Building prescribed orbit equivalence with the group of integers

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Building prescribed quantitative orbit equivalence with $\mathbb{Z}$

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Two groups are *orbit equivalent* if they both admit an action on a same probability space that share the same orbits. In particular the Ornstein-Weiss theorem implies that all infinite amenable groups are orbit equivalent to the group of integers. To refine this notion between infinite amenable groups Delabie, Koivisto, Le Maître and Tessera introduced a quantitative version of orbit equivalence. They furthermore obtained obstructions to the existence of such equivalence using the isoperimetric profile.
In this article we offer to answer the inverse problem (find a group being orbit equivalent to a prescribed group with prescribed quantification) in the case of the group of integers using the so called Følner tiling shifts introduced by Delabie et al. To do so we use the diagonal products defined by Brieussel and Zheng giving groups with prescribed isoperimetric profile.

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1 INTRODUCTION

Two groups are orbit equivalent if they admit free measure-preserving actions on a same standard probability space (X, μ) which share the same orbits. This notion —emerging from the seminal work of Dye [Dye59, Dye63]— can be seen as the *ergodic* version of the famous *measure* equivalence introduced by Gromov [GNR93]. A famous result of Ornstein and Weiss (see Theorem 1.2) implies that all amenable groups are orbit equivalent. In particular —unlike quasi-isometry— orbit equivalence does *not* preserve coarse geometric invariants.

To overcome this issue it is therefore natural to look for some refinements of this orbit equivalence notion. Assume for example that G and H are two finitely generated orbit equivalent groups over a probability space (X, μ). Recall that we can consider the Schreier graph associated to the action of G (resp. H) on X and equip it with the usual metric d_{S_G} (resp. d_{S_H}), fixing the length of an edge to one. A first way to refine the measure equivalence is to quantify how close the two actions are by studying for all $g \in G$ and $h \in H$ the integrability of the two following maps

$$x \mapsto d_{S_G}(x, h \cdot x) \qquad x \mapsto d_{S_H}(x, g \cdot x).$$

When these two maps are L^p we say that the groups are L^p-orbit equivalent (see [BFS13] for more details). In this refined framework a famous result of Bader, Furman and Sauer [BFS13] implies that any group L^1-orbit equivalent to a lattice in $SO(n, 1)$ for some $n \geqslant 2$ is virtually a lattice in $SO(n, 1)$. This refinement also lead Bowen to prove in the appendix of [Aus16] that volume growth was invariant under L^1-orbit equivalence.

Delabie, Koivisto, Le Maître and Tessera offered in [DKLMT20] to extend this quantification to a family of functions larger than $\{x \mapsto x^p,\ p \in [0, +\infty]\}$ (see Definition 1.3). They furthermore showed the monotonicity of the isoperimetric profile under this quantified measure equivalence definition (see Theorem 1.5). In [BZ21] Brieussel and Zheng managed to construct amenable groups with prescribed isoperimetric profile called *diagonal product*. Considering the monotonicity of the isoperimetric profile, the striking result of Brieussel and Zheng thus triggers a new question: instead of trying to quantify the equivalence relation between two given groups, can one find a group that is orbit equivalent to a prescribed group with a prescribed quantification?

This is the problem we address in this article. Using Brieussel-Zheng's construction we exhibit a group that is orbit equivalent to $\mathbb{Z}$ with a prescribed quantification (see Theorem 1.8). Comparing the obtained coupling to the constraints given by Theorem 1.5 we show that our couplings is close to being optimal for a sense of "optimal" that we precise in Section 1.2.

1.1 Quantitative orbit equivalence

Let us recall some material of [DKLMT20].A *measure-preserving action* of a discrete countable group G on a measured space (X, μ) is an action of G on X such that the map $(g, x) \mapsto g \cdot x$ is a Borel map and $\mu(E) = \mu(g \cdot E)$ for all $E \subseteq \mathcal{B}(X)$ and all $g \in G$. We will say that a measure-preserving action of G on (X, μ) is *free* if for almost every $x \in X$ we have $g \cdot x = x$ if and only if $g = e_G$.

We recall below the definition of orbit equivalence and the quantified version as introduced by Delabie, Koivisto, Le Maître and Tessera [DKLMT20]. We conclude by studying the relation between isoperimetric profile and orbit equivalence.

Definition 1.1

Let G and H be two finitely generated groups. We say that G and H are *orbit equivalent* if there exists a probability space (X, μ) and a measure-preserving free action of G (resp. H) on (X, μ) such that for almost every $x \in X$ we have $G \cdot x = H \cdot x$. We call (X, μ) an *orbit equivalence coupling* from G to H.

By the Ornstein Weiss theorem [OW80, Th. 6] below, all infinite amenable groups are in the same equivalence class.

Theorem 1.2 (*[OW80]*)

All infinite amenable groups are orbit equivalent to $\mathbb{Z}$.

To refine this equivalence relation and "distinguish" amenable groups we introduce the quantified version of orbit equivalence.

Recall that if a finitely generated group G acts on a space X and if S_G is a finite generating set of G, we can define the Schreier graph associated to this action as being the graph whose set of vertices is X and set of edges is $\{(x, s \cdot x) \mid s \in S_K\}$. This graph is endowed with a natural metric d_{S_G} fixing the length of an edge to one. Remark that if S'_G is another generating set of G then there exists $C > 0$ such that for all $x \in X$ and $g \in G$

$$\frac{1}{C} d_{S_G}(x, g \cdot x) \leqslant d_{S'_G}(x, g \cdot x) \leqslant C d_{S_G}(x, g \cdot x).$$

Definition 1.3 (*[DKLMT20, Def. 2.18]*)

We say that an orbit equivalence coupling (X, μ) from G to H is (φ, ψ)-*integrable* if for all $g \in G$ (resp. $h \in H$) there exists $c_g > 0$ (resp. $c_h > 0$) such that

$$\int_X \varphi\left(\frac{1}{c_g} d_{S_H}(g \cdot x, x)\right) d\mu(x) < +\infty \quad \text{and} \quad \int_X \psi\left(\frac{1}{c_h} d_{S_G}(h \cdot x, x)\right) d\mu(x) < +\infty.$$

We introduce the constants c_g and c_h in the definition for the integrability to be independent of the choice of generating sets S_G and S_H. If $\varphi(x) = x^p$ we will sometimes talk of (L^p, ψ)-integrability instead of (φ, ψ)-integrability. In particular L^0 means that no integrability assumption is made. Finally, note that every (L^∞, ψ)-integrable coupling is (φ, ψ)-integrable for any increasing map $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. When $\varphi = \psi$ we will say that the coupling is φ-integrable instead of (φ, φ)-integrable.

Examples 1.4 ([DKLMT20]).

1. There exists an orbit equivalence coupling between $\mathbb{Z}^4$ and the Heisenberg group $\mathrm{Heis}(\mathbb{Z})$ that is L^p-integrable for all $p < 1$.

2. Let $k \in \mathbb{N}^*$. Their exists an $(L^\infty, \exp)$-integrable orbit equivalence coupling from the lamplighter group to the Baumslag-Solitar group $\mathrm{BS}(1,k)$.

More examples will be given in Section 3.1. Let us conclude on the quantification by a remark. We chose to refine orbit equivalence using the *integrable* point of view. But it is not the only possible sharpening. For example Kerr and Li [KL21] defined *Shannon orbit equivalence*: instead of looking at the integrability of distance maps they consider the Shannon *entropy* of partitions associated to the coupling.

1.2 Isoperimetric profile

As stated before, the orbit equivalence does not preserve the coarse geometric invariants. But the quantified version defined above allowed Delabie et al. [DKLMT20] to get a relation between the isoperimetric profiles of two orbit equivalent groups which we describe below.

Recall that if G is generated by a finite set S, the *isoperimetric profile* of G is defined as[1]

$$I_G(n) := \sup_{|A| \leqslant n} \frac{|A|}{|\partial A|}.$$

For example the isoperimetric profile of $\mathbb{Z}$ verifies $I_{\mathbb{Z}}(x) \simeq x$. Remark that due to Følner criterion, a group is amenable if and only if its isoperimetric profile is unbounded. Hence we can see the isoperimetric profile as a way to measure the amenability of a group: the faster I_G tends to infinity, the more amenable G is.

The behaviour of the isoperimetric profile under measure equivalence coupling is given by the theorem below. If f and g are two real functions we denote $f \preccurlyeq g$ if there exists some constant $C > 0$ such that $f(x) = \mathcal{O}(g(Cx))$ as x tends to infinity. We write $f \simeq g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

Theorem 1.5 (*[DKLMT20, Th.1]*)

Let G and H be two finitely generated groups admitting a (φ, L^0)-integrable orbit equivalence coupling. If φ and $t/\varphi(t)$ are increasing then

$$\varphi \circ I_H \preccurlyeq I_G.$$

This theorem provides an obstruction for finding φ-integrable couplings with certain functions φ between two amenable groups. For example for a coupling with $H = \mathbb{Z}$ the integrability has to verify $\varphi \preccurlyeq I_G$. This lead the authors of [DKLMT20] to ask the following question.

Question 1.6 ([DKLMT20, Question 1.2])**.** Given an amenable finitely generated group G, does there exist a (I_G, L^0)-integrable orbit equivalence coupling from G to $\mathbb{Z}$?

This interrogation contains actually two questions, starting with the "inverse problem" stated below.

Question 1.7. Given a function φ is there a group G such that there exists a (φ, L^0)-measure equivalent from G to $\mathbb{Z}$?

[1] We chose to adopt the convention of [DKLMT20]. Note that in [BZ21], the isoperimetric profile is defined as $\Lambda_G = 1/I_G$.

We answer the above question for a large family of maps φ in Theorem 1.8.

The second interrogation that Question 1.6 triggers is whether Theorem 1.5 is optimal when $G = \mathbb{Z}$. In other words one can ask if G in Question 1.7 can be chosen such that $\varphi \simeq I_G$. We will see that this relation is verified up to a logarithmic factor.

1.3 Main results

In this paper we show the following main theorem and its corollary below.

Theorem 1.8

For all non-decreasing function $\rho : [1, +\infty[\to [1, +\infty[$ such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing, there exists a group G such that

- $I_G \simeq \rho \circ \log$;
- there exists an orbit equivalence coupling from G to $\mathbb{Z}$ that is $(\varphi_\varepsilon, \exp \circ \rho)$-integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) := \rho \circ \log(x) / (\log \circ \rho \circ \log(x))^{1+\varepsilon}$.

Let us discuss the optimality of this result. Consider a (φ, L^0)-integrable orbit equivalence coupling from some group G to $\mathbb{Z}$. By Theorem 1.5 it verifies $\varphi \circ I_{\mathbb{Z}} \preccurlyeq I_G$. In particular since $I_{\mathbb{Z}}(x) \simeq x$, we can not have a better integrability than $\varphi(x) \simeq I_G$. Since $I_\Delta \simeq \rho \circ \log$ our above theorem is optimal up to a logarithmic error. We discuss this in more length in Section 5.

MAIN INGREDIENTS The main tools of the proof of Theorem 1.8 are Brieussel-Zheng's diagonal products (see Section 2) and Følner tiling shifts (see Section 3). We show that a diagonal product Δ admits a coupling with $\mathbb{Z}$ satisfying Theorem 1.8. To prove it we use the integrability criterion given by Theorem 3.5 and involving Følner tiling shifts.

Therefore we compute in Section 3.2 a Følner tiling shift $(\Sigma_n)_n$ for Δ. We also estimate the tiles' diameter and the proportion of elements in the boundary. We construct a Følner tiling shift for $\mathbb{Z}$ in Section 4.1 and show that these two tiling shifts verify Theorem 3.5.

Let us now consider the possible generalisations of this result to other groups than the group of integers. To do so we can use the *composition* of couplings described in [DKLMT20, Section 2].

Given the above theorem, once we have a measure equivalence coupling from $\mathbb{Z}$ to a group H we can compose the two couplings to obtain a measure equivalence from G to H. If the growth of the isoperimetric profile of H is close to the one of $\mathbb{Z}$, the integrability of the obtained coupling will be close to the optimal one given by Theorem 1.5. It is for example the case when $H = \mathbb{Z}^d$.

Corollary 1.9

Let $d \in \mathbb{N}^*$. For all non-decreasing function $\rho : [1, +\infty[\to [1, +\infty[$ such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing, there exists a group G such that

- $I_G \simeq \rho \circ \log$;
- there exists an orbit equivalence coupling from G to $\mathbb{Z}^d$ that is $(\varphi_\varepsilon, L^0)$-integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) := \rho \circ \log(x) / (\log \circ \rho \circ \log(x))^{1+\varepsilon}$.

