Fibrations by affine lines on rational affine surfaces with irreducible boundaries

Adrien Dubouloz

FIBRATIONS BY AFFINE LINES ON RATIONAL AFFINE SURFACES WITH IRREDUCIBLE BOUNDARIES

ADRIEN DUBOULOZ

ABSTRACT. We consider fibrations by affine lines on smooth affine surfaces obtained as complements of smooth rational curves B in smooth projective surfaces X defined over an algebraically closed field of characteristic zero. We observe that except for two exceptions, these surfaces $X \setminus B$ admit infinitely many families of $\mathbb{A}^1$-fibrations over the projective line with irreducible fibers and a unique singular fiber of arbitrarily large multiplicity. For $\mathbb{A}^1$-fibrations over the affine line, we give a new and essentially self-contained proof that the set of equivalence classes of such fibrations up to composition by automorphisms at the source and target is finite if and only if the self-intersection number B^2 of B in X is less than or equal to 6.

INTRODUCTION

Affine surfaces whose automorphism groups act with a dense orbit with finite complement were studied by M.H. Gizatullin and V.I. Danilov in a series of seminal papers in the seventies [20, 21, 6, 7]. There, they established that except for finitely many special cases, these are the affine surfaces which admit projective completions whose boundaries are chains of smooth proper rational curves. Up to the exception $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1$, such surfaces are equivalently characterized by the property that they admit two fibrations over the affine line $\mathbb{A}^1$, whose general fibers are pairwise distinct and isomorphic to $\mathbb{A}^1$, see [21, 1, 10]. Many of these surfaces actually admit infinitely many such $\mathbb{A}^1$-fibrations with pairwise different general fibers up to the equivalence relation given by composition by automorphisms on the source and the target. This richness of $\mathbb{A}^1$-fibrations contributes in a central way to the complexity of their automorphism groups, see e.g. [2, 19, 3, 13, 23, 24].

In this article, we consider $\mathbb{A}^1$-fibrations on the subclass consisting of affine surfaces S defined over an algebraically closed field k of characteristic zero and which admit smooth projective completions X with boundary $B = X \setminus S$ isomorphic to the projective line $\mathbb{P}^1$. A surface of this type is isomorphic either to the affine plane $\mathbb{A}^2$, or to the complement of a smooth conic in $\mathbb{P}^2$ or to the complement of an ample section of a $\mathbb{P}^1$-bundle $\pi_n : \mathbb{F}_n = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-n)) \to \mathbb{P}^1$ for some $n \geq 0$. Furthermore, the famous Danilov-Gizatullin isomorphism theorem asserts that the isomorphism class of an affine surface of the form $\mathbb{F}_n \setminus B$ depends only on the self-intersection B^2 of the boundary B.

While $\mathbb{A}^2$ and the complements of smooth conics in $\mathbb{P}^2$ only admit $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$, a surface $\mathbb{F}_n \setminus B$ admits an $\mathbb{A}^1$-fibration $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ given by the restriction of the $\mathbb{P}^1$-bundle $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$. These fibrations are actually locally trivial $\mathbb{A}^1$-bundles, and one can check that their equivalence classes are in one-to-one correspondence with the orbits of the natural action of the group $\mathrm{PGL}_2(\mathrm{k})$ on the space $\mathbb{P}(H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-B^2)))$ (see Remark 5). In particular, for $B^2 \geq 3$, these surfaces admit infinitely many equivalence classes of $\mathbb{A}^1$-fibrations over $\mathbb{P}^1$. The geometry and equivalence classes of other families of $\mathbb{A}^1$-fibrations over $\mathbb{P}^1$ on surfaces $\mathbb{F}_n \setminus B$ has been much less studied than those of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$. Our first result, inspired by a construction due to Blanc-van Santen [4] of infinite families of pairwise non-equivalent closed embedding of the affine line in the complement of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, reads as follows:

Theorem 1. *Let $(\mathbb{F}_n, B)$ be a pair consisting of a Hirzebruch surface $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$ and an ample section B of π_n. Then for every $m \geq 4$, there exist infinite families of equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ with a unique singular fiber, irreducible and of multiplicity m.*

The proof of Theorem 1 given in Section 2 actually provides a natural bijection between a set of equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ with a unique singular fiber, irreducible and of multiplicity m, and the elements of the set-theoretic quotient of the set of closed points of a certain Zariski dense open subset of $\mathbb{P}^{m-1}$ by the algebraic action of a linear algebraic group whose general orbits are at most 2-dimensional. This construction strongly suggests that by replacing the consideration of set-theoretic quotients by, for instance, that of GIT quotients of suitable open subsets, one should be able to strengthen Theorem 1 in a form asserting the existence of algebraic families of $\mathbb{A}^1$-fibrations s $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ with a unique singular fiber, irreducible and of multiplicity m parametrized by the closed points of an algebraic variety of dimension $m-3$. Tackling the necessary additional constructions which are needed to give a rigorous and accurate formulation of this coarse moduli viewpoint falls beyond the scope of the present article.

2000 *Mathematics Subject Classification.* 14R25, 14E05, 14R05.

This research was partially funded by ANR Grant "FIBALGA" ANR-18-CE40-0003. The Institute of Mathematics of Burgundy receives support from the EIPHI Graduate School ANR-17- EURE-0002.

In a second step, we consider equivalence classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ on affine surfaces $X \setminus B$. It is a well-known result of Danilov-Gizatullin [6, 7] that every such surface other than the complement of a smooth conic in $\mathbb{P}^2$ has a unique equivalence class of smooth $\mathbb{A}^1$-fibration over $\mathbb{A}^1$. As already observed by Danilov-Gizatullin again, for every $d = 1,\dots,5$, the finiteness of the number of equivalences classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ on surfaces $X \setminus B$ with $B^2 = d$ is then a consequence of the finiteness of isomorphism types of pairs (X,B) with $B^2 = d$. Over the field of complex numbers, equivalence classes of non-smooth $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$, equivalently $\mathbb{A}^1$-fibrations having non-reduced components in their degenerate fibers, have been extensively studied in [19] in a broader context, see especially Corollary 6.3.19 and Corollary 6.3.20 in *loc. cit.*. The techniques there consist in first constructing a finite-to-one correspondence between equivalence classes of such $\mathbb{A}^1$-fibrations and collections of points in a configuration space. The latter encodes the standard construction of a completion of an $\mathbb{A}^1$-fibered smooth affine surface $\pi : S \to \mathbb{A}^1$ into a $\mathbb{P}^1$-fibered smooth projective surface $\bar{\pi} : \bar{S} \to \mathbb{P}^1$ obtained by performing a suitable sequence of blow-ups of closed points starting from a Hirzerbruch surface $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$. The second step consists in describing the possible configurations and determining their respective numbers of moduli. A consequence of this extensive description is that for every $B^2 \geq 7$, the surface $X \setminus B$ admits families of pairwise non-equivalent $\mathbb{A}^1$-fibrations $X \setminus B \to \mathbb{A}^1$ parametrized by a space whose dimension is an increasing function of B^2 (see [19, Example 6.3.2]). Our second result consists of an alternative direct proof of the following theorem, based on the use of a different point of view.

Theorem 2. *Let (X,B) be a pair consisting of a smooth projective surface X and an ample smooth rational curve B on X. Then the following alternative holds:*

a) If $B^2 \leq 6$ then $X \setminus B$ admits at most seven equivalence classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$,

b) If $B^2 \geq 7$ then the set of equivalence classes of $\mathbb{A}^1$-fibrations $X \setminus B \to \mathbb{A}^1$ is infinite, of cardinality larger than or equal to that of the field k.

For $B^2 \leq 6$, the different equivalence classes are derived by an explicit case by case study. For $B^2 \geq 7$, our argument is based on the study of the equivalence classes of a subclass of $\mathbb{A}^1$-fibrations $\mathbb{F}_n \setminus B \to \mathbb{A}^1$ which have an irreducible component of multiplicity two inside their degenerate fiber. We show that for every $B^2 \geq 7$, the set of equivalence classes of $\mathbb{A}^1$-fibration of this type is infinite. More precisely, we actually construct explicit families of equivalences classes of $\mathbb{A}^1$-fibered smooth affine surfaces $S \to \mathbb{A}^1$ with a unique degenerate fiber, irreducible and of multiplicity two, depending algebraically on a parameter varying in an affine space of dimension $\left\lfloor \frac{B^2-5}{2} \right\rfloor$ (see Example 33) and which are all realized as restrictions of $\mathbb{A}^1$-fibrations $\mathbb{F}_n \setminus B \to \mathbb{A}^1$ on suitable Zariski open subsets. This indicates in an indirect fashion that the "number of moduli" of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ on surfaces $X \setminus B$ with $B^2 = d$ is bounded from below by $\left\lfloor \frac{d-5}{2} \right\rfloor$.

The article is organized as follows. In section one, after setting some notations, we review basic properties of smooth affine surfaces completable by smooth rational curves. We then proceed in section two to the proof of Theorem 1. The third section is devoted to the proof of Theorem 2, which combines several known facts together with new results on equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ having an irreducible component of multiplicity two inside their unique degenerate fiber.

1. Preliminaries

All varieties and schemes considered are defined over a fixed algebraically closed field k of characteristic zero.

1.1. Notations and basic definitions.

We briefly recall basic definitions on SNC divisors and standard properties of $\mathbb{A}^1$-fibrations and $\mathbb{P}^1$-fibrations which we use throughout the paper, see e.g [25, Chapter 3] for the details.

1.1.1. *SNC divisors and rationals trees on smooth surfaces.*

(i) Let X be a smooth projective surface. An *SNC divisor* on X is a curve $B \subset X$ with smooth irreducible components and ordinary double points only as singularities. We say that B is *SNC-minimal* if its image by any strictly birational proper morphism $\tau : X \to X'$ onto a smooth projective surface X' with exceptional locus contained in B is not an SNC divisor. A *rational tree* on X is an SNC divisor whose irreducible components are isomorphic to $\mathbb{P}^1$ and whose incidence graph is a tree. A *rational chain* is a rational tree whose incidence graph is a chain. We use the notation $B = B_0 \triangleleft B_1 \triangleleft \cdots \triangleleft B_r$ to indicate a rational chain whose irreducible components B_i are ordered in such a way that for $0 \leq i < j \leq r$, one has $B_i \cdot B_j = 1$ if $j = i+1$ and 0 otherwise. The sequence of self-intersections $[B_0^2,\dots,B_r^2]$ is referred to as the *type* of the oriented rational chain B.

(ii) An *SNC completion* of a smooth quasi-projective surface V is a pair (X,B) consisting of a smooth projective surface X and an SNC divisor $B \subset V$ such that $X \setminus B \simeq V$. The completion is said to be SNC-minimal if B is SNC-minimal and to be smooth if B is smooth.

1.1.2. *Recollection on $\mathbb{A}^1$-fibrations and $\mathbb{P}^1$-fibrations on smooth surfaces.*

(i) A $\mathbb{P}^1$-*fibration* on a smooth projective surface X is a morphism $\overline{\rho}: X \to C$ onto a smooth projective curve C whose generic fiber is isomorphic to the projective line over the function field of C. Every $\mathbb{P}^1$-fibration $\overline{\rho}: X \to C$ is obtained from a Zariski locally trivial $\mathbb{P}^1$-bundle over C by performing a finite sequence of blow-ups of points. In particular, every $\mathbb{P}^1$-fibration has a section and its singular fibers are supported by rational trees on X. If X is rational, then for every smooth proper rational curve F with self-intersection 0, the complete linear system $|F|$ of effective divisors on X linearly equivalent to F defines a $\mathbb{P}^1$-fibration $\overline{\rho}_{|F|}: X \to \mathbb{P}^1_k$ having F as a smooth fiber.

(ii) An $\mathbb{A}^1$-*fibration* on a smooth quasi-projective surface V is a surjective morphism $\rho: V \to A$ onto a smooth curve A whose generic fiber is isomorphic to the affine line over the function field of A. Every $\mathbb{A}^1$-fibration is the restriction of a $\mathbb{P}^1$-fibration $\overline{\rho}: X \to C$ over the smooth projective model C of A, defined on an SNC completion (X,B) of V with boundary $B = \bigcup_{c\in C\setminus A} F_c \cup H \cup \bigcup_{a\in A} G_a$ where, $F_c = \overline{\rho}^{-1}(c) \simeq \mathbb{P}^1$ for every $c \in C\setminus A$, H is a section of $\overline{\rho}$, and for every $a \in A$, G_a is a union of SNC-minimal rational subtree of the rational tree $(\overline{\rho}^{-1}(a))_{\mathrm{red}}$. The pair (X,B) is called a relatively minimal $\mathbb{P}^1$-fibered completion of $\rho: V \to A$. If $\rho: V \to A$ is affine, every nonempty G_a is connected and has an irreducible component intersecting H, and the closure in X of every irreducible component of $\rho^{-1}(a)$ intersects G_a transversely in a unique point. In particular, every irreducible component of $\rho^{-1}(a)$ is isomorphic to $\mathbb{A}^1$ when equipped with its reduced structure. A scheme-theoretic closed fiber of $\rho: V \to A$ which is not isomorphic to $\mathbb{A}^1$ is called *degenerate*.

