



HAL
open science

Asymptotic behavior of blowing-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids

Ahmed Bachir, Jacques Giacomoni, Guillaume Warnault

► **To cite this version:**

Ahmed Bachir, Jacques Giacomoni, Guillaume Warnault. Asymptotic behavior of blowing-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids. *Differential and integral equations*, 2022, 35 (9/10), pp.511 - 530. 10.57262/die035-0910-511 . hal-03678617

HAL Id: hal-03678617

<https://hal.science/hal-03678617v1>

Submitted on 6 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Asymptotic behavior of blowing-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids

Ahmed Bachir, Jacques Giacomoni and Guillaume Warnault

December 30, 2021

Abstract

In this paper, we deal with the following quasilinear elliptic system involving gradient terms in the form:

$$\begin{cases} \Delta_p u = v^m |\nabla u|^\alpha & \text{in } \Omega \\ \Delta_p v = v^\beta |\nabla u|^q & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is either equal to \mathbb{R}^N or equal to a ball B_R centered at the origin and having radius $R > 0$, $1 < p < \infty$, $m, q > 0$, $\alpha \geq 0$, $0 \leq \beta \leq m$ and $\delta := (p-1-\alpha)(p-1-\beta) - qm \neq 0$. Our aim is to establish the asymptotics of the blowing-up radial solutions to the above system. Precisely, we provide the accurate asymptotic behavior at the boundary for such blowing-up radial solutions. For that, we prove a strong maximal principle for the problem of independent interest and study an auxiliary asymptotically autonomous system in \mathbb{R}^3 .

1 Introduction and main results

This paper deals with a class of quasilinear elliptic system of the following type:

$$\boxed{\text{otoo23}} \quad (1.1) \quad \begin{cases} \Delta_p u = v^m |\nabla u|^\alpha & \text{in } \Omega \\ \Delta_p v = v^\beta |\nabla u|^q & \text{in } \Omega, \end{cases}$$

Here the operator Δ_p represents the standard p -Laplace operator, and $\Omega \subset \mathbb{R}^N (N \geq 2)$ is either equal to \mathbb{R}^N if the solution is global or equal to a ball B_R if the solution blows up at some $\hat{R} \geq R$ with $\hat{R} < \infty$, and the parameters involved in (1.1) satisfy

$$1 < p < \infty, \quad m, q > 0, \quad \alpha \geq 0, \quad 0 \leq \beta \leq m.$$

The global homogeneity parameter related to the system (1.1), δ , verifies additionally

$$\boxed{\text{otoo24}} \quad (1.2) \quad \delta := (p-1-\alpha)(p-1-\beta) - mq \neq 0.$$

Our goal is to determine the behaviour of radial and positive blowing-up solutions at boundary. In this regard, the present paper can be considered as the subsequent continuation of [8], where several results about existence, global behavior and uniqueness are obtained for global solutions to the system (1.2) (see also [25] and [7] for the particular case $p = 2$). The semilinear case $p = 2$, $\alpha = \beta = 0$, $m = 1$, $q = 2$ of problem (1.1) was considered in details by Diaz, Lazzo and Schmidt [6] as a model arising from the study of dynamics of a viscous, heat-conducting fluid. Considering a unidirectional flow, independent of distance in the flow direction under the Boussinesq approximation, the velocity u and the temperature θ satisfy the following system

$$\boxed{\text{otoo140}} \quad (1.3) \quad \begin{cases} u_t - \Delta u = \theta & \text{in } \Omega \times (0, T), \\ \theta_t - \Delta \theta = |\nabla u|^2 & \text{in } \Omega \times (0, T). \end{cases}$$

The source terms θ and $|\nabla u|^2$ represent the buoyancy force and viscous heating, respectively. With the change of variable $v = -\theta$, steady states of (1.3) satisfy the following coupled elliptic equations:

otoo140b

$$(1.4) \quad \begin{cases} \Delta u = v & \text{in } \Omega \times (0, T), \\ \Delta v = |\nabla u|^2 & \text{in } \Omega \times (0, T), \end{cases}$$

which corresponds to the semilinear case $p = 2$, $\alpha = \beta = 0$, $m = 1$, $q = 2$ of Problem (1.1).

Elliptic and parabolic equations with gradient absorption term arise in different models : reaction diffusion equations (see [16]) in non newtonian fluids models, turbulent flows in porous media (see [27]). We refer to [8] for further explanations of physical models. In [8] the authors studied nonconstant positive radial solutions of (1.1), that is, solutions (u, v) which satisfy

- $u, v \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{0\})$ are positive and radially symmetric;
- u and v are not constant in any neighbourhood of the origin;
- u and v satisfy (1.1).

If $\Omega = \mathbb{R}^N$, solutions of (1.1) will be called global solutions.

Theorem 1.1 given below is one of main results in [8], that provides a Keller-Osserman type condition (see [18] and [21]) for system (1.1) and then classifies all nonconstant positive radial solutions in a ball B_R according to their asymptotic behavior at the boundary.

khaled4

Theorem 1.1. *Assume that $\Omega = B_R$, $1 < p < \infty$, $m, q > 0$, $0 \leq \alpha < p - 1$, $0 \leq \beta \leq m$ and $\delta \neq 0$. Then,*

- (i) *There are no positive radial solutions (u, v) with $u(R^-) = \infty$ and $v(R^-) < \infty$.*
- (ii) *All positive radial solutions of (1.1) are bounded i.i.f. $\delta > 0$*
- (iii) *There are positive radial solutions (u, v) of (1.1) with $u(R^-) < \infty$ and $v(R^-) = \infty$ i.i.f. $\delta < -mp - (p - 1 - \beta)$.*
- (iv) *There are positive radial solutions (u, v) of (1.1) with $u(R^-) = v(R^-) = \infty$ i.i.f. $\delta \in [-mp - (p - 1 - \beta), 0)$.*

From the seminal works of [2], Keller-Osserman condition was investigated for a very large class of elliptic equations, systems and inequations. We refer to [9] and [23] for a survey of the corresponding results. The second result in [8] concerns the existence of nonconstant global positive radial solutions of (1.1), which shows that Theorem (1.1)-(ii) is sharp regarding the existence of global solutions:

khaled5

Theorem 1.2. *Assume that $\Omega = \mathbb{R}^N$, $p > 1$, $m, q > 0$, $\alpha \geq 0$, $0 \leq \beta \leq m$ and $\delta \neq 0$. Then, (1.1) admits nonconstant global positive radial solutions if and only if*

otoo31

$$(1.5) \quad 0 \leq \alpha < p - 1 \quad \text{and} \quad \delta > 0.$$

In [8], the asymptotic behavior and the uniqueness of global solutions are also investigated. As it is observed in [8], by taking advantage of the homogeneity of system (1.1), we have that any nonconstant positive radial solution (u, v) of (1.1) in a ball B_R satisfies the following system

otoo35

$$(1.6) \quad \begin{cases} ((u')^{p-1})' + \frac{N-1}{r}(u')^{p-1} = v^m(u')^\alpha & \text{in } (0, R), \\ ((v')^{p-1})' + \frac{N-1}{r}(v')^{p-1} = v^\beta(u')^q & \text{in } (0, R), \\ u'(0) = v'(0) = 0, \quad u, v > 0 & \text{in } (0, R). \end{cases}$$

and u' and v' are increasing on $(0, R)$.

