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Optimality conditions for optimal control problems with respect to the initial condition *via* a Laplace type method and two-scales like expansions

Idriss Mazari, Grégoire Nadin

May 24, 2022

Abstract

We propose a fine analysis of second order optimality conditions for the optimal control of semi-linear parabolic equations with respect to the initial condition. More precisely, we investigate the following problem: maximise with respect to $u_0 \in L^\infty(\Omega)$ the cost functional $J(u_0) = \iint_{(0;T) \times \Omega} j_1(t, x, u) + \int_{\Omega} j_2(x, u(T, \cdot))$ where $\partial_t u - \Delta u = f(t, x, u)$, $u(0, \cdot) = u_0$ with some classical boundary conditions, under constraints of the form $-\kappa_0 \leq u_0 \leq \kappa_1$ a.e., $\int_{\Omega} u_0 = V_0$. This class of problems arises in several application fields. A challenging feature of these problems is the study of the so-called abnormal set $\{-\kappa_0 < u_0^* < \kappa_1\}$ where u_0^* is an optimiser. This set is in general non-empty and it is important (for instance for numerical applications) to understand the behaviour of u_0^* in this set: which values can u_0^* take? In this paper, we introduce a Laplace-type method to provide some answers to this question. This Laplace type method is of independent interest.

Keywords: Reaction-diffusion equation, semi-linear parabolic equation, optimal control, second order optimality conditions, shape optimisation, two-scale expansions.

AMS classification: 35Q92,49J99,34B15.

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1 Introduction and main result

1.1 Scope and objective of the article

An ubiquitous query in PDE constrained optimisation is that of *optimisation with respect to the initial condition*. While several works [5, 18, 19, 20] tackle the delicate issue of analysing second (and first) order optimality conditions under a wide class of constraints and penalisations, these works often fail to offer conclusive information in the context of $L^\infty - L^1$ control problems. These type of constraints arise naturally in the context of population dynamics [13], and the recent activity in the analysis of these optimal control problems, whether it be in the elliptic [15] or in the parabolic setting [7, 10], has underlined the underlying mathematical challenges. While previous works are discussed in section 1.4 let us underline here that, in the present paper, motivated by such applications in the optimal control of population dynamics, we consider a general optimisation problem for heterogeneous semi-linear parabolic equations. Besides being relevant for the numerical approximation of such optimal control problems [14], this sheds a new light on the qualitative properties of solutions of optimal control problems. Furthermore, in exploiting these optimality conditions, we develop a *Laplace-type method* that deals with the limit behaviour of solutions to linear parabolic equations when the initial condition is a sum of highly oscillating frequencies. Our result is related to *two-scale asymptotic expansions*. What is notable here is that we prove a result that does not assume a scale separation, unlike what is usually done in this context [2].

1.2 The optimal control problem

We begin by describing our optimal control problem.

State equation Throughout the paper, $\Omega \subset \mathbb{R}^d$ is a domain with a \mathcal{C}^2 boundary. We chose a boundary condition operator \mathcal{B} as follows

$$\mathcal{B} : u \mapsto \begin{cases} u & \text{(Dirichlet case)} \\ \frac{\partial u}{\partial \nu} & \text{(Neumann case)}. \end{cases}$$

We work with a non-linearity $f = f(t, x, u)$ that satisfies for the time being (additional assumptions related to its higher derivatives are detailed below)

$$\begin{cases} \text{For any compact } K \subset \mathbb{R}, f, \partial_u f, \text{ are bounded over } [0, T] \times \Omega \times \mathbb{R}, \\ \exists M > 0, \forall u \geq M, \forall (t, x) \in (0; T) \times \Omega, f(t, x, u) \leq 0, f(t, x, -u) \geq 0. \end{cases} \quad (\mathbf{H}_f^{\text{exist}})$$

For any initial condition $u_0 \in L^\infty(\Omega)$ we define u_{u_0} as the solution of

$$\begin{cases} \partial_t u_{u_0} - \Delta u_{u_0} = f(t, x, u_{u_0}) & \text{in } (0; T) \times \Omega, \\ \mathcal{B}u_{u_0} = 0 & \text{on } (0; T) \times \partial\Omega, \\ u_{u_0}(0, \cdot) = u_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $T > 0$ is a fixed time horizon. By the standard theory of non-linear parabolic equations [12], for any initial condition $u_0 \in L^\infty(\Omega)$ (where M is given in $(\mathbf{H}_f^{\text{exist}})$), there exists a unique solution u_{u_0} to (1.1). More regularity properties of the solution u_{u_0} are given in Lemma 9.

The optimal control problem We fix two cost functions $j_1 = j_1(t, x, u)$, $j_2 = j_2(x, u)$ and define the functional to optimise

$$J : L^\infty(\Omega) \ni u_0 \mapsto \iint_{(0;T) \times \Omega} j_1(t, x, u_{u_0}(t, x)) dx dt + \int_{\Omega} j_2(x, u_{u_0}(T, x)) dx.$$

We need to define the class of admissible controls we work with. As is often the case in applications, we enforce two constraints, an L^∞ one and an L^1 - one. In other words we fix three constants $0 \leq \kappa_0 < \kappa_1$ and $V_0 \in (0; 1)$, and we define the class of admissible controls as

$$\mathcal{A} := \left\{ u_0 \in L^\infty(\Omega) : -\kappa_0 \leq u_0 \leq \kappa_1 \text{ almost everywhere, } \int_{\Omega} u_0 = V_0 \right\}. \quad (\mathbf{Adm})$$

The optimisation problem under scrutiny throughout this article is

$$\boxed{\max_{u_0 \in \mathcal{A}} J(u_0)} \quad (\mathbf{P})$$

We used a max instead of sup; indeed, under mild assumptions (typically, the continuity of f, j_1, j_2 in u) the existence of an optimal initial datum u_0^* is immediate. It should be noted that working in the class \mathcal{A} is justified in the context of mathematical biology by modelling considerations, but also from the theoretical point of view, as \mathcal{A} corresponds to the natural relaxation of the set $\{\kappa_1 \mathbb{1}_E - \kappa_0 \mathbb{1}_{E^c}, E \subset \Omega, \text{Vol}(E) = \frac{V_0 + \kappa_0}{\kappa_0 + \kappa_1} \text{Vol}(\Omega)\}$ in the $L^\infty - *$ topology.

Optimality conditions for (P) The main contribution of this article is the analysis of the optimality conditions of the optimal control problem (P). For this reason, we first need to describe these optimality conditions (we explain in section 1.4 prior works and why they fail, in the present case, to give satisfactory answers). The first (and second) order Gateaux-differentiability of the functional J hinges on that of the map $u_0 \mapsto u_{u_0}$. Similar to [16, Lemma 3.2], we have the following Lemma.

Lemma 1. *The map $S : \mathcal{A} \ni u_0 \mapsto u_{u_0}$ is twice Gateaux-differentiable at u_0 . For any $u_0 \in \mathcal{A}$, for any perturbation $h \in L^\infty(\Omega)$, the first order Gateaux-derivative of the functional J at u_0 in the direction h is given by*

$$\dot{J}(u_0)[h] = \iint_{(0;T) \times \Omega} \dot{u}_{u_0} \frac{\partial j_1}{\partial u}(t, x, u_{u_0}) + \int_{\Omega} \dot{u}_{u_0} \frac{\partial j_2}{\partial u}(x, u_{u_0}(T, \cdot)) = \int_{\Omega} p_{u_0}(0, x) h(x) dx \quad (1.2)$$

where p_{u_0} is the solution p_{u_0} of

$$\begin{cases} \partial_t p_{u_0} + \Delta p_{u_0} = -\frac{\partial j_1}{\partial u}(t, x, u_{u_0}) - \partial_u f(t, x, u_{u_0}) p_{u_0} & \text{in } (0; T) \times \Omega, \\ \mathcal{B} p_{u_0} = 0 & \text{on } (0; T) \times \partial\Omega, \\ p_{u_0}(T, \cdot) = \frac{\partial j_2}{\partial u}(x, u_{u_0}). \end{cases} \quad (1.3)$$

Similarly, the second order Gateaux-derivative of the functional J at u_0 in the direction (h, h) is given by

$$\ddot{J}(u_0)[h, h] = \iint_{(0;T) \times \Omega} \left(\dot{u}_{u_0}^2 p_{u_0} \frac{\partial^2 f}{\partial u^2}(t, x, u_{u_0}) + \dot{u}_{u_0}^2 \frac{\partial^2 j_1}{\partial u^2}(t, x, u_{u_0}) \right) + \int_{\Omega} \dot{u}_{u_0}^2(T, x) \frac{\partial^2 j_2}{\partial u^2}(x, u_{u_0}), \quad (1.4)$$

where \dot{u}_{u_0} is the unique solution of

$$\begin{cases} \partial_t \dot{u}_{u_0} - \Delta \dot{u}_{u_0} = \partial_u f(t, x, u_{u_0}) \dot{u}_{u_0} & \text{in } (0; T) \times \Omega, \\ \mathcal{B} \dot{u}_{u_0} = 0 & \text{on } (0; T) \times \partial\Omega, \\ \dot{u}_{u_0}(0, \cdot) = h & \text{in } \Omega. \end{cases} \quad (1.5)$$

The solution p_{u_0} of (1.3) is called the *adjoint* of (P); it encodes the first order optimality conditions for (P), as shown by the following result, which is a straightforward generalization of results from [16]:

Proposition 2. *Let u_0^* be a solution of (P). Then there exists $c \in \mathbb{R}$ such that*

$$\begin{cases} u_0^*(x) = \kappa_1 & \text{if } p_{u_0}(0, x) > c, \\ u_0^*(x) = -\kappa_0 & \text{if } p_{u_0}(0, x) < c, \\ \{p_{u_0}(0, \cdot) = c\} \subset \{-\kappa_0 < u_0^* < \kappa_1\}. \end{cases}$$

where p_{u_0} is defined by (1.3).

