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Asymptotic behavior of blowing-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids

Ahmed Bachir, Jacques Giacomoni and Guillaume Warnault

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Abstract

In this paper, we deal with the following quasilinear elliptic system involving gradient terms in the form:

$$\begin{cases} \Delta_p u = v^m |\nabla u|^\alpha & \text{in } \Omega \\ \Delta_p v = v^\beta |\nabla u|^q & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is either equal to \mathbb{R}^N or equal to a ball B_R centered at the origin and having radius $R > 0$, $1 < p < \infty$, $m, q > 0$, $\alpha \geq 0$, $0 \leq \beta \leq m$ and $\delta := (p-1-\alpha)(p-1-\beta) - qm \neq 0$. Our aim is to establish the asymptotics of the blowing-up radial solutions to the above system. Precisely, we provide the accurate asymptotic behavior at the boundary for such blowing-up radial solutions. For that, we prove a strong maximal principle for the problem of independent interest and study an auxiliary asymptotically autonomous system in \mathbb{R}^3 .

1 Introduction and main results

This paper deals with a class of quasilinear elliptic system of the following type:

$$\boxed{\text{otoo23}} \quad (1.1) \quad \begin{cases} \Delta_p u = v^m |\nabla u|^\alpha & \text{in } \Omega \\ \Delta_p v = v^\beta |\nabla u|^q & \text{in } \Omega, \end{cases}$$

Here the operator Δ_p represents the standard p -Laplace operator, and $\Omega \subset \mathbb{R}^N (N \geq 2)$ is either equal to \mathbb{R}^N if the solution is global or equal to a ball B_R if the solution blows up at some $\hat{R} \geq R$ with $\hat{R} < \infty$, and the parameters involved in (1.1) satisfy

$$1 < p < \infty, \quad m, q > 0, \quad \alpha \geq 0, \quad 0 \leq \beta \leq m.$$

The global homogeneity parameter related to the system (1.1), δ , verifies additionally

$$\boxed{\text{otoo24}} \quad (1.2) \quad \delta := (p-1-\alpha)(p-1-\beta) - mq \neq 0.$$

Our goal is to determine the behaviour of radial and positive blowing-up solutions at boundary. In this regard, the present paper can be considered as the subsequent continuation of [8], where several results about existence, global behavior and uniqueness are obtained for global solutions to the system (1.2) (see also [25] and [7] for the particular case $p = 2$). The semilinear case $p = 2$, $\alpha = \beta = 0$, $m = 1$, $q = 2$ of problem (1.1) was considered in details by Diaz, Lazzo and Schmidt [6] as a model arising from the study of dynamics of a viscous, heat-conducting fluid. Considering a unidirectional flow, independent of distance in the flow direction under the Boussinesq approximation, the velocity u and the temperature θ satisfy the following system

$$\boxed{\text{otoo140}} \quad (1.3) \quad \begin{cases} u_t - \Delta u = \theta & \text{in } \Omega \times (0, T), \\ \theta_t - \Delta \theta = |\nabla u|^2 & \text{in } \Omega \times (0, T). \end{cases}$$

The source terms θ and $|\nabla u|^2$ represent the buoyancy force and viscous heating, respectively. With the change of variable $v = -\theta$, steady states of (1.3) satisfy the following coupled elliptic equations:

otoo140b

$$(1.4) \quad \begin{cases} \Delta u = v & \text{in } \Omega \times (0, T), \\ \Delta v = |\nabla u|^2 & \text{in } \Omega \times (0, T), \end{cases}$$

which corresponds to the semilinear case $p = 2$, $\alpha = \beta = 0$, $m = 1$, $q = 2$ of Problem (1.1).

Elliptic and parabolic equations with gradient absorption term arise in different models : reaction diffusion equations (see [16]) in non newtonian fluids models, turbulent flows in porous media (see [27]). We refer to [8] for further explanations of physical models. In [8] the authors studied nonconstant positive radial solutions of (1.1), that is, solutions (u, v) which satisfy

- $u, v \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{0\})$ are positive and radially symmetric;
- u and v are not constant in any neighbourhood of the origin;
- u and v satisfy (1.1).

If $\Omega = \mathbb{R}^N$, solutions of (1.1) will be called global solutions.

Theorem 1.1 given below is one of main results in [8], that provides a Keller-Osserman type condition (see [18] and [21]) for system (1.1) and then classifies all nonconstant positive radial solutions in a ball B_R according to their asymptotic behavior at the boundary.

khaled4

Theorem 1.1. *Assume that $\Omega = B_R$, $1 < p < \infty$, $m, q > 0$, $0 \leq \alpha < p - 1$, $0 \leq \beta \leq m$ and $\delta \neq 0$. Then,*

- (i) *There are no positive radial solutions (u, v) with $u(R^-) = \infty$ and $v(R^-) < \infty$.*
- (ii) *All positive radial solutions of (1.1) are bounded i.i.f. $\delta > 0$*
- (iii) *There are positive radial solutions (u, v) of (1.1) with $u(R^-) < \infty$ and $v(R^-) = \infty$ i.i.f. $\delta < -mp - (p - 1 - \beta)$.*
- (iv) *There are positive radial solutions (u, v) of (1.1) with $u(R^-) = v(R^-) = \infty$ i.i.f. $\delta \in [-mp - (p - 1 - \beta), 0)$.*

From the seminal works of [2], Keller-Osserman condition was investigated for a very large class of elliptic equations, systems and inequations. We refer to [9] and [23] for a survey of the corresponding results. The second result in [8] concerns the existence of nonconstant global positive radial solutions of (1.1), which shows that Theorem (1.1)-(ii) is sharp regarding the existence of global solutions:

khaled5

Theorem 1.2. *Assume that $\Omega = \mathbb{R}^N$, $p > 1$, $m, q > 0$, $\alpha \geq 0$, $0 \leq \beta \leq m$ and $\delta \neq 0$. Then, (1.1) admits nonconstant global positive radial solutions if and only if*

otoo31

$$(1.5) \quad 0 \leq \alpha < p - 1 \quad \text{and} \quad \delta > 0.$$

In [8], the asymptotic behavior and the uniqueness of global solutions are also investigated. As it is observed in [8], by taking advantage of the homogeneity of system (1.1), we have that any nonconstant positive radial solution (u, v) of (1.1) in a ball B_R satisfies the following system

otoo35

$$(1.6) \quad \begin{cases} ((u')^{p-1})' + \frac{N-1}{r}(u')^{p-1} = v^m(u')^\alpha & \text{in } (0, R), \\ ((v')^{p-1})' + \frac{N-1}{r}(v')^{p-1} = v^\beta(u')^q & \text{in } (0, R), \\ u'(0) = v'(0) = 0, \quad u, v > 0 & \text{in } (0, R). \end{cases}$$

and u' and v' are increasing on $(0, R)$.