In the present paper, to complement the above results, we focus on the asymptotic behaviour of blowing-up solutions. Thanks to the homogeneity property of the system, (1.6) can be reduced to the problem

otoo36

$$(1.7) \quad \begin{cases} ((u')^{p-1-\alpha})' + \frac{\gamma}{r}(u')^{p-1-\alpha} = \frac{\gamma}{N-1}v^m & \text{in } (0, R), \\ ((v')^{p-1})' + \frac{N-1}{r}(v')^{p-1} = v^\beta(u')^q & \text{in } (0, R), \\ u'(0) = v'(0) = 0, \quad u, v > 0 & \text{in } (0, R), \end{cases}$$

where

$$\text{otoo37} \quad (1.8) \quad \gamma = \frac{(N-1)(p-1-\alpha)}{p-1}.$$

Let (u, v) be a nonconstant positive radially symmetric solution of (1.7) in the ball B_R , then, the pair

$$\text{scal} \quad (1.9) \quad (u_\lambda, v_\lambda) = \left(\lambda^{1-\alpha_0} u\left(\frac{x}{\lambda}\right), \lambda^{-\beta_0} v\left(\frac{x}{\lambda}\right) \right)$$

where $\alpha_0 = \frac{1+\beta-p(m+1)}{\delta}$ and $\beta_0 = -\frac{p(p-1-\alpha)+q}{\delta}$, provides a nonconstant positive radially symmetric solution of (1.7) in the ball $B_{\lambda R}$. This guarantees that in any ball of positive radius there is nonconstant positive radially symmetric solution of (1.7). Hence, thanks to this scaling property, R can be arbitrary, then without loss of generality we fix in the sequel $R = 1$.

Considering $R = 1$, the main result of the present paper yields the asymptotic behavior of the blowing-up solutions of (1.1) namely when (iii) or (iv) holds.

theorem1 **Theorem 1.3.** *Under the conditions of Theorem 1.2, assume $\delta < 0$ and $\alpha_0 > 0$. Then,*

$$u'(r) \sim \frac{\lambda}{(1-r)^{\alpha_0}} \quad \text{and} \quad v(r) \sim \frac{\mu}{(1-r)^{\beta_0}} \quad \text{as } r \rightarrow 1^-$$

where λ and μ are positive constants given in (3.8).

Remark 1.1. *From Theorem 1.3, we get the asymptotics of u when $\alpha_0 \in [1, \infty[$, that is as $r \rightarrow 1^-$, $u(r) \sim \frac{\lambda}{\alpha_0-1}(1-r)^{-\alpha_0+1}$ if $\alpha_0 > 1$ and $u(r) \sim -\lambda \ln(1-r)$ if $\alpha_0 = 1$. $\alpha_0 < 1$ holds when $\delta < -mp - (p-1-\beta)$ and then one falls in the case of (iii) of Theorem 1.1 whereas assertion (iv) is verified as $\alpha_0 \geq 1$.*

To prove theorem 1.3, we perform several changes of variables that reduce (1.6) to an asymptotically autonomous dynamical system for which we appeal the rich theory of cooperative systems (see [11], [15], [12] for further details on the subject). We also use chain-recurrence properties of asymptotically-autonomous flows (see [20]) to identify the ω -limit set of the non autonomous dynamical system. We also need properties concerning equation (1.6) or (1.7) as comparison principle and continuity with respect the initial data. Section 2 is dedicated to prove these tools whereas Section 3 deals the proof of Theorem 1.3. We also give at the end of Section 3 some numerical simulations associated to asymptotics proved in Theorem 1.3.

2 Technical results

More precisely, we study in this section the pairs of positive functions (U, v) which satisfy, for some $R > 0$

$$\text{uniq} \quad (2.1) \quad \begin{cases} (U^{p-1-\alpha})' + \frac{\gamma}{r} U^{p-1-\alpha} = \frac{\gamma}{N-1} v^m & \text{in } (0, R), \\ ((v')^{p-1})' + \frac{N-1}{r} (v')^{p-1} = v^\beta U^q & \text{in } (0, R), \\ v'(0) = U(0) = 0 \text{ and } v \rightarrow +\infty \text{ as } r \rightarrow R. \end{cases}$$

We start with a comparison principle satisfied by (2.1).

lemn **Proposition 2.1.** *Let (U, v) and $(\widehat{U}, \widehat{v})$ be positive functions which satisfy*

$$\text{otooe1} \quad (2.2) \quad \begin{cases} (r^\gamma U^{p-1-\alpha})' \geq \frac{\gamma}{N-1} r^\gamma v^m & \text{for any } r \in (0, R), \\ (r^{N-1} (v')^{p-1})' \geq r^{N-1} v^\beta U^q & \text{for any } r \in (0, R), \\ v(0) = v_0 \text{ and } v'(0) = U(0) = 0 \\ U > 0, v' > 0 \text{ in } (0, R) \end{cases}$$

and

$$(2.3) \quad \begin{cases} \left(r^\gamma \widehat{U}^{p-1-\alpha} \right)' \leq \frac{\gamma}{N-1} r^\gamma \widehat{v}^m & \text{for any } r \in (0, \widehat{R}), \\ \left(r^{N-1} (\widehat{v}')^{p-1} \right)' \leq r^{N-1} \widehat{v}^\beta \widehat{U}^q & \text{for any } r \in (0, \widehat{R}), \\ \widehat{v}(0) = \widehat{v}_0 \text{ and } \widehat{v}'(0) = \widehat{U}(0) = 0 \\ \widehat{U} > 0, \widehat{v}' > 0 \text{ in } (0, \widehat{R}) \end{cases}$$

for some $R, \widehat{R} > 0$.

If $\widehat{v}(0) < v(0)$, then $\widehat{v} < v$ and $\widehat{U} < U$ in $(0, \min(R, \widehat{R}))$.

In addition, assuming that \widehat{v} blows up at \widehat{R} , we have $R < \widehat{R}$.

Proof. Define $\rho^* = \sup\{\rho > 0 \mid \widehat{v}(r) < v(r) \text{ on } (0, \rho)\}$. Since $\widehat{v}(0) < v(0)$, $\rho^* > 0$ hence from the first equations of (2.2)-(2.3), we deduce $U > \widehat{U}$ on $(0, \rho^*)$. Moreover using the second equations of (2.2)-(2.3), we get $v' > \widehat{v}'$ on $(0, \rho^*)$. Thus the mapping $v - \widehat{v}$ is increasing $(0, \rho^*)$ and we deduce $\rho^* = \min(R, \widehat{R})$ and $\widehat{v} < v$ on $(0, \rho^*)$.

Now, assume that \widehat{v} blows up at \widehat{R} . Since $\widehat{v} < v$ on $(0, \min(R, \widehat{R}))$, we have $R \leq \widehat{R}$.

Consider the pair (U_λ, v_λ) where $U_\lambda = \lambda^{-\alpha_0} U(\frac{\cdot}{\lambda})$ and v_λ is defined in (1.9). Thus (U_λ, v_λ) satisfies (2.2) in $(0, \lambda R)$.

