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Null-forms of conic systems in \mathbb{R}^3 are determined by their symmetries

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ABSTRACT

We address the problem of characterisation of null-forms of conic 3-dimensional systems, that is, control-affine systems whose field of admissible velocities forms a conic (without parameters) in the tangent space. Those systems have been previously identified as the simplest control systems under a conic nonholonomic constraint or as systems of zero curvature. In this work, we propose a direct characterisation of null-forms of conic systems among all control-affine systems by studying the Lie algebra of infinitesimal symmetries. Namely, we show that the Lie algebra of infinitesimal symmetries characterises uniquely null-forms of conic systems.

1. Introduction

In Schmoderer and Respondek (2021), we propose a characterisation and a classification of single input control-affine systems which admit conic sets as admissible velocities. In particular, we give a characterisation of all 3-dimensional control-affine systems with scalar control that are feedback equivalent, locally around $(x_0, y_0, w_0) \in \mathbb{R}^3$, to one of the following conic null-forms

$$\Sigma_E : \begin{cases} \dot{x} = \cos(w) \\ \dot{y} = \sin(w) \\ \dot{w} = u \end{cases}, \quad \Sigma_H : \begin{cases} \dot{x} = \cosh(w) \\ \dot{y} = \sinh(w) \\ \dot{w} = u \end{cases},$$
$$\text{and } \Sigma_P : \begin{cases} \dot{x} = w^2 \\ \dot{y} = w \\ \dot{w} = u \end{cases}.$$

The parabolic system Σ_P actually depends on the nature of the point (x_0, y_0, w_0) around which the systems Σ_P is considered: an equilibrium or not. As a consequence, Σ_P has two distinct normal forms, for $w_0 = 0$ and $w_0 \neq 0$, and we will specify this distinction just before our main result Theorem 1.

We call the systems Σ_E , Σ_H , and Σ_P , *null-forms* because of the absence of any parameters, functional or real-continuous, in their expression (in particular, observe that those system are trivial in the sense of Serres (2009)). We call the systems Σ_E , Σ_H , and Σ_P , elliptic, hyperbolic, and parabolic, respectively, because in the manifold \mathbb{R}^2 , equipped with local coordinates (x, y) , the trajectories $(x(t), y(t))$ of Σ_E , Σ_H , and Σ_P satisfy a nonholonomic constraint of the form

$$S_E = \dot{x}^2 + \dot{y}^2 - 1 = 0, \quad S_H = \dot{x}^2 - \dot{y}^2 - 1 = 0,$$
$$\text{and } S_P = \dot{y}^2 - \dot{x} = 0,$$

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respectively. And, clearly, S_E , resp. S_H , and resp. S_P , describes ellipses, resp. hyperbolas, and resp. parabolas, in the tangent bundle $T\mathbb{R}^2$; see Schmoderer and Respondek (2021) for a detailed analysis of the link between quadratic equations on $T\mathbb{R}^2$ and control systems. The elliptic and hyperbolic systems Σ_E and Σ_H show up in the seminal trend-setting paper Agrachev (1998) as control-affine systems of zero curvature. Moreover, Σ_E describes the well-known Dubins model of a car Dubins (1957), while Σ_H is its hyperbolic counterpart Monroy-Pérez (1998). The parabolic system Σ_P is studied in Schmoderer and Respondek (2021). We will denote by Σ_Q the set $\{\Sigma_E, \Sigma_H, \Sigma_P\}$ of the three above null-forms conic systems when we discuss properties that do not require to distinguish a specific system.

In the present work, we propose a direct local characterisation of those null-forms conic systems within the class of all control-affine systems on a 3-dimensional manifold with scalar control. In Schmoderer and Respondek (2021), we analysed systems under quadratic nonholonomic constraints and, in particular, studied the null forms Σ_Q but the obtained results differ from those of the present paper in two aspects. In Schmoderer and Respondek (2021), first, we characterise Σ_E , Σ_H , and Σ_P among conic systems and, second, that goal is achieved by identifying invariants of the classification. On the other hand, in the present paper, first, we propose a characterisation that directly applies to any control-affine system (on a 3D-manifold with scalar control). Second, we use completely different tools. Namely, the methodology that we apply to describe our null-forms is based on the study of the Lie algebra of infinitesimal symmetries (see below for a precise definition), which turns out to uniquely determine all null-forms Σ_Q (with additional regularity conditions for Σ_P).

1.1. Outline of the paper

The paper is organised as follows. In the next subsection, we will introduce all notions of control theory that we will need for our results. In particular, we give a definition of feedback equivalence and of infinitesimal symmetries. Next, we will first compute the Lie algebra of infinitesimal symmetries of null-forms of conic systems Σ_Q and identify those

Lie algebras in the well-known Bianchi classification of 3-dimensional Lie algebras. Afterwards, we will enunciate and prove our main result, which shows that the Lie algebras of infinitesimal symmetries of null-forms of conic systems completely determine corresponding systems.

1.2. Preliminaries

Throughout the paper, the word "smooth" will always mean C^∞ -smooth, and all objects (vector fields, differential forms, functions, manifolds) are assumed to be smooth.

