rule shares the cost of this pipeline equally among the first demand units of all consumers regardless of their position on the network. Then, assume that this pipeline is upgraded to meet a demand of 2. This 2nd upgrade generates a positive incremental cost. The Uniform rule shares this incremental cost equally among the second demand units of all consumers whose demand is at least 2. This procedure continues until the demand for which this pipeline is designed (i.e. its highest downstream demand) is reached. We apply this procedure to each pipeline. This way, the Uniform rule recovers the total cost of the network and it respects the uniformity principle and the independence of higher demands principle.

The connection principle and the uniformity principle, though incompatible, are desirable for a network operator. To reach a compromise between these two principles, we propose a family of rules. Each rule in this family achieves a compromise by means of convex combinations between the Connection rule and the Uniform rule. These rules are called the **Mixed rules** and are computed as follows. Pick any pipeline. Assume that this pipeline is designed to meet a demand of 1. The Mixed rule charges the cost of this pipeline to the first demand unit of each consumer. This share is computed as a convex combination between the share that would have been allocated by the Connection rule and the share that would have been allocated by the Uniform rule. Then, assume that this pipeline is upgraded in order to meet a demand of 2. The incremental cost generated by this 2nd upgrade is shared among the second demand units of all consumers whose demand is at least 2. This share is computed as a convex combination (possibly different from the first combination) between the Connection rule and the Uniform rule. This procedure continues until the demand for which this pipeline is designed (i.e., its highest downstream demand) is reached. This procedure is applied to each pipeline. Observe that the Mixed rules allow to reach different compromises depending on the level of demand.

We formalize the independence of higher demands principle, the connection principle and the uniformity principle into axioms for rules. Additionally, we introduce other axioms for rules that do not necessarily fit into the idea of the three principles but can still be viewed as desirable properties in the context of gas distribution. Combining these axioms, we provide an axiomatic characterization for each rule introduced in this paper.

1.4. Comparison with multi-choice games

An appropriate game theoretic tool for modeling gas distribution problems are multi-choice (cooperative) games. Multi-choice games, introduced by Hsiao and Raghavan (1992) and van den Nouweland (1993), are a natural extension of TU-games in which each player is endowed with a certain number of activity levels. A (multi-choice) coalition is a profile describing each player's activity level within this coalition. The worth of each coalition is measured by a characteristic function. Given a gas distribution problem, we derive a special multi-choice game associated with this problem. This game is called the gas distribution game. The player set represents the set of consumers, and the activity levels represent the demands of the consumers. The worth of a coalition corresponds to the total cost of the network designed to meet the demands specified in this coalition. The Connection rule applied to a gas distribution problem corresponds to the multi-choice Shapley value, introduced by Lowing and Techer (2022), of the corresponding gas distribution game. Similarly, the Uniform rule corresponds to the multi-choice Equal division value and the Mixed rules to the multi-choice Egalitarian Shapley values. Both values are introduced by Lowing and Techer (2022). Moreover, for each gas distribution problem, the multi-choice Shapley value of the corresponding gas distribution game is in the Core of this gas distribution game.

1.5. Related literature

From its extraction to its distribution to consumers, natural gas goes through three major steps before being consumed: the production step, the transmission step and the distribution step. Some papers address cost or resource sharing problems in the context of gas transmission. For instance, Junqueira et al. (2007) study how to share the cost of an energy transmission network and Bergantiños et al. (2017) study how to share the energy losses in transmission networks. To our knowledge, no paper addresses this kind of problems in the context of gas production or gas distribution. In addition, several other papers address the cooperative aspect of gas-related problems. For instance, Massol and Tchong-Ming (2010) study cooperation among liquefied gas suppliers and Csercsik et al. (2019) model transfer profit as externalities using partition function form games.

The gas distribution problem shares some similarities with the polluted river problem introduced by Ni and Wang (2007) and studied by Dong et al. (2012). In such problem, a river carries pollutants to agents living along it. To make the water in the river clean, some costs are incurred and must be shared among the agents. The main difference between our model and theirs is that we take into account the demands of consumers (who are the analog of agents in their model) into account. Consequently, the connection principle is in the same spirit than the upstream responsibility principle that comes from the unlimited territorial integrity theory for polluted river problems. In short, the upstream responsibility principle states that agents, located on a polluted river network, should pay for the cost of cleaning the portion of the river that connects them to the source of the river (see Dong et al. (2012) for details). Furthermore, the Connection rule is conceptually close to the Downstream equal sharing solution, introduced by Dong et al. (2012) for polluted river problems.

In the same way that we make an analogy between gas distribution problems and multi-choice games, van den Brink et al. (2018) makes an analogy between polluted river problem and TU-game with a permission structure (see Gilles et al. (1992) for details on games with a permission structure). In particular, the authors show that the permission value (see van den Brink and Gilles (1996)) applied to a special TU-game constructed from the polluted river problem coincides with the Downstream equal sharing solution.

1.6. Overview of the paper

The rest of the paper is organized as follows. After dealing with preliminaries on the gas distribution framework in Section 2.1, Section 2.2 defines gas distribution problems. Section 3 defines the above mentioned rules and provide their axiomatic characterizations. Section 4.1 recalls some definitions from multi-choice games and defines gas distribution games. Then, Section 4.2 shows how solution concepts for multi-choice games relate to our rule. Section 5 concludes and provides leads for future research. Section 6 is an appendix containing all the proofs of the results.

2. The model

2.1. Notation and definitions

Fix $N = \{a, b, \dots, n\}$, $|N| \geq 3$, a finite set of **consumers**. These consumers are connected to a source \mathcal{S} through **pipelines**. The consumers and the pipelines form a fixed **network**, which is modeled by a rooted tree graph. A rooted tree graph is a couple $(N \cup \mathcal{S}, P)$ where P is a map $P : N \cup \mathcal{S} \rightarrow 2^{N \cup \mathcal{S}}$ representing the connections (pipelines) between the nodes. The relationship $i' \in P(i)$ means that $i' \in N$ is supplied right after consumer i . Equivalently, denote $i \in P^{-1}(i')$ if and only if $i' \in P(i)$. In this case, consumer i is the **predecessor** of i' while i' is the **successor** of i . Put differently, i is the **head** of the pipeline and i' is the **tail** of the pipeline. In a rooted tree graph, each node has at most one predecessor and only one node (the source) has no predecessor, i.e., $P^{-1}(\mathcal{S}) = \emptyset$. The transitive closure of a rooted graph P is a rooted graph \hat{P} such that, for each $i \in N$, $i' \in \hat{P}(i)$ if and only if there exists a path $i = h_1, h_2, \dots, h_k = i'$ such that $h_k \in P(h_{k-1}), \dots, h_2 \in P(h_1)$. The consumers in $\hat{P}(i)$ are called the **subordinates** of $i \in N$ in P , and the consumers in $\hat{P}^{-1}(i) := \{i' \in N : i \in \hat{P}(i')\}$ are called the **superiors** of $i \in N$ in P . Similarly, the set $\hat{P}(E)$ represents the subordinates of the players in $E \subseteq N$ and $\hat{P}^{-1}(E)$ represents the superiors of the players in $E \subseteq N$. For each $i \in N$, we call **pipeline i** the pipeline having consumer i at its tail. Consumer $i \in N$ and its subordinates are called the **downstream consumers** of pipeline i . If no confusion arises, simply denote the network by P .

Fix $K \in \mathbb{N}$ an upper bound for consumer's effective demands. Each consumer $i \in N$ has an **effective demand** $q_i \leq K$ and is endowed with a discrete set $M_i = \{0, \dots, q_i\}$ describing all of its available **demand units**. The **profile of effective demands** is denoted by $q = (q_a, \dots, q_n)$. Without loss of generality, assume that $q_n \geq q_i$, for each $i \in N$. The set of all consumers with an effective demand of at least j is denoted by $Q(j) = \{i \in N : q_i \geq j\}$. A **cost function** is a map

$$C : N \times \{0, \dots, K\} \rightarrow \mathbb{R}_+$$

that measures the cost of any pipeline when designed to meet any demand. Formally, for each $i \in N$ and $j \in \{1, \dots, K\}$, $C(i, j)$ represents the cost of operating the pipeline i when it is designed to meet a demand of j . We use the convention $C(i, 0) = 0$ for each $i \in N$. We assume that each map

$$\forall i \in N, \quad C(i, \cdot) : \{1, \dots, K\} \rightarrow \mathbb{R}_+$$

is non-decreasing: the larger the demand, the larger is the pipeline's capacity, which leads to higher costs. On the other hand, for any $j \in \{1, \dots, K\}$ and any two $i, i' \in N$, we do not necessarily have $C(i, j) = C(i', j)$. Indeed, the cost of operating a pipeline may differ depending on the pipeline. These differences are due to exogenous features of the gas distribution network such as the length or the geographical location.

Equivalently, a cost function can be expressed as a matrix of incremental costs. We denote by A_{ij}^C the **incremental cost** generated by the j -th upgrade of pipeline i , i.e., when pipeline i is upgraded by one unit to meet a demand of j instead of $j - 1$. Formally, it is defined as

$$\forall i \in N, \forall j \in \{1, \dots, K\}, \quad A_{ij}^C = C(i, j) - C(i, j - 1).$$

Obviously, $A_{ij}^C \geq 0$ for each $i \in N$ and each $j \in \{1, \dots, K\}$, since $C(i, \cdot)$ is a non-decreasing map. The (n, K) **matrix of incremental costs** collects all the incremental costs, and is denoted by A^C . Take any $i \in N$ and any $j \leq K$. The **unit cost matrix** I^{ij} is the (n, K) matrix defined as

$$\forall k \in N, l \leq K, \quad I_{kl}^{ij} = \begin{cases} 1 & \text{if } k = i, l = j, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

In such matrix, only the j -th upgrade of pipeline i generates non-null incremental costs. Unit cost matrices can be useful if one wants to focus on a specific upgrade of a specific pipeline. A matrix of incremental costs A^C can be expressed as the linear combination of $n \times K$ unit cost matrices:

$$A^C = \sum_{i \in N} \sum_{j \leq K} A_{ij}^C I^{ij}. \quad (2)$$

Alternatively, the (n, q_n) sub-matrix A^{C, q_n} collects all the incremental costs generated by the q_n -th and lower upgrades. Such sub-matrix can be used in situation where the highest effective demand of the consumers in N is at most q_n . Similarly, a sub-matrix I^{ij, q_n} is a (n, q_n) matrix where only the j -th upgrade of pipeline i generates non-null incremental costs. In the following, we are only interested in the incremental costs contained within A^{C, q_n} . Thus, we are going to focus on the sub-matrix A^{C, q_n} instead of the whole matrix A^C . For the sake of clarity, we keep the notation A^C to refer to A^{C, q_n} . Similarly, for each $i \in N$ and each $j \leq q_n$, we keep the notation I^{ij} to refer to I^{ij, q_n} .