(iii) A smooth $\mathbb{A}^1$-fibered surface is a pair (V,π) consisting of a smooth quasi-projective surface V and an $\mathbb{A}^1$-fibration $\pi: V \to A$ onto a smooth curve A. The $\mathbb{A}^1$-fibration π is said to be of affine type if A is affine and of complete type otherwise. Two $\mathbb{A}^1$-fibered surfaces $(V,\pi: V \to A)$ and $(V',\pi': V' \to A')$ are said to be *equivalent* if there exist an isomorphism $\Psi: V \to V'$ and an isomorphism $\psi: A \to A'$ such that $\pi' \circ \Psi = \psi \circ \pi$.

1.2. **Models of smooth affine surfaces with irreducible rational boundaries.** We review known basic properties of smooth affine surfaces admitting smooth completions (X,B) with boundaries $B \cong \mathbb{P}^1$. Recall [22, Theorem 2] that for such a pair (X,B), the affineness of $X\setminus B$ implies that B is the support of an ample effective divisor on X.

Lemma 3. [20, Proposition 1] *A pair (X,B) consisting of a smooth projective surface X and a divisor $B \cong \mathbb{P}^1$ such that $X\setminus B$ is affine is isomorphic to one of the following:*

a) $(\mathbb{P}^2,B)$ where B is either a line L or a smooth conic Q,

b) $(\mathbb{F}_n,B)$ where $\pi_n: \mathbb{F}_n = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-n)) \to \mathbb{P}^1$, $n \geq 0$, is a $\mathbb{P}^1$-bundle and B is an ample section of π_n.

Proof. The log-canonical divisor $K_X + B$ is not nef since $(K_X + B)\cdot B = -2$ by adjunction formula. Given a $K_X + B$ extremal smooth rational curve R on X, the conditions $B\cdot R > 0$ and $(K_X + B)\cdot R < 0$ imply that $R^2 \geq 0$. If $R^2 > 0$ then X is a smooth log del Pezzo surface of Picard rank 1, hence is isomorphic to $\mathbb{P}^2$, and B is either a line or a smooth conic. If $R^2 = 0$, then the associated extremal contraction is a Zariski locally trivial $\mathbb{P}^1$-bundle $h: X \to C$ over a smooth projective curve C and B is a section of h. Thus, $C \cong B \cong \mathbb{P}^1$ and $(X,h) \cong (\mathbb{F}_n,\pi_n)$ for some $n \geq 0$. □

For a pair $(\mathbb{F}_n,B)$ as in Lemma 3 b), we denote by $C_0 \subset \mathbb{F}_n$ a section of π_n with self-intersection $C_0^2 = -n$ and by F a closed fiber of π_n. Recall [8, Corollary V.2.18] that for $m \geq 1$, the complete linear system $|C_0 + mF|$ contains prime divisors if and only if $m \geq n$. Since B is a section of π_n, we have $B \sim C_0 + \frac{1}{2}(B^2 + n)F$, where $B^2 \geq n+2$ because B is ample. For a fixed $d \geq 2$, the Hirzebruch surfaces $\mathbb{F}_n$ containing an ample section B with $B^2 = d$ are those of the form $\mathbb{F}_{d-2i}$, $i = 1,\ldots,\lfloor \frac{d}{2} \rfloor$, with B belonging to the complete linear system $|C_0 + (d-i)F|$.

Since the divisor class group of $\mathbb{F}_n$ is freely generated by the classes of F and of a section of π_n, the divisor class group of $\mathbb{F}_n\setminus B$ is freely generated by the class of $F|_{\mathbb{F}_n\setminus B}$. A canonical divisor $K_{\mathbb{F}_n}$ of $\mathbb{F}_n$ being linearly equivalent to $-2C_0 - (n+2)F$, we have $K_{\mathbb{F}_n} \sim -2B + (B^2-2)F$ and hence $K_{\mathbb{F}_n\setminus B} \sim (B^2-2)F|_{\mathbb{F}_n\setminus B}$. A result due to Danilov-Gizatullin asserts conversely that the integers B^2 are a complete invariant of the isomorphism classes of surfaces $\mathbb{F}_n\setminus B$, namely:

Theorem 4. [7, Theorem 5.8.1] *(see also* [5, Corollary 4.8], [12, §3.1], [18] *and* [19, Corollary 6.2.4]*) The isomorphism class of the complement of an ample section B in a Hirzebruch surface $\mathbb{F}_n$ depends only on the self-intersection B^2 of B.*

Remark 5. For a pair $(\mathbb{F}_n,B)$ as in Lemma 3 b) with $B^2 = d$, the closed immersion $B \hookrightarrow \mathbb{F}_n$ is determined by a surjection $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-n) \to \mathscr{L}$ onto an invertible sheaf $\mathscr{L}$ on $\mathbb{P}^1$, with kernel $\mathscr{K}$ isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-\frac{1}{2}(d+n))$. The locally trivial $\mathbb{A}^1$-bundle $\nu = \pi_n|_{\mathbb{F}_n\setminus B}: \mathbb{F}_n\setminus B \to \mathbb{P}^1$ thus carries the additional structure of a non-trivial torsor under the line bundle associated to the invertible sheaf $\mathscr{L}^\vee \otimes \mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(-d)$. Isomorphism classes of such $\mathbb{A}^1$-bundles are in one-to-one correspondence with the elements of the projective space $\mathbb{P}(\mathrm{Ext}^1(\mathscr{L},\mathscr{K})) \cong \mathbb{P}(H^1(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(-d))) \cong \mathbb{P}^{d-2}$. By [7, Remark 4.8.6] (see also [12, §3.1] and [11, Proposition 2]), every non-trivial $\mathscr{O}_{\mathbb{P}^1}(-d)$-torsor arises as a restriction $\pi_n|_{\mathbb{F}_n\setminus B}: \mathbb{F}_n\setminus B \to \mathbb{P}^1$ for some pair $(\mathbb{F}_n,B)$ as in Lemma 3 b) with $B^2 = d$.

Example 6. The pairs (X,B) of Lemma 3 a) are unique up to isomorphism. In particular every affine surface isomorphic to the complement of a smooth conic in $\mathbb{P}^2$ is isomorphic to the complement of the conic $Q_0=\{xz+y^2=0\}$ in $\mathbb{P}^2_{[x;y:;z]}$. A model of an affine surface $\mathbb{F}_n\setminus B$ with $B^2=d$ is given for $d=2e\geq 2$ by the complement in $\mathbb{F}_0=\mathbb{P}^1_{[u_0:u_1]}\times\mathbb{P}^1_{[v_0:v_1]}$ of the section $\Delta_e=\{u_1^e v_0-u_0^e v_1=0\}$ of $\pi_0=\mathrm{pr}_1$, and for $d=2e+1\geq 3$ by the complement in $\pi_1:\mathbb{F}_1\to\mathbb{P}^1$ viewed as the blow-up of $\mathbb{P}^2_{[x:y:z]}$ at the point $p=[0:1:0]$ of the proper transform of the rational cuspidal curve $C_e=\{yz^e+x^{e+1}=0\}$.

The next examples illustrate some other representatives of isomorphism classes of affine surfaces $\mathbb{F}_n\setminus B$.

Example 7. [12] For every $d\geq 2$ and every pair of integers $p,q\geq 1$ such that $p+q=d$, the geometric quotient S_d of the smooth affine threefold $X_{p,q}=\{x^pv-y^qu=1\}$ in $\mathbb{A}^4$ by the free $\mathbb{G}_m$-action defined by $\lambda\cdot(x,y,u,v)=(\lambda x,\lambda y,\lambda^{-q}u,\lambda^{-p}v)$ is a representative of the isomorphism class of surfaces of the form $\mathbb{F}_n\setminus B$ such that $B^2=d$. Indeed, S_d is isomorphic to the complement in the geometric quotient

$$\pi_{|p-q|}:\mathbb{F}_{|p-q|}\cong\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(p)\oplus\mathscr{O}_{\mathbb{P}^1}(q))\to\mathbb{P}^1=\mathrm{Proj}(k[x,y])$$

of $(\mathbb{A}^2\setminus\{0\})\times(\mathbb{A}^2\setminus\{0\})$ by the $\mathbb{G}_m^2$-action $(\lambda,\mu)\cdot(x,y,u,v)=(\lambda x,\lambda y,\lambda^{-p}\mu u,\lambda^{-q}\mu v)$ of the ample section B of $\pi_{|p-q|}$ with self-intersection d determined by the vanishing of the polynomial x^pv-y^qu of bi-homogeneous degree $(0,1)$.

Example 8. [11] For every $d\geq 2$, the surface W_d in $\mathbb{A}^4=\mathrm{Spec}(k[x_1,x_2,x_2,x_4])$ defined by the equations

$$x_1x_3-x_2(x_2+1)=0,\ x_2^{d-2}x_4-x_3^{d-1}=0,\ x_1^{d-2}x_4-(x_2+1)^{d-2}x_3=0$$

is a representative of the isomorphism class of surfaces of the form $\mathbb{F}_n\setminus B$ such that $B^2=d$. Indeed, the morphism $\nu:W_d\to\mathbb{P}^1$, $(x_1,x_2,x_3,x_4)\mapsto[x_1:x_2+1]=[x_2:x_3]$ is a locally trivial $\mathbb{A}^1$-bundle with local trivializations

$$\nu^{-1}(\mathbb{P}^1\setminus[0:1])\cong\mathrm{Spec}(k[w][x_4])\quad\text{and}\quad\nu^{-1}(\mathbb{P}^1\setminus[1:0])\cong\mathrm{Spec}(k[w'][x_1])$$

and gluing isomorphism $(w,x_4)\mapsto(w',x_1)=(w^{-1},w^dx_4-w^{d-1})$, hence is a torsor under the line bundle associated to $\mathscr{O}_{\mathbb{P}^1}(-d)$. By Remark 5, the surface W_d isomorphic to $\mathbb{F}_n\setminus B$ for some pair $(\mathbb{F}_n,B)$ as in Lemma 3 b) with $B^2=d$.

The surface W_2 is isomorphic to the surface in $\mathbb{A}^3=\mathrm{Spec}(k[x,y,z])$ given by the equation $xy=z(z+1)$. For $d\geq 3$, the morphism $W_d\to\mathbb{A}^3$, $(x_1,x_2,x_3,x_4)\mapsto(x_1,x_4,x_2)$ has image equal to the non-normal surface V_d given by the equation $x^{d-1}y=z(z+1)^{d-1}$ and the induced morphism $\nu_d:W_d\to V_d$ is finite and birational, expressing W_d as the normalization of V_d. This recovers the other description of representative of surfaces $\mathbb{F}_n\setminus B$ such that $B^2=d$ as normalizations of surfaces of the form V_d given in [19, §1.0.8].

2. Families of $\mathbb{A}^1$-fibrations of complete type

Since they have torsion class groups, the affine plane $\mathbb{A}^2=\mathbb{P}^2\setminus L$ and the complements of smooth conics in $\mathbb{P}^2$ do not admit $\mathbb{A}^1$-fibrations over complete curves. In contrast, a surface $\mathbb{F}_n\setminus B$ admits a smooth $\mathbb{A}^1$-fibration $\pi_n|_{\mathbb{F}_n\setminus B}:\mathbb{F}_n\setminus B\to\mathbb{P}^1$. In this section, we are interested in the properties of certain $\mathbb{A}^1$-fibrations $\mathbb{F}_n\setminus B\to\mathbb{P}^1$ with multiple fibers.

Lemma 9. *Let $(\mathbb{F}_n,B)$ be a pair as in Lemma 3 b), let q be a point of B and let $m\geq 1$. Then the linear subsystem $\mathscr{Z}_q(m)$ of the complete linear system $|B+mF|$ consisting of divisors intersecting B with multiplicity B^2+m at q has dimension m. Furthermore, the open subset $\mathscr{U}_q(m)$ of $\mathscr{Z}_q(m)$ consisting of prime divisors is Zariski dense.*

Proof. Put $d=B^2$ and $F_q=\pi_n^{-1}(\pi_n(q))$. Let $\mathscr{I}_B\cong\mathscr{O}_{\mathbb{F}_n}(-B)$ denote the ideal sheaf of B and consider the long exact sequence of cohomology associated to the short exact sequence of coherent sheaves

$$0\to\mathscr{I}_B\otimes\mathscr{O}_{\mathbb{F}_n}(B+mF)\cong\mathscr{O}_{\mathbb{F}_n}(mF)\to\mathscr{O}_{\mathbb{F}_n}(B+mF)\to\mathscr{O}_{\mathbb{F}_n}(B+mF)|_B\cong\mathscr{O}_B(d+m)\to 0$$

Since $H^1(\mathbb{F}_n,\mathscr{O}_{\mathbb{F}_n}(mF))=0$, the map $H^0(\mathbb{F}_n,\mathscr{O}_{\mathbb{F}_n}(B+mF))\to H^0(B,\mathscr{O}_B(d+m))$ is surjective, with kernel of dimension $\dim H^0(\mathbb{F}_n,\mathscr{O}_{\mathbb{F}_n}(mF))=m+1$. It follows that $\mathscr{Z}_q(m)\cong\mathbb{P}^m$. A singular element of $\mathscr{Z}_q(m)$ decomposes as the sum of a prime member $B_{m'}$ of the complete linear system $|B+m'F|$, $0\leq m'<m$, intersecting B with multiplicity $d+m'$ at q and of $(m-m')F_q$. By the same computation as above, these elements form a closed linear subspace $\mathscr{Z}_q(m')\cong\mathbb{P}^{m'}$ of $\mathscr{Z}_q(m)$ and so, $\mathscr{U}_q(m)=\mathscr{Z}_q(m)\setminus\bigcup_{m'=0}^{m-1}\mathscr{Z}_q(m')$ is a dense open subset of $\mathscr{Z}_q(m)$. □

Example 10. Let $\mathbb{F}_0=\mathbb{P}^1_{[u_0:u_1]}\times\mathbb{P}^1_{[v_0:v_1]}$, $\Delta=\{u_1v_0-u_0v_1=0\}$ and let $q=([0:1],[0:1])$. For every $m\geq 1$, denote by $\mathscr{V}_m\subset k[t]$ the m-dimensional vector space of monic polynomials of degree m. Writing $P(u,v)=p(\frac{v}{u})u^m$ for the homogenization of a polynomial $p(t)\in k[t]$, the map which associates to $p\in\mathscr{V}_m$ the section

$$B_{m,p}=\{u_0P(u_0,u_1)v_1-(u_0^{m+1}+u_1P(u_0,u_1))v_0=0\}$$

of $\pi_0=\mathrm{pr}_1$ defines an open immersion $\mathscr{V}_m\to\mathscr{U}_q(m)\subset\mathscr{Z}_q(m)\cong\mathbb{P}^m$. These curves $B_{m,p}$ were considered by Blanc-van Santen [4, Section 2] for the fact that their restrictions $B_{m,p}\cap(\mathbb{F}_0\setminus\Delta)\cong\mathbb{A}^1$ provide examples of non-equivalent closed embeddings of the affine line into the smooth affine quadric surface $\mathbb{F}_0\setminus\Delta$.

Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 b), let $m \geq 2$ and let B_m be a section of π_n corresponding to a closed point of the scheme $\mathscr{U}_q(m)$ for some $q \in B$. Let $\mathscr{L}_{q,B_m} \subset |B+mF|$ be the pencil generated by the divisors B_m and $B+mF_q$.

Lemma 11. *Every member of $\mathscr{L}_{q,B_m}$ other than $B+mF_q$ is a smooth rational curve.*

Proof. Every divisor in the complete linear system $|B+mF|$ has self-intersection B^2+2m. The minimal resolution $\tau : \tilde{\mathbb{F}}_n \to \mathbb{F}_n$ of the rational map $\gamma : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$ defined by $\mathscr{L}_{q,B_m}$ is obtained by performing B^2+2m successive blow-ups with center at q on the successive proper transforms of B_m, with exceptional divisors $E_1, \ldots, E_{B^2+2m}$. Letting $\tau_*^{-1}B_m$ be the proper transform of B_m, the composition $\tilde{\gamma} = \gamma \circ \tau : \tilde{\mathbb{F}}_n \to \mathbb{P}^1$ is the $\mathbb{P}^1$-fibration defined by the complete linear system $|\tau_*^{-1}B_m|$. The total transform of B_m is a rational chain $E_1 \lhd \cdots \lhd E_{B^2+2m} \lhd \tau_*^{-1}B_m$, where E_{B^2+2m} is a section of $\tilde{\gamma}$. Since every singular member of $\mathscr{L}_{q,B_m}$ is the sum of a prime member B' of the linear system $|B+m'F|$ for some $0 \leq m' < m$ and of $(m-m')F_q$, every fiber of $\tilde{\gamma}$ other than that containing the proper transform of $B \cup F_q$ is smooth. This implies that every member of $\mathscr{L}_{q,B_m}$ other than $B+mF_q$ is a smooth rational curve. □

For every $q \in B$, the space of pencils $\mathscr{L}_{q,B_m}$ identifies with a dense open subset $\mathscr{S}_q(m)$ of the projective space $\mathbb{P}^{m-1}$ of lines passing through the point of $\mathscr{Z}_q(m) \setminus \mathscr{U}_q(m)$ corresponding to the reducible divisor $B+mF_q$ and a point of $\mathscr{U}_q(m)$. The linear action on $H^0(\mathbb{F}_n, \mathscr{O}_{\mathbb{F}_n}(B+mF))$ of the algebraic subgroup $\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$ of automorphisms of $\mathbb{F}_n$ preserving $B \cup F_q$ induces an action of $\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$ on $\mathscr{S}_q(m)$. On the other hand, the rational map $\gamma_{q,B_m} : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$ associated to a pencil $\mathscr{L}_{q,B_m}$ restricts to a surjective $\mathbb{A}^1$-fibration $\delta_{q,B_m} : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ having $F_q \cap (\mathbb{F}_n \setminus B) \cong \mathbb{A}^1$ as a unique degenerate fiber, of multiplicity m. We have the following characterization:

Lemma 12. *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 b), let $m \geq 3$ and let B_m and B'_m be sections of $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$ corresponding to points of the scheme $\mathscr{U}_q(m)$ for some $q \in B$. Then the $\mathbb{A}^1$-fibered surfaces $(\mathbb{F}_n \setminus B, \delta_{q,B_m})$ and $(\mathbb{F}_n \setminus B, \delta_{q,B'_m})$ are equivalent if and only if the pencils $\mathscr{L}_{q,B_m}$ and $\mathscr{L}_{q,B'_m}$ belong to the same $\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$-orbit.*

Proof. Let $\Psi : (\mathbb{F}_n \setminus B, \delta_{q,B_m}) \to (\mathbb{F}_n \setminus B, \delta_{q,B'_m})$ be an equivalence of $\mathbb{A}^1$-fibered surfaces, let $\bar{\Psi}$ be its extension to a birational automorphism of $\mathbb{F}_n$ and let $\mathbb{F}_n \xleftarrow{\sigma} Y \xrightarrow{\sigma'} \mathbb{F}_n$ be the minimal resolution of $\bar{\Psi}$. If σ' is not an isomorphism, then σ is not an isomorphism and Ψ contracts B onto a point. By the minimality assumption, the proper transform $\sigma_*^{-1}(B)$ of B is the only σ'-exceptional (-1)-curve contained in $\sigma^{-1}(B)$. It follows that σ has B^2+1 proper or infinitely near base points on B and hence, since $B \cdot B_m = B^2+m$, that $\sigma_*^{-1}B_m \cdot \sigma_*^{-1}(B) \geq m-1 \geq 2$. But then, $\sigma'(\sigma_*^{-1}B_m)$ is an irreducible singular member of $\mathscr{L}_{q,B'_m}$, which is impossible by Lemma 11. Thus, σ' is an isomorphism and $\bar{\Psi}$ is an automorphism of $\mathbb{F}_n$, which preserves B and the closure F_q of the unique common degenerate fiber $F_q \cap (\mathbb{F}_n \setminus B)$ of δ and δ'. Furthermore, $\bar{\Psi}$ maps B_m onto a certain smooth member of $\mathscr{L}_{q,B'_m}$, hence maps $\mathscr{L}_{q,B_m}$ onto $\mathscr{L}_{q,B'_m}$. The converse implication is clear. □

Remark 13. For every $m \geq 2$ and every point $q \in B$, one can find distinct points B_m and B'_m in the scheme $\mathscr{U}_q(m)$ such that the pencils $\mathscr{L}_{q,B_m}$ and $\mathscr{L}_{q,B'_m}$ have distinct general members. The associated $\mathbb{A}^1$-fibrations δ_{q,B_m} and δ_{q,B'_m} have distinct general fibers but share $F_q \cap (\mathbb{F}_n \setminus B)$ as a degenerate fiber. This contrasts with $\mathbb{A}^1$-fibrations of affine type for which no curve can be contained simultaneously in fibers of two $\mathbb{A}^1$-fibrations with distinct general fibers, see [10, Corollary 2.22].

Proposition 14. *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 b). Then for every $m \geq 4$, there exist infinitely many equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ with a unique degenerate fiber of multiplicity m.*

Proof. By Theorem 4, it suffices to construct such families from the two pairs $(\mathbb{F}_n, B) = (\mathbb{F}_0, \Delta_e)$ and $(\mathbb{F}_1, \sigma_*^{-1}C_e)$ of Example 6. If $B^2 = 2e \geq 2$, let $q = ([1:0],[0:1]) \in \Delta_e = \{u_1^e v_0 - u_0^e v_1 = 0\} \subset \mathbb{F}_0$. The group $\mathrm{Aut}(\mathbb{F}_0, \Delta_e \cup F_q)$ is isomorphic to the affine group $\mathbb{G}_m \ltimes \mathbb{G}_a$ acting by

$$(\lambda, t) \cdot ([u_0:u_1],[v_0:v_1]) = ([\lambda u_0 + t u_1 : u_1], [v_0 + \lambda^{-1} t v_1 : \lambda^{-1} v_1])$$

if $e=1$ and for every $e \geq 2$ to the group $\mathbb{G}_m$ acting by $\lambda \cdot ([u_0:u_1],[v_0:v_1]) = ([\lambda u_0 : u_1], [v_0 : \lambda^{-e} v_1])$. If $B^2 = 2e+1 \geq 3$, viewing $\mathbb{F}_1$ as the blow-up $\sigma : \mathbb{F}_1 \to \mathbb{P}^2_{[x:y:z]}$ of the point $p = [0:1:0]$ with exceptional divisor C_0, let q be the intersection point of $\sigma_*^{-1}C_e$ with the proper transform of the tangent line $L = \{z=0\}$ to $C_e = \{yz^e + x^{e+1} = 0\}$ at p. The group $\mathrm{Aut}(\mathbb{F}_0, \sigma_*^{-1}C_e \cup F_q)$ is then isomorphic to the group $\mathrm{Aut}(\mathbb{P}^2, C_e \cup L)$. The latter is isomorphic to $\mathbb{G}_m \ltimes \mathbb{G}_a$ acting by $(\lambda, t) \cdot [x:y:z] = [\lambda x + tz : \lambda^2 y - 2\lambda t x - t^2 z : z]$ if $e=1$ and for every $e \geq 2$ to $\mathbb{G}_m$ acting by $\lambda \cdot [x:y:z] = [\lambda x : \lambda^{e+1} y : z]$.

In both cases, the $\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$-orbit of a point of the open subset $\mathscr{S}_q(m) \subset \mathbb{P}^{m-1}$ is at most 2-dimensional. Since $m-1 \geq 3$, the set-theoretic orbit space $\mathscr{S}_q(m)/\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$ is infinite and the assertion follows from Lemma 12. □

3. Equivalence classes of $\mathbb{A}^1$-fibrations of affine type

3.1. **Special pencils of rational curves and associated $\mathbb{A}^1$-fibrations of affine type.** Let (X, B) be a pair as in Lemma 3. For every point $q \in B$, denote by $\mathscr{P}_q$ the linear subsystem of the complete linear system $|B|$ on X consisting of curves

with local intersection number with B at q equal to B^2. If $(X,B)\cong(\mathbb{P}^2,L)$ where L is a line, then $\mathscr{P}_q$ is simply the pencil of lines through q. More generally, if $B^2\geq 2$ then the same type of computation as in the proof of Lemma 11 implies that $\mathscr{P}_q$ is a pencil. The minimal resolution $\sigma:\tilde{X}\to X$ of the rational map $\rho_q:X\dashrightarrow\mathbb{P}^1$ defined by $\mathscr{P}_q$ is obtained by performing B^2 successive blow-ups with center at q on the successive proper transforms of B, with respective exceptional divisor $E_1,\ldots,E_{B^2}$. The total transform of B in $\tilde{X}$ is a rational chain $\sigma_*^{-1}B\lhd E_{B^2}\lhd E_{B^2-1}\lhd\cdots E_1$ of type $[0,-1,-2,\ldots,-2]$ and the morphism $\tilde{\rho}_q=\rho_q\circ\sigma:\tilde{X}\to\mathbb{P}^1$ is the $\mathbb{P}^1$-fibration defined by the complete linear system $|\sigma_*^{-1}B|$.

Example 15. For every point q on a smooth conic $Q\subset\mathbb{P}^2$, $\mathscr{P}_q$ is the pencil of conics intersecting Q with multiplicity 4 at q, generated by Q and twice its tangent line T_qQ at q. The total transform of $Q\cup T_qQ$ by σ is the rational tree

$$\begin{array}{ccccccc}(\sigma_*^{-1}Q,0)&\lhd&(E_4,-1)&\lhd&(E_3,-2)&\lhd&(E_2,-2)-(\sigma_*^{-1}T_qQ,-1)\\&&&&&&|\\&&&&&&(E_1,-2),\end{array}$$

where the displayed numbers in the parenthesis are the self-intersection numbers of the corresponding irreducible components. The $\mathbb{P}^1$-fibration $\tilde{\rho}_q:\tilde{\mathbb{P}}^2\to\mathbb{P}^1$ has $\tilde{\rho}_q^{-1}(\rho_q(T_qQ))=\bigcup_{i=1}^3E_i\cup\sigma_*^{-1}T_qQ$ as a unique singular fiber.

For pairs $(\mathbb{F}_n,B)$ as in Lemma 3 b), we have the following description (see also [13, Section 3]):

Lemma 16. *Let $(\mathbb{F}_n,B)$ be a pair as in Lemma 3 b). Then for every point $q\in B$, the following hold:*

a) The pencil $\mathscr{P}_q$ has a unique singular member consisting of a divisor of the form $C+m_qF_q$, where C is a section of π_n, $F_q=\pi_n^{-1}(\pi_n(q))$ and $m_q\in\{1,\ldots,B^2-1\}$.

b) The $\mathbb{P}^1$-fibration $\tilde{\rho}_q:\tilde{\mathbb{F}}_n\to\mathbb{P}^1$ has $\tilde{\rho}_q^{-1}(\rho_q(C\cup F_q))=\bigcup_{i=1}^{B^2-1}E_i\cup\sigma_*^{-1}(C\cup F_q)$ as a unique singular fiber.

c) For a general point $q\in B$, the unique singular member of $\mathscr{P}_q$ is reduced.