Control-affine systems. We consider control-affine systems Σ of the form

$$\Sigma : \dot{\xi} = f(\xi) + g(\xi)u, \quad u \in \mathbb{R},$$

where the state ξ belongs to a smooth 3-dimensional manifold \mathcal{X} (or an open subset of \mathbb{R}^3 since all our results are local), and f and g are smooth vector fields on \mathcal{X} (smooth sections of the tangent bundle $T\mathcal{X}$). We denote a control-affine system by the pair $\Sigma = (f, g)$, we set $\mathcal{G} = \text{span}\{g\}$, the distribution spanned by the vector field g , and we set $\mathcal{G}^1 = \text{span}\{g, [f, g]\}$. We call two control affine systems $\Sigma = (f, g)$ and $\tilde{\Sigma} = (\tilde{f}, \tilde{g})$ on \mathcal{X} feedback equivalent, if there exists a diffeomorphism $\phi : \mathcal{X} \rightarrow \mathcal{X}$ and smooth functions $\alpha(\xi)$ and $\beta(\xi)$, satisfying $\beta(\xi) \neq 0$, and such that

$$\tilde{f} = \phi_*(f + g\alpha) \quad \text{and} \quad \tilde{g} = \phi_*(g\beta),$$

where ϕ_* denotes the tangent map of ϕ , that is $(\phi_*f)(\tilde{\xi}) = \frac{\partial \phi}{\partial \xi}(\phi^{-1}(\tilde{\xi})) \cdot f(\phi^{-1}(\tilde{\xi}))$. If ϕ is defined locally around ξ_0 and $\tilde{\xi}_0 = \phi(\xi_0)$, then we say that Σ and $\tilde{\Sigma}$ are locally feedback equivalent at ξ_0 and $\tilde{\xi}_0$, respectively. A feedback transformation will be denoted by the triple (ϕ, α, β) or simply (α, β) if $\phi = \text{Id}$.

Infinitesimal symmetries. We introduce the notion of symmetries in the case of single input control-affine system (see Respondek and Tall (2002); Grizzle and Marcus (1985) for a detailed introduction). For a control-affine system $\Sigma = (f, g)$, we define the field of admissible velocities \mathcal{A} as

$$\mathcal{A}(\xi) = \{f(\xi) + g(\xi)u : u \in \mathbb{R}\} \subset T_\xi \mathcal{X}.$$

We say that a diffeomorphism $\phi : \mathcal{X} \rightarrow \mathcal{X}$ is a *symmetry* of Σ if it preserves the field of affine lines \mathcal{A} (equivalently, the affine distribution $\mathcal{A} = f + \mathcal{G}$), that is,

$$\phi_*\mathcal{A} = \mathcal{A}.$$

We say that a vector field v on \mathcal{X} is an infinitesimal symmetry of $\Sigma = (f, g)$ if the (local) flow γ_t^v of v is a local symmetry, for any t for which it exists, that is, $(\gamma_t^v)_*\mathcal{A} = \mathcal{A}$. Consider the system $\Sigma = (f, g)$ and recall that \mathcal{G} is the distribution spanned by the vector field g . Since $(\gamma_t^v)_*(f + \mathcal{G}) = f + \mathcal{G}$, it follows that $(\gamma_t^v)_*$ preserves \mathcal{G} and preserves f modulo \mathcal{G} , which immediately gives the following characterisation of infinitesimal symmetries.

Proposition 1. *A vector field v is an infinitesimal symmetry of the control-affine system $\Sigma = (f, g)$ if and only if*

$$[v, g] = 0 \pmod{\mathcal{G}} \quad \text{and} \quad [v, f] = 0 \pmod{\mathcal{G}}.$$

By the Jacobi identity, it is easy to see that if v_1 and v_2 are infinitesimal symmetries, then so is $[v_1, v_2]$, hence the set of all infinitesimal symmetries forms a real Lie algebra. Notice that the Lie algebra of infinitesimal symmetries is attached to the affine distribution $\mathcal{A} = f + \mathcal{G}$ and not to a particular pair (f, g) . Different pairs (f, g) related via feedback transformations (α, β) define the same \mathcal{A} and thus give rise to the same Lie algebra of infinitesimal symmetries which, therefore, is a feedback invariant object attached to Σ .

2. Main results

In this section, we prove that the Lie algebras of infinitesimal symmetries of the systems Σ_E, Σ_H , and Σ_P , determine the corresponding systems among all control-affine systems. With the help of Proposition 1, we give in the next lemma the Lie algebras of infinitesimal symmetries of the systems Σ_E, Σ_H , and Σ_P .

Lemma 1 (Lie algebra of infinitesimal symmetries of Σ_Q). *The Lie algebra of infinitesimal symmetries of Σ_E, Σ_H , and Σ_P are respectively:*

$$\mathfrak{L}_E = \text{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} - \frac{\partial}{\partial w} \right\},$$

$$\mathfrak{L}_H = \text{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial w} \right\},$$

$$\mathfrak{L}_P = \text{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + w \frac{\partial}{\partial w} \right\}.$$

We will denote by \mathfrak{L}_Q the set of the three above Lie algebras in order to state general facts about them. Those three Lie algebras share the property of having the abelian Lie ideal $\text{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$, which corresponds to the fact that the systems Σ_Q are invariant under translations $(x, y) \mapsto (x + c, y + d)$, with $c, d \in \mathbb{R}$. The third vector field plays the role of an Euler vector field for \mathfrak{L}_P and is an infinitesimal rotation (trigonometric or hyperbolic) in the case of \mathfrak{L}_E and \mathfrak{L}_H .