In the present paper, to complement the above results, we focus on the asymptotic behaviour of blowing-up solutions. Thanks to the homogeneity property of the system, (1.6) can be reduced to the problem

otoo36

$$(1.7) \quad \begin{cases} ((u')^{p-1-\alpha})' + \frac{\gamma}{r}(u')^{p-1-\alpha} = \frac{\gamma}{N-1}v^m & \text{in } (0, R), \\ ((v')^{p-1})' + \frac{N-1}{r}(v')^{p-1} = v^\beta(u')^q & \text{in } (0, R), \\ u'(0) = v'(0) = 0, \quad u, v > 0 & \text{in } (0, R), \end{cases}$$

where

$$\text{otoo37} \quad (1.8) \quad \gamma = \frac{(N-1)(p-1-\alpha)}{p-1}.$$

Let (u, v) be a nonconstant positive radially symmetric solution of (1.7) in the ball B_R , then, the pair

$$\text{scal} \quad (1.9) \quad (u_\lambda, v_\lambda) = \left(\lambda^{1-\alpha_0} u\left(\frac{x}{\lambda}\right), \lambda^{-\beta_0} v\left(\frac{x}{\lambda}\right) \right)$$

where $\alpha_0 = \frac{1+\beta-p(m+1)}{\delta}$ and $\beta_0 = -\frac{p(p-1-\alpha)+q}{\delta}$, provides a nonconstant positive radially symmetric solution of (1.7) in the ball $B_{\lambda R}$. This guarantees that in any ball of positive radius there is nonconstant positive radially symmetric solution of (1.7). Hence, thanks to this scaling property, R can be arbitrary, then without loss of generality we fix in the sequel $R = 1$.

Considering $R = 1$, the main result of the present paper yields the asymptotic behavior of the blowing-up solutions of (1.1) namely when (iii) or (iv) holds.

theorem1 **Theorem 1.3.** *Under the conditions of Theorem 1.2, assume $\delta < 0$ and $\alpha_0 > 0$. Then,*

$$u'(r) \sim \frac{\lambda}{(1-r)^{\alpha_0}} \quad \text{and} \quad v(r) \sim \frac{\mu}{(1-r)^{\beta_0}} \quad \text{as } r \rightarrow 1^-$$

where λ and μ are positive constants given in (3.8).

Remark 1.1. *From Theorem 1.3, we get the asymptotics of u when $\alpha_0 \in [1, \infty[$, that is as $r \rightarrow 1^-$, $u(r) \sim \frac{\lambda}{\alpha_0-1}(1-r)^{-\alpha_0+1}$ if $\alpha_0 > 1$ and $u(r) \sim -\lambda \ln(1-r)$ if $\alpha_0 = 1$. $\alpha_0 < 1$ holds when $\delta < -mp - (p-1-\beta)$ and then one falls in the case of (iii) of Theorem 1.1 whereas assertion (iv) is verified as $\alpha_0 \geq 1$.*

To prove theorem 1.3, we perform several changes of variables that reduce (1.6) to an asymptotically autonomous dynamical system for which we appeal the rich theory of cooperative systems (see [11], [15], [12] for further details on the subject). We also use chain-recurrence properties of asymptotically-autonomous flows (see [20]) to identify the ω -limit set of the non autonomous dynamical system. We also need properties concerning equation (1.6) or (1.7) as comparison principle and continuity with respect the initial data. Section 2 is dedicated to prove these tools whereas Section 3 deals the proof of Theorem 1.3. We also give at the end of Section 3 some numerical simulations associated to asymptotics proved in Theorem 1.3.

2 Technical results

More precisely, we study in this section the pairs of positive functions (U, v) which satisfy, for some $R > 0$

$$\text{uniq} \quad (2.1) \quad \begin{cases} (U^{p-1-\alpha})' + \frac{\gamma}{r} U^{p-1-\alpha} = \frac{\gamma}{N-1} v^m & \text{in } (0, R), \\ ((v')^{p-1})' + \frac{N-1}{r} (v')^{p-1} = v^\beta U^q & \text{in } (0, R), \\ v'(0) = U(0) = 0 \text{ and } v \rightarrow +\infty \text{ as } r \rightarrow R. \end{cases}$$

We start with a comparison principle satisfied by (2.1).

lemn **Proposition 2.1.** *Let (U, v) and $(\widehat{U}, \widehat{v})$ be positive functions which satisfy*

$$\text{otooe1} \quad (2.2) \quad \begin{cases} (r^\gamma U^{p-1-\alpha})' \geq \frac{\gamma}{N-1} r^\gamma v^m & \text{for any } r \in (0, R), \\ (r^{N-1} (v')^{p-1})' \geq r^{N-1} v^\beta U^q & \text{for any } r \in (0, R), \\ v(0) = v_0 \text{ and } v'(0) = U(0) = 0 \\ U > 0, v' > 0 \text{ in } (0, R) \end{cases}$$

and

$$(2.3) \quad \begin{cases} \left(r^\gamma \widehat{U}^{p-1-\alpha} \right)' \leq \frac{\gamma}{N-1} r^\gamma \widehat{v}^m & \text{for any } r \in (0, \widehat{R}), \\ \left(r^{N-1} (\widehat{v}')^{p-1} \right)' \leq r^{N-1} \widehat{v}^\beta \widehat{U}^q & \text{for any } r \in (0, \widehat{R}), \\ \widehat{v}(0) = \widehat{v}_0 \text{ and } \widehat{v}'(0) = \widehat{U}(0) = 0 \\ \widehat{U} > 0, \widehat{v}' > 0 \text{ in } (0, \widehat{R}) \end{cases}$$

for some $R, \widehat{R} > 0$.

If $\widehat{v}(0) < v(0)$, then $\widehat{v} < v$ and $\widehat{U} < U$ in $(0, \min(R, \widehat{R}))$.

In addition, assuming that \widehat{v} blows up at \widehat{R} , we have $R < \widehat{R}$.

Proof. Define $\rho^* = \sup\{\rho > 0 \mid \widehat{v}(r) < v(r) \text{ on } (0, \rho)\}$. Since $\widehat{v}(0) < v(0)$, $\rho^* > 0$ hence from the first equations of (2.2)-(2.3), we deduce $U > \widehat{U}$ on $(0, \rho^*)$. Moreover using the second equations of (2.2)-(2.3), we get $v' > \widehat{v}'$ on $(0, \rho^*)$. Thus the mapping $v - \widehat{v}$ is increasing $(0, \rho^*)$ and we deduce $\rho^* = \min(R, \widehat{R})$ and $\widehat{v} < v$ on $(0, \rho^*)$.