STRUCTURE OF THE PAPER In Section 2 we present the diagonal products introduced by Brieussel and Zheng. We recall some of the properties shown in [BZ21] and compute Følner sequences. Section 3 is devoted to Følner tiling shifts. These tools built by Delabie et al. [DKLMT20] allow us to construct and quantify an orbit equivalence coupling between two groups. In this section we also construct Følner tiling shifts for diagonal products Δ. We show our main theorem in Section 4 combining the results of the two previous sections. Finally we discuss the limits of this construction and some open problems in Section 5.

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2 DIAGONAL PRODUCTS OF LAMPLIGHTER GROUPS

We recall here necessary material from [BZ21] concerning the definition of *Brieussel-Zheng's diagonal products*. We give the definition of such a group, recall and prove some results concerning the range (see Definition 2.7) of an element and use it to identify a Følner sequence. Finally we present in Section 2.3 the tools needed to recover such a diagonal product starting with a prescribed isoperimetric profile.

2.1 Definition of diagonal products

Recall that the wreath product of a group G with $\mathbb{Z}$ denoted $G \wr \mathbb{Z}$ is defined as $G \wr \mathbb{Z} := \oplus_{m\in\mathbb{Z}} G \rtimes \mathbb{Z}$. An element of $G \wr \mathbb{Z}$ is a pair (f, t) where f is a map from $\mathbb{Z}$ to G with finite support and t belongs to $\mathbb{Z}$. We refer to f as the *lamp configuration* and t as the *cursor*.

2.1.1 General definition

Let A and B be two finite groups. Let $(\Gamma_m)_{m\in\mathbb{N}}$ be a sequence of finite groups such that each Γ_m admits a generating set of the form $A_m \cup B_m$ where A_m and B_m are finite subgroups of Γ_m isomorphic respectively to A and B. For $a \in A$ we denote a_m the copy of a in A_m and similarly for B_m.

Finally let $(k_m)_{m\in\mathbb{N}}$ be a sequence of integers such that $k_{m+1} \geqslant 2k_m$ for all m. We define $\Delta_m = \Gamma_m \wr \mathbb{Z}$ and endow it with the generating set

$$S_{\Delta_m} := \Big\{(\mathrm{id}, 1)\Big\} \cup \Big\{\big(a_m\delta_0, 0\big) \mid a_m \in A_m\Big\} \cup \Big\{\big(b_m\delta_{k_m}, 0\big) \mid b_m \in A_m\Big\}.$$

Definition 2.1

The *Brieussel-Zheng's diagonal product* associated to $(\Gamma_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$ is the subgroup Δ of $(\prod_m \Gamma_m) \wr \mathbb{Z}$ generated by

$$S_\Delta := \Big\{\Big(\big(\mathrm{id}\big)_m, 1\Big)\Big\} \cup \Big\{\big((a_m\delta_0)_m, 0\big) \mid a \in A\Big\} \cup \Big\{\big((b_m\delta_{k_m})_m, 0\big) \mid b \in B\Big\}.$$

The group Δ is uniquely determined by the sequences $(\Gamma_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$. Let us give an illustration of what an element in such a group looks like. We will denote by $\mathbf{g}$ the sequence $(g_m)_{m\in\mathbb{N}}$.

Example 2.2. We represent in Figure 1 the element (g, t) of Δ verifying

$$(g, t) = \big((g_m)_{m\in\mathbb{N}}, t\big) := \big((a_m\delta_0)_m, 0\big)\big((b_m\delta_{k_m})_m, 0\big)(0, 3),$$

when $k_m = 2^m$. The cursor is represented by the blue arrow at the bottom of the figure. The only value of g_0 different from the identity is $g_0(0) = (a_0, b_0)$. Now if $m > 0$ then the only values of g_m different from the identity are $g_m(0) = a_m$ and $g_m(k_m) = b_m$.

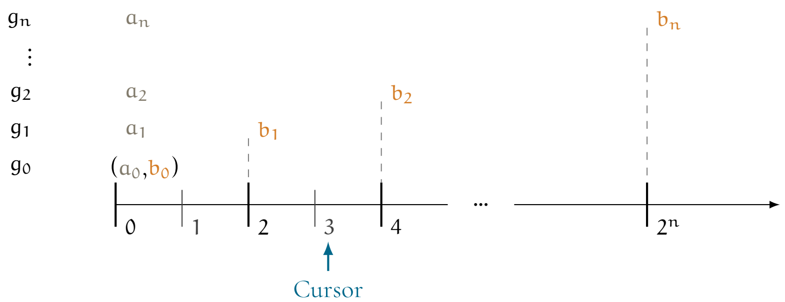

Figure 1: Representation of $(g, t) = \big((a_m\delta_0)_m, 0\big)\big((b_m\delta_{k_m})_m, 0\big)(0, 3)$ when $k_m = 2^m$.

2.1.2 The expanders case

In this article we will restrict ourselves to a particular familiy of groups $(\Gamma_m)_{m\in\mathbb{N}}$ called *expanders*. Recall that $(\Gamma_m)_{m\in\mathbb{N}}$ is said to be a sequence of *expanders* if the sequence of diameters $(\operatorname{diam}(\Gamma_m))_{m\in\mathbb{N}}$ is unbounded and if there exists $c_0 > 0$ such that for all $m \in \mathbb{N}$ and all $n \leqslant |\Gamma_m|/2$ the isoperimetric profile verifies $I_{\Gamma_m}(n) \leqslant c_0$.

When talking about diagonal products we will always make the following assumptions. We refer to [BZ21, Example 2.3] for an explicit example of diagonal product verifying **(H)**.

Hypothesis (H)

- $(k_m)_m$ and $(l_m)_m$ are sub-sequences of geometric sequences.
- $k_{m+1} \geqslant 2k_m$ for all $m \in \mathbb{N}$;
- $(\Gamma_m)_{m\in\mathbb{N}}$ is a sequence of expanders such that Γ_m is a quotient of $A * B$ and there exists $c > 0$ such that $1/cl_m \leqslant \operatorname{diam}(\Gamma_m) \leqslant cl_m$ for all $m \in \mathbb{N}$;
- $k_0 = 0$ and $\Gamma_0 = A_0 \times B_0$;
- $\langle\langle [A_m, B_m] \rangle\rangle \backslash \Gamma_m \simeq A_m \times B_m$ where $\langle\langle [A_m, B_m] \rangle\rangle$ denotes the normal closure of $[A_m, B_m]$.

Recall (see [BZ21, page 9]) that in this case there exist $c_1, c_2 > 0$ such that, for all m

$$c_1 l_m - c_2 \leqslant \ln|\Gamma_m| \leqslant c_1 l_m + c_2. \tag{2.1}$$

Finally we adopt the convention of [BZ21, Notation 2.2] and allow $(k_m)_m$ to take the value $+\infty$. In this case Δ_s is the trivial group. In particular when $k_1 = +\infty$ the diagonal product Δ corresponds to the usual lamplighter $(A \times B) \wr \mathbb{Z}$.

2.1.3 Relative commutators subgroups

For all $m \in \mathbb{N}$ let $\theta_m : \Gamma_m \to \langle\langle [A_m, B_m] \rangle\rangle \backslash \Gamma_m \simeq A_m \times B_m$ be the natural projection. Let θ_m^A and θ_m^B denote the composition of θ_m with the projection to A_m and B_m respectively.

Now let $m \in \mathbb{N}$ and define $\Gamma'_m := \langle\langle [A_m, B_m] \rangle\rangle$. If (g_m, t) belongs to Δ_m then there exists a unique $g'_m : \mathbb{Z} \to \Gamma'_m$ such that $g_m = g'_m \theta_m(g_m)$.

Example 2.3. Let $(\mathbf{g}, 3)$ be the element described in Figure 1. Then the only non-trivial value of $\theta_0(g_0)$ is $\theta_0(g_0(0)) = (a_0, b_0)$. If $m > 0$ then the only non trivial values of $\theta_m(g_m)$ are $\theta_m(g_m(0)) = (a_m, e)$ and $\theta_m(g_m(k_m)) = (e, b_m)$. Finally for all m we have $g'_m = \mathrm{id}$ since there are no commutators appearing in the decomposition of $(\mathbf{g}, 0)$.

Example 2.4. Assume that $k_m = 2^m$ and consider first the element $(\mathbf{f}, 0)$ of Δ defined by $(\mathbf{f}, 0) := (0, -k_1)((a_m\delta_0)_m, 0)(0, k_1)$. Now define the commutator

$$(\mathbf{g}, 0) = (\mathbf{f}, 0) \cdot ((b_m \delta_{k_m})_m, 0) \cdot (\mathbf{f}, 0)^{-1} \cdot ((b_m^{-1} \delta_{k_m})_m, 0)$$

and let us describe the values taken by $\mathbf{g}$ and the induced maps $\theta_m(g_m)$ and g'_m (see Figure 2 for a representation of $\mathbf{g}$). The only non-trivial commutator appearing in the values taken by $\mathbf{g}$ is $g_1(k_1)$ which is equal to $a_1 b_1 a_1^{-1} b_1^{-1}$. In other words g_0 is the identity, thus $\theta_0 = \mathrm{id}$. Moreover when $m = 1$ we have $\theta_1 = \mathrm{id}$ and the only value of $g'_1(x)$ different from e is $g'_1(k_1) = a_1 b_1 a_1^{-1} b_1^{-1}$ (on a blue background in Figure 2). Finally if $m > 1$ then g_m is the identity thus $\theta_m = \mathrm{id}$ and $g'_m = \mathrm{id}$.

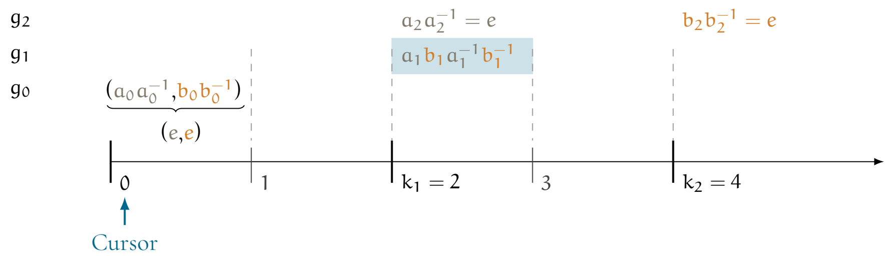

Figure 2: Representation of $(\mathbf{g}, 0)$ defined in Example 2.4

Let us study the behaviour of this decomposition under product of lamp configurations.

Claim 2.5. If $g_m, f_m : \mathbb{Z} \to \Gamma_m$ then $(g_m f_m)' = g'_m \theta_m(g_m) f'_m \left(\theta_m(g_m)\right)^{-1}$.

Proof. Since $g_m = \theta_m(g_m) g'_m$ and $f_m = \theta_m(f_m) f'_m$ we can write

$$g_m f_m \;=\; g'_m \theta_m(g_m) \cdot f'_m \theta_m(f_m) \;=\; g'_m \theta_m(g_m) f'_m \theta_m(g_m)^{-1} \theta_m(g_m) \theta_m(f_m).$$

But $\theta_m(g_m)\theta_m(f_m)$ takes values in $A_m \times B_m$ and Γ'_m is a normal subgroup of Γ_m thus the map $g'_m \theta_m(g_m) f'_m \theta_m(g_m)^{-1}$ takes values in Γ'_m. Hence the claim. $\square$

Combining Lemma 2.7 and Fact 2.9 of [BZ21], we get the following result.

Lemma 2.6

Let $(\mathbf{g}, t) \in \Delta$. For all $m \in \mathbb{N}$ and $x \in \mathbb{Z}$

$$g_m(x) = g'_m(x) \theta_m^A(g_0(x)) \theta_m^B(g_0(x - k_m)).$$

In particular the sequence $\mathbf{g} = (g_m)_{m \in \mathbb{N}}$ is uniquely determined by g_0 and $(g'_m)_{m \in \mathbb{N}}$.

In the next subsection we are going to see that we actually need only a *finite* number of elements of the sequence $(g'_m)_{m \in \mathbb{N}}$ to characterize $\mathbf{g}$.

2.2 Range and support

In this subsection we introduce the notion of *range* of an element $(\mathfrak{g}, t)$ in Δ and link it to the supports of the lamp configurations $(\mathfrak{g}_m)_{m\in\mathbb{N}}$.