However, this is not a fully satisfactory characterisation of optimisers. Essentially, it can be proved that this optimality conditions entail the existence of a function \tilde{f} such that, for any optimiser u_0^* , we have, on $\{-\kappa_0 < u_0^* < \kappa_1\}$, an equation of the form $f'(0, x, u_{u_0^*}) = \tilde{f}(p_{u_0}(0, x), \partial_t p_{u_0}(0, x))$, see [16]. However when f is neither concave, nor convex, this equation can have multiple roots. As was observed in [16, 14] this is problematic when dealing with numerical approximations of the problem. For this reason it is necessary to exploit second order optimality conditions. Of course, the main difficulty with the expression of \ddot{J} given in Lemma 1 is that the expression is distributed over $(0; T) \times \Omega$ while we would need a localised information, at $t = 0$. This is the purpose of Theorem I.

1.3 Main results about second order optimality conditions

Regularity assumptions As we hinted at earlier, obtaining our result requires more regularity on f, j_1, j_2 . We assume that the nonlinearity f and the cost functions j_1, j_2 satisfy

$$\begin{cases} \text{for any compact } K \subset \mathbb{R}, f, j_1, \partial_u f, \partial_u j_1, \partial_{uu} f, \partial_{tu} f, \nabla \partial_u f, \Delta \partial_u f, \partial_{uuu} f, j_2, \partial_u j_2, \partial_{uu}^2 j_2 \\ \text{are bounded over } [0, T] \times \Omega \times K, \\ \frac{\partial^2 f}{\partial u^2}, \partial_{uu} j_1 \in \mathcal{C}^0([0; T]; L_{\text{loc}}^1(\Omega \times \mathbb{R})), \\ \exists M > 0, \forall u \geq M, \forall (t, x) \in (0; T) \times \Omega, f(t, x, u) \leq 0, f(t, x, -u) \geq 0. \end{cases} \quad (\mathbf{H}_{\text{reg}})$$

Typical examples of functions satisfying these regularity assumptions are functions f, j_1, j_2 taking the form $h(t, x)g(u)$ with $h \in L^\infty(\Omega)$ and $g \in \mathcal{C}^3(\mathbb{R})$. Natural examples in the field of population dynamics would be $f = f(u) = u(u - \theta)(1 - u)$ (bistable nonlinearity), $j_1 = 0$ and $j_2 = j_2(x, u) = \mathbb{1}_\omega(x)u$, which corresponds to optimising a proportion of sane mosquitoes within a global population [3, 14].

Our main result Our main result is the following:

Theorem I. *Let u_0^* be a maximiser of J over \mathcal{A} . Let $\omega := \{-\kappa_0 < u_0^* < \kappa_1\}$ and assume that $\text{Vol}(\omega) > 0$. Then*

$$W_{u_0^*}(0, \cdot) \leq 0 \text{ a.e. in } \omega$$

where for any $u_0 \in \mathcal{A}$:

$$W_{u_0}(t, x) := p_{u_0} \frac{\partial^2 f}{\partial u^2}(t, x, u_{u_0}) + \frac{\partial^2 j_1}{\partial u^2}(t, x, u_{u_0}).$$

Comments on Theorem I This result is a non-trivial generalisation of [14], where it was obtained on the interior of the set $\{-\kappa_0 < u_0^* < \kappa_1\}$, in the one-dimensional case for $j_1 \equiv 0$ and $j_2 \equiv u$, and was then used to considerably speed-up the running time of numerical simulations. However, as is often the case in optimal control theory, there is no guarantee that $\{-\kappa_0 < u_0^* < \kappa_1\}$ actually has a non-empty interior: it might a priori be a very irregular set such as Cantor set for example. In this paper, we prove that $W_{u_0^*}(0, \cdot) \leq 0$ holds almost everywhere without such strong topological assumption on the abnormal set $\omega := \{-\kappa_0 < u_0^* < \kappa_1\}$.

The following corollary immediately follows from Theorem I:

Corollary 3. *If $j_1(0, x, \cdot)$ and $f(0, x, \cdot)$ are convex in u for all $x \in \Omega$, with either $\frac{\partial^2 f}{\partial u^2}(0, x, u) > 0$ or $\frac{\partial^2 j_1}{\partial u^2}(0, x, u) > 0$ for all $(x, u) \in \Omega \times \mathbb{R}$, then any maximiser u_0^* of J over \mathcal{A} is bang-bang, in the sense that $u_0^*(x) \in \{-\kappa_0, \kappa_1\}$ for almost every $x \in \Omega$.*

Another interesting consequence of Theorem I is that the changes of concavity in f over the course of time does not matter in the following sense: assume $\partial_u j_1, \partial_u j_2 \geq 0$ (and non-identically zero) so that by the strong maximum principle applied to (1.3) we obtain $p_{u_0} > 0$. Assume that $f = f(t, x, u)$ is chosen so that $f(0, x, \cdot)$ is convex in u and such that there exists $\varepsilon > 0$ satisfying that for any $t \geq \varepsilon$ $f(t, x, \cdot)$ is concave. Then any solution u_0^* of the optimisation problem is bang-bang, in the sense that $\{-\kappa_0 < u_0^* < 1\}$ has zero Lebesgue measure. This property is not at all obvious from the distributed expression of \dot{J} .

Finally, observe that Theorem I gives an explicit description of the competition between the concavity/convexity of the non-linearity of the equation, and that of the cost function j_1 .

1.4 Bibliographical references for (P)

We investigated the optimisation problem (P) in an earlier paper with Toledo [14] the case where $d = 1$, $\Omega = (0, 1)$, f only depends on u and is convex, with $\kappa_0 = 0$, $\kappa_1 = 1$, $f(0) = f(1) = 0$, $j_1 \equiv 0$ and $j_2(x, u) \equiv u$. We have proved that in that case $u_0^* \equiv 1_{(0, V_0)}$ is a global maximiser of J . Apart from this example, it is not true in general that the maximisers are bang-bang. Indeed, if f is concave in u , then it was proved in [16] that the constant function $u_0^* \equiv V_0 \in (0, 1)$ is a global maximiser. Similar results were derived when $j_2(x, u) \equiv -(1 - u)^2$ in [7]. This emphasises the need for understanding the behaviour of the maximiser u_0^* on the abnormal set $\{-\kappa_0 < u_0^* < \kappa_1\}$. We have proved with Toledo in the simple framework of [14] that for any interior point x of $\{-\kappa_0 < u_0^* < \kappa_1\}$, one has $f''(u_0^*(x)) \leq 0$. The question of regularity of optimal controls is a difficult one, and we can not rule out that the interior of the abnormal set $\{-\kappa_0 < u_0^* < \kappa_1\}$ is empty. The aim of the present paper is to generalise this result to a more general framework, and to derive a result that holds almost everywhere on $\{-\kappa_0 < u_0^* < \kappa_1\}$, and not only on its interior. One of the reasons such information are important is the numerical approximation of these $L^\infty - L^1$ -constrained optimal control problems, a standard and powerful algorithm is the *thresholding scheme*, akin to a gradient ascent method. Roughly speaking, it is expected that optimisers u_0^* can be described using the level-sets of the so-called "adjoint state". When optimisers u_0^* are bang-bang, it is expected that this scheme can be defined and used with the knowledge of first order optimality conditions only. That an optimiser u_0^* is not bang-bang essentially amounts to saying that the adjoint state has a level-set of positive measure, which leads to using second-order optimality conditions in the definition of this scheme. Thus, having tractable information about the behaviour of optimisers u_0^* in the set $\{-\kappa_0 < u_0^* < \kappa_1\}$ is essential in implementing a cost-efficient algorithm. Finally, let us mention the recent [10], in which the same problem is discussed from another qualitative point of view: the authors study the influence of adding advection terms to the main equation on the value of the functional to optimise.

In order to further characterise u_0^* on the abnormal set $\{-\kappa_0 < u_0^* < \kappa_1\}$, one needs to extract information from the first and second order optimality conditions. Let us now explain why we

could not use earlier results on optimal control for parabolic equations and what our contribution to this field of research is. There is a vast literature on this topic, and we will only focus here on earlier works that are close to the problem we consider here, that is, second order optimality conditions for a control on the initial datum.

First order optimality conditions, in other words Pontryagin maximum principle, for semi-linear parabolic equations have been established in a very general framework in [19]. In this paper, three types of controls are considered: one acts on the initial datum, as in the present article, one acts as a source term in $(0, T) \times \Omega$, and one acts on the boundary $(0, T) \times \partial\Omega$. Another difference with the present paper is that L^1 constraints are not covered by their framework. Here, we consider a much simpler problem, since our control only acts on the initial datum. The reason for this is that we want to isolate the phenomenon we exhibit.

