

# Lack of controllability of the viscous Burgers equation. Part II: The $L^2$ setting, with a detour into the well-posedness of unbounded entropy solutions to scalar conservation laws

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**Abstract.** We provide a series of partial negative answers to the question raised in [Coron, Contemp. Math 2007] concerning the controllability, in the framework of  $L^2$  solutions, of the viscous Burgers equation  $u_t + (u^2/2)_x = u_{xx}$  for initial and terminal data prescribed for  $x \in (0, 1)$ . We adapt the scaling and compactness technique of [Andreianov, Ghoshal, Koumatos, Part I] conceived for  $L^\infty$  solutions and based upon non-controllability, by Kruzhkov entropy solutions, of the inviscid Burgers equation. To this end, we develop a basic well-posedness theory of unbounded entropy solutions to the Cauchy problem for multi-dimensional scalar conservation laws with pure  $L^p$  data and polynomial growth up to the critical power  $p$  of the flux function. The case of the Cauchy-Dirichlet problem for the Burgers equation on an interval is also addressed, in the  $L^2$  solution framework which is considerably simpler than the established  $L^1$  theory of renormalized solutions to such problems. Local regularity of unbounded entropy solutions is discussed in the one-dimensional case with convex flux. We deduce non-controllability results for the viscous Burgers equation under the bounded amplification restriction of the form  $\|u\|_2 \leq LT|C|$  and an additional  $L^2 - L^3_{loc}$  regularization assumption.

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## 1. Introduction

We extend our study, conducted in [5], of the controllability of the viscous Burgers equation

$$u_t + (u^2/2)_x = u_{xx} \text{ in } D, \tag{BE}$$

where  $(x, t) \in D = \mathbb{R} \times (0, T)$  (“the strip setting”) or  $(x, t) \in D = (0, 1) \times (0, T)$  (“the box setting”), with a given  $T > 0$ . Our primary motivation comes from [15], where J.-M. Coron asked the following question (Open Problem 4). Let  $T > 0$  and  $C \in \mathbb{R} \setminus \{0\}$ .

*Question:* Does there exist  $u \in L^2((0, 1) \times (0, T))$  satisfying (BE) such that for all  $x \in (0, 1)$ ,  $u(\cdot, 0) = 0$  and  $u(\cdot, T) = C$ ? (Q)

We refer to the Introduction of the paper [5] for a detailed account on previously obtained results and on related work. Let us only recall here that positive controllability results, obtained in particular in [15, 16, 19], require the assumption  $|C|T > 1$ . As in our work [5], in the present paper we focus on non-controllability issues in the opposite case  $|C|T \leq 1$  and under additional “bounded amplification” assumptions, which essentially mean that the size of the desired solutions is limited relative to the size of the target datum; this can be witnessed through the “amplification factor”  $L$  present in the

main statements. Note that there is a serious difference with the original question, as highlighted in [5, Remark 1.1]. In [5], we highlighted the existence of many triples  $(M, C, T)$ , with  $0 < |C| \leq M$  and  $0 < T \leq 1/|C|$ , such that the target state  $u_T = C$  is not reachable by solutions of **(BE)** satisfying  $\|u\|_\infty \leq M$ . In the present paper, we will work with  $L^2$  solutions as suggested in [15].

The argument of [5] relies in an essential way on properties of Kruzhkov entropy solutions to the inviscid Burgers equation, therefore it is formulated within the  $L^\infty$  framework. In order to adapt the approach of [5] to the original  $L^2$  setting of question **(Q)**, we need to rely upon a uniqueness theory for *unbounded entropy solutions*. While such theories exist already (those of kinetic and of renormalized solutions, cf. [30] and [10], respectively), we are led to develop a more elementary well-posedness study - for data of critical growth, like  $L^2$  solutions of the Burgers equation - for scalar conservation laws. Therefore, the scope of the present article is two-fold.

### 1.1. The detour into unbounded solutions of scalar conservation laws

First, motivated by the one-dimensional Burgers equation with  $L^2$  data in the strip, we take this opportunity to develop, in Section 3, a rather elementary theory of unbounded  $L^\infty(0, T; L^p(\mathbb{R}^d))$  entropy solutions of multi-dimensional scalar conservation laws, under the critical order  $p$  polynomial growth assumption on the flux. This should be seen as a complement to the well-established pure  $L^1$  theories (kinetic solutions, cf. [30] and the references therein; renormalized solutions, see [10]) and to the deep recent work [33] on  $L^p - L^q$ ,  $p < q \leq \infty$ , regularizing estimates for multi-dimensional Burgers equations. Note that in our setting of unbounded entropy solutions, the subcritical case has already been developed a long time ago (cf. [34], see also [9] for a more recent development). Also note that although we do not rely on the more sophisticated renormalized solutions theory [10] we follow very similar guidelines in what concerns the existence issue (cf. [2]).

While we are able to develop a complete existence and uniqueness theory for the Cauchy problem with  $L^p$  data for a flux growing at infinity like  $|u|^p$  (in particular,  $L^2(\mathbb{R})$  data for the Burgers case), the boundary-value analogue corresponding, for example, to  $D = (0, 1) \times (0, T)$  is more delicate. In the Cauchy-Dirichlet setting, we focus on the basic case of the inviscid Burgers equation to develop existence results for, roughly speaking,  $L^3$  boundary data. We provide a uniqueness result based upon the strong trace properties for “quasi-solutions” ([26]). Boundary data are included in a way which is well-suited for  $L^2$  solutions and compatible with the Bardos-LeRoux-Nédélec interpretation of Dirichlet conditions. The resulting  $L^2$  theory is less general than the  $L^1$  theory of renormalized solutions [1, 31], but it has the advantage of relative technical simplicity.

Finally, for the Burgers case (and more generally, for the strictly convex one-dimensional flux) we give an elementary proof of the  $L_{loc}^\infty$  regularization effect put forward in [33].

### 1.2. Non-controllability of **(BE)**, under $L^2$ bounded amplification assumptions

Second, slightly adapting the techniques of [5] in the setting of  $D = \mathbb{R} \times (0, T)$ , we assess non-controllability of large constant states in  $(0, 1)$  (i.e.,  $u(\cdot, T)|_{(0,1)} = C$  with  $|C| > C_0(L)$ ) for times  $T < 1/|C|$  (or a bit longer times) by solutions of the viscous Burgers equation **(BE)** with  $L^2(\mathbb{R})$  initial data, with  $\|u_0\|_{L^2} \leq L|C|$  taken as the bounded amplification assumption on the initial data. This constitutes a partial, and a quantitative ( $L$ -dependent) negative answer to the question **(Q)** in its strip formulation (**Pb<sub>strip</sub>**) recalled in Section 2.

Finally, we also address the setting  $D = (0, 1) \times (0, T)$ , i.e., the box formulation (**Pb<sub>box</sub>**) recalled in Section 2. We deal with  $L^2((0, 1) \times (0, T)) \cap L_{loc}^3((0, 1) \times (0, T))$  solutions (see Corollary 4.11), under the  $L^2$  amplification assumption

$$\|u\|_{L^2((0,1) \times (0,T))} \leq LT|C|$$

on the solutions and an  $L^2 - L_{loc}^3$  regularization assumption. Because of this additional  $L_{loc}^3$  regularity restriction, we do not reach the pure  $L^2((0, 1) \times (0, T))$  setting of **(Q)**, but we get particularly close to it. Indeed, the  $L^2 - L_{loc}^\infty$  regularization can be expected for sufficiently robust notions of solution for the viscous Burgers equation, see Remark 4.12. Details can be found in Section 4.

### 1.3. The outline and the key arguments of the paper

Let us describe the structure of the paper and provide an insight into the method and the techniques in use. Precise statements are given in Sections 3 and 4. We start by fixing, in Section 2, the framework of our studies.

In Section 3, we conduct an investigation of unbounded solutions of critical integrability for the Burgers equation. It is closely related to the theory of renormalized solutions (and to the one of kinetic solutions), the latter ones being more general but also technically and conceptually more involved. Section 3 is devoted to the well-posedness analysis of the Cauchy problem for unbounded entropy solutions to scalar conservation laws far more general than the Burgers equation. The Cauchy-Dirichlet problem in  $L^2((0, 1) \times (0, T))$  for the Burgers flux is also addressed in the context of the *ad hoc* boundary formulation well-adapted to the  $L^2$  integrability of the data. In many aspects of the theory, we heavily rely upon fine regularity properties of the solution and compactness tools put forward by Panov [26, 27, 28], based upon the techniques of parametrized  $H$ -measures as introduced in [24]. We also exploit some of the techniques employed in the past 30 years in the study of renormalized solutions to elliptic and parabolic PDEs (cf. [2, 12, 23]). While the latter were successfully extended to the hyperbolic framework (see in particular [10, 1, 31]), here we avoid the use of truncations found in the renormalization approach, due to the sharp integrability assumption on the flux.

In Section 4 we sharpen the answer, brought in [5], to question **(Q)**. We take the advantage of the results, proved in Section 3, on uniqueness of unbounded entropy solutions and their approximation by the standard bounded Kruzhkov entropy solutions. The key observation is the non-attainability of the constant target state  $u_T = C$  at time  $T$ , for the *inviscid* Burgers equation. This occurs, roughly speaking, for couples  $(C, T)$  with  $|C|T < 1$ ; in [5] we indicate wider families of non-attainable, not necessarily constant, terminal states. The lack of attainability relies upon the generalized characteristics of [17], which remain meaningful in our setting of unbounded entropy solutions. Then, we show that the viscous Burgers equation “inherits” the non-attainability, in a specific asymptotic regime. Our approach, put forward in [5], relies on the scaling

$$\begin{aligned} (t, v) &\mapsto \left(\frac{t}{\epsilon}, \mathcal{S}_\epsilon v\right), \\ \mathcal{S}_\epsilon v &:= \epsilon v(\cdot, \epsilon \cdot), \quad \text{i.e. } t = \epsilon \tau, \quad \mathcal{S}_\epsilon(v)(x, \tau) = \epsilon v(x, \epsilon \tau), \end{aligned} \tag{Zoom}$$

where  $v$  is a function of  $(x, t) \in (0, 1) \times (0, T)$ . This scaling permits to link **(BE)** to the viscous Burgers equation with viscosity parameter  $\epsilon > 0$ :

$$\mathcal{S}_\epsilon(v)_t + \left(\frac{(\mathcal{S}_\epsilon(v))^2}{2}\right)_x = \epsilon \mathcal{S}_\epsilon(v)_{xx}.$$

Under the scaling (Zoom) of solutions of the viscous Burgers equation **(BE)** with  $\epsilon = T$ , the conclusion on non-controllability for **(BE)** follows whenever one can provide uniform in  $\epsilon$  bounds ensuring compactness of sequences of solutions  $(u_\epsilon)$ . Therefore an *a priori* bound on solutions is required in our argument (cf. [5, Remark 1.1]); we interpret it as an amplification assumption, thus limiting the size of the solutions in terms of the size of the target data and the “bounded amplification factor” denoted by  $L$ . The construction leads to non-controllability of couples  $(C, T)$  for small enough times  $T$  and the accordingly large states  $C$ , with roughly speaking  $0 < \delta \leq |C|T \leq 1$ . In the box setting **(Pb<sub>box</sub>)**, an additional  $L^2 - L^3_{loc}$  regularization assumption arises, as a byproduct of our investigation of unbounded entropy solutions to the Cauchy-Dirichlet problem in Section 3.2.

## 2. Notion of solution and main notation

Let us detail the framework(s) we explore. As in [5], we discuss the two following situations:

$$\begin{cases} u_t + (u^2/2)_x = u_{xx} & \text{in } (0, 1) \times (0, T), \\ u(\cdot, 0) = 0 \text{ and } u(\cdot, T) = u_T & \text{for } x \in (0, 1), \end{cases} \tag{Pb<sub>box</sub>}$$

and

$$\begin{cases} u_t + (u^2/2)_x = u_{xx} \text{ in } \mathbb{R} \times (0, T), \\ u(\cdot, 0) = u_0 \text{ with } u_0 = 0 \text{ for } x \in (0, 1), \\ \text{and } u(\cdot, T) = u_T \text{ for } x \in (0, 1). \end{cases} \quad (\mathbf{Pb}_{strip})$$

As in [5], we use the term *weak solution of the viscous Burgers equation* (supplemented with initial and terminal data) meaning the following.

**Definition 2.1 (Adopted notion of solution for the viscous Burgers equation).**

Let  $D = I \times (0, T)$  with  $I = (0, 1)$  or  $I = \mathbb{R}$ . Let  $u_0$  and  $u_T$  belong to  $L^2_{loc}(I)$ . A function  $u \in L^2(D)$  is called a *weak solution of (BE)* with initial data  $u_0$  and terminal data  $u_T$  if  $u \in L^2(0, T; H^1_{loc}(I))$  and for all  $\xi \in C_c^\infty(I \times [0, T])$ , there holds

$$\int_0^T \int_0^1 \left( u \xi_t + \frac{u^2}{2} \xi_x - u_x \xi_x \right) dx dt + \int_{\mathbb{R}} u_0(x) \xi(x, 0) dx - \int_{\mathbb{R}} u_T(x) \xi(x, T) dx = 0 \quad (1)$$

and, furthermore, for all  $\xi \in C_c^\infty(I \times [0, T])$ ,  $\xi \geq 0$ , for all  $k \in \mathbb{R}$  there holds

$$- \int_0^T \int_0^1 \left( |u - k| \xi_t + |u - k| \frac{u + k}{2} \xi_x - |u - k|_x \xi_x \right) dx dt - \int_{\mathbb{R}} |u_0 - k| \xi(x, 0) dx \leq 0. \quad (2)$$

In particular, solutions to  $(\mathbf{Pb}_{box})$  or  $(\mathbf{Pb}_{strip})$  are understood in the sense of Definition 2.1 in the sequel; they will be supplemented with additional bounds in  $L^p(D)$  for different choices of  $p$ .