2.2.1 Range

We denote by $\pi_2 : \Delta \to \mathbb{Z}$ the projection on the second factor and for all $n \in \mathbb{N}$ denote by $\mathfrak{l}(n)$ the integer such that $k_{\mathfrak{l}(n)} \leqslant n < k_{\mathfrak{l}(n)+1}$.

Definition 2.7

If $w = s_1 \dots s_m$ is a word over S_Δ we define its *range* as

$$\mathrm{range}(w) := \left\{ \pi_2 \left(\prod_{j=1}^{i} s_j \right) \mid i = 1, \dots, n \right\}.$$

The range is a finite subinterval of $\mathbb{Z}$. It represents the set of sites visited by the cursor.

Definition 2.8

The *range* of an element $\delta \in \Delta$ is defined as the minimal diameter interval obtained as the range of a word over S_Δ representing δ. We denote it $\mathrm{range}(\delta)$.

When there is no ambiguity we will denote $\mathrm{range}(\delta)$ the diameter of this interval.

Example 2.9. Let $(\mathfrak{g}, 0) \in \Delta$ such that $\mathrm{range}(\mathfrak{g}, 0) = [0, 6]$, that is to say: the cursor can only visit sites between 0 and 6. Then the map $\mathfrak{g}_m$ can "write" elements of A_m only on sites visited by the cursor, that is to say from 0 to 6, and it can write elements of B_m only from k_m to $6 + k_m$. Thus $\mathfrak{g}_0$ is supported on $[0, 6]$, since $k_0 = 0$. Moreover, commutators (and hence elements of Γ'_m) can only appear between k_m and 6, thus $\mathrm{supp}(\mathfrak{g}'_m) \subseteq [k_m, 6]$. In particular $\mathrm{supp}(\mathfrak{g}'_m)$ is empty when $k_m > 6$.

Such a $(\mathfrak{g}, 0)$ is represented in Figure 3 for $k_m = 2^m$.

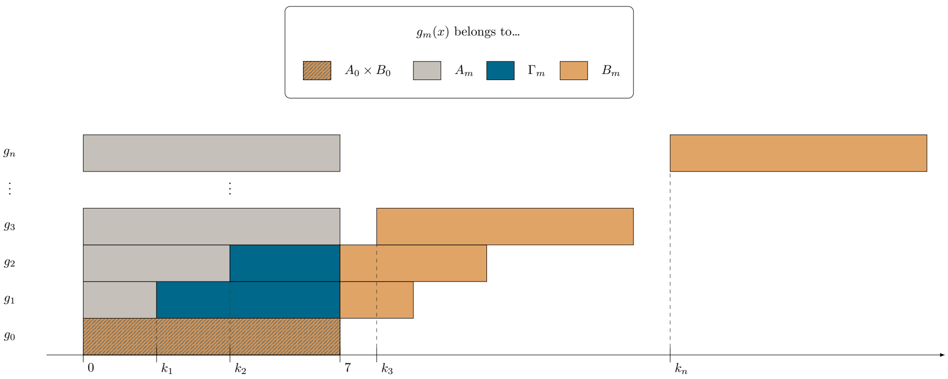

Figure 3: An element of Δ

Recall that $\mathfrak{g}_m : \mathbb{Z} \to \Gamma_m$. If $m \leqslant \mathfrak{l}(6)$, then $\mathfrak{g}_m(x)$ belongs to A_m if $x \in [0, k_m - 1]$, it belongs to Γ_m if $x \in [k_m, 6]$ and to B_m if $x \in [7, 6 + k_m]$ and equals e elsewhere. If $m > \mathfrak{l}(6)$ then $\mathfrak{g}_m(x)$ belongs to A_m if $x \in [0, 6]$ and to B_m if $x \in [k_m, 6 + k_m]$ and equals e elsewhere.

Let us now recall a useful fact proved in [BZ21].

Claim 2.10 ([BZ21, Fact 2.9]). An element $(\mathbf{g}, t) \in \Delta$ is uniquely determined by t, g_0 and the sequence $(g'_m)_{m \leqslant l(\text{range}(\mathbf{g},t))}$.

Example 2.11. Consider again $(\mathbf{g}, 0) \in \Delta$ such that $\text{range}(\mathbf{g}, 0) = [0, 6]$, which was illustrated in Figure 3. Since $k_3 = 8 > 6$, the element $(\mathbf{g}, 0)$ is uniquely determined by the data g_0 (that is to say, the values read in the bottom line) and the values of g'_i for $i = 1, 2$ (namely, the value taken in the blue area). Figure 4 represents the aforementioned characterizing data.

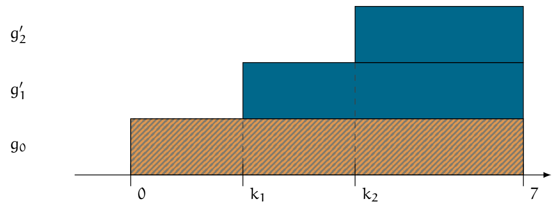

Figure 4: Data needed to characterized $\mathbf{g}$ such that $\text{range}(\mathbf{g}) \subset [0, 6]$ when $k_m = 2^m$

2.2.2 Relation between range and support

Recall that for all $m \in \mathbb{N}$ we can write $g_m(x) = g'_m(x)\theta^A_m(g_0(x))\theta^B_m(g_0(x - k_m))$.

To work with the Følner sequence we compute in Section 2.2.3 and deduce a Følner *tiling* shift from it, we will need to link the range of $(\mathbf{g}, t)$ in Δ with the support of g_0 and the sequence of supports of $(g'_m)_{m\in\mathbb{N}}$. This is what the following lemma formalises.

Lemma 2.12

Let $n \in \mathbb{N}$ and take $(\mathbf{g}, t) \in \Delta$. Then $\text{range}(\mathbf{g}, t)$ is included in $[0, n]$ if and only if

$$\begin{cases} t \in [0, n] \\ \text{supp}(g_0) \subset [0, n] \\ \text{supp}(g'_m) \subseteq [k_m, n] & \forall 1 \leqslant m \leqslant l(n) \\ g'_m \equiv e & \forall m > l(n). \end{cases}$$

Proof. Let $n \in \mathbb{N}$ andf first assume that $\text{range}(\mathbf{g}, t) \subseteq [0, n]$, that is to say: the cursor can only visit sites between 0 and n. Let $(\mathbf{g}, t) = \prod_{i=0}^{l} s_i$ be a decomposition in a product of elements of S_Δ of minimal length. Let $m \in \mathbb{N}$, then by definition of S_Δ an element s_i can "write" elements of A_m only between 0 and n, and it can write elements of B_m only between k_m and $n + k_m$. Thus g_0 is supported on $[0, n]$, since $k_0 = 0$. And commutators can only appear between k_m and n, hence $\text{supp}(g'_m) \subseteq [k_m, n]$. In particular if $k_m > n$ then $g'_m \equiv e$. Finally we obtain that t belongs to $[0, n]$ by noting that $t = \pi_2\left(\prod_{j=1}^{l} s_j\right)$.

Now let us prove the other way round. Consider $m \in [1, l(n)]$ then $g'_m(x) \in \Gamma'_m$. It is therefore a product of conjugates of commutators of the form $[a_m, b_m]$, where $a_m \in A_m$ and $b_m \in B_m$. Applying Example 2.4 with x instead of k_1 we can show that we can write $[a_m, b_m]$ at $g_m(x)$ without changing any other entry in $\mathbf{g}$ (see also Figure 2). In a similar way, we can write a conjugate of $[a_m, b_m]$ at $g_m(x)$ without changing any other entry in $\mathbf{g}$. Finally writing (a_0, b_0) at the entry $g_0(x)$ writes a_m at $g_m(0)$ and b_m at $g_m(k_m)$ (see also Figure 1). Therefore using Lemma 2.6 we can obtain $(\mathbf{g}, 0)$ by first considering the word in Δ that writes all the values of g_0, then multiplying it on the left by a word that writes the value of g'_1, and continue this process to write all g'_m for $m \leqslant l(n)$.

Let us now check that the cursor remains in $[0, n]$ when writing g_0 and g'_m. Take $m \in [1, l(n)]$, then $k_m \leqslant n$ and $\text{supp}(g'_m)$ is contained in $[k_m, n]$. Now let $x \in \text{supp}(g'_m) \subseteq [k_m, n]$. Since $\Gamma'_m \subseteq \Gamma_m$ which is generated by $A_m \times B_m$ we can decompose $g'_m(x)$ as a product of elements in A_m and B_m. To write some $a_m \in A_m$ at the position x the cursor needs to visit sites in $[0, x]$. To write some $b_m \in B_m$ it needs to visit sites in $[0, x - k_m]$. Therefore, the cursor remains in $[0, n]$ when writing $g_m(x)$ at position x. Finally, for all x the cursor needs only to visit position x in order to write $g_0(x)$. Since $\text{supp}(g_0)$ is contained in $[0, n]$ then the cursor needs only to visit sites between 0 and n.

Combining what precedes with Lemma 2.6 and the hypothesis that $t \in [0, n]$, we get that the cursor needs only to visit cites between $[0, n]$ to write (g, t). Hence the lemma.

□

2.2.3 Følner sequence

In this subsection we describe a Følner sequence $(F_n)_{n \in \mathbb{N}}$ for Δ. Recall that $l(n)$ denotes the integer such that $k_{l(n)} \leqslant n < k_{l(n)+1}$.

Proposition 2.13

The following sequence is a Følner sequence of Δ

$$F_n := \left\{(f, t) \mid \quad \text{range}\,(f, t) \subseteq \{0, \dots, n-1\}\right\}.$$

Proof. Let $n \in \mathbb{N}$ and $\delta \in F_n$ and let $s_1, \dots, s_l \in S_\Delta$ such that $\delta = s_1 \cdots s_l$. Now take $s_{l+1} \in S_\Delta$. If $s_{l+1} = ((a_m \delta_0), 0)$ for some $a \in A$ or if $s_{l+1} = ((b_m \delta_{k_m}), 0)$ for some $b \in B$ then since the cursor of s_{l+1} equals 0,

$$\text{range}(\delta s_{l+1}) = \left\{\pi_2\left(\prod_{j=1}^{i} s_j\right) \mid i = 1, \dots, l+1\right\} = \text{range}(\delta).$$

Thus $\delta s_{l+1} \in F_n$. Finally denote by $[x, y]$ the range of δ. Using the same formula as above we get

$$\begin{aligned} \text{range}(\delta \cdot (\text{id}, 1)) &\subseteq [x, y+1] \quad \text{if } t = y, \\ \text{range}(\delta \cdot (\text{id}, 1)) &\subseteq [x, y] \quad \text{if } t < y. \end{aligned}$$

Hence for all $t < n-1$ we have $\text{range}(\delta \cdot (\text{id}, 1)) \subseteq [0, n-1]$. Now if $t = n-1$ then the cursor of $\delta(\text{id}, 1)$ visits the site n, thus $\text{range}(\delta \cdot (\text{id}, 1))$ is not included in $[0, n-1]$ and therefore $\delta(\text{id}, 1)$ does not belong to F_n.

A similar argument shows that $\delta(0, -1)$ belongs to F_n if and only if $t \neq 0$. Hence $\partial F_n = \{(f, t) \in F_n \;:\; t = 0, n\}$ and thus

$$|\partial F_n| / |F_n| = 2/n \underset{n \to \infty}{\longrightarrow} 0.$$

□

2.3 From the isoperimetric profile to the group

We saw how to define a diagonal product from two sequences $(k_m)_m$ and $(l_m)_m$. In this section we recall the definition given in [BZ21, Appendice B] of a Brieussel-Zheng's group from its isoperimetric profile. We conclude with some useful results concerning the metric of these groups.

2.3.1 Definition of Δ

Recall that in the particular case of expanders (see Section 2.1.2) a Brieussel-Zheng's group is uniquely determined by the sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ (where l_m corresponds to the diameter of Γ_m). Thus, starting from a prescribed function ρ, we will define sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ such that the corresponding Δ verifies $I_\Delta \simeq \rho\circ\log$. Let

$$\mathcal{C} := \left\{ \zeta : [1,+\infty) \to [1,+\infty) \;\middle|\; \begin{array}{c} \zeta \text{ continue}, \quad \zeta(1)=1 \\ \zeta \text{ and } x\mapsto x/\zeta(x) \text{non-decreasing} \end{array} \right\}.$$

Equivalently this is the set of functions ζ satisfying $\zeta(1)=1$ and

$$(\forall x, c \geqslant 1) \qquad \zeta(x) \leqslant \zeta(cx) \leqslant c\zeta(x). \tag{2.2}$$

So let $\rho\in\mathcal{C}$. Combining [BZ21, Proposition B.2 and Theorem 4.6] we can show the following result (remember that with our convention the isoperimetric profile considered in [BZ21] corresponds to $1/I_\Delta$).