Sufficient second order conditions guaranteeing local optimality are discussed when the control acts on $(0, T) \times \Omega$ and/or on the boundary $(0, T) \times \partial\Omega$ (see [5, 18, 20]). Let us also mention a wide literature on second order conditions for optimal control of semi-linear elliptic equations (see for example [4]). The general approach of these papers is to derive the necessary second order optimality condition $\ddot{J}[u_0] \leq 0$, and to provide sufficient conditions in order to characterise a local maximiser. A Hamiltonian $H = H(t, x, u, p)$ is often derived from the first order conditions (see [19]) such that, if u_0^* is a maximising initial datum for example, then $H(0, x, u_0^*(x), p(0, x)) = \max_{u_0 \in \mathcal{A}} H(0, x, u_0(x), p(0, x))$ where p is the adjoint state of the equation. The second order necessary optimality conditions then read $\partial_{uu}H(0, x, u_0^*(x), p(0, x)) \leq 0$ for all $x \in \Omega$ on the abnormal set, that is, for all x such that $\partial_u H(0, x, u_0^*(x), p(0, x)) = 0$. A way to include L^1 constraints on the initial datum in order to compare this earlier framework to the present one is to consider a penalised cost function

$$G(u_0) := J(u_0) - c \int_{\Omega} u_0$$

over the wider class of admissible controls

$$\mathcal{B} := \{u_0 \in L^\infty(\Omega) : -\kappa_0 \leq u_0 \leq \kappa_1 \text{ almost everywhere}\}.$$

In that case, deriving the Hamiltonian F from [19], one gets $\partial_{uu}H \equiv 0$. Hence, it is hopeless to extract any information from second order optimality conditions using these earlier approaches in that case. More generally, we believe that these earlier works are not well-fitted to L^1 constraints. In the present, we push further the second order optimality conditions using a Laplace-type method that allows to concentrate the relevant information at $t = 0$. We do not investigate sufficient conditions and leave it for a future work.

1.5 A Laplace-type method

To prove Theorem I, we rely on a new technique, which we dub a Laplace-type method. This is a combination of the technique we developed with Toledo in [14], which relied on Laplace-type arguments for a simple perturbation of the initial datum which is only well-fitted on interior points of the abnormal set $\{x, -\kappa_0 < u_0^*(x) < \kappa_1\}$, and of the technique developed with Privat in [15] in another framework, in order to construct well-fitted perturbations regardless of any regularity assumption on the abnormal set.

Statement of the result For any function $h \in L^2(\Omega)$ we denote its support by $\text{supp}(h)$; it is a closed set. We adopt the same notation for the support of a distribution.

We consider the sequence of eigenvalues $\{\lambda_{k,\mathcal{B}}\}_{k \in \mathbb{N}}$, associated with eigenfunctions $\{\varphi_{k,\mathcal{B}}\}_{k \in \mathbb{N}}$ of the Laplace operator:

$$\begin{cases} -\Delta \varphi_{k,\mathcal{B}} = \lambda_{k,\mathcal{B}} \varphi_{k,\mathcal{B}} & \text{in } \Omega, \\ \mathcal{B} \varphi_{k,\mathcal{B}} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} \varphi_{k,\mathcal{B}}^2 = 1. \end{cases} \quad (1.6)$$

It satisfies

$$0 \leq \lambda_{1,\mathcal{B}} \leq \lambda_{2,\mathcal{B}} \leq \dots \leq \lambda_{k,\mathcal{B}} \xrightarrow{k \rightarrow \infty} \infty.$$

We are now in position to state our main technical result.

Theorem II. *Let $q \in L^\infty((0;T) \times \Omega)$ be a fixed potential and let $\omega \subset \Omega$ be a closed subset of Ω with positive measure. We assume that for any $r \in [1; +\infty)$ there holds*

$$\partial_t q - \Delta q, \nabla q \in L^r(0, T; L^r(\Omega)). \quad (\mathbf{H}_q)$$

Additionally, if \mathcal{B} is of Neumann type, we assume that $\mathcal{B}q = 0$. We consider a sequence $(h_K)_{K \in \mathbb{N}} \in L^2(\Omega)^\mathbb{N}$ such that, for any $K \in \mathbb{N}$, h_K writes

$$h_K := \sum_{k=K}^{\infty} a_{K,k} \varphi_{k,\mathcal{B}},$$

where the sequence $(a_{K,k})_{k \in \mathbb{N}}$ satisfies

$$\sum_{k=K}^{\infty} a_{K,k}^2 = 1,$$

and such that

$$\text{supp}(h_K) \subset \omega.$$

Define v_K as the solution of the heat equation

$$\begin{cases} \frac{\partial v_K}{\partial t} - \Delta v_K = q v_K & \text{in } (0; T) \times \Omega, \\ \mathcal{B} v_K = 0 & \text{on } (0; T) \times \partial\Omega, \\ v_K(0, \cdot) = h_K & \text{in } \Omega. \end{cases}$$

Consider the unit ball X of the dual of the space $\mathcal{C}^0([0; T]; L^1(\Omega))$. Finally, define the sequence of probability measures $(\nu_K)_{K \in \mathbb{N}} \in X^\mathbb{N}$ by

$$\forall K \in \mathbb{N}, \nu_K := \frac{v_K^2}{\iint_{(0;T) \times \Omega} v_K^2}.$$

Then any closure point ν_∞ of the sequence $(\nu_K)_{K \in \mathbb{N}} \in X^\mathbb{N}$ satisfies

$$\text{supp}(\nu_\infty) \subset \{t = 0\} \times \omega, \iint_{(0;T) \times \Omega} \nu_\infty = 1, \nu_\infty \geq 0 \text{ in the sense of measures.}$$

In this last equality, $\text{supp}(\nu_\infty) \subset \{t = 0\} \times \omega$ should be understood as follows: for any $\varphi \in \mathcal{C}^0([0; T]; L^1(\Omega))$ such that $\text{supp}(\varphi) \subset (\{t = 0\} \times \omega)^c$, $\langle \nu_\infty, \varphi \rangle = 0$ where $\langle \cdot, \cdot \rangle$ stands for the duality bracket on $\mathcal{C}^0([0; T]; L^1(\Omega))$.

Regarding our terminology In this paragraph we justify the terminology of the title of this paper. First of all, we claim that Theorem II is an extension of the standard two-scales expansion technique for parabolic equations. Namely, consider, as done in [14], the solution w_K of the equation

$$\begin{cases} \partial_t w_K - \partial_{xx} w_K = q w_K & \text{in } (0; T) \times \mathbb{T}, \\ w_K(0, x) = \theta(x) \sin(Kx) & \text{in } \mathbb{T}, \end{cases} \quad (1.7)$$

where \mathbb{T} is the one-dimensional torus and θ is a smooth bump function in \mathbb{T} . In [14] it is proved that

$$w_K \underset{K \rightarrow \infty}{\sim} \theta(x) \sin(Kx) e^{-K^2 t}$$

in the $L^2((0; T) \times \mathbb{T})$ sense. Consequently, using the fact that

$$\sin(Kx)^2 \underset{K \rightarrow \infty}{\rightharpoonup} \frac{1}{2}$$

and the Laplace method, this implies that, for any smooth test function ϕ ,

$$\iint_{(0; T) \times \Omega} w_K^2 \phi \underset{K \rightarrow \infty}{\sim} \frac{1}{2K^2} \int_{\mathbb{T}} \theta(x)^2 \phi(0, x)^2 dx.$$

In other words, in the limit $K \rightarrow \infty$, we only see (up to a proper rescaling) the support of the initial condition. In higher dimensional situations, Theorem II establishes the same kind of qualitative behaviours, but we highlight here several non-trivial difficulties. First and foremost, it is not true that $\varphi_{K, \mathcal{B}}^2 \underset{K \rightarrow \infty}{\rightharpoonup} \frac{1}{2}$, since in many domains we may have a so-called localisation phenomenon [9]. Second, this type of expansion only holds under strong regularity assumptions on the function θ . In particular, this result assumes, in a sense, that we are considering highly oscillating initial conditions, with a regular support (for instance, that has non-empty interior). When considering applications to the optimal control of reaction-diffusion equations, it is extremely difficult (and, in general, a completely open question) to obtain this type of regularity.

Regarding the first difficulty, as a byproduct of the proof of Lemma 6 below, we obtain that

$$v_K \underset{K \rightarrow \infty}{\sim} \sum_{k=K}^{\infty} a_{K,k} \varphi_{k, \mathcal{B}} e^{-t \lambda_{k, \mathcal{B}}}.$$

This is not yet enough to conclude as to the support in space of the limit ν_{∞} as this would only yield, for any smooth function ϕ ,

$$\langle v_K^2, \phi \rangle \underset{K \rightarrow \infty}{\sim} \sum_{k, k'=K}^{\infty} \frac{a_{K,k} a_{K,k'}}{\lambda_{k, \mathcal{B}} + \lambda_{k', \mathcal{B}}} \int_{\Omega} \phi(0, \cdot) \varphi_{k, \mathcal{B}} \varphi_{k', \mathcal{B}},$$

and it is then unclear from this expression to derive a meaningful information about the support of ν_{∞} .

2 Proof of Theorem II

We begin with the proof of Theorem II, as Theorem I is a corollary of it.

