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Abstract

A new hierarchy of non-hydrostatic layer-averaged models for the non-stationary Euler equations is presented in this work, with improved dispersion relations. They are a generalisation of the layer-averaged models introduced in [9], named LDNH models, where the vertical profile of the horizontal velocity is layerwise constant. This assumption implies that solutions of LDNH can be seen as a first order Galerkin approximation of Euler equations. This work focuses on particular weak solutions where the horizontal velocity is layerwise linear on z and possibly discontinuous across layer interfaces. Several closure relations of the layer-averaged system with non-hydrostatic pressure are presented. The resulting models are named LIN-NH$_p$ models, with $p = 0, 1, 2$. Parameter p indicates the degree of the vertical velocity profile considered in the approximation of the vertical momentum equation. All the introduced models satisfy a dissipative energy balance. Finally, an analysis and a comparison of the dispersive properties of each model are carried out. We show that Models LIN-NH$_1$ and LIN-NH$_2$ provide a better dispersion relation, group velocity and shoaling than LDNH models.

Contents

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1 Introduction

Many efforts have been devoted to including dispersive effects in models in fluid mechanics, and in particular, they have been widely studied in the context of shallow flows in the literature. There are essentially two approaches to consider dispersive effects in this framework: Boussinesq type systems and non-hydrostatic systems.

Boussinesq type systems are mainly based on classic shallow water system, whose unknowns are the total height of the fluid and the horizontal velocity. The system is then extended by introducing high order derivatives of the variables. The two pioneering works on Boussinesq models were proposed by Boussinesq [3] and Peregrine [21]. Some other models were proposed afterwards: Madsen & Sorensen [18,19], Nwogu [20], Serre-Green-Nagdhi [12,24], Schäffer & Madsen [23], Lannes [15], among many others. For a review on Boussinesq type dispersive models, see [13, 14].

Contrary to Boussinesq systems, non-hydrostatic models only consider first order derivatives of the variables but with a larger number of unknowns. Usually, in addition to the water depth and horizontal velocity, we find the vertical velocity, non-hydrostatic pressure, and other unknowns related to the vertical profile of the different variables. One of the essential points of these models is the presence of constraint equations related to the incompressibility condition, see for example [25] and [4].

Although both approaches seem quite different, they are equivalent, as it was shown in [7]. They introduced a general formulation that relates classic Boussinesq systems and non-hydrostatic models. It was shown that classic Boussinesq systems might be written as non-hydrostatic models. We want to point out that the non-hydrostatic formulation has several advantages from the numerical point of view. In particular, avoiding high order derivatives makes the numerical approximation easier. Moreover, the treatment of boundary conditions is also more straightforward.

A strategy to improve the dispersion relation of such systems is to increase the accuracy of the vertical profile for both velocity and pressure components. For this purpose, several attempts have also been introduced in the literature. The so-called multilayer approach considers non-material vertical interfaces, dividing the fluid into virtual vertical layers. Different profiles for the velocity and pressure within each layer may be assumed. Notice that the term multilayer has been used sometimes for stratified flow models, where one considers a constant density within each layer. This has no relation with the approach we are considering in this paper. To avoid any possible confusion, in [9] it was proposed to name the former (non-stratified flows) as layer-averaged models. We shall do the same in what follows.

In [22] a non-hydrostatic layer-averaged model is proposed, which corresponds to the extension of the hydrostatic model in [1]. The momentum equations are approximated by considering a constant

vertical profile for horizontal and vertical velocity components. Then, velocity may be discontinuous at the interfaces, although the pressure profile is supposed to be continuous. However, a linear profile of the vertical velocity within each layer is considered to approximate the incompressibility condition, which is a compatible condition of strong solutions in each layer for this equation.

A similar model was proposed in [2], which assumes a constant profile for the horizontal velocity and linear profiles for the vertical velocity and pressure. The common ground for both references is that the horizontal velocity has a constant vertical profile within each layer and may be discontinuous at the interfaces. A different assumption is found in [16], where a layer-averaged model is derived by using a continuous global profile of the horizontal velocity, which is quadratic within each layer.

In [9] a hierarchy of models is presented, with an associated energy balance, where again a constant profile is assumed for the horizontal velocity inside each layer. At the same time, several degrees of freedom are introduced for the vertical velocity and pressure, accounting for the vertical approximation of such variables. In that work, the linear dispersion relation was studied for the different models proposed. It was shown that the dispersion relation converges to the exact dispersion relation for Euler equations when the number of layers goes to infinity. A numerical strategy to solve these models is proposed in [8]. Let us recall the notation of the models introduced in that paper as it will be helpful in what follows. These models are named as LDNH_k where LDNH stands for *Layerwise Discretisation Non-Hydrostatic*, and k is the order of approximation. More precisely, LDNH_0 may be seen as a generalisation of models [25] and [4], while LDNH_2 is as a generalisation of the Serre-Green-Nagdhi model. LDNH_1 is an in-between intermediate model. Similarly, in [5] authors derived a weighted-averaged non-hydrostatic pressure model, under the assumption of linear horizontal velocity, whereas the vertical velocity and the non-hydrostatic pressure was assumed to have a quadratic profile.

This paper focuses on the derivation of 2D layer-averaged models where the vertical profile of the horizontal velocity is supposed to be layerwise linear and discontinuous at the interfaces. The extension to 2D is straightforward. Concretely, a hierarchy of novel models is introduced, named as LIN-NH_k models for $k = 0, 1, 2$, where the index k corresponds to the degree of approximation of the vertical velocity in the vertical momentum equation. Therefore, three models are proposed, based on three different profiles for the pressure, which are related to the approximation of the vertical momentum equation and the incompressibility condition. In addition, the pressure profile is considered as a polynomial of degree $k + 1$. All the proposed non-hydrostatic models also satisfy a dissipative energy balance. As we show in this paper, since more unknowns are introduced, linear dispersion relation, group velocity, and linear shoaling are greatly improved for such models. In particular, the LIN-NH_1 and LIN-NH_2 models exhibit excellent results for these dispersive properties. Moreover, another contribution is a general procedure to compute the group velocity and shoaling gradient from the dispersion relation for the wave celerity.

The paper is organised as follows. In Section 2, we state the notation and the layer-averaged procedure for the layerwise linear horizontal velocity case. Subsection 2.1 focuses on the discrete spaces and hypotheses over the vertical profile of each variable (horizontal and vertical velocities and pressure), as well as the normal jump condition at the interfaces associated with the weak formulation of the Euler system. In Subsection 2.2 we describe the vertical averaging of the mass and horizontal momentum equations without detailing the pressure terms. Section 3 is devoted to derive the cascade of LIN-NH_k non-hydrostatic models, for $k = 0, 1, 2$. The dispersive properties for these models are analysed in Section 4, and some conclusions are presented in Section 5. Finally, in Appendix A we give a detailed proof of the energy balance satisfied by the LIN-NH_2 non-hydrostatic model.

2 Initial system and layer-averaged approach

This section introduces the general settings associated with the layer-averaged approach for the Euler equations. In this work we shall assume a piecewise linear approximation of the horizontal velocity.

This layer-averaged approach can be seen as a technique to approximate the solution of the Euler equations in the framework of the Discontinuous Galerkin methods. The final model that we obtain is a system of partial differential equations. The solution of this system may be seen as a particular weak solution of Euler equations, in the sense that it corresponds to an approximated piecewise smooth weak solution that may be discontinuous at the internal interfaces. We remark that these interfaces

are not physical or material interfaces. They represent a virtual decomposition or partition of the domain vertically. The procedure is as follows: first, we consider a vertical discretisation. Second, a layer-averaging approach is considered assuming an appropriate structure of the weak solutions. In particular, here, we shall assume that the velocity vector has horizontal components that are linear in the vertical z-direction within each layer. Third, some closure relations (or constraints) are needed, and they will be presented later in Subsection 3.1.

For the sake of simplicity, we shall consider here a 2-dimensional space, where the space variables are denoted by $(x, z) \in \mathbb{R}^2$. Nevertheless, it could be extended easily to the 3D case. In what follows, $\nabla = (\partial_x, \partial_z)$ represents the usual differential operator. Let us consider now an incompressible fluid with constant density $\rho \in \mathbb{R}$. Let us denote $g \in \mathbb{R}$ the gravity acceleration, $p \in \mathbb{R}$ the pressure, $\boldsymbol{U} = (u, w)^{'} \in \mathbb{R}^2$ the velocity vector, and a moving domain

$$\Omega(t) = \Big\{(x, z) : \ z_b(x) < z < z_b(x) + H(t, x)\Big\},$$

where z_b assumed independent from time is the topography and H the water height (see Figure 1). The Euler system is written, for $(x, z) \in \Omega(t)$, as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{U}) = 0, \\ \partial_t (\rho \boldsymbol{U}) + \nabla \cdot (\rho \boldsymbol{U} \otimes \boldsymbol{U} + p \boldsymbol{I}) = \rho \boldsymbol{g}, \end{cases} \tag{1}$$

where $\boldsymbol{g} = (0, -g)^{'} \in \mathbb{R}^2$ and $\boldsymbol{I}$ is the identity matrix. The system is completed with initial and boundary conditions, and the following kinematic conditions:

$$\partial_t H + \boldsymbol{U}|_{z=z_b+H} \cdot \nabla (z_b + H - z) = 0, \tag{2}$$

$$\boldsymbol{U}|_{z=z_b} \cdot \nabla (z_b - z) = 0. \tag{3}$$

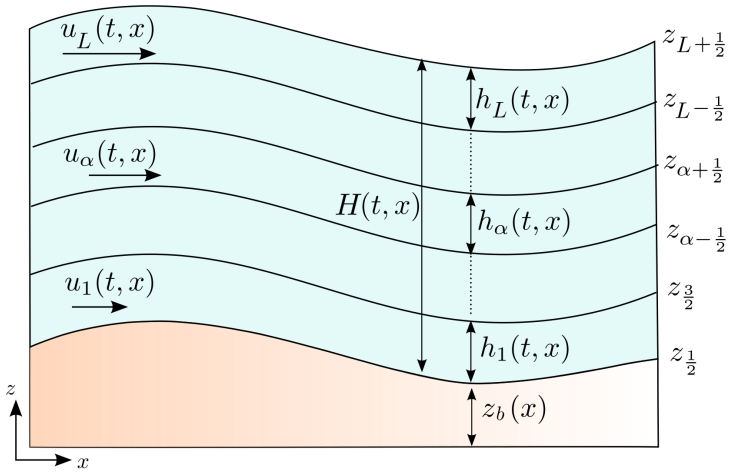

Figure 1: *Sketch of the vertical partition of the fluid domain.*

Now, in order to apply the layer-averaged approach, the domain is subdivided along the vertical direction into $L \in \mathbb{N}$ layers with thickness $h_\alpha(t, x)$, which are denoted by $\Omega_\alpha(t)$. The layers are separated by $L+1$ interfaces $\mathcal{L}_{\alpha+1/2}(t)$, defined by $z = z_{\alpha+1/2}(t, x)$ for $\alpha = 0, 1, ..., L$. The interfaces are assumed sufficiently smooth, at least C^1 (see Figure 1). More explicitly,

$$\Omega_\alpha(t) = \Big\{(x, z) : \ z_{\alpha-1/2}(t, x) < z < z_{\alpha+1/2}(t, x)\Big\}.$$

Notice then that

$$z_{\alpha+1/2} = z_b + \sum_{\beta=1}^{\alpha} h_\beta, \qquad \text{for } \alpha = 1, \dots, L,$$

where $z_b = z_{1/2}$ and $h_\alpha = z_{\alpha+1/2} - z_{\alpha-1/2}$. Moreover, the total height of the fluid is equal to

$$H = \sum_{\alpha=1}^{L} h_\alpha, \quad \text{with } h_\alpha = \ell_a H, \text{ where } (\ell_\alpha)_{\{\alpha\in 1,\dots,L\}} \text{ are such that } \ell_\alpha \in [0,1], \sum_{\alpha=1}^{L} \ell_\alpha = 1. \tag{4}$$

Let us also denote by z_α the midpoint of the layer Ω_α, i.e.,

$$z_\alpha = \frac{z_{\alpha-1/2} + z_{\alpha+1/2}}{2} = z_{\alpha-1/2} + \frac{h_\alpha}{2}, \qquad \text{for } \alpha = 1, \dots, L.$$

Furthermore, let us introduce the following notations: for any function $f(t,x,z)$, we define:

- Its approximations at the interfaces $\mathcal{L}_{\alpha+1/2}$ for $\alpha = 0, \dots, L$

  $$f^-_{\alpha+1/2} = \lim_{\substack{z\to z_{\alpha+1/2}\\ z<z_{\alpha+1/2}}} f_{|\Omega_\alpha}, \qquad f^+_{\alpha+1/2} = \lim_{\substack{z\to z_{\alpha+1/2}\\ z>z_{\alpha+1/2}}} f_{|\Omega_{\alpha+1}}.$$

  In the case of continuous functions, we shall simply write $f_{\alpha+1/2} = f^+_{\alpha-1/2} = f^-_{\alpha+1/2}$;

- The average value over the layer Ω_α

  $$\overline{f}_\alpha(t,x) = \frac{1}{h_\alpha} \int_{z_{\alpha-1/2}}^{z_{\alpha-1/2}} f(t,x,z)\,\mathrm{d}z; \tag{5a}$$

- The linear average of f over the layer Ω_α

  $$\widehat{f}_\alpha = \frac{f^+_{\alpha-1/2} + f^-_{\alpha+1/2}}{2}; \tag{5b}$$

- The variation through the layer Ω_α

  $$(\delta f)_\alpha = f^-_{\alpha+1/2} - f^+_{\alpha-1/2}; \tag{5c}$$

- The average value through the interface $\mathcal{L}_{\alpha+1/2}$

  $$\widetilde{f}_{\alpha+1/2} = \frac{f^+_{\alpha+1/2} + f^-_{\alpha+1/2}}{2}; \tag{5d}$$

- The jump across the interface $\mathcal{L}_{\alpha+1/2}$

  $$[f]_{\alpha+1/2} = f^+_{\alpha+1/2} - f^-_{\alpha+1/2}. \tag{5e}$$

Note that, in general, $\overline{f}_\alpha \neq \widehat{f}_\alpha$, but the equality holds if f is constant or linear in z over the layer Ω_α.

2.1 Layerwise approximation of the variables and normal jump conditions

This subsection deals with the main hypotheses upon velocity and pressure fields in the layer-averaged framework. In particular, we detail here the vertical profile for the variables considered in this work and its notation. We also write the normal jump condition associated with the weak formulation.

2.1.1 Choice of discrete spaces for velocity and pressure fields

The layer-averaged models presented in [10, 11] assume that the horizontal velocity is constant in the z-direction within each layer. In practice, this will formally limit us to first order approximations of the solution for the Euler equations. In this work we shall consider a piecewise linear profile in z for the horizontal velocity. Concerning the other variables, a piecewise parabolic profile in z is supposed for the vertical velocity, and a piecewise third order polynomial in z is used for the non-hydrostatic pressure. Nevertheless, some other alternative simplified equations will be presented afterwards, resulting in simplified models.

More explicitly, let us specify the discretisation assumptions for the velocity unknowns

$$\boldsymbol{U}_\alpha := \boldsymbol{U}_{|\Omega_\alpha} := (u_\alpha, w_\alpha)^{'},$$

where u_α and w_α are the horizontal and vertical components of the velocity field in layer Ω_α. Then, the following profiles and notation are chosen for this model:

Horizontal velocity: a linear profile on the vertical direction is assumed within each layer

$$u_\alpha(t,x,z) = \overline{u}_\alpha(t,x) + \lambda_\alpha(t,x)\big(z - z_\alpha(t,x)\big), \text{ for } z \in [z_{\alpha-1/2}(t,x), z_{\alpha+1/2}(t,x)], \tag{6}$$

where $\overline{u}_\alpha$ and λ_α are the averaged horizontal velocity and the slope respectively in the z direction. In what follows and for the sake of simplicity, we shall not write the dependence of the variables on (t,x) explicitly unless necessary. It follows then that the limit at the interfaces $u^-_{\alpha+1/2}$ and $u^+_{\alpha-1/2}$ are obtained by evaluating (6) at $z_{\alpha+1/2}$ and $z_{\alpha-1/2}$, respectively, i.e.,

$$u^-_{\alpha+1/2} = \overline{u}_\alpha + \frac{h_\alpha\lambda_\alpha}{2}, \qquad u^+_{\alpha-1/2} = \overline{u}_\alpha - \frac{h_\alpha\lambda_\alpha}{2}. \tag{7}$$

Vertical velocity: focusing on the incompressibility condition and given the piecewise linear profile (6) for the horizontal velocity, a compatibility criterion leads us to consider a piecewise parabolic polynomial in z, that is

$$w_\alpha(z) = w_\alpha(z_\alpha) + \varphi_\alpha\,(z - z_\alpha) + \frac{\psi_\alpha}{2}\,(z - z_\alpha)^2, \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]. \tag{8}$$

Hence, there are three parameters to set. On the one hand, the integration of (8) over the layer yields

$$w_\alpha(z_\alpha) = \overline{w}_\alpha - \frac{h_\alpha^2\psi_\alpha}{24}. \tag{9}$$

Note that $w_\alpha(z_\alpha) \neq \overline{w}_\alpha$ in this case because of the contribution of the second order term.

On the other hand, a second order Taylor expansion, which is exact for a quadratic function, yields

$$w_\alpha(z) = w_\alpha(z_\alpha) + (\partial_z w_\alpha)_{|z=z_\alpha}\,(z - z_\alpha) + \big(\partial^2_{zz} w_\alpha\big)_{|z=z_\alpha}\,\frac{(z-z_\alpha)^2}{2}, \quad \text{for } \alpha = 1,\dots,L.$$

As we want w_α to approximate $w_{|\Omega_\alpha}$, let us mimic the incompressibility constraint $\partial_z w = -\partial_x u$ using (6). Hence we impose

$$(\partial_z w_\alpha)_{|z=z_\alpha} = -\partial_x\overline{u}_\alpha + \lambda_\alpha\partial_x z_\alpha, \quad \text{and} \quad \big(\partial^2_{zz} w_\alpha\big)_{|z=z_\alpha} = -\partial_x\lambda_\alpha. \tag{10}$$

Therefore, combining (8), (9) and (10), we get

$$w_\alpha(z) = \overline{w}_\alpha + \varphi_\alpha(z - z_\alpha) + \frac{\psi_\alpha}{2}\left((z-z_\alpha)^2 - \frac{h_\alpha^2}{12}\right), \tag{11}$$

for $\alpha = 1, \dots, L$, with

$$\begin{cases} \varphi_\alpha = -\partial_x \overline{u}_\alpha + \lambda_\alpha \partial_x z_\alpha, & \text{(12a)} \\ \psi_\alpha = -\partial_x \lambda_\alpha. & \text{(12b)} \end{cases}$$

Evaluating (11) at $z_{\alpha\pm1/2}$, we obtain

$$w^-_{\alpha+1/2} = \overline{w}_\alpha + \frac{h_\alpha \varphi_\alpha}{2} + \frac{h_\alpha^2 \psi_\alpha}{12}, \qquad w^+_{\alpha-1/2} = \overline{w}_\alpha - \frac{h_\alpha \varphi_\alpha}{2} + \frac{h_\alpha^2 \psi_\alpha}{12}. \tag{13}$$

Pressure: the total pressure is decomposed into a hydrostatic part $g\,(z_b + H - z)$, and a non-hydrostatic counterpart q:

$$p = \rho\,(g\,(z_b + H - z) + q)\,,$$

where we assume that the pressure is known at the surface, which is usually set to zero.