Proof. With the notation of subsection 1.2, put $d=B^2\geq 2$ and $B\sim C_0+\ell F$ with $\ell=\frac{1}{2}(d+n)\leq d-1$. Since $d\geq 2$ and $E_{d-1}^2=-2$, $\tilde{\rho}_q^{-1}(\tilde{\rho}_q(E_{d-1}))$ is a singular fiber of $\tilde{\rho}_q$ and its image by σ is a singular member of $\mathscr{P}_q$. Since a singular member of $\mathscr{P}_q$ decomposes as the union of a section $C\sim C_0+\ell' F$ of π_n for some $0\leq\ell'<\ell\leq d-1$ intersecting B with multiplicity $d-(\ell-\ell')$ at q and of $(\ell-\ell')F_q$, it follows that $\sigma(\tilde{\rho}_q^{-1}(\tilde{\rho}_q(E_{d-1})))=C+(\ell-\ell')F_q$ is the unique singular member of $\mathscr{P}_q$. This proves a) and b). For assertion c), see [7, Proposition 4.8.11] or [12, Lemma 3.2]. □

For a pencil $\mathscr{P}_q$ on a pair (X,B) as in Lemma 3, the rational map $\rho_q:\mathbb{F}_n\dashrightarrow\mathbb{P}^1$ defined by $\mathscr{P}_q$ restricts to an $\mathbb{A}^1$-fibration $\pi_q:X\setminus B\to\mathbb{A}^1=\mathbb{P}^1\setminus\rho_q(B)$. In the case where $(X,B)=(\mathbb{P}^2,L)$ for some line L, π_q is a trivial $\mathbb{A}^1$-bundle on $\mathbb{A}^2=\mathbb{P}^2\setminus L$, and in the case where $(X,B)=(\mathbb{P}^2,Q)$, π_q has a unique degenerate fiber consisting of $T_qQ\cap(\mathbb{P}^2\setminus Q)\cong\mathbb{A}^1$ occurring with multiplicity 2 (see Example 15). For pairs $(\mathbb{F}_n,B)$, it follows from Lemma 16 a) that $\pi_q:\mathbb{F}_n\setminus B\to\mathbb{A}^1$ has unique degenerate fiber which is reducible, consisting of the disjoint union of $C\cap(\mathbb{F}_n\setminus B)\cong\mathbb{A}^1$ occurring with multiplicity 1 and of $F_q\cap(\mathbb{F}_n\setminus B)\cong\mathbb{A}^1$ occurring with a certain multiplicity $m_q\in\{1,\ldots,B^2-1\}$. The following lemma shows conversely that for a pair (X,B) as above, every $\mathbb{A}^1$-fibration $\pi:X\setminus B\to\mathbb{A}^1$ is induced by a pencil $\mathscr{P}_q$ on a suitable smooth completion (X',B') of $X\setminus B$.

Lemma 17. *Let (X,B) be a pair as in Lemma 3 and let $\pi:X\setminus B\to\mathbb{A}^1$ be an $\mathbb{A}^1$-fibration. Then there exists a smooth completion (X',B') of $X\setminus B$ by some pair as in Lemma 3, an isomorphism $\Psi:X\setminus B\to X'\setminus B'$ and a point $q'\in B'$ such that $\pi=\pi_{q'}\circ\Psi$.*

Proof. Note that $X\setminus B$ admits an SNC completion by a rational chain of type $[0,-1]$ if $(X,B)\cong(\mathbb{P}^2,L)$ and of type $[0,-1,-2,\ldots,-2]$ with $B^2-1\geq 1$ curves of self-intersection number -2 otherwise. Now let (Y,D) be a relatively minimal SNC completion of $\pi:X\setminus B\to\mathbb{A}^1$ into a $\mathbb{P}^1$-fibration $\bar{\pi}:Y\to\mathbb{P}^1$ as in subsection 1.1.2 (ii). By [10, Proposition 2.15], D is a rational chain $F_\infty\lhd H\lhd E$, where $F_\infty\cong\mathbb{P}^1$ is the fiber of $\bar{\pi}$ over the point $\mathbb{P}^1\setminus\mathbb{A}^1$, H is a section of $\bar{\pi}$ and E is either the empty divisor or a rational chain $E_1\lhd\cdots\lhd E_{d-1}$ consisting of curves with self-intersection number ≤-2 contained in a fiber of $\bar{\pi}$. By making elementary transformations consisting of the blow-up of a point of F_∞ followed by the contraction of the proper transform of F_∞, we can further assume from the beginning that $H^2=-1$. By [6, Corollary 2] (see also [16, Corollary 3.32] or [2, Corollary 3.2.3]), the number of irreducible components of E and their self-intersection numbers are independent on (Y,D). Thus, D is a chain of one of the types listed above and so, letting $\tau:Y\to X'$ be the contraction of the subchain $H\lhd E$ onto a smooth point $q'\in B'=\tau(F_\infty)\cong\mathbb{P}^1$, we obtain a smooth completion (X',B') of $X\setminus B$ and an isomorphism $\Psi=\tau|_{Y\setminus D}:X\setminus B\cong Y\setminus D\to X'\setminus B'$ such that $\pi=\pi_{q'}\circ\Psi$. □

Corollary 18. *The affine plane $\mathbb{A}^2$, the complement $\mathbb{P}^2\setminus Q$ of a smooth conic $Q\subset\mathbb{P}^2$ and the affine quadric surface $\mathbb{P}^1\times\mathbb{P}^1\setminus\Delta$ all have a unique equivalence class of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$.*

Proof. In each case, given an $\mathbb{A}^1$-fibration $\pi:X\setminus B\to\mathbb{A}^1$, Lemma 17 provides a smooth completion (X',B') of $X\setminus B$ such that $\pi=\pi_{q'}\circ\Psi$ for some isomorphism $\Psi:X\setminus B\to X'\setminus B'$ and some point $q'\in B'$. If $(X,B)=(\mathbb{P}^2,L)$ or $(\mathbb{P}^2,Q)$

then $X' = \mathbb{P}^2$ and B' is respectively a line L' or a smooth conic Q'. If $(X,B) = (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ then $X' = \mathbb{P}^1 \times \mathbb{P}^1$ and B' is a prime divisor of type $(1,1)$. The assertion then follows from the fact that in each case, the automophism group of the surface $X' = X$ acts transitively on the set of pairs (B', q'). □

Corollary 19. *Every $\mathbb{A}^1$-fibration $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ on an affine surface $\mathbb{F}_n \setminus B$ has a unique degenerate fiber which consists of the disjoint union of a reduced irreducible component and an irreducible component of multiplicity $m \in \{1, \ldots, B^2 - 1\}$.*

In contrast to Corollary 18, the following lemma, whose proof reproduces that of [9, Theorem 16.2.1] (in french), implies that every affine surface $\mathbb{F}_n \setminus B$ with $B^2 \geq 3$ admits more than one equivalence class of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$.

Lemma 20. *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 b). Then for every integer $m \in \{1, \ldots, B^2 - 1\}$, there exists an $\mathbb{A}^1$-fibration $\pi_m : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ whose degenerate fiber has a reduced component and a component of multiplicity m.*[1]

Proof. We use the notation introduced in subsection 1.2 and put $d = B^2 \geq 2$. Let $n' = d - 2i$ for some $i = 1, \ldots \lfloor \frac{d}{2} \rfloor$, let $C_{n'}$ be a prime member of the complete linear system $|C_0 + n'F|$ and let $q_0 \in C_0$ and $q_{n'} \in C_{n'}$ be a pair of closed points contained in two different fibers F_{q_0} and $F_{q_{n'}}$ of $\pi_{n'} : \mathbb{F}_{n'} \to \mathbb{P}^1$. Note that $B' \cdot C_0 = i$ and $B' \cdot C_{n'} = d - i$ for every member B' of $|C_0 + (d-i)F|$. Applying a sequence of i elementary transformations with center at q_0 followed by a sequence of $d - i$ elementary transformations with center at $q_{n'}$ yields a birational map $\beta : \mathbb{F}_{n'} \dashrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ such that $\pi_{n'} = \mathrm{pr}_1 \circ \beta$. The composition $\mathrm{pr}_2 \circ \beta : \mathbb{F}_{n'} \dashrightarrow \mathbb{P}^1$ is given by a pencil $\mathscr{L} \subset |C_0 + (d-i)F|$ whose general members are sections B' of $\pi_{n'}$ which satisfy $B' \cap C_0 = q_0$ and $B' \cap C_{n'} = q_{n'}$. Since $(B')^2 = 2(d-i) - (d-2i) = d$, $\mathbb{F}_{n'} \setminus B'$ is isomorphic to $\mathbb{F}_n \setminus B$ by Theorem 4. On other hand, the pencil $\mathscr{P}_{q_0}$ has a unique singular member equal to $C_0 + (d-i)F_{q_0}$ whereas the pencil $\mathscr{P}_{q_{n'}}$ has a unique singular member equal to $C_{n'} + iF_{q_{n'}}$. The degenerate fibers of the associated $\mathbb{A}^1$-fibrations $\pi_{q_0} : \mathbb{F}_{n'} \setminus B' \to \mathbb{A}^1$ and $\pi_{q_{n'}} : \mathbb{F}_{n'} \setminus B' \to \mathbb{A}^1$ have $F_{q_0} \cap (\mathbb{F}_{n'} \setminus B')$ and $F_{q_{n'}} \cap (\mathbb{F}_{n'} \setminus B')$ as irreducible components of multiplicity $d - i$ and i respectively. Since i ranges from 1 to $\lfloor \frac{d}{2} \rfloor$, the assertion follows. □

3.2. **Some classes of $\mathbb{A}^1$-fibrations of affine type on surfaces $\mathbb{F}_n \setminus B$.** By Corollary 19 and Lemma 20, the classification of equivalences classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ is divided into that of each type according to the multiplicity $m \in \{1, \ldots, B^2 - 1\}$ of the possibly non-reduced irreducible component of their unique degenerate fiber. Hereafter, we first recall known results on the two extremal cases: $\mathbb{A}^1$-fibrations with a component of maximal multiplicity $B^2 - 1$ on the one hand, and smooth $\mathbb{A}^1$-fibrations on the other hand. We then proceed to the study of equivalence classes of $\mathbb{A}^1$-fibrations with a component of multiplicity two in their unique degenerate fiber.

3.2.1. *Equivalence classes of $\mathbb{A}^1$-fibrations with maximal multiplicity.*

Proposition 21. *For every pair $(\mathbb{F}_n, B)$ as in Lemma 3 b), the affine surface $\mathbb{F}_n \setminus B$ has a unique equivalence class of $\mathbb{A}^1$-fibration $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ with a degenerate fiber containing an irreducible component of multiplicity $B^2 - 1$.*

Proof. A pair $(\mathbb{F}_n, B)$ with $B^2 = d + 2 \geq 2$ such that B contains a point q for which the singular member of the pencil $\mathscr{P}_q$ has the form $C + (d+1)F_q$ for some irreducible curve C is necessarily equal to $(\mathbb{F}_d, B)$ for some section $B \sim C_0 + (d+1)F$ of π_d intersecting C_0 transversely at the point q, the curve C being then equal to C_0. Then the assertion follows from the fact that the group $\mathrm{Aut}(\mathbb{F}_d)$ acts transitively on the set of sections $B \sim C_0 + (d+1)F$ of π_d. Let us recall the argument. Since the restriction homomorphism $\mathrm{Aut}(\mathbb{F}_d, C_0) \to \mathrm{Aut}(C_0)$ is surjective, it suffices to show that for some chosen fiber F_0 of π_d, the action of $\mathrm{Aut}(\mathbb{F}_d, C_0 \cup F_0)$ on the set of sections $B_0 \sim C_0 + (d+1)F$ intersecting C_0 at the point $F_0 \cap C_0$ is transitive. Identifying $\mathbb{F}_d \setminus (C_0 \cup F_0)$ to $\mathbb{A}^2 = \mathrm{Spec}(k[x,y])$ in such a way that $\pi_d|_{\mathbb{A}^2} = \mathrm{pr}_x$ and that the closures in $\mathbb{F}_d$ of the level sets of y are sections of π_d linearly equivalent to $C_0 + dF$, these sections B_0 are the closures in $\mathbb{F}_d$ of curves $\Gamma_p \subset \mathbb{A}^2$ defined by equations of the form $y = p(x)$ where $p(x) \in k[x]$ is a polynomial of degree $d+1$. Since every automorphism of $\mathbb{A}^2$ of the form $(x,y) \mapsto (\lambda x + \mu, \nu y + r(x))$, where $r(x) \in k[x]$ is a polynomial of degree at most d, extends to an element of $\mathrm{Aut}(\mathbb{F}_d, C_0 \cup F_0)$, it follows that every section B_0 belongs to the $\mathrm{Aut}(\mathbb{F}_d, C_0 \cup F_0)$-orbit of the closure of the curve $\Gamma_{x^{d+1}} = \{y = x^{d+1}\}$. □

3.2.2. *Equivalence classes of smooth $\mathbb{A}^1$-fibrations.* The following lemma is a reformulation of [7, Lemma 5.5.5], which appeared, stated in a different language, in the Appendix of [12].