Choosing λ such that $\frac{\widehat{v}(0)}{v(0)} \leq \lambda^{-\beta_0} < 1$ then $v_\lambda(0) > \widehat{v}(0)$ and from the first part, we deduce $v_\lambda > \widehat{v}$ on $(0, \min(\lambda R, \widehat{R}))$ and $\lambda R \leq \widehat{R}$. \square

lem2 **Proposition 2.2.** *The mapping $\Phi : v_0 \rightarrow R$, where $(0, R)$ is the maximal interval of a solution (U, v) of (2.1) with $v(0) = v_0$, is continuous on $(0, +\infty)$.*

Proof. Let $(v_{0,n}) \subset \mathbb{R}$ such that $v_{0,n} \rightarrow v_0$ and \cdot . We define (U, v) and R (respectively (U_n, v_n) and R_n) solutions of (2.1) with $v(0) = v_0$ defined on the maximal interval $(0, R)$ (respectively with $v_n(0) = v_{0,n}$ defined on the maximal interval $(0, R_n)$).

Let $\varepsilon > 0$, Then, we define

$$\lambda_\pm = \left(1 \pm \frac{\varepsilon}{v_0} \right)^{-\frac{1}{\beta_0}}$$

and the pair $(U_{\lambda_\pm}, v_{\lambda_\pm}) = \left(\lambda_\pm^{-\alpha_0} U\left(\frac{\cdot}{\lambda_\pm}\right), \lambda_\pm^{-\beta_0} v\left(\frac{\cdot}{\lambda_\pm}\right) \right)$ is solution of (2.1) on $(0, \lambda_\pm R)$ with $v_{\lambda_\pm}(0) = v_0 \pm \varepsilon$. For n large enough, $v_{0,n} \in (v_0 - \varepsilon, v_0 + \varepsilon)$, hence from Lemma 2.1, we have $\lambda_+ R < R_n < \lambda_- R$ and we deduce $R_n \rightarrow R$ as $n \rightarrow +\infty$. \square

Corollary 2.1. *For any $R > 0$, there exist an unique $v_0 > 0$ and an unique pair of positive functions (U, v) solution to (2.1) such that $v(0) = v_0$.*

Proof. Lemma 2.1-2.2 imply that the mapping Φ is decreasing and continuous on $(0, +\infty)$. Thus Φ is invertible on $(0, +\infty)$ and we deduce the uniqueness. \square

form **Remark 2.1.** *Using the scaling property and the previous results, we have, for any $R > 0$, $\Phi^{-1}(R) = \Phi^{-1}(1)R^{-\beta_0}$ and the unique solution (U_R, v_R) on $(0, R)$ of (2.1) is given by*

$$U_R = R^{-\alpha_0} U_1 \left(\frac{\cdot}{R} \right) \quad \text{and} \quad v_R = R^{-\beta_0} v_1 \left(\frac{\cdot}{R} \right)$$

where (U_1, v_1) is the unique solution of (2.1) on $(0, 1)$.

stabi **Remark 2.2.** *Let $\varepsilon > 0$ and let (U, v) and $(U_\varepsilon, v_\varepsilon)$ two pairs of solutions of (2.1) such that $v(0) - \tilde{v}(0) = \pm\varepsilon$. From Remark 2.1, there exists λ_ε such that $\lambda_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$, $\tilde{v}_\varepsilon = \lambda_\varepsilon^{-\beta_0} v\left(\frac{\cdot}{\lambda_\varepsilon}\right)$ and $\tilde{U}_\varepsilon = \lambda_\varepsilon^{-\alpha_0} U\left(\frac{\cdot}{\lambda_\varepsilon}\right)$. By the proof of Lemma 2.1, the sequences $(v_\varepsilon)_\varepsilon$, $(U_\varepsilon)_\varepsilon$ and $(v'_\varepsilon)_\varepsilon$ are monotone and converge everywhere to v , U and v' on the compact sets of $[0, R)$. By Dini's Theorem, we deduce the uniform convergence on the compact sets of $[0, R)$.*

3 Asymptotic behavior and proof of Theorem 1.3

Through this section, we only consider $\delta < 0$, $\alpha_0 > 0$ and $R = 1$. Moreover we have $\alpha_0(p-1-\alpha) + 1$, β_0 and γ_0 are positive. At the end of the section, we give the proof of Theorem 1.3.

Let (u, v) be a pair of nonconstant positive radial solutions of (1.1) given by Theorem 1.1. Then, setting $U = u'$ and $V = v'$ in (1.7), U and V are positive, increasing on $(0, 1)$ and satisfy

$$\text{eq1} \quad (3.1) \quad \begin{cases} (U^{p-1-\alpha})' + \frac{\gamma}{r} U^{p-1-\alpha} = \frac{\gamma}{N-1} v^m & \text{for } r \in (0, 1), \\ (V^{p-1})' + \frac{N-1}{r} V^{p-1} = v^\beta U^q & \text{for } r \in (0, 1), \\ U(0) = V(0) = 0. \end{cases}$$

Now, set

$$U(r) = \frac{\lambda}{(1-r)^{\alpha_0}} a(r), \quad v(r) = \frac{\mu}{(1-r)^{\beta_0}} b(r) \quad \text{and} \quad V(r) = \frac{\nu}{(1-r)^{\gamma_0}} c(r)$$

where $\lambda, \mu, \nu > 0$ to be fixed later. Thus, (3.1) leads to

$$\text{eq2} \quad (3.2) \quad \begin{cases} \left(\left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}} \right)^{p-1-\alpha} \right)' + \frac{\gamma}{r} \left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}} \right)^{p-1-\alpha} = \frac{p-1-\alpha}{p-1} \left(\frac{\mu b(r)}{(1-r)^{\beta_0}} \right)^m & \text{for } r \in (0, 1), \\ \left(\left(\frac{\nu c(r)}{(1-r)^{\gamma_0}} \right)^{p-1} \right)' + \frac{N-1}{r} \left(\frac{\nu c(r)}{(1-r)^{\gamma_0}} \right)^{p-1} = \left(\frac{\mu b(r)}{(1-r)^{\beta_0}} \right)^\beta \left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}} \right)^q & \text{for } r \in (0, 1). \end{cases}$$

Simple computations give $\alpha_0(p-1-\alpha) + 1 = m\beta_0$ and

$$\left(\left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}} \right)^{p-1-\alpha} \right)' = (p-1-\alpha) \lambda^{p-1-\alpha} \frac{(1-r)a'(r) + \alpha_0 a(r)}{(1-r)^{m\beta_0}} a(r)^{p-2-\alpha}.$$

Thus, multiplying the first equation of (3.2) by $(1-r)^{m\beta_0}$ choosing λ and μ such that $\lambda^{p-1-\alpha} \alpha_0(p-1-\alpha) = \frac{\gamma}{N-1} \mu^m$, this yields for any $r \in (0, 1)$

$$\text{eq5} \quad (3.3) \quad (1-r) a^{p-\alpha-2} ((p-1-\alpha)a' + \frac{\gamma}{r} a) = \alpha_0(p-1-\alpha) (b^m - a^{p-1-\alpha}).$$

In the same way, noting that $\gamma_0(p-1)+1 = \beta\beta_0 + \alpha_0 q$ and multiplying the second equation of (3.2) by $(1-r)^{\beta\beta_0 + \alpha_0 q}$, we get

$$\text{eq8} \quad (3.4) \quad (1-r) c^{p-2} ((p-1)c' + \frac{N-1}{r} c) = \gamma_0(p-1) (b^\beta a^q - c^{p-1})$$

where λ, μ and ν satisfy $\mu^\beta \lambda^q = \gamma_0(p-1)\nu^{p-1}$.