2.2. The problem

We propose an approach to compute the total cost of operating a gas distribution network from N , q , P and C . Alternative approaches are discussed at the end of this paper. Assume that each pipeline is always designed to meet the highest effective demand of its downstream consumers. Then, for each $i \in N$, the cost of operating pipeline i is given by

$$\forall i \in N, \quad C(i, \bar{q}_i), \quad \text{where } \bar{q}_i = \max_{k \in \hat{P}(i) \cup i} q_k. \quad (3)$$

Equivalently, observe that the cost of operating a pipeline can be expressed using incremental costs.

$$\forall i \in N, \quad C(i, \bar{q}_i) = \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C.$$

The **total cost** of operating the network is computed as the sum of the costs of all the pipelines designed in this way (3), which is given by

$$\sum_{i \in N} C(i, \bar{q}_i), \quad \text{or equivalently, } \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C. \quad (4)$$

Definition 1 (The cost sharing problem of gas distribution). A cost sharing problem of gas distribution (gas distribution problem for short) is denoted by (N, q, A^C, P) , or by (q, A^C) for short, since N and P are fixed and C and A^C are equivalent. The problem is to determine a way to share the total cost among consumers based on their demands and their location on the distribution network. The set of gas distribution problems is denoted by GDP .

Example 1. Consider the set of consumers $N = \{a, b, c, d, e\}$ and a source \mathcal{S} in gas. In this example, we consider the distribution network P and the profile of effective demands $q = (2, 1, 3, 1, 3)$.

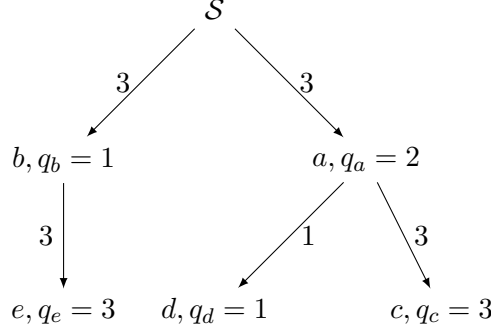


Figure 2: Gas distribution network

We introduce the lengths of the pipelines, which is one of the possible exogenous features of a network. The profile $L = (1, 2, 1, 2, 1)$ collects these lengths. Consider the cost function given by

$$C : \{a, b, c, d, e, f\} \times \{1, 2, 3\} \rightarrow \mathbb{R}_+$$

$$(i, j) \mapsto L_i + j.$$

This cost function specifies the cost of each pipeline when designed to meet a certain demand. It is computed as the sum of its length plus the demand it needs to meet. This cost function is obviously overly simplified for the clarity of the example. But the cost of a pipeline is increasing with respect to both parameters, which is consistent with reality.

Consider pipeline a . This pipeline is designed to meet its highest downstream demand. In this case, the highest downstream demand of pipeline a is given by

$$\max_{k \in \hat{P}(a) \cup \{a\}} q_k = q_c = 3.$$

Therefore, this pipeline costs $C(a, q_c) = L_a + q_c = 4$. Applying the same reasoning to each pipeline, we obtain the total cost of operating this network, which is given by

$$\begin{aligned} \sum_{i \in N} C(i, \bar{q}_i) &= C(a, q_c) + C(b, q_e) + C(c, q_c) + C(d, q_d) + C(e, q_e) \\ &= 4 + 5 + 4 + 3 + 4 \\ &= 20. \end{aligned}$$

3. Rules and characterizations

We address gas distribution problems by defining **cost sharing rules** (rules for short). This section is devoted to the study of the Connection rule, the Uniform rule and the Mixed rules. An axiomatic characterization for each one of these rules is provided.

Definition 2 (Cost sharing rule). A rule on GDP is a map

$$f : GDP \rightarrow \mathbb{R}_+^{n \times K}.$$

Pick any $(q, A^C) \in GDP$. A rule f assigns a non-negative real number $f_{ij}(q, A^C) \in \mathbb{R}_+$ to each demand $j \in \{1, \dots, q_i\}$ of each consumer $i \in N$. This number represents the amount that i must pay for its j -th unit, assuming that it already pays a certain amount for its $(j-1)$ -th unit, its $(j-2)$ -th unit, and so on. In other words, it can be interpreted as the additional amount i has to pay to increase its demand from $j-1$ to j . For each $j \notin \{1, \dots, q_i\}$, we use the convention $f_{ij}(q, A^C) = 0$. The total cost share of a consumer $i \in N$ is given by

$$F_i(q, A^C) = \sum_{j \leq q_j} f_{ij}(q, A^C).$$

A rule satisfies the **budget balanced** condition, which states that a rule recovers the total cost of operating the network. Formally, the budget balanced condition is given by

$$\begin{aligned} \forall (q, A^C) \in GDP, \quad \sum_{i \in N} F_i(q, A^C) &= \sum_{i \in N} \sum_{j \in \{1, \dots, q_i\}} f_{ij}(q, A^C) \\ &= \sum_{i \in N} C(i, \bar{q}_i) \\ &= \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C. \end{aligned} \quad (5)$$

3.1. The Connection rule

We define a rule on gas distribution problems that satisfies the connection principle and the independence of higher demands principle. This rule is called the Connection rule, it ensures that each consumer pays a fair share of the portion of the network that connects it to the source. Formally, for each pipeline k and each demand j such that some downstream consumers of pipeline k are in $Q(j)$, the Connection rule shares the incremental cost A_{kj}^C equally among the downstream consumers of pipeline k that are in $Q(j)$.

Definition 3 (Connection rule). The Connection rule Ψ is defined, for each $(q, A^C) \in GDP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Psi_{ij}(q, A^C) = \sum_{k \in \hat{P}^{-1}(i) \cup i} \frac{A_{kj}^C}{(|\hat{P}(k) \cup k| \cap Q(j))}. \quad (6)$$

An axiomatic characterization of the Connection rule is provided. It invokes two independence axioms and one equal loss axiom. First, a formal expression of the independence of higher demands principle is translated into an axiom. It ensures that the amount charged to a consumer for a certain demand unit is independent from any other greater demand unit. Formally, pick a gas distribution problem. Assume that the network operator must limit its gas supply due to some shortage. Consequently, it does not allow consumers to demand more than a certain amount of gas $l \leq q_n$. Then, the axiom guarantees that this limitation will not affect the costs allocated to demand units below l of any consumer.

Independence to Higher Demands (IHD) For each $(q, A^C) \in GDP$ and each $l \in \{0, \dots, q_n\}$,

$$\forall (i, j) \in M^+ : j \leq l, \quad f_{ij}(q, A^C) = f_{ij}((l \wedge q_k)_{k \in N}, A^C).$$

We say that a pipeline is **irrelevant** to a consumer if it is not a pipeline that helps to connect this consumer to the source. For instance, in Example 1, pipeline b is irrelevant for consumer a . The next axiom is in line with the Connection principle as it ensures that a consumer is not charged for the costs generated by irrelevant pipelines. Formally, the axiom states that the cost charged to a demand unit of a consumer should not be impacted by the increase of an irrelevant incremental cost (an incremental cost generated by an irrelevant pipeline).

Independence to Irrelevant Cost (IIC): Pick any $(q, A^C) \in GDP$. For each $j \leq q_n$, each $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon \in \mathbb{R}$,

$$\forall h \in Q(j), h \notin (\hat{P}(i) \cup \{i\}), \quad f_{hj}(q, A^C) = f_{hj}(q, A^C + \varepsilon I^{ij}).$$

In contrast to irrelevant costs, the incremental costs generated by a pipeline are relevant to any of its downstream consumers. The next axiom is an equal loss requirement, arguing that the cost sharing rule must treat each downstream consumer equally regarding a given upstream cost.

Formally, the axiom states that the amount charged to the demand unit j of two downstream consumers should be equally impacted by the increase of an incremental cost generated by the j -th upgrade of an upstream pipeline.

Equal Loss for Downstream Consumers (ELDC): Pick any $(q, A^C) \in GDP$. For each $j \leq q_n$, each $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon \in \mathbb{R}$,

$$\forall h, h' \in (\hat{P}(i) \cup \{i\}) \cap Q(j), \quad f_{hj}(q, A^C + \varepsilon I^{ij}) - f_{hj}(q, A^C) = f_{h'j}(q, A^C + \varepsilon I^{ij}) - f_{h'j}(q, A^C).$$

We now have the necessary material to characterize the connection rule.

Theorem 1. *A rule f on GDP satisfies (IHD), (IIC) and (ELDC) if and only if $f = \Psi$.*

Proof. See Appendix 6.1 □

Observe that the axioms invoked in this characterization are conceptually close to those used in Dong et al. (2012) to characterize the Downstream equal sharing rule for polluted river problems. In short, their characterization uses axioms equivalent to (IIC), (ELDC), the Budget balanced condition and an Additivity axiom. However, two major differences can be highlighted between their characterization and ours. First, due to the presence of demands in our model, we invoke the (IHD) axiom. Second, we do not use an Additivity axiom.

3.2. The Uniform rule

This section defines a rule for gas distribution problems that follows the independence of higher demands principle and the uniformity principle. This rule is called the Uniform rule, it ensures that two consumers should always be charged with the same amount for the same demand, regardless of any other parameter of the gas distribution problem. Formally, consider a demand $j \leq q_n$ and the consumers in $Q(j)$. Recall that, for each consumer $h \in Q(j)$, the pipelines connecting it to the source are upgraded at least j times. This implies that each pipeline k , where $k \in \hat{P}^{-1}(h)$, generates the incremental cost A_{kj}^C for its j -th upgrade. Observe that k is not necessarily in $Q(j)$, if $k \notin Q(j)$ then k is the superior of a consumer in $Q(j)$. The Uniform rule shares each incremental cost A_{kj}^C equally among each consumer in $Q(j)$.

Definition 4 (Uniform rule). The Uniform rule Υ is defined, for each $(q, A^C) \in GDP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Upsilon_{ij}(q, A^C) = \frac{1}{|Q(j)|} \sum_{k \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{kj}^C. \quad (7)$$

Observe that the Uniform rule satisfies (ELDC). We discuss an additional axiom for rules. Let f be a rule on GDP . Assume that one or several pipelines generate additional costs due to exogenous reasons (incident, natural disaster, etc), which leads to an increase of the incremental costs. Such additional costs should not increase the inequalities between the cost shares of the consumers. Formally, pick any demand j and assume that all the incremental costs generated by the j -th upgrades of the pipelines increase. Compare the difference between the highest cost share and the lowest cost share charged to the consumers in $Q(j)$ for their unit of demand j . This difference should not increase to aggravate inequalities when the incremental costs increase.