Further, we will use the tags  $(\mathbf{Pb}_{box}^0)$ ,  $(\mathbf{Pb}_{strip}^0)$  for the inviscid problems (analogues of the problems  $(\mathbf{Pb}_{box})$ ,  $(\mathbf{Pb}_{strip})$  with zero right-hand side in the respective PDEs) understood in the sense that generalizes the Kruzhkov setting of entropy solutions; we refer to Section 3 for the precise definitions in the unbounded entropy solution setting.

### 3. Entropy solutions of scalar conservation laws in critical $L^p$ setting

Question (Q) raised in the survey [15] concerns  $L^2$  solutions of the one-dimensional viscous Burgers equation. Adaptation of our strategy of analysis to this case requires an extension of the theory of the inviscid Burgers equation to  $L^2$  solutions; this is the purpose of the present section, where we put forward a basic theory covering the desired case. Let us point out that for the (multidimensional) Burgers equation, a pure  $L^p$  theory for the Cauchy problem was constructed in the recent work [33] of Serre and Silvestre. Well-posedness and instantaneous regularizing effect (from  $L^p$  to  $L^\infty$ ) are shown, with the help of dispersion estimates.

The goal of the present section is two-fold.

On the one hand, we provide an elementary theory of the Cauchy problem for scalar multidimensional conservation laws for general flux functions exhibiting polynomial (of degree  $p$ ) growth in the critical case of  $L^p$  initial data. The criticality is to be understood in comparison with the results of Szepessy [34] and Bendahmane, Karlsen [9], where for a flux of degree  $(p - \alpha)$  polynomial growth,  $\alpha > 0$ , an  $L^1 \cap L^p$  theory is constructed. It should be stressed that the critical case requires neither adaptation of the notion of solution (cf. [10] and [30] for the pure  $L^1$  theories of, respectively, renormalized and kinetic solutions), nor the fine regularization estimates of the super-critical case [33]. Compared to the subcritical case of [34, 9], it requires a careful control of approximations in the existence proof. The precise well-posedness result for the Cauchy problem is given in Theorem 3.2 below. Further, we also investigate the analogous Cauchy-Dirichlet problem in the one-dimensional case, limiting our attention to the Burgers equation. We provide sufficient conditions on boundary data for existence of  $L^2$  entropy solutions, and give the uniqueness result based upon the automatic boundary regularity of such solutions. Concerning the Cauchy-Dirichlet version of our problem, let us point out the related work [3] of Ancona and Marson where semigroup solutions with initial and boundary data of appropriate integrability are considered, though not fully characterized by entropy inequalities, and the works [31] of Porretta and Vovelle, [1] of Ammar, Carrillo and Wittbold where

the problem is treated in the renormalized solutions setting, i.e. for a more involved truncated entropy formulation.

On the other hand, we provide an elementary argument for the  $L^2 - L^\infty$  regularization effect observed in [33] in the case of the one-dimensional Burgers equation. In addition to being elementary, it applies also to the viscous Burgers equation (cf. Remark 4.12). The precise instantaneous regularization result is given in Propositions 3.11 and 3.14 below. We stress that this result permits us to apply the tools of generalized characteristics, in arguments like [5, Prop. 3.7, 4.4], to  $L^2$  solutions of the inviscid Burgers equation.

### 3.1. Well-posedness of the Cauchy problem in the critical integrability setting

We consider the problem

$$\begin{cases} u_t + \operatorname{div} F(u) = 0 & \text{in } \mathbb{R}^d \times (0, T), \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R}^d, \end{cases} \quad (3)$$

for  $u_0 \in L^p(\mathbb{R}^d)$ ,  $1 \leq p < \infty$ , and the nonlinearity  $F \in C(\mathbb{R}; \mathbb{R}^d)$  verifying for some  $C > 0$  the critical growth assumption

$$\forall u \in \mathbb{R} \quad |F(u)| \leq C(1 + |u|^p). \quad (4)$$

Up to the issue of integrability, the notion of solution for problem (3) is the standard notion in the entropy framework of Kruzhkov [22]:

**Definition 3.1.** *A function  $u \in L^\infty(0, T; L^p(\mathbb{R}^d))$  is said to be an entropy solution of (3) with  $F$  satisfying (4) and  $u_0 \in L^p(\mathbb{R}^d)$  if for all  $\xi \in C_c^\infty(\mathbb{R}^d \times [0, T])$ ,  $\xi \geq 0$  and all  $k \in \mathbb{R}$  there holds*

$$- \int_0^T \int_{\mathbb{R}^d} |u - k| \xi_t + \operatorname{sign}(u - k)(F(u) - F(k)) \cdot \nabla \xi \, dx \, dt - \int_{\mathbb{R}^d} |u_0 - k| \xi(x, 0) \, dx \leq 0. \quad (5)$$

We remark that the above extension of the Kruzhkov [22] definition can be compared to the approach and the techniques of renormalized solutions of conservation laws (Bénilan, Carrillo and Wittbold [10]). In our setting, there is no need to rely upon truncations of  $u$  to define solutions. This is due to (4) and the integrability assumption on  $u$ , making all terms in the above formulation well defined. However, like in the renormalized solution theory, it is not immediately clear from the entropy inequalities (5) (even letting  $k \rightarrow \pm\infty$ ) that an entropy solution is also a weak one; we therefore include this point into the well-posedness result below. Like in the renormalized solutions theory, we can link entropy solutions in the sense of Definition 3.1 to their approximation by classical  $L^\infty \cap L^1$  Kruzhkov entropy solutions. Actually, this approximation - in a monotone way, following Ammar and Wittbold [2] - is in the heart of the existence proof; it also permits to recover easily the structural properties highlighted in Proposition 3.11 and Corollary 3.13 below.

**Theorem 3.2.** *Under assumption (4), for all  $u_0 \in L^p(\mathbb{R}^d)$  there exists a unique entropy solution  $u$  to problem (3). The solutions for different data verify the  $L^1$  contraction and comparison property (with possibly infinite right-hand side)*

$$\|(u - \hat{u})^\pm\|_{L^\infty(0, T; L^1)} \leq \|(u_0 - \hat{u}_0)^\pm\|_{L^1}. \quad (6)$$

Moreover, the solution  $u$  is the pointwise (a.e.) limit, as  $M \rightarrow +\infty$ , of the sequence  $(u^M)_M$  of Kruzhkov entropy solutions of the Cauchy problem for the same conservation law with the truncated initial data  $u_0^M = \operatorname{sign}(u_0) \min\{|u_0|, M\}$ .

Further, the solution  $u$  satisfies the weak formulation of problem (3), it is time-continuous, i.e.  $u \in C([0, T]; L^1_{loc}(\mathbb{R}^d))$ , and  $\sup_{t>0} \|u - u^M\|_{L^1(\mathbb{R}^d)} \rightarrow 0$  as  $M \rightarrow +\infty$ . In particular,  $(u^M)$  converges to  $u$  in  $C([0, T]; L^1_{loc}(\mathbb{R}^d))$ .

*Proof.* The proof follows [22] and assembles several techniques developed, for different purposes in related literature.

**Uniqueness and comparison property**<sup>1</sup>. Let  $u, \hat{u}$  be solutions of (3) in the sense of Definition 3.1 with the same initial data  $u_0 = \hat{u}_0$ .

The first step is to obtain from the entropy inequalities (5) the so-called Kato inequality: for all  $\xi \in C_c^\infty(\mathbb{R}^d \times [0, T))$ ,  $\xi \geq 0$  there holds

$$-\int_0^T \int_{\mathbb{R}^d} |u - \hat{u}| \xi_t + q(u, \hat{u}) \cdot \nabla \xi \, dx \, dt \leq 0, \quad (7)$$

where

$$q(z, r) := \text{sign}(z - r)(F(z) - F(r)).$$

This is done by the classical doubling of variables argument [22]. We stress that the properties  $u, \hat{u} \in L^1_{loc}(\mathbb{R}^d \times [0, T))$ ,  $F(u), F(\hat{u}) \in L^1_{loc}(\mathbb{R}^d \times [0, T))$  are precisely the ones that are required in this argument (one can use the technique based upon the upper semicontinuity of the  $L^1$  bracket<sup>2</sup> in [4, Lemma 6, Prop. 3]; see also [8, Remark 4]). By the standard density argument (in the weak-\* topology of  $W^{1, \infty}$ ), it is immediate to extend (7) to Lipschitz continuous test functions with compact support in  $\mathbb{R}^d \times [0, T)$ .

The second step is to pass to the limit with a well-chosen sequence of test functions  $\xi_k$  verifying  $\xi_k(x, t) \rightarrow (e^{-t} - e^{-T})$  as  $k \rightarrow +\infty$ . Note that we cannot rely on the Kruzhkov choice of localized in space and time test functions, because in our setting the conservation law exhibits infinite speed of propagation. We pick the simple technique suitable in the case of infinite speed of propagation and integrable flux (cf. [11]). According to the preceding density remark, one can choose

$$\xi_R(x, t) = \min\{1, (|x| - R)^+\}(e^{-t} - e^{-T}) \quad (8)$$

in the Kato inequality (7). Then  $|\nabla \xi|$  is bounded by 1 and supported in  $\{R \leq |x| \leq R + 1\}$ ; therefore the fact that

$$q(u(x, t), \hat{u}(x, t)) \in L^\infty(0, T; L^1(\mathbb{R}^d))$$

(due to  $u, \hat{u} \in L^\infty(0, T; L^p(\mathbb{R}^d))$ ) and the growth assumption (4) permits to pass to the limit  $R \rightarrow \infty$  in (7) and infer  $\int_0^T \int_{\mathbb{R}^d} e^{-t} |u(x, t) - \hat{u}(x, t)| \, dx \, dt \leq 0$ . This proves uniqueness.

The contraction and comparison properties (6) can be proved similarly, taking  $(u_0 - \hat{u}_0)^+ \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$  and using the half-entropies  $(\cdot - k)^\pm$  ([32, 14]) instead of the Kruzhkov entropies and weights  $e^{-ct}$ ,  $c \rightarrow 0^+$ , instead of  $e^{-t}$ .

**Existence.** Let us introduce for  $M, N > 0$  the truncation functions

$$T^{M, N} : \mathbb{R} \rightarrow \mathbb{R}, \quad T^{M, N}(z) = \max\{\min\{z, M\}, -N\}. \quad (9)$$

The existence argument is based on the established theory of (bounded) Kruzhkov entropy solutions (with the comparison property (6)) and on the use of bi-monotone sequences of approximations of initial data put forward by Ammar and Wittbold in [2]. Let  $u_0^{M, N} = T^{M, N}(u_0)$ . It is clear that for all  $M, N > 0$  we have  $u_0^{M, N} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  with the uniform in  $M, N$  bounds

$$\|u_0^{M, N}\|_{L^p(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)}. \quad (10)$$

The corresponding Kruzhkov entropy solutions  $u^{M, N}$  exist and the classical Kruzhkov theory also yields the bound  $\|u^{M, N}\|_{L^\infty(0, T; L^p(\mathbb{R}^d))} \leq \|u_0\|_{L^p(\mathbb{R}^d)}$ , in view of the above bounds on the initial data and because  $\eta : u \mapsto |u|^p$  is an entropy. By construction, we have the pointwise convergence

$$\forall N > 0 \quad u_0^{M, N} \uparrow u_0^{\infty, N} := \max\{u_0, -N\} \quad \text{as } M \rightarrow +\infty.$$

The comparison property contained in (6) ensures that  $(u^{M, N})_M$  is a non-decreasing sequence uniformly bounded in  $L^\infty(0, T; L^p(\mathbb{R}^d))$ . Therefore it admits a pointwise limit (which is  $\mathbb{R} \cup \{+\infty\}$ -valued) that we denote  $u^{\infty, N}$ , and the monotone convergence theorem ensures that  $u^{\infty, N} \in L^\infty(0, T; L^p(\mathbb{R}^d))$ ,

<sup>1</sup>Note that the uniqueness proof in [9], written for  $L^1 \cap L^p$  data and subcritical (polynomial of degree  $p - \alpha$ ,  $\alpha > 0$ ) growth of  $F$ , actually extends to our case of the critical growth and pure  $L^p$  data. Here we point out simpler techniques for dealing with the Kato inequality, because we are concerned with a less general family of PDEs.