Moreover, since $V = v'$, we deduce that

$$\text{eq9} \quad (3.5) \quad \mu(b'(r)(1-r) + \beta_0 b(r)) = \nu c(r).$$

Choosing μ and ν such that $\beta_0 \mu = \nu$, (3.5) becomes

$$\text{eq10} \quad (3.6) \quad b'(r)(1-r) = \beta_0 (c(r) - b(r)).$$

Hence, we deduce that the triplet (a, b, c) satisfies

$$\text{eq11} \quad (3.7) \quad \begin{cases} (1-r) a^{p-\alpha-2} ((p-1-\alpha)a' + \frac{\gamma}{r} a) = \alpha_0(p-1-\alpha) (b^m - a^{p-1-\alpha}), \\ (1-r) c^{p-2} ((p-1)c' + \frac{N-1}{r} c) = \gamma_0(p-1) (b^\beta a^q - c^{p-1}), \\ (1-r)b' = \beta_0 (c-b) \end{cases}$$

under the following relations

$$\lambda^{p-1-\alpha} \alpha_0 = \frac{\mu^m}{p-1}, \quad \mu^\beta \lambda^q = \gamma_0(p-1)\nu^{p-1}, \quad \beta_0 \mu = \nu.$$

which leads to

$$\text{eq13} \quad (3.8) \quad \begin{cases} \lambda = \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{p-1-\beta}{m}} \right)^{-\frac{m}{\delta}}, \\ \mu = \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{q}{p-1-\alpha}} \right)^{-\frac{p-1-\alpha}{\delta}}, \\ \nu = \beta_0 \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{q}{p-1-\alpha}} \right)^{-\frac{p-1-\alpha}{\delta}}. \end{cases}$$

Consider now the following change of variable $r = 1 - e^{-t}$ i.e. $t = -\ln(1-r)$ and we define \mathcal{X} , \mathcal{Y} and \mathcal{Z} as follows

$$\mathcal{X}(t) = a^{p-1-\alpha}(r), \mathcal{Y}(t) = b(r) \text{ and } \mathcal{Z}(t) = c^{p-1}(r).$$

Hence, $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ are positive functions satisfying the following C^1 -asymptotically autonomous system on $[t_0, +\infty)$ for some $t_0 > 0$:

$$\text{eq14} \quad (3.9) \quad \begin{cases} X'(t) + \frac{\gamma}{e^t - 1} X(t) = \alpha_0(p-1-\alpha)(|Y|^{m-1}(t)Y(t) - X(t)) \\ Y'(t) = \beta_0(|Z|^{\frac{1}{p-1}-1}(t)Z(t) - Y(t)) \\ Z'(t) + \frac{N-1}{e^t - 1} Z(t) = \gamma_0(p-1)(|Y|^{\beta-1}(t)Y(t)|X|^{\frac{q}{p-1-\alpha}-1}(t)X(t) - Z(t)) \end{cases}$$

for some $(X(t_0), Y(t_0), Z(t_0)) \in \mathbb{R}^3$. The asymptotic dynamical system associated to (3.9) is given by:

$$\text{2eq14} \quad (3.10) \quad \begin{cases} X'(t) = \alpha_0(p-1-\alpha)(|Y|^{m-1}Y - X) \\ Y'(t) = \beta_0(|Z|^{\frac{1}{p-1}-1}Z - Y) \\ Z'(t) = \gamma_0(p-1)(|Y|^{\beta-1}Y|X|^{\frac{q}{p-1-\alpha}-1}X - Z) \end{cases}$$

The system (3.10) can be rewritten as $\zeta'(t) = g(\zeta)$, $\zeta(t_0) = (X(t_0), Y(t_0), Z(t_0)) \in \mathbb{R}^3$ where

$$\text{eq16} \quad (3.11) \quad \zeta(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} \text{ and } g(\zeta) = \begin{pmatrix} \alpha_0(p-1-\alpha)(|Y|^{m-1}Y - X) \\ \beta_0(|Z|^{\frac{1}{p-1}-1}Z - Y) \\ \gamma_0(p-1)(|Y|^{\beta-1}Y|X|^{\frac{q}{p-1-\alpha}-1}X - Z) \end{pmatrix}.$$

The system (3.10) is cooperative, $\text{div } g < 0$ and we have the following properties.

Proposition 3.1. *The following assertions hold:*

1. $(0, 0, 0)$ and $(1, 1, 1)$ are the only equilibrium points of the system (3.10).
2. $(0, 0, 0)$ is asymptotically stable (and then a sink).
3. $(1, 1, 1)$ is an hyperbolic saddle point.

Proof. Let $P = (X_e, Y_e, Z_e)$ be an equilibrium point of the system (3.10), then $g(P) = 0$ which implies that

$$\text{zizo1} \quad (3.12) \quad \begin{cases} |Y_e|^{m-1}Y_e = X_e, \\ |Z_e|^{\frac{1}{p-1}-1}Z_e = Y_e, \\ |Y_e|^{\beta-1}|X_e|^{\frac{q}{p-1-\alpha}-1}Y_eX_e = Z_e. \end{cases}$$

Proof of 1.: Obviously, $Y_e = 0$ yields $X_e = Z_e = 0$ and solve the system (3.12).

If $Y_e \neq 0$, from (3.12), we deduce that X_e, Y_e and Z_e have the same sign and that $|X_e|^{\frac{q}{p-1-\alpha}-1}X_e = |Y_e|^{p-2-\beta}Y_e$. Thus, the first equation of (3.12) implies that $Y_e^{\frac{\delta}{q}} = 1$. Hence, we get $Y_e = X_e = Z_e = 1$.

Proof of 2.: The assertion follows directly looking the linearized matrix at $(0, 0, 0)$ given by:

$$M_0 = \begin{pmatrix} -\alpha_0(p-1-\alpha) & 0 & 0 \\ 0 & -\beta_0 & 0 \\ 0 & 0 & -\gamma_0(p-1) \end{pmatrix}.$$

Proof of β .: The linearized matrix at $(1, 1, 1)$ is given by

$$M_1 = \begin{pmatrix} -\alpha_0(p-1-\alpha) & m\alpha_0(p-1-\alpha) & 0 \\ 0 & -\beta_0 & \frac{\beta_0}{p-1} \\ \frac{q\gamma_0(p-1)}{p-1-\alpha} & \beta_0\gamma_0(p-1) & -\gamma_0(p-1) \end{pmatrix}.$$

Thus, $\det(\lambda I - M_1) = \lambda^3 + C_1\lambda^2 + C_2\lambda + C_3$ where

$$\begin{aligned} C_1 &= \alpha_0(p-1-\alpha) + \beta_0 + \gamma_0(p-1), \\ C_2 &= \beta_0\gamma_0(p-1-\beta) + \alpha_0(p-1-\alpha)(\beta_0 + \gamma_0(p-1)), \\ C_3 &= \alpha_0\beta_0\gamma_0\delta. \end{aligned}$$

Since $\beta_0 + 1 = \gamma_0$, $\alpha_0(p-1-\alpha) + 1 = \beta_0m$ and $\gamma_0(p-1) + 1 = \beta\beta_0 + \alpha_0q$, we get $C_1 + C_2 + C_3 = -1$. Hence $\lambda_1 = 1$ is an eigenvalue of M_1 .