As mentioned above, we choose a layerwise linear horizontal velocity and a layerwise parabolic vertical velocity (potentially discontinuous across the interfaces). Consequently, the pressure must be a layerwise cubic function due to the vertical momentum equation. We assume in the present work that the pressure is continuous across interfaces.

Hence, we are looking for a 3rd-order polynomial q_α which satisfies the following requirements:

$$q_\alpha(z_{\alpha\pm1/2}) = q_{\alpha\pm1/2}, \qquad \text{and} \qquad \partial_z q_\alpha(z_\alpha) = \frac{\pi_\alpha}{h_\alpha}.$$

Together with the definitions (5), this leads to

$$q_\alpha(z) = \frac{3\overline{q}_\alpha - \widehat{q}_\alpha}{2} + \pi_\alpha \frac{z - z_\alpha}{h_\alpha} + 6\,(\widehat{q}_\alpha - \overline{q}_\alpha)\,\frac{(z - z_\alpha)^2}{h_\alpha^2} + 4\big((\delta q)_\alpha - \pi_\alpha\big)\frac{(z - z_\alpha)^3}{h_\alpha^3}, \tag{14}$$

for $z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]$.

2.1.2 Jump conditions and evolution equation for the layer midpoints

Following [11], the model is deduced by looking for a particular piecewise smooth weak solution $(\rho, \boldsymbol{U}, p)$ of System (1). More explicitly, we search for a solution that satisfies the weak formulation of the system for a particular set of tests functions. We refer to [11] for further details. In particular, they correspond to a classic, smooth solution inside each layer Ω_α, while it should satisfy the normal flux jump conditions across the interfaces $\mathcal{L}_{\alpha+1/2}$, $\alpha = 0, \dots, L$.

The jump condition for the mass conservation law reads

$$[(\rho; \rho\boldsymbol{U})]_{\alpha+1/2} \cdot \big(\partial_t z_{\alpha+1/2}, \partial_x z_{\alpha+1/2}, -1\big) = 0, \tag{15a}$$

and for the momentum conservation law

$$[(\rho\boldsymbol{U}; \rho\boldsymbol{U} \otimes \boldsymbol{U} - p\boldsymbol{I})]_{\alpha+1/2} \cdot \big(\partial_t z_{\alpha+1/2}, \partial_x z_{\alpha+1/2}, -1\big) = 0. \tag{15b}$$

We are considering here a constant density for the fluid. Hence, following [11], (15a) implies

$$\Gamma_{\alpha+1/2} := \Gamma^-_{\alpha+1/2} = \Gamma^+_{\alpha+1/2}, \tag{16a}$$

where $\Gamma_{\alpha+1/2}$ is the mass transfer term through the interface $\mathcal{L}_{\alpha+1/2}$ given by

$$\Gamma^\pm_{\alpha+1/2} = -\partial_t z_{\alpha+1/2} - u^\pm_{\alpha+1/2} \partial_x z_{\alpha+1/2} + w^\pm_{\alpha+1/2}. \tag{16b}$$

In particular, we infer that

$$\partial_t z_\alpha + \overline{u}_\alpha \partial_x z_\alpha = \widehat{w}_\alpha - \frac{h_\alpha}{4} \lambda_\alpha \partial_x h_\alpha - \frac{\Gamma_{\alpha-1/2} + \Gamma_{\alpha+1/2}}{2}, \tag{17}$$

which gives the evolution of the midpoint at each layer Ω_α.

2.2 Layer-averaged approximation: mass and horizontal momentum equations

In this section we obtain the mass and horizontal momentum equations of the target model by a layer-averaged process. In what follows, we use a general depth-integration process for mass and horizontal momentum equations.

Mass conservation. The incompressibility condition is integrated over each layer, which leads to

$$\begin{aligned} 0 &= \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \nabla \cdot \boldsymbol{U}_\alpha \, \mathrm{d}z \\ &= \partial_x \left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha(t,x,z) \, \mathrm{d}z \right) - u^-_{\alpha+1/2} \partial_x z_{\alpha+1/2} + u^+_{\alpha-1/2} \partial_x z_{\alpha-1/2} + w^-_{\alpha+1/2} - w^+_{\alpha-1/2}. \end{aligned}$$

Taking into account the definition of the mass transfer terms (16), we obtain the mass conservation laws

$$\partial_t h_\alpha + \partial_x (h_\alpha \overline{u}_\alpha) = \Gamma_{\alpha-1/2} - \Gamma_{\alpha+1/2}, \qquad \alpha = 1, \dots, L, \tag{18}$$

where we recall that $\Gamma_{\alpha\pm1/2}$ account for mass transfer across interfaces $\mathcal{L}_{\alpha\pm1/2}$. Actually, they can be expressed in terms of the velocities by combining previous equations, getting, for $\alpha = 1, \dots, L-1$,

$$\Gamma_{\alpha+1/2} = \sum_{\beta=\alpha+1}^{L} \partial_x \left(h_\beta \left(\overline{u}_\beta - \overline{\overline{u}} \right) \right), \qquad \text{with} \qquad \overline{\overline{u}} := \sum_{\alpha=1}^{L} \ell_\alpha \overline{u}_\alpha, \tag{19}$$

for ℓ_α defined in (4). Moreover, summing Equation (18) over α we get

$$\partial_t H + \partial_x \left(H \overline{\overline{u}} \right) = 0. \tag{20}$$

Therefore, it is equivalent to considering Equation (18) and the set of equations defined by (19) and (20).

Note that $\Gamma_{1/2}$ and $\Gamma_{L+1/2}$ correspond to the mass exchange at the free surface and the bottom, respectively, and they are set to zero unless specified. In any case, both of them should be provided as data.

Averaged horizontal momentum conservation. The approximation of the horizontal momentum equation is deduced in two steps. First, a layer-integration of the horizontal momentum equation provides the horizontal momentum conservation law equation related to the averaged velocity, $\overline{u}_\alpha$.

$$\partial_t (h_\alpha \overline{u}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + \frac{h_\alpha^3 \lambda_\alpha^2}{12} \right) + \frac{1}{\rho} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \partial_x p_\alpha \, \mathrm{d}z = -\widetilde{u}_{\alpha+1/2} \Gamma_{\alpha+1/2} + \widetilde{u}_{\alpha-1/2} \Gamma_{\alpha-1/2}, \tag{21}$$

see (5) for notations. Note that here we have used the jump condition (15b), otherwise a term $u^\mp_{\alpha\pm1/2}$ would appear instead of $\widetilde{u}_{\alpha\pm1/2}$. This is done in what follows when the limit values of the horizontal and vertical velocities at the interfaces appear.

Secondly, we evaluate the mean deviation of the horizontal velocity and use (6) to derive the conservation law associated with λ_α. To do so, the horizontal momentum equation is multiplied by $(z - z_\alpha)$ and then integrated inside the layer.

To this aim, let us first compute the horizontal momentum equation multiplied by z and integrate it inside the layer.

$$\begin{aligned} &\partial_t \left(h_\alpha \left(z_\alpha \overline{u}_\alpha + \frac{\lambda_\alpha h_\alpha^2}{12} \right) \right) + \partial_x \left(h_\alpha z_\alpha \overline{u}_\alpha^2 + \lambda_\alpha^2 z_\alpha \frac{h_\alpha^3}{12} + 2 \overline{u}_\alpha \lambda_\alpha \frac{h_\alpha^3}{12} \right) + \frac{1}{\rho} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} z \partial_x p_\alpha \, \mathrm{d}z \\ &\qquad = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u_\alpha w_\alpha \, dz - z_{\alpha+1/2} \widetilde{u}_{\alpha+1/2} \Gamma_{\alpha+1/2} + z_{\alpha-1/2} \widetilde{u}_{\alpha-1/2} \Gamma_{\alpha-1/2}. \end{aligned} \tag{22}$$

Finally, by using (17) and subtracting (21) multiplied by z_α to (22), we obtain (after dividing by h_α)

$$\partial_t\left(\frac{h_\alpha^2\lambda_\alpha}{12}\right)+\partial_x\left(\frac{h_\alpha^2\lambda_\alpha u_\alpha}{12}\right)+\frac{h_\alpha^2\overline{u}_\alpha}{12}\partial_x\lambda_\alpha+\frac{h_\alpha^2\lambda_\alpha^2}{12}\partial_x z_\alpha+\frac{1}{\rho h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}(z-z_\alpha)\partial_x p_\alpha\,\mathrm{d}z$$
$$=\Gamma_{\alpha+1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}+\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha+1/2}}{2}\right)-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}-\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha-1/2}}{2}\right)$$
$$-\overline{u}_\alpha\widehat{w}_\alpha+\frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}u_\alpha w_\alpha\,\mathrm{d}z.$$

The last two terms in the right hand side are obtained due to the definition of the vertical velocity (11)

$$-\overline{u}_\alpha\widehat{w}_\alpha+\frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}u_\alpha w_\alpha\,\mathrm{d}z=\frac{\overline{u}_\alpha h_\alpha^2}{12}\partial_x\lambda_\alpha+\frac{\lambda_\alpha h_\alpha^2}{12}\left(-\partial_x\overline{u}_\alpha+\lambda_\alpha\partial_x z_\alpha\right).$$

Combining the last two equations, we finally get

$$\partial_t\left(\frac{h_\alpha^2\lambda_\alpha}{12}\right)+\partial_x\left(\frac{h_\alpha^2\lambda_\alpha}{12}\overline{u}_\alpha\right)+\frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{u}_\alpha+\frac{1}{\rho h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}(z-z_\alpha)\partial_x p_\alpha\,\mathrm{d}z$$
$$=\Gamma_{\alpha+1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}+\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha+1/2}}{2}\right)-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}-\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha-1/2}}{2}\right),\quad(23)$$

for $\alpha=1,\dots,L$.

Note that the integrals involving the pressure in (21) and (23) have not been specified. These terms will be computed later on depending on the assumption on the pressure profile. The simplest choice corresponds to the assumption of hydrostatic pressure, and this model is presented in the following subsection. Non-hydrostatic pressure models are deduced in Section 3.

2.3 LIN-H model: linear horizontal velocity and hydrostatic pressure

Let us mention the case of hydrostatic pressures. In that case, the final system is defined by (18), (21), (23), where, as it is usual in the framework of hydrostatic shallow flows, the pressure is defined by

$$p(z)=\rho g\left(z_b+H-z\right).$$

Now, the integrals associated to the pressure in (21) and (23) are

$$\frac{1}{\rho}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\partial_x p\,dz=gh_\alpha\partial_x\left(z_b+H\right),\qquad\text{and}\qquad\frac{1}{\rho}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}(z-z_\alpha)\partial_x p\,dz=0.$$

Therefore, the final system for hydrostatic pressure is defined by the following set of equations:

$$\begin{cases}\partial_t h_\alpha+\partial_x(h_\alpha\overline{u}_\alpha)=\Gamma_{\alpha-1/2}-\Gamma_{\alpha+1/2},\\[2mm]\partial_t(h_\alpha\overline{u}_\alpha)+\partial_x\left(h_\alpha\overline{u}_\alpha^2+\dfrac{h_\alpha^3\lambda_\alpha^2}{12}\right)+gh_\alpha\partial_x(z_b+H)=-\widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2}+\widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2},\\[2mm]\partial_t\left(\dfrac{h_\alpha^2\lambda_\alpha}{12}\right)+\partial_x\left(\dfrac{h_\alpha^2\lambda_\alpha\overline{u}_\alpha}{12}\right)+\dfrac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{u}_\alpha\\[2mm]\qquad=\Gamma_{\alpha+1/2}\left(\dfrac{h_\alpha\lambda_\alpha}{12}+\dfrac{\overline{u}_\alpha-\widetilde{u}_{\alpha+1/2}}{2}\right)-\Gamma_{\alpha-1/2}\left(\dfrac{h_\alpha\lambda_\alpha}{12}-\dfrac{\overline{u}_\alpha-\widetilde{u}_{\alpha-1/2}}{2}\right),\end{cases}\quad(24)$$

for $\alpha=1,\dots,L$, where

$$\widetilde{u}_{\alpha+1/2}=\frac{\overline{u}_\alpha+\overline{u}_{\alpha+1}}{2}+\frac{h_\alpha\lambda_\alpha-h_{\alpha+1}\lambda_{\alpha+1}}{4}.$$

This model verifies exactly a dissipative energy balance, stated by the following theorem.

Theorem 1 *System (24) satisfies the dissipative energy balance*

$$\partial_t \left(\sum_{\alpha=1}^{L} E_\alpha \right) + \partial_x \left(\sum_{\alpha=1}^{L} \overline{u}_\alpha \left(E_\alpha + g h_\alpha \frac{H}{2} + \frac{h_\alpha^3 \lambda_\alpha^2}{12} \right) \right) \leq 0,$$

where

$$E_\alpha := h_\alpha \left(\frac{\overline{u}_\alpha^2 + \overline{w}_\alpha^2}{2} + \frac{h_\alpha^2 \lambda_\alpha^2}{24} + g \left(z_b + \frac{H}{2} \right) \right).$$

The proof can be seen as a particular case of the result stated in Thereom 2.

3 A hierarchy of non-hydrostatic layer-averaged systems with linear horizontal velocity: LIN-NH$_p$ models

We consider now a non-hydrostatic pressure. Therefore, we must define the vertical profile for the vertical velocity and, consequently, for the non-hydrostatic pressure. Note that the incompressibility condition implies that inside each layer, the vertical velocity is a polynomial on z of degree $d+1$, where d is the degree of the horizontal velocity (in this paper, we consider $d=1$). Thus, a hierarchy of models is presented, based on the degree of the polynomial employed to approximate the vertical momentum equation. We consider a polynomial of degree $p \leq d+1$, on z for the vertical velocity. We also consider that the pressure profile is a polynomial on z of degree $p+1$, which becomes an essential hypothesis to obtain a system with an associated dissipative energy balance. In [9], authors introduced a hierarchy of non-hydrostatic layer-averaged models for Euler equations for $d=0$ and $p \leq 1$.

In the following subsections, firstly, the incompressibility condition within each layer is deduced. Secondly, three models are proposed, corresponding to different approximations of the vertical velocity and non-hydrostatic pressure counterparts. All of them satisfy a dissipative energy balance. We will detail the more complex model, where the vertical velocity is a layerwise quadratic polynomial according to the incompressibility condition, and therefore the non-hydrostatic pressure is a layerwise cubic function. Later, we see that if the vertical structure of the vertical velocity is simplified just for the vertical momentum equation, then the model degenerates to simpler models, but keeping still an analogous energy balance. This hierarchy of non-hydrostatic models will be denoted by Multilayer Horizontal Linear discretisation Non-Hydrostatic p model (from now on LIN-NH$_p$), where p is the degree of the polynomial approximating the vertical velocity in the vertical momentum conservation equation, that is, $p = 0, 1, 2$.

3.1 Averaged incompressibility condition

In the non-hydrostatic framework, we need some extra information or constraints to solve the resulting system, which now includes the degrees of freedom related to the non-hydrostatic pressure. Note that for each layer, there are three unknowns that define the non-hydrostatic pressure profile, namely $q_{\alpha+1/2}$, $\overline{q}_\alpha$ and π_α. Therefore, one would need three constraints at each layer.

On the one hand, two restrictions are associated to the relationship between the first and second order vertical derivative of the vertical and horizontal velocities. Concretely, in Subsection 2.1 the profile for the vertical velocity was obtained, where from the incompressiblity inside each layer we got (12).

On the other hand, the jump condition (15a) reads

$$w^+_{\alpha+1/2} - w^-_{\alpha+1/2} = \left(u^+_{\alpha+1/2} - u^-_{\alpha+1/2} \right) \partial_x z_{\alpha+1/2}, \qquad \alpha = 1, \ldots, L-1. \tag{25}$$

By substituting (7) and (13) in (25), we obtain

$$\begin{aligned} \overline{w}_{\alpha+1} - \frac{h_{\alpha+1}\varphi_{\alpha+1}}{2} + \frac{h_{\alpha+1}^2 \psi_{\alpha+1}}{12} - \overline{w}_\alpha - \frac{h_\alpha \varphi_\alpha}{2} - \frac{h_\alpha^2 \psi_\alpha}{12} \\ = \left(\overline{u}_{\alpha+1} - \frac{h_{\alpha+1}\lambda_{\alpha+1}}{2} - \overline{u}_\alpha - \frac{h_\alpha \lambda_\alpha}{2} \right) \partial_x z_{\alpha+1/2}, \qquad \alpha = 1, \ldots, L-1. \end{aligned}$$

To get the Lth constraint, let us integrate $\nabla \cdot \mathbf{U}_1 = 0$ between z_1 and $z_{1/2}$ (see [9, 25]). We obtain thanks to (9) applied to $\alpha = 1$

$$\partial_x \overline{u}_1 - \lambda_1 \partial_x z_1 - \frac{h_1}{6}\partial_x \lambda_1 + \frac{\overline{w}_1 - w^+_{1/2}}{h_1/2} = 0$$

Finally, the incompressibility conditions read

$$\begin{cases} \varphi_\alpha = -\partial_x \overline{u}_\alpha + \lambda_\alpha \partial_x z_\alpha, & \alpha = 1, \dots, L. \\ \psi_\alpha = -\partial_x \lambda_\alpha, & \alpha = 1, \dots, L, \\ \overline{w}_{\alpha+1} - \dfrac{h_{\alpha+1}\varphi_{\alpha+1}}{2} + \dfrac{h^2_{\alpha+1}\psi_{\alpha+1}}{12} - \overline{w}_\alpha - \dfrac{h_\alpha \varphi_\alpha}{2} - \dfrac{h^2_\alpha \psi_\alpha}{12} & \\ \qquad = \left(\overline{u}_{\alpha+1} - \dfrac{h_{\alpha+1}\lambda_{\alpha+1}}{2} - \overline{u}_\alpha - \dfrac{h_\alpha \lambda_\alpha}{2}\right)\partial_x z_{\alpha+1/2}, & \alpha = 1, \dots, L-1. \\ \partial_x \overline{u}_1 - \lambda_1 \partial_x z_1 - \dfrac{h_1}{6}\partial_x \lambda_1 + \dfrac{\overline{w}_1 - w^+_{1/2}}{h_1/2} = 0. & \end{cases} \tag{26}$$

where, in order to verify the kinematic condition (3), we set

$$w^+_{1/2} = w_b = \left(\overline{u}_1 - \lambda_1 \frac{h_1}{2}\right)\partial_x z_b.$$

3.2 LIN-NH$_2$ model: non-hydrostatic model with $w_\alpha \in \mathbb{P}_2[z]$

In what follows we shall detail the first non-hydrostatic model that we propose here. It will be denoted by LIN-NH$_2$ model. Its final formulation is (39) below with the subsequent notations. Later on, in Subsection 3.3 we shall introduce as well two extra simplified models, named LIN-NH$_1$ and LIN-NH$_0$ models.