Lemma 22. *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 b) and let $q \in B$ be a point such that the singular member of the pencil $\mathscr{P}_q$ is reduced. Then the isomorphism type of the $\mathbb{A}^1$-fibered surface $\pi_q : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ depends only on the integer B^2.*

Proof. Put $d = B^2$ and $S = \mathbb{F}_n \setminus B$. The singular member of $\mathscr{P}_q$ has the form $C + F_q$ where C is a prime member of the complete linear system $|B - F_q|$. Without loss of generality, we can fix an isomorphism $A = \mathbb{P}^1 \setminus \rho_q(B) \cong \mathrm{Spec}(k[x])$

[1] In particular, there exist at least $B^2 - 1$ equivalence classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ on $\mathbb{F}_n \setminus B$. The lower bound $\left\lfloor \frac{B^2-1}{2} \right\rfloor$ was discovered earlier by Peter Russell (unpublished) and was proven by Flenner-Kaliman-Zaidenberg [17, Corollary 5.16 a)] using a closely related construction.

so that $\rho_q(C\cup F_q)=\{0\}\in A$. With the notation of subsection 3.1, the total transform of $B\cup C\cup F_q$ in the minimal resolution $\sigma:\tilde{\mathbb{F}}_n\to\mathbb{F}_n$ of the rational map map $\rho_q:\mathbb{F}_n\dashrightarrow\mathbb{P}^1$ defined by $\mathscr{P}_q$ is a rational tree of the form

$$\begin{array}{ccccccccccc}(\sigma_*^{-1}B,0) & \triangleleft & (E_d,-1) & \triangleleft & (E_{d-1},-2) & \triangleleft & (E_{d-2},-2) & \triangleleft & \cdots\cdots & \triangleleft & (E_1,-2)-\!\!-(\sigma_*^{-1}F_q,-1)\\ & & & & \vert & & & & & & \\ & & & & (\sigma_*^{-1}C,-1). & & & & & & \end{array}$$

Furthermore, the singular fiber $\tilde{\rho}_q^{-1}(\rho_q(C\cup F_q))=\bigcup_{i=1}^{d-1}E_1\cup\sigma_*^{-1}C\cup\sigma_*^{-1}F_q$ of the $\mathbb{P}^1$-fibration $\tilde{\rho}_q=\rho_q\circ\sigma:\tilde{\mathbb{F}}_n\to\mathbb{P}^1$ is reduced. By contracting successively $\sigma_*^{-1}F_q$, $E_1,\ldots,E_{d-2}$ and $\sigma_*^{-1}C$, we get a birational morphism $\overline{\tau}:\tilde{\mathbb{F}}_n\to\mathbb{F}_1$ of $\mathbb{P}^1$-fibered surfaces over $\mathbb{P}^1$. The later restricts to a morphism

$$\tau:S\cong\tilde{\mathbb{F}}_n\setminus\sigma^{-1}(B)\longrightarrow\mathbb{F}_1\setminus\overline{\tau}(\sigma_*^{-1}(B)\cup E_d)\simeq A\times\mathbb{A}^1$$

of schemes over A, inducing an isomorphism $S\setminus(C\cup F_q)\xrightarrow{\sim}A\setminus\{0\}\times\mathbb{A}^1$ and contracting $C\cap S$ and $F_q\cap S$ to a pair of distinct points supported on $\{0\}\times\mathbb{A}^1$. One can choose a coordinate on the second factor of $A\times\mathbb{A}^1$ and a pair of isomorphisms of A-schemes $S\setminus F_q\simeq A\times\mathrm{Spec}(k[u])$ and $S\setminus C\simeq A\times\mathrm{Spec}([u'])$ so that the restrictions of τ to $S\setminus F_q$ and $S\setminus C$ coincide respectively with the morphisms

$$S\setminus F_q\to A\times\mathbb{A}^1,\ (x,u)\mapsto(x,xu+1)\quad\text{and}\quad S\setminus C\to A\times\mathbb{A}^1,\ \left(x,u'\right)\mapsto(x,x^{d-1}u').$$

Thus, $\pi_q:S\to A$ is A-isomorphic to the surface W_d obtained by gluing two copies $U_\pm=\mathrm{Spec}(k[x][v_\pm])$ of $A\times\mathbb{A}^1$ along the open subset $(A\setminus\{0\})\times\mathbb{A}^1$ by the isomorphism $U_+\ni(x,v_+)\mapsto(x,x^{2-d}v_++x^{1-d})\in U_-$, endowed with the $\mathbb{A}^1$-fibration $\xi_d:W_d\to A$ induced by the first projections on each of the open subsets $U_\pm$. □

Corollary 23. *For every pair $(\mathbb{F}_n,B)$ as in Lemma 3 b), the affine surface $\mathbb{F}_n\setminus B$ has a unique equivalence class of smooth $\mathbb{A}^1$-fibration $\pi:\mathbb{F}_n\setminus B\to\mathbb{A}^1$.*

Remark 24. The proofs of Lemma 16 and Lemma 22 do not depend on the Danilov-Gizatullin isomorphism theorem, and, when combined together, they actually provide a proof of Theorem 4. Indeed, Lemma 16 c) asserts in particular that for every pair $(\mathbb{F}_n,B)$ there exists a point $q\in B$ such that the $\mathbb{A}^1$-fibration $\pi_q:\mathbb{F}_n\setminus B\to\mathbb{A}^1$ associated to the pencil $\mathscr{P}_q$ is a smooth morphism. On the other hand, Lemma 22 implies that the isomorphism type of $\pi_q:\mathbb{F}_n\setminus B\to\mathbb{A}^1$ as an $\mathbb{A}^1$-fibered surface over $\mathbb{A}^1$, hence in particular as an abstract affine surface, depends only on the integer B^2.

3.2.3. *Equivalence classes $\mathbb{A}^1$-fibrations with an irreducible component of multiplicity two.* Given a pair $(\mathbb{F}_n,B)$ as in Lemma 3 b), denote by $\mathscr{A}_2(B^2)$ the set of equivalence classes of $\mathbb{A}^1$-fibrations $\pi:\mathbb{F}_n\setminus B\to\mathbb{A}^1$ whose unique degenerate fiber has an irreducible component of multiplicity two. By Corollary 19 and Lemma 20, $\mathscr{A}_2(2)=\emptyset$ and $\mathscr{A}_2(d)\neq\emptyset$ for every $d\geq 3$. The aim of this subsection is to establish the following more precise description of the sets $\mathscr{A}_2(d)$ for $d\geq 3$.

Proposition 25. *With the notation above, the following hold:*
a) The sets $\mathscr{A}_2(3)$ and $\mathscr{A}_2(4)$ both consist of a single element,
b) The sets $\mathscr{A}_2(5)$ and $\mathscr{A}_2(6)$ both consist of two elements,
c) For every $d\geq 7$, $\mathscr{A}_2(d)$ has cardinality larger than or equal to that of the field k.

The proof follows from a combination of several intermediate results established below. By Lemma 17, for a surface $S=\mathbb{F}_n\setminus B$ every element of $\mathscr{A}_2(B^2)$ is represented by an $\mathbb{A}^1$-fibration $\pi_q:S\to\mathbb{A}^1$ associated to a pencil $\mathscr{P}_q$ on some smooth completion $(\mathbb{F}_{n'},B')$ of S whose unique singular member is a divisor of the form $C+2F_q$ for some prime element C of the complete linear system $|B'-2F|$. The unique degenerate fiber of π_q then consists of the disjoint union of $C\cap S$ with multiplicity one and of $F_q\cap S$ with multiplicity two.

Lemma 26. *Let $(\mathbb{F}_n,B)$ be pair as in Lemma 3 b) with $d=B^2\geq 3$ and such that there exists a point $q\in B$ for which the singular member of the pencil $\mathscr{P}_q$ is a divisor of the form $C+2F_q$ for some prime element C of the complete linear system $|B-2F|$. Then the total transform $\sigma_*^{-1}B\cup D_q\cup\sigma_*^{-1}(C)\cup\sigma_*^{-1}(F_q)$ of $B\cup C\cup F_q$ in the minimal resolution $\sigma:\tilde{\mathbb{F}}_n\to\mathbb{F}_n$ of the rational map $\rho_q:\mathbb{F}_n\dashrightarrow\mathbb{P}^1$ defined by $\mathscr{P}_q$ is a rational tree of the form*

$$\begin{array}{ccccccccccc}(\sigma_*^{-1}B,0) & \triangleleft & (E_d,-1) & \triangleleft & (E_{d-1},-2) & \triangleleft & (E_{d-2},-2) & \triangleleft & \cdots\cdots & \triangleleft & (E_1,-2)-\!\!-(\sigma_*^{-1}F_q,-1)\\ & & & & & & \vert & & & & \\ & & & & & & (\sigma_*^{-1}C,-2). & & & & \end{array}$$

Proof. The assertion is straightforward to verify from the description of $\sigma:\tilde{\mathbb{F}}_n\to\mathbb{F}_n$ given in subsection 3.1. □

Notation 27. Let $(\mathbb{F}_n,B)$ be a pair as in Lemma 3 b) with $d=B^2\geq 3$ and such that there exists a point $q\in B$ for which the singular member of $\mathscr{P}_q$ is a divisor of the form $C+2F_q$. We denote by $\hat{S}_q$ the affine open subset $\mathbb{F}_n\setminus(B\cup C)$ of $S=\mathbb{F}_n\setminus B$ and we denote by $\hat{\pi}_q:\hat{S}_q\to\mathbb{A}^1$ the $\mathbb{A}^1$-fibration with degenerate fiber $F_q\cap\hat{S}_q$ of multiplicity two induced by

$\pi_q : S \to \mathbb{A}^1$. With the notation of Lemma 26, the $\mathbb{P}^1$-fibered surface $\tilde{\rho}_q = \rho_q \circ \sigma : \tilde{\mathbb{F}}_n \to \mathbb{P}^1$ is a relatively minimal SNC completion of $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ with boundary divisor $\hat{D}_q = \sigma_*^{-1}(B) \cup D_q \cup \sigma_*^{-1}C = \sigma_*^{-1}B \cup \bigcup_{i=1}^d E_i \cup \sigma_*^{-1}C$.

Let $(\mathbb{F}_n, B)$ and $(\mathbb{F}_{n'}, B')$ with $B^2 = (B')^2 = d \geq 3$ be smooth completions of an affine surface S such that for some points $q \in B$ and $q' \in B'$, the pencils $\mathscr{P}_q$ and $\mathscr{P}'_q$ have a singular member of the form $C + 2F_q$ and $C' + 2F'_{q'}$ respectively. Let $\pi_q : S \to \mathbb{A}^1$, $\hat{\pi}_q : \hat{S}_q = S \setminus C \to \mathbb{A}^1$, $\pi_{q'} : S \to \mathbb{A}^1$ and $\hat{\pi}_{q'} : \hat{S}_{q'} = S \setminus C' \to \mathbb{A}^1$ be the associated $\mathbb{A}^1$-fibrations. The next lemma reduces the study of equivalence classes of $\mathbb{A}^1$-fibered surfaces (S, π_q) to those of the simpler ones $(\hat{S}_q, \hat{\pi}_q)$.

Lemma 28. *The $\mathbb{A}^1$-fibered surfaces (S, π_q) and $(S, \pi_{q'})$ are equivalent if and only if $(\hat{S}_q, \hat{\pi}_q)$ and $(\hat{S}_{q'}, \hat{\pi}_{q'})$ are equivalent.*

Proof. Since $C \cap S$ (resp. $C' \cap S$) is a reduced fiber of π_q (resp. $\pi_{q'}$) whereas $F_q \cap S$ (resp. $F_{q'} \cap S$) is a fiber of multiplicity two of it, every equivalence of $\mathbb{A}^1$-fibered surface $\Psi : (S, \pi_q) \to (S, \pi_{q'})$ maps $C \cap S$ onto $C' \cap S$ and $F_q \cap S$ onto $F_{q'} \cap S$, hence restricts to equivalence between $\hat{\Psi} : (\hat{S}_q, \hat{\pi}_q) \to (\hat{S}_{q'}, \hat{\pi}_{q'})$. Now assume conversely that there exists an equivalence of $\mathbb{A}^1$-fibered surfaces $\hat{\Psi} : (\hat{S}_q, \hat{\pi}_q) \to (\hat{S}_{q'}, \hat{\pi}_{q'})$. With the notation of Lemma 26, let

$$(\tilde{\mathbb{F}}_n, \hat{D}_q = \sigma_*^{-1}B \cup D_q \cup \sigma_*^{-1}C) \quad \text{and} \quad (\tilde{\mathbb{F}}_{n'}, \hat{D}'_{q'} = {\sigma'}_*^{-1}B' \cup D'_{q'} \cup {\sigma'}_*^{-1}C')$$