Now, assume that \widehat{v} blows up at \widehat{R} . Since $\widehat{v} < v$ on $(0, \min(R, \widehat{R}))$, we have $R \leq \widehat{R}$.

Consider the pair (U_λ, v_λ) where $U_\lambda = \lambda^{-\alpha_0} U(\frac{\cdot}{\lambda})$ and v_λ is defined in (1.9). Thus (U_λ, v_λ) satisfies (2.2) in $(0, \lambda R)$.

Choosing λ such that $\frac{\widehat{v}(0)}{v(0)} \leq \lambda^{-\beta_0} < 1$ then $v_\lambda(0) > \widehat{v}(0)$ and from the first part, we deduce $v_\lambda > \widehat{v}$ on $(0, \min(\lambda R, \widehat{R}))$ and $\lambda R \leq \widehat{R}$. \square

lem2 **Proposition 2.2.** *The mapping $\Phi : v_0 \rightarrow R$, where $(0, R)$ is the maximal interval of a solution (U, v) of (2.1) with $v(0) = v_0$, is continuous on $(0, +\infty)$.*

Proof. Let $(v_{0,n}) \subset \mathbb{R}$ such that $v_{0,n} \rightarrow v_0$ and \cdot . We define (U, v) and R (respectively (U_n, v_n) and R_n) solutions of (2.1) with $v(0) = v_0$ defined on the maximal interval $(0, R)$ (respectively with $v_n(0) = v_{0,n}$ defined on the maximal interval $(0, R_n)$).

Let $\varepsilon > 0$, Then, we define

$$\lambda_\pm = \left(1 \pm \frac{\varepsilon}{v_0} \right)^{-\frac{1}{\beta_0}}$$

and the pair $(U_{\lambda_\pm}, v_{\lambda_\pm}) = \left(\lambda_\pm^{-\alpha_0} U\left(\frac{\cdot}{\lambda_\pm}\right), \lambda_\pm^{-\beta_0} v\left(\frac{\cdot}{\lambda_\pm}\right) \right)$ is solution of (2.1) on $(0, \lambda_\pm R)$ with $v_{\lambda_\pm}(0) = v_0 \pm \varepsilon$. For n large enough, $v_{0,n} \in (v_0 - \varepsilon, v_0 + \varepsilon)$, hence from Lemma 2.1, we have $\lambda_+ R < R_n < \lambda_- R$ and we deduce $R_n \rightarrow R$ as $n \rightarrow +\infty$. \square

Corollary 2.1. *For any $R > 0$, there exist an unique $v_0 > 0$ and an unique pair of positive functions (U, v) solution to (2.1) such that $v(0) = v_0$.*

Proof. Lemma 2.1-2.2 imply that the mapping Φ is decreasing and continuous on $(0, +\infty)$. Thus Φ is invertible on $(0, +\infty)$ and we deduce the uniqueness. \square

form **Remark 2.1.** *Using the scaling property and the previous results, we have, for any $R > 0$, $\Phi^{-1}(R) = \Phi^{-1}(1)R^{-\beta_0}$ and the unique solution (U_R, v_R) on $(0, R)$ of (2.1) is given by*

$$U_R = R^{-\alpha_0} U_1 \left(\frac{\cdot}{R} \right) \quad \text{and} \quad v_R = R^{-\beta_0} v_1 \left(\frac{\cdot}{R} \right)$$

where (U_1, v_1) is the unique solution of (2.1) on $(0, 1)$.

stabi **Remark 2.2.** *Let $\varepsilon > 0$ and let (U, v) and $(U_\varepsilon, v_\varepsilon)$ two pairs of solutions of (2.1) such that $v(0) - \tilde{v}(0) = \pm \varepsilon$. From Remark 2.1, there exists λ_ε such that $\lambda_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$, $\tilde{v}_\varepsilon = \lambda_\varepsilon^{-\beta_0} v\left(\frac{\cdot}{\lambda_\varepsilon}\right)$ and $\tilde{U}_\varepsilon = \lambda_\varepsilon^{-\alpha_0} U\left(\frac{\cdot}{\lambda_\varepsilon}\right)$. By the proof of Lemma 2.1, the sequences $(v_\varepsilon)_\varepsilon$, $(U_\varepsilon)_\varepsilon$ and $(v'_\varepsilon)_\varepsilon$ are monotone and converge everywhere to v , U and v' on the compact sets of $[0, R)$. By Dini's Theorem, we deduce the uniform convergence on the compact sets of $[0, R)$.*

3 Asymptotic behavior and proof of Theorem 1.3

Through this section, we only consider $\delta < 0$, $\alpha_0 > 0$ and $R = 1$. Moreover we have $\alpha_0(p-1-\alpha) + 1$, β_0 and γ_0 are positive. At the end of the section, we give the proof of Theorem 1.3.

Let (u, v) be a pair of nonconstant positive radial solutions of (1.1) given by Theorem 1.1. Then, setting $U = u'$ and $V = v'$ in (1.7), U and V are positive, increasing on $(0, 1)$ and satisfy

$$\text{eq1} \quad (3.1) \quad \begin{cases} (U^{p-1-\alpha})' + \frac{\gamma}{r} U^{p-1-\alpha} = \frac{\gamma}{N-1} v^m & \text{for } r \in (0, 1), \\ (V^{p-1})' + \frac{N-1}{r} V^{p-1} = v^\beta U^q & \text{for } r \in (0, 1), \\ U(0) = V(0) = 0. \end{cases}$$

Now, set

$$U(r) = \frac{\lambda}{(1-r)^{\alpha_0}} a(r), \quad v(r) = \frac{\mu}{(1-r)^{\beta_0}} b(r) \quad \text{and} \quad V(r) = \frac{\nu}{(1-r)^{\gamma_0}} c(r)$$

where $\lambda, \mu, \nu > 0$ to be fixed later. Thus, (3.1) leads to

$$\text{eq2} \quad (3.2) \quad \begin{cases} \left(\left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}} \right)^{p-1-\alpha} \right)' + \frac{\gamma}{r} \left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}} \right)^{p-1-\alpha} = \frac{p-1-\alpha}{p-1} \left(\frac{\mu b(r)}{(1-r)^{\beta_0}} \right)^m & \text{for } r \in (0, 1), \\ \left(\left(\frac{\nu c(r)}{(1-r)^{\gamma_0}} \right)^{p-1} \right)' + \frac{N-1}{r} \left(\frac{\nu c(r)}{(1-r)^{\gamma_0}} \right)^{p-1} = \left(\frac{\mu b(r)}{(1-r)^{\beta_0}} \right)^\beta \left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}} \right)^q & \text{for } r \in (0, 1). \end{cases}$$