Proposition 2.14

Let $\kappa,\lambda \geqslant 2$. For any $\rho\in\mathcal{C}$ there exists a subsequence $(k_m)_{m\in\mathbb{N}}$ of $(\kappa^n)_{n\in\mathbb{N}}$ and a subsequence $(l_m)_{m\in\mathbb{N}}$ of $(\lambda^n)_{n\in\mathbb{N}}$ such that the group Δ defined in Section 2.1.2 verifies $I_\Delta(x)\simeq\rho\circ\log$.

Example 2.15 ([BZ21, Example 4.5]). Let $\alpha>0$. If $\rho(x) := x^{1/(1+\alpha)}$ then the diagonal product Δ defined by $k_m=\kappa^m$ and $l_m=\kappa^{\alpha m}$ verifies $I_\Delta\simeq\rho\circ\log$.

2.3.2 Technical tools

We recall the intermediate functions defined in [BZ21, Appendix B] and some of their properties.

Let $\rho\in\mathcal{C}$ and let f such that $\rho(x)=x/f(x)$. The construction of a group corresponding to the given isoperimetric profile $\rho\circ\log$ is based on the approximation of f by a piecewise linear function $\bar{f}$. For the quantification of orbit equivalence, many of our computations will use $\bar{f}$ and some of its properties. We recall below all the needed results, beginning with the definition of $\bar{f}$.

Lemma 2.16

Let $\rho\in\mathcal{C}$ and f such that $\rho(x)=x/f(x)$. Let (k_m) and (l_m) given by Proposition 2.14 above and Δ the corresponding diagonal product. The function $\bar{f}$ defined by

$$\bar{f}(x) := \begin{cases} l_m & \text{if } x\in[k_m l_m, k_{m+1}l_m], \\ \frac{x}{k_{m+1}} & \text{if } x\in[k_{m+1}l_m, k_{m+1}l_{m+1}], \end{cases} \tag{2.3}$$

verifies $\bar{f}\simeq f$. In particular the map $\bar{\rho}$ defined by $\bar{\rho}(x)=x/\bar{f}(x)$ verifies $\bar{\rho}\simeq\rho$.

Example 2.17. If $\rho(x)=x$ then $f(x)=1$ leads to $l_m=1$ for all m and $k_m=+\infty$ for all $m\geqslant 1$. In this case $\Delta=(A\times B)\wr\mathbb{Z}$.

Remark that both $\bar{f}$ and $\bar{\rho}$ belong to $\mathcal{C}$. In particular they verify eq. (2.2), which is only true when c and x are greater than 1. When $c<1$ we get the following inequality.

Claim 2.18. If $0<c'<1$ and $x'\geqslant 1/c'$ then $c'\bar{\rho}(x')\leqslant\bar{\rho}(c'x')$.

Proof. If $0 < c' < 1$ then $1/c' > 1$, thus we can apply eq. (2.2) with $c = 1/c'$ and $x = c'x$ to obtain $\bar{\rho}(x') = \bar{\rho}\left(\frac{1}{c'}c'x'\right) = \bar{\rho}(cx) \leqslant c\bar{\rho}(x) = \frac{1}{c'}\bar{\rho}(c'x')$. □

2.3.3 Metric

We recall here some useful material about the metric of Δ and refer to [BZ21, Section 2.2] for more details. First, let $(x)_+ := \max\{x, 0\}$.

Definition 2.19

For $j \in \mathbb{Z}$ and $m \in \mathbb{N}$ let $I_j^m := [jk_m/2, (j+1)k_m/2 - 1]$. Let $f_m : \mathbb{Z} \to \Gamma_m$. The *essential contribution* of f_m is defined as

$$E_m(f_m) := k_m \sum_{j:\mathrm{range}(f_m,t)\cap I_j^m \neq \emptyset} \max_{x\in I_j^m} (|f_m(x)|_{\Gamma_m} - 1)_+ .$$

The following proposition sums up [BZ21, Lemma 2.13, Proposition 2.14].

Proposition 2.20

For any $\delta = (f,t) \in \Delta$ we have

$$|(f,t)|_\Delta \leqslant 500 \sum_{m=0}^{\iota(\mathrm{range}(\delta))} |(f_m,t)|_{\Delta_m},$$
$$|(f_m,t)|_{\Delta_m} \leqslant 9\left(\mathrm{range}(f_m,t) + E_m(f_m)\right).$$

3 FØLNER TILING SHIFTS

We start by recalling some material of [DKLMT20] about Følner tiling shifts and then construct such a tiling for diagonal products.

3.1 Følner tiling shifts

The tools we are going to use to build orbit equivalence are *Følner tiling shifts*[2]. These sequences lead to Følner sequences defined recursively: the term of rank $(n+1)$ is composed of a finite number of translates of the n-th term of the sequence.

Definition 3.1

Let G be an amenable group and $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of G. Define by induction the sequence $(T_n)_{n\in\mathbb{N}}$ by $T_0 := \Sigma_0$ and $T_{n+1} := T_n\Sigma_{n+1}$. We say that $(\Sigma_n)_{n\in\mathbb{N}}$ is a (left) *Følner tiling shift* if

- $(T_n)_{n\in\mathbb{N}}$ is a left Følner sequence, *viz.*

$$(\forall g \in G) \qquad \lim_{n\to\infty} \frac{|gT_n \setminus T_n|}{|T_n|} = 0,$$

- $T_{n+1} = \bigsqcup_{\sigma\in\Sigma_{n+1}} \sigma T_n$.

We call Σ_n the set of *shifts* and $(T_n)_{n\in\mathbb{N}}$ the *tiles*.

[2] Delabie et al. [DKLMT20] use the term "Folner tiling sequence". We chose to call $(\Sigma_n)_n$ a tiling *shift* in order to avoid confusion with usual Følner sequences.

We can also consider *right* Følner tiling shifts, that is to say sequences $(\Sigma_n)_n$ such that $T_{n+1} := \Sigma_{n+1} T_n$ defines a right Følner sequence.

Definition 3.2

Let S be a generating part of G. We say that $(\Sigma_n)_{n\in\mathbb{N}}$ is a (R_n, ε_n)-Folner tiling shift if for all n we have

$$\operatorname{diam}(T_n) \leqslant R_n, \qquad |sT_n \backslash T_n| \leqslant \varepsilon_n |T_n| \quad (\forall s \in S).$$

Delabie et al. showed in [DKLMT20] the two following examples.

Example 3.3. If $G = \mathbb{Z}$ the sequence defined by $\Sigma_{n+1} := \{0, 2^n\}$ is a $(2^n, 2^{1-n})$-Følner tiling shift and the sequence (T_n) thus defined verifies $T_n = [0, 2^n - 1]$.

Example 3.4. If $G = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ then the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} \Sigma_0 & := \left\{(f, 0) \in G \mid \operatorname{supp}(f) \subseteq \{0, 1\}\right\}, \\ \Sigma_{n+1} & := \left\{(f, 0) \in G \mid \operatorname{supp}(f) \subseteq [2^n, 2^{n+1} - 1]\right\} \\ & \quad \cup \left\{(f, 2^n) \in G \mid \operatorname{supp}(f) \subseteq [0, 2^n - 1]\right\}, \end{cases}$$

is a right $(3 \cdot 2^n, 2^{-n})$-Følner tiling shift. Moreover the tiling $(T_n)_{n\in\mathbb{N}}$ thus defined verifies $T_n = \left\{(f, m) \in G \mid \operatorname{supp}(f) \subseteq [0, 2^n - 1],\ m \in [0, 2^n - 1]\right\}$.

In [DKLMT20] the authors gave a condition for two amenable groups admitting both Følner tiling shifts to be orbit equivalent. Indeed if G admits a Følner tiling shift $(\Sigma_n)_{n\in\mathbb{N}}$ then we can define $X := \prod_{n\in\mathbb{N}} \Sigma_n$ and endow it with an action of G. Up to measure zero, two elements of X will be in the same orbit under that action if and only if they differ by a finite number of indices. The equivalence relation thus induced is called the *cofinite equivalence relation*. Now if G' admits a Følner tiling shift $(\Sigma'_n)_{n\in\mathbb{N}}$ verifying $|\Sigma_n| = |\Sigma'_n|$ for all integer n, then there exists a natural bijection between X and $X' := \prod_{n\in\mathbb{N}} \Sigma'_n$ which preserves the cofinite equivalence relation. That is to say G and H are orbit equivalent. Furthermore they showed that if we know the diameter and the ratio of elements in the boundary of each tile then we can deduce the integrability of the coupling. This is what the following proposition sums up.

Theorem 3.5 (*[DKLMT20, Prop. 6.6]*)

Let G and G' be two discrete amenable groups and let $(\Sigma_n)_n$ be an (ε_n, R_n)-Følner tiling shift for G and $(\Sigma'_n)_n$ be an (ε'_n, R'_n)-Følner tiling shift for G'.

If $|\Sigma_n| = |\Sigma'_n|$, then the groups are orbit equivalent over $X = \prod_{n\in\mathbb{N}} \Sigma_n$. Moreover if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing map such that the sequence $(\varphi(2R'_n)(\varepsilon_{n-1} - \varepsilon_n))_{n\in\mathbb{N}}$ is summable, then the coupling from G to G' is (φ, L^0)-integrable.

Using this tiling technique and the above theorem, Delabie et al. [DKLMT20] obtained the first point of Examples 1.4 and the two following quantifications.

Example 3.6. For all n and m there exists an orbit equivalence coupling from $\mathbb{Z}^m$ to $\mathbb{Z}^n$ which is $(\varphi_\varepsilon, \psi_\varepsilon)$-integrable for every $\varepsilon > 0$ where

$$\varphi_\varepsilon(x) = \frac{x^{n/m}}{\log(x)^{1+\varepsilon}} \qquad \psi_\varepsilon(x) = \frac{x^{m/n}}{\log(x)^{1+\varepsilon}}.$$

Remark that in particular for all $p < n/m$ and $q < m/n$ there exists a (L^p, L^q)-orbit equivalence coupling from $\mathbb{Z}^m$ to $\mathbb{Z}^n$.

Example 3.7. Let $m \geqslant 2$. There exists an orbit equivalence coupling between $\mathbb{Z}$ and $\mathbb{Z}m\mathbb{Z}\wr\mathbb{Z}$ that is $(\exp, \varphi_\varepsilon)$-integrable for all $\varepsilon > 0$ where

$$\varphi_\varepsilon(x) = \frac{\log(x)}{\log(\log(x))^{1+\varepsilon}}.$$

Note that the above example corresponds to the case when $\rho(x) = x$ in our Theorem 1.8.

3.2 Følner tiling shifts of diagonal products

Let $(k_m)_m$ and $(l_m)_m$ be two sequences verifying the conditions of **(H)** and consider Δ the associated diagonal product (see Section 2). We define below a Følner tiling shift for Δ. Our goal is to obtain a tiling verifying $T_n = F_{\kappa^n}$. After defining the shifts sets Σ_n we prove that the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is actually a Følner tiling shift. Finally we precise this last statement by computing $(R_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $(\Sigma_n)_{n\in\mathbb{N}}$ is a (R_n, ε_n)-Følner tiling shift (see Definition 3.1).

3.2.1 Definition of the shifts

For any $n \in \mathbb{N}$, let $\mathfrak{L}(n) = \mathfrak{l}(\kappa^n - 1)$, that is to say $\mathfrak{L}(n)$ is the integer such that $k_{\mathfrak{L}(n)} \leqslant \kappa^n - 1 < k_{\mathfrak{L}(n)+1}$. For example if $k_n := \kappa^n$ for all $n \in \mathbb{N}$, then $\mathfrak{L}(n) = n - 1$.

Before defining our sequence $(\Sigma_n)_{n\in\mathbb{N}}$, let us show some practical results on $\mathfrak{L}$. First remark that since $(k_n)_{n\in\mathbb{N}}$ is a subsequence of $(\kappa^n)_{n\in\mathbb{N}}$, it verifies $k_n \geqslant \kappa^n$ for all $n \in \mathbb{N}$. Thus $\mathfrak{L}(n) \leqslant n$ and

$$k_{\mathfrak{L}(n)} < \kappa^n \leqslant k_{\mathfrak{L}(n)+1}.$$

Claim 3.8. Let $n \geqslant 0$, then either $\mathfrak{L}(n+1) = \mathfrak{L}(n)$ or $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$. Moreover in this second case $k_{\mathfrak{L}(n+1)} = \kappa^n$.