Non-Increasing Inequalities (NII) For each $(q, A^C), (q, A^{C'}) \in GDP$ such that $A_{ij}^{C'} \geq A_{ij}^C$, for each $i \in N$ and $j \leq q_n$,

$$\forall j \in \{1, \dots, q_n\}, \quad \max_{i \in Q(j)} f_{ij}(q, A^{C'}) - \min_{i \in Q(j)} f_{ij}(q, A^{C'}) \leq \max_{i \in Q(j)} f_{ij}(q, A^C) - \min_{i \in Q(j)} f_{ij}(q, A^C).$$

We have the material to characterize the Uniform rule.

Theorem 2. A rule f on GDP satisfies (IHD) and (NII) if and only if $f = \Upsilon$.

Proof. See Appendix 6.2 □

3.3. The Mixed rule

The Uniform rule does not satisfy Independence of Irrelevant Costs. Also, the Connection rule does not satisfy Non-increasing inequalities. This testifies that the uniformity principle and the connection principle are clearly incompatible. However, both principles can be highly desirable in the context of gas distribution problems. For this reason, a trade-off between the two principles is proposed by introducing a family of rules that follow the independence of higher demands principle and reach compromises between the uniformity principle and the connection principle. The rules in this family are called Mixed rules and are computed using convex combinations of the Connection rule and the Uniform rule. Moreover, Mixed rules allow for different compromises between the uniformity principle and the connection principle, depending on the level of demand.

Definition 5 (Mixed rules). Let $\alpha = \{\alpha^j\}_{1 \leq j \leq K}$ be a parameter system such that $\alpha^j \in [0, 1]$, for each $1 \leq j \leq K$. The α -Mixed rule μ^α is defined, for each $(q, A^C) \in GDP$, as

$$\forall j \leq q_n, \forall i \in Q(j), \quad \mu_{ij}^\alpha(q, A^C) = \alpha^j \Psi_{ij}(q, A^C) + (1 - \alpha^j) \Upsilon_{ij}(q, A^C).$$

A α -Mixed rule operates convex combinations between the Connection rule and the Uniform rule. For each demand $j \in \{0, \dots, q_n\}$, a consumer $i \in Q(j)$ is charged a cost share lying between $\Psi_{ij}(q, A^C)$ and $\Upsilon_{ij}(q, A^C)$. If α^j is closer to 1, then this cost share is closer to $\Psi_{ij}(q, A^C)$. On the other hand, if α^j is closer to 0, then this cost share is closer to $\Upsilon_{ij}(q, A^C)$.

To characterize the Mixed rules, we introduce three new axioms. Let f be a rule on GDP . The first axiom ensures that any two distinct consumers are equally impacted by irrelevant costs. Formally, assume that the incremental cost of the j -th upgrade of a pipeline increases. Pick any two consumers $h, h' \in N$ for whom this pipeline is irrelevant. The axiom states that amount paid by h and h' for their demand unit j will be equally impacted by the incremental cost increase.

Equal Impact of Irrelevant Costs (EIIC): Pick any $(q, A^C) \in GDP$. For each $j \leq q_n$, each $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon \in \mathbb{R}$,

$$\forall h, h' \in Q(j), h, h' \notin \hat{P}(i) \cup \{i\}, \quad f_{hj}(q, A^C + \varepsilon I^{ij}) - f_{hj}(q, A^C) = f_{h'j}(q, A^C + \varepsilon I^{ij}) - f_{h'j}(q, A^C).$$

Observe that both the Connection rule and the Uniform rule satisfy Equal impact of irrelevant costs. Moreover, (IIC) implies (EIIC).

The next axiom ensures that no matter where an irrelevant cost is located, the relative impact of that cost on a consumer, for whom that cost is irrelevant, remains the same in any situation. The axiom can be viewed as a two-step process. First, the incremental cost generated by the j -th upgrade of a pipeline $i \in N$ increases by some $\varepsilon \in \mathbb{R}$. Then, the incremental cost generated by the j -th upgrade of another pipeline $i' \in N$ increases by some $\varepsilon' \in \mathbb{R}$. The order does not matter. Consider any consumer $h \in N$ for whom these pipelines are irrelevant. The amount paid by h for its unit j may vary at each step. The axiom states that the ratio between a variation and the corresponding increase must be the same at each step.

Proportional Impact of Irrelevant Costs (PIIC): Pick any $(q, A^C) \in GDP$. For each $j \leq q_n$, each $i, i' \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon, \varepsilon' \in \mathbb{R}$,

$$\begin{aligned} &\forall h \in Q(j), h \notin (\hat{P}(i) \cup \{i\}) \cup (\hat{P}(i') \cup \{i'\}), \\ &\frac{1}{\varepsilon}(f_{hj}(q, A^C + \varepsilon I^{ij}) - f_{hj}(q, A^C)) = \frac{1}{\varepsilon'}(f_{hj}(q, A^C + \varepsilon' I^{i'j} + \varepsilon I^{ij}) - f_{hj}(q, A^C + \varepsilon I^{ij})). \end{aligned}$$

Observe that both the Connection rule and the Uniform rule satisfy (PIIC). Moreover, (IIC) implies (PIIC).

For the last axiom, we assume that the incremental cost of the j -th upgrade of a pipeline increases. Pick any consumers $h \in N$ for whom this cost is relevant. The axiom states that the cost share of h (for its demand unit j) should be more impacted than the cost share of anyone else.

Fairness (F): Pick any $(q, A^C) \in GDP$. For each $j \leq q_n$, each $i \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, and each $\varepsilon \in \mathbb{R}$,

$$\begin{aligned} &\forall h \in (\hat{P}(i) \cup \{i\}) \cap Q(j), \forall h' \in Q(j), \\ &f_{hj}(q, A^C + \varepsilon I^{ij}) - f_{hj}(q, A^C) \geq f_{h'j}(q, A^C + \varepsilon I^{ij}) - f_{h'j}(q, A^C). \end{aligned}$$

Observe that both the Connection rule and the Uniform rule satisfy (F). Moreover, (F) implies (ELDC).

The following characterization results holds on a sub-class of GDP . This sub-class is defined by the following assumptions. We assume that the source has at least three successors, i.e., $|P(\mathcal{S})| \geq 3$. We also assume that, for each $k \in P(\mathcal{S})$, there exists at least one consumer $i \in \hat{P}(k)$ such that $q_i = q_n$. The sub-class of gas distribution problems satisfying these two assumptions is denoted by \overline{GDP} . We now have the material to characterize the Mixed rules.

Theorem 3. *A rule on \overline{GDP} satisfies (IHD), (EIIC), (PIIC) and (F) if and only if $f = \mu^\alpha$, for some parameter system α .*

Proof. See Appendix 6.3 □

Example 2. Consider Example 1. Let us illustrate our rules by computing the cost shares of consumer c . Recall that the profile $L = (1, 2, 1, 2, 1)$ collects the length of each pipeline. The cost function is given by

$$C : \{a, b, c, d, e, f\} \times \{1, 2, \dots, 10\} \rightarrow \mathbb{R}_+$$

$$(i, j) \mapsto L_i + j.$$

The corresponding matrix of incremental costs is given by the following table:

A^C	a	b	c	d	e
1	2	3	2	3	2
2	1	1	1	1	1
3	1	1	1	1	1

The incremental costs in light gray are the ones that are not used in the computation of the total costs. The sum of the other incremental costs is equal to 20, which corresponds to the total cost of operating the distribution network.

By the definition of the Connection rule,

$$\begin{aligned} \Psi_{c1}(q, A^C) &= \sum_{k \in \hat{P}^{-1}(c) \cup c} \frac{A_{k1}^C}{(|\hat{P}(k) \cup k| \cap Q(1))} \\ &= \frac{A_{a1}^C}{(|\hat{P}(a) \cup a| \cap Q(1))} + \frac{A_{c1}^C}{(|\hat{P}(c) \cup c| \cap Q(1))} \\ &= \frac{2}{3} + \frac{2}{1} \\ &\approx 2.67. \end{aligned}$$

Similarly,

$$\Psi_{c2}(q, A^C) = 1.5 \quad \text{and} \quad \Psi_{c3}(q, A^C) = 2.$$

By the definition of the Uniform rule,

$$\begin{aligned} \Upsilon_{c1}(q, A^C) &= \frac{1}{|Q(1)|} \sum_{k \in \hat{P}^{-1}(Q(1)) \cup Q(1)} A_{k1}^C \\ &= \frac{1}{|Q(1)|} (A_{a1}^C + A_{b1}^C + A_{c1}^C + A_{d1}^C + A_{e1}^C) \\ &= \frac{12}{5} \\ &= 2.4 \end{aligned}$$

Similarly,

$$\Upsilon_{c2}(q, A^C) \approx 1.33 \quad \text{and} \quad \Upsilon_{c3}(q, A^C) = 2.$$

According to the Connection rule, consumer c must pay 2.67 for its first demand unit. Then, to increase its demand by one unit, consumer c must pay 1.5 more, making her pay a total of $2.67 + 1.5 =$

4.17. Then, to reach its effective demand, c must pay an additional 2. The total amount charged to c by the Connection rule is then

$$\begin{aligned} F_c &= \sum_{l \leq q_c} \Psi_{cl}(q, A^C) \\ &= 6.17. \end{aligned}$$

According to the Uniform rule, consumer c must pay 2.4 for its first demand unit. Then, to increase its demand by one unit, consumer c must pay 1.33 more, making her pay a total of 3.73. Then, to reach its effective demand, c must pay an additional 2. The total amount charged to c by the Connection rule is then

$$\begin{aligned} F_c &= \sum_{l \leq q_c} \Upsilon_{cl}(q, A^C) \\ &= 5.73. \end{aligned}$$

Pick $\alpha = (0, 0.5, 1)$. According to the associated Mixed rule μ^α , it holds that

$$\mu_{c1}^\alpha(q, A^C) = 2.4, \quad \mu_{c2}^\alpha(q, A^C) = 1.415 \quad \text{and} \quad \mu_{c3}^\alpha(q, A^C) = 2.$$

This mixed rule favors the uniformity principle for the demand units 2. It compromises between the uniformity principle and the connection principle for the demand unit 2. Finally, it favors the connection principle for the demand unit 3.

We represent the aggregated cost share of c by Figure 3. The squares represent the Connection rule and the stars represent the Uniform rule. Any point located between a square and the corresponding star may represent the cost share obtained by using a Mixed rule.