Proof. The proof is a direct calculation using the characterisation of infinitesimal symmetries given in Proposition 1. Let $v = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial w}$, where $v_i = v_i(x, y, w)$, be an infinitesimal symmetries of Σ_Q . Then, using that v is a symmetry of \mathcal{G} , we deduce that $v_1 = v_1(x, y)$ and $v_2 = v_2(x, y)$. We now distinguish the three conic systems.

For Σ_E , we have $f = \cos(w) \frac{\partial}{\partial x} + \sin(w) \frac{\partial}{\partial y}$, hence $[v, f] \in \mathcal{G}$ implies

$$\cos(w) \frac{\partial v_1}{\partial x} + \sin(w) \frac{\partial v_1}{\partial y} = -\sin(w) v_3,$$

$$\cos(w) \frac{\partial v_2}{\partial x} + \sin(w) \frac{\partial v_2}{\partial y} = \cos(w) v_3.$$

Multiplying the first equation by $\cos(w)$ and inserting the second equation, we obtain

$$\begin{aligned} \left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} \right) \cos(2w) + \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) \sin(2w) \\ + \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) = 0, \end{aligned}$$

which yields, first, $v_1 = v_1(y)$ and $v_2 = v_2(x)$, and, second, $v_1(y) = ay + b$ and $v_2(x) = -ax + c$, where $a, b, c \in \mathbb{R}$. Finally, taking $(a, b, c) = (1, 0, 0)$, $(a, b, c) = (0, 1, 0)$, and $(a, b, c) = (0, 0, 1)$ we obtain \mathfrak{L}_E .

For Σ_H , it is an analogous calculation as above but with hyperbolic functions.

For Σ_P , we have $f = w^2 \frac{\partial}{\partial x} + w \frac{\partial}{\partial y}$, hence $[v, f] \in \mathcal{G}$ implies

$$\begin{aligned} w^2 \frac{\partial v_1}{\partial x} + w \frac{\partial v_1}{\partial y} &= 2wv_3, \\ w^2 \frac{\partial v_2}{\partial x} + w \frac{\partial v_2}{\partial y} &= v_3. \end{aligned}$$

Inserting the second equation in the first, we obtain a polynomial of degree 3 in w that identically vanishes. Thus each of its coefficient vanishes, implying $\frac{\partial v_2}{\partial x} = 0$, $\frac{\partial v_1}{\partial y} = 0$, and $\frac{\partial v_1}{\partial x} = 2 \frac{\partial v_2}{\partial y}$. Therefore, we conclude that $v = (2ax + b) \frac{\partial}{\partial x} + (ay + c) \frac{\partial}{\partial y} + aw \frac{\partial}{\partial w}$, where $a, b, c \in \mathbb{R}$. Finally, taking $(a, b, c) = (1, 0, 0)$, $(a, b, c) = (0, 1, 0)$, and $(a, b, c) = (0, 0, 1)$ we obtain \mathfrak{L}_P . \square

Based on the commutativity relations of the generators of \mathfrak{L}_Q , we identify the three Lie algebras in the classification of 3-dimensional Lie algebras as presented by Winternitz in Bowers (2005). First, $\mathfrak{L}_E \cong L(3, 4, 0)$ is the Lie algebra of the euclidean group $E(2)$, that is, the group of affine transformations of \mathbb{R}^2 which preserve the euclidean metric. In other words, \mathfrak{L}_E is VII₀ in the Bianchi (1898, 2001) classification of 3-dimensional Lie algebras. Second, $\mathfrak{L}_H \cong L(3, 2, -1)$ is the Lie algebra of the Poincaré group $P(1, 1)$, of affine transformations of \mathbb{R}^2 which preserve the Lorentz metric, i.e. \mathfrak{L}_H is VI₀ in the Bianchi classification. Finally, $\mathfrak{L}_P \cong L(3, 2, 2)$ with no immediate interpretation and \mathfrak{L}_P is one of VI-algebras of the Bianchi classification, see Proposition 1 for its characterisation. The notation emphasises that \mathfrak{L}_P and \mathfrak{L}_H fall into the same class of Bowers (2005) but for different parameters (2 and -1 , respectively), while \mathfrak{L}_E remains apart. When considering complex Lie algebras we would have $\mathfrak{L}_E \cong L(3, 2, -1)$ and thus the three Lie algebras \mathfrak{L}_Q would fall into the same class but in that case we would not be able to distinguish between \mathfrak{L}_E and \mathfrak{L}_H as those complex Lie algebras would be isomorphic.

The following proposition gives checkable conditions to identify the Lie algebras \mathfrak{L}_Q among all 3-dimensional Lie algebras.