Proof. Recall that by definition $\mathfrak{L}(m) = \max\{i \in \mathbb{N} \,|\, k_i \leqslant \kappa^m - 1\}$ for all $m \in \mathbb{N}$.

Let $n \in \mathbb{N}$, then $\mathfrak{L}(n+1) \geqslant \mathfrak{L}(n)$. Moreover if $k_{\mathfrak{L}(n)+1} \geqslant \kappa^{n+1}$ then $\mathfrak{L}(n+1) < \mathfrak{L}(n) + 1$. That is to say $\mathfrak{L}(n+1) \leqslant \mathfrak{L}(n)$ and thus $\mathfrak{L}(n+1) = \mathfrak{L}(n)$.

On the contrary, if $k_{\mathfrak{L}(n)+1} < \kappa^{n+1}$ then $\mathfrak{L}(n+1) \geqslant \mathfrak{L}(n)+1$. But, by definition of $\mathfrak{L}(n)$ it verifies $k_{\mathfrak{L}(n)+1} \geqslant \kappa^n$ and by construction of $(k_m)_{m\in\mathbb{N}}$ we also have $k_{\mathfrak{L}(n)+2} \geqslant \kappa k_{\mathfrak{L}(n)+1}$ thus $k_{\mathfrak{L}(n)+2} \geqslant \kappa^{n+1}$. Hence $\mathfrak{L}(n+1) < \mathfrak{L}(n) + 2$ and the first assertion.

Finally if $\mathfrak{L}(n+1) = \mathfrak{L}(n) + 1$ then by definition of $\mathfrak{L}$

$$k_{\mathfrak{L}(n)} < \kappa^n \leqslant k_{\mathfrak{L}(n)+1} = k_{\mathfrak{L}(n+1)} \leqslant \kappa^{n+1} - 1.$$

But $(k_m)_{m\in\mathbb{N}}$ is a subsequence of κ^m thus the above inequality implies $k_{\mathfrak{L}(n+1)} = \kappa^n$. $\square$

Now, let us define the shifts. First let $\Sigma_0 := F_0$, then if $n \geqslant 0$ we distinguish two cases depending on whether $\mathfrak{L}(n+1) = \mathfrak{L}(n)$ or $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$ and in both cases we split the set of shifts Σ_{n+1} in κ parts.

If $\mathfrak{L}(n+1) = \mathfrak{L}(n)$, let for all $j \in \{0, \dots, \kappa - 1\}$

$$\Sigma_{n+1}^{j} := \left\{ (g, j\kappa^n) \in \Delta \;\middle|\; \begin{array}{l} \qquad \operatorname{supp}(g_0) \subseteq [0, j\kappa^n - 1] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right], \\ \forall m \in [1, \mathfrak{L}(n)] \\ \qquad \operatorname{supp}(g'_m) \subseteq [k_m, j\kappa^n + k_m - 1] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right], \\ \forall m \notin [0, \mathfrak{L}(n)] \\ \qquad \operatorname{supp}(g'_m) = \emptyset. \end{array} \right\}.$$

Now if $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$ we add the condition that $g'_{\mathfrak{L}(n)+1}$ has support contained in $[k_{\mathfrak{L}(n+1)}, \kappa^{n+1}-1]$, namely

$$\Sigma^j_{n+1} := \left\{ (g, j\kappa^n) \in \Delta \;\middle|\; \begin{array}{l} \quad\quad \operatorname{supp}(g_0) \subseteq [0, j\kappa^n - 1] \cup \left[(j+1)\kappa^n, \kappa^{n+1}-1\right] \\ \quad \forall m \in [1, \mathfrak{L}(n)] \\ \quad\quad \operatorname{supp}(g'_m) \subseteq [k_m, j\kappa^n + k_m - 1] \cup \left[(j+1)\kappa^n, \kappa^{n+1}-1\right], \\ \operatorname{supp}\left(g'_{\mathfrak{L}(n)+1}\right) \subseteq \left[k_{\mathfrak{L}(n)+1}, \kappa^{n+1}-1\right], \\ \forall m \notin [0, \mathfrak{L}(n+1)] \ \operatorname{supp}(g'_m) = \emptyset. \end{array} \right\}.$$

Finally, in both cases we define $\Sigma_{n+1} := \cup_{j=0}^{\kappa-1} \Sigma^j_{n+1}$.

Let (g, t) be an element of some Σ^j_{n+1}. We represent in Figure 5 the supports and the sets where the maps $g_0, {g'}_1, \dots, {g'}_{\mathfrak{L}(n+1)}$ take their values. The light-blue rectangle with dotted outline is in Σ^j_{n+1} if and only if $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$.

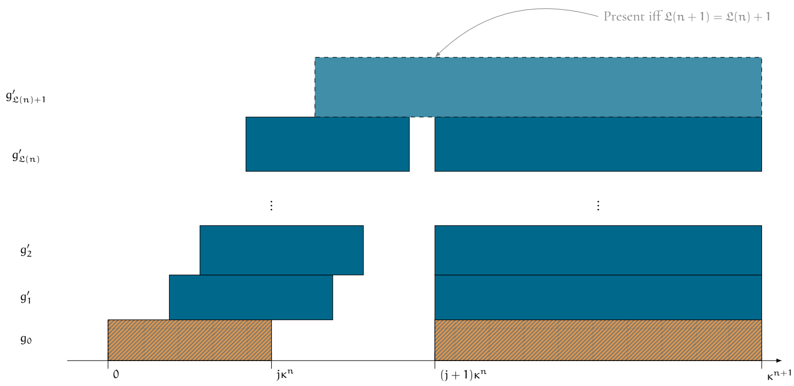

Figure 5: Support and values taken by $(g, t) \in \Sigma^j_n$

Now that we have the shifts sequence, let us turn to the definition of the tiles.

3.2.2 Tiling

Recall that $(F_n)_{n\in\mathbb{N}}$ denotes the Følner sequence of Δ defined in Proposition 2.13. The aim of this section is to show the theorem below.

Theorem 3.9

The sequence $(\Sigma_n)_{n\in\mathbb{N}}$ defined in Section 3.2.1 is a Følner tiling shift of Δ.

Before showing that the sequence of tiles $(T_n)_{n\in\mathbb{N}}$ thus induced verifies indeed the conditions of Definition 3.1, let us show the following lemma.

Lemma 3.10

The sequence $(T_n)_{n\in\mathbb{N}}$ defined by $T_0 := F_0$ and $T_{n+1} := \Sigma_{n+1} T_n$ for all $n > 0$ verifies

$$(\forall n \in \mathbb{N}) \quad T_n = F_{\kappa^n}.$$

Let us discuss the idea of the proof. We proceed by induction and use a double inclusion argument to prove the induction step. To show that $\Sigma_{n+1} T_n$ is included in $F_{\kappa^{n+1}}$ we rely on Lemma 2.12, that is to say we verify that every element of $\Sigma_{n+1} T_n$ has range included in $[0, \kappa^{n+1}-1]$. For the reversed inclusion we consider an element (h, t) of $F_{\kappa^{n+1}}$ and explicit the elements $(g, j\kappa^n)$ of Σ_{n+1} and (f, t') of T_n such that $(h, t) = (g, j\kappa^n)(f, t')$.

Mind the involved maps here: we study the values of g_m and f_m instead of the "derived" functions g'_m, f'_m usually considered.

Proof of the lemma. The assertion is true for T_0. Now let $n \geqslant 0$ and assume that $T_n = F_{\kappa^n}$. We show the induction step by double inclusion.

First inclusion

Let us prove that $\Sigma_{n+1}T_n \subseteq F_{\kappa^{n+1}}$. Recall that $\Sigma_{n+1} = \cup_{j=0}^{\kappa-1}\Sigma_{n+1}^j$.

Let $(\mathbf{f}, t) \in T_n$ and $j \in \{0, \dots, \kappa-1\}$. Take $(\mathbf{g}, j\kappa^n) \in \Sigma_{n+1}^j$, then the following product

$$(\mathbf{g}, j\kappa^n)(\mathbf{f}, t) = \left(\left(g_m f_m(\cdot - j\kappa^n)\right)_m, t + j\kappa^n\right)$$

verifies $t + j\kappa^n \in [j\kappa^n, \kappa^n - 1 + j\kappa^n]$ which is contained in $\left[0, \kappa^{n+1} - 1\right]$ since $j \leqslant \kappa - 1$. Moreover

$$g_0(x)f_0(x - j\kappa^n) = \begin{cases} g_0(x) & \text{if } x \in [0, j\kappa^n] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right] \\ f_0(x - j\kappa^n) & \text{if } x \in \left[j\kappa^n, (j+1)\kappa^n - 1\right] \\ 0 & \text{else.} \end{cases}$$

Thus $\operatorname{supp}(g_0 f_0(\cdot - j\kappa^n)) \subseteq \left[0, \kappa^{n+1} - 1\right]$. Furthermore, for all $m \in \{1, \dots, \mathfrak{L}(n)\}$

$$\operatorname{supp}(g'_m) \subset [k_m, j\kappa^n + k_m - 1] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right]$$
$$\operatorname{supp}\left(f'_m(\cdot - j\kappa^n)\right) \subseteq [j\kappa^n + k_m, (j+1)\kappa^n - 1],$$

hence by Claim 2.5 the support of $(g_m f_m(\cdot - j\kappa^m))'$ is contained in $[k_m, \kappa^{n+1} - 1]$.

Now if $\mathfrak{L}(n+1) = \mathfrak{L}(n) + 1$ consider $m = \mathfrak{L}(n) + 1$. In that case $f'_m \equiv e$ since $m > \mathfrak{L}(n)$. Thus $(g_m f_m(\cdot - j\kappa^m))' = g'_n$ whose support is contained in $[k_{\mathfrak{L}(n)+1}, \kappa^{n+1} - 1]$.

Finally $(g_m f_m(\cdot - j\kappa^m))' \equiv 0$ for all $m \notin [0, \mathfrak{L}(n+1)]$. Hence by Lemma 2.12 the product $(\mathbf{g}, j\kappa^n)(\mathbf{f}, t)$ has range included in $\left[0, \kappa^{n+1} - 1\right]$ and thus belongs to $F_{\kappa^{n+1}}$.

Second Inclusion

Let us show that $F_{\kappa^{n+1}}$ is contained in $\Sigma_{n+1}T_n$. So take $(\mathbf{h}, t)$ in $F_{\kappa^{n+1}}$. We want to define $(\mathbf{f}, t') \in T_n$ and $(\mathbf{g}, j\kappa^n) \in \Sigma_{n+1}$ such that $(\mathbf{g}, j\kappa^n)(\mathbf{f}, t') = (\mathbf{h}, t)$. First remark that $t < \kappa^{n+1}$ since $(\mathbf{h}, t)$ belongs to $F_{\kappa^{n+1}}$. Thus there exists $t_0, \dots, t_n$ in $[0, \kappa - 1]$ such that $t = \sum_{i=0}^{n} t_i \kappa^i$. Let $j = t_n$ and $t' = \sum_{i=0}^{n-1} t_i \kappa^i$. Then j does belong to $[0, \kappa - 1]$ and t' to $[0, \kappa^n - 1]$. We now have to define $\mathbf{f}$ and $\mathbf{g}$ such that

$$\left((g_m f_m(\cdot - j\kappa^n))_m, t' + j\kappa^n\right) = (\mathbf{h}, t).$$

We refer to Figure 6 for an illustration of the different supports. Let

$$f_0(x) := \begin{cases} h_0(x + j\kappa^n) & \text{if } x \in [0, \kappa^n - 1], \\ e & \text{else,} \end{cases}$$

$$g_0(x) := \begin{cases} h_0(x) & \text{if } x \in [0, j\kappa^n - 1] \cup [(j+1)\kappa^n, \kappa^{n+1} - 1], \\ e & \text{else.} \end{cases}$$

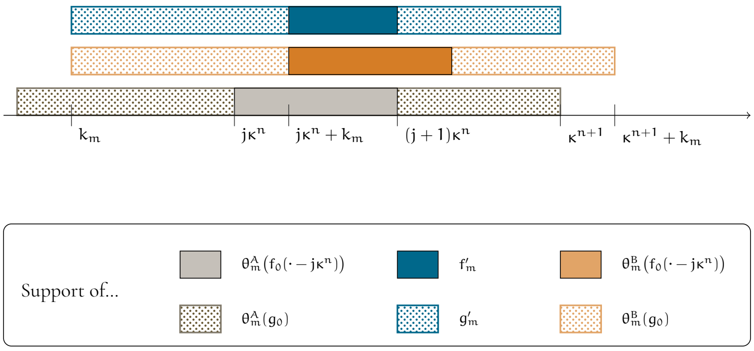

Figure 6: Supports

One can verify immediately that $g_0 f_0(\cdot - j\kappa^n) = h_0$. Then take $m \in [1, \mathfrak{L}(n)]$ and let

$$f'_m(x) := \begin{cases} h'_m(x + j\kappa^n) & \text{if } x \in [k_m, \kappa^n - 1], \\ e & \text{else,} \end{cases}$$

$$g'_m(x) := \begin{cases} h'_m(x) & \text{if } x \in [k_m, j\kappa^n + k_m - 1] \cup [(j+1)\kappa^n, \kappa^{n+1} - 1] \\ e & \text{else.} \end{cases}$$

Now if $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$ then $k_{\mathfrak{L}(n+1)} \geqslant \kappa^n$ and in that case define $g'_{\mathfrak{L}(n+1)} = h'_{\mathfrak{L}(n+1)}$. Finally let $f'_{\mathfrak{L}(n+1)} \equiv e$ and if $m > \mathfrak{L}(n+1)$ let $g'_m \equiv e \equiv f'_m$.