2.1 Steps of the proof

The proof is divided up in several steps. As each can be technical and sometimes long, we summarise them here:

- First we give some basic preliminary results related to parabolic regularity and the Laplace method. We refer to Propositions 4 and 5.
- Second, we prove an estimate of the L^2 -norm of v_K under the form

$$\iint_{(0;T) \times \Omega} v_K^2 \geq c_0 \sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}.$$

We refer to Lemma 6 below.

- Third, we prove that $\text{supp}(\nu_\infty) \subset \{t = 0\} \times \Omega$, see Lemma 7.
- Finally, we prove that $\text{supp}(\nu_\infty) \subset (0;T) \times \Omega$, thereby concluding the proof.

2.2 Step 1: Preliminaries on the parabolic regularity and the Laplace method

A preliminary parabolic regularity result We recall the following parabolic regularity result:

Proposition 4. *Let $q, g \in L^\infty((0;T) \times \Omega)$. For any $\theta_0 \in L^\infty(\Omega)$, the solution θ of*

$$\begin{cases} \partial_t \theta - \Delta \theta - q\theta = g & \text{in } (0;T) \times \Omega, \\ \mathcal{B}\theta = 0 & \text{on } (0;T) \times \partial\Omega, \\ \theta(0, \cdot) = \theta_0 \end{cases} \quad (2.1)$$

satisfies

$$\sup_{t \in [0;T]} \|\theta\|_{L^2(\Omega)} \leq C \left(\int_0^T \|g(t, \cdot)\|_{L^2(\Omega)} dt + \|\theta_0\|_{L^2(\Omega)} \right)$$

where the constant C only depends on $\|q\|_{L^\infty((0;T) \times \Omega)}$ and T .

As this result is instrumental in deriving our estimates we prove it here.

Proof of Proposition 4. Multiplying (2.1) by θ and integrating by parts in time we obtain

$$\frac{d}{dt} \int_{\Omega} \frac{\theta(t, \cdot)^2}{2} + \int_{\Omega} |\nabla \theta|^2 - \|q\|_{L^\infty((0;T) \times \Omega)} \int_{\Omega} \theta^2 \leq \int_{\Omega} g\theta \leq \|g(t, \cdot)\|_{L^2(\Omega)} \|\theta(t, \cdot)\|_{L^2(\Omega)}$$

whence

$$\frac{d}{dt} (\|\theta(t, \cdot)\|_{L^2(\Omega)}) - \|q\|_{L^\infty((0;T) \times \Omega)} \|\theta(t, \cdot)\|_{L^2(\Omega)} \leq \|g(t, \cdot)\|_{L^2(\Omega)}.$$

It suffices to apply the Gronwall lemma to conclude. □

Background on the Laplace method We recall here the following result:

Proposition 5. For any $m \in \mathbb{N}$,

$$\int_0^T t^m e^{-kt} dt \sim_{k \rightarrow \infty} \frac{C_m}{k^{m+1}}.$$

Proof of Proposition 5. Integrating by parts $(m+1)$ -times we have

$$\int_0^T t^m e^{-kt} = \frac{m!}{k^{m+1}} (1 - e^{-kT})$$

whence the conclusion. □

2.3 Step 2: Asymptotic of the L^2 norm of the solution

The goal of this paragraph is to prove the following result:

Lemma 6. There exists a constant $c_0 > 0$ such that

$$\forall K \in \mathbb{N}, \iint_{(0;T) \times \Omega} v_K^2 \geq c_0 \sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}.$$

Proof of Lemma 6. Let us introduce, for any $k \in \mathbb{N}$, the function

$$w_{0,K}(t, x) := \sum_{k=K}^{\infty} a_{K,k} \varphi_{k,\mathcal{B}} e^{-\lambda_{k,\mathcal{B}} t}.$$

It is expected that there should hold

$$v_K \approx w_{0,K}$$

in a certain sense. In order to establish this approximation we first compute:

$$\iint_{(0;T) \times \Omega} w_{0,K}^2 = \sum_{k=K}^{\infty} \frac{a_{K,k}^2}{2\lambda_{k,\mathcal{B}}} (1 - e^{-2T\lambda_{k,\mathcal{B}}}) \geq \frac{1}{4} \sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}} \quad (2.2)$$

whenever $1 - e^{-2T\lambda_{k,\mathcal{B}}} \geq \frac{1}{2}$. To control the distance between $w_{0,K}$ and v_K , consider the remainder term

$$T_{0,K} := \bar{v}_K - w_{0,K}.$$

It is clear that $T_{0,K}$ satisfies

$$\partial_t T_{0,K} - \Delta T_{0,K} - qT_{0,K} = qw_{0,K}.$$

But this is not yet enough. Indeed, if we were to apply Proposition 4 directly, we would need to estimate $\int_0^T \|qw_{0,K}\|_{L^2(\Omega)}$ but we could *a priori* only bound it as

$$\int_0^T \|qw_{0,K}\|_{L^2(\Omega)} \leq \|q\|_{L^\infty((0;T) \times \Omega)} \left(\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}} \right)^{\frac{1}{2}},$$

that is, by a term of order $\|w_{0,K}\|_{L^2((0;T) \times \Omega)}$, which is not strong enough. We thus have to take more care when handling this. For this reason, introduce the function

$$z_{0,K} : (0;T) \times \Omega \ni (t, x) \mapsto tq(t, x)w_{0,K}(t, x)$$

and define

$$R_{0,K} := v_K - w_{0,K} - z_{0,K}.$$

As $z_{0,K}$ satisfies

$$\partial_t z_{0,K} - \Delta z_{0,K} = qw_{0,K} + (\partial_t q - \Delta q)(tw_{0,K}) - 2t\langle \nabla q, \nabla w_{0,k} \rangle$$

we obtain

$$\partial_t R_{0,K} - \Delta R_{0,K} - qR_{0,K} = \underbrace{qz_{0,K}}_{=:V_{0,K}} - \underbrace{(\partial_t q - \Delta q)(tw_{0,K})}_{=:V_{1,K}} + 2t \underbrace{\langle \nabla q, \nabla w_{0,k} \rangle}_{=:V_{2,K}}.$$

Moreover, notice that there holds

$$\mathcal{B}z_{0,K} = 0$$

which in turn implies

$$\mathcal{B}R_{0,K} = 0.$$

Finally, we have

$$R_{0,K}(0, \cdot) = 0 \text{ in } \Omega$$

by construction. From Proposition 4 there holds, for some constant $C > 0$ independent of K ,

$$\sup_{t \in [0;T]} \|R_{0,K}(t, \cdot)\|_{L^2(\Omega)} \leq C \left(\int_0^T \|V_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt + \int_0^T \|V_{1,K}(t, \cdot)\|_{L^2(\Omega)} dt + \int_0^T \|V_{2,K}(t, \cdot)\|_{L^2(\Omega)} dt \right).$$

Now observe that

$$\begin{aligned} \int_0^T \|V_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt &= \int_0^T t \|q(t, \cdot)w_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt \\ &\leq \|q\|_{L^\infty((0;T) \times \Omega)} \int_0^T t \|w_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt \\ &\leq \|q\|_{L^\infty((0;T) \times \Omega)} \left(\iint_{(0;T) \times \Omega} t^2 w_{0,K}^2 \right)^{\frac{1}{2}} \\ &\leq \|q\|_{L^\infty} \sqrt{\sum_{k=K}^{\infty} a_{K,k}^2 \int_0^T t^2 e^{-2t\lambda_{k,\mathcal{B}}} dt} \\ &\leq \frac{M}{\lambda_{K,\mathcal{B}}} \sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}}. \end{aligned}$$

In the last step, we applied Proposition 5 with $m = 3$. We have thus proved

$$\int_0^T \|V_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt \leq \frac{C}{\lambda_{K,\mathcal{B}}} \sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}}. \quad (2.3)$$

Similarly, we can estimate $\int_0^T \|V_{1,K}(t, \cdot)\|_{L^2(\Omega)} dt$. Define $Q := \partial_t q - \Delta q$. Then there holds

$$\begin{aligned} \int_0^T \|V_{1,K}(t, \cdot)\|_{L^2(\Omega)} dt &= \int_0^T \|Q(t, \cdot) t w_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt \\ &\leq C \int_0^T t \|Q(t, \cdot)\|_{L^{r_0}(\Omega)} \|w_{0,K}(t, \cdot)\|_{L^{p_0}(\Omega)} \\ &\quad \text{from the Hölder inequality with } 1/r_0 + 1/p_0 = 1/2 \\ &\leq C \sqrt{\int_0^T \|Q(t, \cdot)\|_{L^{r_0}(\Omega)}^2} \sqrt{\int_0^T t^2 \|w_{0,K}\|_{L^{p_0}(\Omega)}^2} \\ &\quad \text{from the Cauchy-Schwarz inequality.} \end{aligned}$$

We now choose $p_0 > 2$ such that

$$W^{1,2}(\Omega) \hookrightarrow L^{p_0}(\Omega)$$

and fix the corresponding exponent r_0 . Then, up to a constant $C > 0$ we have

$$\begin{aligned} \int_0^T \|V_{1,K}(t, \cdot)\|_{L^2(\Omega)} dt &\leq C \sqrt{\int_0^T \|Q(t, \cdot)\|_{L^{r_0}(\Omega)}^2} \sqrt{\int_0^T t^2 \|w_{0,K}\|_{L^{p_0}(\Omega)}^2} \\ &\leq C \sqrt{\int_0^T \|Q(t, \cdot)\|_{L^{r_0}(\Omega)}^2} \sqrt{\int_0^T t^2 \|\nabla w_{0,K}\|_{L^2(\Omega)}^2}. \end{aligned}$$