be the relatively minimal SNC completions of $\hat{\pi}_q$ and $\hat{\pi}_{q'}$ respectively. The isomorphism $\hat{\Psi}$ induces a birational map of $\mathbb{P}^1$-fibered surfaces $\hat{\Psi} : (\tilde{\mathbb{F}}_n, \tilde{\rho}_q) \dashrightarrow (\tilde{\mathbb{F}}_{n'}, \tilde{\rho}_{q'})$. In particular, $\hat{\Psi}$ maps the section E_d of $\tilde{\rho}_q$ isomorphically onto the section E'_d of $\tilde{\rho}_{q'}$, which implies that any proper base point of $\hat{\Psi}$ is supported either on $D_q \cup \sigma_*^{-1}C$ or on $\sigma_*^{-1}B$. Assume that $\hat{\Psi}$ has a proper base point supported on $D_q \cup \sigma_*^{-1}C$ and let $\tilde{\mathbb{F}}_n \xleftarrow{\eta} Z \xrightarrow{\eta'} \tilde{\mathbb{F}}_{n'}$ be the minimal resolution of $\hat{\Psi}$. Since $\hat{\Psi}$ maps E_d isomorphically onto E'_d and F_q onto $F_{q'}$, it follows that η' contracts $\eta^{-1}(D_q \cup \sigma_*^{-1}C)$ onto $D'_{q'} \cup {\sigma'}_*^{-1}C'$. Since $D_q \cup \sigma_*^{-1}C$ and $D'_{q'} \cup {\sigma'}_*^{-1}C'$ are SNC divisors with the same number of irreducible components, the minimality assumption implies that the proper transform in Z of $D_q \cup \sigma_*^{-1}C$ contains an η'-exceptional (-1)-curve. But this is impossible since all these curves have self-intersection ≤ -2 in Z. For the same reason, $\hat{\Psi}^{-1}$ has no proper base point on $D'_{q'} \cup {\sigma'}_*^{-1}C'$. So, $\hat{\Psi}$ is well-defined on an open neighborhood U of $\hat{D}_q \setminus \sigma_*^{-1}B$ in $\tilde{\mathbb{F}}_n$ and $\hat{\Psi}|_U$ is an isomorphism onto an open neighborhood U' of $\hat{D}_{q'} \setminus {\sigma'}_*^{-1}B'$ in $\tilde{\mathbb{F}}_{n'}$. The geometry of the divisors $\hat{D}_q \setminus \sigma_*^{-1}B$ and $\hat{D}_{q'} \setminus {\sigma'}_*^{-1}B'$ and the fact that $\hat{\Psi}$ maps $\sigma_*^{-1}F_q$ onto ${\sigma'}_*^{-1}F'_{q'}$ imply that $\hat{\Psi}(\sigma_*^{-1}C) = {\sigma'}_*^{-1}C'$ and hence, that $\hat{\Psi}$ induces an equivalence of $\mathbb{A}^1$-fibered surfaces $\Psi : (S, \pi_q) \to (S, \pi_{q'})$. □

To study equivalence classes of $\mathbb{A}^1$-fibered surfaces $(\hat{S}_q, \hat{\pi}_q)$, we now introduce two auxiliary families of surfaces.

Notation 29. For every integer $\ell \geq 1$ and every polynomial $s \in k[x^2] \subset k[x]$ of degree $< \ell$ with $s(0) = 1$, denote by $\tilde{S}_{\ell,s}$ the surface in $\mathbb{A}^3 = \mathrm{Spec}(k[x,y,z])$ with equation $x^\ell z = y^2 - s^2(x)$. The morphism $\tilde{\pi}_{\ell,s} = \mathrm{pr}_x : \tilde{S}_{\ell,s} \to \mathbb{A}^1$ is a smooth $\mathbb{A}^1$-fibration with unique degenerate fiber $\tilde{\pi}_{\ell,s}^{-1}(0)$ consisting of two irreducible components $\{x = y \pm 1 = 0\}$. The morphism $\tilde{\pi}_{\ell,s}$ is equivariant for the actions of the group $\mu_2 = \{\pm 1\}$ given by $(-x, -y, (-1)^\ell z)$ on $\tilde{S}_{\ell,s}$ and by $x \mapsto -x$ on $\mathbb{A}^1$. As a scheme over $\mathbb{A}^1$, $\tilde{S}_{\ell,s}$ is μ_2-equivariantly isomorphic to the surface $W_{2x^\ell s(x)}^{(-1)^{1-\ell}}$ obtained by gluing two copies

$$U_\pm = \tilde{S}_{\ell,s} \setminus \{x = y \mp s(x) = 0\} = \mathrm{Spec}(k[x][u_\pm]), \quad \text{where } u_\pm = x^{-\ell}(y - s(x)) = (y + s(x))^{-1}z$$

of $\mathbb{A}^1 \times \mathbb{A}^1$ over $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$ by the isomorphism $U_+ \ni (x, u_+) \mapsto (x, u_+ + 2x^{-\ell}s(x)) \in U_-$, endowed with the μ_2-action $U_+ \ni (x, u_+) \mapsto (-x, (-1)^{1-\ell}u_+) \in U_-$ and with the $\mathbb{A}^1$-fibration $\theta_{2x^\ell s(x)} : W_{2x^\ell s(x)}^{(-1)^{1-\ell}} \to \mathbb{A}^1$ induced by the first projections on each of the open subsets $U_\pm$.

Notation 30. The categorical quotient $S_{\ell,s} = \tilde{S}_{\ell,s}//\mu_2$ in the category of affine schemes of the fixed point free μ_2-action on $\tilde{S}_{\ell,s}$ is a geometric quotient and the quotient morphism $\Phi_{\ell,s} : \tilde{S}_{\ell,s} \to S_{\ell,s}$ is a nontrivial μ_2-torsor, in particular, $S_{\ell,s}$ is a smooth affine surface. The $\mathbb{A}^1$-fibration $\tilde{\pi}_{\ell,s} : \tilde{S}_{\ell,s} \to \mathbb{A}^1$ descends to an $\mathbb{A}^1$-fibration $\pi_{\ell,s} : S_{\ell,s} \to \mathbb{A}^1//\mu_2 = \mathrm{Spec}(k[x^2])$ with $\pi_{\ell,s}^{-1}(0)$ as a unique degenerate fiber and we have a commutative diagram

$$\begin{array}{ccc} \tilde{S}_{\ell,s} & \xrightarrow{\Phi_{\ell,s}} & S_{\ell,s} = \tilde{S}_{\ell,s}//\mu_2 \\ \downarrow{\scriptstyle \tilde{\pi}_{\ell,s}} & & \downarrow{\scriptstyle \pi_{\ell,s}} \\ \mathbb{A}^1 = \mathrm{Spec}(k[x]) & \xrightarrow{\phi} & \mathbb{A}^1 = \mathrm{Spec}(k[x^2]), \end{array}$$

where $\phi : \mathbb{A}^1 \to \mathbb{A}^1//\mu_2$ is the quotient morphism induced by the inclusion $k[x^2] \subset k[x]$. Since $\Phi_{\ell,s}^{-1}(\pi_{\ell,s}^{-1}(0)) = \tilde{\pi}_{\ell,s}^{-1}(0)$ consists of two component which are exchanged by the μ_2-action and since Φ is étale whereas ϕ is totally ramified

of ramification index 2 over 0, it follows that $\pi_{\ell,s}^{-1}(0)$ is irreducible, of multiplicity 2. The Picard group $\mathrm{Pic}(S_{\ell,s})$ is isomorphic to $\mathbb{Z}_2$, generated by the class of the ideal sheaf of the curve $F_{\ell,s} = (\pi_{\ell,s}^{-1}(0))_{\mathrm{red}}$ of $S_{\ell,s}$.

Lemma 31. *Every smooth affine surface S endowed with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$ whose unique degenerate fiber is irreducible and of multiplicity two is equivalent to an $\mathbb{A}^1$-fibered surface $\pi_{\ell,s} : S_{\ell,s} \to \mathbb{A}^1$ for some $\ell \geq 1$.*

Proof. We can assume that $\mathbb{A}^1 = \mathrm{Spec}(k[t])$ and that $\pi^{-1}(0)$ is the degenerate fiber of π. Let $\phi : A = \mathrm{Spec}(k[x]) \to \mathbb{A}^1$ be the ramified μ_2-cover $x \mapsto t = x^2$ and let $\nu : \tilde{S} \to S \times_{\pi,\mathbb{A}^1,\phi} A$ be the normalization of $S \times_{\pi,\mathbb{A}^1,\phi} A$, endowed with the μ_2-action lifting that on the second factor. By [15, Example 1.6 and Theorem 1.7], $\tilde{\pi} = \mathrm{pr}_1 \circ \nu : \tilde{S} \to A$ is μ_2-equivariantly isomorphic as an A-scheme to an affine surface W_f^ε obtained by gluing two copies $U_\pm = \mathrm{Spec}(k[x][u_\pm])$ of $A \times \mathbb{A}^1$ over $(A \setminus \{0\}) \times \mathbb{A}^1$ by an isomorphism of the form $U_+ \ni (x, u_+) \mapsto (x, u_+ + f(x)) \in U_-$ for some $f \in k[x^{-1}] \setminus k$, endowed with a μ_2-action of the form $U_+ \ni (x, u_+) \mapsto (-x, \varepsilon u_+) \in U_-$, where $\varepsilon = 1$ or -1, viewed as a scheme over A via the $\mathbb{A}^1$-fibration $\theta_f : W_f^\varepsilon \to A$ induced by the first projections on each of the open subsets $U_\pm$. If $\varepsilon = 1$, we have $f(-x) = -f(x)$, which implies that the pole order $\ell = -\mathrm{ord}_0 f$ of f at 0 is odd. The polynomial $\sigma = \frac{1}{2}x^\ell f(x) \in k[x^2] \setminus x^2 k[x^2]$ can be written in the form $\sigma = \lambda(s(x) + x^\ell r(x))$ for some $s \in k[x^2]$ of degree $< \ell$ with $s(0) = 1$ and $\lambda \in k^*$ and the local isomorphisms

$$U_\pm = \mathrm{Spec}(k[x][u_\pm]) \to U'_\pm = \mathrm{Spec}(k[x][u'_\pm]),\ (x, u_\pm) \to (x, \lambda^{-1}(u_\pm \pm r(x)))$$

glue to a μ_2-equivariant isomorphism $W_f^1 \to W^1_{2x^{-\ell}s(x)} \cong \tilde{S}_{\ell,s}$ of A-schemes. The latter induces in turn an isomorphism $S = \tilde{S}//\mu_2 = W_f^1 \cong \tilde{S}_{\ell,s}//\mu_2 = S_{\ell,s}$ of $\mathbb{A}^1$-fibered surfaces over $\mathbb{A}^1$. If $\varepsilon = -1$, then $f(-x) = f(x)$, $\ell = -\mathrm{ord}_0 f$ is even and the same argument shows that $\pi : S \to \mathbb{A}^1$ is isomorphic as a scheme over $\mathbb{A}^1$ to $\pi_{\ell,s} : S_{\ell,s} = W^{-1}_{2x^{-\ell}s(x)}//\mu_2 \to \mathbb{A}^1$ where s is the unique polynomial of degree $< \ell$ with $s(0) = 1$ such that $\frac{1}{2}x^\ell f(x) = \lambda(s(x) + x^\ell r(x)) \in k[x^2] \setminus x^2 k[x^2]$. □

Lemma 32. *Let $(S_{\ell_i,s_i}, \pi_{\ell_i,s_i})$, $i = 1, 2$ be $\mathbb{A}^1$-fibered affine surfaces as in Notation 30. Then the following are equivalent:*

a) The $\mathbb{A}^1$-fibered surfaces $(S_{\ell_1,s_1}, \pi_{\ell_1,s_1})$ and $(S_{\ell_2,s_2}, \pi_{\ell_2,s_2})$ are equivalent,

b) The surfaces S_{ℓ_1,s_1} and S_{ℓ_2,s_2} are isomorphic,

c) There exists $\lambda \in k^*$ such that $s_2(\lambda x) = s_1(x)$.

Proof. The implication a)⇒b) is clear. Now put $S_1 = S_{\ell_1,s_1}$, $S_2 = S_{\ell_2,s_2}$ and assume that there exists an isomorphism $\Psi : S_1 \to S_2$. Since $\mathrm{Pic}(S_2) \cong H^1_{\text{ét}}(S_2, \mathscr{O}^*_{S_2}) \cong \mathbb{Z}_2$ and $H^0(S_2, \mathscr{O}^*_{S_2}) = k^*$, the long exact sequence of étale cohomology associated to the short exact sequence of sheaves of abelian groups

$$1 \to \mu_2 \to \mathscr{O}^*_{S_2} \xrightarrow{f \mapsto f^2} \mathscr{O}^*_{S_2} \to 1$$

implies that $H^1_{\text{ét}}(S_2, \mu_2) = \mathbb{Z}_2$, generated by the class of the μ_2-torsor $\Phi_{\ell,s} : \tilde{S}_{\ell_2,s_2} \to S_2$. Since $\Psi \circ \Phi_{\ell_1,s_1} : \tilde{S}_{\ell_1,s_1} \to S_2$ is a nontrivial μ_2-torsor, it follows that there exists a unique μ_2-equivariant isomorphism $\tilde{\Psi} : \tilde{S}_{\ell_1,s_1} \to \tilde{S}_{\ell_2,s_2}$ such that $\tilde{\Psi} \circ \Phi_{\ell_1,s_1} = \Phi_{\ell_2,s_2} \circ \Psi$. We deduce from [14, Proposition 3.6] that $\ell_1 = \ell_2$ and that there exists a pair $(\lambda, \mu) \in k^* \times k^*$ such that $s_2^2(\lambda x) = \mu^2 s_1^2(x)$. Since $s_1(0) = s_2(0) = 1$, the only possibility is that $\mu = \pm 1$ and the implication b)⇒c) follows. The last implication c)⇒a) follows from the observation that for $\ell_1 = \ell_2 = \ell$ and $s_2(\lambda x) = s_1(x)$, the morphism $\tilde{\Psi} : \tilde{S}_{\ell,s_1} \to \tilde{S}_{\ell,s_2}$ defined by $(x, y, z) \mapsto (\lambda x, y, \lambda^{-\ell} z)$ is a μ_2-equivariant equivalence between the $\mathbb{A}^1$-fibered surfaces $(\tilde{S}_{\ell,s_1}, \tilde{\pi}_{\ell,s_1})$ and $(\tilde{S}_{\ell,s_2}, \tilde{\pi}_{\ell,s_2})$ which descends to an equivalence between $(S_{\ell,s_1}, \pi_{\ell,s_1})$ and $(S_{\ell,s_2}, \pi_{\ell,s_2})$. □

Example 33. For every $\ell \geq 5$, put $m = \left\lfloor \frac{\ell-3}{2} \right\rfloor$, $R = k[a_1, \ldots a_m]$ and $\mathfrak{s}(x) = 1 + x^2 + \sum_{i=2}^m a_i x^{2i} \in R[x^2]$. Let $V = \mathrm{Spec}(R) \cong \mathbb{A}^m$ and let $\mathfrak{S}_\ell$ be the quotient of the closed subscheme $\tilde{\mathfrak{S}}_\ell \subset V \times \mathbb{A}^3$ with equation $x^\ell z = y^2 - \mathfrak{s}^2(x)$ by the $\mu_{2,V}$-action $(x, y, z) \mapsto (-x, -y, (-1)^\ell z)$. By Lemma 32, the closed fibers of the smooth morphism $\Theta : \mathfrak{S}_\ell \to V$ induced by the μ_2-invariant projection $\mathrm{pr}_1 : \tilde{\mathfrak{S}}_\ell \to V$ are pairwise non-isomorphic surfaces of the form $S_{\ell,s}$.