With the above definitions $\mathbf{f}$ and $\mathbf{g}$ are uniquely defined. Moreover, by definition $(\mathbf{g}, j\kappa^n)$ belongs to Σ^j_{n+1} and by Lemma 2.12 we have $\text{range}(\mathbf{f}, t) \subseteq [0, \kappa^n - 1]$ thus $(\mathbf{f}, t')$ belongs to T_n.

Now, using Lemma 2.6 we verify that $g_m f_m(\cdot - j\kappa^n) = h_m$ thus $(\mathbf{h}, t) \in \Sigma_{n+1} T_n$.

Hence, combining the first and second inclusion we get $F_{\kappa^{n+1}} = T_n$. $\square$

We now know that $(T_n)_{n\in\mathbb{N}}$ is a Følner sequence. To prove Theorem 3.9 we have to show that $(\Sigma_n)_{n\in\mathbb{N}}$ a Følner *tiling* shift.

Proof of Theorem 3.9. The sequence $(T_n)_{n\in\mathbb{N}}$ is a Følner sequence, by the last lemma. Thus we only have to show that for all $\sigma \neq \tilde{\sigma} \in \Sigma_{n+1}$, $\sigma T_n \cap \tilde{\sigma} T_n = \emptyset$. So let us denote by $(\mathbf{h}, t)$ an element of $\sigma T_n \cap \tilde{\sigma} T_n$. We distinguish two cases.

First if $\sigma \in \Sigma^j_{n+1}$ and $\tilde{\sigma} \in \Sigma^i_{n+1}$ for some $i \neq j$, then the cursor of σ is equal to $j\kappa^n$ and the one of $\tilde{\sigma}$ to $i\kappa^n$. Thus

$$(\mathbf{h}, t) \in \sigma T_n \Rightarrow t \in [j\kappa^n, (j+1)\kappa^n - 1],$$
$$(\mathbf{h}, t) \in \tilde{\sigma} T_n \Rightarrow t \in [i\kappa^n, (i+1)\kappa^n - 1].$$

But since $i \neq j$ these two intervals are disjoint, thus $\sigma T_n \cap \tilde{\sigma} T_n = \emptyset$.

Now fix $j \in \{0, \dots, \kappa - 1\}$ and take $\sigma, \tilde{\sigma} \in \Sigma^j_{n+1}$. Let $\sigma := (\mathbf{g}, j\kappa^n)$ and $\tilde{\sigma} := (\tilde{\mathbf{g}}, j\kappa^n)$. Assume that there exists $(\mathbf{f}, t), (\tilde{\mathbf{f}}, \tilde{t}) \in T_n$ such that $(\mathbf{g}, j\kappa^n)(\mathbf{f}, t) = (\tilde{\mathbf{g}}, j\kappa^n)(\tilde{\mathbf{f}}, \tilde{t})$. Then

$$\forall m \in \mathbb{N} \quad g_m f_m(\cdot - j\kappa^n) = \tilde{g}_m \tilde{f}_m(\cdot - j\kappa^n). \tag{3.1}$$

First remark that

$$\begin{aligned} \sigma, \tilde{\sigma} \in \Sigma^j_{n+1} \quad &\Longrightarrow \quad \text{supp}(g_0), \text{supp}(\tilde{g}_0) \subseteq [0, j\kappa^n - 1] \cup [(j+1)\kappa^n, \kappa^{n+1} - 1] \\ (\mathbf{f}, t), (\tilde{\mathbf{f}}, \tilde{t}) \in T_n \quad &\Longrightarrow \quad \text{supp}(f_0(\cdot - j\kappa^n)), \text{supp}(\tilde{f}_0(\cdot - j\kappa^n)) \subseteq [j\kappa^n, (j+1)\kappa^n - 1]. \end{aligned}$$

In other word the support of g_0 (resp $\tilde{g}_0$) is disjoint from the one of $f_0(\cdot - j\kappa^n)$ (resp $\tilde{f}_0(\cdot - j\kappa^n)$). Combining this with eq. (3.1) we obtain that $g_0 = \tilde{g}_0$ and $f_0 = \tilde{f}_0$.

Now let $m > 0$ and let us show that $g_m = \tilde{g}_m$. Due to supports overlap (see Figure 7) we need to decompose $[0, \kappa^{n+1} - 1]$ in five subintervals, namely

$$\begin{aligned}\left[0, \kappa^{n+1} - 1\right] = &\left[0, j\kappa^n - 1\right] \sqcup \left[j\kappa^n, j\kappa^n + k_m - 1\right] \sqcup \left[j\kappa^n + k_m, (j+1)\kappa^n - 1\right], \\ &\sqcup \left[(j+1)\kappa^n, (j+1)\kappa^n + k_m - 1\right] \sqcup \left[(j+1)\kappa^n + k_m, \kappa^{n+1} - 1\right].\end{aligned}$$

If $x \leqslant j\kappa^n - 1$ or $x \geqslant (j+1)\kappa^n + k_m$, then $f_m(x - j\kappa^n) = e = \tilde{f}_m(x - j\kappa^n)$ and thus $g_m(x) = \tilde{g}_m(x)$ by eq. (3.1).

If $x \in [j\kappa^n, j\kappa^n + k_m - 1]$ then using Lemma 2.6 and the fact that on that subinterval $f_0 = \tilde{f}_0$, we get

$$f_m(x - j\kappa^n) = \theta_0^A\left(f_0\left(x - j\kappa^n\right)\right) = \theta_0^A\left(\tilde{f}_0\left(x - j\kappa^n\right)\right) = \tilde{f}_m(x - j\kappa^n).$$

Hence by eq. (3.1) we get $g_m(x) = \tilde{g}_m(x)$.

If x belongs to $[j\kappa^n + k_m, (j+1)\kappa^n - 1]$ then $g_m(x) = \tilde{g}_m(x) = e$ and thus eq. (3.1) implies that $f_m(x - j\kappa^n) = \tilde{f}_m(x - j\kappa^n)$, that is to say f_m and $\tilde{f}_m$ coïncide on $[k_m, \kappa^n - 1]$.

Finally if $x \in [(j+1)\kappa^n, (j+1)\kappa^n + k_m - 1]$ then using Lemma 2.6 and the fact that $f_0 = \tilde{f}_0$ on that subinterval, we get

$$f_m(x - j\kappa^n) = \theta_0^B\left(f_0\left(x - j\kappa^n - k_m\right)\right) = \theta_0^B\left(\tilde{f}_0\left(x - j\kappa^n - k_m\right)\right) = \tilde{f}_m(x).$$

Hence by eq. (3.1), we have $g_m(x) = \tilde{g}_m(x)$.

Thus $\mathbf{g} = \tilde{\mathbf{g}}$ and then $\sigma = \tilde{\sigma}$. Which concludes the proof of the theorem. □

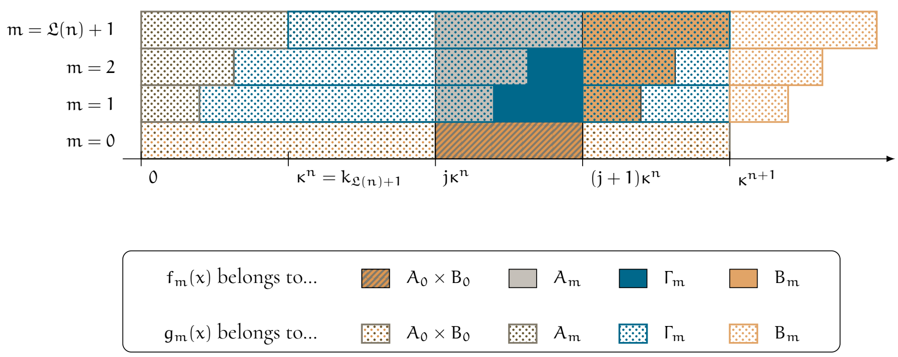

Figure 7: Supports overlap

3.2.3 Diameter and boundary

Let us now quantify our shifts sequence.

Proposition 3.11

The sequence $(\Sigma_n)_{n\in\mathbb{N}}$ defined in Section 3.2.1 is a (R_n, ε_n)-Følner tiling shift where

$$R_n = C_R \kappa^n l_{\mathfrak{L}(n)} \qquad \varepsilon_n = \frac{2}{\kappa^n},$$

for some strictly positive constant C_R.

First we prove the following lemma.

Lemma 3.12

There exists $C_R > 0$ depending only on Δ such that $\operatorname{diam}(F_n) \leqslant C_R n l_{\iota(n-1)}$ for all $n \in \mathbb{N}$.

To show this result, we use Proposition 2.20.

Proof. Let $n \in \mathbb{N}$ and $(f,t) \in F_n$. First, take $m \leqslant \iota(n-1)$ and let us bound E_m by above. Recall that $I_j^m = [jk_m/2, (j+1)k_m/2 - 1]$. Since (f,t) belongs to F_n its range is included in $[0, n-1]$, thus

$$\begin{aligned}\left|\left\{j \in \mathbb{Z} : \text{range}(f_m, t) \cap I_j^m \neq \emptyset\right\}\right| &\leqslant \left|\left\{j \in \mathbb{Z} : [0, n-1] \cap [jk_m/2, (j+1)k_m/2 - 1] \neq \emptyset\right\}\right|, \\ &\leqslant \left|\left\{j \in \mathbb{Z} : jk_m/2 \leqslant n-1 \text{ and } (j+1)k_m/2 \geqslant 1\right\}\right|, \\ &\leqslant \frac{2(n-2)}{k_m} + 1.\end{aligned}$$

Moreover remark that $|f_m(x)|_{\Gamma_m} \leqslant \operatorname{diam}(\Gamma_m) \leqslant c l_m$ for all x, thus

$$\begin{aligned}E_m(f_m) &= k_m \sum_{j : \text{range}(f_m, t) \cap I_j^m \neq \emptyset} \max_{x \in I_j^m} \left(|f_m(x)|_{\Gamma_m} - 1\right)_+, \\ &\leqslant k_m \sum_{j : \text{range}(f_m, t) \cap I_j^m \neq \emptyset} l_m, \\ &\leqslant k_m l_m \left(\frac{2(n-2)}{k_m} + 1\right) = l_m(2(n-2) + k_m).\end{aligned}$$

Thus, applying the second part of Proposition 2.20 we get

$$|(f_m, t)|_{\Delta_m} \leqslant 9\left(\text{range}(f_m, t) + E_m(f_m)\right) \leqslant 9\left(n + l_m(2(n-2) + k_m)\right).$$

But if $m \leqslant \iota(n-1)$ then $k_m \leqslant n-1 \leqslant n$ thus we can bound $|(f_m, t)|_{\Delta_m}$ by above by $9n(3l_m + 1)$. Now remark that $\iota\left(\text{range}(f,t)\right) \leqslant \iota(n-1)$. Thus, using the preceding inequality and the first part of Proposition 2.20, we get

$$\begin{aligned}|(f,t)|_{\Delta} \leqslant 500 \sum_{m=0}^{\iota(\text{range}(f,t))} |(f_m, t)|_{\Delta_m} &\leqslant 500 \sum_{m=0}^{\iota(n-1)} 9n\left(3l_m + 1\right), \\ &\leqslant 4500n \sum_{m=0}^{\iota(n-1)} \left(3l_m + 1\right)\end{aligned}$$

Finally, since l_m is a subsequence of a geometric sequence, there exists $C_l > 0$ such that $\sum_{m=0}^{\iota(n-1)} (3l_m + 1) \leqslant C_l l_{\iota(n-1)}$. Denoting $C_R := 4500C_l$ we get the lemma. □

Let us now show the wanted proposition.