<sup>2</sup>For  $u, v \in L^1(\Omega)$ , their bracket in  $L^1(\Omega)$  is  $[u, v]_{L^1} := \int_\Omega \text{sign}(u)v \, dy + \int_{\{y \in \Omega \mid u(y)=0\}} |v| \, dy$ , see, e.g., [8].

in particular  $u^{\infty,N}$  is a.e. finite. Note that the monotonicity implies the uniform in  $M$  domination  $|u^{M,N}| \leq \max\{(u^{1,N})^-, (u^{\infty,N})^+\} \in L^\infty(0, T; L^p(\mathbb{R}^d))$ . Moreover, it follows by assumption (4) that

$$|F(u^{M,N})| \leq C(1 + \max\{(u^{1,N})^-, (u^{\infty,N})^+\}^p) \in L^1(\mathbb{R}^d \times (0, T)). \quad (11)$$

This makes it possible to apply the dominated convergence theorem, as  $M \rightarrow +\infty$ , in entropy inequalities (5) written for  $u^{M,N}$ . We deduce that  $u^{\infty,N}$  is an entropy solution in the sense of Definition 3.1 for the initial datum  $u^{\infty,N}$ . By construction,

$$u_0^{\infty,N} \downarrow u_0 \text{ as } N \rightarrow +\infty.$$

The same monotonicity argument ensures that the resulting sequence  $(u^{\infty,N})_N$  converges pointwise to a limit  $\bar{u} \in L^\infty(0, T; L^p(\mathbb{R}^d))$ , which implies the domination analogous to (11). Consequently, the entropy inequalities (5) satisfied by  $u^{\infty,N}$  are inherited by  $\bar{u}$ . This concludes the existence proof.

In addition, exchanging the roles of  $M$  and  $N$  in the above argument, we construct  $\underline{u} = \lim_{M \rightarrow +\infty} \lim_{N \rightarrow +\infty} u^{M,N}$ . We have by construction

$$\begin{aligned} u^{M,\infty} &\leq u^{M,N} \leq u^{\infty,N}, \\ u^{M,\infty} &\rightarrow \underline{u} \text{ as } M \rightarrow +\infty \text{ and } u^{\infty,N} \rightarrow \bar{u} \text{ as } N \rightarrow +\infty. \end{aligned}$$

Since  $\underline{u}, \bar{u}$  are entropy solutions of (3) corresponding to the same  $L^p$  datum  $u_0$ , they coincide according to the above uniqueness analysis. Therefore  $u^{M,N}$  converges to the unique entropy solution  $\underline{u} \equiv \bar{u}$  of (3) as  $M, N \rightarrow +\infty$ , and in particular the sequence  $(u^{M,M})_M$  of Kruzhkov entropy solutions corresponding to initial data  $T^{M,M}(u_0)$  converge pointwise to  $\underline{u} \equiv \bar{u}$  as  $M \rightarrow +\infty$ .

**Weak formulation and time continuity.** It is classical that Kruzhkov entropy solutions in the standard  $L^\infty$  setting satisfy the weak formulation of the problem. It is enough to point out that the weak formulation passes to the limit as  $u = \bar{u}$  is approximated by  $u^{\infty,N}$  which is in turn approximated by  $u^{M,N}$ , due to the dominated convergence argument of the above the existence proof.

As for the time continuity of  $u$ , we apply the  $L^1$  contraction principle of the uniqueness claim to solutions  $u, u^M$ , being understood that the  $L^1 \cap L^\infty$  solutions  $u^M$  fulfill as well the formulation of Definition 3.1. Combining Hölder and Chebyshev inequalities we find

$$\begin{aligned} \sup_{t>0} \|u(\cdot, t) - u^M(\cdot, t)\|_{L^1(\mathbb{R}^d)} &\leq \|u_0 - u_0^M\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (|u_0| - M)^+ dx \\ &\leq \|(|u_0| - M)^+\|_{L^2(\mathbb{R}^d)} \left( \text{meas}(\{x \mid |u_0(x)| > M\}) \right)^{1/2} \leq \|u_0\|_{L^2(\mathbb{R}^d)} \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{M} \rightarrow 0 \end{aligned}$$

as  $M \rightarrow \infty$ . This uniform-in-time convergence ensures that the time continuity of functions  $u^M$  with values in  $L^1_{loc}(\mathbb{R}^d)$  (see [5, Remark 3.4]) is inherited by  $u$ ; moreover, the difference  $(u - u^M)$ , as a function of  $t$ , vanishes in  $L^1(\mathbb{R}^d)$  uniformly in time, as  $M \rightarrow +\infty$ .  $\square$

**Remark 3.3.** *While proving Theorem 4.4 below, we sketch the analogous theory in  $L^\infty(0, T; L^2(\mathbb{R}))$  for the viscous Burgers equation, with the additional requirement that  $u \in L^2(0, T; H^1(\mathbb{R}))$  needed to ensure uniqueness.*

*Let us point out that one can address such theory of unbounded entropy solutions in the more general degenerate anisotropic diffusion case. For the subcritical case, well-posedness is established by Bendahmane and Karlsen in [9]; in order to extend their work to the critical case, one needs to insert the bi-monotonicity construction of [2] within the argumentation and employ sharp  $L^1$  arguments for the derivation of the Kato inequalities, similarly to the proof of Theorem 3.2.*

### 3.2. On $L^2$ solutions of the Cauchy-Dirichlet problem for the inviscid Burgers equation

Here, our goal is to sketch an  $L^2$  theory of entropy solutions to the inviscid Burgers equations in  $(0, 1) \times (0, T)$ , with assigned initial and boundary data. This issue appeared as a byproduct of question (Q), and the resulting theory can be seen as a simpler but less general alternative to the setting of renormalized solutions to Cauchy-Dirichlet problems, as developed by Ammar, Carrillo and Wittbold [1] and Porretta and Vovelle [31].

We set the following definition. The notation  $z^\pm$  refers to the decomposition of  $z \in \mathbb{R}$  into the difference  $z = z^+ - z^-$  of two positive numbers, whereas  $\text{sign}^\pm(z)$  stands for  $\pm \text{sign}(z^\pm)$ . We also write  $z \top w$  for  $\max\{z, w\}$ .

**Definition 3.4.** Assume  $u_0 \in L^2((0, 1))$  and let  $b_1, b_2$  be measurable functions such that the positive part  $b_0^+$  of  $b_0$  and the negative part  $b_1^-$  of  $b_1$  belong to  $L^2((0, T))$ . The function  $u \in L^2((0, 1) \times (0, T))$  is an unbounded entropy solution of the Cauchy-Dirichlet problem

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } D = (0, 1) \times (0, T), \\ u(x, 0) = u_0 & \text{on } (0, 1), \\ u(0, t) = b_0 & \text{on } (0, T), \\ u(1, t) = b_1 & \text{on } (0, T), \end{cases} \quad (12)$$

if for all  $k \in \mathbb{R}$ , for all  $\xi \in C_c^\infty([0, 1] \times [0, T])$ ,  $\xi \geq 0$  there holds

$$\begin{aligned} & - \int_0^T \int_0^1 (u - k)^\pm \xi_t + \text{sign}^\pm(u - k) \frac{u^2 - k^2}{2} \xi_x \, dx \, dt - \int_0^1 (u_0 - k)^\pm \xi(x, 0) \, dx \\ & \leq \int_0^T \frac{1}{2} \left( (k^- \top b_0^+)^2 - k^2 \right)^\pm \xi(0, t) \, dt + \int_0^T \frac{1}{2} \left( (k^+ \top b_1^-)^2 - k^2 \right)^\mp \xi(1, t) \, dt. \end{aligned} \quad (13)$$

**Remark 3.5.** The unusual form of the boundary terms in the above definition is motivated by the formalism of the work [6] of Sbihi and the first author, the precise form of these expressions being adapted to our setting of unbounded solutions. In the expression  $k \mapsto \frac{1}{2}(k^- \top b_0^+)^2$  (respectively  $k \mapsto \frac{1}{2}(k^+ \top b_1^-)^2$ ), one can recognize the projected graph in the sense of [6] corresponding to the Dirichlet data  $b_0$  imposed at  $x = 0$  (respectively, to the Dirichlet data  $b_1$  imposed at  $x = 1$ ).

We first discuss several technical issues related to the above definition. To start with, we note that all terms in (13) make sense, due to the  $L^2$  assumptions on  $u$  and on  $(b_0)^+, (b_1)^-$ . In the sequel we are able to establish existence of  $L^2$  solutions to (12) in the sense of Definition 3.4 only under the  $L^3$  assumption on  $(b_0)^+, (b_1)^-$ , which comes quite naturally from the *a priori* estimates. Because in our application in § 4.3 we will need to go slightly beyond the setting of  $L^3$  boundary data (see Remark 3.15), we will link our notion of solution to the well-established but far more technical (in what concerns both the definitions and the proofs) notion of renormalized solution [1, 31]. In this relation, let us also mention the early work [3] of Ancona and Marson, where semigroup type solutions with  $L^1$  initial data and  $L^2$  boundary data (for the case of the left boundary alone) are constructed. We remark that by the theory of [3] these solutions satisfy a representation of the Lax-Oleinik type and they have the  $L_{loc}^\infty$  regularity; these two features are the essential ingredients of our analysis in § 4.3. Unfortunately, it is not clear whether the solutions of [3] can be fully characterized by entropy inequalities; whereas in the context of § 4.3, we rely solely upon such local entropy characterization. Therefore the theory of [3] is not instrumental for the application we have in mind.

We stress that the boundary conditions at  $x = 0, 1$  are interpreted in a relaxed sense, in the spirit of Bardos, LeRoux and Nédélec [7]. Note that (13) is a variant of the expression of the BLN conditions put forward in [6, Proposition 3.3(iv)], and here we adapt its expression to the Burgers flux and to the  $L^2$  setting. To show that this adaptation is consistent with the established theory of [7, 35], we prove the existence of strong boundary traces for unbounded entropy solutions of (12).

For convenience of application of the results of [26, 28], we introduce a notation closely related to (9): for  $a, b \in \mathbb{R}$ ,  $a < b$ , the  $[a, b]$ -valued truncation of the identity map is

$$s_{a,b} : r \mapsto \min\{b, \max\{r, a\}\} \equiv T^{b,-a}(r). \quad (14)$$

**Lemma 3.6.** For  $a, b \in \mathbb{R}$ ,  $a < b$ , there exist nonnegative locally bounded in  $[0, 1] \times [0, T]$  measures  $\gamma_b^+, \gamma_a^+$  such that there holds in  $\mathcal{D}'((0, 1) \times (0, T))$ :

$$s_{a,b}(u)_t + \frac{1}{2}(s_{a,b}(u)^2)_x = \gamma_b^+ - \gamma_a^+. \quad (15)$$



for some locally bounded in  $[0, 1] \times [0, T)$  measure  $\mu_\ell^+$  with vanishing total variation<sup>3</sup> as  $\ell \rightarrow +\infty$ . An analogous property should be shown for  $(u \top (-\ell) - k)^-$  with some measure  $\mu_{-\ell}^-$  of vanishing total variation, as  $-\ell \rightarrow -\infty$ . We concentrate on (17).

For the proof, we first observe that  $(u \perp \ell - k)^+ = (u - k)^+ - (u - \ell)^+$  because  $k < \ell$ . With the arguments and the notation of Lemma 3.6, we readily write the following analogue of (17):

$$(u \perp \ell - k)_i^+ + \frac{1}{2} q^+(u \perp \ell, k)_x \leq \gamma_\ell^+ + \frac{1}{2} \sum_{i=0,1} R_i \delta_i,$$

with

$$R_0 = ((k^- \top b_0^+)^2 - k^2)^+ - ((\ell^- \top b_0^+)^2 - \ell^2)^+, \quad R_1 = ((k^+ \top b_1^-)^2 - k^2)^- - ((\ell^+ \top b_1^-)^2 - \ell^2)^-. \quad (18)$$

Therefore we will prove (17) with  $\mu_\ell^+ := \gamma_\ell^+$ . To check the vanishing, as  $\ell \rightarrow +\infty$ , of this measure's total variation we examine closer the expressions of measures  $\gamma_k^+$  obtained in the proof of Lemma 3.6. We fix  $\tau < T$  and take  $K = [0, 1] \times [0, \tau]$ . Because  $u$  and  $(b_0)^+$ ,  $(b_1)^-$  are assumed to belong to  $L^2$  in their domains of definition, (16) and the absolute continuity of the Lebesgue integral ensure the vanishing property  $|\gamma_\ell^+|([0, 1] \times [0, \tau]) \rightarrow 0$ , as  $\ell \rightarrow +\infty$ . From this we get the formulation of the form (17) in any time domain  $(0, \tau)$ ,  $\tau < T$ , but with  $(b_i \perp \ell - k)^+$  replaced with  $R_i$  in (18). A case study shows that  $R_0 \leq 2\ell(b_0 \perp \ell - k)^+$  and that  $R_1 \leq 0$ , thus concluding the proof.  $\square$

After these preliminaries, we turn to the well-posedness analysis of (12) in the setting of  $L^2$  entropy solutions with  $L^2$  or  $L^3$  boundary data.