Now observe that by the Jensen inequality we have, up to a constant still denoted C for notational convenience

$$\int_0^T \|Q(t, \cdot)\|_{L^{r_0}(\Omega)}^2 dt \leq C \left(\iint_{(0;T) \times \Omega} |Q|^{2r_0} \right)^{\frac{1}{r_0}} = C \|Q\|_{L^{2r_0}(0,T;L^{2r_0})}^4 = C_{r_0} < \infty.$$

Here we used Assumption (\mathbf{H}_q) . All in all, up to a multiplicative constant (once again denoted C) we have obtained

$$\begin{aligned} \int_0^T \|V_{1,K}(t, \cdot)\|_{L^2(\Omega)} dt &\leq C \sqrt{\int_0^T t^2 \|\nabla w_{0,K}\|_{L^2(\Omega)}^2} \\ &= C \sqrt{\sum_{k=K}^{\infty} \lambda_{k,\mathcal{B}} a_{K,k}^2 \int_0^T t^2 e^{-2t\lambda_{k,\mathcal{B}}} dt} \\ &\leq \frac{C}{\sqrt{\lambda_{K,\mathcal{B}}}} \sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \end{aligned}$$

We have thus obtained

$$\int_0^T \|V_{1,K}(t, \cdot)\|_{L^2(\Omega)} dt \leq \frac{C}{\sqrt{\lambda_{K,\mathcal{B}}}} \sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}}. \quad (2.4)$$

Let us finally estimate $V_{2,K}$. Define $Q_1 := \nabla q$. We need to estimate

$$\int_0^T t \|\langle Q_1, \nabla w_{0,k} \rangle\|_{L^2(\Omega)} dt.$$

Applying the same Hölder and Cauchy-Schwarz inequalities as above, we have

$$\begin{aligned} \int_0^T t \|\langle Q_1, \nabla w_{0,k} \rangle\|_{L^2(\Omega)} dt &\leq \int_0^T t \|Q_1(t, \cdot)\|_{L^{r_0}(\Omega)} \|\nabla w_{0,K}(t, \cdot)\|_{L^{p_0}(\Omega)} dt \\ &\leq C \sqrt{\int_0^T t^2 \|\nabla w_{0,K}(t, \cdot)\|_{L^{p_0}(\Omega)}^2} \end{aligned}$$

where $1/r_0 + 1/p_0 = 1/2$. We are thus left with estimate

$$\int_0^T t^2 \|\nabla w_{0,K}\|_{L^{p_0}(\Omega)}^2 dt$$

for some $p' > 2$. However, by the fractional Sobolev embedding [1, Theorem 7.57] (see also [6, Theorem 3.4, Lemma 4.11]) $H^{1+\gamma} \hookrightarrow W^{1,p_0}(\Omega)$, for some $\gamma \in]0; 1[$ and $p_0 > 2$, we have

$$\|\nabla w_{0,k}(t, \cdot)\|_{L^{p_0}(\Omega)}^2 \leq \|w_{0,k}(t, \cdot)\|_{H^{1+\gamma}(\Omega)}^2 = \sum_{k=K}^{\infty} a_{K,k}^2 \lambda_{k,\mathcal{B}}^{1+\gamma} e^{-t\lambda_{k,\mathcal{B}}}$$

so that the last term can be estimated as

$$\int_0^T t^2 \sum_{k=K}^{\infty} a_{K,k}^2 \lambda_{k,\mathcal{B}}^{1+\gamma} e^{-t\lambda_{k,\mathcal{B}}} = \sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}^{2-\gamma}}$$

whence we obtain

$$\int_0^T \|V_{2,K}(t, \cdot)\|_{L^2(\Omega)} dt \leq \frac{C}{\lambda_{K,\mathcal{B}}^{\frac{1-\gamma}{2}}} \sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \quad (2.5)$$

Summing estimates (2.3)-(2.4)-(2.5) we get that for some constant C and some $\beta > 0$ there holds

$$\sup_{t \in [0; T]} \|R_{0,K}(t, \cdot)\|_{L^2(\Omega)} \leq \frac{C}{\lambda_{K,\mathcal{B}}^{\beta}} \sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}}. \quad (2.6)$$

We now turn back to the function v_K . Developing the square root we obtain

$$\begin{aligned} \iint_{(0;T) \times \Omega} v_K^2 &= \iint_{(0;T) \times \Omega} R_{0,K}^2 + \iint_{(0;T) \times \Omega} (R_{0,K} - v_K)^2 \\ &= \iint_{(0;T) \times \Omega} R_{0,K}^2 + \iint_{(0;T) \times \Omega} (w_{0,K} + z_{0,K})^2 \\ &= \underset{K \rightarrow \infty}{o} \left(\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}} \right) \quad \text{from (2.6)} \\ &+ \iint_{(0;T) \times \Omega} w_{0,K}^2 + 2 \iint_{(0;T) \times \Omega} w_{0,K} z_{0,K} + \iint_{(0;T) \times \Omega} z_{0,K}^2. \end{aligned}$$

Observe that from (2.2) we have

$$\iint_{(0;T) \times \Omega} w_{0,K}^2 \underset{K \rightarrow \infty}{\sim} \sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{K,k}}.$$

Remembering that $z_{0,K} = tq(t, \cdot)w_{0,K}$ we have

$$\begin{aligned}
\iint_{(0;T) \times \Omega} z_{0,K}^2 &\leq \|q\|_{L^\infty((0;T) \times \Omega)} \iint_{(0;T) \times \Omega} t^2 w_{0,K}^2 \\
&= \|q\|_{L^\infty((0;T) \times \Omega)} \sum_{k=K}^\infty \int_0^T t^2 a_{K,k}^2 e^{-2t\lambda_{K,k}} dt \\
&= 2\|q\|_{L^\infty((0;T) \times \Omega)} \sum_{k=K}^\infty \frac{a_{K,k}^2}{\lambda_{K,k}^3} \\
&= \underset{K \rightarrow \infty}{o} \left(\|w_{0,K}\|_{L^2((0;T) \times \Omega)}^2 \right).
\end{aligned} \tag{2.7}$$

As

$$\iint_{(0;T) \times \Omega} w_{0,K} z_{0,K} \leq \|w_{0,K}\|_{L^2((0;T) \times \Omega)} \|z_{0,K}\|_{L^2((0;T) \times \Omega)}$$

we can conclude that

$$\begin{aligned}
\iint_{(0;T) \times \Omega} v_K^2 &= \underset{K \rightarrow \infty}{o} \left(\sum_{k=K}^\infty \frac{a_{K,k}^2}{\lambda_{k,B}} \right) \\
&\quad + \iint_{(0;T) \times \Omega} w_{0,K}^2 + 2 \iint_{(0;T) \times \Omega} w_{0,K} z_{0,K} + \iint_{(0;T) \times \Omega} z_{0,K}^2 \\
&\underset{K \rightarrow \infty}{\sim} \iint_{(0;T) \times \Omega} w_{0,K}^2 \underset{K \rightarrow \infty}{\sim} \sum_{k=K}^\infty \frac{a_{K,k}^2}{\lambda_{K,k}}.
\end{aligned} \tag{2.8}$$

The proof is finished. \square

2.4 Step 3: Controlling the support in time

The goal of this paragraph is the following lemma:

Lemma 7. *For any closure point ν_∞ of the sequence $(\nu_K)_{K \in \mathbb{N}}$ (defined in the statement of Theorem II) there holds*

$$\text{supp}(\nu_\infty) \subset \{t = 0\} \times \Omega. \tag{2.9}$$

As we shall see, this is an almost straightforward consequence of the computations carried out in the proof of Lemma 6.

Proof of Lemma 7. From Lemma 6 we know that for some constant $c_0 > 0$ we have

$$\iint_{(0;T) \times \Omega} v_K^2 \geq c_0 \sum_{k=K}^\infty \frac{a_{K,k}^2}{\lambda_{k,B}}.$$

To prove (2.9) it suffices to prove that, for any $\varepsilon > 0$,

$$\iint_{(\varepsilon;T) \times \Omega} v_K^2 = \underset{K \rightarrow \infty}{o} \left(\sum_{k=K}^\infty \frac{a_{K,k}^2}{\lambda_{k,B}} \right).$$

Using the same notations as in the proof of Lemma 6 we have

$$\begin{aligned}
\iint_{(\varepsilon;T) \times \Omega} v_K^2 &= \iint_{(\varepsilon;T) \times \Omega} (v_K - w_{0,K} - z_{0,K})^2 && (=: I_{1,K}) \\
&\quad + 2 \iint_{(\varepsilon;T) \times \Omega} (v_{0,K} - w_{0,K} - z_{0,K})(w_{0,K} + z_{0,K}) && (=: I_{2,K}) \\
&\quad + \iint_{(\varepsilon;T) \times \Omega} (w_{0,K} + z_{0,K})^2 && (=: I_{3,K}).
\end{aligned}$$

As in the proof of Lemma 6 we have

$$I_{1,K}, I_{2,K} = \underset{K \rightarrow \infty}{o} \left(\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}} \right).$$