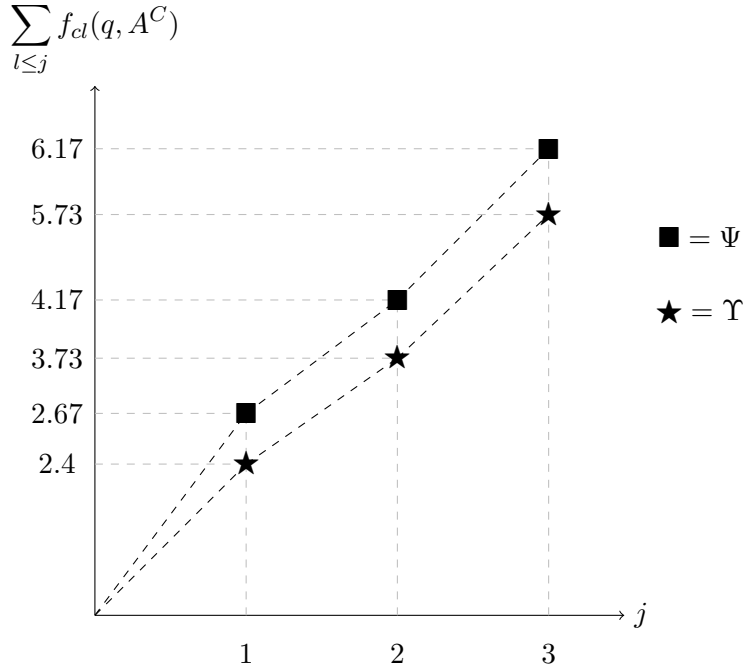


Figure 3: Evolution of the cost share of c

4. Rules and Multi-choice games

This section presents multi-choice games as introduced by Hsiao and Raghavan (1992) and van den Nouweland (1993). For each gas distribution problem, we derive a specific multi-choice game called the gas distribution game. Using this gas distribution game, we study how our rules relate to some solution concepts from multi-choice games. In particular, we show that the Connection rule is a **stable rule**, in the sense that consumers have an interest in becoming customers of the operator.

4.1. The gas distribution game

In order to remain consistent with gas distribution problems, some notation coincide with Section 2.1. Let $N = \{a, b, \dots, n\}$ be a fixed set of players and $K \in \mathbb{N}$. Each player $i \in N$ has a finite set of pairwise distinct activity levels $M_i := \{0, \dots, q_i\}$, where $q_i \leq K$. Define the set of players capable of playing at least activity level j as $Q(j) = \{i \in N : q_i \geq j\}$. Denote by \mathcal{M} the Cartesian product $\prod_{i \in N} M_i$. Each element $s = (s_1, \dots, s_n) \in \mathcal{M}$ specifies a participation profile for players and is referred to as a (multi-choice) coalition. So, a coalition indicates each player's activity level. Then, $q = (q_1, \dots, q_n) \in \mathcal{M}$ is the players' maximal participation profile that plays the role of the grand coalition, whereas $\Theta = (0, \dots, 0)$ plays the role of the empty coalition. We use the notations $M_i^+ := M_i \setminus \{0\}$ for each $i \in N$ and $M^+ := \bigcup_{i \in N} (\{i\} \times M_i^+)$. A pair $(i, j) \in M^+$ represents a player and one of its activity levels. A (cooperative) **multi-choice game** on N is a couple (q, v) where $v : \mathcal{M} \rightarrow \mathbb{R}$ is a characteristic function, with $v(\Theta) = 0$, that specifies a worth, $v(s)$, for each $s \in \mathcal{M}$. The full class of multi-choice games is denoted by \mathcal{G} . A multi-choice game is **sub-modular** if, for each $s, t \in \mathcal{M}$,

$$v(s \vee t) + v(s \wedge t) \leq v(s) + v(t).$$

Consider $(q, v) \in \mathcal{G}$. A payoff vector $x \in \mathbb{R}^{M^+}$ for the game (q, v) assigns a payoff $x_{ij} \in \mathbb{R}$ to each pair $(i, j) \in M^+$. A set-valued solution on \mathcal{G} is a map F that assigns a collection of payoff vectors $F(q, v)$ to each $(q, v) \in \mathcal{G}$. A value f is a single-valued solution on \mathcal{G} , that assigns a unique payoff vector $f(q, v)$ to each $(q, v) \in \mathcal{G}$. We define gas distribution games, which are specific multi-choice games based on gas distribution problems. A gas distribution game, derived from a gas distribution problem (q, A^C) , measures the total cost of each gas distribution problem (s, A^C) in which each consumer $i \in N$ has a demand $s_i \leq q_i$. Put differently, a gas distribution problem is a collection of all the total costs that could be generated by networks smaller than the one they are derived from.

Definition 6 (The gas distribution game). For each $(q, A^C) \in GDP$, its associated gas distribution (multi-choice) game $(q, v^{C,P})$ is defined as

$$\forall s \leq q, \quad v^{C,P}(s) = \sum_{i \in N} C(i, \bar{s}_i) = \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{s}_i\}} A_{ij}^C \quad \text{where } \forall i \in N, \quad \bar{s}_i = \max_{k \in \hat{P}(i) \cup i} s_k. \quad (8)$$

In a gas distribution game, the players represent the consumers and the activity levels represent the demands of the consumers. The worth $v^{C,P}(s)$ represents the cost of operating a network (s, A^C) in which each consumer $i \in N$ has an effective demand of s_i . Obviously, $v^{C,P}(q)$ coincides with the total cost of operating the network and $v^{C,P}(0, \dots, 0) = 0$ since $C(i, 0) = 0$, for each $i \in N$. Since $C(i, \cdot)$ is non-decreasing, for each $i \in N$, it follows that a gas distribution game is also a non-decreasing map, meaning that for each $(q, A^C) \in GDP$, each $s \geq t$, we have $v^{C,P}(s) \geq v^{C,P}(t)$.

4.2. Rules and solution concepts for multi-choice games

This section presents several solution concepts for multi-choice games and compare them with the rules of Section 3. First, we present the multi-choice Shapley value as introduced by Lowing and Techer (2022) for multi-choice games. To that end, we introduce **restricted orders** over the set of pairs M^+ as defined by Grabisch and Xie (2007). Consider a gas distribution game $(q, v^{C,P})$. Restricted orders are such that no pair $(i, j) \in M^+$ is ordered before a pair $(i', j') \in M^+$ with a strictly lower activity level $j' < j$. This means that no consumer can demand $j + 1$ unless each consumer $i \in Q(j)$ demands at least j . Formally, a restricted order over the set of pairs is a bijection $\sigma : M^+ \rightarrow \{1, \dots, \sum_{i \in N} q_i\}$ defined as

$$\forall (i, j), (i', j') \in M^+, \quad [j < j'] \implies [\sigma(i, j) < \sigma(i', j')].$$

Denote by \bar{O} the set of all restricted orders over the set of pairs. Let $\sigma \in \bar{O}$ be a restricted order and $h \in \{1, \dots, \sum_{i \in N} q_i\}$. Denote by $s^{\sigma, h}$ the coalition formed after h steps. We use the convention $s^{\sigma, 0} = \emptyset$. Formally, it is defined as

$$\forall i \in N, \quad s_i^{\sigma, h} = \max \{j \in \{1, \dots, q_i\} : \sigma(i, j) \leq h\} \cup \{0\}. \quad (9)$$

For each $\sigma \in \bar{O}$, the marginal vector $\eta^\sigma(q, v^{C,P})$ is defined as

$$\forall (i, j) \in M^+, \quad \eta_{ij}^\sigma(q, v^{C,P}) = v^{C,P}(s^{\sigma, \sigma(i, j)}) - v^{C,P}(s^{\sigma, \sigma(i, j) - 1}).$$

Each $\eta_{ij}^\sigma(q, v^{C,P})$ is called the marginal contribution of the pair (i, j) to the coalition $s^{\sigma, \sigma(i, j) - 1}$, which is formed after $\sigma(i, j) - 1$ steps with respect to the restricted order σ . The marginal contribution of the pair (i, j) can be interpreted as the additional costs generated when consumer i increases its demand from $j - 1$ to j , according to the restricted order σ . We have the material to define the multi-choice Shapley value for gas distribution games. This value assigns to each pair $(i, j) \in M^+$ its expected marginal contribution assuming that each restricted orders over the set of pairs occurs with the same probability.

Definition 7 (The multi-choice Shapley value). For each gas distribution game $(q, v^{C,P})$, the multi-choice Shapley value is defined as

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(q, v^{C,P}) = \frac{1}{\prod_{j \in \{1, \dots, q_n\}} |Q(j)|!} \sum_{\sigma \in \bar{O}} \eta_{ij}^\sigma(q, v^{C,P}). \quad (10)$$

The next result states that, for each gas distribution problem, the Connection rule applied to the problem coincides with the multi-choice Shapley value applied to the gas distribution game associated with the problem.

Theorem 4. For each gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$,

$$\varphi(q, v^{C,P}) = \Psi(q, A^C). \quad (11)$$

Proof. See Appendix 6.4 □

Next, we present the multi-choice Equal division value as introduced by Lowing and Techer (2022) for multi-choice games. Assume that all consumers agree on forming a coalition in which everyone has the same demand, let us say j . Each consumer $i \in N$ with an effective demand $q_i \leq j$ demands q_i . We call such coalition a j -synchronized coalition. For each activity level $j \in \{1, \dots, q_n\}$, the multi-choice Equal division value shares the difference in worth between the j -synchronized coalition and the $j - 1$ -synchronized coalition equally among the pairs containing the activity level j .

Definition 8 (The multi-choice Equal division value). For each gas distribution game $(q, v^{C,P})$, the multi-choice Equal division value ξ is defined as

$$\forall (i, j) \in M^+, \quad \xi_{ij}(q, v^{C,P}) = \frac{1}{|Q(j)|} \left[v^{C,P}((j \wedge q_k)_{k \in N}) - v^{C,P}(((j-1) \wedge q_k)_{k \in N}) \right]. \quad (12)$$

The next result states that, for each gas distribution problem, the Uniform rule applied to the gas distribution problem coincides with the multi-choice Equal division value applied to the gas distribution game associated with the problem.

Theorem 5. For each gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$,

$$\xi(q, v^{C,P}) = \Upsilon(q, A^C). \quad (13)$$

Proof. The proof of Theorem 5 follows directly from (8) and (12). \square

Next, we present the multi-choice Egalitarian Shapley values as defined by Lowing and Techer (2022) for multi-choice games. These values operate a trade-off between the multi-choice Shapley value and the multi-choice Equal division value by mean of convex combinations between the two values. In particular, a convex combination is done for each demand, meaning that the trade-off between the two values can differ depending on the demand.

Definition 9 (The multi-choice Egalitarian Shapley values). Let $\alpha = \{\alpha^j\}_{1 \leq j \leq K}$ be a parameter system such that $\alpha^j \in [0, 1]$ for each $1 \leq j \leq K$. For each gas distribution game $(q, v^{C,P})$, the multi-choice Egalitarian Shapley value χ^α is defined as

$$\forall (i, j) \in M^+, \quad \chi_{ij}^\alpha(q, v^{C,P}) = \alpha^j \varphi_{ij}(q, v^{C,P}) + (1 - \alpha^j) \xi_{ij}(q, v^{C,P}). \quad (14)$$

The next result states that, for each gas distribution problem, the set of Mixed rules applied to the problem coincides with the set of multi-choice Egalitarian Shapley values applied to the gas distribution game associated with the problem.