We now relate the family of surfaces $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ of Notation 27 to those $\pi_{\ell,s} : S_{\ell,s} \to \mathbb{A}^1$ of Notation 30.

Lemma 34. *An $\mathbb{A}^1$-fibered affine surface $\pi_{\ell,s} : S_{\ell,s} \to \mathbb{A}^1$ admits a relatively minimal SNC completion $(Y_{\ell,s}, D_{\ell,s})$ into a $\mathbb{P}^1$-fibered surface $\bar{\pi}_{\ell,s} : Y_{\ell,s} \to \mathbb{P}^1$ such that the union of $D_{\ell,s}$ and of the closure $\overline{F}_{\ell,s}$ of $F_{\ell,s}$ is a rational tree of the form*

$$\begin{array}{ccccccccccc} (F_\infty, 0) & \triangleleft & (H, -1) & \triangleleft & (G_0, -2) & \triangleleft & (G_2, -2) & \triangleleft & \cdots\cdots\cdots & \triangleleft & (G_{\ell+1}, -2) - (\bar{F}_{\ell,s}, -1) \\ & & & & & & | & & & & \\ & & & & & & (G_1, -2), & & & & \end{array}$$

where F_∞ is the fiber of $\bar{\pi}_{\ell,s}$ over $\mathbb{P}^1 \setminus \mathbb{A}^1$, H is a section of $\bar{\pi}_{\ell,s}$ and $\bar{\pi}_{\ell,s}^{-1}(0) = \overline{F}_{\ell,s} \cup \bigcup_{i=0}^{\ell+1} G_i$.

Proof. For every $\ell \geq 2$ and every polynomial $s_\ell \in k[x^2]$ of degree $< \ell$ with $s(0) = 1$, write $s_\ell = s_{\ell-1} + a x^{\ell-1}$ where $s_{\ell-1} \in k[x^2]$ is a polynomial of degree $< \ell - 1$ and $a \in k$. The endomorphism $(x, y, z) \mapsto (x, y, xz + 2a s_{\ell-1} + a^2 x^{\ell-1})$ of $\mathbb{A}^3$ induces a μ_2-equivariant birational morphism $\tilde{\sigma} : \tilde{S}_{\ell,s_\ell} \to \tilde{S}_{\ell-1,s_{\ell-1}}$ of $\mathbb{A}^1$-fibered surfaces. It descends to a birational

morphism $\sigma : S_{\ell,s_\ell} \to S_{\ell-1,s_{\ell-1}}$ of $\mathbb{A}^1$-fibered surfaces restricting to an isomorphism over $\mathbb{A}^1 \setminus \{0\}$ and contracting F_{ℓ,s_ℓ} onto a point x_{ℓ,s_ℓ} of $F_{\ell-1,s_{\ell-1}}$. This morphism σ expresses S_{ℓ,s_ℓ} as the surface obtained from $S_{\ell-1,s_{\ell-1}}$ by blowing-up the point x_{ℓ,s_ℓ} and then removing the proper transform of F_{ℓ,s_ℓ}. Assume that $(Y_{\ell-1,s_{\ell-1}}, D_{\ell-1,s_{\ell-1}})$ is a relatively minimal SNC completion of $\pi_{\ell-1,s_\ell} : S_{\ell-1,s_{\ell-1}} \to \mathbb{A}^1$ into a $\mathbb{P}^1$-fibered surface $\overline{\pi}_{\ell-1,s_{\ell-1}} : Y_{\ell-1,s_{\ell-1}} \to \mathbb{P}^1$ which satisfies the claimed properties. Then the pair $(Y_{\ell,s_\ell}, D_{\ell,s_\ell})$, where $\tau : Y_{\ell,s_\ell} \to Y_{\ell-1,s_{\ell-1}}$ is the blow-up of the point $x_{\ell,s_\ell} \in \overline{F}_{\ell-1,s_{\ell-1}}$ and $D_{\ell,s}$ is the proper transform of $D_{\ell-1,s_{\ell-1}} \cup \overline{F}_{\ell-1,s_{\ell-1}}$, endowed the $\mathbb{P}^1$-fibration $\bar{\pi}_{\ell,s_\ell} = \overline{\pi}_{\ell-1,s_{\ell-1}} \circ \tau$ is a relatively minimal SNC completion of $\pi_{\ell,s_\ell} : S_{\ell,s_\ell} \to \mathbb{A}^1$ which also satisfies the claimed properties. Now the assertion follows by induction from Example 15 and the fact that $\pi_{1,1} : S_{1,1} = \{xz = y^2 - 1\}//\mu_2 \to \mathbb{A}^1$ is isomorphic to the complement of the smooth conic $Q = \{-xz + y^2 = 0\}$ in $\mathbb{P}^2_{[x:y,z]}$, endowed with the $\mathbb{A}^1$-fibration associated to the pencil $\mathscr{P}_{[0:0:1]}$. □

By combining Lemma 26, Lemma 31 and Lemma 34, we obtain that every $\mathbb{A}^1$-fibered surface $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ as in Notation 27 is equivalent to some surface of the form $\pi_{d-2,s} : S_{d-2,s} \to \mathbb{A}^1$ of Notation 30, and, conversely, that every equivalence class of $\mathbb{A}^1$-fibered surface $\pi_{d-2,s} : S_{d-2,s} \to \mathbb{A}^1$ is realized by an $\mathbb{A}^1$-fibration $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ induced by restriction of an $\mathbb{A}^1$-fibration on an affine surface $S = \mathbb{F}_n \setminus B$ with $B^2 = d$. Combining in turn this result with Lemma 28, we obtain the following:

Corollary 35. *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 b) with $d = B^2 \geq 3$. Then equivalence classes of $\mathbb{A}^1$-fibered surfaces $((\mathbb{F}_n \setminus B)_q, \pi_q)$ where q ranges through the set of closed points of the boundaries B' of smooth completions $(\mathbb{F}_{n'}, B')$ of $\mathbb{F}_n \setminus B$ such that $\mathscr{P}_q$ has a singular member of the form $C + 2F_q$ are in one-to-one correspondence with equivalence classes of $\mathbb{A}^1$-fibered surfaces $(S_{d-2,s}, \pi_{d-2,s})$ of Notation 30.*

Proposition 25 is now a straightforward consequence of Corollary 35 and of the description of equivalence classes of $\mathbb{A}^1$-fibered surfaces $(S_{d-2,s}, \pi_{d-2,s})$ given in Lemma 32. Namely, for $d = 3, 4$, the unique equivalences classes are those of $(S_{1,1}, \pi_{1,1})$ and $(S_{2,1}, \pi_{2,1})$ respectively. For $d = 5$, the two equivalence classes are those of the surfaces $(S_{3,1}, \pi_{3,1})$ and $(S_{3,x^2+1}, \pi_{3,x^2+1})$. The case $d = 6$ is similar. Finally, if $d \geq 7$, then Example 33 provides a family pairwise non-equivalent $\mathbb{A}^1$-fibered surfaces $(S_{d-2,s}, \pi_{d-2,s})$ parametrized by the elements of k^m, where $m = \lfloor \frac{d-5}{2} \rfloor \geq 1$, showing in particular that the cardinality of $\mathscr{A}_2(d)$ is at least equal to that of k.

Remark 36. The "number of moduli" $m = \lfloor \frac{d-5}{2} \rfloor \geq 1$ for equivalence classes of $\mathbb{A}^1$-fibered surfaces $(S_{d-2,s}, \pi_{d-2,s})$ with a unique singular fiber of multiplicity two deduced from the explicit family in Example 33 is the same as that computed by different techniques in [19], as can be seen by taking $k = 2$ in Corollary 6.3.20 of *loc. cit.*. The results in [19] apply more generally, in particular, to any smooth affine $\mathbb{A}^1$-fibered surface $S \to \mathbb{A}^1$ having a unique singular fiber, irreducible of arbitrary multiplicity $e \geq 2$. On the other hand, it follows from [15] that similarly as in the case $e = 2$ described above, every such surface can be realized as a quotient of smooth affine surface $\tilde{S}$ endowed with a smooth $\mathbb{A}^1$-fibration $\tilde{\pi} : \tilde{S} \to \mathbb{A}^1$ by a suitable free action of a cyclic group μ_e of e-th roots of unity. This suggests the possibility to construct for every $e \geq 2$ explicit families as in Example 33 over a base scheme V whose dimension equals the number of moduli computed in [19, Corollary 6.3.20].

3.3. **Proof of Theorem 2.** In this subsection, we finish the proof of Theorem 2. Let $S = \mathbb{F}_n \setminus B$ for some pair $(\mathbb{F}_n, B)$ as in Lemma 3 b) with $d = B^2 \geq 2$. If $d \geq 7$, then, by Proposition 25, S has infinitely many equivalence classes of $\mathbb{A}^1$-fibrations $\pi : S \to \mathbb{A}^1$. It remains to show that for every $d \leq 6$, the number of equivalence classes is finite. For every $d \geq 2$ and every $m \in 1, \ldots, d-1$, denote by $\mathscr{A}_m(d)$ the set of equivalence classes of $\mathbb{A}^1$-fibrations $\pi : S \to \mathbb{A}^1$ whose unique degenerate fiber has an irreducible component of multiplicity m . The following table summarizes the properties of the sets $\mathscr{A}_m(d)$:

	$\sharp\mathscr{A}_1(d)$	$\sharp\mathscr{A}_2(d)$	$\sharp\mathscr{A}_3(d)$	$\sharp\mathscr{A}_4(d)$	$\sharp\mathscr{A}_5(d)$	$\sum\sharp\mathscr{A}_i(d)$
$d=2$	1	0	0	0	0	1
$d=3$	1	1	0	0	0	2
$d=4$	1	1	1	0	0	3
$d=5$	1	2	1	1	0	5
$d=6$	1	2	2	1	1	7

TABLE 1. Numbers of equivalence classes of $\mathbb{A}^1$-fibrations

Indeed, we have $\mathscr{A}_m(d) = \emptyset$ if $m \geq d$ by Corollary 19. On the other hand, the cardinal $\sharp\mathscr{A}_m(d)$ of $\mathscr{A}_m(d)$ is larger than or equal to 1 for every $1 \leq m \leq d-1$ by Lemma 20. The sets $\mathscr{A}_{d-1}(d)$ and $\mathscr{A}_1(d)$ both consist of a single element by Proposition 21 and Corollary 23 respectively. By Proposition 25, we have $\sharp\mathscr{A}_2(d) = 1$ for $d = 3, 4$ and $\sharp\mathscr{A}_2(d) = 2$ for $d = 5, 6$. These observations settle the cases $d = 2$, 3 and 4. In the next paragraphs, we determine the remaining numbers of equivalence classes of $\mathbb{A}^1$-fibrations displayed in the table. We refer the reader to [13, Section 4] for the details of the reductions to the chosen particular models of pairs which are used in the argument.

3.3.1. *The case $d=5$.* The $\mathbb{A}^1$-fibrations on S representing elements of $\mathscr{A}_3(5)$ can only arise form pencils $\mathscr{P}_q$ on pairs $(\mathbb{F}_1,B)$ for which $B\sim C_0+3F$ intersects C_0 with multiplicity two at a single point q. Up to isomorphism, there is a unique such pair, which is given, under the identification of $\mathbb{F}_1$ with the blow-up $\sigma:\mathbb{F}_1\to\mathbb{P}^2$ of $\mathbb{P}^2$ with homogeneous coordinates $[x:y:z]$ at the point $p=[0:1:0]$ with exceptional divisor C_0, by taking for B the proper transform of the cuspidal cubic $C=\{x^3-z^2y=0\}$ in $\mathbb{P}^2$. The section $B=\sigma_*^{-1}C\sim C_0+3F$ intersects C_0 with multiplicity two at the intersection point q of C_0 with the proper transform of the tangent line $T_pC=\{z=0\}$ to C at p and the pencil $\mathscr{P}_q$ is the proper transform of the pencil generated by C and $3T_pC$. The associated $\mathbb{A}^1$-fibration $\pi_q:S\to\mathbb{A}^1$ has $F_q\cap S$ as a component of multiplicity three in its degenerate fiber. We conclude that $\sharp\mathscr{A}_3(5)=1$.