Let us recall that, up to this point, we have evolution equations (21) and (23) together with the restrictions (26) and the jump conditions (19) and (20).

In order to close the system, some extra equations are needed, which are obtained by successive integration on the vertical momentum conservation equation, as shown in what follows.

Let us first rewrite evolution equations in the non-hydrostatic framework by computing explicitly pressure terms. For (21), we have

$$\frac{1}{\rho}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \partial_x p_\alpha \,\mathrm{d}z = g h_\alpha \partial_x (z_b + H) + \partial_x \left(h_\alpha \overline{q}_\alpha\right) - q_{\alpha+1/2}\partial_x z_{\alpha+1/2} + q_{\alpha-1/2}\partial_x z_{\alpha-1/2},$$

and then

$$\begin{aligned} \partial_t(h_\alpha \overline{u}_\alpha) + \partial_x\left(h_\alpha \overline{u}^2_\alpha + \frac{h^3_\alpha \lambda^2_\alpha}{12} + h_\alpha \overline{q}_\alpha\right) + g h_\alpha \partial_x(z_b + H) & \\ = q_{\alpha+1/2}\partial_x z_{\alpha+1/2} - q_{\alpha-1/2}\partial_x z_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2} + \widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2}. & \end{aligned} \tag{27}$$

For (23), we have

$$\begin{aligned} \frac{1}{\rho h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} (z - z_\alpha)\partial_x p_\alpha \,\mathrm{d}z = \frac{1}{10}\partial_x\left(h_\alpha\left[\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right]\right) + \overline{q}_\alpha \partial_x z_\alpha & \\ + \frac{\partial_x h_\alpha}{10}\left(h_\alpha\left[\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right]\right) - \frac{q_{\alpha+1/2}\partial_x z_{\alpha+1/2} + q_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2}. & \end{aligned}$$

and then

$$\partial_t\left(\frac{h_\alpha^2\lambda_\alpha}{12}\right)+\partial_x\left(\frac{h_\alpha^2\lambda_\alpha\overline{u}_\alpha}{12}\right)+\frac{1}{10}\partial_x\left(h_\alpha\left[\frac{(\delta q)_\alpha}{2}+\frac{\pi_\alpha}{3}\right]\right)+\frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{u}_\alpha+\overline{q}_\alpha\partial_x z_\alpha$$
$$=-\frac{\partial_x h_\alpha}{10}\left(\frac{(\delta q)_\alpha}{2}+\frac{\pi_\alpha}{3}\right)+\frac{q_{\alpha+1/2}\partial_x z_{\alpha+1/2}+q_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2}$$
$$+\Gamma_{\alpha+1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}-\frac{\widetilde{u}_{\alpha+1/2}-\overline{u}_\alpha}{2}\right)-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}-\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha-1/2}}{2}\right),\quad(28)$$

for $\alpha=1,\dots,L$.

Let us focus now on the equations for the variables involved in the vertical velocity. To do so, we consider a basis in $\mathbb{P}_2[z]$ of test functions

$$\chi_\alpha(z)\in\left\{1,z-z_\alpha,\frac{(z-z_\alpha)^2}{2}-\frac{h_\alpha^2}{24}\right\},$$

and compute the averaged equations

$$\partial_t\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}w_\alpha\chi_\alpha\,\mathrm{d}z\right)+\partial_x\left(\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}u_\alpha w_\alpha\chi_\alpha\,\mathrm{d}z\right)$$
$$+\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\left(\chi_\alpha\partial_z q_\alpha-w_\alpha\partial_t\chi_\alpha-u_\alpha w_\alpha\partial_x\chi_\alpha-w_\alpha^2\partial_z\chi_\alpha\right)\mathrm{d}z$$
$$=-\chi_{\alpha|_{z_{\alpha+1/2}}}\widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2}+\chi_{\alpha|_{z_{\alpha-1/2}}}\widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2}.\quad(29)$$

- When considering $\chi_\alpha(z)=1$ in (29), we obtain a standard evolution equation for $\overline{w}_\alpha$:

$$\partial_t\left(h_\alpha\overline{w}_\alpha\right)+\partial_x\left(h_\alpha\overline{u}_\alpha\overline{w}_\alpha+\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{12}\right)$$
$$=q_{\alpha-1/2}-q_{\alpha+1/2}-\widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2}+\widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2}.\quad(30)$$

- If $\chi_\alpha(z)=z-z_\alpha$ in (29), using (17), we obtain

$$\partial_t\left(\frac{h_\alpha^3\varphi_\alpha}{12}\right)+\partial_x\left(\frac{h_\alpha^3\varphi_\alpha u_\alpha}{12}+\frac{h_\alpha^5\lambda_\alpha\psi_\alpha}{360}\right)+\frac{h_\alpha^3\varphi_\alpha}{12}\partial_x\overline{u}_\alpha+\frac{h_\alpha^3\lambda_\alpha}{12}\partial_x\overline{w}_\alpha-\frac{h_\alpha^5\psi_\alpha^2}{720}+h_\alpha\left(\widehat{q}_\alpha-\overline{q}_\alpha\right)$$
$$=\frac{h_\alpha}{2}\left(\Gamma_{\alpha-1/2}\left(\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}\right)-\Gamma_{\alpha+1/2}\left(\widetilde{w}_{\alpha+1/2}-\overline{w}_\alpha\right)\right),\quad(31)$$

  where we recall that $\widehat{q}_\alpha$ is defined by (5). Taking into account the mass conservation equation (18) and dividing by h_α, it is rewritten as

$$\partial_t\left(\frac{h_\alpha^2\varphi_\alpha}{12}\right)+\partial_x\left(\frac{h_\alpha^2\varphi_\alpha u_\alpha}{12}+\frac{h_\alpha^4\lambda_\alpha\psi_\alpha}{360}\right)+\frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{w}_\alpha-\frac{h_\alpha^4\psi_\alpha^2}{720}+\frac{h_\alpha^3\lambda_\alpha\psi_\alpha}{360}\partial_x h_\alpha+\widehat{q}_\alpha-\overline{q}_\alpha$$
$$=\Gamma_{\alpha+1/2}\left(\frac{h_\alpha\varphi_\alpha}{12}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{2}\right)-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha\varphi_\alpha}{12}-\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{2}\right).\quad(32)$$

- For $\chi_\alpha(z)=(z-z_\alpha)^2/2-h_\alpha^2/24$, one gets

$$\partial_t\left(\frac{h_\alpha^5\psi_\alpha}{720}\right)+\partial_x\left(\frac{h_\alpha^5\psi_\alpha\overline{u}_\alpha}{720}+\frac{h_\alpha^5\varphi_\alpha\lambda_\alpha}{360}\right)$$
$$-\frac{h_\alpha^4\lambda_\alpha\varphi_\alpha}{72}\partial_x h_\alpha+\frac{h_\alpha^5\varphi_\alpha\psi_\alpha}{720}+\frac{h_\alpha^5\psi_\alpha\lambda_\alpha\partial_x z_\alpha}{360}+\frac{h_\alpha^2}{30}\left((\delta q)_\alpha-\pi_\alpha\right)$$
$$=h_\alpha^2\Gamma_{\alpha+1/2}\left(\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{12}\right)-h_\alpha^2\Gamma_{\alpha-1/2}\left(-\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{12}\right).\quad(33)$$

Again, we use the mass equation (18) and divide by h_α^2, so that it writes

$$\partial_t\left(\frac{h_\alpha^3\psi_\alpha}{720}\right)+\partial_x\left(\frac{h_\alpha^3\psi_\alpha\overline{u}_\alpha}{720}+\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{360}\right)-\frac{h_\alpha^2\lambda_\alpha\varphi_\alpha}{120}\partial_x h_\alpha+\frac{h_\alpha^3\varphi_\alpha\psi_\alpha}{240}$$
$$+\frac{(\delta q)_\alpha-\pi_\alpha}{30}=\Gamma_{\alpha+1/2}\left(\frac{h_\alpha^2\psi_\alpha}{360}+\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{12}\right)$$
$$-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha^2\psi_\alpha}{360}-\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{12}\right). \quad (34)$$

For the sake of simplicity, let us introduce the following notations:

$$\Lambda_\alpha=\frac{h_\alpha\lambda_\alpha}{2\sqrt{3}},\quad \Phi_\alpha=\frac{h_\alpha\varphi_\alpha}{2\sqrt{3}},\quad \Psi_\alpha=\frac{h_\alpha^2\psi_\alpha}{12\sqrt{5}}. \quad (35)$$

Hence the LIN-NH$_2$ model, which is made of (20,26,27,28,30,32,34), reads

$$\begin{cases}
\partial_t H+\partial_x\left(H\bar{\overline{u}}\right)=0,\\
\partial_t(h_\alpha\overline{u}_\alpha)+\partial_x\left(h_\alpha\overline{u}_\alpha^2+h_\alpha\Lambda_\alpha^2+h_\alpha\overline{q}_\alpha\right)+gh_\alpha\partial_x(z_b+H)\\
\qquad=q_{\alpha+1/2}\partial_x z_{\alpha+1/2}-q_{\alpha-1/2}\partial_x z_{\alpha-1/2}-\widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2}+\widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2},\\
\partial_t\left(h_\alpha\Lambda_\alpha\right)+\partial_x\left(h_\alpha\Lambda_\alpha\overline{u}_\alpha+h_\alpha\frac{\sqrt{3}}{5}\left(\frac{q_{\alpha+1/2}-q_{\alpha-1/2}}{2}+\frac{\pi_\alpha}{3}\right)\right)+h_\alpha\Lambda_\alpha\partial_x\overline{u}_\alpha\\
\qquad+\frac{\sqrt{3}}{5}\left(\frac{q_{\alpha+1/2}-q_{\alpha-1/2}}{2}+\frac{\pi_\alpha}{3}\right)\partial_x h_\alpha-\sqrt{3}\left(q_{\alpha+1/2}\partial_x z_{\alpha+1/2}+q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\right)\\
\qquad+2\sqrt{3}\,\overline{q}_\alpha\partial_x z_\alpha=\Gamma_{\alpha+1/2}\left(\Lambda_\alpha-\sqrt{3}(\widetilde{u}_{\alpha+1/2}-\overline{u}_\alpha)\right)\\
\qquad\qquad-\Gamma_{\alpha-1/2}\left(\Lambda_\alpha-\sqrt{3}(\overline{u}_\alpha-\widetilde{u}_{\alpha-1/2})\right),\\
\partial_t\left(h_\alpha\overline{w}_\alpha\right)+\partial_x\left(h_\alpha\overline{u}_\alpha\overline{w}_\alpha+h_\alpha\Lambda_\alpha\Phi_\alpha\right)=-q_{\alpha+1/2}+q_{\alpha-1/2}-\widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2}+\widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2},\\
\partial_t\left(h_\alpha\Phi_\alpha\right)+\partial_x\left(h_\alpha\Phi_\alpha\overline{u}_\alpha+\frac{2\sqrt{5}}{5}h_\alpha\Lambda_\alpha\Psi_\alpha\right)+h_\alpha\Lambda_\alpha\partial_x\overline{w}_\alpha-2\sqrt{3}\Psi_\alpha^2+\frac{2\sqrt{5}}{5}\Lambda_\alpha\Psi_\alpha\partial_x h_\alpha\\
\qquad+2\sqrt{3}\left(\frac{q_{\alpha+1/2}+q_{\alpha-1/2}}{2}-\overline{q}_\alpha\right)=\Gamma_{\alpha+1/2}\left(\Phi_\alpha-\sqrt{3}(\widetilde{w}_{\alpha+1/2}-\overline{w}_\alpha)\right)\\
\qquad\qquad-\Gamma_{\alpha-1/2}\left(\Phi_\alpha-\sqrt{3}(\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2})\right),\\
\partial_t\left(h_\alpha\Psi_\alpha\right)+\partial_x\left(h_\alpha\Psi_\alpha\overline{u}_\alpha+\frac{2\sqrt{5}}{5}h_\alpha\Phi_\alpha\Lambda_\alpha\right)-\frac{6\sqrt{5}}{5}\Phi_\alpha\Lambda_\alpha\partial_x h_\alpha+6\sqrt{3}\Phi_\alpha\Psi_\alpha\\
\qquad+\frac{2\sqrt{5}}{5}\left(q_{\alpha+1/2}-q_{\alpha-1/2}-\pi_\alpha\right)=\Gamma_{\alpha+1/2}\left(2\Psi_\alpha+\sqrt{15}\Phi_\alpha+\sqrt{5}(\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2})\right)\\
\qquad\qquad-\Gamma_{\alpha-1/2}\left(2\Psi_\alpha-\sqrt{15}\Phi_\alpha+\sqrt{5}(\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2})\right),
\end{cases} \quad (36)$$

for $\alpha = 1, \dots, L$, combined with the following constraints,

$$\begin{cases} h_\alpha \partial_x \overline{u}_\alpha + 2\sqrt{3}\Phi_\alpha - 2\sqrt{3}\Lambda_\alpha \partial_x z_\alpha = 0, & \alpha = 1, \dots, L, \\ \dfrac{2\sqrt{5}}{5}\Psi_\alpha + \dfrac{\sqrt{3}}{15}\left(h_\alpha \partial_x \Lambda_\alpha - \Lambda_\alpha \partial_x h_\alpha\right) = 0, & \alpha = 1, \dots, L, \\ \overline{w}_{\alpha+1} - \sqrt{3}\Phi_{\alpha+1} + \sqrt{5}\Psi_{\alpha+1} - \overline{w}_\alpha - \sqrt{3}\Phi_{\alpha+1} - \sqrt{5}\Psi_{\alpha+1} & \\ \qquad = (\overline{u}_{\alpha+1} - \sqrt{3}\Lambda_{\alpha+1}\overline{u}_\alpha - \sqrt{3}\Lambda_{\alpha+1})\partial_x z_{\alpha+1/2} & \alpha = 1, \dots, L-1, \\ \partial_x \overline{u}_1 - \dfrac{2\sqrt{3}}{h_1}\Lambda_1 \partial_x z_1 - \dfrac{1}{\sqrt{3}}(\partial_x \Lambda_1 - \Lambda_1 \partial_x h_1) + \dfrac{\overline{w}_1 - w^+_{1/2}}{h_1/2} = 0, & \end{cases} \tag{37}$$

where

$$w^+_{1/2} = (\overline{u}_1 - \sqrt{3}\,\Lambda_1)\partial_x z_b.$$

In System (36) the velocities at the interfaces are given by

$$\begin{cases} w^{\mp}_{\alpha\pm1/2} = \overline{w}_\alpha \pm \sqrt{3}\Phi_\alpha + \sqrt{5}\Psi_\alpha, \\ u^{\mp}_{\alpha\pm1/2} = \overline{u}_\alpha \pm \sqrt{3}\Lambda_\alpha, \end{cases} \tag{38}$$

and therefore

$$\widetilde{u}_{\alpha+1/2} = \frac{\overline{u}_\alpha + \overline{u}_{\alpha+1}}{2} + \sqrt{3}\,\frac{\Lambda_\alpha - \Lambda_{\alpha+1}}{2}$$

and

$$\widetilde{w}_{\alpha+1/2} = \frac{\overline{w}_\alpha + \overline{w}_{\alpha+1}}{2} + \sqrt{3}\,\frac{\Phi_\alpha - \Phi_{\alpha+1}}{2} + \sqrt{5}\,\frac{\Psi_\alpha + \Psi_{\alpha+1}}{2}.$$

The system is thus composed of $8\,L + 1$ equations for $8\,L + 1$ unknowns,

$$\left(H, \{\,\overline{u}_\alpha, \Lambda_\alpha, \overline{w}_\alpha, \Phi_\alpha, \Psi_\alpha, \overline{q}_\alpha, q_{\alpha-1/2}, \pi_\alpha\}_{\alpha=1}^{L}\right).$$