Theorem 6. For each gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$, and for each parameter system α ,

$$\chi^\alpha(q, v^{C,P}) = \mu^\alpha(q, A^C). \quad (15)$$

Proof. The proof of Theorem 6 follows directly from Theorem 5 and Theorem 6. \square

We introduce one last solution concept: the Core of a gas distribution game, inspired from the Core for multi-choice games introduced by Grabisch and Xie (2007). The Core is defined by two conditions. The first condition (16) states that no coalition can achieve, by itself, a better outcome than the one prescribed by the payoff vectors in the Core. The second condition (17) states that any h -synchronized coalition achieves the same outcome than the one prescribed by the payoff vectors in the Core.

Definition 10 (The Core). The Core of a gas distribution game $(q, v^{C,P})$, denoted by $\mathcal{C}(q, v^{C,P})$, is the set of payoff vectors $x \in \mathbb{R}^{\sum_{i \in N} q_i}$ defined as

$$x \in \mathcal{C}(q, v^{C,P}) \iff \begin{cases} \forall s \leq q, & \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \leq v^{C,P}(s), & (16) \\ \forall l \leq q_n, & \sum_{i \in N} \sum_{j=1}^{l \wedge q_i} x_{ij} = v^{C,P}((l \wedge q_i)_{i \in N}). & (17) \end{cases}$$

If we re-interpret the Core conditions in terms of rule for gas distribution problems, condition (16) ensures that consumers always pay less than the cost of supplying themselves in gas and (17) states that if a group of consumers, which synchronize their demands in the sense of a synchronized coalition, decide to supply themselves without resorting to the network operator, then they should pay the same amount as they would have been charged by the operator. Thus, the Core can be viewed as the set of **stable rules**, in the sense that consumers have an interest in becoming customers of the operator. We point out that condition (16) is the opposite from the original definition of the Core provided by Grabisch and Xie (2007). For each gas distribution problem, the multi-choice Shapley value applied to a gas distribution game is always in the Core. Therefore, the Connection rule is a stable rule.

Theorem 7. For each gas distribution problem $(q, v^{C,P}) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$,

$$\varphi(q, v^{C,P}) \in \mathcal{C}(q, v^{C,P}). \quad (18)$$

Proof. See Appendix 6.5 □

5. Conclusion

5.1. Synthesis

In this paper, we define gas distribution problems and determine three rules for these problems on the basis of ethical principles. Applying the connection principle and the independence of higher demands principle, we propose the Connection rule. Applying the uniformity principle and the independence of higher demands principle, we propose the Uniform rule. To compromise between the connection principle and the uniformity principle, we propose the Mixed rules. We provide an axiomatic characterization for each of these rules. In addition, these rules coincide with solution concepts from multi-choice games. In particular, the Connection rule applied to a gas distribution problem belongs to the Core of a specific multi-choice game derived from this problem.

5.2. Future research

In this paper, we assume that each pipeline is designed to meet its highest downstream demand. Let us call this approach the **optimistic approach** to pipeline design. The main drawback of this approach is that a pipeline can only satisfy a few effective demands at a time. In particular, the highest downstream demand of a pipeline already saturates its capacity, preventing any other additional demand from being satisfied. However, this approach has its advantages since it can be implemented at a low cost while ensuring a minimal service to consumers.

An alternative is the **pessimistic approach** to pipeline design: the network operator must be able to satisfy all effective demand at any time. Therefore each pipeline should be designed to meet the sum of the effective demands of the consumers located at the tail and downstream of this pipeline. Unlike the previous approach, this approach ensures a flawless service to consumers. The main drawback of this approach is the resulting total cost of operating the network. Indeed, a network designed this way would be huge and too expensive to operate.

The two approaches have their advantages and their drawbacks. In practice, a gas distribution network is rarely design according to the optimistic or the pessimistic approach. It may be interesting to investigate a new approach that compromises between these two. Such an approach would ensure that the network is capable of handling multiple effective demands at the same time, while limiting the total cost of the network.

Acknowledgments The author wants to thank Justin Leroux, Stéphane Gonzalez and Philippe Solal for useful comments, and GRDF (Gaz Réseau Distribution France) for their financial support through the ANRT (Association Nationale Recherche Technologie) doctoral program CIFRE (Conventions Industrielles de Formation par la REcherche). The author also wants to thank Sylvain Béal for letting him present this paper at the CRESE seminar (University of Besançon).

6. Appendix

If a rule satisfies (IHD), then the sum of all cost shares charged to consumers in $Q(j)$ for their demand unit j is equal to the sum of all incremental costs, generated by the j -th upgrades of the pipelines that connect the consumers in $Q(j)$ to the source.

Lemma 1. *If a rule f on GDP satisfies (IHD) then, for each $(q, A^C) \in GDP$,*

$$\forall l \leq q_n, \quad \sum_{i \in Q(l)} f_{il}(q, A^C) = \sum_{i \in \hat{P}^{-1}(Q(l)) \cup Q(l)} A_{il}^C.$$

Proof. Consider a rule f on GDP that satisfies (IHD). By the Budget Balanced condition (see (5)), for each $(q, A^C) \in GDP$,

$$\begin{aligned} \sum_{i \in N} \sum_{j \in \{1, \dots, q_i\}} f_{ij}(q, A^C) &= \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C \\ \Leftrightarrow \sum_{j \in \{1, \dots, q_n\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &= \sum_{i \in N} \sum_{j \in \{1, \dots, \bar{q}_i\}} A_{ij}^C \end{aligned} \quad (19)$$

Recall that, for each $i \in N$,

$$\bar{q}_i = \max_{h \in \hat{P}(i) \cup i} q_h.$$

Observe that, for each $l \leq q_n$, and each $i \in N$,

$$\begin{aligned}
\overline{((l \wedge q)_{k \in N})}_i &= \max_{h \in \hat{P}(i) \cup i} ((l \wedge q)_{k \in N})_h \\
&= \max_{h \in \hat{P}(i) \cup i} (l \wedge q_h) \\
&= \begin{cases} l & \text{if } \max_{h \in \hat{P}(i) \cup i} q_h \geq l \\ \max_{h \in \hat{P}(i) \cup i} q_h & \text{otherwise.} \end{cases} \\
&= l \wedge \bar{q}_i.
\end{aligned} \tag{20}$$

By (IHD), (19) and (20), for each $l \leq q_n$, it holds that

$$\begin{aligned}
\sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &\stackrel{(IHD)}{=} \sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}((l \wedge q_k)_{k \in N}, A^C) \\
&\stackrel{(19)}{=} \sum_{i \in N} \sum_{j \in \{1, \dots, \overline{((l \wedge q)_{k \in N})}_i\}} A_{ij}^C \\
&\stackrel{(20)}{=} \sum_{i \in N} \sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C
\end{aligned}$$

and similarly
$$\begin{aligned}
\sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) &= \sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(((l-1) \wedge q_k)_{k \in N}, A^C) \\
&= \sum_{i \in N} \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C.
\end{aligned}$$

Therefore, for each $l \leq q_n$,

$$\begin{aligned}
\sum_{i \in Q(l)} f_{il}(q, A^C) &= \sum_{j \in \{1, \dots, l\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) - \sum_{j \in \{1, \dots, l-1\}} \sum_{i \in Q(j)} f_{ij}(q, A^C) \\
&= \sum_{i \in N} \sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C - \sum_{i \in N} \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C \\
&= \sum_{i \in N} \left[\sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C - \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C \right].
\end{aligned}$$

To conclude this proof, observe that, for each $i \in N$ such that $l \wedge \bar{q}_i = (l-1) \wedge \bar{q}_i$ (i.e., $\bar{q}_i \leq l-1$),

$$\sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C - \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C = 0.$$

To the contrary, for each $i \in N$ such that $l \wedge \bar{q}_i > (l-1) \wedge \bar{q}_i$ (i.e., $\bar{q}_i \geq l$),

$$\sum_{j \in \{1, \dots, l \wedge \bar{q}_i\}} A_{ij}^C - \sum_{j \in \{1, \dots, (l-1) \wedge \bar{q}_i\}} A_{ij}^C = A_{il}^C.$$

Observe that $\bar{q}_i \geq l$ if and only if $i \in \hat{P}^{-1}(Q(l)) \cup Q(j)$. Therefore, for each $l \leq q_n$, we obtain the desired result

$$\sum_{i \in Q(l)} f_{il}(q, A^C) = \sum_{i \in \hat{P}^{-1}(Q(l))} A_{il}^C.$$

□

6.1. Proof of Theorem 1

Existence: We show that Ψ satisfies the axioms of the statement of Theorem 1.

(IHD): The proof follows directly from the definition of the Connection rule. The cost share charged to a consumer i for a demand unit j is computed using only the incremental costs generated by j -th upgrade of the pipelines. Therefore, we directly have that Ψ satisfies the axiom.

(IIC): The proof follows directly from the definition of the Connection rule. The cost share charged to a consumer i for a demand unit j is computed using only the incremental costs generated by the pipelines that connect i to the source. Therefore, the incremental costs generated by the pipelines irrelevant to i are not taken into account by the Connection rule in the computation of the cost share of i for its demand unit j .

(ELDC): Pick any $(q, A^C) \in GDP$, any $i \in N$, any $j \leq q_n$, and any $\varepsilon \in \mathbb{R}$. Assume that there exists two distinct consumers $h, h' \in (\hat{P}(i) \cup \{i\}) \cap Q(j)$. For h , it holds that

$$\begin{aligned} \Psi_{hj}(q, A^C + \varepsilon I^{ij}) - \Psi_{hj}(q, A^C) &= \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{A_{kj}^C + \varepsilon I_{kj}^{ij}}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} - \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{A_{kj}^C}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} \\ &= \sum_{k \in \hat{P}^{-1}(h) \cup h} \frac{\varepsilon I_{kj}^{ij}}{|(\hat{P}(k) \cup \{k\}) \cap Q(j)|} \\ &= \frac{\varepsilon}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|}. \end{aligned}$$

Similarly, for h' , it holds that

$$\Psi_{h'j}(q, A^C + \varepsilon I^{ij}) - \Psi_{h'j}(q, A^C) = \frac{\varepsilon}{|(\hat{P}(i) \cup \{i\}) \cap Q(j)|},$$

which shows that Ψ satisfies the axiom.