Simple computations give $\alpha_0(p-1-\alpha) + 1 = m\beta_0$ and

$$\left(\left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}} \right)^{p-1-\alpha} \right)' = (p-1-\alpha) \lambda^{p-1-\alpha} \frac{(1-r)a'(r) + \alpha_0 a(r)}{(1-r)^{m\beta_0}} a(r)^{p-2-\alpha}.$$

Thus, multiplying the first equation of (3.2) by $(1-r)^{m\beta_0}$ choosing λ and μ such that $\lambda^{p-1-\alpha} \alpha_0(p-1-\alpha) = \frac{\gamma}{N-1} \mu^m$, this yields for any $r \in (0, 1)$

$$\text{eq5} \quad (3.3) \quad (1-r)a^{p-\alpha-2}((p-1-\alpha)a' + \frac{\gamma}{r}a) = \alpha_0(p-1-\alpha)(b^m - a^{p-1-\alpha}).$$

In the same way, noting that $\gamma_0(p-1)+1 = \beta\beta_0 + \alpha_0q$ and multiplying the second equation of (3.2) by $(1-r)^{\beta\beta_0 + \alpha_0q}$, we get

$$\text{eq8} \quad (3.4) \quad (1-r)c^{p-2}((p-1)c' + \frac{N-1}{r}c) = \gamma_0(p-1)(b^\beta a^q - c^{p-1})$$

where λ, μ and ν satisfy $\mu^\beta \lambda^q = \gamma_0(p-1)\nu^{p-1}$.

Moreover, since $V = v'$, we deduce that

$$\text{eq9} \quad (3.5) \quad \mu(b'(r)(1-r) + \beta_0 b(r)) = \nu c(r).$$

Choosing μ and ν such that $\beta_0 \mu = \nu$, (3.5) becomes

$$\text{eq10} \quad (3.6) \quad b'(r)(1-r) = \beta_0(c(r) - b(r)).$$

Hence, we deduce that the triplet (a, b, c) satisfies

$$\text{eq11} \quad (3.7) \quad \begin{cases} (1-r)a^{p-\alpha-2}((p-1-\alpha)a' + \frac{\gamma}{r}a) = \alpha_0(p-1-\alpha)(b^m - a^{p-1-\alpha}), \\ (1-r)c^{p-2}((p-1)c' + \frac{N-1}{r}c) = \gamma_0(p-1)(b^\beta a^q - c^{p-1}), \\ (1-r)b' = \beta_0(c-b) \end{cases}$$

under the following relations

$$\lambda^{p-1-\alpha} \alpha_0 = \frac{\mu^m}{p-1}, \quad \mu^\beta \lambda^q = \gamma_0(p-1)\nu^{p-1}, \quad \beta_0 \mu = \nu.$$

which leads to

$$\text{eq13} \quad (3.8) \quad \begin{cases} \lambda = \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{p-1-\beta}{m}} \right)^{-\frac{m}{\delta}}, \\ \mu = \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{q}{p-1-\alpha}} \right)^{-\frac{p-1-\alpha}{\delta}}, \\ \nu = \beta_0 \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{q}{p-1-\alpha}} \right)^{-\frac{p-1-\alpha}{\delta}}. \end{cases}$$

Consider now the following change of variable $r = 1 - e^{-t}$ i.e. $t = -\ln(1-r)$ and we define \mathcal{X} , \mathcal{Y} and \mathcal{Z} as follows

$$\mathcal{X}(t) = a^{p-1-\alpha}(r), \mathcal{Y}(t) = b(r) \text{ and } \mathcal{Z}(t) = c^{p-1}(r).$$

Hence, $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ are positive functions satisfying the following C^1 -asymptotically autonomous system on $[t_0, +\infty)$ for some $t_0 > 0$:

$$\text{eq14} \quad (3.9) \quad \begin{cases} X'(t) + \frac{\gamma}{e^t - 1} X(t) = \alpha_0(p-1-\alpha)(|Y|^{m-1}(t)Y(t) - X(t)) \\ Y'(t) = \beta_0(|Z|^{\frac{1}{p-1}-1}(t)Z(t) - Y(t)) \\ Z'(t) + \frac{N-1}{e^t - 1} Z(t) = \gamma_0(p-1)(|Y|^{\beta-1}(t)Y(t)|X|^{\frac{q}{p-1-\alpha}-1}(t)X(t) - Z(t)) \end{cases}$$

for some $(X(t_0), Y(t_0), Z(t_0)) \in \mathbb{R}^3$. The asymptotic dynamical system associated to (3.9) is given by:

$$\text{2eq14} \quad (3.10) \quad \begin{cases} X'(t) = \alpha_0(p-1-\alpha)(|Y|^{m-1}Y - X) \\ Y'(t) = \beta_0(|Z|^{\frac{1}{p-1}-1}Z - Y) \\ Z'(t) = \gamma_0(p-1)(|Y|^{\beta-1}Y|X|^{\frac{q}{p-1-\alpha}-1}X - Z) \end{cases}$$

The system (3.10) can be rewritten as $\zeta'(t) = g(\zeta)$, $\zeta(t_0) = (X(t_0), Y(t_0), Z(t_0)) \in \mathbb{R}^3$ where

$$\text{eq16} \quad (3.11) \quad \zeta(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} \text{ and } g(\zeta) = \begin{pmatrix} \alpha_0(p-1-\alpha)(|Y|^{m-1}Y - X) \\ \beta_0(|Z|^{\frac{1}{p-1}-1}Z - Y) \\ \gamma_0(p-1)(|Y|^{\beta-1}Y|X|^{\frac{q}{p-1-\alpha}-1}X - Z) \end{pmatrix}.$$

The system (3.10) is cooperative, $\text{div } g < 0$ and we have the following properties.

Proposition 3.1. *The following assertions hold:*

1. $(0, 0, 0)$ and $(1, 1, 1)$ are the only equilibrium points of the system (3.10).
2. $(0, 0, 0)$ is asymptotically stable (and then a sink).
3. $(1, 1, 1)$ is an hyperbolic saddle point.

Proof. Let $P = (X_e, Y_e, Z_e)$ be an equilibrium point of the system (3.10), then $g(P) = 0$ which implies that

$$\text{zizo1} \quad (3.12) \quad \begin{cases} |Y_e|^{m-1}Y_e = X_e, \\ |Z_e|^{\frac{1}{p-1}-1}Z_e = Y_e, \\ |Y_e|^{\beta-1}|X_e|^{\frac{q}{p-1-\alpha}-1}Y_eX_e = Z_e. \end{cases}$$

Proof of 1.: Obviously, $Y_e = 0$ yields $X_e = Z_e = 0$ and solve the system (3.12).