3.3.2. *The case $d=6$.* The numbers to be computed are $\sharp\mathscr{A}_3(6)$ and $\sharp\mathscr{A}_4(6)$. The possible smooth completions $(\mathbb{F}_n,B)$ of S are either of the form $(\mathbb{F}_0,B)$ where $B\sim C_0+3F$ is a section of π_0, or of the form $(\mathbb{F}_2,B)$ where $B\sim C_0+4F$ is a section of π_2, or the form $(\mathbb{F}_4,B)$ where $B\sim C_0+5F$ is a section of π_4.

The $\mathbb{A}^1$-fibrations on S representing elements of $\mathscr{A}_4(6)$ can arise only from pairs $(\mathbb{F}_2,B)$ for which $B\sim C_0+4F$ intersects C_0 with multiplicity two in a single point. Up to isomorphism, there exists a unique such pair which is given, after fixing a fiber F_∞ of π_2 and an identification $\mathbb{F}_2\setminus(C_0\cup F_\infty)\cong\mathbb{A}^2=\mathrm{Spec}(k[x,y])$ in such way that $\pi_2|_{\mathbb{A}^2}=\mathrm{pr}_x$ and that the closures in $\mathbb{F}_2$ of the level sets of y are sections of π_2 linearly equivalent to C_0+2F, by taking for B the closure of the curve $\Gamma_{x^4}=\{y=x^4\}\subset\mathbb{A}^2$. For the point $q=B\cap C_0=B\cap F_\infty$, the singular member of the pencil $\mathscr{P}_q$ is equal to C_0+4F_∞. The unique degenerate fiber of the corresponding $\mathbb{A}^1$-fibration $\pi_{q_4}:S\to\mathbb{A}^1$ has $F_\infty\cap S$ as a component of multiplicity four and we conclude that $\sharp\mathscr{A}_4(6)=1$.

The $\mathbb{A}^1$-fibrations representing elements of $\mathscr{A}_3(6)$ can arise only from pairs $(\mathbb{F}_0,B)$ on $\pi_0=\mathrm{pr}_1:\mathbb{F}_0=\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$ for which $B\sim C_0+3F$ intersects a fiber of the second projection with multiplicity three at some point q. Up to isomorphisms, there are exactly two such pairs $(\mathbb{F}_0,B_1)$ and $(\mathbb{F}_0,B_2)$ which, using bi-homogeneous coordinates $([u_0:u_1],[v_0:v_1])$ on $\mathbb{P}^1\times\mathbb{P}^1$, are given by the curves $B_1=\{u_1^3v_0+u_0^2(u_0+u_1)v_1=0\}$ and $B_2=\{u_1^3v_0+u_0^3v_1=0\}$. The only fiber of pr_2 which intersects B_1 in a single point is the curve $C_{[1:0]}=\{v_1=0\}$ with $q=C_{[1,0]}\cap B_1=([1:0],[1:0])$. This yields an $\mathbb{A}^1$-fibration $\pi_q:S\to\mathbb{A}^1$ which has $F_{q_1}\cap S$ as component of multiplicity three in its degenerate fiber. In contrast, there are two fibers of pr_2 which intersects B_2 in a single point: the curve $C_{[0:1]}=\{v_0=0\}$ at the point $q_0=([0:1],[0:1])$ and the curve $C_{[1:0]}$ at the point $q_\infty=([1:0],[1:0])$. The $\mathbb{A}^1$-fibrations $\pi_{q_0}:S\to\mathbb{A}^1$ and $\pi_{q_\infty}:S\to\mathbb{A}^1$ associated to the pencils $\mathscr{P}_{q_0}$ and $\mathscr{P}_{q_\infty}$ have $F_{q_0}\cap S$ and $F_{q_\infty}\cap S$ as components of multiplicity three of their respective degenerate fibers, and since the points q_0 and q_∞ belongs to the same orbit of the action of the group $\mathrm{Aut}(\mathbb{F}_0,B_2)\cong\mathbb{G}_m\times\mathbb{Z}_2$, these $\mathbb{A}^1$-fibrations represents a same element of $\mathscr{A}_3(6)$. The next lemma shows that $\mathscr{A}_3(6)$ consists of two elements and completes the proof.

Lemma 37. *The $\mathbb{A}^1$-fibration $\pi_q:S\to\mathbb{A}^1$ is not equivalent to $\pi_{q_0}:S\to\mathbb{A}^1$ (hence not equivalent to $\pi_{q_\infty}:S\to\mathbb{A}^1$).*

Proof. The curve $C_{[1:0]}\cap S\cong\mathbb{A}^1$ is a 3-section of $\pi_{q_0}:S\to\mathbb{A}^1$ which intersects the multiple irreducible component $F_{q_0}\cap S$ of the degenerate fiber of π_{q_0} transversely in a single point. To verify that $\pi_q:S\to\mathbb{A}^1$ is not equivalent to $\pi_{q_0}:S\to\mathbb{A}^1$, it suffices to show that there is no 3-section of $\pi_q:S\to\mathbb{A}^1$ isomorphic to $\mathbb{A}^1$ and intersecting F_q transversely in a single point. Suppose on the contrary that such a 3-section D exists. Let $\sigma:\tilde{\mathbb{F}}_0\to\mathbb{F}_0$ be the minimal resolution of the rational map $\rho_q:\mathbb{F}_0\dashrightarrow\mathbb{P}^1$ defined by $\mathscr{P}_q$. The closure $\bar{D}$ of D in $\tilde{\mathbb{F}}_0$ is a rational 3-section of the $\mathbb{P}^1$-fibration $\tilde{\rho}_q=\rho_q\circ\sigma$ which intersects the proper transform $\sigma_*^{-1}B_1$ of B_1 with multiplicity 3 in a single point p. The total transform of $B_1\cup F_q\cup C_{[1:0]}$ in $\tilde{\mathbb{F}}_0$ being rational tree of the form

$$\begin{array}{ccccccccc}(\sigma_*^{-1}B,0)&\triangleleft&(E_6,-1)&\triangleleft&(E_5,-2)&\triangleleft&(E_4,-2)&\triangleleft&(E_3,-2)\\ &&&&&&&&\vert\\ &&&&&&&&(\sigma_*^{-1}C_{[1:0]},-3),\end{array}\quad\triangleleft\quad(E_2,-2)\triangleleft(E_1,-2)-(\sigma_*^{-1}F_q,-1)$$

there exists a unique birational morphism of $\mathbb{P}^1$-fibered surface $\tau:\tilde{\mathbb{F}}_0\to\mathbb{F}_1$ which contracts $\sigma_*^{-1}F_q\cup\sigma_*^{-1}C_{[1:0]}\cup\bigcup_{i=1}^4E_i$ onto a point $s\in\tau(E_5)\setminus\tau(E_6)$. The curve $\tau(\bar{D})$ is a 3-section of $\pi_1:\mathbb{F}_1\to\mathbb{P}^1$ which has a cusp of multiplicity 2 at s and intersects $\tau(E_5)$ with multiplicity 3 at s. Let C be the image of $\tau(\bar{D})$ by the contraction $\alpha:\mathbb{F}_1\to\mathbb{P}^2$ of $\tau(E_6)$ to a point p'. Assume that $m=\tau(\bar{D})\cdot\tau(E_6)\geq 1$. Then C is a curve of degree $m+3$ which intersects the line $\alpha(\tau(\sigma_*^{-1}B_1))$ with multiplicity $m+3$ at p' and the line $\alpha(\tau(E_5))$ with multiplicity m at p' and multiplicity 3 at $\alpha(s)$. Choosing homogeneous coordinates $[x:y:z]$ on $\mathbb{P}^2$ so that $\alpha(\tau(\sigma_*^{-1}B_1))=\{z=0\}$, $\alpha(\tau(E_5))=\{x=0\}$ and $\alpha(s)=[0:0:1]$, the curve C is thus given by an equation of the form $\lambda x^{m+3}-\mu y^3z^m=0$ for some $\lambda,\mu\in k^*$. But this is impossible since on the other hand $C=\alpha(\tau(\bar{D}))$ has multiplicity 2 at $\alpha(s)$. So $m=0$ and hence, C is a cubic with a cusp at $\alpha(s)$ and intersecting $\tau(\sigma_*^{-1}B)$ with multiplicity 3 at a point other than p'. It follows that $\sigma(\bar{D})$ is a smooth rational curve which intersects F_q transversely at unique point of $F_q\setminus\{q\}$ and B_1 at a unique point of $B_1\setminus\{q\}$, with multiplicity 3. Thus, $\sigma(\bar{D})$ is a fiber of $\mathrm{pr}_2:\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$ which intersects B_1 with multiplicity 3 at a point other than q, which is impossible. □

Acknowledgement. The author is very grateful to Mikhail Zaidenberg and Takashi Kishimito for their helpful comments and valuable suggestions which helped to improve the completeness and the readability of the article.

REFERENCES

1. J. Bertin, *Pinceaux de droites et automorphismes des surfaces affines*, J. Reine Angew. Math. 341 (1983), 32-53.
2. J. Blanc and A. Dubouloz, *Automorphisms of $\mathbb{A}^1$-fibered affine surfaces*, Trans. Amer. Math. Soc. 363 (2011), no. 11, 5887-5924.
3. J. Blanc and A. Dubouloz *Affine surfaces with a huge group of automorphisms*, Int. Math. Res. Not. IMRN 2015, no. 2, 422-459.
4. J. Blanc and I. van Santen, *Embeddings of affine spaces into quadrics*, Trans. Amer. Math. Soc. 371 (2019), no. 12, 8429-8465.
5. P. Cassou-Noguès and P. Russell, *Birational morphisms $\mathbb{C}^2 \to \mathbb{C}^2$ and affine ruled surfaces*, Affine algebraic geometry, 57-105, Osaka Univ. Press, Osaka, 2007.
6. V. I. Danilov, M. H. Gizatullin, *Automorphisms of affine surfaces, I*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 3, 523-565.
7. V. I. Danilov, M. H. Gizatullin, *Automorphisms of affine surfaces, II*, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 1, 54-103.
8. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.
9. A. Dubouloz, *Sur une classe de schémas avec actions de fibrés en droites*, PhD thesis, Grenoble, 2004.
10. A. Dubouloz, *Completions of normal affine surfaces with a trivial Makar-Limanov invariant*, Michigan Math. J. 52 (2004), no. 2, 289-308.
11. A. Dubouloz, *Complements of hyperplane sub-bundles in projective spaces bundles over $\mathbb{P}^1$*. Math. Ann. 361 (2015), no. 1-2, 259-273.
12. A. Dubouloz, D.R. Finston, *On exotic affine 3-spheres*, J. Algebraic Geom. 23 (2014), no. 3, 445-469.
13. A. Dubouloz and S. Lamy, *Automorphisms of open surfaces with irreducible boundary*, Osaka J. Math. 52 (2015), no. 3, 747-791.
14. A. Dubouloz and P.-M. Poloni, *On a class of Danielewski surfaces in affine 3-space*, J. Algebra 321 (2009), no. 7, 1797-1812.
15. K.-H. Fieseler, *On complex affine surfaces with $\mathbb{C}_+$-action*, Comment. Math. Helv. 69 (1994), no. 1, 5-27.
16. H. Flenner, S. Kaliman and M. Zaidenberg, *Birational transformations of weighted graphs*, Affine algebraic geometry, 107–147, Osaka Univ. Press, Osaka, 2007.
17. H. Flenner, S. Kaliman and M Zaidenberg, *Completions of $\mathbb{C}^*$-surfaces*, Affine algebraic geometry, 149-201, Osaka Univ. Press, Osaka, 2007.
18. H. Flenner, S. Kaliman and M. Zaidenberg, *On the Danilov-Gizatullin isomorphism theorem*, Enseign. Math. (2) 55 (2009), no. 3-4, 275-283.
19. H. Flenner, S. Kaliman and M. Zaidenberg, *Smooth affine surfaces with non-unique $\mathbb{C}^*$-actions*, J. Algebraic Geom. 20 (2011), no. 2, 329-398.
20. M. H. Gizatullin, *On affine surfaces that can be completed by a nonsingular rational curve*, Math. USSR Izv. 4 (1970), 787-810.
21. M. H. Gizatullin, *Quasihomogeneous affine surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 1047-1071.
22. J. E. Goodman, *Affine open subsets of algebraic varieties and ample divisors*, Ann. of Math. (2) 89 (1969), 160-183.
23. S. Kovalenko, *Transitivity of automorphism groups of Gizatullin surfaces*, Int. Math. Res. Not. IMRN 2015, no. 21, 11433-11484.
24. S. Kovalenko, A. Perepechko and M. Zaidenberg, *On automorphism groups of affine surfaces*, Algebraic varieties and automorphism groups, 207-286, Adv. Stud. Pure Math., 75, Math. Soc. Japan, Tokyo, 2017.
25. M. Miyanishi, *Open Algebraic Surfaces*, CRM Monogr. Ser., 12, Amer. Math. Soc., Providence, RI, 2001.

IMB UMR5584, CNRS, UNIV. BOURGOGNE FRANCHE-COMTÉ, F-21000 DIJON, FRANCE.
Email address: adrien.dubouloz@u-bourgogne.fr