Uniqueness: We show that the Connection rule is the only rule on GDP that satisfies all the axioms of the statement of Theorem 1. Pick any $(q, A^C) \in GDP$. Let f be a rule that satisfies all the axioms of the statement of Theorem 1 on GDP . Let us show that $f(q, A^C)$ coincides with the Connection rule. In other words, we show that

$$\forall i \in N, \forall j \leq q_i, \quad f(q, A^C) = \Psi(q, A^C).$$

Pick any $i \in N$ and any $j \leq q_i$. Consider the problem $(q, A^j) \in GDP$, in which the matrix of incremental costs is defined as

$$\forall k \in N, \forall l \leq q_n, \quad A_{kl}^j = \begin{cases} 0 & \text{if } l = j \text{ and } k \in \hat{P}^{-1}(Q(j)) \cup Q(j) \\ A_{kl}^C & \text{otherwise.} \end{cases}$$

By Lemma 1,

$$\begin{aligned} \sum_{i \in Q(j)} f_{ij}(q, A^j) &= \sum_{i \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{ij}^j \\ &= 0 \end{aligned}$$

By definition of a rule a cost share must be non-negative. It follows that

$$\forall i \in Q(j), \quad f_{ij}(q, A^j) = 0.$$

In (q, A^j) , the incremental cost A_{kj}^j of each pipeline $k \in \hat{P}^{-1}(Q(j)) \cup Q(j)$ has been nullified. We have shown that $f_{ij}(q, A^j) = 0$ for each $i \in Q(j)$. In this proof we consider, step by step, the pipelines in $\hat{P}^{-1}(Q(j)) \cup Q(j)$. For each pipeline $k \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, we will un-nullified the incremental cost A_{kj}^j into A_{kj} and discuss the impact on the cost shares of the consumers in $Q(j)$ for their demand unit j . This way, we reconstruct the problem (q, A) to show the desired result.

Pick any $a \in \hat{P}^{-1}(Q(j)) \cup Q(j)$. If $a \in \hat{P}^{-1}(i) \cup \{i\}$, then by (ELDC),

$$\begin{aligned} \forall h \in (\hat{P}(a) \cup \{a\}) \cap Q(j), \quad f_{ij}(q, A^j + A_{aj}^C I^{aj}) - f_{ij}(q, A^j) &= f_{hj}(q, A^j + A_{aj}^C I^{aj}) - f_{hj}(q, A^j) \\ \iff f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= f_{hj}(q, A^j + A_{aj}^C I^{aj}). \end{aligned} \quad (21)$$

By (IIC),

$$\begin{aligned} \forall h \in Q(j), h \notin \hat{P}(a) \cup \{a\}, \quad f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= f_{hj}(q, A^j) \\ &= 0. \end{aligned} \quad (22)$$

By Lemma 1, (21) and (22), we obtain

$$\begin{aligned} \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= \sum_{i \in \hat{P}^{-1}(Q(j)) \cup Q(j)} (A^j + A_{aj}^C I^{aj}) \\ \iff \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= A_{aj}^C \\ \iff \sum_{h \in (\hat{P}(a) \cup \{a\}) \cap Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= A_{aj}^C \\ \implies f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|}. \end{aligned}$$

If $a \notin \hat{P}^{-1}(i) \cup \{i\}$, then by (IIC), it holds that

$$\begin{aligned} f_{ij}(q, A^j + A_{aj}^C I^{aj}) &= f_{ij}(q, A^j) \\ &= 0. \end{aligned}$$

To summarize, we obtain

$$f_{ij}(q, A^j + A_{aj}^C I^{aj}) = \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ 0 & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases}$$

Pick $b \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, $b \neq a$. If $b \in \hat{P}^{-1}(i) \cup \{i\}$, then by (ELDC), it holds that

$$\begin{aligned} \forall h \in (\hat{P}(b) \cup \{b\}) \cap Q(j), \quad & f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{ij}(q, A^j + A_{aj}^C I^{aj}) \\ & = f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{hj}(q, A^j + A_{aj}^C I^{aj}) \\ & = \lambda, \end{aligned} \tag{23}$$

for some $\lambda \in \mathbb{R}$. By (IIC),

$$\forall h \in Q(j), h \in \hat{P}(b) \cup \{b\}, \quad f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = f_{hj}(q, A^j + A_{aj}^C I^{aj}). \tag{24}$$

By Lemma 1, (23) and (24), we obtain

$$\begin{aligned} & \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = A_{aj}^C + A_{bj}^C \\ \iff & \sum_{h \in (\hat{P}(b) \cup \{b\}) \cap Q(j)} (f_{hj}(q, A^j + A_{aj}^C I^{aj}) + \lambda) \\ & + \sum_{h \notin (\hat{P}(b) \cup \{b\}) \cap Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) = A_{aj}^C + A_{bj}^C \\ \iff & |(\hat{P}(b) \cup \{b\}) \cap Q(j)| \lambda + \sum_{h \in N} f_{hj}(q, A^j + A_{aj}^C I^{aj}) = A_{aj}^C + A_{bj}^C \\ \iff & |(\hat{P}(b) \cup \{b\}) \cap Q(j)| \lambda + A_{aj}^C = A_{aj}^C + A_{bj}^C \\ \iff & |(\hat{P}(b) \cup \{b\}) \cap Q(j)| \lambda = A_{bj}^C \\ \iff & \lambda = \frac{A_{bj}^C}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|}. \end{aligned}$$

We obtain

$$\begin{aligned} f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) & = f_{ij}(q, A^j + A_{aj}^C I^{aj}) + \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|} \\ & = \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} + \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ \frac{A_{bj}^C}{|\hat{P}(b) \cup \{b\} \cap Q(j)|} & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases} \end{aligned}$$

If $b \notin \hat{P}^{-1}(i) \cup \{i\}$, then

$$\begin{aligned} f_{ij}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) & = f_{ij}(q, A^j + A_{aj}^C I^{aj}) \\ & = \begin{cases} \frac{A_{aj}^C}{|\hat{P}(a) \cup \{a\} \cap Q(j)|} & \text{if } a \in \hat{P}^{-1}(i) \cup \{i\} \\ 0 & \text{if } a \notin \hat{P}^{-1}(i) \cup \{i\}. \end{cases} \end{aligned}$$

We have considered the pipelines a and b in $\hat{P}^{-1}(Q(j)) \cup Q(j)$. Continue this reasoning until all the pipelines in $\hat{P}^{-1}(Q(j)) \cup Q(j)$ have been considered once. Then, we obtain

$$\begin{aligned} f_{ij}(q, A^C) &= f_{ij}(q, \sum_{h \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{hj}^j I^{hj}) \\ &= \sum_{k \in \hat{P}^{-1}(i) \cup i} \frac{A_{kj}^C}{(|\hat{P}(k) \cup k| \cap Q(j))} \\ &= \Psi_{ij}(q, A^C). \end{aligned}$$

We have shown that $f_{ij}(q, A^C) = \Psi_{ij}(q, A^C)$ for any $i \in N$ and any $j \leq q_i$. This concludes the proof of Theorem 1. \square

6.2. Proof of Theorem 2

Existence: We show that Υ satisfies the axioms of the statement of Theorem 2.

(IHD): The proof follows directly from the definition of the Uniform rule. The cost share charged to a consumer i for a demand j is computed using only the incremental costs generated by demand j of the consumers. Therefore, we directly have that Υ satisfies the axiom.

(NII): The proof is straightforward since each consumer is charged with the same amount for a given demand.

Uniqueness: We show that the Uniform rule is the only rule on GDP that satisfies all the axioms of the statement of Theorem 2. Consider $(q, A^C) \in GDP$ and f a rule that satisfies all the axioms of the statement of Theorem 2 on GDP . Let us show that $f(q, A^C)$ is uniquely determined. By Lemma 1, for each $j \leq q_n$,

$$\sum_{i \in Q(j)} f_{ij}(q, A^C) = \sum_{i \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{ij}^C. \quad (25)$$

Consider the matrix of incremental costs 0^C in which each incremental cost is null. By definition of a rule, each cost share is non-negative. Thus, by (25),

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}(q, 0^C) = 0.$$

Observe that the difference between any two cost shares is always null. Therefore, by Non-Increasing Inequalities and the non-negativity of a rule, we get

$$\forall j \leq q_n, \forall i, i' \in Q(j), \quad f_{ij}(q, A^C) = f_{i'j}(q, A^C). \quad (26)$$

Combining (25) and (26), $f(q, A^C)$ is uniquely determined, which concludes the proof. \square

6.3. Proof of Theorem 3

Existence: Pick any parameter system α and consider the Mixed rule μ^α . By definition, and the fact that μ^α is computed as a convex combination of Ψ and Υ , μ^α satisfies all the axioms of the statement

of Theorem 3.

Uniqueness: We show that the Mixed rules are the only rules that satisfy all the axioms of the statement of Theorem 3. Pick any $(q, A^C) \in \overline{GDP}$. Let f be a rule that satisfies all the axioms of the statement of Theorem 3 on \overline{GDP} . Let us show that $f(q, A^C)$ coincides with the a Mixed rule. In other words, we show that

$$\forall i \in N, \forall j \leq q_i, \quad f(q, A^C) = \mu^\alpha(q, A^C),$$

for some parameter system α . Pick any $j \leq q_n$. Consider the problem $(q, A^j) \in \overline{GDP}$, in which the matrix of incremental costs is defined as

$$\forall k \in N, \forall l \leq q_n, \quad A_{kl}^j = \begin{cases} 0 & \text{if } l = j \text{ and } k \in \hat{P}^{-1}(Q(j)) \cup Q(j) \\ A_{kj}^C & \text{otherwise.} \end{cases}$$

Similarly to the proof of Theorem 1, by Lemma 1 and by definition of a rule,

$$\forall i \in Q(j), \quad f_{ij}(q, A^j) = 0.$$

In this proof, we procede in a similar manner than in proof of Theorem 1.