Let us call λ_2, λ_3 the two other eigenvalues of M_1 . Then we have $\lambda_2 + \lambda_3 = -C_1 - 1 < 0$ and $\lambda_2\lambda_3 = -C_3 = -\alpha_0\beta_0\gamma_0\delta > 0$.

Thus, $Re(\lambda_2), Re(\lambda_3) < 0$ and we deduce that $(1, 1, 1)$ is hyperbolic point with 2-dimensional stable manifold. \square

We have the following properties about the dynamical system (3.9):

pos **Proposition 3.2.** *Let $T > 0$ and $t_0 \in (0, T)$. Assume (X, Y, Z) a triplet of functions satisfying (3.9) on $[t_0, T)$ with $(X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}_+^3$. Then, for any $t \in [t_0, T)$, $(X(t), Y(t), Z(t)) \in \text{Int } \mathbb{R}_+^3$.*

Proof. Define $T^* = \sup\{t > t_0 \mid X(s) > 0, Y(s) > 0 \text{ and } Z(s) > 0 \text{ for any } s \in (0, t)\}$ and assume that $T^* < T$.

From the second equation of (3.9), we deduce that for any $t \in [t_0, T^*)$, $(e^{\beta_0 t} Y(t))' > 0$ and hence $e^{\beta_0 t} Y(t) > e^{\beta_0 t_0} Y(t_0) > 0$. This implies that $Y(T^*) > 0$.

Assuming $X(T^*) = 0$, the first equation of (3.9) yields for $t = T^*$, $X'(T^*) = \alpha_0(p-1-\alpha)Y^m(T^*) > 0$. This implies there exists a small neighbourhood $(T^* - \eta, T^* + \eta)$ of T^* such that X' is positive on $(T^* - \eta, T^* + \eta)$. Since $X(T^*) = 0$, we deduce $X(t) < 0$ on $(T^* - \eta, T^*)$ which contradicts the definition of T^* .

In the same, using the third equation of (3.9), we can not have $Z(T^*) = 0$ and hence we conclude $T^* = T$. \square

marchencore **Remark 3.1.** *Proposition 3.2 holds considering a triplet of functions satisfying the autonomous system (3.10).*

inf1 **Proposition 3.3.** *Let $T > 0$ and $t_0 \in (0, T)$. Let (X, Y, Z) be a triplet of functions satisfying (3.9) on $[t_0, T)$ with $(X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}_+^3$. Assume $X(t_0) < 1, Y(t_0) < 1$ and $Z(t_0) < 1$. Then, for any $t \in [t_0, T)$, $X(t) < 1, Y(t) < 1$ and $Z(t) < 1$.*

Proof. By Proposition 3.2, for any $t \in [t_0, T)$, $(X(t), Y(t), Z(t)) \in \text{Int } \mathbb{R}_+^3$ and hence we deduce

$$\begin{cases} (X-1)'(t) < \alpha_0(p-1-\alpha)(Y^m - 1 - (X-1)), \\ (Y-1)'(t) = \beta_0 \left(Z^{\frac{1}{p-1}} - 1 - (Y-1) \right), \\ (Z-1)'(t) < \gamma_0(p-1) \left(Y^\beta X^{\frac{q}{p-1-\alpha}} - 1 - (Z-1) \right) \end{cases}$$

and we get

name (3.13)
$$\begin{cases} e^{-\alpha_0(p-1-\alpha)t} (e^{\alpha_0(p-1-\alpha)t} (X-1))' < \alpha_0(p-1-\alpha)(Y^m - 1), \\ e^{-\beta_0 t} (e^{\beta_0 t} (Y-1))' = \beta_0 \left(Z^{\frac{1}{p-1}} - 1 \right), \\ e^{-\gamma_0(p-1)t} (e^{\gamma_0(p-1)t} (Z-1))' < \gamma_0(p-1) \left(Y^\beta X^{\frac{q}{p-1-\alpha}} - 1 \right). \end{cases}$$

Define $\tau = \sup\{t > t_0 \mid X(t) < 1, Y(t) < 1 \text{ and } Z(t) < 1 \text{ on } (t_0, t)\}$.

Assume $\tau < T$ then $X(\tau) = 1$ or $Y(\tau) = 1$ or $Z(\tau) = 1$.

In any cases, the third equation of (3.13) at $t = \tau$ implies that $Z(t) < 1$ on the interval $[\tau, \tau + \eta]$ for

some $\eta > 0$. Hence, if $Z(\tau) = 1$, we get a contradiction.

Otherwise, using the previous estimate in the second equation of (3.13), we get $Y(\tau) < 1$ which would contradict $Y(\tau) = 1$. Finally, using $Y(\tau) < 1$ in the first equation of (3.13), this yields $X(\tau) < 1$ and a contradiction with $X(\tau) = 1$. \square

For t_0 small enough, $(\mathcal{X}(t_0), \mathcal{Y}(t_0), \mathcal{Z}(t_0)) \in \text{Int } \mathbb{R}_+^3$, hence the triplet $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ satisfies the C^1 -asymptotically autonomous system

$$\text{eq14b} \quad (3.14) \quad \begin{cases} X'(t) + \frac{\gamma}{e^t - 1} X(t) = \alpha_0(p-1-\alpha)(Y^m(t) - X(t)) & \text{on } [t_0, \infty) \\ Y'(t) = \beta_0(Z^{\frac{1}{p-1}}(t)Z(t) - Y(t)) & \text{on } [t_0, \infty) \\ Z'(t) + \frac{N-1}{e^t - 1} Z(t) = \gamma_0(p-1)(Y^\beta(t)X^{\frac{q}{p-1-\alpha}}(t) - Z(t)) & \text{on } [t_0, \infty) \end{cases}$$

with $(X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}_+^3$. Hence we get the following asymptotic dynamical system:

$$\text{2eq14b} \quad (3.15) \quad \begin{cases} X' = \alpha_0(p-1-\alpha)(Y^m - X) \\ Y' = \beta_0(Z^{\frac{1}{p-1}} - Y) \\ Z' = \gamma_0(p-1)(Y^\beta X^{\frac{q}{p-1-\alpha}} - Z). \end{cases}$$

Now, we study the trajectories $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. The first result is the relatively compactness:

prop1 **Proposition 3.4.** *The solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is bounded as $t \rightarrow +\infty$.*

Proof. Firstly, assume that \mathcal{Y} is bounded, then we have from the first equation of (3.14)

$$\mathcal{X}'(t) + \alpha_0(p-1-\alpha)\mathcal{X} \leq C,$$

which is equivalent to

$$(e^{\alpha_0(p-1-\alpha)t} \mathcal{X})' \leq C e^{\alpha_0(p-1-\alpha)t}.$$

Integrating this last inequality from t_0 to t leads to

$$e^{\alpha_0(p-1-\alpha)t} \mathcal{X}(t) - e^{\alpha_0(p-1-\alpha)t_0} \mathcal{X}(t_0) \leq \frac{C}{\alpha_0(p-1-\alpha)} (e^{\alpha_0(p-1-\alpha)t} - e^{\alpha_0(p-1-\alpha)t_0}).$$

Since \mathcal{X} is positive, this implies that \mathcal{X} is bounded. A similar argument implies that \mathcal{Z} is bounded.