Proposition 2 (Characterisation of \mathfrak{L}_Q). *A 3-dimensional Lie algebra \mathfrak{L} is isomorphic to \mathfrak{L}_Q if and only if $\mathfrak{L} = I \oplus L$, with I an abelian ideal of dimension 2, and the action $\text{ad}_\ell : I \rightarrow I$ of any $\ell \in L$ on I satisfies*

- (i) *either ad_ℓ is diagonalisable with two non-zero real eigenvalues related by $\lambda_1 = 2\lambda_2$, in which case $\mathfrak{L} \cong \mathfrak{L}_P$.*
- (ii) *or ad_ℓ has two purely imaginary eigenvalues, in which case $\mathfrak{L} \cong \mathfrak{L}_E$.*
- (iii) *or ad_ℓ is diagonalisable with two non-zero real eigenvalues related by $\lambda_1 = -\lambda_2$, in which case $\mathfrak{L} \cong \mathfrak{L}_H$.*

Proof. See Bowers (2005). \square

Before giving our main result, notice that null-forms elliptic and hyperbolic systems Σ_E and Σ_H do not possess any equilibrium point, i.e. $0 \notin \mathcal{A}(\xi)$. On the other hand, the drift $f_P = w^2 \frac{\partial}{\partial x} + w \frac{\partial}{\partial y}$ of the null-form parabolic system Σ_P considered locally around (x_0, y_0, w_0) possesses an equilibrium if $w_0 = 0$ and not if $w_0 \neq 0$. As a consequence, the system Σ_P possesses two non-equivalent local normal forms, which can be represented around $w_0 = 0$ by

$$\begin{aligned} \Sigma_P^0 : \begin{cases} \dot{x} &= w^2 \\ \dot{y} &= w \\ \dot{w} &= u \end{cases} \\ \text{and } \Sigma_P^1 : \begin{cases} \dot{x} &= (w+1)^2 \\ \dot{y} &= w+1 \\ \dot{w} &= u \end{cases}, \text{ respectively.} \end{aligned}$$

We now formulate and prove that the Lie algebra \mathfrak{L} of Σ , being isomorphic to \mathfrak{L}_Q , completely characterises those 3-dimensional control-affine systems $\Sigma = (f, g)$, with scalar control, that are locally feedback equivalent to Σ_Q . Recall that we attach to a control affine system the distributions $\mathcal{G} = \text{span}\{g\}$ and $\mathcal{G}^1 = \text{span}\{g, [f, g]\}$.

Theorem 1 (Σ_Q are characterised by \mathfrak{L}_Q). *Consider the control-affine system $\Sigma : \dot{\xi} = f(\xi) + g(\xi)u$ on a smooth 3-dimensional manifold \mathcal{X} with a scalar control, and let \mathfrak{L} be its Lie algebra of infinitesimal symmetries.*

- (i) *Σ is locally feedback equivalent to Σ_E , around $\xi_0 \in \mathcal{X}$, if and only if*

$$\mathfrak{L} \cong \mathfrak{L}_E \quad \text{and} \quad I(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0} \mathcal{X}.$$

- (ii) *Σ is locally feedback equivalent to Σ_H , around $\xi_0 \in \mathcal{X}$, if and only if*

$$\mathfrak{L} \cong \mathfrak{L}_H \quad \text{and} \quad I(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0} \mathcal{X}.$$

- (iii) *Σ is, locally around $\xi_0 \in \mathcal{X}$, feedback equivalent to Σ_P^1 , around $(x_0, y_0, 0)$, if and only if*

$$\begin{aligned} \mathfrak{L} \cong \mathfrak{L}_P, \quad I(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0} \mathcal{X}, \\ \text{and} \quad f(\xi_0) \notin \mathcal{G}(\xi_0). \end{aligned}$$

(iii)' Σ is, locally around $\xi_0 \in \mathcal{X}$, feedback equivalent to Σ_P^0 , around $(x_0, y_0, 0)$, if and only if

$$\begin{aligned} \mathfrak{L} &\cong \mathfrak{L}_P, \quad I(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0} \mathcal{X}, \\ f(\xi_0) &\in \mathcal{G}(\xi_0), \quad \text{and} \quad \dim \mathcal{G}^1(\xi_0) = 2. \end{aligned}$$

Remark 1. Two systems Σ_E (resp. Σ_H) around any two points ξ_0 and $\tilde{\xi}_0$ are locally feedback equivalent to each other. On the other hand, equilibrium points $f(\xi_0) \in \mathcal{G}(\xi_0)$ and non-equilibrium points $f(\xi_0) \notin \mathcal{G}(\xi_0)$ are distinguished for Σ_P : the system is feedback equivalent to Σ_P^0 in the former case and to Σ_P^1 in the latter as assert statements (iii)' and (iii), respectively.

Notice that in statement (iii)' the condition on the pointwise rank of the distribution \mathcal{G}^1 can be replaced by $(g \wedge [f, g])(\xi_0) \neq 0$, compare this statement with the assumptions of Proposition 3 below.