It remains to estimate $I_{3,K}$. However, up to a multiplicative constant C that also depends on $\|q\|_{L^\infty((0;T) \times \Omega)}$

$$\begin{aligned} I_{3,K} &= \iint_{(\varepsilon;T) \times \Omega} (w_{0,K} + z_{0,K})^2 \\ &\leq C \left(\iint_{(\varepsilon;T) \times \Omega} w_{0,K}^2 + \iint_{(\varepsilon;T) \times \Omega} z_{0,K}^2 \right) \\ &\leq C \left(\iint_{(\varepsilon;T) \times \Omega} w_{0,K}^2 + \iint_{(0;T) \times \Omega} z_{0,K}^2 \right) \\ &\leq C \left(\iint_{(\varepsilon;T) \times \Omega} w_{0,K}^2 + \underset{K \rightarrow \infty}{o} (\|w_{0,K}\|_{L^2((0;T) \times \Omega)}^2) \right) \quad \text{from (2.7)} \end{aligned}$$

Moreover, for a constant C

$$\begin{aligned} \iint_{(\varepsilon;T) \times \Omega} w_{0,K}^2 &= \sum_{k=K}^{\infty} a_k^2 \int_{\varepsilon}^T e^{-t\lambda_k} dt \\ &\leq C e^{-\varepsilon\lambda_{K,\mathcal{B}}} \sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}} \\ &= \underset{K \rightarrow \infty}{o} \left(\|w_{0,K}\|_{L^2((0;T) \times \Omega)}^2 \right) \end{aligned}$$

so that

$$I_{3,K} = \underset{K \rightarrow \infty}{o} \left(\|w_{0,K}\|_{L^2((0;T) \times \Omega)}^2 \right).$$

Summarising, we have obtained

$$\iint_{(\varepsilon;T) \times \Omega} v_K^2 = \underset{K \rightarrow \infty}{o} (\|w_{0,K}\|_{L^2((0;T) \times \Omega)}^2) = \underset{K \rightarrow \infty}{o} \left(\|v_K\|_{L^2((0;T) \times \Omega)}^2 \right).$$

Thus we obtain, for any test function $\phi \in \mathcal{C}^\infty((0;\varepsilon) \times \Omega)$, (the limit is taken along a subsequence)

$$\begin{aligned} \langle \nu_\infty, \phi \rangle &= \lim_{K \rightarrow \infty} \iint_{(0;T) \times \Omega} \nu_K \phi \\ &= \lim_{K \rightarrow \infty} \iint_{(\varepsilon;T) \times \Omega} \nu_K \phi \\ &\leq \| \phi \|_{L^\infty((0;T) \times \Omega)} \lim_{K \rightarrow \infty} \frac{\iint_{(\varepsilon;T) \times \Omega} v_K^2}{\iint_{(0;T) \times \Omega} v_K^2} \\ &= 0. \end{aligned}$$

The conclusion follows. □

2.5 Step 4: Controlling the support in space

The goal of this paragraph is the following result:

Lemma 8. *For any closure point ν_∞ of the sequence $(\nu_K)_{K \in \mathbb{N}}$ (defined in the statement of Theorem II) there holds*

$$\text{supp}(\nu_\infty) \subset (0; T) \times \omega. \quad (2.10)$$

Proof of Lemma 8. To prove (2.10) it suffices to prove the following: for any open set $F \subset \Omega$ such that $\text{dist}(\overline{F}, \omega) > 0$ (remember that ω is closed), for any $\phi \in \mathcal{C}_c^\infty((0; T) \times F)$, there holds

$$\langle \nu_\infty, \phi \rangle = 0.$$

Here ν_∞ is a closure point of the sequence $(\nu_K)_{K \in \mathbb{N}}$. Hence, fix an open set $F \subset \Omega$ such that $\text{dist}(\overline{F}, \omega) > 0$. We consider a smooth function $\theta \in \mathcal{C}_c^\infty(\Omega)$ such that

$$\theta h_K = h_K.$$

This amounts to requiring that $\text{supp}(h_K) \subset \{\theta = 1\}$. Furthermore, we require that

$$\theta \equiv 0 \text{ in } \overline{F}.$$

We now look for a two-scale like asymptotic expansion of the solution v_K in terms of θ . Introduce (with the notations of Lemma 6)

$$\eta_{0,K,\theta} := \theta(x) \sum_{k=K}^{\infty} a_{K,k} \varphi_{k,\mathcal{B}} e^{-t\lambda_{k,\mathcal{B}}} = \theta(x) w_{0,K}(t, x)$$

and

$$R_{0,K,\theta} := v_K - \eta_{0,K,\theta}.$$

The function $R_{0,K,\theta}$ satisfies

$$\partial_t R_{0,K,\theta} - \Delta R_{0,K,\theta} - q R_{0,K,\theta} = 2\langle \nabla \theta, \nabla w_{0,K} \rangle + (\Delta \theta) w_{0,K} + q \eta_{0,K,\theta}.$$

Define

$$G := \Delta \theta + q \theta.$$

The equation on $R_{0,K,\theta}$ rewrites

$$\partial_t R_{0,K,\theta} - \Delta R_{0,K,\theta} = 2\langle \nabla \theta, \nabla w_{0,K} \rangle + G w_{0,K}.$$

We can hence split $R_{0,K,\theta}$ as

$$R_{0,K,\theta} = r_{1,K,\theta} + 2r_{2,K,\theta}$$

where

$$\begin{cases} \partial_t r_{1,K,\theta} - \Delta r_{1,K,\theta} = G w_{0,K} & \text{in } (0; T) \times \Omega, \\ \partial_t r_{2,K,\theta} - \Delta r_{2,K,\theta} = \langle \nabla \theta, \nabla w_{0,K} \rangle & \text{in } (0; T) \times \Omega, \\ \mathcal{B} r_{j,K,\theta} = 0 & \text{on } (0; T) \times \partial \Omega, \quad (j = 1, 2), \\ r_{j,K,\theta}(0, \cdot) = 0 & \text{in } \Omega, \quad (j = 1, 2). \end{cases}$$

We estimate $r_{1,K,\theta}$ and $r_{2,K,\theta}$ separately.

Estimate on $r_{1,K,\theta}$ Introducing

$$z_{1,K,\theta} := tGw_{0,K}$$

we show, exactly as in the proof of Lemma 6, that

$$\sup_{t \in [0;T]} \|r_{1,K,\theta}(t, \cdot) - z_{1,K,\theta}(t, \cdot)\|_{L^2(\Omega)} = \underset{K \rightarrow \infty}{o} \left(\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}} \right).$$

Indeed, it suffices to observe that with the assumptions on q , and as $\theta \in \mathcal{C}_c^\infty(\Omega)$, G also satisfies Assumption **(H_q)**. Furthermore, for any $t \in [0;T]$,

$$\begin{aligned} \|z_{1,K,\theta}(t, \cdot)\|_{L^2((0;T) \times \Omega)} &\leq \|G\|_{L^\infty(\Omega)} \sqrt{\sum_{k=K}^{\infty} a_{K,k}^2 \int_0^T t^2 e^{-2t\lambda_{k,\mathcal{B}}} dt} \\ &= \underset{K \rightarrow \infty}{o} \left(\sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \right). \end{aligned}$$

Thus,

$$\|r_{1,K,\theta}\|_{L^2((0;T) \times \Omega)} = \underset{K \rightarrow \infty}{o} \left(\sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \right).$$

Estimate on $r_{2,K,\theta}$ Let us first reason heuristically. Formally, we should have

$$r_{2,K,\theta} \approx t \langle \nabla \theta, \nabla w_{0,K} \rangle = \left\langle \nabla \theta, t \sum_{k=K}^{\infty} a_{K,k} \nabla \varphi_k e^{-t\lambda_{k,\mathcal{B}}} \right\rangle =: \tilde{r}_{2,K,\theta}.$$

Let us first estimate $\tilde{r}_{2,K,\theta}$. We have, up to a multiplicative constant C ,

$$\begin{aligned} \int_0^T \|\tilde{r}_{2,K,\theta}\|_{L^2(\Omega)} dt &\leq C \left(\sum_{k=K}^{\infty} a_{K,k}^2 \lambda_{k,\mathcal{B}} \int_0^T t^2 e^{-2t\lambda_{k,\mathcal{B}}} dt \right)^{1/2} \\ &\leq C \left(\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}^2} \right)^{\frac{1}{2}} \\ &= \underset{K \rightarrow \infty}{o} \left(\sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \right). \end{aligned}$$

Consider now

$$r_K := r_{2,K,\theta} - \tilde{r}_{2,K,\theta}$$

The function r_K satisfies

$$\partial_t r_K - \Delta r_K - q r_K = \underbrace{q \tilde{r}_{2,K,\theta} + \langle \nabla \Delta \theta, t \nabla w_{0,K} \rangle}_{=: J_K} + \underbrace{2t (\nabla^2 \theta \odot \nabla^2 w_{0,K})}_{=: I_K}$$

where \odot denotes the Hadamard product of matrices. Adapting the computation that led to estimating $\tilde{r}_{2,K,\theta}$ we see that the solution $\beta_{1,K}$ of

$$\partial_t \beta_{1,K} - \Delta \beta_{1,K} - q \beta_{1,K} = J_K$$

satisfies

$$\|\beta_{1,K}\|_{L^2((0;T)\times\Omega)} = \underset{K\rightarrow\infty}{o} \left(\sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \right). \quad (2.11)$$