It remains to prove that \mathcal{Y} is actually bounded and for this, we proceed by contradiction.

Consider (\hat{u}, \hat{v}) the solution of (1.7) with $\hat{u}(0) = u(0)$ and $\hat{v}(0) > v(0)$ defined in $(0, \hat{R})$. From Proposition 2.1, it follows that $\hat{R} < 1$. We define $(\hat{X}, \hat{Y}, \hat{Z})$ on $(0, -\ln(1 - \hat{R}))$ the transform of (\hat{u}, \hat{v}) constructed as previously.

We also define $\tilde{X}(t) = \mathcal{X}(t + T_0)$, $\tilde{Y}(t) = \mathcal{Y}(t + T_0)$ and $\tilde{Z}(t) = \mathcal{Z}(t + T_0)$ with T_0 large enough such that $\mathcal{Y}(T_0) > \hat{Y}(0)$ since \mathcal{Y} is unbounded. Then (\tilde{u}, \tilde{v}) the reverse transform of $(\tilde{X}, \tilde{Y}, \tilde{Z})$ satisfies

$$\begin{cases} ((\tilde{u}')^{p-1-\alpha})' + \frac{\gamma}{r + \tilde{r}_0} (\tilde{u}')^{p-1-\alpha} = \frac{\gamma}{N-1} \tilde{v}^m & \text{in } (0, 1), \\ ((\tilde{v}')^{p-1})' + \frac{N-1}{r + \tilde{r}_0} (\tilde{v}')^{p-1} = v^\beta (\tilde{u}')^q & \text{in } (0, 1), \\ \tilde{u}'(0) \geq 0, \tilde{v}'(0) \geq 0, \tilde{u}, \tilde{v} > 0 & \text{in } (0, 1), \end{cases}$$

where $\tilde{r}_0 = \frac{r_0}{1-r_0}$ with $r_0 = 1 - e^{-T_0}$.

Noting (\tilde{u}', \tilde{v}) satisfies (2.2) and $\tilde{v}(0) > \hat{v}(0)$, we apply Proposition 2.1 with (\hat{u}', \hat{v}) , we deduce $\hat{v} < \tilde{v}$ in $(0, \min(1, \hat{R})) = (0, \hat{R})$. Since \hat{v} blows up at \hat{R} , we get the contradiction. \square

From proposition 3.4, we deduce the convergence of the trajectories $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$:

prop2 **Proposition 3.5.** *We have*

$$\lim_{t \rightarrow +\infty} (\mathcal{X}(t), \mathcal{Y}(t), \mathcal{Z}(t)) = (1, 1, 1).$$

For the proof of Proposition 3.5, we need to recall two notions of dynamical systems:

Definition 3.1. A circuit is a finite sequence of equilibria $z_1, z_2, \dots, z_K = z_1$, ($K \geq 2$) such that $W^u(z_i) \cap W^s(z_{i+1}) \neq \emptyset$ where W^u, W^s denote the stable and unstable manifolds of each equilibria.

Definition 3.2. Let $X \subset \mathbb{R}^3$ be a nonempty positively invariant subset for an autonomous semiflow ϕ and $x, y \in X$.

- (i) For $\epsilon > 0$ and $t > 0$, an (ϵ, t) -chain from $x \in X$ to $y \in X$ is a sequence of points in X , $x = x_1, x_2, \dots, x_n, x_{n+1} = y$ and of times $t_1, t_2, \dots, t_n \geq t$ such that $|\phi(t_i, x_i) - x_{i+1}| < \epsilon$.
- (ii) A point $x \in X$ is called chain recurrent if for every $\epsilon > 0$, $t > 0$ there is an (ϵ, t) -chain from x to x in X .
- (iii) The set X is said to be chain recurrent if every point $x \in X$ is chain recurrent in X .

Proof. Setting $\zeta_0 = (\mathcal{X}(t_0), \mathcal{Y}(t_0), \mathcal{Z}(t_0)) \in \text{Int } \mathbb{R}_+^3$, we define the ω -limit set $\omega_0 = \omega(t_0, \zeta_0) \subset \mathbb{R}_+^3$ of ϕ the semiflow of the asymptotically autonomous system (3.14).

From Proposition 3.4, ω_0 is bounded. Thus Theorem 1.8 in [20] implies ω_0 is nonempty, compact and connected. Moreover ω_0 is invariant for the semiflow denoted by ϕ_A associated to the asymptotic autonomous system (3.15), *i.e.* for any $t \geq 0$

$$\phi_A(t, \omega_0) = \omega_0$$

and ω_0 is chain recurrent for ϕ_A .

Finally, ω_0 satisfies $\text{dist}(\phi(t, t_0, \zeta_0), \omega_0) \rightarrow 0$ as $t \rightarrow +\infty$.

Step 1: $\omega_0 \subset \{(0, 0, 0); (1, 1, 1)\}$.

For that we argue by contradiction: assume that there exists $P_0 \in \omega_0 \setminus \{(0, 0, 0), (1, 1, 1)\}$.

Then, $\phi_A(t, P_0) \in \omega_0$ for any $t \geq 0$ thus the trajectory $t \rightarrow \phi_A(t, P_0)$ is bounded as $t \rightarrow +\infty$. From Proposition 1.2 in [20], we have $\omega_{\phi_A}(P_0)$ is invariant. Hence, we deduce for any $t < 0$:

$$\begin{aligned} \omega_{\phi_A}(P_0) &= \phi_A(0, \omega_{\phi_A}(P_0)) = \phi_A(t, \phi_A(-t, \omega_{\phi_A}(P_0))) \\ &= \phi_A(t, \omega_{\phi_A}(P_0)). \end{aligned}$$

Thus we deduce that $t \rightarrow \phi_A(t, P_0)$ is bounded as $t \rightarrow -\infty$ and the limit sets of P_0 associated to ϕ_A is bounded.

Then, applying Theorem 10 in [12] together with $\text{div } g < 0$ and Proposition 3.1, we infer that

$$\boxed{*} \quad (3.16) \quad \lim_{|t| \rightarrow +\infty} \phi_A(t, P_0) \in \{(0, 0, 0), (1, 1, 1)\}.$$

Now, since $(0, 0, 0)$ is asymptotically stable, this implies $\phi_A(t, P_0) \rightarrow (1, 1, 1)$ as $t \rightarrow -\infty$ and we have two following cases:

Case 1: heteroclinic orbit *i.e.* $\phi_A(t, P_0) \rightarrow (0, 0, 0)$ as $t \rightarrow +\infty$

or

Case 2: homoclinic orbit *i.e.* $\phi_A(t, P_0) \rightarrow (1, 1, 1)$ as $t \rightarrow +\infty$.

Since ω_0 is invariant by the flow ϕ_A , both cases imply that $\omega(\zeta_0)$ contains either a connecting orbit between the two different equilibria ($P_0 \in W_u(1, 1, 1)$) or a cycle with respect to ϕ_A .

In the first case, the heteroclinic orbit included in ω_0 is not chain recurrent since $(0, 0, 0)$ is asymptotically stable which contradicts ω_0 is chain recurrent.