Proof. We show the sufficiency part of the statements only as the necessity follows immediately from the list of infinitesimal symmetries given by Lemma 1. For all three cases, the beginning of the proof is the same.

Consider a control-affine system Σ , given by vector fields f and g , and let three vector fields v_1, v_2, v_3 span the 3-dimensional Lie algebra $\mathfrak{L} = \text{vect}_{\mathbb{R}} \{v_1, v_2, v_3\}$ of infinitesimal symmetries, which by assumption is isomorphic to \mathfrak{L}_Q . We can assume that the abelian ideal of \mathfrak{L} is $I = \text{vect}_{\mathbb{R}} \{v_1, v_2\}$ and that v_1, v_2, v_3 satisfy the commutativity relations of \mathfrak{L}_Q (we express them below for each case separately). Since $I(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0} \mathcal{X}$, it follows that v_1, v_2 , and g are independent, locally around ξ_0 . We apply a local diffeomorphism $\psi(\xi) = (\tilde{x}, \tilde{y}, \tilde{w})$, $\psi(\xi_0) = 0 \in \mathbb{R}^3$, such that $\tilde{v}_1 = \psi_* v_1 = \frac{\partial}{\partial \tilde{x}}$, $\tilde{v}_2 = \psi_* v_2 = \frac{\partial}{\partial \tilde{y}}$, and $\tilde{g} = \psi_* g = g^1 \frac{\partial}{\partial \tilde{x}} + g^2 \frac{\partial}{\partial \tilde{y}} + g^3 \frac{\partial}{\partial \tilde{w}}$, for some smooth functions g^i , satisfying $g^3(0) \neq 0$. Replacing \tilde{g} by $\frac{1}{g^3} \tilde{g}$, we may assume that $\tilde{g} = g^1 \frac{\partial}{\partial \tilde{x}} + g^2 \frac{\partial}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{w}}$. Then, since \tilde{v}_1 and \tilde{v}_2 are symmetries of $\mathcal{G} = \text{span} \left\{ g^1 \frac{\partial}{\partial \tilde{x}} + g^2 \frac{\partial}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{w}} \right\}$, we have $[\tilde{v}_i, \tilde{g}] \in \mathcal{G}$, for $i = 1, 2$, which implies $g^1 = g^1(\tilde{w})$ and $g^2 = g^2(\tilde{w})$. Therefore, we in fact have $[\tilde{v}_1, \tilde{g}] = [\tilde{v}_2, \tilde{g}] = 0$ and thus there exists a local diffeomorphism $(x, y, w) = \phi(\tilde{x}, \tilde{y}, \tilde{w})$ such that $\phi_* \tilde{v}_1 = \frac{\partial}{\partial x}$, $\phi_* \tilde{v}_2 = \frac{\partial}{\partial y}$ and $\phi_* \tilde{g} = \frac{\partial}{\partial w}$. Denote $v_3 = v_3^1 \frac{\partial}{\partial x} + v_3^2 \frac{\partial}{\partial y} + v_3^3 \frac{\partial}{\partial w}$, the third infinitesimal symmetry, where $v_3^1 = v_3^1(x, y)$ and $v_3^2 = v_3^2(x, y)$ since v_3 is a symmetry of $\mathcal{G} = \text{span} \left\{ \frac{\partial}{\partial w} \right\}$. We now separate the case of elliptic, hyperbolic, and parabolic systems.

(i) Assume that $\mathfrak{L} \cong \mathfrak{L}_E$ and, since diffeomorphisms do not change the commutation relations, we have $[v_1, v_3] = -v_2$ and $[v_2, v_3] = v_1$. Hence,

$$v_3 = (y + c) \frac{\partial}{\partial x} + (-x + d) \frac{\partial}{\partial y} + v_3^3(w) \frac{\partial}{\partial w},$$

where c, d are real constants. To simplify our computations, we replace v_3 by $v_3 - cv_1 - dv_2 \in \mathfrak{L}$,

which does not change the commutativity relations, so we can get rid of both constants. Now consider the vector field $f = f^1 \frac{\partial}{\partial x} + f^2 \frac{\partial}{\partial y} + f^3 \frac{\partial}{\partial w}$, then by the fact that v_1 and v_2 are symmetries of Σ we get that $f^1 = f^1(w)$ and $f^2 = f^2(w)$. Using the symmetry $v_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + v_3^3(w) \frac{\partial}{\partial w}$, calculate

$$\begin{aligned} [v_3, f] &= \left(v_3^3 \frac{\partial f^1}{\partial w} - f^2 \right) \frac{\partial}{\partial x} \\ &\quad + \left(v_3^3 \frac{\partial f^2}{\partial w} + f^1 \right) \frac{\partial}{\partial y} \quad \text{mod } \mathcal{G}, \end{aligned}$$

which implies the following system of ODEs, where the derivatives are taken with respect to w :

$$(Sys_E) : \begin{cases} v_3^3(f^1)' - f^2 = 0 \\ v_3^3(f^2)' + f^1 = 0 \end{cases}.$$