Proof of Proposition 3.11. First remark that by the proof of Proposition 2.13 we have

$$\varepsilon_n = \frac{|\partial T_n|}{|T_n|} = \frac{|\partial F_{\kappa^n}|}{|F_{\kappa^n}|} = \frac{2}{\kappa^n}.$$

Now by Lemma 3.12 we have $\operatorname{diam}(T_n) = \operatorname{diam}(F_{\kappa^n}) \leqslant C_R \kappa^n l_{\mathfrak{L}(n)}$. □

4 COUPLING WITH $\mathbb{Z}$

Our aim in this section is to show Theorem 1.8. What we actually show is that a diagonal product Δ admits a coupling with $\mathbb{Z}$ satisfying Theorem 1.8. We start by defining a Følner tiling shift for $\mathbb{Z}$ in Section 4.1. We compute in Section 4.2 an estimate the diameter of such tiles, namely the cardinal $|T_n|$. We conclude by showing the integrability of the coupling using the criterion given by Theorem 3.5. And then show that Δ thus considered satisfies Theorem 1.8.

4.1 Tiles for $\mathbb{Z}$

We will denote by $(\Sigma'_n)_{n\in\mathbb{N}}$ a Følner tiling shift of $\mathbb{Z}$ and by $(T'_n)_n$ the corresponding tiles.

Consider $(\Sigma_n)_n$ and $(T_n)_n$ as defined in Section 3.2.1 and Lemma 3.10 respectively. In order to use Theorem 3.5 to get an orbit equivalence coupling between $\mathbb{Z}$ and Δ we need Σ_{n+1} and Σ'_{n+1} to have the same number of elements. We thus define

$$\begin{cases} & \Sigma'_0 = [0, |T_0| - 1] \\ \forall n \in \mathbb{N} & \Sigma'_{n+1} := \{0, |T_n|, 2|T_n|, \dots, (|\Sigma_{n+1}| - 1)\,|T_n|\}. \end{cases} \tag{4.1}$$

It induces a sequence $(T'_n)_{n\in\mathbb{N}}$ defined by $T'_0 = \Sigma'_0$ and $T'_{n+1} = \Sigma'_{n+1}T'_n$ for all $n \geqslant 0$. We are going to prove that $(\Sigma'_n)_{n\in\mathbb{N}}$ is a Følner tiling shift for $\mathbb{Z}$.

Proposition 4.1

The sequence $(\Sigma'_n)_{n\in\mathbb{N}}$ defined by eq. (4.1) is a (R'_n, ε'_n)-Følner tiling shifts for $\mathbb{Z}$ with

$$R'_n = |T_n| \qquad \varepsilon'_n = 2/|T_n|.$$

Moreover the induced sequence $(T'_n)_{n\in\mathbb{N}}$ verifies $T'_n = [0, |T_n| - 1]$ for all $n \in \mathbb{N}$.

Proof. Let $(\Sigma'_n)_{n\in\mathbb{N}}$ be as defined by eq. (4.1) and recall that the induced tiling $(T'_n)_{n\in\mathbb{N}}$ is the sequence defined by $T'_0 := \Sigma'_0$ and $T'_{n+1} = \Sigma'_{n+1}T'_n$ for all $n \in \mathbb{N}$. One can easily prove that for all $n \geqslant 0$

$$T'_n = [0, |T_n| - 1]\,. \tag{4.2}$$

It is now immediate to check that $\operatorname{diam}(T'_n) = |T_n|$ and $|\partial T'_n|/|T'_n| = 2/|T_n|$. Furthermore note that if $\sigma, \sigma' \in \Sigma'_{n+1}$ such that $\sigma \neq \sigma'$ then $d_{\mathbb{Z}}(\sigma, \sigma') \geqslant |T_n| = \operatorname{diam}(T'_n)$. Thus for such σ and σ' we get $\sigma T'_n \cap \sigma' T_n = \emptyset$. Therefore $(\Sigma_n)_{n\in\mathbb{N}}$ is a Følner tiling shift and the proposition follows from the above quantifications on T_n. $\square$

4.2 Estimates: diameter and boundary

The integrability of the coupling between $\mathbb{Z}$ and Δ depends on (R_n, ε_n) and (R'_n, ε'_n) but by the above proposition, that last couple depends on the value of the cardinality of the tiles $(T_n)_{n\in\mathbb{N}}$. The aim of this section is to give estimates of $|T_n|$ involving only terms of $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$. First let us precise the value of $|T_n|$.

Lemma 4.2

The sequence $(T_n)_n$ defined in Theorem 3.9 verifies

$$|T_n| = \kappa^n \left(|A||B|\right)^{\kappa^n} \prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}.$$

Proof. Recall that $T_n = F_{\kappa^n} = \{(f,t) \mid \operatorname{range}(f,t) \subseteq \{0,\dots,\kappa^n-1\}\}$ for all $n \in \mathbb{N}$. We use here Lemma 2.12 linking range and supports. Let $n \in \mathbb{N}$ and take $(f,t) \in T_n$, then there are exactly κ^n values of t possible. Moreover f is uniquely determined by f_0 and $f'_1,\dots,f'_{\mathfrak{L}(n)}$ (see Lemma 2.6). But f_0 is supported on $[0,\kappa^n-1]$ which is set of cardinal κ^n so there are exactly $(|A||B|)^{\kappa^n}$ possible values for f_0. Moreover if $m>0$ then remark that f'_m is supported on $[k_m,\kappa^n-1]$ which has κ^n-k_m elements so there are exactly $\left|\Gamma'_m\right|^{\kappa^n-k_m}$ possible values for f'_m. Thus the number of elements in T_n is

$$\kappa^n\left(|A||B|\right)^{\kappa^n}\prod_{m=1}^{\mathfrak{L}(n)}\left|\Gamma'_m\right|^{\kappa^n-k_m}.$$

□

Now let us bound $|T_n|$ such that the bounds depend only on $(\kappa^m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$.

Proposition 4.3

There exists two constants $C_2, C_3 > 0$ such that for all $n \in \mathbb{N}$,

$$C_2\kappa^{n-1}l_{\mathfrak{L}(n)} \leqslant \ln|T_n| \leqslant C_3\kappa^n l_{\mathfrak{L}(n)}.$$

Before showing the above proposition let us give an estimate of the right factor of the expression of $|T_n|$.

Lemma 4.4

There exists two constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$,

$$C_2\kappa^{n-1}l_{\mathfrak{L}(n)} \leqslant \ln\left(\prod_{m=1}^{\mathfrak{L}(n)}\left|\Gamma'_m\right|^{\kappa^n-k_m}\right) \leqslant C_1\kappa^n l_{\mathfrak{L}(n)}.$$

Proof. Recall that by eq. (2.1) there exists c_1, $c_2 > 0$ such that, for all m

$$c_1 l_m - c_2 \leqslant \ln|\Gamma_m| \leqslant c_1 l_m + c_2.$$

Since $\Gamma'_m \leqslant \Gamma_m$ we thus have

$$\begin{aligned}\ln\left(\prod_{m=1}^{\mathfrak{L}(n)}\left|\Gamma'_m\right|^{\kappa^n-k_m}\right) &\leqslant \sum_{m=1}^{\mathfrak{L}(n)}(\kappa^n-k_m)\ln|\Gamma_m|,\\ &\leqslant \sum_{m=1}^{\mathfrak{L}(n)}(\kappa^n-k_m)\left(c_1 l_m + c_2\right).\end{aligned}$$

But we can bound $\kappa^n - k_m$ from above by κ^n and since $(l_m)_{m\in\mathbb{N}}$ is a subsequence of a sequence having geometric growth, the sum $\sum_{m=1}^{\mathfrak{L}(n)}(c_1 l_m + c_2)$ is bounded from above by its last term up to a multiplicative constant. That is to say: there exists $C_1 > 0$ such that

$$\ln\left(\prod_{m=1}^{\mathfrak{L}(n)}\left|\Gamma'_m\right|^{\kappa^n-k_m}\right) \leqslant C_1\kappa^n l_{\mathfrak{L}(n)}.$$

Hence the upper bound. Now, using that $[\Gamma_m : \Gamma'_m] = |A||B|$ we have

$$\ln\left(\prod_{m=1}^{\mathfrak{L}(n)}\left|\Gamma'_m\right|^{\kappa^n-k_m}\right) = \sum_{m=1}^{\mathfrak{L}(n)}(\kappa^n-k_m)\ln|\Gamma'_m| = \sum_{m=1}^{\mathfrak{L}(n)}(\kappa^n-k_m)\ln\left(\frac{|\Gamma_m|}{|A||B|}\right).$$

Bounding the sum from below by its last term and using once more eq. (2.1), we get

$$\begin{aligned}\ln\left(\prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}\right) &\geqslant (\kappa^n - k_{\mathfrak{L}(n)}) \ln\left(\frac{|\Gamma_{\mathfrak{L}(n)}|}{|A||B|}\right), \\ &\geqslant (\kappa^n - k_{\mathfrak{L}(n)})\left(c_1 l_{\mathfrak{L}(n)} - c_2 - \ln(|A||B|)\right), \\ &\geqslant C_2(\kappa^n - k_{\mathfrak{L}(n)}) l_{\mathfrak{L}(n)},\end{aligned}$$

for some $C_2 > 0$. We get the wanted inequality by noting that $\kappa^n - k_{\mathfrak{L}(n)} \geqslant \kappa^{n-1}$. ☐

Proof of Proposition 4.3. Applying Lemma 4.4 to the cardinal of T_n given by Lemma 4.2 we obtain that there exists $C_3 > 0$ such that $\ln|T_n| \leqslant C_3 \kappa^n l_{\mathfrak{L}(n)}$. Hence the upper bound. The minoration comes imediately from Lemma 4.4. ☐

Equipped with these bounds on $|T_n|$ we can now show the wanted integrability for the coupling.

4.3 Integrability of the coupling

We will show that Δ is the group satisfying Theorem 1.8, but first let us quantify the integrability of the orbit equivalence coupling with $\mathbb{Z}$ induced by the Følner tiling shifts we built. Recall that $\mathcal{C}$ denotes the set of non-decreasing functions $\rho\colon [1, +\infty[\to [1, +\infty[$ such that $x/\rho(x)$ is non-decreasing.

Theorem 4.5

Let $\rho \in \mathcal{C}$ and take Δ to be the Brieussel-Zheng's diagonal product defined from ρ. Let $\varepsilon > 0$ and $\Psi := \exp \circ \rho$ and let

$$\varphi_\varepsilon(x) := \frac{\rho \circ \ln(x)}{\left(\ln \circ \rho \circ \ln(x)\right)^{1+\varepsilon}}.$$

There exists an orbit equivalence coupling from Δ to $\mathbb{Z}$ that is $(\varphi_\varepsilon, \Psi)$-integrable.

Let us discuss the strategy of the proof. The demonstration is based on Theorem 3.5, thus we first prove that $(\Psi(2R_n)\varepsilon'_{n-1})_n$ is summable and then that $(\varphi_\varepsilon(2R'_n)\varepsilon_{n-1})_n$ is. In both cases we use Proposition 4.3 to get upper bounds. So far, we have the following quantifications.

$$\begin{aligned} R_n &= C_R \kappa^n l_{\mathfrak{L}(n)} & \qquad R'_n &= |T_n| \\ \varepsilon_n &= 2\kappa^{-n} & \varepsilon'_n &= 2/|T_n| \end{aligned}$$

Proof of Theorem 4.5. Let $\rho \in \mathcal{C}$ and take Δ to be the diagonal product defined from ρ as described in Section 2.3.