Finally, let us rewrite System (36) and Constraints (37) in a compact way which will have the same structure for all models LIN-NH$_k$:

$$\begin{cases} \partial_t H + \partial_x\left(H\bar{\bar{u}}\right) = 0, & \text{(39a)} \\ \partial_t(h_\alpha \mathbf{X}_\alpha) + \partial_x(h_\alpha \mathbf{X}_\alpha \overline{u}_\alpha) + \mathbf{F}_\alpha + \nabla_{NH}\mathbf{Q}_\alpha = \mathbf{S}_\alpha \partial_x\left(z_b + H\right) + \Gamma_{\alpha+1/2}\mathbf{G}^+_\alpha - \Gamma_{\alpha-1/2}\mathbf{G}^-_\alpha, & \text{(39b)} \\ \nabla_{NH} \cdot \mathbf{X}_\alpha = 0, & \text{(39c)} \end{cases}$$

where we introduced some notations, namely the vectors of unknowns

$$\mathbf{X}_\alpha = \begin{pmatrix} \overline{u}_\alpha \\ \Lambda_\alpha \\ \overline{w}_\alpha \\ \Phi_\alpha \\ \Psi_\alpha \end{pmatrix}, \qquad \mathbf{Q}_\alpha = \begin{pmatrix} \overline{q}_\alpha \\ q_{\alpha-1/2} \\ \pi_\alpha \end{pmatrix},$$

where

$$\mathbf{F}_\alpha = \begin{pmatrix} \partial_x(h_\alpha \Lambda_\alpha^2) \\ h_\alpha \Lambda_\alpha \partial_x \overline{u}_\alpha \\ \partial_x(h_\alpha \Lambda_\alpha \Phi_\alpha) \\ \frac{2\sqrt{5}}{5}\partial_x(h_\alpha \Lambda_\alpha \Psi_\alpha) + h_\alpha \Lambda_\alpha \partial_x \overline{w}_\alpha + \frac{2\sqrt{5}}{5}\Lambda_\alpha \Psi_\alpha \partial_x h_\alpha \\ \frac{2\sqrt{5}}{5}\partial_x(h_\alpha \Lambda_\alpha \Phi_\alpha) - \frac{6\sqrt{5}}{5}\Lambda_\alpha \Phi_\alpha \partial_x h_\alpha + 6\sqrt{3}\Phi_\alpha \Psi_\alpha \end{pmatrix}, \qquad \mathbf{S}_\alpha = \begin{pmatrix} -g h_\alpha \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{G}_{\alpha}^{\pm} = \begin{pmatrix} -\widetilde{u}_{\alpha\pm1/2} \\ \Lambda_\alpha \pm \sqrt{3}(\overline{u}_\alpha - \widetilde{u}_{\alpha\pm1/2}) \\ -\widetilde{w}_{\alpha\pm1/2} \\ \Phi_\alpha \pm \sqrt{3}(\overline{w}_\alpha - \widetilde{w}_{\alpha\pm1/2}) \\ 2\Psi_\alpha \pm \sqrt{15}\Phi_\alpha + \sqrt{5}(\overline{w}_\alpha - \widetilde{w}_{\alpha\pm1/2}) \end{pmatrix}$$

and the following differential operators, for $\alpha \in \{1, \dots, L\}$

$$\nabla_{NH}\mathbf{Q}_\alpha = \begin{pmatrix} \partial_x(h_\alpha \overline{q}_\alpha) - \left(\delta\left(q\partial_x z\right)\right)_\alpha \\ \frac{\sqrt{3}}{5}\left[\partial_x\left(h_\alpha\left(\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right)\right) + 10\overline{q}_\alpha\partial_x z_\alpha + \left(\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right)\partial_x h_\alpha - 10\widehat{(q\partial_x z)}_\alpha\right] \\ (\delta q)_\alpha \\ 2\sqrt{3}\left(\widehat{q}_\alpha - \overline{q}_\alpha\right) \\ \frac{2\sqrt{5}}{5}\left((\delta q)_\alpha - \pi_\alpha\right) \end{pmatrix}.$$

Then for $\alpha \in \{2, \dots, L\}$ we define

$$\nabla_{NH}\cdot\mathbf{X}_\alpha = \left(\begin{array}{c} h_\alpha\partial_x\overline{u}_\alpha + 2\sqrt{3}\Phi_\alpha - 2\sqrt{3}\Lambda_\alpha\partial_x z_\alpha \\ \hline \begin{array}{c}\overline{w}_\alpha - \overline{w}_{\alpha-1} - (\overline{u}_\alpha - \overline{u}_{\alpha-1})\partial_x z_{\alpha-1/2} - \sqrt{3}(\Phi_\alpha + \Phi_{\alpha-1}) \\ +\frac{2\sqrt{5}}{5}(\Psi_\alpha - \Psi_{\alpha-1}) + \sqrt{3}(\Lambda_\alpha + \Lambda_{\alpha-1})\partial_x z_{\alpha-1/2} \\ +\frac{\sqrt{3}}{10}(\Lambda_\alpha\partial_x h_\alpha - h_\alpha\partial_x\Lambda_\alpha - \Lambda_{\alpha-1}\partial_x h_{\alpha-1} + h_{\alpha-1}\partial_x\Lambda_{\alpha-1})\end{array} \\ \hline \frac{1}{5}\left[2\sqrt{5}\Psi_\alpha + \frac{\sqrt{3}}{3}\left(h_\alpha\partial_x\Lambda_\alpha - \Lambda_\alpha\partial_x h_\alpha\right)\right] \end{array}\right)$$

and finally

$$\nabla_{NH}\cdot\mathbf{X}_1 = \left(\begin{array}{c} h_1\partial_x\overline{u}_1 + 2\sqrt{3}\Phi_1 - 2\sqrt{3}\Lambda_1\partial_x z_1 \\ \hline \overline{w}_1 - \overline{u}_1\partial_x z_b + \sqrt{3}(\Lambda_1\partial_x z_b - \Phi_1) + \frac{\sqrt{3}}{10}(\Lambda_1\partial_x h_1 - h_1\partial_x\Lambda_1) + \frac{2\sqrt{5}}{5}\Psi_1 \\ \hline \frac{2\sqrt{5}}{5}\Psi_1 + \frac{\sqrt{3}}{15}\left(h_1\partial_x\Lambda_1 - \Lambda_1\partial_x h_1\right) \end{array}\right).$$

These definitions of the NH-gradient and the NH-divergence operators satisfy the following duality relation,

$$\sum_{\alpha=1}^{L}\mathbf{X}_\alpha\cdot\left(\nabla_{NH}\mathbf{Q}_\alpha\right) = -\sum_{\alpha=1}^{L}\mathbf{Q}_\alpha\cdot\left(\nabla_{NH}\cdot\mathbf{X}_\alpha\right) + \partial_x\left(\sum_{\alpha=1}^{L}h_\alpha\left[\overline{q}_\alpha\overline{u}_\alpha + \frac{\sqrt{3}}{5}\Lambda_\alpha\left(\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right)\right]\right). \quad (40)$$

Remark 1 *Notice that setting $\Lambda_\alpha = 0$, $\Psi_\alpha = 0$ and $\pi_\alpha = 0$, System (39) reduces to the $LDNH_2(L)$ model described in [8] where the assumptions were: a layerwise-constant horizontal velocity, a layerwise-linear vertical velocity and a quadratic pressure.*

An interesting property of such a system is that it satisfies a dissipative energy balance, as stated by the following theorem.

Theorem 2 *The proposed LIN-NH$_2$ model, defined by (39) satisfies the dissipative energy balance*

$$\partial_t\left(\sum_{\alpha=1}^{L} h_\alpha E_\alpha\right) + \partial_x\left(\sum_{\alpha=1}^{L} h_\alpha\left[\overline{u}_\alpha\left(E_\alpha + g\frac{H}{2} + \Lambda_\alpha^2 + \overline{q}_\alpha\right) + \Phi_\alpha\Lambda_\alpha\overline{w}_\alpha \right.\right.$$
$$\left.\left. + \frac{2\sqrt{5}}{5}\Phi_\alpha\Psi_\alpha\Lambda_\alpha + \frac{\sqrt{3}}{5}\Lambda_\alpha\left(\frac{(\delta q)_\alpha}{2} + \frac{\pi_\alpha}{3}\right)\right]\right) \leq 0,$$

where

$$E_\alpha := \frac{|\mathbf{X}_\alpha|^2}{2} + g\left(z_b + \frac{H}{2}\right).$$

Proof: The proof relies on algebraic relations. Multiplying (39b) by $\mathbf{X}_\alpha$, we then use the duality relation (40), the divergence constraint (39c) and the following equalities:

$$\mathbf{X}_\alpha \cdot [\partial_t(h_\alpha\mathbf{X}_\alpha) + \partial_x(h_\alpha\mathbf{X}_\alpha\overline{u}_\alpha)] = \partial_t\left(h_\alpha\frac{|\mathbf{X}_\alpha|^2}{2}\right) + \partial_x\left(h_\alpha\frac{|\mathbf{X}_\alpha|^2}{2}\overline{u}_\alpha\right) + \frac{|\mathbf{X}_\alpha|^2}{2}\left(\Gamma_{\alpha-1/2} - \Gamma_{\alpha+1/2}\right),$$
$$\mathbf{F}_\alpha \cdot \mathbf{X}_\alpha = \partial_x\left(h_\alpha\Lambda_\alpha\left[\Lambda_\alpha\overline{u}_\alpha + \Phi_\alpha\overline{w}_\alpha + \frac{2\sqrt{5}}{5}\Phi_\alpha\Psi_\alpha\right]\right),$$

and

$$\mathbf{S}_\alpha \cdot \mathbf{X}_\alpha\partial_x\left(z_b + H\right) = \partial_x\left(\mathbf{S}_\alpha \cdot \mathbf{X}\left(z_b + H\right)\right) + g\left(z_b + H\right)\partial_x\left(h_\alpha\overline{u}_\alpha\right)$$
$$= \partial_x\left(\mathbf{S}_\alpha \cdot \mathbf{X}\left(z_b + H\right)\right) + g\left(z_b + H\right)\left(\Gamma_{\alpha-1/2} - \Gamma_{\alpha+1/2}\right) - g\left(z_b + H\right)\partial_t h_\alpha,$$

where

$$g\left(z_b + H\right)\partial_t h_\alpha = g\left(z_b + \frac{H}{2}\right)\partial_t h_\alpha + g\frac{H}{2}\partial_t h_\alpha = \partial_t\left(h_\alpha\, g\left(z_b + \frac{H}{2}\right)\right) - \frac{g}{2}\left(h_\alpha\partial_t H - H\partial_t h_\alpha\right).$$

and, given the expressions of interface velocities (38),

$$\Gamma_{\alpha+1/2}\mathbf{G}_\alpha^+ - \Gamma_{\alpha-1/2}\mathbf{G}_\alpha^- + \frac{|\mathbf{X}_\alpha|^2}{2}(\Gamma_{\alpha+1/2} - \Gamma_{\alpha-1/2})$$
$$= \Gamma_{\alpha+1/2}\left[u^-_{\alpha+1/2}\left(\frac{u^-_{\alpha+1/2}}{2} - \widetilde{u}_{\alpha+1/2}\right) + w^-_{\alpha+1/2}\left(\frac{w^-_{\alpha+1/2}}{2} - \widetilde{w}_{\alpha+1/2}\right)\right]$$
$$- \Gamma_{\alpha-1/2}\left[u^+_{\alpha-1/2}\left(\frac{u^+_{\alpha-1/2}}{2} - \widetilde{u}_{\alpha-1/2}\right) + w^+_{\alpha-1/2}\left(\frac{w^+_{\alpha-1/2}}{2} - \widetilde{w}_{\alpha-1/2}\right)\right].$$

Summing the last equality over all layers, we get

$$\sum_{\alpha=1}^{L}\Gamma_{\alpha-1/2}\left[\left(\widetilde{u}_{\alpha-1/2} - \frac{u^-_{\alpha-1/2} + u^+_{\alpha-1/2}}{2}\right)[u]_{\alpha-1/2} + \left(\widetilde{w}_{\alpha-1/2} - \frac{w^-_{\alpha-1/2} + w^+_{\alpha-1/2}}{2}\right)[w]_{\alpha-1/2}\right] = 0$$

due to the definition[1] of $\widetilde{u}_{\alpha\pm1/2}$, $\widetilde{w}_{\alpha\pm1/2}$ and

$$\sum_{\alpha=1}^{L} g\left(z_b + H\right)\left(\Gamma_{\alpha-1/2} - \Gamma_{\alpha+1/2}\right) = 0, \qquad \sum_{\alpha=1}^{L}\frac{g}{2}\left(h_\alpha\partial_t H - H\partial_t h_\alpha\right) = 0,$$

which completes the proof. □

Notice that thanks to the fact of writing the compact form (39) and the duality relation (40) the proof of Theorem 2 has been notably shortened. However, for the sake of clarity and completeness, we give the detailed proof in Appendix A.

[1]Here, the average at interfaces is chosen to be the convex combination $(\frac{1}{2}, \frac{1}{2})$. A more general combination $(\gamma_{\alpha-1/2}, 1 - \gamma_{\alpha-1/2})$ would provide a negative term, which would ensure the decrease of the energy.

1-layer model

In the 1-layer case, the model may be written in Boussinesq form by expressing all variables in terms of the averaged horizontal velocity and the fluid height as the only unknowns of the system. As we have said previously, this will result in a system of equations that contains third order derivatives in space and time. See for example, [9], where it is shown that the LDNH$_2$ model with 1 layer coincides with the Serre-Green-Naghdi model. More explicitly, the model proposed in this work for 1-layer case may be written as follows,

$$
\begin{cases}
\partial_t H + \partial_x(H\overline{u}) = 0, \\
\partial_t(H\overline{u}) + \partial_x(H\overline{u}^2) + \partial_x(H\Lambda^2) + gH\partial_x(z_b + H) = -\partial_x(H\overline{q}) - q_{1/2}\partial_x z_b, \\
\partial_t(H\Lambda) + \partial_x(H\Lambda\overline{u}) + H\Lambda\partial_x\overline{u} = \dfrac{\sqrt{3}}{5}\partial_x\left(H\left(\dfrac{q_{1/2}}{2} - \dfrac{\pi}{3}\right)\right) - 2\sqrt{3}\,\overline{q}\partial_x\left(z_b + \dfrac{H}{2}\right) \\
\qquad + \dfrac{\sqrt{3}}{5}\left(\dfrac{q_{1/2}}{2} - \dfrac{\pi}{3}\right)\partial_x H + \sqrt{3}q_{1/2}\partial_x z_b, \\
\overline{w} = w_b - \dfrac{H}{2}\partial_x\overline{u} + \sqrt{3}\Lambda\partial_x\left(z_b + \dfrac{H}{3}\right) + \dfrac{H}{2\sqrt{3}}\partial_x\Lambda, \\
\Phi = -\dfrac{H\partial_x\overline{u}}{2\sqrt{3}} + \Lambda\partial_x\left(z_b + \dfrac{H}{2}\right), \\
\Psi = \dfrac{1}{2\sqrt{15}}\left(\Lambda\partial_x H - H\partial_x\Lambda\right), \\
q_{1/2} = H(\partial_t\overline{w} + \overline{u}\partial_x\overline{w}) + \partial_x(H\Lambda\Phi), \\
\overline{q} = \dfrac{q_{1/2}}{2} + \dfrac{1}{2\sqrt{3}}\left[H(\partial_t\Phi + \overline{u}\partial_x\Phi) + \dfrac{2\sqrt{5}}{5}\partial_x(H\Lambda\Psi) + H\Lambda\partial_x\overline{w} - 2\sqrt{3}\Psi^2 + \dfrac{2\sqrt{5}}{5}\Lambda\Psi\partial_x H\right], \\
\pi = -q_{1/2} + \dfrac{\sqrt{5}}{2}\left[H(\partial_t\Psi + \overline{u}\partial_x\Psi) + 6\sqrt{3}\Phi\Psi\right] + \partial_x(H\Phi\Lambda) - 3\Phi\Lambda\partial_x H.
\end{cases}
$$

Note that by a subsequent substitution of the different unknowns of the system into the second and third equations, we easily obtain a system formed by three equations with the unknowns H, $\overline{u}$ and Λ, which includes terms of third order derivatives in space and time.

3.3 Simplified non-hydrostatic models

We propose now to obtain two simplified models in order to reduce the complexity of the previous one. The main objective is to reduce the number of unknowns related to the pressure and consequently the number of constraints. From a numerical point of view, this would be important to reduce the computational cost. For example, if a projection method is considered, it will be needed to solve a linear system for the pressure where the number of equations is proportional to the number of pressure unknowns and layers. Then, for the simplified models, we consider a more straightforward structure for the vertical velocity and, therefore, the non-hydrostatic pressure. We remark that it is essential that, to obtain models with an associated exact energy balance, we must always consider the incompressibility equation without any simplification on the vertical velocity profile.

We mention that among the two simplified models presented below, named LIN-NH$_1$ and LIN-NH$_0$, the one corresponding to a layerwise constant vertical velocity (LIN-NH$_0$) does not produce good results concerning the dispersive properties. However, we find interesting to show these results for the sake of completeness in the study, since it is natural to wonder whether it is a reasonable simplification or not. A possible reason for the poor results for that model could be that there should be a relation between the degrees of the polynomials approximating horizontal and vertical velocities. This fact would suggest that the degree of the polynomial approximation for the vertical velocity should be at least the one for the horizontal velocity, and it cannot be not be smaller.

The derivation of these simplified models can be obtained by following a similar approach as the one carried out in Subsection 3.2 for LIN-NH$_2$ model. This results in a similar compact form (39). Hence analogous energy balances as in Theorem 2 are obtained. For the sake of brevity, we do not

give here these proofs, which can be easily carried out following the analogous steps described in the proof of Theorem 2.

3.3.1 LIN-NH$_1$ model: non-hydrostatic model with $w_\alpha \in \mathbb{P}_1[z]$

We consider first that the vertical velocity is a layerwise linear polynomial in z, $w_\alpha \in \mathbb{P}_1[z]$ to approximate the vertical momentum equation, that is,

$$w_\alpha(z) = \overline{w}_\alpha + \varphi_\alpha\,(z - z_\alpha) \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]. \tag{41}$$

Concerning the pressure, we consider a piecewise parabolic function, $q_\alpha \in \mathbb{P}_2[z]$, given by

$$q_\alpha(z) = \frac{3\overline{q}_\alpha - \widehat{q}_\alpha}{2} + (\delta q)_\alpha \frac{(z - z_\alpha)}{h_\alpha} + 6\,(\widehat{q}_\alpha - \overline{q}_\alpha)\,\frac{(z - z_\alpha)^2}{h_\alpha^2}, \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]. \tag{42}$$

Note that this pressure profile coincides with (14) by setting $\pi_\alpha = (\delta q)_\alpha$. Then, the unknowns related to the non-hydrostatic pressure are $\overline{q}_\alpha$, and $q_{\alpha-1/2}$ for $\alpha = 1, \ldots, L$. Recall that $q_{L+1/2}$ is a given data, usually assumed to vanish at the surface.