By multiplying the first equation of (Sys_E) by f_1 and the second by f_2 and adding them, we obtain the following relation $v_3^3((f^1)^2 + (f^2)^2)' = 0$. From (Sys_E) , it is clear that $v_3^3 \neq 0$ otherwise we would have $f^1 \equiv f^2 \equiv 0$ and the Lie algebra of symmetries would be much larger than \mathfrak{L} . Observe that the points w , where $v_3^3(w) = 0$, satisfy also $f^1(w) = f^2(w) = 0$. We will show that we always have $((f^1)^2 + (f^2)^2)' = 0$. If $v_3^3(0) \neq 0$, then it is clear that the conclusion holds, if $v_3^3(0) = 0$ but in an open neighbourhood around 0 there are no other points where v_3^3 vanishes, then it is also clear that the conclusion holds. Finally if $v_3^3(0) = 0$ and in any open neighbourhood of zero there are other zeros of v_3^3 , then consider the open segment $]w_1, w_2[$ between two consecutive zeros. For all $w \in]w_1, w_2[$ we have $((f^1)^2 + (f^2)^2)' = 0$ thus $(f^1)^2 + (f^2)^2 = R$ ($R \in \mathbb{R}$) but since for $w = w_1$ and $w = w_2$ we have $f^1(w_i) = f^2(w_i) = 0$, for $i = 1, 2$, therefore, by continuity, $R = 0$. But then, we have $((f^1)^2 + (f^2)^2) = 0$ implying $f^1 = f^2 = 0$ on $]w_1, w_2[$ and repeating this process for all consecutive roots around 0 implies that $f^1 = f^2 = 0$ around 0 but then the Lie algebra \mathfrak{L} of infinitesimal symmetries would be of infinite dimension and this situation is excluded. Finally, we have $((f^1)^2 + (f^2)^2)' = 0$ around zero, implying $(f^1)^2 + (f^2)^2 = R$, $R > 0$. Therefore $f^1 = \sqrt{R} \cos(w)$ and $f^2 = \sqrt{R} \sin(w)$ and, by normalising the coordinates x and y , we obtain the form Σ_E .

(ii) The proof is essentially the same so we omit some details. Using the multiplication table $[v_1, v_3] = v_2$ and $[v_2, v_3] = v_1$, we obtain

$$v_3 = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + v_3^3(w) \frac{\partial}{\partial w},$$

which, used as a symmetry of f , yields the system

$$(Sys_H) : \begin{cases} v_3^3(f^1)' - f^2 = 0 \\ v_3^3(f^2)' - f^1 = 0 \end{cases}.$$

We derive the relation $v_3^3((f^1)^2 - (f^2)^2)' = 0$ and by a similar reasoning we get $((f^1)^2 - (f^2)^2)' = 0$ around 0. Thus, $((f^1)^2 - (f^2)^2) = R$ with $R \in \mathbb{R}$, but necessarily $R \neq 0$ because the Lie algebra of infinitesimal symmetries of Σ with $f^1 = \pm\sqrt{|R|}$ is of infinite dimension. Finally, we have $f^1 = \sqrt{|R|} \cosh(w)$ and $f^2 = \sqrt{|R|} \sinh(w)$ and using a coordinate normalisation we get $f^1 = \cosh(w)$ and $f^2 = \sinh(w)$.

- (iii) Using the multiplication table of \mathfrak{L}_P given by $[v_1, v_3] = 2v_1$ and $[v_2, v_3] = v_2$, which has not been changed by applying diffeomorphisms, we obtain

$$v_3 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + v_3(w) \frac{\partial}{\partial w},$$

which is a symmetry of f and thus yields the system

$$(Sys_P) : \begin{cases} v_3^3(f^1)' - 2f^1 = 0 \\ v_3^3(f^2)' - f^2 = 0 \end{cases},$$

recall that $f^1 = f^1(w)$ and $f^2 = f^2(w)$. We will now distinguish two cases, namely we separate between existence or not of an equilibrium at $w_0 = 0$.

- (a) Assume that $f(0) \notin \mathcal{G}(0)$, that is $(f^1, f^2)(0) \neq (0, 0)$ and thus by (Sys_P) we have $((f^1)', (f^2)')(0) \neq (0, 0)$. Assume $f^2(0) \neq 0$, thus $f^2(w) = c + h(w)$, where $c = f^2(0)$ and $h(0) = 0$. Replacing y by y/c we may assume that $f^2(w) = 1 + h(w)$, where $h'(0) \neq 0$, if not the second equation of (Sys_P) is not satisfied at $0 \in \mathbb{R}$. Set $\hat{w} = h(w)$ and denote the transformed vector fields \hat{f} and \hat{v}_3 , for which (Sys_P) implies $\hat{v}_3^3 = 1 + \hat{w}$ and

$$(\hat{f}^1)' (1 + \hat{w}) = 2\hat{f}^1.$$

Solving this equation gives $\hat{f}^1(\hat{w}) = c(1 + \hat{w})^2$ with $c \in \mathbb{R}$. But c can not be 0, otherwise the Lie algebra \mathfrak{L} of infinitesimal symmetries would be of infinite dimension, thus not isomorphic to \mathfrak{L}_P . Finally, introducing $\hat{x} = x/c$ we obtain Σ_P^1 . If $f^2(0) = 0$, then $f^1(0) \neq 0$ implying that $(f^1)'(0) \neq 0$ and leading to the normalisation $\hat{f}^1(\hat{w}) = 1 + \hat{w}$ giving $\hat{v}_3^3 = 2(1 + \hat{w})$ and $\hat{f}^2 = (1 + \hat{w})^{1/2}$. This forms is equivalent to Σ_P^1 by the local diffeomorphism $w = (1 + \hat{w})^{1/2} - 1$, sending 0 into 0.