If $Y_e \neq 0$, from (3.12), we deduce that X_e, Y_e and Z_e have the same sign and that $|X_e|^{\frac{q}{p-1-\alpha}-1}X_e = |Y_e|^{p-2-\beta}Y_e$. Thus, the first equation of (3.12) implies that $Y_e^{\frac{\delta}{q}} = 1$. Hence, we get $Y_e = X_e = Z_e = 1$.

Proof of 2.: The assertion follows directly looking the linearized matrix at $(0, 0, 0)$ given by:

$$M_0 = \begin{pmatrix} -\alpha_0(p-1-\alpha) & 0 & 0 \\ 0 & -\beta_0 & 0 \\ 0 & 0 & -\gamma_0(p-1) \end{pmatrix}.$$

Proof of β .: The linearized matrix at $(1, 1, 1)$ is given by

$$M_1 = \begin{pmatrix} -\alpha_0(p-1-\alpha) & m\alpha_0(p-1-\alpha) & 0 \\ 0 & -\beta_0 & \frac{\beta_0}{p-1} \\ \frac{q\gamma_0(p-1)}{p-1-\alpha} & \beta_0\gamma_0(p-1) & -\gamma_0(p-1) \end{pmatrix}.$$

Thus, $\det(\lambda I - M_1) = \lambda^3 + C_1\lambda^2 + C_2\lambda + C_3$ where

$$\begin{aligned} C_1 &= \alpha_0(p-1-\alpha) + \beta_0 + \gamma_0(p-1), \\ C_2 &= \beta_0\gamma_0(p-1-\beta) + \alpha_0(p-1-\alpha)(\beta_0 + \gamma_0(p-1)), \\ C_3 &= \alpha_0\beta_0\gamma_0\delta. \end{aligned}$$

Since $\beta_0 + 1 = \gamma_0$, $\alpha_0(p-1-\alpha) + 1 = \beta_0m$ and $\gamma_0(p-1) + 1 = \beta\beta_0 + \alpha_0q$, we get $C_1 + C_2 + C_3 = -1$. Hence $\lambda_1 = 1$ is an eigenvalue of M_1 .

Let us call λ_2, λ_3 the two other eigenvalues of M_1 . Then we have $\lambda_2 + \lambda_3 = -C_1 - 1 < 0$ and $\lambda_2\lambda_3 = -C_3 = -\alpha_0\beta_0\gamma_0\delta > 0$.

Thus, $Re(\lambda_2), Re(\lambda_3) < 0$ and we deduce that $(1, 1, 1)$ is hyperbolic point with 2-dimensional stable manifold. \square

We have the following properties about the dynamical system (3.9):

pos **Proposition 3.2.** *Let $T > 0$ and $t_0 \in (0, T)$. Assume (X, Y, Z) a triplet of functions satisfying (3.9) on $[t_0, T)$ with $(X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}_+^3$. Then, for any $t \in [t_0, T)$, $(X(t), Y(t), Z(t)) \in \text{Int } \mathbb{R}_+^3$.*

Proof. Define $T^* = \sup\{t > t_0 \mid X(s) > 0, Y(s) > 0 \text{ and } Z(s) > 0 \text{ for any } s \in (0, t)\}$ and assume that $T^* < T$.

From the second equation of (3.9), we deduce that for any $t \in [t_0, T^*)$, $(e^{\beta_0 t} Y(t))' > 0$ and hence $e^{\beta_0 t} Y(t) > e^{\beta_0 t_0} Y(t_0) > 0$. This implies that $Y(T^*) > 0$.

Assuming $X(T^*) = 0$, the first equation of (3.9) yields for $t = T^*$, $X'(T^*) = \alpha_0(p-1-\alpha)Y^m(T^*) > 0$. This implies there exists a small neighbourhood $(T^* - \eta, T^* + \eta)$ of T^* such that X' is positive on $(T^* - \eta, T^* + \eta)$. Since $X(T^*) = 0$, we deduce $X(t) < 0$ on $(T^* - \eta, T^*)$ which contradicts the definition of T^* .

In the same, using the third equation of (3.9), we can not have $Z(T^*) = 0$ and hence we conclude $T^* = T$. \square

marchencore **Remark 3.1.** *Proposition 3.2 holds considering a triplet of functions satisfying the autonomous system (3.10).*

inf1 **Proposition 3.3.** *Let $T > 0$ and $t_0 \in (0, T)$. Let (X, Y, Z) be a triplet of functions satisfying (3.9) on $[t_0, T)$ with $(X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}_+^3$. Assume $X(t_0) < 1, Y(t_0) < 1$ and $Z(t_0) < 1$. Then, for any $t \in [t_0, T)$, $X(t) < 1, Y(t) < 1$ and $Z(t) < 1$.*

Proof. By Proposition 3.2, for any $t \in [t_0, T)$, $(X(t), Y(t), Z(t)) \in \text{Int } \mathbb{R}_+^3$ and hence we deduce

$$\begin{cases} (X-1)'(t) < \alpha_0(p-1-\alpha)(Y^m - 1 - (X-1)), \\ (Y-1)'(t) = \beta_0 \left(Z^{\frac{1}{p-1}} - 1 - (Y-1) \right), \\ (Z-1)'(t) < \gamma_0(p-1) \left(Y^\beta X^{\frac{q}{p-1-\alpha}} - 1 - (Z-1) \right) \end{cases}$$

and we get

name (3.13)
$$\begin{cases} e^{-\alpha_0(p-1-\alpha)t} (e^{\alpha_0(p-1-\alpha)t} (X-1))' < \alpha_0(p-1-\alpha)(Y^m - 1), \\ e^{-\beta_0 t} (e^{\beta_0 t} (Y-1))' = \beta_0 \left(Z^{\frac{1}{p-1}} - 1 \right), \\ e^{-\gamma_0(p-1)t} (e^{\gamma_0(p-1)t} (Z-1))' < \gamma_0(p-1) \left(Y^\beta X^{\frac{q}{p-1-\alpha}} - 1 \right). \end{cases}$$

Define $\tau = \sup\{t > t_0 \mid X(t) < 1, Y(t) < 1 \text{ and } Z(t) < 1 \text{ on } (t_0, t)\}$.

Assume $\tau < T$ then $X(\tau) = 1$ or $Y(\tau) = 1$ or $Z(\tau) = 1$.