The system thus comprises $6\,L + 1$ equations for $6\,L + 1$ unknowns,

$$\left(H, \{\,\overline{u}_\alpha, \Lambda_\alpha, \overline{w}_\alpha, \Phi_\alpha, \overline{q}_\alpha, q_{\alpha-1/2}\}_{\alpha=1}^{L}\right).$$

Analogously to what is done for the LIN-NH$_2$ model, the LIN-NH$_1$ model can be written in a compact form (39), where in this case

$$\mathbf{X}_\alpha = \begin{pmatrix} \overline{u}_\alpha \\ \Lambda_\alpha \\ \overline{w}_\alpha \\ \Phi_\alpha \end{pmatrix}, \qquad \mathbf{Q}_\alpha = \begin{pmatrix} \overline{q}_\alpha \\ q_{\alpha-1/2} \end{pmatrix}, \tag{43}$$

with

$$\mathbf{F}_\alpha = \begin{pmatrix} \partial_x(h_\alpha \Lambda_\alpha^2) \\ h_\alpha \Lambda_\alpha \partial_x \overline{u}_\alpha \\ \partial_x(h_\alpha \Lambda_\alpha \Phi_\alpha) \\ h_\alpha \Lambda_\alpha \partial_x \overline{w}_\alpha + \frac{2\sqrt{5}}{5} \Lambda_\alpha \Psi_\alpha \partial_x h_\alpha \end{pmatrix}, \quad \mathbf{G}_\alpha^{\pm} = \begin{pmatrix} -\widetilde{u}_{\alpha\pm1/2} \\ \Lambda_\alpha \pm \sqrt{3}(\overline{u}_\alpha - \widetilde{u}_{\alpha\pm1/2}) \\ -\widetilde{w}_{\alpha\pm1/2} \\ \Phi_\alpha \pm \sqrt{3}(\overline{w}_\alpha - \widetilde{w}_{\alpha\pm1/2}), \end{pmatrix}, \quad \mathbf{S}_\alpha = \begin{pmatrix} -g h_\alpha \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$\widetilde{u}_{\alpha+1/2} = \frac{\overline{u}_\alpha + \overline{u}_{\alpha+1}}{2} + \sqrt{3}\,\frac{\Lambda_\alpha - \Lambda_{\alpha+1}}{2}, \quad \text{and} \quad \widetilde{w}_{\alpha+1/2} = \frac{\overline{w}_\alpha + \overline{w}_{\alpha+1}}{2} + \sqrt{3}\,\frac{\Phi_\alpha - \Phi_{\alpha+1}}{2}.$$

The following NH-gradient operator read in this case, for $\alpha \in \{1, \ldots, L\}$,

$$\nabla_{NH}\mathbf{Q}_\alpha = \begin{pmatrix} \partial_x(h_\alpha \overline{q}_\alpha) - (\delta\,(q\partial_x z))_\alpha \\ \frac{\sqrt{3}}{6}\left[\partial_x\left(h_\alpha(\delta q)_\alpha\right) + 12\overline{q}_\alpha \partial_x z_\alpha + (\delta q)_\alpha \partial_x h_\alpha - 12\widehat{(q\partial_x z)}_\alpha\right] \\ (\delta q)_\alpha \\ 2\sqrt{3}\,(\widehat{q}_\alpha - \overline{q}_\alpha) \end{pmatrix}$$

and the NH-divergence opertator, for $\alpha \in \{2, \dots, L\}$

$$\nabla_{NH} \cdot \mathbf{X}_\alpha = \begin{pmatrix} h_\alpha \partial_x \overline{u}_\alpha + 2\sqrt{3}\Phi_\alpha - 2\sqrt{3}\Lambda_\alpha \partial_x z_\alpha \\ \overline{w}_\alpha - \overline{w}_{\alpha-1} - (\overline{u}_\alpha - \overline{u}_{\alpha-1})\partial_x z_{\alpha-1/2} - \sqrt{3}(\Phi_\alpha + \Phi_{\alpha-1}) \\ +\sqrt{3}(\Lambda_\alpha + \Lambda_{\alpha-1})\partial_x z_{\alpha-1/2} + \dfrac{\sqrt{3}}{6}(\Lambda_\alpha \partial_x h_\alpha - h_\alpha \partial_x \Lambda_\alpha - \Lambda_{\alpha-1}\partial_x h_{\alpha-1} + h_{\alpha-1}\partial_x \Lambda_{\alpha-1}) \end{pmatrix}$$

and for $\alpha = 1$,

$$\nabla_{NH} \cdot \mathbf{X}_1 = \begin{pmatrix} h_1 \partial_x \overline{u}_1 + 2\sqrt{3}\Phi_1 - 2\sqrt{3}\Lambda_1 \partial_x z_1 \\ \overline{w}_1 - \overline{u}_1 \partial_x z_b + \sqrt{3}(\Lambda_1 \partial_x z_b - \Phi_1) + \dfrac{\sqrt{3}}{6}(\Lambda_1 \partial_x h_1 - h_1 \partial_x \Lambda_1) \end{pmatrix}.$$

These differential operators verify in this case the following duality relation,

$$\sum_{\alpha=1}^{L} \mathbf{X}_\alpha \cdot (\nabla_{NH} \mathbf{Q}_\alpha) = -\sum_{\alpha=1}^{L} \mathbf{Q}_\alpha \cdot (\nabla_{NH} \cdot \mathbf{X}_\alpha) + \partial_x \left(\sum_{\alpha=1}^{L} h_\alpha \left[\overline{q}_\alpha \overline{u}_\alpha + \frac{\sqrt{3}}{6} \Lambda_\alpha (\delta q)_\alpha \right] \right).$$

This simplified system satisfies a dissipative energy balance, that coincides with the result stated at Theorem 2 by setting $\Psi_\alpha = 0$ and $\pi_\alpha = (\delta q)_\alpha$. We obtain in this case the following result.

Theorem 3 *The proposed LIN-NH$_1$ model satisfies the dissipative energy balance*

$$\partial_t \left(\sum_{\alpha=1}^{L} h_\alpha E_\alpha \right) + \partial_x \left(\sum_{\alpha=1}^{L} h_\alpha \left[\overline{u}_\alpha \left(E_\alpha + g\frac{H}{2} + \Lambda_\alpha^2 + \overline{q}_\alpha \right) + \Phi_\alpha \Lambda_\alpha \overline{w}_\alpha + \frac{\sqrt{3}}{6} \Lambda_\alpha (\delta q)_\alpha \right] \right) \leq 0,$$

where

$$E_\alpha := \frac{|\mathbf{X}_\alpha|^2}{2} + g\left(z_b + \frac{H}{2} \right).$$

with $\mathbf{X}_\alpha$ is defined by (43).

The proof is analogous to LIN-NH$_2$ model.

3.3.2 LIN-NH$_0$ model: non-hydrostatic model with $w_\alpha \in \mathbb{P}_0[z]$

Finally, we consider even a simpler case where a layerwise constant vertical velocity, $w_\alpha \in \mathbb{P}_0[z]$, is assumed in the vertical momentum equation. Therefore, we have the structure

$$w_\alpha(z) = \overline{w}_\alpha \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}], \tag{44}$$

for the vertical velocity, and the pressure, which is linear $q_\alpha \in \mathbb{P}_1[z]$, is given by

$$q_\alpha(z) = \widehat{q}_\alpha + (\delta q)_\alpha \frac{(z - z_\alpha)}{h_\alpha}, \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]. \tag{45}$$

Now, we have only one pressure unknow by layer, $q_{\alpha-1/2}$. Making such assumptions is equivalent to considering

$$\overline{q}_\alpha = \widehat{q}_\alpha = \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} \quad \text{for } \alpha = 1, \dots, L. \tag{46}$$

LIN-NH$_0$ model can be written in a compact form (39), where in this case

$$\mathbf{X}_\alpha = \begin{pmatrix} \overline{u}_\alpha \\ \Lambda_\alpha \\ \overline{w}_\alpha \end{pmatrix}, \qquad \mathbf{Q}_\alpha = q_{\alpha-1/2}, \tag{47}$$

with

$$\mathbf{F}_\alpha = \begin{pmatrix} \partial_x(h_\alpha \Lambda_\alpha^2) \\ h_\alpha \Lambda_\alpha \partial_x \overline{u}_\alpha \\ \partial_x(h_\alpha \Lambda_\alpha \Phi_\alpha) \end{pmatrix}, \quad \mathbf{G}_\alpha^\pm = \begin{pmatrix} -\widetilde{u}_{\alpha\pm1/2} \\ \Lambda_\alpha \pm \sqrt{3}(\overline{u}_\alpha - \widetilde{u}_{\alpha\pm1/2}) \\ -\widetilde{w}_{\alpha\pm1/2} \end{pmatrix}, \quad \mathbf{S}_\alpha = \begin{pmatrix} -g h_\alpha \\ 0 \\ 0 \end{pmatrix},$$

where

$$\widetilde{u}_{\alpha+1/2} = \frac{\overline{u}_\alpha + \overline{u}_{\alpha+1}}{2} + \sqrt{3}\,\frac{\Lambda_\alpha - \Lambda_{\alpha+1}}{2}, \qquad \text{and} \qquad \widetilde{w}_{\alpha+1/2} = \frac{\overline{w}_\alpha + \overline{w}_{\alpha+1}}{2},$$

with the following NH-gradient operator for $\alpha \in \{1, \dots, L\}$

$$\nabla_{NH}\mathbf{Q}_\alpha = \begin{pmatrix} \partial_x(h_\alpha \widehat{q}_\alpha) - (\delta\,(q\partial_x z))_\alpha \\ \dfrac{\sqrt{3}}{6}\left[\partial_x\left(h_\alpha(\delta q)_\alpha\right) + 12\widehat{q}_\alpha \partial_x z_\alpha + (\delta q)_\alpha \partial_x h_\alpha - 12\widehat{(q\partial_x z)}_\alpha\right] \\ (\delta q)_\alpha \end{pmatrix}$$

and the NH-divergence opertator, for $\alpha \in \{2, \dots, L\}$

$$\begin{aligned} \nabla_{NH}\cdot\mathbf{X}_\alpha = \; & \overline{w}_\alpha - \overline{w}_{\alpha-1} - (\overline{u}_\alpha - \overline{u}_{\alpha-1})\partial_x z_{\alpha-1/2} - \frac{h_\alpha}{2}\partial_x \overline{u}_a - \frac{h_{\alpha-1}}{2}\partial_x \overline{u}_{\alpha-1} \\ & + \sqrt{3}\left(-\frac{5}{6}\Lambda_\alpha \partial_x h_\alpha - h_\alpha \partial_x \Lambda_\alpha - \frac{5}{6}\Lambda_{\alpha-1}\partial_x h_{\alpha-1} + h_{\alpha-1}\partial_x \Lambda_{\alpha-1}\right) \end{aligned}$$

and for $\alpha = 1$,

$$\nabla_{NH}\cdot\mathbf{X}_1 = \overline{w}_1 - \overline{u}_1 \partial_x z_b - h_\alpha \partial_x \overline{u}_1.$$

These differential operators verify in this case the following duality relation,

$$\sum_{\alpha=1}^{L} \mathbf{X}_\alpha \cdot (\nabla_{NH}\mathbf{Q}_\alpha) = -\sum_{\alpha=1}^{L} \mathbf{Q}_\alpha \cdot (\nabla_{NH}\cdot\mathbf{X}_\alpha) + \partial_x\left(\sum_{\alpha=1}^{L} h_\alpha \left[\widehat{q}_\alpha \overline{u}_\alpha + \frac{\sqrt{3}}{6}\Lambda_\alpha(\delta q)_\alpha\right]\right).$$

This simplified system satisfies a dissipative energy balance, that coincides with the result stated at Theorem 2 by setting $\Psi_\alpha = \Phi_\alpha = 0$, $\pi_\alpha = (\delta q)_\alpha$, $\overline{q}_\alpha = \widehat{q}_\alpha$.

4 Dispersion relations

In this section, we study the linear dispersion relations of the LIN-NH$_p$ systems, following the approach used in [6,17]. In particular, we focus on the linear dispersion relations for the wave celerity, the group velocity and the linear shoaling. In other words, a formal study is performed for the PDE systems introduced previously, focusing on the propagation of dispersive waves.

To do so, a flat bottom is assumed as usual (z_b constant), and the governing equations are linearised around a steady-state solution. Then, a standard Stokes-type Fourier analysis is used to obtain the wave and group velocities. Finally, a shoaling analysis of the linearised equations is carried out.

4.1 Linear dispersion relation for the LIN-NH$_2$ model

Let us begin with the LIN-NH$_2$ model that is described by (36). Let us linearise the system around the steady state

$$z_b = -H_0 = cst,\; u_0 = \lambda_0 = w_0 = \varphi_0 = \psi_0 = 0,$$

and let us consider $\eta = z_b + H$ the free-surface. We consider then the following asymptotic expansion

$$H = H_0 + \epsilon\eta^{(1)} + O(\epsilon^2), \tag{48}$$

and for any variable $\phi_\alpha \in \{\overline{u}_\alpha, \lambda_\alpha, \overline{w}_\alpha, \varphi_\alpha, \psi_\alpha\}$,

$$\phi_\alpha = \epsilon \phi_\alpha^{(1)} + O(\epsilon^2). \tag{49}$$

For the sake of simplicity, we shall neglect in this section the superindex (1) and bars $(\bar{\cdot})$ so that the notation is less cumbersome. Using this linearisation in (36), we shall neglect $O(\epsilon^2)$ terms and keep the system at first order:

$$\begin{cases}
\partial_t \eta + H_0 \partial_x \left(\displaystyle\sum_{\beta=1}^{\alpha} \ell_\beta u_\beta \right) = 0 & \text{(50a)} \\
\partial_t u_\alpha + \partial_x q_\alpha + g \partial_x \eta = 0, & \text{(50b)} \\
\dfrac{\ell_\alpha^2 H_0^2}{12} \partial_t \lambda_\alpha + \dfrac{\ell_\alpha H_0}{20} \partial_x (\delta q)_\alpha + \dfrac{\ell_\alpha H_0}{30} \partial_x \pi_\alpha = 0, & \text{(50c)} \\
\ell_\alpha H_0 \partial_t w_\alpha + (\delta q)_\alpha = 0, & \text{(50d)} \\
\dfrac{\ell_\alpha^2 H_0^2}{12} \partial_t \varphi_\alpha + \dfrac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} - q_\alpha = 0, & \text{(50e)} \\
\dfrac{\ell_\alpha^3 H_0^3}{720} \partial_t \psi_\alpha + \dfrac{(\delta q)_\alpha - \pi_\alpha}{30} = 0, & \text{(50f)} \\
\varphi_\alpha = -\partial_x u_\alpha, & \text{(50g)} \\
\psi_\alpha = -\partial_x \lambda_\alpha, & \text{(50h)} \\
w_\alpha = \dfrac{\ell_\alpha H_0}{2} \partial_x u_\alpha + \dfrac{\ell_\alpha H_0^2}{12} \partial_x \lambda_\alpha - \displaystyle\sum_{\beta=1}^{\alpha} \ell_\beta H_0 \partial_x u_\beta. & \text{(50i)}
\end{cases}$$

Remark that combining (50c), (50d), and (50i), we obtain

$$-(\delta q)_\alpha = \frac{\ell_\alpha^2 H_0^2}{2} \partial_{tx} u_\alpha - \ell_\alpha H_0 \sum_{\beta=1}^{\alpha} \ell_\beta H_0 \partial_{tx} u_\beta - \frac{\ell_\alpha^2 H_0^2}{20} \partial_{xx} (\delta q)_\alpha - \frac{\ell_\alpha^2 H_0^2}{30} \partial_{xx} \pi_\alpha. \tag{51}$$

Moreover, given that $q_{L+1/2} = 0$ and $(\delta q)_\alpha = q_{\alpha+1/2} - q_{\alpha-1/2}$ we have

$$q_{\alpha-1/2} = -\sum_{\beta=\alpha}^{L} (\delta q)_\beta.$$

Let us consider now a plane-wave solution of the linearised system in the form

$$(\widehat{\eta}, \widehat{u}_\alpha, \widehat{\lambda}_\alpha, \widehat{w}_\alpha, \widehat{\varphi}_\alpha, \widehat{\delta q}_\alpha, \widehat{\pi}_\alpha)^T e^{i(kx - \omega t)},$$

where here the hat notation $(\widehat{\cdot})$ is new and is not related to (5b). Inserting the plane wave into (51) we get

$$-(1 + d_\alpha)\, \widehat{\delta q}_\alpha = \frac{\ell_\alpha^2 H_0^2}{2} k\omega \widehat{u}_\alpha - \ell_\alpha H_0^2 k\omega \sum_{\beta=1}^{\alpha} \ell_\beta \widehat{u}_\beta, \quad d_\alpha = \frac{\ell_\alpha^2 H_0^2 k^2}{12 \left(1 + \frac{\ell_\alpha^2 H_0^2 k^2}{60}\right)}.$$

Doing so in (50e) gives

$$\widehat{q}_\alpha = -\omega i \frac{\ell_\alpha^2 H_0^2}{12} \widehat{\varphi}_\alpha + \widehat{q}_{\alpha+1/2} - \frac{\widehat{\delta q}_{\alpha+1}}{2} = -\omega i \frac{\ell_\alpha^2 H_0^2}{12} \widehat{\varphi}_\alpha - \sum_{\beta=\alpha+1}^{L} \widehat{\delta q}_\beta - \frac{\widehat{\delta q}_\alpha}{2}.$$

The same may be done in the other equations and we obtain, using the relation $\widehat{\varphi}_\alpha = -ik\widehat{u}_\alpha$,

$$\begin{cases} \omega\widehat{\eta} = kH_0 \sum_{\alpha=1}^{L} \ell_\alpha \widehat{u}_\alpha, \\ \omega\widehat{u}_\alpha = k\widehat{q}_\alpha + gk\widehat{\eta}, \\ (1+d_\alpha)\,\widehat{\delta q}_\alpha = -\frac{\ell_\alpha^2 H_0^2}{2} k\omega\widehat{u}_\alpha + \ell_\alpha H_0^2 k\omega \sum_{\beta=1}^{\alpha} \ell_\beta \widehat{u}_\beta \\ \widehat{q}_\alpha = -\omega k \frac{\ell_\alpha^2 H_0^2}{12}\widehat{u}_\alpha - \sum_{\beta=\alpha+1}^{L} \widehat{\delta q}_\beta - \frac{\widehat{\delta q}_\alpha}{2}. \end{cases} \tag{52}$$