- (b) Assume $f(0) \in \mathcal{G}(0)$ and $g \wedge \text{ad}_g f(0) \neq 0$. If $(f^2)'(0) \neq 0$, take $(\hat{x}, \hat{y}, \hat{w}) = (x, y, f^2(w))$ as a local diffeomorphism around $0 \in \mathbb{R}^3$ that maps f^1, f^2 and v_3^3 into \hat{f}^1, \hat{f}^2 and \hat{v}_3^3 , respectively. We have $\hat{f}^2 = \hat{w}$, so the system (Sys_P) implies $\hat{v}_3^3 = \hat{w}$ and

$$\hat{w} (\hat{f}^1)' = 2\hat{f}^1.$$

Solving this equation gives $\hat{f}^1(\hat{w}) = c(\hat{w})^2$ with $c \in \mathbb{R}$. But c can not be 0, otherwise the Lie algebra \mathfrak{L} of infinitesimal symmetries would be of infinite dimension. Finally, introducing $\hat{x} = x/c$ we obtain Σ_P^0 . If $(f^2)'(0) = 0$, then $(f^1)'(0) \neq 0$ and by applying the local diffeomorphism $(\hat{x}, \hat{y}, \hat{w}) = (x, y, f^1(w))$ we get $\hat{f}^1 = \hat{w}$ yielding $\hat{v}_3^3(\hat{w}) = 2\hat{w}$ and $2\hat{w}(\hat{f}^2)' = \hat{f}^2$. Hence, $|\hat{f}^2| = d|\hat{w}|^{1/2}$ and the only smooth solution, around $\hat{w} = 0$, is given by $d = 0$ but then the Lie algebra \mathfrak{L} of infinitesimal symmetries would be of infinite dimension contradicting our assumption. \square

All systems Σ that are locally feedback equivalent to either Σ_E or Σ_H are completely characterised by their symmetry algebras being isomorphic to \mathfrak{L}_E or \mathfrak{L}_H , respectively. This is not the case for Σ_P . Namely, there are systems that have \mathfrak{L}_P as the symmetry algebra although they are not locally feedback equivalent to Σ_P .

Proposition 3. *Let $\Sigma : \dot{\xi} = f(\xi) + g(\xi)u$ be a control-affine system on a smooth 3-dimensional manifold \mathcal{X} with a scalar control, and let \mathfrak{L} be its Lie algebra of infinitesimal symmetries. Assume $f(\xi_0) \in \mathcal{G}(\xi_0)$ and, additionally, that there exists $k \geq 1$, the smallest integer such that $g \wedge \text{ad}_g^k f(\xi_0) \neq 0$. Then,*

$$\mathfrak{L} \cong \mathfrak{L}_P \quad \text{and} \quad I(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0} \mathcal{X}$$

if and only if Σ is feedback equivalent to

$$\Sigma_P^{0,k} : \begin{cases} \dot{z} = w^{2k} \\ \dot{y} = w^k \\ \dot{w} = u \end{cases}$$

around $(x_0, y_0, 0) \in \mathbb{R}^3$.

Moreover, it is a classical fact that (under the above assumptions) the integer k is an invariant of feedback transformations, hence if $k \neq k'$, $\Sigma_P^{0,k}$ and $\Sigma_P^{0,k'}$ are not locally feedback equivalent around $w_0 = 0$. To be consistent with the notation of the above proposition, the previously considered normal form Σ_P^0 should actually be denoted $\Sigma_P^{0,1}$.

Proof. We prove the necessity only, as there are no difficulties to show (repeating the arguments of Lemma 1) that the Lie algebra of infinitesimal symmetries of $\Sigma_P^{0,k}$ is isomorphic to \mathfrak{L}_P . We adapt the point (iii)-(b) of the proof of Theorem 1. Assume that $f(0) \in \mathcal{G}(0)$ and that $k \geq 1$ is the smallest integer such that $g \wedge \text{ad}_g^k f(0) \neq 0$, that is $\frac{d^k}{dw^k}(f^1, f^2)(0) \neq (0, 0)$. If $(f^2)^{(k)}(0) \neq 0$, then by the Taylor expansion we can write $f^2(w) = w^k H(w)$, where $H(0) \neq 0$. We can suppose $H(0) > 0$, if not, replace y by $-y$, and we apply around $0 \in \mathbb{R}^3$ the local diffeomorphism $(\hat{x}, \hat{y}, \hat{w}) = (x, y, w(H(w))^{1/k})$ that maps f^1, f^2 and v_3^3 into