In any cases, the third equation of (3.13) at $t = \tau$ implies that $Z(t) < 1$ on the interval $[\tau, \tau + \eta]$ for

some $\eta > 0$. Hence, if $Z(\tau) = 1$, we get a contradiction.

Otherwise, using the previous estimate in the second equation of (3.13), we get $Y(\tau) < 1$ which would contradict $Y(\tau) = 1$. Finally, using $Y(\tau) < 1$ in the first equation of (3.13), this yields $X(\tau) < 1$ and a contradiction with $X(\tau) = 1$. \square

For t_0 small enough, $(\mathcal{X}(t_0), \mathcal{Y}(t_0), \mathcal{Z}(t_0)) \in \text{Int } \mathbb{R}_+^3$, hence the triplet $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ satisfies the C^1 -asymptotically autonomous system

$$\text{eq14b} \quad (3.14) \quad \begin{cases} X'(t) + \frac{\gamma}{e^t - 1} X(t) = \alpha_0(p-1-\alpha)(Y^m(t) - X(t)) & \text{on } [t_0, \infty) \\ Y'(t) = \beta_0(Z^{\frac{1}{p-1}}(t)Z(t) - Y(t)) & \text{on } [t_0, \infty) \\ Z'(t) + \frac{N-1}{e^t - 1} Z(t) = \gamma_0(p-1)(Y^\beta(t)X^{\frac{q}{p-1-\alpha}}(t) - Z(t)) & \text{on } [t_0, \infty) \end{cases}$$

with $(X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}_+^3$. Hence we get the following asymptotic dynamical system:

$$\text{2eq14b} \quad (3.15) \quad \begin{cases} X' = \alpha_0(p-1-\alpha)(Y^m - X) \\ Y' = \beta_0(Z^{\frac{1}{p-1}} - Y) \\ Z' = \gamma_0(p-1)(Y^\beta X^{\frac{q}{p-1-\alpha}} - Z). \end{cases}$$

Now, we study the trajectories $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. The first result is the relatively compactness:

prop1 **Proposition 3.4.** *The solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is bounded as $t \rightarrow +\infty$.*

Proof. Firstly, assume that \mathcal{Y} is bounded, then we have from the first equation of (3.14)

$$\mathcal{X}'(t) + \alpha_0(p-1-\alpha)\mathcal{X} \leq C,$$

which is equivalent to

$$(e^{\alpha_0(p-1-\alpha)t} \mathcal{X})' \leq C e^{\alpha_0(p-1-\alpha)t}.$$

Integrating this last inequality from t_0 to t leads to

$$e^{\alpha_0(p-1-\alpha)t} \mathcal{X}(t) - e^{\alpha_0(p-1-\alpha)t_0} \mathcal{X}(t_0) \leq \frac{C}{\alpha_0(p-1-\alpha)} (e^{\alpha_0(p-1-\alpha)t} - e^{\alpha_0(p-1-\alpha)t_0}).$$

Since \mathcal{X} is positive, this implies that \mathcal{X} is bounded. A similar argument implies that \mathcal{Z} is bounded.

It remains to prove that \mathcal{Y} is actually bounded and for this, we proceed by contradiction.

Consider (\hat{u}, \hat{v}) the solution of (1.7) with $\hat{u}(0) = u(0)$ and $\hat{v}(0) > v(0)$ defined in $(0, \hat{R})$. From Proposition 2.1, it follows that $\hat{R} < 1$. We define $(\hat{X}, \hat{Y}, \hat{Z})$ on $(0, -\ln(1 - \hat{R}))$ the transform of (\hat{u}, \hat{v}) constructed as previously.

We also define $\tilde{X}(t) = \mathcal{X}(t + T_0)$, $\tilde{Y}(t) = \mathcal{Y}(t + T_0)$ and $\tilde{Z}(t) = \mathcal{Z}(t + T_0)$ with T_0 large enough such that $\mathcal{Y}(T_0) > \hat{Y}(0)$ since \mathcal{Y} is unbounded. Then (\tilde{u}, \tilde{v}) the reverse transform of $(\tilde{X}, \tilde{Y}, \tilde{Z})$ satisfies

$$\begin{cases} ((\tilde{u}')^{p-1-\alpha})' + \frac{\gamma}{r + \tilde{r}_0} (\tilde{u}')^{p-1-\alpha} = \frac{\gamma}{N-1} \tilde{v}^m & \text{in } (0, 1), \\ ((\tilde{v}')^{p-1})' + \frac{N-1}{r + \tilde{r}_0} (\tilde{v}')^{p-1} = v^\beta (\tilde{u}')^q & \text{in } (0, 1), \\ \tilde{u}'(0) \geq 0, \tilde{v}'(0) \geq 0, \tilde{u}, \tilde{v} > 0 & \text{in } (0, 1), \end{cases}$$

where $\tilde{r}_0 = \frac{r_0}{1-r_0}$ with $r_0 = 1 - e^{-T_0}$.

Noting (\tilde{u}', \tilde{v}) satisfies (2.2) and $\tilde{v}(0) > \hat{v}(0)$, we apply Proposition 2.1 with (\hat{u}', \hat{v}) , we deduce $\hat{v} < \tilde{v}$ in $(0, \min(1, \hat{R})) = (0, \hat{R})$. Since \hat{v} blows up at \hat{R} , we get the contradiction. \square

From proposition 3.4, we deduce the convergence of the trajectories $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$:

prop2 **Proposition 3.5.** *We have*

$$\lim_{t \rightarrow +\infty} (\mathcal{X}(t), \mathcal{Y}(t), \mathcal{Z}(t)) = (1, 1, 1).$$

For the proof of Proposition 3.5, we need to recall two notions of dynamical systems:

Definition 3.1. A circuit is a finite sequence of equilibria $z_1, z_2, \dots, z_K = z_1$, ($K \geq 2$) such that $W^u(z_i) \cap W^s(z_{i+1}) \neq \emptyset$ where W^u, W^s denote the stable and unstable manifolds of each equilibria.

Definition 3.2. Let $X \subset \mathbb{R}^3$ be a nonempty positively invariant subset for an autonomous semiflow ϕ and $x, y \in X$.

- (i) For $\epsilon > 0$ and $t > 0$, an (ϵ, t) -chain from $x \in X$ to $y \in X$ is a sequence of points in X , $x = x_1, x_2, \dots, x_n, x_{n+1} = y$ and of times $t_1, t_2, \dots, t_n \geq t$ such that $|\phi(t_i, x_i) - x_{i+1}| < \epsilon$.
- (ii) A point $x \in X$ is called chain recurrent if for every $\epsilon > 0$, $t > 0$ there is an (ϵ, t) -chain from x to x in X .
- (iii) The set X is said to be chain recurrent if every point $x \in X$ is chain recurrent in X .