Let us write the previous system under a linear system. First, consider the following vectors in $\mathbb{R}^L$

$$U = (\widehat{u}_1, \dots, \widehat{u}_L), \quad Q = (\widehat{q}_1, \dots, \widehat{q}_L)^T, \quad \delta Q = (\widehat{\delta q}_1, \dots, \widehat{\delta q}_L)^T, \quad l = (\ell_1, \dots, \ell_L)^T, \quad e = (1, \dots, 1)^T,$$

and the matrices of size $L \times L$

$$\mathcal{L} = \mathrm{diag}(\ell_1, \dots, \ell_L), \quad \mathcal{I} = \mathrm{diag}(1, \dots, 1), \quad \mathcal{I}_d = \mathrm{diag}(d_1, \dots, d_L),$$

$$\mathcal{T} = \begin{pmatrix} \ell_1 & 0 & 0 & \dots & 0 \\ \ell_1 & \ell_2 & 0 & \dots & 0 \\ \ell_1 & \ell_2 & \ell_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_1 & \ell_2 & \ell_3 & \dots & \ell_L \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Then (52) may be written as

$$\omega^2 U = k\omega Q + gk^2 H_0 \langle l, U\rangle e,$$

$$Q = \left(\frac{1}{2}\mathcal{I} - \mathcal{S}\right)\delta Q - \omega k \frac{H_0^2}{12}\mathcal{L}^2 U,$$

$$(\mathcal{I} + \mathcal{I}_d)\,\delta Q = -k\omega \frac{H_0^2}{2}\mathcal{L}^2 U + k\omega \mathcal{L}\mathcal{T} U,$$

where $\langle\cdot,\cdot\rangle$ represents the scalar product in $\mathbb{R}^L$ and, performing a progressive substitution from bottom to top, we obtain

$$\omega^2 \mathcal{A}_{2,L} U = gk^2 H_0 \langle l, U\rangle e,$$

with

$$\mathcal{A}_{2,L} = k^2 H_0^2 \left(\frac{1}{2}\mathcal{I} - \mathcal{S}\right)(\mathcal{I} + \mathcal{I}_d)^{-1}\left(\frac{1}{2}\mathcal{L}^2 - \mathcal{L}\mathcal{T}\right) + (\mathcal{I} + \mathcal{I}_d).$$

This means that, provided $\mathcal{A}_{2,L}$ is invertible,

$$\omega^2 \langle e, U\rangle = gk^2 H_0 \langle l, U\rangle \langle \mathcal{A}_{2,L}^{-1} e, e\rangle.$$

In the particular case $\ell_\alpha = \dfrac{1}{L}$, $\mathcal{A}_{2,L}$ reduces to

$$\mathcal{A}_{2,L} = \frac{k^2 H_0^2}{L^2}(1+d)^{-1}\left(\frac{1}{2}\mathcal{I} - \mathcal{S}\right)\left(\frac{1}{2}\mathcal{I} - \mathcal{S}^T\right) + (1+d)\,\mathcal{I}, \quad d = \frac{k^2 H_0^2}{12\left(L^2 + \dfrac{k^2 H_0^2}{60}\right)}, \tag{53}$$

and assuming $\langle e, U\rangle \neq 0$ we get

$$\omega^2 = g\frac{k^2 H_0}{L}\langle \mathcal{A}_{2,L}^{-1} e, e\rangle,$$

Therefore, the following dispersion relation for the wave celerity $c := \dfrac{\omega}{k}$ is obtained for LIN-NH$_2$ model

$$\frac{c_{2,L}^2(k, H_0)}{gH_0} = \frac{\omega^2}{gk^2 H_0} = \frac{1}{L}\langle \mathcal{A}_{2,L}^{-1} e, e\rangle.$$

4.2 Linear dispersion relation for the LIN-NH$_1$ model

In the particular case of LIN-NH$_1$, the equations that govern the system are those corresponding to (36) under the assumption of $(\delta q)_\alpha = \pi_\alpha$, as well as the restrictions (26).

Doing a linearisation as in (48) and (49) for $\phi_\alpha \in \{\overline{u}_\alpha, \lambda_\alpha, \overline{w}_\alpha, \varphi_\alpha\}$, keeping the first order terms and after inserting the plane wave solution

$$(\widehat{\eta}, \widehat{u}_\alpha, \widehat{\lambda}_\alpha, \widehat{w}_\alpha, \widehat{\varphi}_\alpha, \widehat{\delta q}_\alpha)^T e^{i(kx-\omega t)},$$

we arrive to System (52) where, in this case, $d_\alpha = \dfrac{\ell_\alpha^2 H_0^2 k^2}{12}$. That leads us, for the particular case $l_\alpha = \dfrac{1}{L}$, to the following dispersion relation for the wave celerity $c := \dfrac{\omega}{k}$:

$$\frac{c_{1,L}^2(k, H_0)}{gH_0} = \frac{\omega^2}{gk^2 H_0} = \frac{1}{L}\langle \mathcal{A}_{1,L}^{-1} e, e\rangle, \tag{54}$$

where $\mathcal{A}_{1,L}$ is similar to (53), but with a change on the value of d, more precisely:

$$\mathcal{A}_{1,L} = \frac{k^2 H_0^2}{L^2}(1+d)^{-1}\left(\frac{1}{2}\mathcal{I} - \mathcal{S}\right)\left(\frac{1}{2}\mathcal{I} - \mathcal{S}^T\right) + (1+d)\,\mathcal{I}, \quad d = \frac{k^2 H_0^2}{12L^2}. \tag{55}$$

4.3 Linear dispersion relation for the LIN-NH$_0$ model

Remark that LIN-NH$_0$ is quite similar to LIN-NH$_1$ model, where some equations are removed and

$$q_\alpha = \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2}.$$

This means that the linearisation (48) and (49) for $\phi_\alpha \in \{\overline{u}_\alpha, \lambda_\alpha, \overline{w}_\alpha\}$, keeping the first order terms, will result in a reduced version of (50). Then, following a similar procedure as in the previous subsections, we get that a plane wave solution

$$(\widehat{\eta}, \widehat{u}_\alpha, \widehat{\lambda}_\alpha, \widehat{w}_\alpha, \widehat{\delta q}_\alpha)^T e^{i(kx-\omega t)},$$

of this reduced linearised system should satisfy

$$\begin{cases} \omega\widehat{\eta} = kH_0 \displaystyle\sum_{\alpha=1}^{L} \ell_\alpha \widehat{u}_\alpha, \\ \omega\widehat{u}_\alpha = k\widehat{q}_\alpha + gk\widehat{\eta}, \\ \left(1 + \dfrac{\ell_\alpha^2 H_0^2 k^2}{12}\right)\widehat{\delta q}_\alpha = -\frac{\ell_\alpha^2 H_0^2}{2} k\omega\widehat{u}_\alpha + \ell_\alpha H_0^2 k\omega \sum_{\beta=1}^{\alpha} \ell_\beta \widehat{u}_\beta \\ \widehat{q}_\alpha = -\sum_{\beta=\alpha}^{L} \widehat{\delta q}_\beta - \frac{\widehat{\delta q}_\alpha}{2}, \end{cases}$$

that is the same as (52) except for the term $-\omega k\frac{\ell_\alpha^2 H_0^2}{12}\widehat{u}_\alpha$, which is now missing in the last equation. As a consequence, for the particular case of $\ell_\alpha = 1/L$, we will obtain a matrix $\mathcal{A}_{0,L}$ similar to $\mathcal{A}_{1,L}$ in (55), but with the only difference that the last term $(1+d)\,\mathcal{I}$ is replaced by $\mathcal{I}$. That is, the following dispersion relation for the wave celerity $c := \dfrac{\omega}{k}$ for LIN-NH$_0$ model is satisfied:

$$\frac{c_{0,L}^2(kH_0)}{gH_0} = \frac{\omega^2}{gk^2H_0} = \frac{1}{L}\langle \mathcal{A}_{0,L}^{-1}e, e\rangle,$$

where

$$\mathcal{A}_{0,L} = \frac{k^2H_0^2}{L^2}\left(1+\frac{k^2H_0^2}{12L^2}\right)^{-1}\left(\frac{1}{2}\mathcal{I}-\mathcal{S}\right)\left(\frac{1}{2}\mathcal{I}-\mathcal{S}^T\right)+\mathcal{I}.$$

4.4 Computation of the linear group velocity and the linear shoaling of dispersive systems

Let us introduce a general procedure that allows to compute the linear group velocity and shoaling for dispersive systems presented in this work.

Moreover, we provide a very general methodology that can be applied to compute the group velocity and the linear shoaling of any dispersive PDE system once the wave celerity $c^2(k, H_0)$ of the system is known.

4.4.1 Group velocity

The group velocity $c_g(k, H_0)$ is essentially computed by taking the derivative of the wave celerity c, and is defined as

$$c_g = c + k\partial_k c.$$

Let us express the ratio of the wave celerity of a given dispersive system and the shallow water celerity as a function of kH_0 :

$$\frac{c^2(k,H_0)}{gH_0} = f(kH_0). \tag{56}$$

For instance, for the LIN-NH$_1$ model, f is given in (54) as:

$$f(kH_0) = \frac{1}{L}\langle \mathcal{A}_{1,L}^{-1}e, e\rangle.$$

Therefore, the group velocity can be written as follows

$$c_g = \sqrt{gH_0}\Upsilon(kH_0), \quad \Upsilon(kH_0) = \frac{2f(kH_0)+kH_0f'(kH_0)}{2\sqrt{f(kH_0)}}, \tag{57}$$

where f' means the derivative of the function f w.r.t. its argument.

4.4.2 Linear shoaling

The linear shoaling gradient γ is a non-dimensional quantity to measure the change in wave height in the presence of a bottom slope. It was firstly introduced in [18] and can be expressed as

$$\frac{\partial_x\eta}{\eta} = -\gamma\frac{\partial_x H_0}{H_0}.$$

As usual, we will assume the dependency on the $x-$direction for H_0, η and the wave number k. On the contrary, we suppose that the frequency ω does not depends on x, and therefore $\partial_x\omega = 0$.

The linear shoaling gradient can be determined for a given dispersive PDE system by assuming the constancy of the energy flux

$$\partial_x\left(\eta^2 c_g\right) = 0,$$

and therefore

$$\frac{\partial_x \eta}{\eta} = -\frac{1}{2}\frac{\partial_x c_g}{c_g}. \tag{58}$$

Now, from (57), it yields

$$\frac{\partial_x c_g}{c_g} = \left(\frac{1}{2} + kH_0\frac{\Upsilon'(kH_0)}{\Upsilon(kH_0)}\right)\frac{\partial_x H_0}{H_0} + kH_0\frac{\Upsilon'(kH_0)}{\Upsilon(kH_0)}\frac{\partial_x k}{k}, \tag{59}$$

where again Υ' stands for the derivative of the function Υ w.r.t. its argument.

Given the definition of the celerity $c = \dfrac{\omega}{k}$ and the constancy of ω with respect to x, it follows that

$$\frac{\partial_x k}{k} = -\frac{\partial_x c}{c},$$

and from (56), we get

$$\frac{\partial_x k}{k} = -\frac{\partial_x c}{c} = -\left(\frac{1}{2} + kH_0\frac{f'(kH_0)}{f(kH_0)}\right)\frac{\partial_x H_0}{H_0} - kH_0\frac{f'(kH_0)}{f(kH_0)}\frac{\partial_x k}{k}.$$

Therefore, from the above relation we can express $\dfrac{\partial_x k}{k}$ as

$$\frac{\partial_x k}{k} = \left(\frac{1}{2}\frac{f(kH_0)}{f(kH_0) + kH_0 f'(kH_0)} - 1\right)\frac{\partial_x H_0}{H_0}. \tag{60}$$

Finally, we can use (58), (59) and (60) to obtain the linear shoaling gradient of a given dispersive PDE system:

$$\frac{\partial_x \eta}{\eta} = -\gamma(kH_0)\frac{\partial_x H_0}{H_0}, \quad \gamma(kH_0) = \frac{1}{4}\left(1 + kH_0\frac{\Upsilon'(kH_0)}{\Upsilon(kH_0)}\cdot\frac{f(kH_0)}{f(kH_0) + kH_0 f'(kH_0)}\right). \tag{61}$$

4.5 Comparison of the dispersion relations

Let us assess the different models in the framework of the Stokes linear theory, also called Airy theory, which establishes

$$\frac{c_{Airy}^2}{gH_0} = \frac{\tanh(kH_0)}{kH_0}, \qquad \frac{c_{g,Airy}^2}{gH_0} = \frac{(2kH_0 + \sinh(2kH_0))^2}{2kH_0(2\sinh(2kH_0) + \sinh(4kH_0))},$$

and

$$\gamma_{Airy} = kH_0\tanh(kH_0)\frac{(1 - kH_0\tanh(kH_0))\left(1 - \tanh^2(kH_0)\right)}{\left(\tanh(kH_0) + kH_0\left(1 - \tanh^2(kH_0)\right)\right)^2}.$$

In what follows, we consider the relative dispersion error for the celerity and group celerity of the computed systems

$$\frac{c^2 - c_{Airy}^2}{c_{Airy}^2}, \qquad \frac{c_g^2 - c_{g,Airy}^2}{c_{Airy}^2},$$

where c and c_g stands for the wave celerity and group velocity of a given dispersive system respectively. Concerning the shoaling gradient, we directly compare, as usually done (see [6]), the coefficients from the Airy theory, γ_{Airy}, and the one given in the previous subsection γ for a given dispersive system.

For the sake of clarity, we recall in Table 1 the models that are analysed here, as well as the approximation spaces for the velocity a pressure fields.

First, let us remark that it can be stated from Figures 2 and 3 that the dispersion relations of the models studied here converge to the one given by the Stokes linear theory for the increasing number

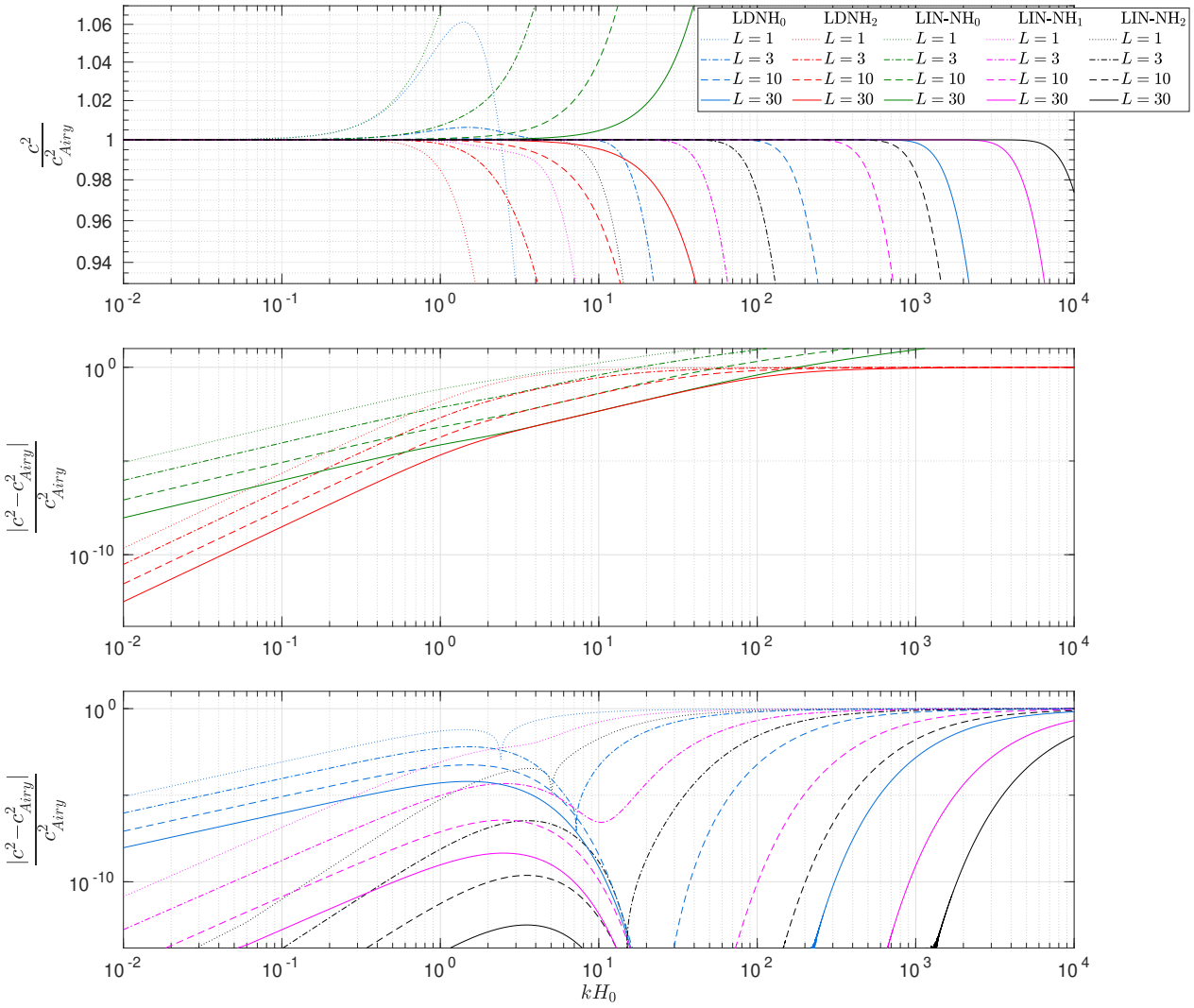

Figure 2: Comparison of the relative dispersion ratio and errors in logarithmic scale.

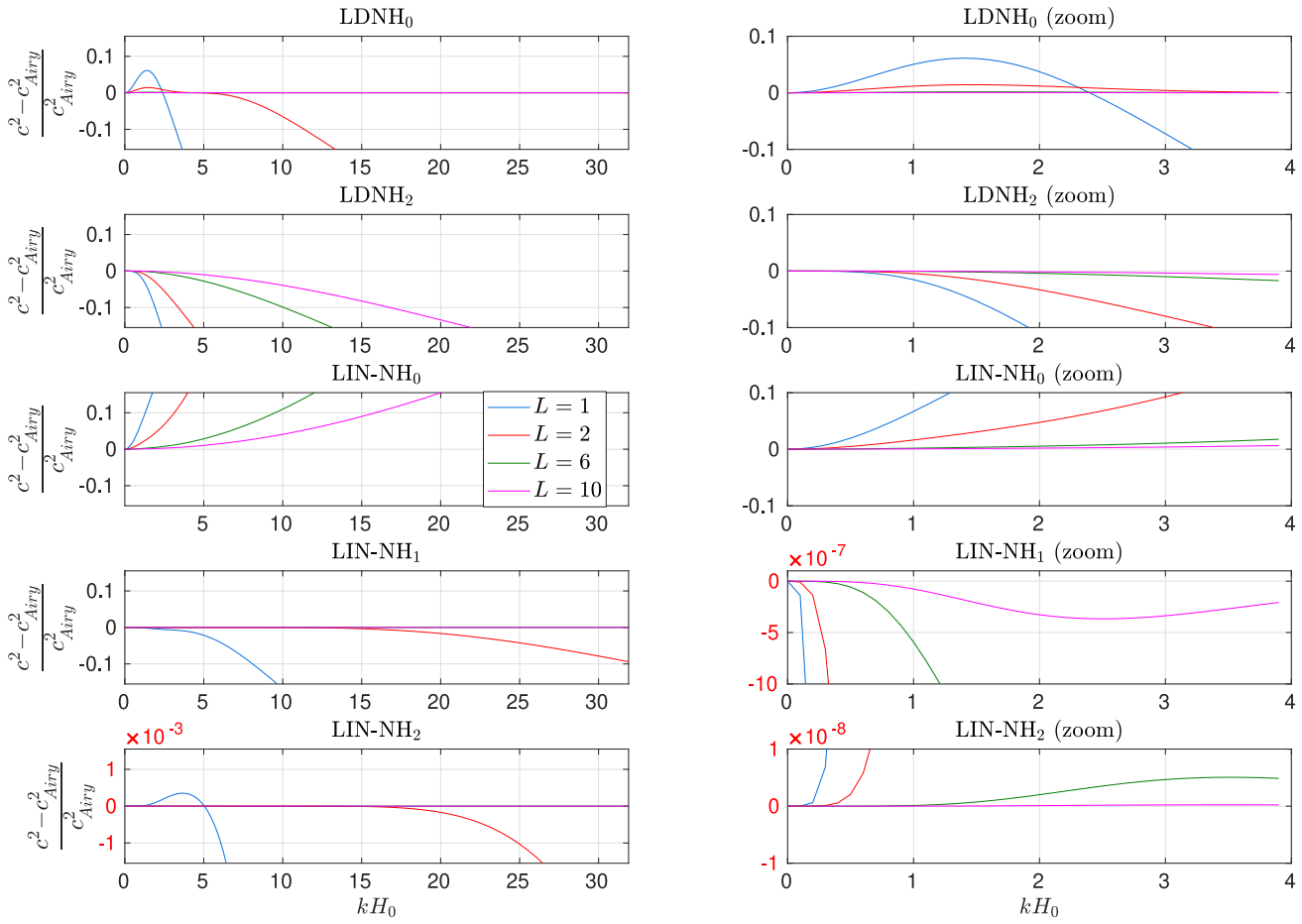

Figure 3: Comparison of the relative dispersion errors for the wave celerity for each model and different number of layers.