Thus the only term that should be estimated is the solution \tilde{r}_K of

$$\partial_t \tilde{r}_K - \Delta \tilde{r}_K - q \tilde{r}_K = t (\nabla^2 \theta \odot \nabla^2 w_{0,K})$$

We introduce two last auxiliary functions, namely,

$$\tilde{r}_{3,K,\theta} := \frac{t^2}{2} (\nabla^2 \theta \odot \nabla^2 w_{0,K}), \quad \tilde{T}_K := \tilde{r}_K - \tilde{r}_{3,K,\theta}.$$

On the one hand we have

$$\partial_t \tilde{T}_K - \Delta \tilde{T}_K - q \tilde{T}_K = q \tilde{r}_{3,K,\theta} + \frac{t^2}{2} \nabla^2 \Delta \theta \odot \nabla^2 w_{0,K} + t^2 \nabla \nabla^2 \theta \odot \nabla^2 \nabla w_{0,K}.$$

On the other hand, up to a multiplicative constant C ,

$$\begin{aligned} \int_0^T t^2 \|\nabla^2 \theta \odot \nabla^2 w_{0,K}\|_{L^2(\Omega)} dt &\leq C \int_0^T t^2 \|\theta\|_{\mathcal{C}^2(\Omega)} \|\nabla^2 w_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt \\ &\leq C \int_0^T t^2 \|\Delta w_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt \text{ by elliptic regularity} \\ &\leq C \left(\int_0^T t^4 \sum_{k=K}^{\infty} a_{K,k}^2 \lambda_{k,\mathcal{B}}^2 e^{-2t\lambda_{k,\mathcal{B}}} \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}^{\frac{3}{2}}} \right)^{\frac{1}{2}} \\ &= \underset{K\rightarrow\infty}{o} \left(\sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \right). \end{aligned}$$

Finally, up to a multiplicative constant, we have

$$\begin{aligned} \int_0^T t^2 \|\nabla \nabla^2 \theta \odot \nabla^2 \nabla w_{0,K}(t, \cdot)\|_{L^2(\Omega)} dt &\leq C \int_0^T t^2 \|\Delta \nabla w_{0,K}(t, \cdot)\|_{L^2(\Omega)}^2 dt \text{ by elliptic regularity} \\ &\leq C \left(\int_0^T \sum_{k=K}^{\infty} a_{K,k}^2 t^4 \lambda_{k,\mathcal{B}}^3 e^{-2t\lambda_{k,\mathcal{B}}} dt \right)^{1/2} \\ &\leq C \left(\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}^2} \right)^{\frac{1}{2}} \\ &= \underset{K\rightarrow\infty}{o} \left(\sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \right). \end{aligned}$$

We can hence conclude that

$$\|r_{2,K,\theta}\|_{L^2((0;T)\times\Omega)} = \underset{K\rightarrow\infty}{o} \left(\sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \right) \quad (2.12)$$

and, thus, that

$$\|v_K - \eta_{0,K,\theta}\|_{L^2((0;T)\times\Omega)} = \underset{K\rightarrow\infty}{\overset{o}{K}} \left(\sqrt{\sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}} \right). \quad (2.13)$$

Recall now from Lemma 6 that

$$\iint_{(0;T)\times\Omega} v_K^2 \geq c_0 \sum_{k=K}^{\infty} \frac{a_{K,k}^2}{\lambda_{k,\mathcal{B}}}.$$

Now let us turn back to the set F , and take any $\phi \in \mathcal{C}_c^\infty((0;T) \times F)$. Fix a closure point ν_∞ of the sequence $(\nu_K)_{K \in \mathbb{N}}$. Then

$$\begin{aligned} \langle \nu_\infty, \phi \rangle &= \lim_{K \rightarrow \infty} \langle \nu_K, \phi \rangle \\ &= \lim_{K \rightarrow \infty} \frac{\iint_{(0;T)\times\Omega} v_K^2 \phi}{\iint_{(0;T)\times\Omega} v_K^2} \\ &= \lim_{K \rightarrow \infty} \frac{\iint_{(0;T)\times\Omega} \eta_{0,K,\theta}^2 \phi + \iint_{(0;T)\times\Omega} (v_K - \eta_{0,K,\theta})^2 \phi + 2 \iint_{(0;T)\times\Omega} \eta_{0,K,\theta} (v_K - \eta_{0,K,\theta}) \phi}{\iint_{(0;T)\times\Omega} v_K^2}. \end{aligned}$$

As $\eta_{0,K,\theta} = \theta w_{0,K} \equiv 0$ on F by the definition of θ , and as ϕ is supported in F ,

$$\iint_{(0;T)\times\Omega} \eta_{0,K,\theta}^2 \phi = 0.$$

Thus

$$\begin{aligned} \langle \nu_\infty, \phi \rangle &= \lim_{K \rightarrow \infty} \frac{\iint_{(0;T)\times\Omega} \eta_{0,K,\theta}^2 \phi + \iint_{(0;T)\times\Omega} (v_K - \eta_{0,K,\theta})^2 \phi + 2 \iint_{(0;T)\times\Omega} \eta_{0,K,\theta} (v_K - \eta_{0,K,\theta}) \phi}{\iint_{(0;T)\times\Omega} v_K^2} \\ &= \lim_{K \rightarrow \infty} \frac{\iint_{(0;T)\times\Omega} (v_K - \eta_{0,K,\theta})^2 \phi + 2 \iint_{(0;T)\times\Omega} \eta_{0,K,\theta} (v_K - \eta_{0,K,\theta}) \phi}{\iint_{(0;T)\times\Omega} v_K^2} \\ &\leq \|\phi\|_{L^\infty((0;T)\times\Omega)} \lim_{K \rightarrow \infty} \frac{\|v_K - \eta_{0,K,\theta}\|_{L^2((0;T)\times\Omega)}^2 + \|\eta_{0,K,\theta}\|_{L^2(\Omega)} \|v_K - \eta_{0,K,\theta}\|_{L^2(\Omega)}}{\iint_{(0;T)\times\Omega} v_K^2} \end{aligned}$$

By definition of $\eta_{0,K,\theta}$,

$$\|\eta_{0,K,\theta}\|_{L^2((0;T)\times\Omega)} \leq \|\theta\|_{L^\infty(\Omega)} \|w_{0,K}\|_{L^2((0;T)\times\Omega)} \leq C \|v_K\|_{L^2((0;T)\times\Omega)}$$

for a constant C . In the last inequality we used Lemma 6. Combined with Estimate (2.13) this gives

$$\begin{aligned} \langle \nu_\infty, \phi \rangle &\leq \|\phi\|_{L^\infty((0;T)\times\Omega)} \lim_{K \rightarrow \infty} \frac{\|v_K - \eta_{0,K,\theta}\|_{L^2((0;T)\times\Omega)}^2 + \|\eta_{0,K,\theta}\|_{L^2(\Omega)} \|v_K - \eta_{0,K,\theta}\|_{L^2(\Omega)}}{\iint_{(0;T)\times\Omega} v_K^2} \\ &= 0, \end{aligned}$$

whence the conclusion. \square

3 Proof of Theorem I

3.1 Preliminary analysis of the maximisation problem

We first prove Lemma 1, which deals with the first and second order derivatives of J .

Proof of Lemma 1. We just briefly sketch the proof, the detailed arguments being the same as in [16, Lemma 3.2]. The first Gateaux-derivative of \dot{u}_{u_0} at u_0 in the direction h is the unique solution of

$$\begin{cases} \partial_t \dot{u}_{u_0} - \Delta \dot{u}_{u_0} = \frac{\partial f}{\partial u}(t, x, u_{u_0}) \dot{u}_{u_0} & \text{in } (0; T) \times \Omega, \\ \mathcal{B} \dot{u}_{u_0} = 0 & \text{on } (0; T) \times \partial\Omega, \\ \dot{u}_{u_0}(0, \cdot) = h & \text{in } \Omega. \end{cases}$$

It easily follows that the first order Gateaux-derivative of the functional J at u_0 in the direction h is given by

$$\dot{J}(u_0)[h] = \iint_{(0;T) \times \Omega} \dot{u}_{u_0} \frac{\partial j_1}{\partial u}(t, x, u_{u_0}) + \int_{\Omega} \dot{u}_{u_0} \frac{\partial j_2}{\partial u}(x, u_{u_0}(T, \cdot)). \quad (3.1)$$

Multiplying (1.5) by p_{u_0} and integrating by parts in time and space yields

$$\iint_{(0;T) \times \Omega} \dot{u}_{u_0} \frac{\partial j_1}{\partial u}(t, x, u_{u_0}) + \int_{\Omega} \dot{u}_{u_0}(T, \cdot) \frac{\partial j_2}{\partial u}(x, u_{u_0}(T, \cdot)) = \iint_{(0;T) \times \Omega} h(x) p(0, x) dx.$$

We thus conclude that

$$\dot{J}(u_0)[h] = \int_{\Omega} h(x) p(0, x) dx.$$

Similarly, the second order Gateaux-derivative of \ddot{u}_{u_0} at u_0 in the direction (h, h) is the unique solution of

$$\begin{cases} \partial_t \ddot{u}_{u_0} - \Delta \ddot{u}_{u_0} = \frac{\partial f}{\partial u}(t, x, u_{u_0}) \ddot{u}_{u_0} + \frac{\partial^2 f}{\partial u^2}(t, x, u_{u_0}) \dot{u}_{u_0}^2 & \text{in } (0; T) \times \Omega, \\ \mathcal{B} \ddot{u}_{u_0} = 0 & \text{on } (0; T) \times \partial\Omega, \\ \ddot{u}_{u_0}(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (3.2)$$