**Theorem 3.10.** *Assume that  $u_0 \in L^2((0, 1))$  and let  $b_0, b_1$  be measurable functions with  $b_0^+, b_1^- \in L^2((0, T))$ . Then problem (12) admits at most one solution in the sense of Definition 3.4.*

*Moreover, if we reinforce the assumption on the boundary data to  $b_0^+, b_1^- \in L^3((0, T))$ , then there exists a unique solution to (12). Moreover, the solution  $u$  is the pointwise (a.e.) limit, as  $M \rightarrow +\infty$ , of the sequence of  $(u^M)_M$  of Kruzhkov entropy solutions of the Cauchy-Dirichlet problem for the same conservation law with the truncated initial data  $u_0^M = \text{sign}(u_0) \min\{|u_0|, M\}$  and boundary data  $b_{0,1}^M = \text{sign}(b_{0,1}) \min\{|b_{0,1}|, M\}$ .*

*Further, the solution  $u$  belongs to  $L^\infty(0, T; L^2((0, 1))) \cap C(0, T; L^1((0, 1)))$ , it satisfies the weak formulation of problem (3) in  $\mathcal{D}'((0, 1) \times [0, T])$ , and it assumes the boundary data in the pointwise BLN sense of Proposition 3.8.*

One can also infer from the below proof that solutions for different data verify the comparison property. The  $L^1$  contraction property was explored by Porretta and Vovelle in [31] in the related setting of renormalized solutions; here we do not extend on this as it is not needed for the application we have in mind in Section 4. We note the fact that, in view of Proposition 3.9, the uniqueness claim could be deduced from the one of [31]. Also existence could be justified along the guidelines of [31], we would need however to guarantee the  $L^2$  integrability of the solutions and pass to the limit  $\ell \rightarrow \infty$  in (17). We will instead follow a more direct and much simpler than [31] approach since, due to Proposition 3.8 and the  $L^1$  integrability of the flux term  $u^2/2$ , we can avoid truncations.

*Proof of Theorem 3.10.* The arguments follow those of the proof of Theorem 3.2.

**Uniqueness.** Let  $u, \hat{u}$  be solutions of (3) in the sense of Definition 3.4 with the same initial and boundary data. Precisely as in the proof of Theorem 3.2, from the entropy formulation we obtain the Kato inequality (7) with  $C_c^\infty((0, 1) \times [0, T])$  test functions (so that the domain of integration in space is  $(0, 1)$  in place of  $\mathbb{R}^d$  in (7)). Taking instead of (8) smooth approximations of the Lipschitz test functions

$$\xi_h(x, t) = \frac{1}{h} \min\{h, x, 1 - x\}(e^{-t} - e^{-T})$$

<sup>3</sup>While the authors of [31] require the vanishing total variation on  $\bar{Q} := [0, 1] \times [0, T]$ , here we work locally on  $[0, 1] \times [0, T)$ , i.e., we fix some time horizon  $\tau < T$ . This is enough to apply the well-posedness results of [31].

and using Corollary 3.7, we find with the notation  $q(u, k) = \text{sign}(u - k)(u^2 - k^2)/2$ ,

$$\int_0^1 e^{-t}|u - \hat{u}|(x, t) dx \leq - \int_0^t (e^{-\tau} - e^{-T}) \left( q(u(0^+, \tau), \hat{u}(0^+, \tau)) + q(u(1^-, \tau), \hat{u}(1^-, \tau)) \right) d\tau. \quad (19)$$

It is well known (see e.g. [6] for an interpretation in terms of monotone graphs) that the BLN conditions stated in Proposition 3.8 ensure that the right-hand side of (19) is non-positive, thus ensuring that  $u = \hat{u}$  a.e. in  $(0, 1) \times (0, T)$ . Note that the comparison principle is shown in the same way, using the  $(\cdot - k)^\pm$  variants of the entropy and the Kato inequalities.

**Existence.** Recalling (9), we take truncated data  $u_0^{M,N} = T^{M,N}(u_0)$ ,  $b_{0,1}^{M,N} = T^{M,N}(b_{0,1})$  and consider the inviscid Burgers equation in the usual BLN setting for these truncated initial and boundary data. We denote  $u^{M,N}$  the unique solution of this problem and recall that the comparison principle (6) holds true. This permits to mimick the reasoning of the existence proof in Theorem 3.2, but we have to ensure that the construction leads to  $u^{M,N}$  satisfying a uniform in  $M, N$  bound in  $L^2((0, 1) \times (0, T))$ ; at this point, the  $L^3$  assumption on  $b_0^+, b_1^-$  appears from the estimate. To be precise, because  $u^{M,N}$  is a bounded entropy solution we can apply the classical entropy inequalities with the entropy  $\eta : z \mapsto z^2/2$  and the associated flux  $q : z \mapsto z^3/3$ . Taking  $\xi$  approaching the indicator function of the time interval  $[0, t]$ , bearing in mind the time continuity of  $u^{M,N}$  ([5, Remark 3.4]) we infer that for all  $t \in (0, T)$

$$\int_{(0,1)} |u^{M,N}(t, x)|^2 dx \leq \int_{(0,1)} |u_0^{M,N}(x)|^2 dx + \frac{2}{3} \int_0^t \left( (u^{M,N}(0^+, \tau))^3 - (u^{M,N}(1^-, \tau))^3 \right) d\tau.$$

Bearing in mind the BLN interpretation (see Proposition 3.8) of the boundary data  $b_{0,1}^{M,N}$ , we further find

$$\begin{aligned} \|u^{M,N}\|_{L^\infty(0,T;L^2((0,1)))}^2 &\leq \|u_0^{M,N}\|_{L^2((0,1))}^2 + \frac{2}{3} \int_0^t \left( (b_0(\tau)^+)^3 - (b_1(\tau)^-)^3 \right) d\tau \\ &\leq \|u_0\|_{L^2((0,1))}^2 + \frac{2}{3} \left( \|b_0^+\|_{L^3((0,T))}^3 + \|b_1^-\|_{L^3((0,T))}^3 \right). \end{aligned}$$

This bound ensures that the limit  $\underline{u} = \lim_{M \rightarrow +\infty} \lim_{N \rightarrow +\infty} u^{M,N}$  belongs to  $L^2$  in the box – actually to  $L^\infty(0, T; L^2(0, 1))$  – and permits the passage to the limit in the half-entropy inequalities satisfied by  $u^{M,N}$ . We stress that the boundary conditions can be included into the formulation under the form (13), due to their relation to the pointwise BLN conditions (cf. [6, Proposition 3.3] for the converse statement of Proposition 3.8). Therefore,  $\underline{u}$  is an unbounded entropy solution of (12) in the sense of Definition 3.4. Like in the proof of Theorem 3.2, we introduce analogously  $\bar{u} = \lim_{N \rightarrow +\infty} \lim_{M \rightarrow +\infty} u^{M,N}$ , use the already proved uniqueness claim for (12) in our  $L^2$  setting to prove that  $\underline{u} = \bar{u}$ , and conclude that also  $(u^M)$  (as defined in the statement of the theorem) converges to  $u$  as  $M \rightarrow +\infty$ . This concludes the proof of existence.

**Weak formulation, time continuity and boundary data.** Going back to the above existence proof, writing the weak formulation for  $u^{M,N}$ , we infer the one for  $u^{\infty,N}$  and then the one for  $\bar{u} = u$  with the help of the dominated convergence arguments, like for the entropy formulation. Recall that we also inferred that  $u \in L^\infty(0, T; L^2((0, 1)))$  from the uniform bound on  $u^{M,N}$  in this space. Further, the boundary formulation in the pointwise BLN sense is contained in Proposition 3.8, which includes the existence of strong boundary traces of the unbounded entropy solution  $u$  due to Corollary 3.7. Here we point out that analogously to this corollary, existence of strong traces  $u(\cdot, t^\pm)$  in the  $L^1_{loc}((0, 1))$  sense, for all  $t \in [0, T)$  (with one-sided trace at  $t = 0^+$ ) follows from the quasi-solution formulation of Lemma 3.6 ([25, 26]). Formally extending the boundary data by 0 on  $(T, T + \delta)$ , we extend the construction of the solution beyond the time horizon  $T$ , so the argument actually applies up to  $t = T^-$ . For  $t \in (0, T)$ , the existence of  $u(\cdot, t^\pm)$  and the already justified weak formulation of (12) guarantee that  $u \in C([0, T]; L^1_{loc}((0, 1)))$ . Then the  $L^\infty(0, T; L^2((0, 1)))$  integrability of  $u$  yields the uniform in  $t$  estimate

$$\int_{(0,\delta) \cup (1-\delta,1)} |u(x, t)| dx \leq \sqrt{2\delta} \|u\|_{L^\infty(0,T;L^2((0,1)))}.$$

In particular,  $u \in C([0, T]; L^1((0, 1)))$ .  $\square$

### 3.3. One-dimensional instantaneous regularization effect

In this paragraph we point out that in the case of dimension  $N = 1$  and strictly convex flux, the solutions constructed in the previous paragraph have the same qualitative properties as the classical Kruzhkov entropy solutions as soon as we regard them locally inside  $I \times (0, T]$ ,  $I = \mathbb{R}$  or  $I = (0, 1)$ .

**Proposition 3.11.** *Let  $p \in (1, +\infty)$ ,  $p' = p/(p - 1)$  and  $f \in C^2(\mathbb{R}; \mathbb{R})$  a strictly convex function satisfying the growth restrictions*

$$\exists A > 0 \forall u \in \mathbb{R} \quad |f(u)| \leq A(1 + |u|^p), \quad |f'(u)| \leq A(1 + |u|^{p-1}).$$

*Then the unique entropy solution  $u$  of the problem  $u_t + (f(u))_x = 0$  in  $\mathbb{R} \times (0, T)$ ,  $u(0, \cdot) = u_0 \in L^p(\mathbb{R})$  admits the representation*

$$u(x, t) = (f')^{-1}\left(\frac{x - y(x, t)}{t}\right) \text{ for a.e. } (x, t) \in \mathbb{R} \times (0, +\infty) \quad (20)$$

where for a.e.  $t > 0$ ,  $y(\cdot, t)$  is a non-decreasing function.

*In particular, it follows that for all  $t_0 > 0$  there holds  $u \in L^\infty(\mathbb{R} \times (t_0, T))$ , more precisely, there exists a constant  $B$  that depends on  $t_0, T, \|u_0\|_{L^p(\mathbb{R})}$  and  $f$  such that  $\|u\|_{L^\infty(\mathbb{R} \times (t_0, T))} \leq B$ .*

*Proof.* First, note that the Lax-Oleinik representation (20) is inherited by  $u = \lim_{M \rightarrow +\infty} u^M$  (the limit is taken in the pointwise sense, see Theorem 3.2) because the  $L^\infty$  entropy solutions  $u^M$  satisfy the Lax-Oleinik representation with  $y^M(x, t) = x - tf'(u^M(x, t))$  and  $f'$  is a continuous function. Indeed, it is enough to define

$$y(x, t) := x - tf'(u(x, t))$$

and observe that  $y = \lim_{M \rightarrow +\infty} y^M$  is monotone in  $x$ , for a.e.  $t$ .

Observe that the information from (10) along with our assumptions on  $f$  imply that  $f'(u) \in L^\infty(0, T; L^p_{loc}(\mathbb{R}))$ , with the bound depending on the  $f$ -specific constant  $A$  and on  $\|u_0\|_{L^p(\mathbb{R})}$ . Hence, Hölder's inequality implies the property

$$\exists C > 0 \int_{(x, x+1)} |f'(u(z, t))| dz \leq \int_{(x, x+1)} |f'(u(z, t))|^{p'} dz \leq C \quad (21)$$

uniformly with respect to  $t \in (0, T)$  and  $x \in \mathbb{R}$ , with  $C$  that depends on  $f, \|u_0\|_{L^p(\mathbb{R})}$ .

Now, fix  $R > 0$  and restrict our attention to  $x \in [-R, R]$ . Bearing in mind the expression (20) of  $u$  given  $y$ , in order to bound  $u$  in  $L^\infty([-R, R] \times [t_0, T])$  it is enough to show that an  $L^\infty$  bound by a constant depending on  $f, T$  and  $\|u_0\|_{L^p(\mathbb{R})}$  is valid for  $y$ ; the claimed  $L^\infty_{loc}$  result for  $u$ , with the additional dependence on the minimal allowed time  $t_0$ , then follows. We are intended to prove an upper bound on  $y(x, t)$ ; the lower bound is proved analogously. The functions  $y(\cdot, t)$  are nondecreasing and in particular,

$$y(z, t) \geq y(t, x) \text{ for all } z \in [x, x + 1].$$

Then, by integrating and using the fact that  $y(z, t) = z - tf'(u(z, t))$ , we infer that

$$y(x, t) \leq \int_x^{x+1} y(z, t) dz \leq t \|f'(u(\cdot, t))\|_{L^1([x, x+1])} + \int_x^{x+1} z dz \leq TC + R + 1 \quad (22)$$

due to (21). This yields the desired bound, whence we conclude that  $\|u\|_{L^\infty([-R, R] \times [t_0, T])} \leq B$  with the constant  $B$  depending on  $f, t_0, T, \|u_0\|_{L^p(\mathbb{R})}$  and the chosen fixed value  $R > 0$ . It remains to extend this bound from  $x \in [-R, R]$  to  $x \in \mathbb{R}$ . To this end, let us consider  $v$  defined, for some  $r \in \mathbb{R}$ , by

$$v(x, t) := u(x - r, t) \text{ for } (x, t) \in \mathbb{R} \times (0, T). \quad (23)$$

Note that  $v$  solves the same conservation law with  $v_0(\cdot) = u(\cdot - r)$  and thus  $\|u_0\|_{L^p(\mathbb{R})} = \|u_0\|_{L^p(\mathbb{R})}$ . Therefore there holds  $\|v\|_{L^\infty([-R, R] \times [t_0, T])} \leq B$ . Making  $r \in \mathbb{R}$  vary, we transfer the above proved  $L^\infty_{loc}$  bound on  $u$  from  $x \in [-R, R]$  to  $x \in [-R + r, R + r]$ ; in view of the arbitrariness of  $r$ , this yields the required bound  $\|u\|_{L^\infty(\mathbb{R} \times [t_0, T])} \leq B$ .  $\square$

**Remark 3.12.** *Note that combining the  $L^2$ - $L^\infty$  regularization with the classical  $L^\infty$ -BV regularization known for solutions of one-dimensional conservation laws with uniformly convex flux, we observe the instantaneous  $L^2$ -BV regularization.*

The main property that we need from solutions, in the sense of Definition 3.1, of the Burgers equation with  $L^2$  data is the following.