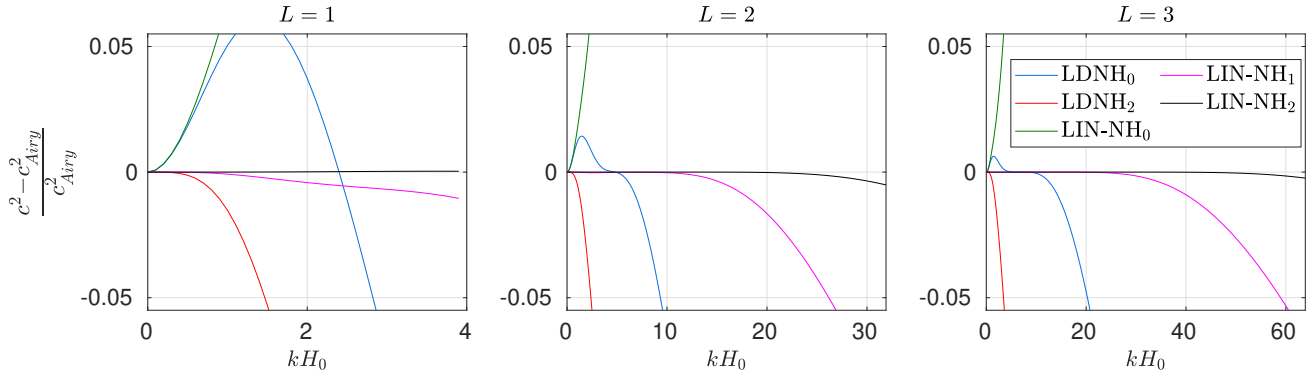

Figure 4: Intercomparison between the relative dispersion errors for the wave celerity of different models, for a fixed number of layers $L = 1, 2, 3$.

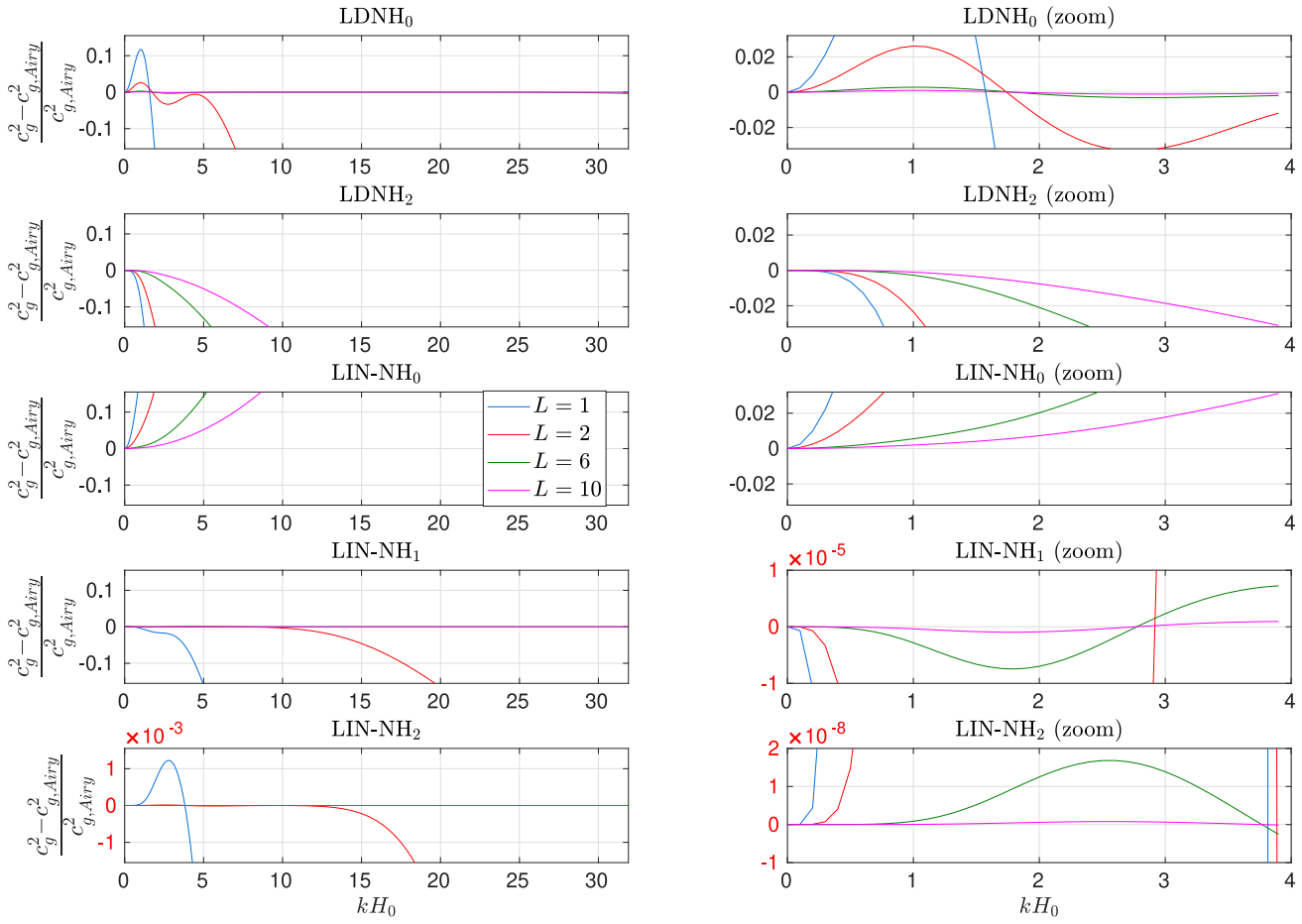

Figure 5: Comparison of the relative dispersion errors for the group velocity for each model and different number of layers.

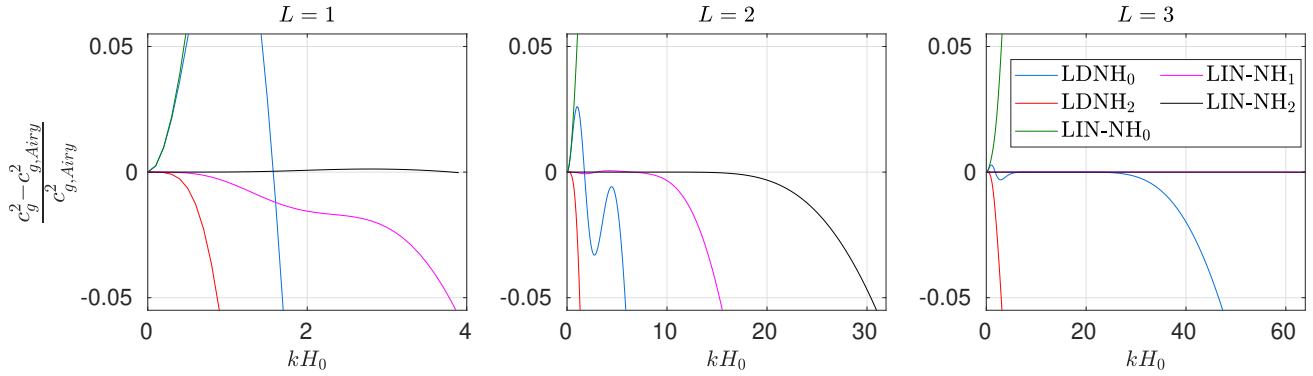

Figure 6: Intercomparison between the relative dispersion errors for the group velocity of different models, for a fixed number of layers $L = 1, 2, 3$.

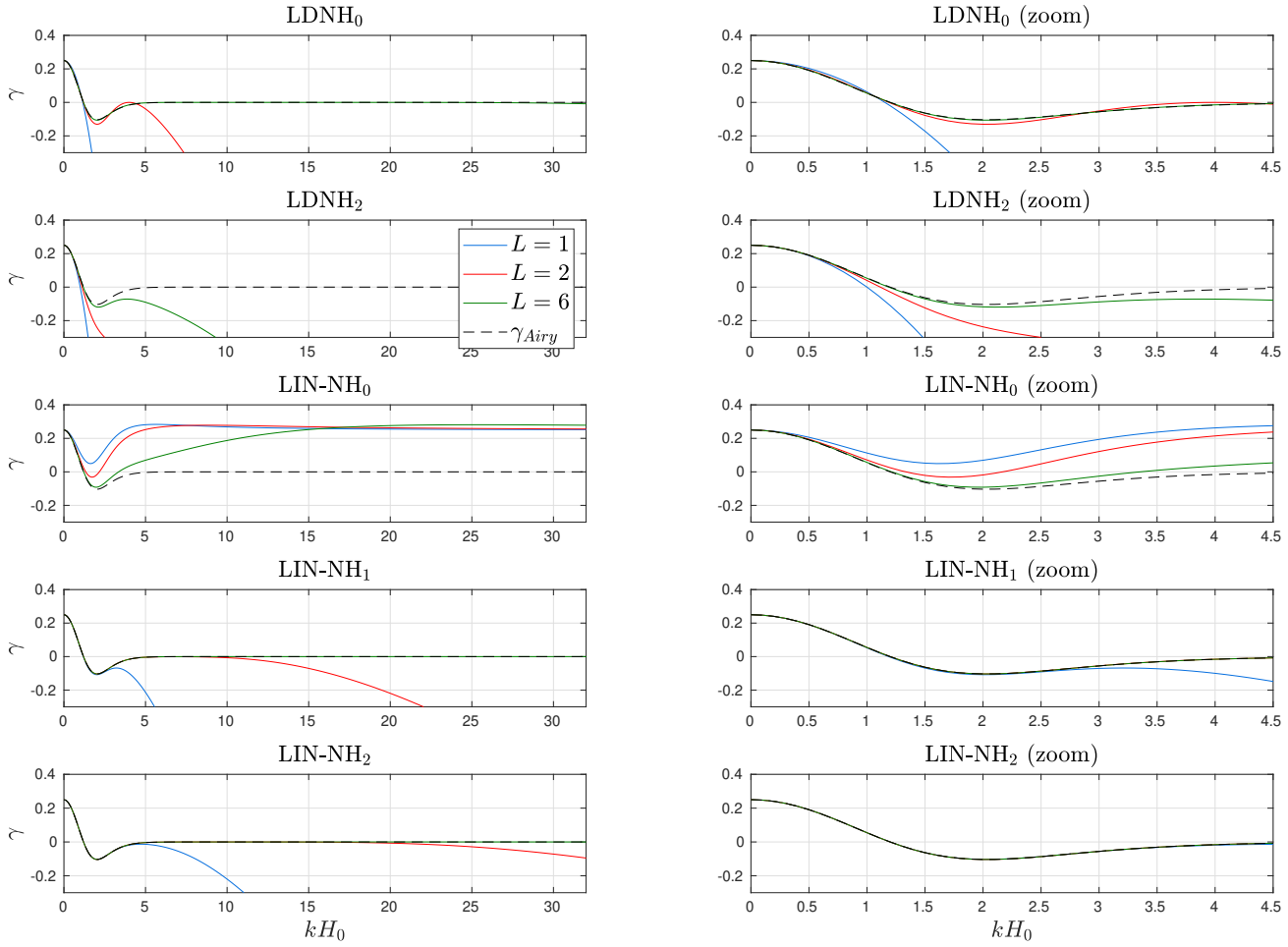

Figure 7: Comparison of the shoaling gradient of each model and different number of layers.

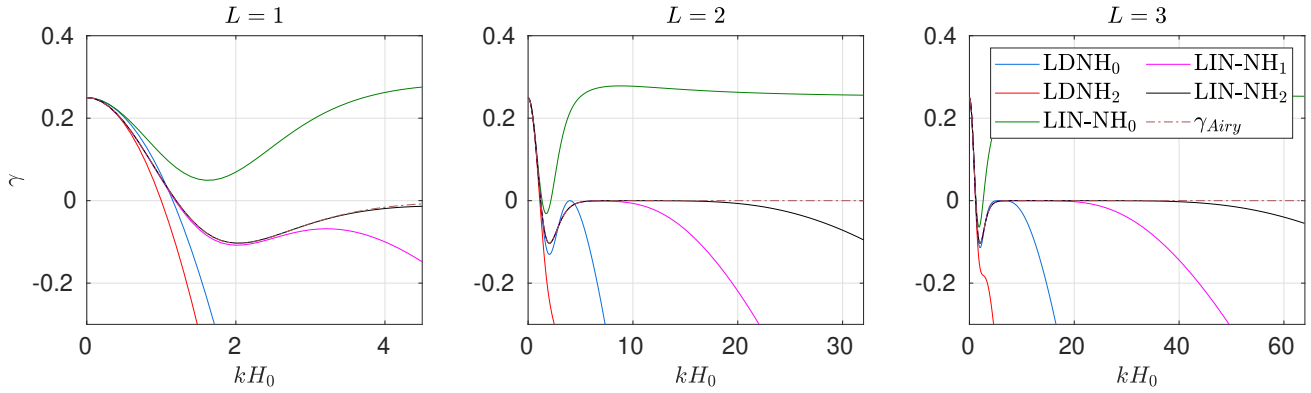

Figure 8: Intercomparison between the shoaling gradient of different models, for a fixed number of layers $L = 1, 2, 3$.

	LDNH$_0$	LDNH$_2$	LIN-NH$_0$	LIN-NH$_1$	LIN-NH$_2$
u_α	$\mathbb{P}_0$	$\mathbb{P}_0$	$\mathbb{P}_1$	$\mathbb{P}_1$	$\mathbb{P}_1$
w_α	$\mathbb{P}_0$	$\mathbb{P}_1$	$\mathbb{P}_0$	$\mathbb{P}_1$	$\mathbb{P}_2$
q_α	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_3$

Table 1: Summary of the model names and the approximation spaces for velocity and pressure variables used.

of layers L. The comparison between the relative dispersion error for the different models reveals that the new models LIN-NH$_1$ and LIN-NH$_2$ are the most accurate with bounded errors by 1 % for an extensive range of kH_0 up to 32. In the same way, we can state that the dispersion accuracy of the new model LIN-NH$_0$ does not improve the one provided by LDNH$_0$ or LDNH$_2$, while the complexity of the model increases.

In Figure 4 we compare the relative dispersion errors for the new models with the corresponding one for LDNH$_0$ and LDNH$_2$ for different numbers of layers and ranges for kH_0. We can see again that both relative dispersion errors converge to 0 as the number of layers L increases. Nevertheless, the dispersion relation for the new model LIN-NH$_2$ is now much better than that of LDNH$_0$. Even for the simple case of one layer, the relative dispersion error is relatively small. Therefore, we can state that LIN-NH$_2$ is an excellent choice in terms of dispersion accuracy. Despite the increased complexity of the model, it gives better results for large values of kH_0 even for a small number of layers.

In the same way, we compare in Figures 5 and 6 the group velocity, and in Figures 7 and 8 the shoaling gradient for the different models considered in this work. We remark the outperformance of the LIN-NH$_1$ and LIN-NH$_2$ models over LIN-NH$_0$, LDNH$_0$ and LDNH$_2$.

In order to complete the dispersion study, we provide in Table 2 the minimum number of layers L that ensures errors smaller than 5 % for the different dispersion relations and for different ranges of $kH_0 \in [0, 2^M]$

$$100 \cdot \left\| \frac{c^2 - c_{Airy}^2}{c_{Airy}^2} \right\|_\infty \leq 5\ \%, \quad 100 \cdot \left\| \frac{c_g^2 - c_{g,Airy}^2}{c_{g,Airy}^2} \right\|_\infty \leq 5\ \%, \quad 100 \cdot \|\gamma - \gamma_{Airy}\|_\infty \leq 5\ \%,$$

where $\|\cdot\|_\infty$ stands for the infinity norm

$$\|f\|_\infty = \max_{s \in [0, 2^M]} |f(s)| .$$

Note that we consider relative errors for the wave celerity and group velocity. However, absolute errors for the shoaling gradient are considered to avoid singularities when dividing by γ_{Airy}.

It can be highlighted the overperformance of the model LIN-NH$_2$ even for large ranges of the parameter kH_0. Despite the complexity of the presented models, the analysis of the dispersive properties for the LIN-NH$_2$ model are outstanding. With no more than 2 layers, it can accurately represent dispersive water waves of very high frequency.

5 Conclusions

A hierarchy of layer-averaged non-hydrostatic models for Euler equations with layerwise linear horizontal velocities, named LIN-NH$_p$ with $p = 0, 1, 2$, has been presented. The derivation of these models follows from a layer-averaging procedure of the mass and momentum conservation equations, where normal jump conditions associated with the weak formulation of the problem are taken into account. In the notation LIN-NH$_p$, the value p indicates the closure relation chosen for the vertical profile of the vertical velocity and pressure. Thus, p stands for the degree of the polynomial approximation of the vertical velocity, being $p + 1$ the degree assumed for the pressure.