Proof. Setting $\zeta_0 = (\mathcal{X}(t_0), \mathcal{Y}(t_0), \mathcal{Z}(t_0)) \in \text{Int } \mathbb{R}_+^3$, we define the ω -limit set $\omega_0 = \omega(t_0, \zeta_0) \subset \mathbb{R}_+^3$ of ϕ the semiflow of the asymptotically autonomous system (3.14).

From Proposition 3.4, ω_0 is bounded. Thus Theorem 1.8 in [20] implies ω_0 is nonempty, compact and connected. Moreover ω_0 is invariant for the semiflow denoted by ϕ_A associated to the asymptotic autonomous system (3.15), i.e. for any $t \geq 0$

$$\phi_A(t, \omega_0) = \omega_0$$

and ω_0 is chain recurrent for ϕ_A .

Finally, ω_0 satisfies $\text{dist}(\phi(t, t_0, \zeta_0), \omega_0) \rightarrow 0$ as $t \rightarrow +\infty$.

Step 1: $\omega_0 \subset \{(0, 0, 0); (1, 1, 1)\}$.

For that we argue by contradiction: assume that there exists $P_0 \in \omega_0 \setminus \{(0, 0, 0), (1, 1, 1)\}$.

Then, $\phi_A(t, P_0) \in \omega_0$ for any $t \geq 0$ thus the trajectory $t \rightarrow \phi_A(t, P_0)$ is bounded as $t \rightarrow +\infty$. From Proposition 1.2 in [20], we have $\omega_{\phi_A}(P_0)$ is invariant. Hence, we deduce for any $t < 0$:

$$\begin{aligned} \omega_{\phi_A}(P_0) &= \phi_A(0, \omega_{\phi_A}(P_0)) = \phi_A(t, \phi_A(-t, \omega_{\phi_A}(P_0))) \\ &= \phi_A(t, \omega_{\phi_A}(P_0)). \end{aligned}$$

Thus we deduce that $t \rightarrow \phi_A(t, P_0)$ is bounded as $t \rightarrow -\infty$ and the limit sets of P_0 associated to ϕ_A is bounded.

Then, applying Theorem 10 in [12] together with $\text{div } g < 0$ and Proposition 3.1, we infer that

$$\boxed{*} \quad (3.16) \quad \lim_{|t| \rightarrow +\infty} \phi_A(t, P_0) \in \{(0, 0, 0), (1, 1, 1)\}.$$

Now, since $(0, 0, 0)$ is asymptotically stable, this implies $\phi_A(t, P_0) \rightarrow (1, 1, 1)$ as $t \rightarrow -\infty$ and we have two following cases:

Case 1: heteroclinic orbit i.e $\phi_A(t, P_0) \rightarrow (0, 0, 0)$ as $t \rightarrow +\infty$

or

Case 2: homoclinic orbit i.e $\phi_A(t, P_0) \rightarrow (1, 1, 1)$ as $t \rightarrow +\infty$.

Since ω_0 is invariant by the flow ϕ_A , both cases imply that $\omega(\zeta_0)$ contains either a connecting orbit between the two different equilibria ($P_0 \in W_u(1, 1, 1)$) or a cycle with respect to ϕ_A .

In the first case, the heteroclinic orbit included in ω_0 is not chain recurrent since $(0, 0, 0)$ is asymptotically stable which contradicts ω_0 is chain recurrent.

For the second case, $(1, 1, 1)$ is hyperbolic with transverse unstable and stable manifolds, there can not be any circuit (see for instance [11] page 1228 or [12] pages 1677-1678) and we have a contradiction.

Step 2: $\omega_0 = (1, 1, 1)$

Since ω_0 is connected then either $\omega_0 = \{(0, 0, 0)\}$ or $\omega_0 = \{(1, 1, 1)\}$.

The first possibility does not hold. Indeed arguing by contradiction, then there exists $T_0 > 0$ such that $\mathcal{X}(T_0) < 1$, $\mathcal{Y}(T_0) < 1$ and $\mathcal{Z}(T_0) < 1$.

Consider (\tilde{u}, \tilde{v}) a couple of blow-up solutions of (1.7) with the initial data $\tilde{v}(0) = v(0) + \varepsilon$ on $(0, R_\varepsilon)$ where $R_\varepsilon < 1$ by Proposition 2.1. Moreover from Proposition 2.2, we have R_ε goes to 1 as $\varepsilon \rightarrow 0$.

Let $R \in (0, 1)$, we define \mathcal{T}_1 , \mathcal{T}_2 and \mathcal{T}_3 from $C([0, R])$ to $C([0, -\ln(1-R)])$ three linear and continuous operators as follows: for any $w \in C([0, R])$, for any $t \in [0, T] = [0, -\ln(1-R)]$,

$$\mathcal{T}_1(w)(t) = \frac{e^{-\alpha_0 t}}{\lambda} w(1 - e^{-t}), \quad \mathcal{T}_2(w)(t) = \frac{e^{-\beta_0 t}}{\mu} w(1 - e^{-t}) \quad \text{and} \quad \mathcal{T}_3(w)(t) = \frac{e^{-\gamma_0 t}}{\nu} w(1 - e^{-t}).$$

By Remark 2.2, we get as $\varepsilon \rightarrow 0$:

$$\sup_{[0,T]} |\mathcal{X} - \tilde{X}| = \sup_{[0,T]} |\mathcal{T}_1(u') - \mathcal{T}_1(\tilde{u}')| \leq C \sup_{[0,R]} |u - \tilde{u}'| \rightarrow 0,$$

$$\sup_{[0,T]} |\mathcal{Y} - \tilde{Y}| = \sup_{[0,T]} |\mathcal{T}_2(v) - \mathcal{T}_2(\tilde{v})| \leq C \sup_{[0,R]} |v - \tilde{v}| \rightarrow 0$$

and

$$\sup_{[0,T]} |\mathcal{Z} - \tilde{Z}| = \sup_{[0,T]} |\mathcal{T}_3(v') - \mathcal{T}_3(\tilde{v}')| \leq C \sup_{[0,R]} |v' - \tilde{v}'| \rightarrow 0.$$

We deduce, for ε small enough that then there exists $R \in (1 - e^{-T_0}, R_\varepsilon)$,

$$\sup_{[0,T]} |\mathcal{X} - \tilde{X}| < 1 - \mathcal{X}(T_0), \quad \sup_{[0,T]} |\mathcal{Y} - \tilde{Y}| < 1 - \mathcal{Y}(T_0) \quad \text{and} \quad \sup_{[0,T]} |\mathcal{Z} - \tilde{Z}| < 1 - \mathcal{Z}(T_0).$$

Hence $\tilde{X}(T_0) < 1$, $\tilde{Y}(T_0) < 1$ and $\tilde{Z}(T_0) < 1$.