For the second case, $(1, 1, 1)$ is hyperbolic with transverse unstable and stable manifolds, there can not be any circuit (see for instance [11] page 1228 or [12] pages 1677-1678) and we have a contradiction.

Step 2: $\omega_0 = (1, 1, 1)$

Since ω_0 is connected then either $\omega_0 = \{(0, 0, 0)\}$ or $\omega_0 = \{(1, 1, 1)\}$.

The first possibility does not hold. Indeed arguing by contradiction, then there exists $T_0 > 0$ such that $\mathcal{X}(T_0) < 1$, $\mathcal{Y}(T_0) < 1$ and $\mathcal{Z}(T_0) < 1$.

Consider (\tilde{u}, \tilde{v}) a couple of blow-up solutions of (1.7) with the initial data $\tilde{v}(0) = v(0) + \varepsilon$ on $(0, R_\varepsilon)$ where $R_\varepsilon < 1$ by Proposition 2.1. Moreover from Proposition 2.2, we have R_ε goes to 1 as $\varepsilon \rightarrow 0$.

Let $R \in (0, 1)$, we define \mathcal{T}_1 , \mathcal{T}_2 and \mathcal{T}_3 from $C([0, R])$ to $C([0, -\ln(1-R)])$ three linear and continuous operators as follows: for any $w \in C([0, R])$, for any $t \in [0, T] = [0, -\ln(1-R)]$,

$$\mathcal{T}_1(w)(t) = \frac{e^{-\alpha_0 t}}{\lambda} w(1 - e^{-t}), \quad \mathcal{T}_2(w)(t) = \frac{e^{-\beta_0 t}}{\mu} w(1 - e^{-t}) \quad \text{and} \quad \mathcal{T}_3(w)(t) = \frac{e^{-\gamma_0 t}}{\nu} w(1 - e^{-t}).$$

By Remark 2.2, we get as $\varepsilon \rightarrow 0$:

$$\sup_{[0,T]} |\mathcal{X} - \tilde{X}| = \sup_{[0,T]} |\mathcal{T}_1(u') - \mathcal{T}_1(\tilde{u}')| \leq C \sup_{[0,R]} |u - \tilde{u}'| \rightarrow 0,$$

$$\sup_{[0,T]} |\mathcal{Y} - \tilde{Y}| = \sup_{[0,T]} |\mathcal{T}_2(v) - \mathcal{T}_2(\tilde{v})| \leq C \sup_{[0,R]} |v - \tilde{v}| \rightarrow 0$$

and

$$\sup_{[0,T]} |\mathcal{Z} - \tilde{Z}| = \sup_{[0,T]} |\mathcal{T}_3(v') - \mathcal{T}_3(\tilde{v}')| \leq C \sup_{[0,R]} |v' - \tilde{v}'| \rightarrow 0.$$

We deduce, for ε small enough that then there exists $R \in (1 - e^{-T_0}, R_\varepsilon)$,

$$\sup_{[0,T]} |\mathcal{X} - \tilde{X}| < 1 - \mathcal{X}(T_0), \quad \sup_{[0,T]} |\mathcal{Y} - \tilde{Y}| < 1 - \mathcal{Y}(T_0) \quad \text{and} \quad \sup_{[0,T]} |\mathcal{Z} - \tilde{Z}| < 1 - \mathcal{Z}(T_0).$$

Hence $\tilde{X}(T_0) < 1$, $\tilde{Y}(T_0) < 1$ and $\tilde{Z}(T_0) < 1$.

From Proposition 3.3, we deduce that, for any $t \in (T_0, -\ln(1 - R_\varepsilon))$, $\tilde{X}(t) < 1$, $\tilde{Y}(t) < 1$ and $\tilde{Z}(t) < 1$. Finally, by construction of \tilde{Y} , we have $\tilde{Y}(t) \rightarrow +\infty$ as $t \rightarrow -\ln(1 - R_\varepsilon)$ and we obtain a contradiction. \square

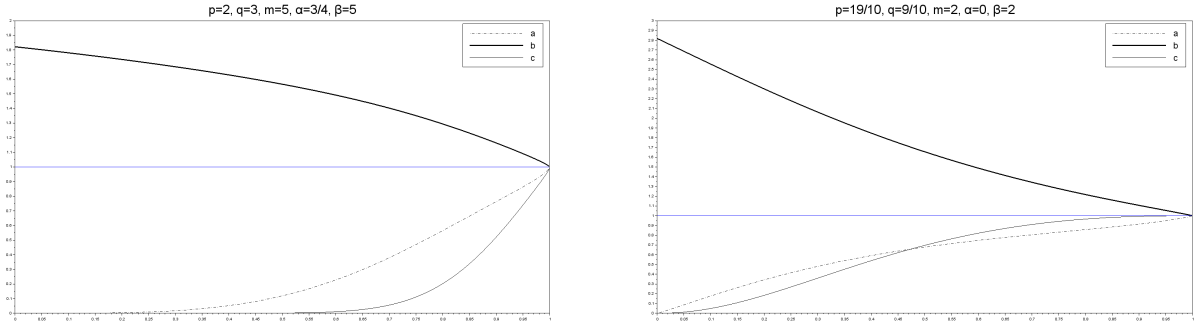


Figure 1: $\alpha_0 < 1$

fig1

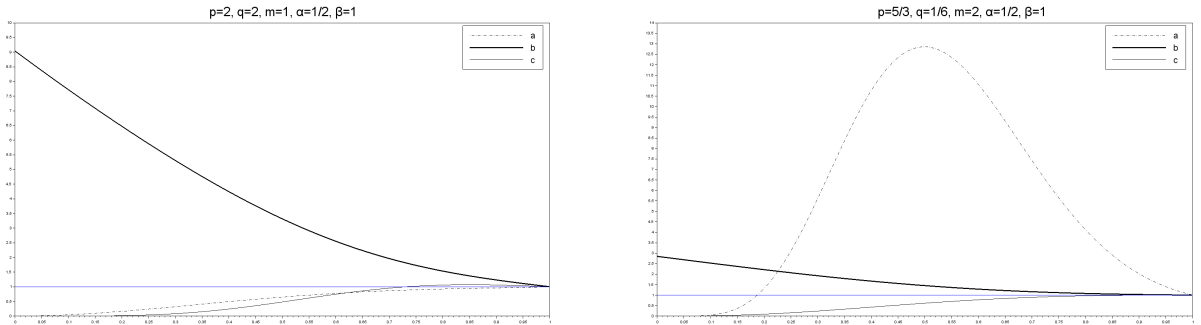


Figure 2: $\alpha_0 \geq 1$

fig2

Remark 3.2. Figures 1-3 give examples of convergence of the triplet functions (a, b, c) to $(1, 1, 1)$ as $r \rightarrow 1$ where (a, b, c) satisfies (3.7). More precisely, we have $\alpha_0 < 1$ in Figure 1 and $\alpha_0 \geq 1$ in Figure 2. We give particular examples for the conditions of Theorem 1.3 in Figure 3: in the left-hand graph, we choose $p > 2$ and $q < p - 1 - \alpha$ and in the right-hand graph, $\beta < 1$ and $m < 1$. For every graph, $\delta < 0$, $b_0 > 0$, $\alpha_0 > 0$ and $N = 3$.