\hat{f}^1 , \hat{f}^2 and \hat{v}_3^3 , respectively. We have $\hat{f}^2 = \hat{w}^k$, so the system (Sys_P) implies $\hat{v}_3^3 = \hat{w}/k$ and

$$\hat{w} (\hat{f}^1)' = 2k\hat{f}^1.$$

Solving this singular equation gives $\hat{f}^1(\hat{w}) = c\hat{w}^{2k}$ with $c \in \mathbb{R}$. However, the solution passing through $\hat{w} = 0$ is not unique so a priori we may have different values of c for $\hat{w} < 0$ and $\hat{w} > 0$ but the only C^∞ solutions are those given by the same value of c (either $c = 0$ or $c \neq 0$) for any \hat{w} . But c can not be 0, otherwise the Lie algebra \mathfrak{L} of infinitesimal symmetries would be of infinite dimension. Finally, introducing $\hat{x} = x/c$ we obtain the desired form $\Sigma_P^{0,k}$.

If $(f^1)^{(k)}(0) \neq 0$, then normalizing $\hat{f}^1 = \hat{w}^k$ and applying an analogous procedure we deduce that $\hat{f}^2(\hat{w}) = c|\hat{w}|^{k/2}$. If $c = 0$ then the Lie algebra of infinitesimal symmetries would be of infinite dimension contradicting our assumptions. In all other cases of k and c the solution is not smooth around $\hat{w} = 0$ except for $k = 2l$ and the same value of c for $\hat{w} < 0$ and $\hat{w} > 0$. But in the latter case we have $(f^2)^{(l)}(0) \neq 0$, with $l < k$, that contradicts the definition of k . \square

Remark 2. Statement (iii) of Theorem 1 and Proposition 3 describe all systems having \mathfrak{L}_P as the symmetry algebra for which k exists (in particular, all analytic systems). In the C^∞ category there are, however, systems for which k does not exist and the symmetry algebra is \mathfrak{L}_P . For example, consider

$$\begin{cases} \dot{x} = \mathfrak{f}(w)^2 \\ \dot{y} = \mathfrak{f}(w) \\ \dot{w} = u \end{cases},$$

with $\mathfrak{f}(w) = \exp(-1/w^2)$ and $\mathfrak{f}(0) = 0$. By a straightforward calculation, its symmetry algebra is, indeed, \mathfrak{L}_P but, obviously, k does not exist at $(x_0, y_0, 0)$.

The fact that the case of existence of an equilibrium, i.e. $f(\xi_0) \in \mathcal{G}(\xi_0)$, exhibits a richer collection of normal forms can be explained as follows. In that case, the fields of admissible velocities $\mathcal{A} = f + \mathcal{G}$ becomes at ξ_0 a linear (and not an affine) subspace $\mathcal{A}(\xi_0) = \mathcal{G}(\xi_0)$ of $T_{\xi_0} \mathcal{X}$ and the symmetries are less constrained by that structure. Conversely, any given symmetry imposes a bit less rigidity on f if $f(\xi_0) \in \mathcal{G}(\xi_0)$.

In Respondek and Tall (2002) it is proved that any control-affine single-input system, with controllable linear approximation around an equilibrium, has at most two symmetries: the identity and, in an exceptional "odd" case, a symmetry conjugated to minus identity. All systems studied in the present paper have a nontrivial 3-dimensional Lie algebra of symmetries but are in a perfect accordance with the result of Respondek and Tall (2002). Namely, the systems Σ_E , Σ_H , and Σ_P^1 do not have equilibria while Σ_P^0 and $\Sigma_P^{0,k}$ are considered around an equilibrium but their linear approximations are not controllable.

3. Conclusions

In this paper, we propose a characterisation of null-forms of conic systems Σ_Q via their Lie algebra of infinitesimal symmetries. We showed that this class of control-affine systems is completely determined by its Lie algebra of symmetries (under some regularity assumptions do the parabolic systems Σ_P). There are few results of that kind for control-affine systems existing in the literature, with a notable exception of Doubrov and Zelenko (2014), and more for the control-linear case, see Anderson and Kruglikov (2011); Doubrov and Kruglikov (2014); Kruglikov (2012). In view of all those results, it would be very interesting to study the following generalisation.

Problem. Let \mathfrak{L} be a finite-dimensional Lie algebra of vector fields acting transitively on a manifold \mathcal{X} . Does \mathfrak{L} uniquely determine a class of control-affine systems (around a generic point) whose Lie algebra of infinitesimal symmetries is \mathfrak{L} ?

We believe that considering control-affine systems instead of control-linear systems in that problem matters. A reason is that the existence of the drift f gives more rigidity on the symmetries and therefore would constraint the structure of systems possessing those symmetries. To start investigating this problem, we could begin with the study of all 2- and 3-dimensional Lie algebras (which are well known) and to characterise the control-affine systems that admit those Lie algebras as symmetries. In the thesis of the first author Schmoderer (2022), Theorem 1 is generalised to the class of paraboloid control-affine systems, that is control-affine system (with $m \geq 1$ controls and the state space being a $(2m + 1)$ -dimensional manifold) whose field of admissible velocities is a paraboloid hypersurface of the tangent bundle.

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