From Proposition 3.3, we deduce that, for any $t \in (T_0, -\ln(1 - R_\varepsilon))$, $\tilde{X}(t) < 1$, $\tilde{Y}(t) < 1$ and $\tilde{Z}(t) < 1$. Finally, by construction of \tilde{Y} , we have $\tilde{Y}(t) \rightarrow +\infty$ as $t \rightarrow -\ln(1 - R_\varepsilon)$ and we obtain a contradiction. \square

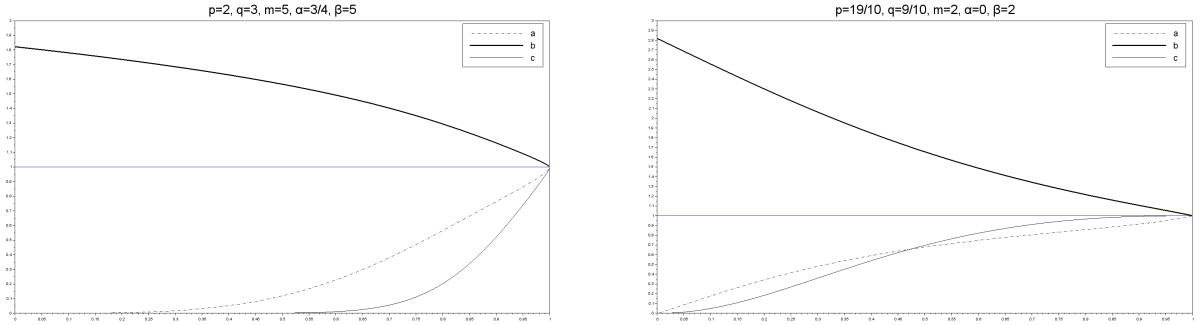


Figure 1: $\alpha_0 < 1$

fig1

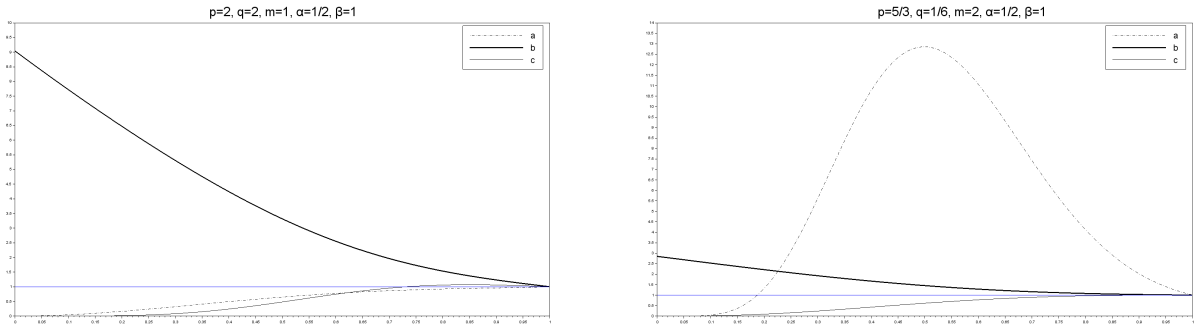


Figure 2: $\alpha_0 \geq 1$

fig2

Remark 3.2. Figures 1-3 give examples of convergence of the triplet functions (a, b, c) to $(1, 1, 1)$ as $r \rightarrow 1$ where (a, b, c) satisfies (3.7). More precisely, we have $\alpha_0 < 1$ in Figure 1 and $\alpha_0 \geq 1$ in Figure 2. We give particular examples for the conditions of Theorem 1.3 in Figure 3: in the left-hand graph, we choose $p > 2$ and $q < p - 1 - \alpha$ and in the right-hand graph, $\beta < 1$ and $m < 1$. For every graph, $\delta < 0$, $b_0 > 0$, $\alpha_0 > 0$ and $N = 3$.

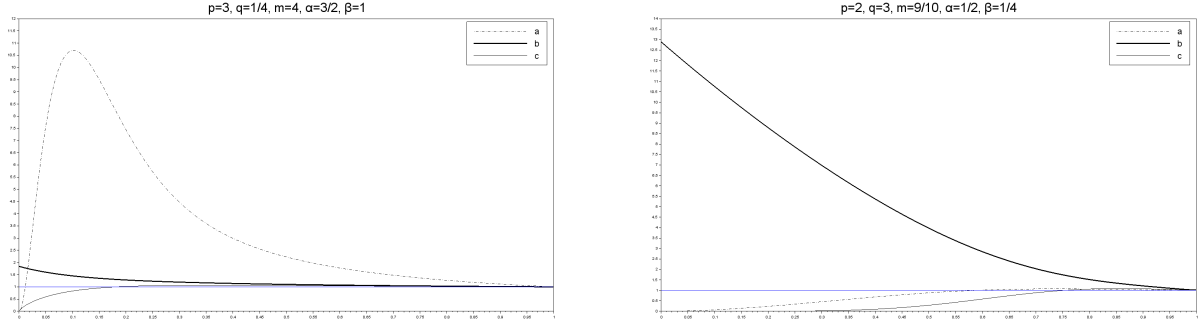


Figure 3: Other examples

fig3

Proof of Theorem 1.3:

We get from Proposition 3.5 that, as $r \rightarrow 1^-$

$$a(r) = \frac{U(r)}{\lambda}(1-r)^{\alpha_0} \rightarrow 1, \quad b(r) = \frac{v(r)}{\mu}(1-r)^{\beta_0} \rightarrow 1 \quad \text{and} \quad c(r) = \frac{V(r)}{\nu}(1-r)^{\gamma_0} \rightarrow 1$$

hence

$$u'(r) \sim \frac{\lambda}{(1-r)^{\alpha_0}} \quad \text{and} \quad v(r) \sim \frac{\mu}{(1-r)^{\beta_0}} \quad \text{as } r \rightarrow 1^-.$$

Since $\beta_0 > 0$, we have obviously $v(1) = \infty$. Finally we deduce

$$u(r) \sim \frac{\lambda}{(\alpha_0 - 1)(1-r)^{\alpha_0 - 1}} \quad \text{as } r \rightarrow 1^-$$

if $\alpha_0 \neq 1$ and

$$u(r) \sim \lambda \ln(1-r) \quad \text{as } r \rightarrow 1^-$$

if $\alpha_0 = 1$.

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