**Corollary 3.13.** *Let  $u$  be the entropy solution of the problem  $u_t + (u^2/2)_x = 0$  in  $\mathbb{R} \times (0, T)$ ,  $u(0, \cdot) = u_0 \in L^2(\mathbb{R})$ . Then for all  $(x, t) \in \mathbb{R} \times (0, T]$  there exist minimal and maximal backward generalized characteristics in the sense of Dafermos [17, 18].*

*Proof.* The result comes from the fact that  $u$  is a classical Kruzhkov entropy solution to the Burgers equation for  $t \geq t_0$ , for all  $t_0$ . The backward characteristics (Dafermos [17], [18, Chapters 10,11.1]) are then defined first for  $t \in [t_0, T]$  and then extended to  $t \in (0, T] = \cup_{t_0 > 0} [t_0, T]$  by the exhausting argument.  $\square$

Note that the analogous result holds for the Cauchy-Dirichlet problem for Burgers equation, as addressed in Theorem 3.10.

**Proposition 3.14.** *Let  $\theta > 0$  and  $u$  be the entropy solution of  $u_t + (\theta u^2/2)_x = 0$  in  $(0, 1) \times (0, T)$ ,  $u(0, \cdot) = u_0 \in L^2(\mathbb{R})$ ,  $u(0, \cdot) = b_0$  and  $u(1, \cdot) = b_1$  on  $(0, T)$  constructed in Theorem 3.10. Then  $u$  admits the representation*

$$u(t, x) = \frac{1}{\theta} \frac{x - y(x, t)}{t} \quad \text{for a.e. } (x, t) \in (0, 1) \times (0, +\infty) \quad (24)$$

where for a.e.  $t > 0$ ,  $y(\cdot, t)$  is a non-decreasing function. In particular, for all  $t_0 > 0$ ,  $0 < \delta < 1/2$  it holds that  $u \in L^\infty((\delta, 1 - \delta) \times (t_0, T))$ .

*Proof.* We treat the Burgers case  $\theta = 1$  which is our main objective, the general case playing only an auxiliary role. We have  $u \in L^2((0, 1) \times (0, T))$  that we regard here as  $L^2(0, 1; L^2((0, T)))$ . Exploiting the notion of Lebesgue points (see, e.g., [29] for details), one finds that for a.e.  $\delta \in (0, 1/2)$  the function  $u$  admits one-sided traces (in the sense of a.e. convergence)  $b_0 \in L^2((0, T))$  at  $x = \delta^+$  and  $b_1 \in L^2((0, T))$  at  $x = (1 - \delta)^-$  (we denote them  $b_{0,1}$  in view of the rescaling that will be operated from now on). In what follows we argue on the restriction  $u|_{(\delta, 1-\delta) \times (0, T)}$  of  $u$ . Upon changing  $x$  into  $(x - \delta)/(1 - 2\delta)$  and rescaling the Burgers flux by the factor  $(1 - 2\delta)$ , we see that the rescaled function that we denote by  $u_\delta$  is an unbounded entropy solution of (12) where the Burgers flux is multiplied by a constant (which changes nothing to the theory developed in Theorem 3.10). By Proposition 3.9,  $u_\delta$  is also a renormalized solution of the same problem. Combining the uniqueness claim and the solution construction of [31] we know that  $u$  is a pointwise limit of  $L^\infty$  (standard Kruzhkov) entropy solutions  $u_\delta^M$  to Cauchy-Dirichlet problems with truncated initial and boundary data  $u_0^M$ ,  $b_{0,1}^M$ . According to Joseph and Gowda [21], each of  $u^M$  admits a representation of the form (24) with the appropriate value  $\theta = \theta_\delta$ . We have already seen that such representations are stable under the pointwise convergence, therefore  $u_\delta$  inherits such representation. As  $\delta \rightarrow 0$  we have  $\theta_\delta \rightarrow 1$  and  $u_\delta \rightarrow u$ , so that  $u$  inherits the representation (24) with  $\theta = 1$ .

With (24) at hand, the proof of the  $L^\infty(0, T; L^2((0, 1))) - L_{loc}^\infty((0, 1) \times (0, T])$  regularization follows the lines of the proof of Proposition 3.11, upon replacing  $[x_j, x_j + 1]$  by the interval  $[x_j, x_j + \delta] \subset (0, 1)$ . As a concluding observation in relation to Remark 3.15, let us point out that under the additional assumption  $u \in L_{loc}^3((0, 1) \times [0, T])$  and using the same arguments, we can reduce the question to the setting of Theorem 3.10 and avoid the arguments referring to renormalized solutions.  $\square$

**Remark 3.15.** *The proof of Proposition 3.14 is the only point in the paper where we have to go beyond the unbounded solutions setting of § 3.2 and refer to fine properties of renormalized solutions in the sense of [31]. We can bypass this ingredient if we assume that  $u \in L_{loc}^3((0, 1) \times [0, T])$ . Here, by  $L_{loc}^3$  on  $(0, 1) \times [0, T]$  we mean the space of functions that are in  $L^3$  on any rectangle of the form  $(\delta, 1 - \delta) \times (0, T)$ ,  $\delta > 0$ .*

Because in our application to non-controllability (Theorem 4.10, Corollary 4.11) we have to require the  $L^3_{loc}((0, 1) \times (0, T))$  integrability of solutions, our unbounded entropy solutions theory for (12) is very close to be enough (but not exactly enough: between  $L^3_{loc}((0, 1) \times [0, T])$  and  $L^3_{loc}((0, 1) \times (0, T))$  there is a gap) in order to attain the optimal result in § 4.3.

Similarly to Corollary 3.13, we derive the following claim that will be instrumental in § 4.3. We stress that it concerns *local* unbounded entropy solutions of the inviscid Burgers equation (without any prescribed boundary behavior). Such solutions appear in our analysis of question (Q).

**Corollary 3.16.** *Let  $u \in L^2((0, 1) \times (0, T))$  be a local unbounded entropy solution (in the sense of (13) but with test functions that vanish on the lateral boundaries of  $(0, 1) \times (0, T)$ ) of the problem  $u_t + (u^2/2)_x = 0$  in  $(0, 1) \times (0, T)$ ,  $u(\cdot, 0) = u_0 \in L^2((0, 1))$ . Then for all  $(x, t) \in (0, 1) \times (0, T]$  there exist minimal and maximal backward generalized characteristics in the sense of Dafermos [17, 18].*

#### 4. Non-attainability for the viscous Burgers equation

In this section we focus on problems  $(\mathbf{Pb}_{strip})$  and  $(\mathbf{Pb}_{box})$  in the setting of  $L^2$  solutions. As in all the results of this paper, we rely upon non-attainability results for the inviscid Burgers problems  $(\mathbf{Pb}_{strip}^0), (\mathbf{Pb}_{box}^0)$ . We prove in particular that non-attainable states for  $(\mathbf{Pb}_{strip})$  belong to

$$\mathcal{NA}_T^{L^2, strip} := \left\{ u_T \in L^2(0, 1) \mid \nexists u \text{ solution in the sense of Definition 3.1} \right. \\ \left. \text{to problem } (\mathbf{Pb}_{strip}^0) \text{ with } u(\cdot, T) = u_T \text{ for } x \in (0, 1) \right\}, \quad (25)$$

while non-attainable states for  $(\mathbf{Pb}_{box})$  belong to the analogously defined set  $\mathcal{NA}_T^{L^2, box}$  based upon Definition 3.4. We recall that in [5], the analogous notation  $\mathcal{NA}_T^{L^\infty, strip}$ ,  $\mathcal{NA}_T^{L^\infty, box}$  was used for sets non-attainable by (bounded) Kruzhkov entropy solutions, under the standard definitions. As in [5], the invariance of Definition 3.1 under the scaling (Zoom) leads (with the analogous statement for Definition 3.4 and the rescaling of  $\mathcal{NA}_T^{L^2, box}$ ) to

$$\mathcal{NA}_T^{L^2, strip} = \left\{ u_T \in L^2(0, 1) \mid Tu_T \in \mathcal{NA}_1^{L^2, strip} \right\} = T^{-1} \mathcal{NA}_1^{L^2, strip}, \quad (26)$$

i.e. we can fix  $T = 1$  in our study of states non-attainable for the inviscid Burgers equation. Also note the following elementary property (cf. [5, Eqn. (10)]):

$$\forall T > 0 \quad \mathcal{NA}_T^{L^2, box} \subset \mathcal{NA}_T^{L^2, strip}, \quad (27)$$

so that the strip setting is simpler than the box one, if one is interested in non-attainability.

At the first step of our analysis, we focus on abstract subsets of  $\mathcal{NA}_1^{L^2, box}$  or  $\mathcal{NA}_1^{L^2, strip}$  (which can be singletons in the simplest applications, cf. the proof of [5, Cor. 3.16]). The statements [5, Thm. 3.12 and Thm. 4.1] concerning non-attainability by  $L^\infty$  solutions for the viscous Burgers equation in its strip and its box formulations are extended to non-attainability, by  $L^2$  solutions of  $(\mathbf{Pb}_{strip})$  or  $(\mathbf{Pb}_{box})$ , of weakly compact subsets of  $\mathcal{NA}_1^{L^2, box}$  or  $\mathcal{NA}_1^{L^2, strip}$ , respectively. As in [5], an appropriate amplification assumption uncovered through the scaling argument is mandatory. Note however that the box setting requires, at this stage, the extra  $L^3_{loc}$  regularization assumption on the solutions. We give the details in § 4.2 (for the strip setting) and § 4.3 (for the box setting).

At the second step, we want to conclude, from the above indicated non-attainability claims, that constants  $C \neq 0$ ,  $|C| \geq C_0$ , are not attainable by  $L^2$  weak solutions of  $(\mathbf{Pb}_{strip})$  or  $(\mathbf{Pb}_{box})$  for times  $T \leq 1/|C|$  (or even  $T < (1 + \Delta)/|C|$  for some  $\Delta = \Delta(L) > 0$ , cf. [5, Prop. 4.2]). Therefore we need to show that constant states  $C \neq 0$  with  $|C| \leq 1$  (respectively,  $|C| < 1 + \Delta$ ) do belong to the sets  $\mathcal{NA}_1^{L^2, box}$ ,  $\mathcal{NA}_1^{L^2, strip}$ . We address this question in two distinct ways.

The first possibility is to invoke the property

$$\text{Int}(\mathcal{NA}_1^{L^\infty, strip}) \subset \mathcal{NA}_1^{L^2, strip} \quad (28)$$

where  $\text{Int}$  denotes, from now on, the topological interior with respect to the  $L^1$  convergence. The analogous statement holds in the box setting. Observation (28) (proved in Lemma 4.3 below) follows from the limit connection between bounded (Kruzhkov) entropy solutions and their unbounded analogues pointed out in the construction of Section 3. Note that both the approximation and the uniqueness claims from Theorem 3.2 are required for this argument. With (28) at hand, it is enough to notice that constant states  $C$  with  $\alpha < |C| < \beta$  belong to the interior of  $\mathcal{NA}_1^{L^\infty, \text{strip}}, \mathcal{NA}_1^{L^\infty, \text{box}}$  in order to negatively answer the strip (respectively, the box) interpretation of the question **(Q)**, in its quantitative interpretation highlighted via [5, Rem. 1.1]. This is done in § 4.1 (cf. [5, Rem. 3.9, 4.5] for generalizations to non-constant target states).

The second possibility, explored in § 4.3, is to directly address the generalization of the key result [5, Prop. 3.7] which claims that the states  $u_1 \in BV_{loc}((0, 1))$  verifying<sup>4</sup>

$$\begin{aligned} & \exists x^* \in [0, 1] \text{ such that} \\ & \text{either } 0 < \overline{u_1}(x^*) \leq x^* , \\ & \text{or } -(1 - x^*) \leq \underline{u_1}(x^*) < 0 , \end{aligned} \tag{NA}$$

fall not only within  $\mathcal{NA}_1^{L^\infty, \text{box}}$  but (provided  $0 < x^* < 1$ ) they also fall within  $\mathcal{NA}_1^{L^2, \text{box}}$  (and consequently, within  $\mathcal{NA}_1^{L^2, \text{strip}}$  as well, see (27)). Note that the constants  $C$ ,  $0 < |C| < 1$ , fulfill **(NA)**. To assess the non-attainability of the states **(NA)** by unbounded entropy solutions, the regularity claims of Proposition 3.14 and Corollary 3.16 stated in Section 3 are instrumental.