To begin, let us recall some preliminary results about ρ. Remember that $\rho \simeq \bar{\rho}$ where $\bar{\rho}$ is defined below eq. (2.3). By definition of $\mathfrak{L}(n)$ we have $k_{\mathfrak{L}(n)} l_{\mathfrak{L}(n)} \leqslant \kappa^n l_{\mathfrak{L}(n)} \leqslant k_{\mathfrak{L}(n)+1} l_{\mathfrak{L}(n)}$, thus by eq. (2.3)

$$\bar{\rho}(\kappa^n l_{\mathfrak{L}(n)}) = \kappa^n. \tag{4.3}$$

Now let us show that the coupling from $\mathbb{Z}$ to Δ is Ψ-integrable. To do so we prove that $(\Psi(2R_n)\varepsilon'_{n-1})$ is summable. First note that by Proposition 4.3 we have the following lower bound on $|T_{n-1}|$

$$|T_{n-1}| \geqslant \exp\left(C_2\kappa^{n-2}l_{\mathfrak{L}(n-1)}\right). \tag{4.4}$$

Moreover recall that $R_n = C_R\kappa^n l_{\mathfrak{L}(n)}$ and $\varepsilon'_{n-1} = 2/|T_{n-1}|$ thus by the inequality above

$$\begin{aligned}\Psi(2R_n)\varepsilon'_{n-1} &= \exp\left[\rho(2C_R\kappa^n l_{\mathfrak{L}(n)})\right]\frac{2}{|T_{n-1}|},\\ &\leqslant 2\exp\left[\rho\left(2C_R\kappa^n l_{\mathfrak{L}(n)}\right) - C_2\kappa^{n-2}l_{\mathfrak{L}(n-1)}\right].\end{aligned}$$

But remember that $\rho \simeq \bar{\rho}$. Thus using eqs. (2.2) and (4.3) we get

$$\rho\left(2C_R\kappa^n l_{\mathfrak{L}(n)}\right) \simeq \bar{\rho}\left(2C_R\kappa^n l_{\mathfrak{L}(n)}\right) \leqslant 2C_R\bar{\rho}\left(\kappa^n l_{\mathfrak{L}(n)}\right) = 2C_R\kappa^n. \tag{4.5}$$

Combining the above result with the previous inequality, we get

$$\begin{aligned}\Psi(2R_n)\varepsilon'_{n-1} &\preccurlyeq 2\exp\left[2C_R\kappa^n - C_2\kappa^{n-2}l_{\mathfrak{L}(n-1)}\right],\\ &= 2\exp\left[\kappa^{n-2}\left(2C_R\kappa^2 - C_2 l_{\mathfrak{L}(n-1)}\right)\right],\end{aligned}$$

which is summable. Indeed $l_{\mathfrak{L}(n)}$ tends to infinity and thus $\left(2C_R\kappa^2 - C_2 l_{\mathfrak{L}(n-1)}\right) < -1$ for n large enough. Hence by Theorem 3.5 the orbit equivalence from $\mathbb{Z}$ to Δ si Ψ-integrable.

Now, let us show that for all $\varepsilon > 0$ the coupling from Δ to $\mathbb{Z}$ is φ_ε-integrable. Based on Theorem 3.5 we only have to prove that $\varphi_\varepsilon(2R'_n)\varepsilon_{n-1}$ is summable. Recall that $R'_n = |T_n|$ and $\varepsilon_{n-1} = 2/\kappa^{n-2}$ and remark that by both the lower and upper bounds given in Proposition 4.3 we have

$$\varphi_\varepsilon(2R'_n)\varepsilon_{n-1} = \frac{2\rho\circ\ln\left(2|T_n|\right)}{\left(\ln\circ\rho\circ\ln\left(2|T_n|\right)\right)^{1+\varepsilon}\kappa^{n-1}} \leqslant \frac{2\rho\left(2C_3\kappa^n l_{\mathfrak{L}(n)}\right)}{\left(\ln\circ\rho\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right)\right)^{1+\varepsilon}\kappa^{n-1}}.$$

Let us give a lower bound for $\rho\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right)$. Recall that $\rho \simeq \bar{\rho}$ furthemore if $2C_2 \geqslant 1$ then by eq. (4.3) and since $\bar{\rho}$ is non-decreasing

$$\kappa^{n-1} = \bar{\rho}\left(\kappa^{n-1}l_{\mathfrak{L}(n)}\right) \leqslant \bar{\rho}\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right) \simeq \rho\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right).$$

Now if $2C_2 < 1$ using Claim 2.18 with $c' = 2C_2$ and $x' = \kappa^{n-1}l_{\mathfrak{L}(n)}$ we get (for n large enough)

$$2C_2\kappa^{n-1} = 2C_2\bar{\rho}(\kappa^{n-1}l_{\mathfrak{L}(n)}) \leqslant \bar{\rho}(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}) \simeq \rho\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right)$$

Hence, in both cases $\kappa^{n-1} \preccurlyeq \rho(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)})$. Finally replacing C_R by C_3 in eq. (4.5) we can show that $\rho\left(2C_3\kappa^n l_{\mathfrak{L}(n)}\right) \leqslant 2C_3\kappa^n$. Thus, combining the two preceding results we obtain

$$\begin{aligned}\varphi_\varepsilon(R'_n)\varepsilon_{n-1} &\leqslant \frac{2\rho\left(C_3\kappa^n l_{\mathfrak{L}(n)}\right)}{\left(\ln\circ\rho\left(C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right)\right)^{1+\varepsilon}\kappa^{n-1}}\\ &\preccurlyeq \frac{\kappa^n}{\left(\ln\left(\kappa^{n-1}\right)\right)^{1+\varepsilon}\kappa^{n-1}} = \frac{\kappa}{\left((n-1)\ln(\kappa)\right)^{1+\varepsilon}},\end{aligned}$$

which is a summable sequence. Hence by Theorem 3.5 the orbit equivalence coupling from Δ to $\mathbb{Z}$ si φ_ε-integrable. □

Remark 4.6. This result is stated in the general case, that is to say for an abstract ρ. Nonetheless, for some particular functions ρ the quantification can be improved. For example the case where $k_n = 2^n$ and $l_n = 2^{\alpha n}$ corresponds to $\rho(x) \simeq x^{1/(1+\alpha)}$. In that case $\mathfrak{L}(n) = n-1$ and we can show that the coupling from $\mathbb{Z}$ to Δ is exp-integrable (instead of $\exp\circ\rho$-integrable). Indeed, let $c_\varphi < C_2/(C_R 2^{3+\alpha})$ and $\Psi(x) := \exp(c_\varphi x)$, then by eq. (4.4)

$$\begin{aligned}\Psi(2R_n)\varepsilon'_{n-1} &= \exp\left[c_\varphi 2C_R k_n l_{n-1}\right]\frac{2}{|T_{n-1}|}\\ &\leqslant \exp\left[c_\varphi 2C_R 2^n 2^{\alpha(n-1)} - C_2 2^{n-2}2^{\alpha(n-2)}\right]2\\ &= 2\exp\left[2^{n-2}2^{\alpha(n-2)}\left(c_\varphi C_R 2^{3+\alpha} - C_2\right)\right].\end{aligned}$$

Which is summable by choice of c_φ.

Remark 4.7. We can verify that the integrability obtained for the coupling from Δ to $\mathbb{Z}$ is “almost” optimal. Indeed if the coupling from Δ to $\mathbb{Z}$ is φ-integrable, then by Theorem 1.5 we have

$$\varphi\circ I_{\mathbb{Z}} \preccurlyeq I_\Delta$$

where we recall that $I_{\mathbb{Z}}(n) \simeq n$ and $I_\Delta(n) \simeq \rho\circ\ln(n)$. Thus using the inequality above, we get $\varphi(n) \preccurlyeq \rho\circ\ln(n)$. Hence the quantification of Theorem 4.5 is optimal up to a logarithmic factor.

It is now easy to prove our first main theorem.

Proof of Theorem 1.8. Let $\rho \in \mathcal{C}$ and Δ to be the group defined in Proposition 2.14. By the aforementioned proposition it verifies $I_\Delta \simeq \rho\circ\log$. Moreover by Theorem 4.5 there exists an orbit equivalence coupling from Δ and $\mathbb{Z}$ that is $(\varphi_\varepsilon, \exp\circ\rho)$-integrable for all $\varepsilon > 0$. □

To prove Corollary 1.9 we use the composition of couplings introduced in [DKLMT20]. We recall below the proposition concerning the integrability of this composition and refer to [DKLMT20, Sections 2.3 and 2.5] for more details on the construction of the corresponding coupling.

Proposition 4.8 (*[DKLMT20, Prop. 2.9 and 2.29]*)

> If (X_1,μ_1) (resp. (X_2,μ_2)) is a (φ, L^0)-integrable (resp. (ψ, L^0)-integrable) orbit equivalence coupling from Γ to Λ (resp. Λ to Σ), the composition of couplings gives a $(\varphi\circ\psi, L^0)$-integrable orbit equivalence coupling from Γ to Σ.

Let us now show Corollary 1.9 concerning the coupling with $\mathbb{Z}^d$.

Proof of Corollary 1.9. Let $\rho \in \mathcal{C}$ and let Δ be the group defined in Proposition 2.14, in particular it verifies $I_\Delta \simeq \rho\circ\log$.

Let $d \geqslant 1$ and recall (see Example 3.6) that for all $p < d$ and all $q < 1/d$ there exists a (L^p, L^q)-integrable orbit equivalence coupling from $\mathbb{Z}$ to $\mathbb{Z}^d$. Hence, using the composition of couplings described in [DKLMT20] we can deduce from Theorem 1.8 and Proposition 4.8 above that there exists a $(\varphi_\varepsilon(\cdot^p), L^0)$-integrable orbit equivalence coupling from Δ to $\mathbb{Z}^d$. Now if $d > p \geqslant 1$ by eq. (2.2)

$$\rho\circ\log(x) \leqslant \rho(p\log(x)) \leqslant p\rho\circ\log(x).$$

Since $\rho(p\log(x)) = \rho\circ\log(x^p)$ we thus have $\rho\circ\log(x^p) \sim \rho\circ\log$. When $p < 1$, using Claim 2.18 instead of eq. (2.2) we obtain a similar equivalence. Thus in both cases $\varphi_\varepsilon(x^p) \simeq \varphi_\varepsilon(x)$. Hence the corollary. □

5 CONCLUSION AND OPEN PROBLEMS

Let us conclude with some questions and remarks.

5.1 Optimality and coupling building techniques

The tiling technique —though inspiring— is not always usable to get orbit equivalence couplings. Indeed the condition that the two Følner tiling shifts must have at each step the same cardinality is very restrictive. Furthemore this technique does not seem to produce couplings with the best quantification: wether it is our coupling with $\mathbb{Z}$ or the one built in [DKLMT20] (Examples 3.6 and 3.7) the integrability is always optimal *up to a logarithmic factor*. One can thus ask: is the optimal integrability reachable? Is the logarithmic error due to the building technique?

5.2 Inverse problem

We studied here the inverse problem for the group of integers (Question 1.7) but one can also ask the same question for other groups than $\mathbb{Z}$.

Question 5.1. Given a function φ and a group H is there a group G such that there exists a (φ, L^0)-measure equivalent from G to H? Can G be chosen such that $\varphi \circ \mathrm{I}_{\mathsf{H}} \simeq \mathrm{I}_{\mathsf{G}}$?

In a future article we plan to answer this question when H is a diagonal product, in particular H can be a lamplighter group. This coupling will be obtained with another building technique than the tiling process and the integrability will be *optimal*, answering the questions of Section 5.1 positively.

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NOTATIONS INDEX

$\preccurlyeq, \simeq$ See above Theorem 1.5.
$|X|$ Cardinal of the set X.
∂F Boundary of the set F.
Δ See Definition 2.1.
Δ_m See Section 2.1.
F_n Følner sequence of Δ.
$\mathbf{g}$ The sequence of maps $(g_m)_{m\in\mathbb{N}}$.
g'_m See Section 2.1.3.
Γ'_m Normal closure of $[A_m, B_m]$.
I_G Isoperimetric profile of G.
R_n Diameter of T_n.
R'_n Diameter of T'_n.
S_G A generating set of the group G.
Σ_n Følner tiling shifts (of Δ).
Σ'_n Følner tiling shifts of $\mathbb{Z}$.
T_n Tile of Δ defined by $T_n = \prod_{i=0}^{n} \Sigma_i$
T'_n Tile of $\mathbb{Z}$ defined by $T'_n = \prod_{i=0}^{n} \Sigma'_i$
$\theta_m^A(f_m)$ Natural projection of f_m on A_m (see Section 2.1.3).
$\theta_m^B(f_m)$ Natural projection of f_m on B_m (see Section 2.1.3).