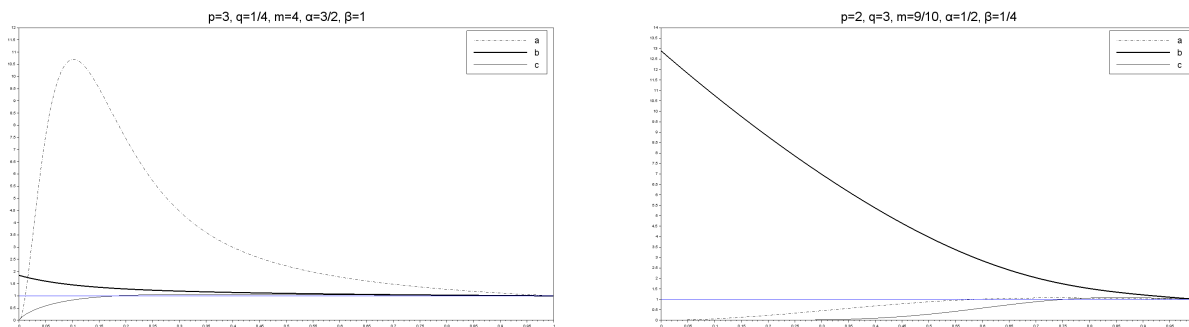


Figure 3: Other examples

fig3

Proof of Theorem 1.3:

We get from Proposition 3.5 that, as $r \rightarrow 1^-$

$$a(r) = \frac{U(r)}{\lambda}(1-r)^{\alpha_0} \rightarrow 1, \quad b(r) = \frac{v(r)}{\mu}(1-r)^{\beta_0} \rightarrow 1 \quad \text{and} \quad c(r) = \frac{V(r)}{\nu}(1-r)^{\gamma_0} \rightarrow 1$$

hence

$$u'(r) \sim \frac{\lambda}{(1-r)^{\alpha_0}} \quad \text{and} \quad v(r) \sim \frac{\mu}{(1-r)^{\beta_0}} \quad \text{as } r \rightarrow 1^-.$$

Since $\beta_0 > 0$, we have obviously $v(1) = \infty$. Finally we deduce

$$u(r) \sim \frac{\lambda}{(\alpha_0 - 1)(1-r)^{\alpha_0 - 1}} \quad \text{as } r \rightarrow 1^-$$

if $\alpha_0 \neq 1$ and

$$u(r) \sim \lambda \ln(1-r) \quad \text{as } r \rightarrow 1^-$$

if $\alpha_0 = 1$.

References

[1] R. Abraham and J. Robbin, *sl Transversal mappings and flows*, An appendix by Al Kelley, W.A. Nijman, Inc., New York-Amsterdam 1967.

[2] L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen, *Math. Ann.*, 77, 1916, 173-212.

[3] C. Conley, *The gradient structure of a flow: I*, IBM research, RC 3939, Yorktown Heights, NJ, July 17, 1972, *Ergodic Theory Dynamical Systems*, 8, (1988), 11-26.

[4] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conf. Ser. in Math., 38, Amer. Math. Soc., Providence, RI, 1978.

[5] J. I. Diaz, J. M. Rakotoson, and P. G. Schmidt, *Local strong solutions of a parabolic system related to the Boussinesq approximation for buoyancy-driven flow with viscous heating*, *Advances in Differential Equations*, Volume 13 (9-10), (2008), 977-1000.

[6] J. I. Diaz, M. Lazzo, and P. G. Schmidt, *Large solutions for a system of elliptic equations arising from fluid dynamics*, *SIAM J. Math. Anal.*, 37 (2), (2005), 490-513.

[7] R. Filippucci and F. Vinti, *Coercive elliptic systems with gradient terms*, *Adv. Nonlinear Anal.*, 6 (2), 2017, 165-182.

[8] M. Ghergu, J. Giacomoni and G. Singh, *Global and blow-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids*, *Nonlinearity*, 32 (4), 2019, 1546-1569.

- [9] M. Ghergu and V. D. Rădulescu, *Nonlinear PDEs. Mathematical Models in Biology, Chemistry and Population Genetics*, Springer Monographs in Mathematics, Springer, Heidelberg (2012).
- [10] P.Hartman, *Ordinary Differential Equations*, SIAM J.Math, 2nd edition, 1982.
- [11] M.W. Hirsch, *Systems of Differential Equations that are Competitive or Cooperative. IV: Structural Stability in Three-Dimensional Systems*, SIAM J.Math. Anal., 21, (1990), 1225-1234.
- [12] M.W. Hirsch, *Systems of Differential Equations that are Competitive or Cooperative. V: Convergence in 3-Dimensional Systems*, J. Differential Equations 80, (1989), 94-106.
- [13] M. W. Hirsch, *Systems of differential equations that are competitive or cooperative. I: Limit sets*, SIAM J. Math. Anal. 13 (1982), 167-179.
- [14] M. W. Hirsch and C. C. Pugh, *Cohomology of chain recurrent sets*, Ergodic Th. Dynam. Sys., 8, (1988), 73-80.
- [15] M.W. Hirsch, *Systems of differential equations that are competitive or cooperative II: convergence almost everywhere*, SIAM J. Math. Analysis, 16, (1985), 423-439.
- [16] R.G. Iagar, Ph. Laurencot, *Positivity, decay and extinction for a singular diffusion equation with gradient absorption*, J. Funct. Anal., 262 (7), (2012), 3186-3239.
- [17] R.G. Iagar, Ph. Laurencot, *Eternal solutions to a singular diffusion equation with critical gradient absorption*, Nonlinearity, 26, (2013), 3169-3195.
- [18] J.B. Keller, *On solutions of $\Delta = f(u)$* , Comm. Pure Appl. Math., 10, (1957), 503-510.
- [19] I. Kupka, *Contribution à la théorie des champs génériques*, Contributions to Differential Equations, 2, (1963), 457-484.
- [20] K. Mischaikow, H. Smith, and H.R. Thieme, *Asymptotically autonomous semiflows: chain recurrence and Lyapunov functions*, Transactions AMS, 347, (1995), 1669-1685.
- [21] R. Osserman, *On the inequality $\Delta u \geq f(u)$* , Pacific J. Math., 7, (1957), 1641-1647.
- [22] M.M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology, 1, (1962) 101-120.
- [23] V.D. Rădulescu, *Singular phenomena in nonlinear elliptic problems. from blow-up boundary solutions to equations with singular nonlinearities*, in: Michel Chipot (Ed.), Handbook of Differential Equations: Stationary Partial Differential Equations, Vol. 4, North-Holland Elsevier Science, Amsterdam, 2007, pp. 483-591.
- [24] W. Rudin, *Principles of Mathematical Analysis*, Int. Series Pure and Applied Mathematics, Third Edition, 1976.
- [25] G. Singh, *Classification of radial solutions for semilinear elliptic systems with nonlinear gradient terms*, Nonlinear Analysis: Theory, Methods and Applications, 129, (2015), 77-103.
- [26] S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 17(3), (1963), 97-116.
- [27] J.L. Vazquez, *The mathematical theories of diffusion: nonlinear and fractional diffusion. Nonlocal and nonlinear diffusions and interactions: new methods and directions*, 205278, Lecture Notes in Math. No. 2186.