#### 4.1. Robustness of non-attainability for the inviscid Burgers equation

The set  $\mathcal{NA}_1^{L^\infty, \text{box}}$  has a large interior with respect to the  $L^1$  (or to the a.e.) convergence, so that one can say that non-attainability for the inviscid Burgers equation is a robust property allowing, for instance, the extension to unbounded entropy solutions.

Specifically, we point out the following observation (cf. [5, Rem. 3.8, 3.9])

**Proposition 4.1.** *The following subsets of  $\mathcal{NA}_1^{L^\infty, \text{box}}$  belong to the interior of  $\mathcal{NA}_1^{L^\infty, \text{box}}$  with respect to the  $L^1((0, 1))$  topology:*

- (i)  $\mathring{\mathcal{K}}_{\alpha, \beta} := \left\{ u : x \mapsto C \mid \alpha < |C| < \beta \right\}$ , for any given  $\alpha, \beta$  with  $0 < \alpha < \beta \leq 1$ ;
- (ii)  $\mathring{\mathcal{K}}_{E, m(\cdot)}^+ := \left\{ u \in L^\infty((0, 1)) \mid \forall x \in E \ m(x) < u(x) < x \right\}$ ,  
for a given  $E \subset (0, 1)$  of non-zero Lebesgue measure and a given measurable  $m : E \rightarrow (0, 1]$ ;
- (iii)  $\mathring{\mathcal{K}}_{E, m(\cdot)}^- := \left\{ u \in L^\infty((0, 1)) \mid \forall x \in E \ -(1 - x) < u(x) < -m(x) \right\}$ , for  $(E, m(\cdot))$  like in (ii).

The elementary inclusion (27) readily yields the analogous statement in the strip:

**Corollary 4.2.** *The sets  $\mathring{\mathcal{K}}_{\alpha, \beta}, \mathring{\mathcal{K}}_{E, m(\cdot)}^\pm$  defined in Proposition 4.1 belong to the interior of the set  $\mathcal{NA}_1^{L^\infty, \text{strip}}$  with respect to the  $L^1((0, 1))$  topology.*

*Proof of Proposition 4.1.* We give the proof for  $\mathring{\mathcal{K}}_{E, m(\cdot)}^+$ , the other cases being entirely similar.

Let  $\mu > 0$  denote the Lebesgue measure  $\text{meas}(E)$  of the set  $E$ . Upon setting for  $\gamma > 0$ ,

$$E^\gamma := \left\{ x \in E \mid m(x) + \gamma < u(x) < x - \gamma \right\},$$

the sets  $E^\gamma$  are nested and their union over all  $\gamma > 0$  gives the whole set  $E$ . It follows by the continuity of the Lebesgue measure that  $\text{meas}(E^\gamma) \rightarrow \text{meas}(E)$  as  $\gamma \rightarrow 0$ , so that  $\text{meas}(E^{\bar{\gamma}}) \geq \mu/2$  for some  $\bar{\gamma} > 0$  which we fix.

Let  $v \in L^1((0, 1))$  be such that  $\|v - u\|_{L^1((0, 1))} \leq \bar{\gamma}\mu/4$ . The Chebyshev inequality yields

$$\text{meas}(F_v^{\bar{\gamma}}) \leq \mu/4, \quad F_v^{\bar{\gamma}} := \{x \in (0, 1) \mid |v(x) - u(x)| \geq \bar{\gamma}\}$$

<sup>4</sup>in this statement,  $\underline{u_1}(\cdot)$  (respectively,  $\overline{u_1}(\cdot)$ ) stands for the left-continuous in  $x$  (respectively, right-continuous in  $x$ ) representative of the  $BV$  function  $x \mapsto u_1(x)$

which implies that  $m(x) < v(x) < x$  at least on  $E_v := E^{\bar{\gamma}} \setminus F_v^{\bar{\gamma}}$ , where  $\text{meas}(E_v) \geq \mu/2 - \mu/4 = \mu/4 > 0$ . Therefore we can conclude that  $v \in \mathring{\mathcal{K}}_{E_v, m(\cdot)}^+ \subset \mathcal{NA}_1^{L^\infty, \text{box}}$ , i.e. the ball of  $L^1((0, 1))$  centered at  $u$ , of radius  $\bar{\gamma}\mu/4$ , lies in  $\mathcal{NA}_1^{L^\infty, \text{box}}$ . This completes the proof.  $\square$

**Lemma 4.3.** *Property (28) holds true, that is*

$$\text{Int}(\mathcal{NA}_1^{L^\infty, \text{strip}}) \subset \mathcal{NA}_1^{L^2, \text{strip}}.$$

We remark that the uniqueness claim of Theorem 3.2 is exploited in the proof of the above lemma. As a matter of fact, we need that unbounded  $L^2$  solutions to the Burgers equation enter some framework where uniqueness is guaranteed in order to connect their properties to classical properties of Kruzhkov entropy solutions. A similar situation occurred in Proposition 3.14 which is crucial for the results of § 4.3.

*Proof of Lemma 4.3.* Recall that for general sets  $A, B$  in a topological space,  $\text{Int}(A) \subset B$  is equivalent to  $B^c \subset \overline{A^c}$ . Therefore it is enough to justify that states  $u_1$  attainable by  $L^2$  unbounded entropy solutions of  $(\mathbf{Pb}_{\text{strip}}^0)$  are limits, in the  $L^1$  topology, of states attainable by bounded (Kruzhkov) entropy solutions of  $(\mathbf{Pb}_{\text{strip}}^0)$ . This follows from the fact that *any* such unbounded solution  $u$  is the  $C(0, T; L_{loc}^1(\mathbb{R}^d))$  limit of bounded solutions  $u^M$  corresponding to truncated data, see the last claim of Theorem 3.2.  $\square$

#### 4.2. Non-attainability: by $L^2$ solutions in the strip

We start by the following – straightforward but quite technical – extension of the claim of [5, Thm. 4.1] to the  $L^2$  setting.

**Theorem 4.4.** *Let  $\mathcal{K}$  be a subset of  $\mathcal{NA}_1^{L^2, \text{strip}}$  compact in the weak topology of  $L^2((0, 1))$ . Let  $L \geq 1$ . Then there exists a constant  $\epsilon_0 > 0$  (depending on  $\mathcal{K}$  and  $L$  only) such that for all couples  $(T, u_T) \in (0, \infty) \times L^2((0, 1))$  satisfying  $Tu_T \in \mathcal{K}$ ,  $T \leq \epsilon_0$  problem  $(\mathbf{Pb}_{\text{strip}})$  has no weak solution  $u \in L^2(\mathbb{R} \times (0, T)) \cap L^2(0, T; H^1(\mathbb{R}))$  with initial data  $u_0$  satisfying the amplification assumption*

$$\|u_0\|_{L^2(\mathbb{R})} \leq L \|u_T\|_{L^2((0, 1))}. \quad (29)$$

**Remark 4.5.** *In the above statement we limit our framework to the  $L^2(\mathbb{R} \times (0, T)) \cap L^2(0, T; H^1(\mathbb{R}))$  weak solutions of the Burgers equation in the strip corresponding to  $L^2(\mathbb{R})$  initial data; the proof indicates that we could as well ask that  $u$  belongs to  $L^\infty(0, T; L^2(\mathbb{R}))$  rather than to  $L^2(\mathbb{R} \times (0, T))$ , since the bounds (30) below are automatically satisfied in our framework.*

*Proof of Theorem 4.4.* The proof follows the lines of the proof of [5, Thms. 3.12, 4.1], the delicate point being to justify the analogue of [5, Lem. 3.10] (recall that it is stated for fixed time  $T = 1$  and variable viscosity parameter  $\epsilon$ ). Arguing by contradiction, we assume targets  $u_1^\epsilon \in \mathcal{K}$  are attainable at time  $T = 1$  by solutions of

$$u_t + (u^2/2)_x = \epsilon u_{xx} \quad \text{in } D = \mathbb{R} \times (0, 1) \quad (\mathbf{Pb}_{\text{strip}}^\epsilon)$$

starting from some initial data  $u_0^\epsilon \in L^2(\mathbb{R})$ ,  $u_0^\epsilon = 0$  in  $(0, 1)$ . We use the boundedness of  $\mathcal{K}$  and the bound (29) to extract a (not relabelled) weakly convergent in  $L^2((0, 1))$  sequence  $(u_1^\epsilon)$  corresponding to the weakly converging in  $L^2(\mathbb{R})$  sequence of initial data  $(u_0^\epsilon)$ ; denote by  $u_1 \in \mathcal{K}$ ,  $u_0 \in \mathcal{K}$  their respective weak limits.

We also need to extract a convergent subsequence from  $(u^\epsilon)$  in the strip  $\mathbb{R} \times (0, 1)$ . At this point, it is essential to observe that initial data  $u_0^\epsilon \in L^2(\mathbb{R})$  give rise to a unique  $L^2(\mathbb{R} \times (0, 1))$  weak solution  $u^\epsilon$  of the viscous Burgers problem  $(\mathbf{Pb}_{\text{strip}}^\epsilon)$  satisfying in addition

$$\|u^\epsilon\|_{L^\infty(0, 1; L^2(\mathbb{R}))} \leq \|u_0^\epsilon\|_{L^2(\mathbb{R})} \leq \text{const}, \quad \|\epsilon |u_x^\epsilon|^2\|_{L^1(\mathbb{R} \times (0, 1))} \leq \|u_0^\epsilon\|_{L^2(\mathbb{R})} \leq \text{const}. \quad (30)$$

The above bounds follow from the combination of two arguments, quite similarly to what is done in the proof of Theorem 3.2 for the inviscid case.

- We rely upon the construction of solutions via approximation by truncated data, being understood that estimates (30) are standard for the resulting  $L^2 \cap L^\infty$  solutions and they are uniform with respect to the approximation (truncation) parameters. Indeed, starting from the estimate

$$\epsilon \iint_K |u_x^\epsilon|^2 dx dt \leq \epsilon \int_0^1 \int_0^1 |u_x^\epsilon|^2 \xi dx dt = \int_0^1 \int_0^1 \frac{1}{2} (u^\epsilon)^2 \xi_t + \frac{(u^\epsilon)^3}{3} \xi_x + \frac{1}{2} \epsilon (u^\epsilon)^2 \xi_{xx} dx dt. \quad (31)$$

(this is [5, Form. (8)]) having in mind the uniform in  $t$  bounds on solutions in both  $L^\infty(\mathbb{R})$  and  $L^1(\mathbb{R})$ , we can pass to the limit and get the analogue of (31) with space-independent test functions (using e.g. (8) like in the proof of Theorem 3.2). To sum up, the convective term in  $(\mathbf{Pb}_{strip}^\epsilon)$  – as soon as a meaning can be given to it – does not contribute to the energy estimates. Let us stress that this feature is closely related to the ideas of Boccardo, Murat et al. that eventually led to the development of the theory of renormalized solutions to elliptic and parabolic PDEs, see in particular [12, 23].

- We invoke the uniqueness of  $L^2(\mathbb{R} \times (0, 1)) \cap L^2(0, T; H^1(\mathbb{R}))$  weak solutions to  $(\mathbf{Pb}_{strip}^\epsilon)$  for given  $u_0^\epsilon \in L^2$ . The uniqueness claim is proved following the argument borrowed from [4]. It is based upon the Kato inequality derived with the method of Carrillo [14] from the entropy formulation of the viscous Burgers equation (inequalities (2) of Definition 2.1). To conclude the argument one picks the specific choice (8) of the test functions in the Kato inequality (see the proof of Theorem 3.2); at this point, the assumption  $u \in L^2(0, 1; H^1(\mathbb{R}))$  contained in the statement of the theorem is used while letting  $R \rightarrow \infty$ .