Pick any $a \in \hat{P}^{-1}(Q(j)) \cup Q(j)$. By the fact that (F) implies (ELDC),

$$\begin{aligned} \forall h, h' \in (\hat{P}(a) \cup \{a\}) \cap Q(j), \quad f_{hj}(q, A^j + A_{aj}^C I^{aj}) - f_{ij}(q, A^j) &= f_{h'j}(q, A^j + A_{aj}^C I^{aj}) - f_{h'j}(q, A^j) \\ \iff f_{hj}(q, A^j + A_{aj}^C I^{aj}) &= f_{h'j}(q, A^j + A_{aj}^C I^{aj}) \\ &= W, \end{aligned} \quad (27)$$

for some $W \in \mathbb{R}$. By (EHC),

$$\forall h, h' \in Q(j), h, h' \notin \hat{P}(a) \cup \{a\}, \quad f_{hj}(q, A^j + A_{aj}^C I^{aj}) = f_{h'j}(q, A^j + A_{aj}^C I^{aj}) \quad (28)$$

$$= X,$$

for some $X \in \mathbb{R}$. By (F),

$$\begin{aligned} \forall h, h' \in Q(j), h \in \hat{P}(a) \cup \{a\}, \forall h' \notin \hat{P}(a) \cup \{a\}, \\ f_{hj}(q, A^j + A_{aj}^C I^{aj}) \geq f_{h'j}(q, A^j + A_{aj}^C I^{aj}). \end{aligned} \quad (29)$$

By (28) and (29), there exists a $\lambda \in \mathbb{R}_+$, such that

$$\begin{aligned} \forall h \in (\hat{P}(a) \cup \{a\}) \cap Q(j), \forall h' \notin (\hat{P}(a) \cup \{a\}) \cap Q(j), \\ f_{hj}(q, A^j + A_{aj}^C I^{aj}) = f_{h'j}(q, A^j + A_{aj}^C I^{aj}) + \lambda \\ = B + \lambda. \end{aligned} \quad (30)$$

By Lemma 1, (29), (28) and (30), we obtain

$$\begin{aligned}
& \sum_{h \in Q(j)} f_{hj}(q, A^j + A_{aj}^C I^{aj}) = A_{aj}^C \\
\iff & \sum_{h \in (\hat{P}(a) \cup \{a\}) \cap Q(j)} X + \sum_{h \notin (\hat{P}(a) \cup \{a\}) \cap Q(j)} X = A_{aj}^C \\
\iff & \sum_{h \in (\hat{P}(a) \cup \{a\}) \cap Q(j)} (X + \lambda) + \sum_{h \notin (\hat{P}(a) \cup \{a\}) \cap Q(j)} X = A_{aj}^C \\
\iff & |Q(j)|X + |(\hat{P}(a) \cup \{a\}) \cap Q(j)|\lambda = A_{aj}^C \\
\iff & \lambda = \frac{A_{aj}^C - |Q(j)|X}{|(\hat{P}(a) \cup \{a\}) \cap Q(j)|}.
\end{aligned} \tag{31}$$

By the Budget balanced condition and by definition of rule, it holds that $|Q(j)|X \leq A_{aj}^C$. Therefore, there exists for some $\alpha^j \in [0, 1]$ such that

$$\lambda = \frac{A_{aj}^C \times \alpha^j}{|(\hat{P}(a) \cup \{a\}) \cap Q(j)|}$$

Replacing λ in (31), we obtain

$$\begin{aligned}
X &= \frac{A_{aj}^C - A_{aj}^C \times \alpha^j}{|Q(j)|} \\
&= (1 - \alpha^j) \frac{A_{aj}^C}{|Q(j)|}.
\end{aligned} \tag{32}$$

Therefore, it holds that

$$\forall h \in Q(j), h \notin \hat{P}(a) \cup \{a\}, \quad f_{hj}(q, A^j + A_{aj}^C I^{aj}) = (1 - \alpha^j) \frac{A_{aj}^C}{|Q(j)|}, \tag{33}$$

and

$$\forall h \in (\hat{P}(a) \cup \{a\}) \cap Q(j), \quad f_{hj}(q, A^j + A_{aj}^C I^{aj}) = (1 - \alpha^j) \frac{A_{aj}^C}{|Q(j)|} + \alpha^j \frac{A_{aj}^C}{|(\hat{P}(a) \cup \{a\}) \cap Q(j)|}.$$

Pick $b \in \hat{P}^{-1}(Q(j)) \cup Q(j)$, $b \neq a$. By (ELDC), it holds that

$$\begin{aligned}
\forall h, h' \in (\hat{P}(b) \cup \{b\}) \cap Q(j), \quad & f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{hj}(q, A^j + A_{aj}^C I^{aj}) \\
&= f_{h'j}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{h'j}(q, A^j + A_{aj}^C I^{aj}) \\
&= D.
\end{aligned} \tag{34}$$

By (EHC),

$$\begin{aligned}
\forall h, h' \in Q(j), h, h' \notin \hat{P}(b) \cup \{b\}, \quad & f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{hj}(q, A^j + A_{aj}^C I^{aj}) \\
&= f_{h'j}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{h'j}(q, A^j + A_{aj}^C I^{aj}) \\
&= E.
\end{aligned} \tag{35}$$

By definition of \overline{GDP} , the source \mathcal{S} has at least three successors. Moreover, each of these successors has at least one subordinates whose effective demand is equal to q_n . Therefore, there exists a consumer $h \in Q(j)$ that is not contained in $\hat{P}(b) \cup \{b\}$ or in $\hat{P}(a) \cup \{a\}$. By (PIIC), this consumer h verifies

$$\begin{aligned} & \frac{1}{A_{bj}^C} \left(f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{hj}(q, A^j + A_{aj}^C I^{aj}) \right) \\ &= \frac{1}{A_{aj}^C} \left(f_{hj}(q, A^j + A_{aj}^C I^{aj}) - f_{hj}(q, A^j) \right). \end{aligned} \quad (36)$$

By (28) and (35), (36) becomes

$$E = \frac{A_{bj}^C}{A_{aj}^C} \times X. \quad (37)$$

By (F), (34), (35) and (36), for each $h, h' \in Q(j)$, $h \in (\hat{P}(b) \cup \{b\})$ and $h' \notin (\hat{P}(b) \cup \{b\})$,

$$\begin{aligned} f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{hj}(q, A^j + A_{aj}^C I^{aj}) &\geq f_{h'j}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{h'j}(q, A^j + A_{aj}^C I^{aj}) \\ \iff D &\geq \frac{A_{bj}^C}{A_{aj}^C} \times X. \end{aligned} \quad (38)$$

By (38), there exists a $\beta \in \mathbb{R}_+$, such that

$$D = \frac{A_{bj}^C}{A_{aj}^C} \times X + \beta. \quad (39)$$

By Lemma 1, (32), (34), (35) and (39), we obtain

$$\begin{aligned}
& \sum_{h \in Q(j)} \left(f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) - f_{hj}(q, A^j + A_{aj}^C I^{aj}) \right) = A_{aj}^C + A_{bj}^C - A_{aj}^C \\
\iff & \sum_{h \in (\hat{P}(b) \cup \{b\}) \cap Q(j)} D + \sum_{h \notin (\hat{P}(b) \cup \{b\}) \cap Q(j)} E = A_{bj}^C \\
\iff & \sum_{h \in (\hat{P}(b) \cup \{b\}) \cap Q(j)} \left(\frac{A_{bj}^C}{A_{aj}^C} \times X + \beta \right) + \sum_{h \notin (\hat{P}(b) \cup \{b\}) \cap Q(j)} \frac{A_{bj}^C}{A_{aj}^C} \times X = A_{bj}^C \\
\iff & |Q(j)| \frac{A_{bj}^C}{A_{aj}^C} \times X + |(\hat{P}(b) \cup \{b\}) \cap Q(j)| \beta = A_{bj}^C \\
\iff & \beta = \frac{A_{bj}^C - |Q(j)| \frac{A_{bj}^C}{A_{aj}^C} \times X}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|} \\
\iff & \beta = \frac{\frac{A_{bj}^C}{A_{aj}^C} (A_{aj}^C - |Q(j)| X)}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|} \\
\iff & \beta = \frac{\frac{A_{bj}^C}{A_{aj}^C} A_{aj}^C \alpha^j}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|} \\
\iff & \beta = \frac{A_{bj}^C \alpha^j}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|}.
\end{aligned}$$

By (32) and (39), it follows that

$$\begin{aligned}
D &= \frac{A_{bj}^C}{A_{aj}^C} \times (1 - \alpha^j) \frac{A_{aj}^C}{|Q(j)|} + \frac{A_{bj}^C \alpha^j}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|} \\
&= (1 - \alpha^j) \frac{A_{bj}^C}{|Q(j)|} + \frac{A_{bj}^C \alpha^j}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|},
\end{aligned}$$

and

$$E = (1 - \alpha^j) \frac{A_{bj}^C}{|Q(j)|}.$$

Therefore, for each $h \in (\hat{P}(b) \cup \{b\}) \cap Q(j)$,

$$\begin{aligned}
& f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) \\
&= f_{hj}(q, A^j + A_{aj}^C I^{aj}) + (1 - \alpha^j) \frac{A_{bj}^C}{|Q(j)|} + \frac{A_{bj}^C \alpha^j}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|} \\
&= \begin{cases} (1 - \alpha^j) \left(\frac{A_{aj}^C}{|Q(j)|} + \frac{A_{bj}^C}{|Q(j)|} \right) + \alpha^j \left(\frac{A_{aj}^C}{|(\hat{P}(a) \cup \{a\}) \cap Q(j)|} + \frac{A_{bj}^C}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|} \right) & \text{if } h \in \hat{P}(a) \cup \{a\} \\ (1 - \alpha^j) \left(\frac{A_{aj}^C}{|Q(j)|} + \frac{A_{bj}^C}{|Q(j)|} \right) + \alpha^j \frac{A_{bj}^C}{|(\hat{P}(b) \cup \{b\}) \cap Q(j)|} & \text{if } h \notin \hat{P}(a) \cup \{a\}. \end{cases}
\end{aligned}$$

Similarly, for each $h \in Q(j)$, $h \notin (\hat{P}(b) \cup \{b\})$,

$$f_{hj}(q, A^j + A_{aj}^C I^{aj} + A_{bj}^C I^{bj}) = \begin{cases} (1 - \alpha^j) \left(\frac{A_{aj}^C}{|Q(j)|} + \frac{A_{bj}^C}{|Q(j)|} \right) + \alpha^j \left(\frac{A_{aj}^C}{|(\hat{P}(a) \cup \{a\}) \cap Q(j)|} \right) & \text{if } h \in \hat{P}(a) \cup \{a\} \\ (1 - \alpha^j) \left(\frac{A_{aj}^C}{|Q(j)|} + \frac{A_{bj}^C}{|Q(j)|} \right) & \text{if } h \notin \hat{P}(a) \cup \{a\}. \end{cases}$$

We have considered the pipelines a and b in $\hat{P}^{-1}(Q(j)) \cup Q(j)$. Continue this reasoning until all the pipelines in $\hat{P}^{-1}(Q(j)) \cup Q(j)$ have been considered once. Then, we obtain

$$\forall i \in Q(j), \quad f_{ij}(q, A^C) = f_{ij}(q, \sum_{h \in \hat{P}^{-1}(Q(j)) \cup Q(j)} A_{hj}^j I^{hj}) \quad (40)$$

$$= \sum_{k \in \hat{P}^{-1}(i) \cup i} \left((1 - \alpha^j) \frac{A_{kj}^C}{|Q(j)|} + \alpha^j \frac{A_{kj}^C}{|(\hat{P}(k) \cup k) \cap Q(j)|} \right) \quad (41)$$

$$= (1 - \alpha^j) \Upsilon_{ij}(q, A^C) + \alpha^j \Psi_{ij}(q, A^C). \quad (42)$$

Observe that α^j only depends on j . Applying this same reasoning to each level $l \leq q_n$, $l \neq j$, we obtain that

$$\forall i \in Q(l), \quad f_{il}(q, A^C) = (1 - \alpha^l) \Upsilon_{il}(q, A^C) + \alpha^l \Psi_{il}(q, A^C),$$

for some $\alpha^l \in [0, 1]$. Therefore, we obtain a parameter system α such that $f_{ij}(q, A^C) = \mu_{ij}^\alpha(q, A^C)$ for any $i \in N$ and any $j \leq q_i$. This concludes the proof of Theorem 3. \square

6.4. Proof of Theorem 4

In order to prove Theorem 4, we recall some definitions on TU-games. Consider a finite set of players $N = \{a, b, \dots, n\}$. A TU-game is a couple (N, v) , where $v : 2^N \rightarrow \mathbb{R}$ is a characteristic function assigning to each coalition of players $S \in 2^N$ its worth $v(S)$. For each (N, v) , the Shapley value is defined as

$$\forall i \in N, \quad Sh_i(N, v) = \frac{1}{n!} \sum_{\sigma \in O} (v(E_i^\sigma \cup i) - v(E_i^\sigma)), \quad (43)$$

where O is the set of all orders over N and E_i^σ is the coalition formed by the players ordered before i according to the order σ .