Hence, the second order Gateaux-derivative of the functional J at u_0 in the direction (h, h) is given by

$$\begin{aligned} \ddot{J}(u_0)[h, h] &= \iint_{(0;T) \times \Omega} \ddot{u}_{u_0} \frac{\partial j_1}{\partial u}(t, x, u_{u_0}) + \int_{\Omega} \ddot{u}_{u_0} \frac{\partial j_2}{\partial u}(x, u_{u_0}(T, \cdot)) \\ &\quad + \iint_{(0;T) \times \Omega} \dot{u}_{u_0}^2 \frac{\partial^2 j_1}{\partial u^2}(t, x, u_{u_0}) + \int_{\mathbb{T}} \dot{u}_{u_0}^2 \frac{\partial^2 j_2}{\partial u^2}(x, u_{u_0}). \end{aligned} \quad (3.3)$$

Multiplying (3.2) by p_{u_0} and integrating by parts in time and space yields

$$\iint_{(0;T) \times \Omega} \ddot{u}_{u_0} \frac{\partial j_1}{\partial u}(t, x, u_{u_0}) + \int_{\Omega} \ddot{u}_{u_0} \frac{\partial j_2}{\partial u}(x, u_{u_0}(T, \cdot)) = \iint_{(0;T) \times \Omega} \dot{u}_{u_0}^2 p_{u_0} \frac{\partial^2 f}{\partial u^2}(t, x, u_{u_0})$$

We eventually get:

$$\ddot{J}(u_0)[h, h] = \iint_{(0;T) \times \Omega} \left(\dot{u}_{u_0}^2 p_{u_0} \frac{\partial^2 f}{\partial u^2}(t, x, u_{u_0}) + \dot{u}_{u_0}^2 \frac{\partial^2 j_1}{\partial u^2}(t, x, u_{u_0}) \right) + \int_{\Omega} \dot{u}_{u_0}^2(T, x) \frac{\partial^2 j_2}{\partial u^2}(x, u_{u_0}).$$

□

3.2 Regularity of the solution u_{u_0}

We begin with a regularity result:

Lemma 9. *Assume f satisfies $(\mathbf{H}_f^{\text{exist}})$. For any $u_0 \in L^\infty(\Omega)$, the solution u_{u_0} of (1.1) satisfies*

$$\forall p \in [1; +\infty), u_{u_0} \in L^p(0, T; W^{1,p}(\Omega)).$$

Proof of Lemma 9. The assumption on f simply allows to write (by the maximum principle) that for any $u_0 \in L^\infty(\Omega)$ we have

$$-M \leq u_{u_0} \leq M$$

for some large M , so that the term $f = f(t, x, u_{u_0})$ can be treated as an L^∞ source term. Define $V := f(t, x, u_{u_0}) \in L^\infty((0; T) \times \Omega)$ and consider the solution v of

$$\begin{cases} \partial_t v - \Delta v = V & \text{in } (0; T) \times \Omega, \\ \mathcal{B}v = 0 & \text{on } (0; T) \times \partial\Omega, \\ v(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

By the maximal parabolic regularity ([8] or [11, Chapter IV, Section 3, pages 289-290]) we know that $v \in L^p(0, T; W^{2,p}(\Omega))$. Then, define w as the solution of

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } (0; T) \times \Omega, \\ \mathcal{B}w = 0 & \text{on } (0; T) \times \partial\Omega, \\ w(0, \cdot) = u_0 & \text{in } \Omega. \end{cases}$$

Here standard parabolic estimates imply that $w \in L^p(0, T; W^{1,p}(\Omega))$. As $u_{u_0} = w + v$ the conclusion follows. In the case of Dirichlet boundary conditions we can write the solution as

$$w(t, \cdot) = e^{-t\Delta} u_0$$

where $e^{-t\Delta}$ is the heat-semigroup in Ω . From [17, Proposition 48.7] we know that, for any $t \in [0; T]$,

$$\|\nabla e^{-t\Delta} u_0\|_{L^\infty(\Omega)} \leq C(1 + t^{-\frac{1}{2}}) \|u_0\|_{L^\infty}.$$

As $t \mapsto 1 + t^{-\frac{1}{2}}$ is integrable we have $w \in L^p(0, T; W^{1,p}(\Omega))$ for any $p \in [1; +\infty)$. Since $u = v + w$, the conclusion follows. For Neumann boundary conditions, it suffices to observe that up to a local flattening of the boundary and to a symmetrisation with respect to each of the axis, we can get rid of the influence of boundary conditions and thus obtain the desired regularity. \square

Corollary 10. *Let $q(t, x) := \partial_u f(t, x, u_{u_0}(t, x))$. Then (\mathbf{H}_q) holds:*

$$\partial_t q - \Delta q, \nabla q \in L^r(0, T; L^r(\Omega)).$$

If \mathcal{B} is of Neumann type then $\mathcal{B}q = 0$.

Proof. We know that $u_{u_0} \in L^r(0, T; W^{1,r}(\Omega))$ for all $r < \infty$ by Lemma 9. Now, we note that

$$\nabla q = \partial_{xu} f + \nabla u_{u_0} \partial_{uu} f,$$

which is in $L^r(0, T; L^r(\Omega))$ since $\partial_{uu} f$ and $\partial_{xu} f$ are bounded by hypothesis $(\mathbf{H}_{\text{reg}})$. Similarly,

$$\begin{aligned} \partial_t q - \Delta q &= \partial_{ut} f - \Delta \partial_u f - 2\nabla \partial_u f \cdot \nabla u_{u_0} + (\partial_t u_{u_0} - \Delta u_{u_0}) \partial_{uu} f - \partial_{uuu} f |\nabla u_{u_0}|^2 \\ &= \partial_{ut} f - \Delta \partial_u f - 2\nabla \partial_u f \cdot \nabla u_{u_0} + f \partial_{uu} f - \partial_{uuu} f |\nabla u_{u_0}|^2, \end{aligned}$$

which is again in $L^r(0, T; L^r(\Omega))$ for all $r < \infty$ since f , $\partial_{uu} f$, $\partial_{tu} f$, $\nabla \partial_u f$, $\Delta \partial_u f$ and $\partial_{uuu} f$ are bounded by $(\mathbf{H}_{\text{reg}})$, and $u_{u_0} \in L^r(0, T; W^{1,r}(\Omega))$ for all $r < \infty$. If \mathcal{B} is of Neumann type then $\partial_\nu q = \partial_\nu u \partial_{uu}^2 f(t, x, u) = 0$. This concludes the proof. \square

3.3 Conclusion of the proof

Proof of Theorem I. We argue by contradiction. Assume that the set

$$\omega := \{-\kappa_0 < u_0^* < \kappa_1\} \cap \{W_{u_{u_0}} > \delta\}$$

has positive measure for some $\delta > 0$. Using the inner regularity of the Lebesgue measure we can find a closed subset ω_δ of ω such that

$$\text{Vol}(\omega_\delta) > 0, \omega_\delta \subset \omega, \omega_\delta \text{ is closed.}$$

From the arguments of [15, Proof of Theorem 1] we know that, for any $K \in \mathbb{N}$, we may choose h_K supported in ω_δ such that h_K writes

$$h_K = \sum_{k \geq K} a_{K,k} \varphi_{k,\mathcal{B}}, \quad \sum_{k=K}^{\infty} a_{K,k}^2 = 1.$$

As u_0^* maximises J over \mathcal{A} , we have

$$\forall K \in \mathbb{N}, \iint_{(0;T) \times \Omega} W_{u_0} \dot{u}_{K,u_0^*}^2 + \int_{\Omega} \partial_{uu}^2 j_2|_{u_{u_0}} \dot{u}_{K,u_0^*}^2 = \ddot{J}(u_0^*)[h_K, h_K] \leq 0, \quad (3.4)$$

where u_{K,u_0^*} is the derivative of $u_0 \mapsto u_{u_0}$ at u_0^* in the direction h_K . Set now

$$\nu_K := \frac{\dot{u}_{K,u_0^*}^2}{\iint_{(0;T) \times \Omega} \dot{u}_{K,u_0^*}^2}$$

and choose ν_∞ to be a closure point (in $\mathcal{C}^0([0;T]; L^1(\Omega))'$) of $(\nu_K)_{K \in \mathbb{N}}$. As $W_{u_{u_0}} \in \mathcal{C}^0([0;T]; L^1(\Omega))$ from standard parabolic regularity, dividing (3.4) by $\iint_{(0;T) \times \Omega} \dot{u}_{K,u_0^*}^2$ and passing to the limit, we obtain

$$\langle \nu_\infty, W_{u_{u_0}} \rangle \leq 0.$$

As ν_∞ is supported in $\{t = 0\} \times \omega_\delta$ from Theorem II, as $W_{u_{u_0}} \geq \delta$ on ω_δ and as $\nu_\infty \geq 0$ we have

$$\delta = \delta \langle \nu_\infty, \mathbb{1}_{[0;T] \times \Omega} \rangle \leq \langle \nu_\infty, W_{u_{u_0}} \rangle \leq 0,$$

a contradiction. This concludes the proof. \square

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