In view of the amplification assumption (29) (the latter being invariant under the scaling (Zoom)), the estimates (30) yield uniform  $L^\infty(0, 1; L^2(\mathbb{R})) \cap L^2(0, 1; H^1(\mathbb{R}))$  estimates on the sequence  $(u^\epsilon)$ . Moreover, the global  $L^1(\mathbb{R} \times (0, 1))$  estimate of  $\epsilon |u_x^\epsilon|^2$  replaces the localized estimate (31). Using the embedding of  $H^1(\mathbb{R})$  into  $L^\infty(\mathbb{R})$  and the interpolation of  $L^\infty(0, T; L^2(\mathbb{R}))$  and  $L^2(0, T; L^\infty(\mathbb{R}))$ , we find that

$$(u^\epsilon) \text{ is bounded in } L^4(0, T; \mathbb{R}). \quad (32)$$

To conclude the proof, we replace the compensated compactness argument of [5, Lem. 3.10] by a more general compactness argument suitable for uniformly bounded in  $L^p$ ,  $p > 2$ , solutions (in place of uniformly bounded in  $L^\infty$  solutions). We choose to exploit the argument based on quasi-solutions, following Panov [28, Remark 3] as detailed in [27, Theorem 6]. To this end, we exploit the  $[a, b]$ -valued truncation of the identity map  $s_{a,b}$ , as introduced in (14). For fixed  $a < b$ , we apply  $s_{a,b}$  to the sequence  $(u^\epsilon)$  and write the resulting weak formulation. To do so, first we combine the weak and the entropy formulations of Definition 2.1 for  $C_c^\infty(\mathbb{R} \times (0, 1))$  test functions to get the local “half-entropy” inequalities (cf. [32, 14, 31, 6]). Using the fact that nonnegative distributions are locally finite measures, we find that for all  $k \in \mathbb{R}$  there exist locally finite nonnegative measures  $\gamma_k^{\epsilon, \pm}$  on  $\mathbb{R} \times (0, 1)$  such that

$$(u^\epsilon - k)^\pm_t + \frac{1}{2} (\text{sign}^\pm(u^\epsilon - k) ((u^\epsilon)^2 - k^2))_x = \epsilon (u^\epsilon - k)^\pm_{xx} - \gamma_k^{\epsilon, \pm} \text{ in } \mathcal{D}'(\mathbb{R} \times (0, 1)). \quad (33)$$

Then, writing  $s_{a,b}(\cdot) = a + (\cdot - a)^+ - (\cdot - b)^+$ , we find that  $u^\epsilon$  is a quasi-solution of the inviscid Burgers equation in the sense introduced by Panov [26, 28]: for all  $a < b$ , there holds

$$s_{a,b}(u^\epsilon)_t + \frac{1}{2} (s_{a,b}(u^\epsilon)^2)_x = \gamma_b^{\epsilon, +} - \gamma_a^{\epsilon, +} - \sqrt{\epsilon} (\sqrt{\epsilon} s_{a,b}(u^\epsilon)_x)_x =: r_{a,b}^\epsilon \quad (34)$$

in the sense of distributions. We intend to apply [27, Theorem 6(a)] (see also [28, Lemma 5, Remark 3]) to ensure that  $u^\epsilon$  converge to a measurable function  $u$  a.e. on  $\mathbb{R} \times (0, 1)$  and also (similarly to the statement of [27, Theorem 6(b)]) in  $L^2(\mathbb{R} \times (0, 1))$ .

The flux  $u \mapsto u^2/2$  of the Burgers equation being non-degenerately nonlinear, in order to apply Panov’s precompactness result it is enough to check that given  $a < b$ , the family  $(r_{a,b}^\epsilon)$  is precompact in  $W_{loc}^{-1,d}(\mathbb{R} \times (0, 1))$  for some  $d > 1$ . To do so, we observe that the uniform  $L^2$  bound on  $(u^\epsilon)$  and the argument of [28, Lemma 5] yield a uniform in  $\epsilon$  bound on the total variation of the measures  $\gamma_b^+$ ,  $\gamma_a^+$  on every compact subset  $K$  of  $\mathbb{R} \times (0, 1)$ . This is analogous to the argument in Lemma 3.6. Indeed,

fixing  $\xi \in C_c^\infty(\mathbb{R} \times (0, 1))$ ,  $\xi \geq 0$  with  $\xi = 1$  on  $K$  and  $\text{supp}(\xi) = F_\xi$ , from (33) we have

$$\begin{aligned} |\gamma_b^{\epsilon,+}|(K) &\leq \int_{\mathbb{R}} \int_0^1 \xi \, d\gamma_b^{\epsilon,+} = \int_{\mathbb{R}} \int_0^1 (u^\epsilon - b)^+ \xi_t + \frac{\text{sign}^+(u^\epsilon - b)((u^\epsilon)^2 - b^2)}{2} \xi_x + \epsilon(u^\epsilon - b)^+ \xi_{xx} \, dx \, dt \\ &\leq \max\{\|\xi_t\|_\infty, \|\xi_x\|_\infty, \|\xi_{xx}\|_\infty\} \left( (1+\epsilon)\|u^\epsilon\|_{L^1(F_\xi)} + \frac{1}{2}\|u^\epsilon\|_{L^2(F_\xi)}^2 + ((1+\epsilon)b + b^2/2)\text{meas}(F_\xi) \right), \end{aligned} \quad (35)$$

which is bounded uniformly in  $\epsilon$ . An analogous estimate holds for  $|\gamma_a^{\epsilon,+}|(K)$ . Writing  $\mathcal{M}_{loc}(\mathbb{R} \times (0, 1))$  for the space of bounded Radon measures, we use the compact embedding  $\mathcal{M}_{loc}(\mathbb{R} \times (0, 1)) \subset W_{loc}^{-1,d}(\mathbb{R} \times (0, 1))$  for  $1 \leq d < 2$  as in the proof of [27, Theorem 1]. In order to justify that  $(r_{a,b}^\epsilon)$  is relatively compact, it remains to recall that from (30) and the obvious bound  $|s_{a,b}(u^\epsilon)_x| \leq |u_x^\epsilon|$ , we find that  $(\sqrt{\epsilon} s_{a,b}(u^\epsilon)_x)$  is bounded in  $L^2(\mathbb{R} \times (0, 1)) \subset L_{loc}^d(\mathbb{R} \times (0, 1))$  so that  $\sqrt{\epsilon}(\sqrt{\epsilon} s_{a,b}(u^\epsilon)_x)_x \rightarrow 0$  in  $W_{loc}^{-1,d}(\mathbb{R} \times (0, 1))$ . At this point, we know that for all  $a < b$ ,  $a, b \in \mathbb{Q}$ ,  $s_{a,b}(u^\epsilon)$  converge in  $L_{loc}^1$  and a.e. on  $\mathbb{R} \times (0, 1)$  up to a subsequence. Following a diagonal argument, and based on the fact that for  $a \leq a' < b' \leq b$  there holds  $s_{a',b'}(r) = s_{a',b'}(s_{a,b}(r))$ , we deduce the a.e. convergence of (a non relabelled subsequence of)  $u^\epsilon$  to some measurable function  $u$  on  $\mathbb{R} \times (0, 1)$ .

In view of the bound (32), the de la Vallée Poussin criterion of equi-integrability and the Vitali convergence theorem imply, like in [27, Theorem 6(b)], that  $(u^\epsilon)$  converges to  $u$  in  $L^p(\mathbb{R} \times (0, 1))$  for all  $p < 4$ , and in particular for  $p = 2$ . Now, we can pass to the limit in the entropy inequalities (2) of Definition 2.1, where we have to limit our attention to test functions vanishing at  $t = 0$ . Using once more the  $L^1$  bound on  $\epsilon|u_x^\epsilon|^2$ , we infer that  $u$  is an (unbounded) entropy solution of the inviscid Burgers equation in the open domain  $\mathbb{R} \times (0, 1)$ . Passing similarly to the limit in the weak formulation (1) with test functions localized to  $(x, t) \in (0, 1) \times [0, 1]$ , we find that  $u$  assumes in the weak sense the initial data  $u_0 = 0$  and the terminal data  $u_1$  (obtained as the weak limit of  $u_1^\epsilon$ ) for  $x \in (0, 1)$ . The delicate point now is to prove that  $u$  is the unbounded entropy solution of the inviscid Burgers equation corresponding to the initial data  $u_0$  (obtained as the weak limit of  $u_0^\epsilon$ ) in the entropy sense, on the whole of  $\mathbb{R}$ . We refer to the example [13, pp. 53-54] to highlight the importance of this issue. Indeed, the initial data – particularly for unbounded solutions – should be imposed in the entropy sense (or in the equivalent  $\text{esslim}_{t \rightarrow 0^+}$  sense, like in the original setting of [22]) in order to guarantee the uniqueness statement of Theorem 3.2. To this end, passing to the limit in (1) of Definition 2.1 with  $\xi \in C_c^\infty(\mathbb{R} \times [0, 1])$  we find that the initial data  $u_0$  are assumed in the weak sense. Now, we exploit the strong traces result of Panov [26] for quasi-solutions. Going back to (34), we can pass to the limit in  $\mathcal{D}'$  using the uniform local bounds like (35), and infer that

$$s_{a,b}(u)_t + \frac{1}{2}(s_{a,b}(u)^2)_x = \gamma_b^+ - \gamma_a^+$$

with some locally finite nonnegative measures  $\gamma_b^+, \gamma_a^+$ . In order to apply the result of [26] and infer the existence of a strong (in the  $L_{loc}^1(\mathbb{R})$  sense) trace  $\tau_{a,b}u$  of  $s_{a,b}(u)$ , it remains to refine (35) by letting  $\xi$  be non-zero at  $t = 0$ . In this case the additional term  $(u_0^\epsilon - b)^+ \xi(\cdot, 0)$  appears in the right-hand side of (35); given  $b$  and  $\xi$  (i.e. given the compact  $K \subset \mathbb{R} \times [0, 1]$  on which the variation of  $\gamma_b^{\epsilon,+}$  is estimated) this term is bounded uniformly in  $\epsilon$ . Thus  $\gamma_b^+$  is locally bounded up to  $t = 0$ , as required by the assumptions of [26], and the same argument applies to  $\gamma_a^+$ . Starting from the traces  $\tau_{a,b}u$ , using once more the diagonal construction while passing to the limit as  $a, b \in \mathbb{Q}$  tend to  $-\infty, +\infty$ , respectively, we infer the existence of a trace  $\tau u$  at  $t = 0$  in the sense of the a.e. convergence. Bearing in mind that  $u \in L^\infty(0, 1; L^2(\mathbb{R}))$  in view of (30), we get a uniform in  $t \in (0, 1)$  bound in  $L^2(\mathbb{R})$  on the family  $(u(t, \cdot))$ , which ensures that the trace  $\tau u$  is also assumed in the  $L_{loc}^1$  sense, like in [26]. It remains to notice that  $\tau u = u_0$  by identification from the weak formulation satisfied by  $u$ . This concludes the passage to the limit in (2) of Definition 2.1 with  $\xi \in C_c^\infty(\mathbb{R} \times [0, 1])$ . Also note that  $u_0 = 0$  in  $(0, 1)$  because  $u_0^\epsilon = 0$  in  $(0, 1)$  for all  $\epsilon > 0$ .

Therefore,  $u$  is an unbounded entropy solution of the inviscid Burgers equation for the initial data

$u_0 \in L^2(\mathbb{R})$  satisfying  $u_0 = 0$  in  $(0, 1)$ , its terminal state at  $T = 1$ , for  $x \in (0, 1)$ , being  $u(\cdot, 1) = u_1 \in$

$\mathcal{NA}_1^{L^2,strip}$ . This contradicts the definition of the non-attainability set  $\mathcal{NA}_1^{L^2,strip}$ , which concludes the justification of the required analogue of [5, Lem. 3.10] and the proof of Theorem 4.4.  $\square$

**Remark 4.6.** *Note that in view of estimate (32), we could address the passage to the limit in  $u^\epsilon$  via the results of Szepessy [34]. Indeed, the  $L^4$  bound ensures that we are in the subcritical setting (cf. the discussion of the above Theorem 3.2). The strong convergence would then come as a consequence of reduction of a measure-valued solution  $\nu_{(x,t)}$  corresponding to some initial data  $u_0$  (which is a trivial Young measure, i.e. a function in the usual sense). The obstacle in doing so is that we need to know, beforehand, that  $(u_0^\epsilon)$  converges to some limit  $u_0$  in the  $L^1_{loc}$  sense. Unfortunately, in the setting of measure-valued solutions employed in [34], strong trace results exploited in the above proof are not available. Therefore the line of argumentation from [34] does not seem to be applicable.*

Combining Theorem 4.4, Lemma 4.3, the result of Corollary 4.2 and the argument of the proof of [5, Cor. 3.16], we get the following negative answer to (the quantitative version of) the question **(Q)**, in its strip interpretation.

**Corollary 4.7.** *For given  $0 < \alpha \leq \beta < 1$  there exists  $\epsilon_0 = \epsilon_0(\alpha, \beta, L)$  such that for all  $T \leq \epsilon_0$  and all  $C$  with  $\alpha T^{-1} \leq |C| \leq \beta T^{-1}$ , the constant state  $C$  is not attainable at time  $T$  for problem  $(\mathbf{Pb}_{strip})$  in the weak  $L^2$  setting, with initial data fulfilling the amplification restriction  $\|u_0\|_{L^2(\mathbb{R})} \leq L|C|$ .*

**Remark 4.8.** *In the spirit of [5, Rem. 4.5], in the statement of Corollary 4.7 one can actually go up to  $\beta = 1$  and even slightly beyond. To this end, one has to replace the  $L^\infty$ -based estimates*

$$|\tilde{u}_0(x)| = |u(x)| \leq L\|u_T\|_\infty, \quad \int_{x_*}^0 |\tilde{u}_0(x)| \, dx \leq L\|u_T\|_\infty |x_*|$$

in the last but one line of the proof of [5, Prop. 4.4] by the  $L^2$ -based analogue

$$\int_{x_*}^0 |\tilde{u}_0(x)| \, dx \leq \sqrt{|x_*|} \left( \int_{x_*}^0 |\tilde{u}_0(x)|^2 \, dx \right)^{1/2} \leq L\|u_T\|_{L^2((0,1))} \sqrt{|x_*|},$$

and follow the guidelines of the proof of [5, Prop. 4.4]. One ends up with the restriction  $C < 1 + \Delta$ , for some  $\Delta = \Delta(L) > 0$  in the case of a constant target  $u_1 = C$  (thus, with  $m = C$  and  $M = 1$  in the notation of [5, Prop. 4.4]).