Next, we introduce an intermediary result. To that end, consider a gas distribution problem $(q, A^C) \in GDP$ and its associated gas distribution game $(q, v^{C,P})$. For each $j \leq q_n$, define the TU-game $(N, w_j^{(q, v^{C,P})})$ as

$$\forall E \subseteq N, \quad w_j^{(q, v^{C,P})}(E) = v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right] - v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right].$$

The worth $w_j^{(m, v^{C,P})}(E)$ can be interpreted as the surplus in cost generated in $v^{C,P}$ when a group of players E decide to increase their activity level from $j-1$ to j while all the other players play the activity level $j-1$ or their maximal feasible activity level if they are unable to play $j-1$. The next Lemma is already proved on the class of non-decreasing multi-choice games in Lowing and Techer (2022), so its is omitted.

Lemma 2. For each gas distribution problem $(q, A^C) \in GDP$, its associated gas distribution game $(q, v^{C,P})$ verifies

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(q, v) = Sh_i(N, w_j^{(q, v^{C,P})}). \quad (44)$$

We have the material to prove Theorem 4. Consider $(q, A^C) \in GDP$ and $j \leq q_n$. For each $k \in N$, define the TU-game $(Q(j), R^k)$ as

$$\forall E \subseteq Q(j), \quad R^k(E) = \begin{cases} 1 & \text{if } E \cap (\hat{P}(k) \cup k) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of the Shapley value,

$$\forall i \in N, \quad Sh_i(Q(j), R^k) = \begin{cases} \frac{1}{|(\hat{P}(k) \cup k) \cap Q(j)|} & \text{if } i \in \hat{P}(k) \cup k, \\ 0 & \text{otherwise.} \end{cases} \quad (45)$$

Consider $(q, v^{C,P}) \in \mathcal{G}$, the gas distribution game associated to (q, A^C) . Consider the TU-game $(Q(j), w^{(q, v^{C,P})})$ and observe that, for each $E \subseteq Q(j)$,

$$\begin{aligned} w_j^{(q, v^{C,P})}(E) &= v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right] - v^{C,P} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right] \\ &\stackrel{(8)}{=} \sum_{i \in N} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right] \end{aligned} \quad (46)$$

$$\begin{aligned} &= \sum_{i \in \hat{P}^{-1}(E) \cup E} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right] \\ &\quad + \sum_{i \notin \hat{P}^{-1}(E) \cup E} \left[C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h \right) \right. \\ &\quad \left. - C \left(i, \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h \right) \right]. \end{aligned} \quad (47)$$

For each $i \in \hat{P}^{-1}(E) \cup E$, there exists at least one $h \in \hat{P}(i) \cup i$ such that $h \in E \subseteq Q(j)$. Therefore, for each $i \in \hat{P}^{-1}(E) \cup E$,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h = j. \quad (48)$$

On the other hand,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h = j-1. \quad (49)$$

For each $i \notin \hat{P}^{-1}(E) \cup E$, there is no $h \in \hat{P}(i) \cup i$ such that $h \in E$. Therefore, for each $i \notin \hat{P}^{-1}(E) \cup E$,

$$\max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k + \sum_{k \in E} e_k \right) \wedge q \right]_h = \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h. \quad (50)$$

For the sake of clarity, for each $i \notin \hat{P}^{-1}(E) \cup E$, we denote

$$K_i = \max_{h \in \hat{P}(i) \cup i} \left[\left(\sum_{k \in N} (j-1)e_k \right) \wedge q \right]_h.$$

Therefore, by (48), (49) and (50), (46) becomes

$$\begin{aligned} w_j^{(q, v^{C,P})}(E) &= \sum_{i \in \hat{P}^{-1}(E) \cup E} \left[C(i, j) - C(i, j-1) \right] + \sum_{i \notin \hat{P}^{-1}(E) \cup E} \left[C(i, K_i) - C(i, K_i) \right] \\ &= \sum_{i \in \hat{P}^{-1}(E) \cup E} A_{ij}^C + 0 \\ &= \sum_{i \in N} A_{ij}^C \times R^i(E). \end{aligned}$$

By Lemma 2, for each $i \in N$ and $j \in \{1, \dots, q_i\}$,

$$\begin{aligned} \varphi_{ij}(q, v^{C,P}) &= Sh_i \left(Q(j), \sum_{l \in N} A_{lj}^C \times R^l \right) \\ &= \sum_{l \in N} A_{lj}^C \times Sh_i(Q(j), R^l) \\ &= \sum_{l \in \hat{P}^{-1}(i) \cup i} \frac{A_{lj}^C}{|(\hat{P}(l) \cup l) \cap Q(j)|} \\ &= \Psi_{ij}(q, A^C). \end{aligned}$$

The second equality follows from the Linearity of the Shapley value and the third equality follows from (45). This concludes the proof. \square

6.5. Proof of Theorem 7

Lowng and Techer (2022) show that the multi-choice Shapley value belongs to the Core of sub-modular (cost) games. Therefore, it suffice to show that gas distribution games are sub-modular games to prove Theorem 8. Consider a gas distribution game $(q, v^{C,P}) \in \mathcal{G}$ associated with a gas distribution problem $(q, A^C) \in GDP$. For each $s, t \in \mathcal{M}$,

$$v^{C,P}(t) + v^{C,P}(s) = \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} s_k) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} t_k),$$

$$\text{and } v^{C,P}(s \vee t) + v^{C,P}(s \wedge t) = \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \vee t_k)) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} s_k \wedge t_k).$$

Take any $i \in N$. Without loss of generality, assume that

$$\max_{k \in \hat{P}(i) \cup i} s_k \geq \max_{k \in \hat{P}(i) \cup i} t_k.$$

Then, on the one hand,

$$\begin{aligned} \max_{k \in \hat{P}(i) \cup i} (s_k \vee t_k) &= \max_{k \in \hat{P}(i) \cup i} s_k \\ \implies C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \vee t_k)) &= C(i, \max_{k \in \hat{P}(i) \cup i} s_k). \end{aligned}$$

On the other hand, since C is non-decreasing, it holds that

$$\begin{aligned} \forall k \in \hat{P}(i) \cup i, \quad s_k \wedge t_k &\leq t_k \\ \iff \max_{k \in \hat{P}(i) \cup i} (s_k \wedge t_k) &\leq \max_{k \in \hat{P}(i) \cup i} t_k \\ \implies C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \wedge t_k)) &\leq C(i, \max_{k \in \hat{P}(i) \cup i} t_k). \end{aligned}$$

This shows that, for each $i \in N$,

$$\sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \vee t_k)) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} (s_k \wedge t_k)) \leq \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} s_k) + \sum_{i \in N} C(i, \max_{k \in \hat{P}(i) \cup i} t_k),$$

and so, we obtain the desired result

$$v^{C,P}(s \vee t) + v^{C,P}(s \wedge t) \leq v^{C,P}(s) + v^{C,P}(t).$$

□

References

- Bergantiños, G., González-Díaz, J., González-Rueda, Á.M., de Córdoba, M.P.F., 2017. Loss allocation in energy transmission networks. *Games and Economic Behavior* 102, 69–97.
- van den Brink, R., Gilles, R.P., 1996. Axiomatizations of the conjunctive permission value for games with permission structures. *Games and Economic Behavior* 12, 113–126.
- van den Brink, R., He, S., Huang, J.P., 2018. Polluted river problems and games with a permission structure. *Games and Economic Behavior* 108, 182–205.
- Csercsik, D., Hubert, F., Sziklai, B.R., Kóczy, L.Á., 2019. Modeling transfer profits as externalities in a cooperative game-theoretic model of natural gas networks. *Energy Economics* 80, 355–365.
- Dong, B., Ni, D., Wang, Y., 2012. Sharing a polluted river network. *Environmental and Resource Economics* 53, 367–387.
- Fleurbaey, M., Trannoy, A., 1998. La péréquation territoriale en question. *FLUX Cahiers scientifiques internationaux Réseaux et Territoires* 14, 91–98.
- Gilles, R.P., Owen, G., van den Brink, R., 1992. Games with permission structures: the conjunctive approach. *International Journal of Game Theory* 20, 277–293.
- Grabisch, M., Xie, L., 2007. A new approach to the core and weber set of multichoice games. *Mathematical Methods of Operations Research* 66, 491–512.
- Hsiao, C.R., Raghavan, T., 1992. Monotonicity and dummy free property for multi-choice cooperative games. *International Journal of Game Theory* 21, 301–312.
- Junqueira, M., da Costa, L.C., Barroso, L.A., Oliveira, G.C., Thomé, L.M., Pereira, M.V., 2007. An Aumann-Shapley approach to allocate transmission service cost among network users in electricity markets. *IEEE Transactions on Power Systems* 22, 1532–1546.

- Lowing, D., Techer, K., 2022. Marginalism, egalitarianism and efficiency in multi-choice games. *Social Choice and Welfare* , 1–47.
- Massol, O., Tchong-Ming, S., 2010. Cooperation among liquefied natural gas suppliers: Is rationalization the sole objective? *Energy Economics* 32, 933–947.
- Moulin, H., Shenker, S., 1992. Serial cost sharing. *Econometrica* 60, 1009–1037.
- Ni, D., Wang, Y., 2007. Sharing a polluted river. *Games and Economic Behavior* 60, 176–186.
- van den Nouweland, C.G.A., 1993. *Games and graphs in economic situations*. tilburg university .