Also note that more general target states of the type **(NA)** can be dealt with, combining the arguments of § 4.1.4.2. In § 4.3, we will address non-constant targets, in the box setting, through a direct justification of their non-attainability by unbounded entropy solutions of  $(\mathbf{Pb}_{box}^0)$ .

### 4.3. Non-attainability by $L^2 \cap L^3_{loc}$ solutions in the box

With the help of Corollary 3.16, we can revisit the non-attainability result of [5, Prop. 3.7] for the inviscid Burgers equation, extending it from the  $L^\infty$  to the  $L^2$  setting, as follows.

**Proposition 4.9.** *Assume that the target state  $u_1$  fulfills **(NA)** with the additional restriction  $0 < x^* < 1$ . Then  $u_1 \in \mathcal{NA}_1^{L^2,box}$  and for all  $T > 0$ ,  $T^{-1}u_1 \in \mathcal{NA}_T^{L^2,box}$ . In particular, for all couples  $(C, T) \in (0, +\infty) \times \mathbb{R}_+$  verifying  $|C|T \leq 1$ , there holds  $C \in \mathcal{NA}_T^{L^2,box}$ .*

Note that the assumption  $x^* < 1$  is needed to ensure, via the application of Corollary 3.16, that the maximal backward generalized characteristics can be utilized; indeed, recall that we can define them in sets  $[\delta, 1 - \delta] \times [\delta, 1]$  where our unbounded entropy solution is actually bounded, and then let  $\delta \rightarrow 0$ .

*Proof.* With reference to the proof of Proposition 3.7 in [5], we pursue the construction of the auxiliary solution  $\tilde{u}$ . The definition of  $\tilde{u}$  in the proof of [5, Prop. 3.7] does not lead to a contradiction in our new setting of unbounded entropy solutions; indeed, we cannot rely any more upon the property of finite speed of propagation, specifically as  $t$  approaches  $0^+$  or as  $x$  approaches  $1^-$  (we stress the local character of the regularization in Proposition 3.11). Therefore we deeper analyze and modify the construction of  $\tilde{u}$ .

To this end, first observe that we can assume that  $u$  in the region below the characteristics  $x = x_* + tu^*$ , normalized by the left continuity (which is possible due to the  $BV_{loc}$  regularity, see Remark 3.12) takes a nonnegative value  $\hat{u}$  at least at some point  $(\hat{x}, \hat{t})$ ,  $0 < \hat{t} < 1$ ,  $x_* + \hat{t}u^* < \hat{x} < 1$ . Indeed, in the contrary case  $\tilde{u}$  undergoes a jump across the line  $x = \gamma(t) := x_* + tu^*$  from the value  $u^* > 0$  to some values  $u_*(t) := \lim_{x \rightarrow x_* + tu^*} u(x, t)$ ,  $u_*(t) \leq 0$ . This is prohibited by the Rankine-Hugoniot conditions (recall that  $\tilde{u}$  is a classical entropy solution locally in the box, due to Proposition 3.11) because we have  $\gamma'(t) = u^* > (u^* + u_*(t))/2$ .

Now we modify  $\tilde{u}$  to the right from the minimal backward characteristic  $x = \hat{x} - (\hat{t} - t)\hat{u}$ , by setting it equal to the constant  $\hat{u}$  for all  $(x, t)$  to the left of the line  $x = \tilde{x} + t\hat{u}/2$  where  $\tilde{x} \in (\hat{x}, +\infty)$  is arbitrary (see Figure 1). The construction is completed by setting  $\tilde{u} = 0$  to the right of the line  $x = \tilde{x} + t\hat{u}/2$ , leading to an entropy-admissible jump across this line.

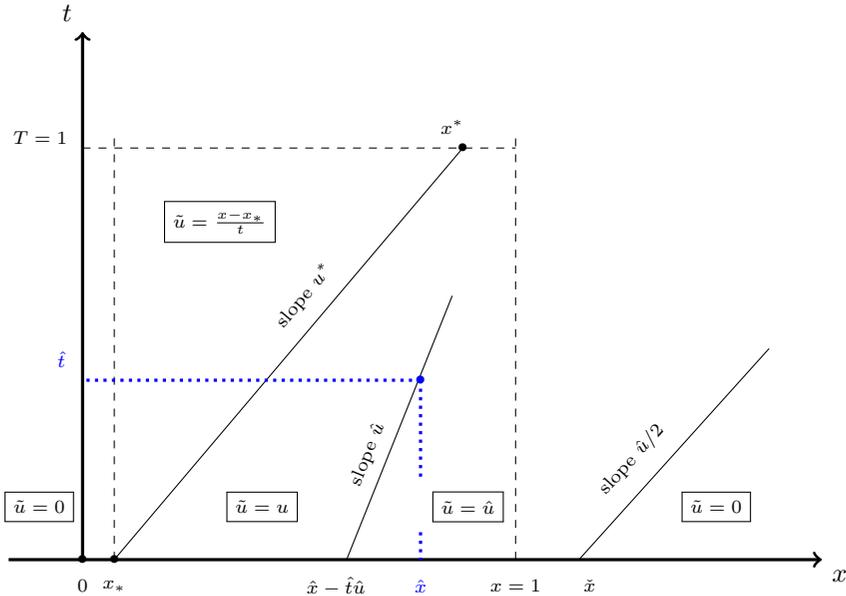


FIGURE 1. Modification of the auxiliary function  $\tilde{u}$  (cf. [5, Fig. 1]): construction based upon the existence of a point  $(\hat{x}, \hat{t}) \in (0, 1) \times (0, 1)$  with  $\hat{x} > x_* - (1 - \hat{t})u^*$  such that  $\hat{u} := u(\hat{x}, \hat{t}) > 0$ ; the point  $\tilde{x} > \hat{x}$  is arbitrary.

The resulting function  $\tilde{u}$  is glued continuously from pieces of (bounded or unbounded) entropy solutions, therefore it is an (unbounded)  $L^\infty(0, \hat{t}; L^2(\mathbb{R}))$  entropy solution to the Burgers equation. Moreover, it assumes the  $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  initial data  $u_0 = \hat{u}$  for  $x \in (\hat{x} - \hat{t}\hat{u}, \tilde{x})$ , while  $u_0 = 0$  otherwise. According to the uniqueness claim of Theorem 3.2,  $\tilde{u}$  should coincide with the classical solution of the Burgers equation with these initial data. However, unlike  $\tilde{u}$ , this classical solution lacks the rarefaction centered at  $(x_*, 0)$ . This contradiction completes the proof.  $\square$

Now, we return to the viscous problem  $(\mathbf{Pb}_{box})$ . We have the following analogue of [5, Thm. 3.12].

**Theorem 4.10.** *Let  $\mathcal{K}$  be a subset of  $\mathcal{NA}_1^{L^2, box}$  compact in the weak topology of  $L^2((0, 1))$ . Let  $L \geq 1$ . Then there exists a constant  $\epsilon_0 > 0$  (depending on  $\mathcal{K}$  and  $L$  only) such that for all couples  $(T, u_T) \in (0, \infty) \times L^2((0, 1))$  satisfying  $Tu_T \in \mathcal{K}$ ,  $T \leq \epsilon_0$  problem  $(\mathbf{Pb}_{box})$  has no weak solution*

$$u \in L^2((0, 1) \times (0, T)) \cap L_{loc}^3((0, 1) \times (0, T)) \quad (36)$$

satisfying the  $L^2$  amplification assumption

$$\|u\|_{L^2((0,1) \times (0,T))} \leq LT \|u_T\|_{L^2((0,1))}. \quad (37)$$

along with the following  $L^3_{loc}$  regularization assumption:

$$\forall \delta \in (0, 1/2) \exists M_\delta \text{ such that } \|u\|_{L^3((\delta, 1-\delta) \times (\delta, T-\delta))} \leq M_\delta \|u\|_{L^2((0,1) \times (0,T))}. \quad (38)$$

*Proof.* We only sketch the proof, pointing out the changes with respect to those of Theorem 3.12 of the paper [5] and Theorem 4.4 of the present paper. As a matter of fact, we only need to check the application of the variant of [5, Lem. 3.10] in the  $L^2$  box setting, which is close to the proof of Theorem 4.4. The bounds, including the scaling-compatible amplification assumption (37), permit to extract  $L^2$  weakly convergent subsequences from  $(u^\epsilon_1)$  and  $(u^\epsilon)$ , while we recall that  $u^\epsilon_0 = 0$  converge to  $u_0 = 0$  strongly on  $(0, 1)$ . In order to apply the compactness arguments of [27, 28] and pass to the weak limit in the entropy formulation (recall that weak solutions of  $(\mathbf{Pb}_{box})$  satisfy (2) of Definition 2.1) we need to show the uniform  $L^1_{loc}$  bound on  $\epsilon|u^\epsilon_x|^2$  and to provide a replacement for the ‘‘higher integrability’’ bound (32). Both issues rely upon the  $L^3_{loc}$  bound on the sequence  $(u^\epsilon)$ . Indeed, first, a bound in  $L^p_{loc}((0, 1) \times (0, 1))$ ,  $p > 2$  is enough to replace the global  $L^4$  bound (32) in the Vitali argument of the proof of Theorem 4.4. Such a bound, with  $p = 3$ , is ensured by our hypotheses (37),(38) (we stress that both of them are scaling-invariant). Second, the estimation of  $\epsilon|u^\epsilon_x|^2$  can be done through (31) where the uniform  $L^3_{loc}$  bound on the sequence  $(u^\epsilon)$  is mandatory. The remaining proof is simpler than the one in Theorem 4.4 because the initial condition  $u_0 = 0$  is enforced in the strong sense by passage to the limit from entropy inequalities (2).  $\square$

We note that in order to apply the above statement to concrete situations, such as constant targets, we need to point out some subsets of  $\mathcal{NA}_1^{L^2, box}$ , in particular, the constant states within it. This could be done via Proposition 4.1 and the box version of property (28). However, in our final result, we prefer to exploit Proposition 4.9 to provide an alternative, more direct argument.

**Corollary 4.11.** *For given  $0 < \alpha \leq \beta < 1$  there exists  $\epsilon_0 = \epsilon_0(\alpha, \beta, L)$  such that for all  $T \leq \epsilon_0$  and all  $C$  with  $\alpha T^{-1} \leq |C| \leq \beta T^{-1}$ , the constant state  $C$  is not attainable at time  $T$  for problem  $(\mathbf{Pb}_{box})$  in the weak  $L^2((0, 1) \times (0, T)) \cap L^3_{loc}((0, 1) \times (0, T))$  setting, with solutions fulfilling the amplification restriction  $\|u\|_{L^2((0,1) \times (0,T))} \leq LT|C|$  and the regularization assumption (38).*

*Proof.* It is enough to apply Proposition 4.9 to establish that  $\mathring{\mathcal{K}}_{\alpha, \beta} \subset \mathcal{NA}_1^{L^2, box}$ , and then apply Theorem 4.10.  $\square$

We do not pursue the goal of stating further and finer results with *ad hoc* choices of  $\mathcal{K}$  in the setting of Theorem 4.10. Interested readers can infer such results from Proposition 4.9, and even go further using the ideas and auxiliary results collected in [5, Prop. 4.4, Rem. 4.5] and in Remark 4.8 of the present paper.

We conclude with the following remark that relates the most general negative answer (for the above indicated couples  $(C, T)$  and under the *ad hoc* amplification bounds) to question **(Q)** of Coron [15] to the precise choice of the underlying notion of solution.

**Remark 4.12.** *Assumption (38) should be seen as an  $L^2 - L^3$  regularizing effect in the interior of  $(0, 1) \times (0, T)$ . Let us underline that this property holds true for sufficiently regular weak solutions of  $(\mathbf{Pb}_{box})$ , with  $(\delta, 1 - \delta) \times (\delta, T - \delta)$  replaced by  $(\delta, 1 - \delta) \times (\delta, T)$ . We have in mind the following point. First, like for the inviscid Burgers equation (cf. Theorems 3.2, 3.10), the  $L^2((0, 1) \times (0, T))$  solutions to the viscous Burgers equation that correspond to some prescribed boundary controls can be expected to belong to  $L^\infty(0, T; L^2(0, 1))$ . Second, it is known since the founding work of Hopf [20] that classical solutions of the viscous Burgers equation satisfy the Lax-Oleinik representation (24). If we assume additionally that our weak solutions are pointwise a.e. limits of such regular solutions, then representation (24) is inherited (see Proposition 3.14, cf. the proof of Proposition 3.11). At least in this setting, the  $L^2 - L^\infty$  local regularization result of Proposition 3.14 – result solely based upon the  $L^\infty(0, T; L^2((0, 1)))$  regularity and the representation (24) with monotone functions  $y(\cdot, t)$  – extends to solutions of the viscous Burgers equation. To sum up, assumption (38) in Theorem 4.10 and Corollary 4.11 may be automatically satisfied in practical control situations.*

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