	Wave celerity c					Group velocity c_g				
kH_0	LIN-NH$_0$	LDNH$_0$	LDNH$_2$	LIN-NH$_1$	LIN-NH$_2$	LIN-NH$_0$	LDNH$_0$	LDNH$_2$	LIN-NH$_1$	LIN-NH$_2$
$[0, 2]$	2	2	2	1	1	4	2	4	1	1
$[0, 4]$	4	2	4	1	1	9	2	9	2	1
$[0, 8]$	8	2	8	2	1	17	3	17	2	2
$[0, 16]$	15	3	15	2	2	33	4	33	3	2
$[0, 32]$	29	4	29	3	2	66	5	64	3	3
$[0, 64]$	58	6	57	4	3	131	8	129	5	3
$[0, 128]$	116	8	113	5	4	261	10	257	6	5

	Shoaling gradient γ				
kH_0	LIN-NH$_0$	LDNH$_0$	LDNH$_2$	LIN-NH$_1$	LIN-NH$_2$
$[0, 2]$	3	2	4	1	1
$[0, 4]$	7	2	7	2	1
$[0, 8]$	12	3	13	2	2
$[0, 16]$	24	4	26	3	2
$[0, 32]$	48	6	50	4	3
$[0, 64]$	95	8	101	5	4
$[0, 128]$	190	11	202	7	5

Table 2: Minimum number of layers (L) that holds $100 \cdot \left\| \frac{f^2 - f^2_{Airy}}{f^2_{Airy}} \right\|_\infty \leq 5\%$, for $f = c, c_g$ the wave celerity and the group velocity, respectively, and $100 \cdot \|\gamma - \gamma_{Airy}\|_\infty \leq 5\%$ for the shoaling gradient.

Once the most complex model (LIN-NH_2) is derived, simplified LIN-$NH_{0,1}$ models are obtained from simple assumptions upon the vertical profile of the vertical velocity, and therefore the pressure, just on the vertical momentum equation. It is important to remark that for the incompressibility constraints, the vertical profile of the velocity is not simplified for any model. For this very equation, the vertical velocity is always a layerwise parabolic polynomial. It is essential to prove a dissipative energy balance for all the proposed models. All the models are written in a compact form, which allows to prove the dissipative energy balance in a few lines. It is done by defining some non-hydrostatic differential operators, which satisfy a crucial duality relation. It is a remarkable issue, especially when designing numerical schemes for such models (see [8]).

An analysis and a comparison of dispersion properties have been performed. In particular, we showed the linear dispersion, shoaling, and group velocity. Interestingly, models LIN-NH_1 and LIN-NH_2 provide accurate dispersion relation, although LIN-NH_1 is a simpler model. Note that the latter model is computationally cheaper since it has $2L$ equations and unknowns less than LIN-NH_2 model. LIN-$NH_{1,2}$ models exhibit excellent dispersion properties for a wide range of wave numbers even for few layers. Then, these models can accurately reproduce dispersive water waves of very high frequency. We also have shown that LIN-$NH_{1,2}$ models improve the results of the models introduced in [9]. On the other hand, LIN-NH_0 model, which considers a layerwise constant vertical velocity, gives poor results on the linear dispersion analysis. That would suggest that there should be a restriction between the degrees of the polynomial approximation assumed for the variables. In particular, we could induce that the degree of the vertical velocity approximation should be greater than for the horizontal velocity. In the future, it would be interesting to propose efficient numerical schemes for LIN-$NH_{1,2}$ models, as well as extend these models to the Navier-Stokes system accounting for the contribution of the deviatoric stress tensor.

Acknowledgements

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A Proof of energy balance for LIN-NH_2 model

This appendix gives detailed proof of the energy balance for LIN-NH_2 model. We remark that, for the sake of clarity, we give the proof of the energy balance for System (36) before the change of variables (35), *i.e.* (20,26,27,28,30,32,34). Later, Theorem 2 is obtained by simply considering these new variables.

Proof:

Using the horiztonal momentum equation as well as the layer height evolution equation, we obtain

$$\begin{aligned} &h_\alpha\left(\partial_t\overline{u}_\alpha+\partial_x\left(\frac{\overline{u}_\alpha^2}{2}\right)\right)+\overline{u}_\alpha\left(\Gamma_{\alpha-1/2}-\Gamma_{\alpha+1/2}\right)+\partial_x\left(h_\alpha\frac{h_\alpha^2\lambda_\alpha^2}{12}+h_\alpha\overline{q}_\alpha\right)+gh_\alpha\partial_x(z_b+H)\\ &\quad=q_{\alpha+1/2}\partial_x z_{\alpha+1/2}-q_{\alpha-1/2}\partial_x z_{\alpha-1/2}-\widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2}+\widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2}. \end{aligned} \tag{62}$$

Now, we sum up (62) multiplied by $\overline{u}_\alpha$ with the equation obtained by multiplying the mass conservation equation by $\overline{u}_\alpha^2/2+g(z_b+H)$, getting

$$\begin{aligned} &\partial_t\left(h_\alpha\frac{\overline{u}_\alpha^2}{2}\right)+g(z_b+H)\partial_t h_\alpha+\partial_x\left(h_\alpha\overline{u}_\alpha\left(\frac{\overline{u}_\alpha^2}{2}+g(z_b+H)\right)\right)+\overline{u}_\alpha\partial_x\left(\frac{h_\alpha^3\lambda_\alpha^2}{12}\right)\\ &\quad+\overline{u}_\alpha\partial_x\left(h_\alpha\overline{q}_\alpha\right)-\overline{u}_\alpha\left(q_{\alpha+1/2}\partial_x z_{\alpha+1/2}-q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\right)\\ &\quad=\ g(z_b+H)\left(\Gamma_{\alpha-1/2}-\Gamma_{\alpha+1/2}\right)+\Gamma_{\alpha+1/2}\left(\frac{\overline{u}_\alpha^2}{2}-\overline{u}_\alpha\widetilde{u}_{\alpha+1/2}\right)-\Gamma_{\alpha-1/2}\left(\frac{\overline{u}_\alpha^2}{2}-\overline{u}_\alpha\widetilde{u}_{\alpha-1/2}\right) \end{aligned} \tag{63}$$

On the another hand, we have

$$
\begin{aligned}
&h_\alpha\left(\partial_t\left(\frac{h_\alpha\lambda_\alpha}{12}\right)+\overline{u}_\alpha\partial_x\left(\frac{h_\alpha\lambda_\alpha}{12}\right)\right)+\frac{h_\alpha\lambda_\alpha}{12}\left(\Gamma_{\alpha-1/2}-\Gamma_{\alpha+1/2}\right)+\frac{h_\alpha^2\lambda_\alpha}{12}\partial_x\overline{u}_\alpha+\overline{q}_\alpha\partial_x z_\alpha\\
&+\partial_x\left(h_\alpha\left(\frac{(\delta q)_\alpha}{20}+\frac{\pi_\alpha}{30}\right)\right)+\left(\frac{(\delta q)_\alpha}{20}+\frac{\pi_\alpha}{30}\right)\partial_x h_\alpha-\frac{1}{2}\left(q_{\alpha+1/2}\partial_x z_{\alpha+1/2}+q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\right)\\
&=\ \Gamma_{\alpha+1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}+\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha+1/2}}{2}\right)-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha\lambda_\alpha}{12}-\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha-1/2}}{2}\right).
\end{aligned}
\tag{64}
$$

Summing up (64) multiplied by $h_\alpha\lambda_\alpha$ with the mass conservation equation multiplied by $\lambda_\alpha^2 h_\alpha^2/24$, we obtain

$$
\begin{aligned}
&\partial_t\left(h_\alpha\frac{(h_\alpha\lambda_\alpha)^2}{24}\right)+\partial_x\left(h_\alpha u_\alpha\frac{(h_\alpha\lambda_\alpha)^2}{24}\right)+\frac{h_\alpha^3\lambda_\alpha^2}{12}\partial_x\overline{u}_\alpha\\
&+\ h_\alpha\lambda_\alpha\overline{q}_\alpha\partial_x z_\alpha+\lambda_\alpha\partial_x\left(h_\alpha^2\left(\frac{(\delta q)_\alpha}{20}+\frac{\pi_\alpha}{30}\right)\right)-\frac{h_\alpha\lambda_\alpha}{2}\left(q_{\alpha+1/2}\partial_x z_{\alpha+1/2}+q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\right)\\
&=\ \Gamma_{\alpha+1/2}\left(\frac{(h_\alpha\lambda_\alpha)^2}{8}+h_\alpha\lambda_\alpha\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha+1/2}}{2}\right)-\Gamma_{\alpha-1/2}\left(\frac{(h_\alpha\lambda_\alpha)^2}{8}-h_\alpha\lambda_\alpha\frac{\overline{u}_\alpha-\widetilde{u}_{\alpha-1/2}}{2}\right).
\end{aligned}
\tag{65}
$$

Finally we sum equations (63) and (65) to get

$$
\begin{aligned}
&\partial_t\left(h_\alpha\frac{\overline{u}_\alpha^2+(h_\alpha\lambda_\alpha)^2/12}{2}\right)+\partial_x\left(h_\alpha\overline{u}_\alpha\left(\frac{\overline{u}_\alpha^2+(h_\alpha\lambda_\alpha)^2/12}{2}+g(z_b+H)+\frac{h_\alpha^2\lambda_\alpha^2}{12}\right)\right)\\
&+\ g(z_b+H)\partial_t h_\alpha+\overline{u}_\alpha\partial_x\left(h_\alpha\overline{q}_\alpha\right)+h_\alpha\overline{q}_\alpha\lambda_\alpha\partial_x z_\alpha+\lambda_\alpha\partial_x\left(h_\alpha^2\left(\frac{(\delta q)_\alpha}{20}+\frac{\pi_\alpha}{30}\right)\right)\\
&-\ q_{\alpha+1/2}\left(\overline{u}_\alpha+\frac{h_\alpha\lambda_\alpha}{2}\right)\partial_x z_{\alpha+1/2}+q_{\alpha-1/2}\left(\overline{u}_\alpha-\frac{h_\alpha\lambda_\alpha}{2}\right)\partial_x z_{\alpha-1/2}\\
&=\ \Gamma_{\alpha+1/2}\left(\frac{1}{2}\left(\overline{u}_\alpha^2+\frac{(h_\alpha\lambda_\alpha)^2}{4}+\overline{u}_\alpha h_\alpha\lambda_\alpha\right)-\widetilde{u}_{\alpha+1/2}\left(\overline{u}_\alpha+\frac{h_\alpha\lambda_\alpha}{2}\right)\right)\\
&-\ \Gamma_{\alpha-1/2}\left(\frac{1}{2}\left(\overline{u}_\alpha^2+\frac{(h_\alpha\lambda_\alpha)^2}{4}-\overline{u}_\alpha h_\alpha\lambda_\alpha\right)-\widetilde{u}_{\alpha+1/2}\left(\overline{u}_\alpha-\frac{h_\alpha\lambda_\alpha}{2}\right)\right)\\
&+\ g\left(z_b+H\right)\left(\Gamma_{\alpha-1/2}-\Gamma_{\alpha+1/2}\right),
\end{aligned}
\tag{66}
$$

Now, similar steps are followed for the equations of the vertical velocity variables. However, for convenience, we are going to use equations (30),(31),(33). Then, from (30) and using the mass equation, we obtain

$$
\begin{aligned}
&h_\alpha\left(\partial_t\overline{w}_\alpha+\overline{u}_\alpha\partial_x\overline{w}_\alpha\right)+\overline{w}_\alpha\left(\Gamma_{\alpha-1/2}-\Gamma_{\alpha+1/2}\right)+\partial_x\left(\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{12}\right)+q_{\alpha+1/2}-q_{\alpha-1/2}\\
&\qquad=-\widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2}+\widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2}.
\end{aligned}
\tag{67}
$$

Previous equation is multiplied by $\overline{w}_\alpha$ and summing the mass equation multiplied by $\overline{w}_\alpha^2/2$, we get

$$
\begin{aligned}
&\partial_t\left(h_\alpha\frac{\overline{w}_\alpha^2}{2}\right)+\partial_x\left(h_\alpha\overline{u}_\alpha\frac{\overline{w}_\alpha^2}{2}\right)+\overline{w}_\alpha\partial_x\left(\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{12}\right)+\overline{w}_\alpha\left(q_{\alpha+1/2}-q_{\alpha-1/2}\right)=\\
&\qquad\Gamma_{\alpha+1/2}\left(\frac{\overline{w}_\alpha^2}{2}-\overline{w}_\alpha\widetilde{w}_{\alpha+1/2}\right)-\Gamma_{\alpha-1/2}\left(\frac{\overline{w}_\alpha^2}{2}-\overline{w}_\alpha\widetilde{w}_{\alpha-1/2}\right).
\end{aligned}
\tag{68}
$$

Making analogous computations with equations (31),(33), we get

$$\begin{aligned}
&\partial_t\left(h_\alpha\frac{(h_\alpha\varphi_\alpha)^2}{24}\right)+\partial_x\left(h_\alpha\overline{u}_\alpha\frac{(h_\alpha\varphi_\alpha)^2}{24}\right)\\
&\quad+\varphi_\alpha\partial_x\left(\frac{h_\alpha^5\lambda_\alpha\psi_\alpha}{360}\right)+\frac{h_\alpha^3\lambda_\alpha\varphi_\alpha}{12}\partial_x\overline{w}_\alpha-\frac{h_\alpha^5\varphi_\alpha\psi_\alpha^2}{720}+\frac{h_\alpha\varphi_\alpha}{2}\left(q_{\alpha-1/2}+q_{\alpha+1/2}\right)-h_\alpha\varphi_\alpha\overline{q}_\alpha \qquad (69)\\
&\quad=\Gamma_{\alpha+1/2}\left(\frac{(h_\alpha\varphi_\alpha)^2}{8}+h_\alpha\varphi_\alpha\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{2}\right)-\Gamma_{\alpha-1/2}\left(\frac{(h_\alpha\varphi_\alpha)^2}{8}-h_\alpha\varphi_\alpha\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t\left(h_\alpha\frac{(h_\alpha^2\psi_\alpha)^2}{1440}\right)+\partial_x\left(h_\alpha\overline{u}_\alpha\frac{(h_\alpha^2\psi_\alpha)^2}{1440}\right)+\psi_\alpha\partial_x\left(\frac{h_\alpha^5\lambda_\alpha\varphi_\alpha}{360}\right)+\frac{h_\alpha^5\varphi_\alpha\psi_\alpha^2}{240}-\frac{h_\alpha^4\varphi_\alpha\lambda_\alpha\psi_\alpha}{72}\partial_x h_\alpha\\
&\quad+\frac{h_\alpha^2\psi_\alpha}{30}\left((\delta q)_\alpha-\pi_\alpha\right)=\frac{h_\alpha^2\psi_\alpha}{12}\Gamma_{\alpha+1/2}\left(\frac{h_\alpha^2\psi_\alpha}{24}+\frac{h_\alpha\varphi_\alpha}{2}+\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}\right) \qquad (70)\\
&\qquad\qquad-\frac{h_\alpha^2\psi_\alpha}{12}\Gamma_{\alpha-1/2}\left(\frac{h_\alpha^2\psi_\alpha}{24}-\frac{h_\alpha\varphi_\alpha}{2}+\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}\right).
\end{aligned}$$

Defining now

$$E_\alpha\;=\;\frac{1}{2}\left(\overline{u}_\alpha^2+\overline{w}_\alpha^2+\frac{(h_\alpha\lambda_\alpha)^2+(h_\alpha\varphi_\alpha)^2}{12}+\frac{(h_\alpha^2\psi_\alpha)^2}{720}\right)+g\left(z_b+\frac{H}{2}\right),$$

summing up now equations (66),(68),(69),(70), after some straightforward computations we obtain

$$\begin{aligned}
\partial_t\left(h_\alpha E_\alpha\right)\;&+\;\partial_x\left(h_\alpha E_\alpha\overline{u}_\alpha+gh_\alpha\frac{H}{2}\overline{u}_\alpha+\frac{h_\alpha^3\lambda_\alpha^2\overline{u}_\alpha}{12}+\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha\overline{w}_\alpha}{12}+\frac{h_\alpha^5\varphi_\alpha\lambda_\alpha\psi_\alpha}{360}\right)\\
&+\;P_{NH,\alpha}=MT_\alpha+\frac{g}{2}\left(h_\alpha\partial_t H-H\partial_t h_\alpha\right),
\end{aligned} \qquad (71)$$

where we have used that

$$g(z_b+H)\partial_t h_\alpha=\partial_t\left(h_\alpha g\left(z_b+\frac{H}{2}\right)\right)+\frac{g}{2}\left(H\partial_t h_\alpha-h_\alpha\partial_t H\right).$$

In equation (71), $P_{NH,\alpha}$ and MT_α collect the terms related to non-hydrostatic pressure and mass transference, respectively, in the layer Ω_α. Let us detail now these terms separately. The non-hydrostatic contributions are

$$\begin{aligned}
P_{NH,\alpha}=\;&\overline{u}_\alpha\partial_x\left(h_\alpha\overline{q}_\alpha\right)+h_\alpha\overline{q}_\alpha\lambda_\alpha\partial_x z_\alpha-h_\alpha\varphi_\alpha\overline{q}_\alpha+\lambda_\alpha\partial_x\left(h_\alpha^2\left(\frac{(\delta q)_\alpha}{20}+\frac{\pi_\alpha}{30}\right)\right)+\frac{h_\alpha^2\psi_\alpha}{30}\left((\delta q)_\alpha-\pi_\alpha\right)\\
&+q_{\alpha+1/2}\left(-\left(\overline{u}_\alpha+\frac{h_\alpha\lambda_\alpha}{2}\right)\partial_x z_{\alpha+1/2}+\overline{w}_\alpha+\frac{h_\alpha\varphi_\alpha}{2}\right)\\
&-q_{\alpha-1/2}\left(-\left(\overline{u}_\alpha-\frac{h_\alpha\lambda_\alpha}{2}\right)\partial_x z_{\alpha-1/2}+\overline{w}_\alpha-\frac{h_\alpha\varphi_\alpha}{2}\right)
\end{aligned}$$

Now we use restriction (12a) to write

$$\overline{u}_\alpha\partial_x\left(h_\alpha\overline{q}_\alpha\right)+h_\alpha\overline{q}_\alpha\lambda_\alpha\partial_x z_\alpha-h_\alpha\varphi_\alpha\overline{q}_\alpha=\partial_x\left(h_\alpha\overline{u}_\alpha\overline{q}_\alpha\right),$$

and restriction (12b) to obtain

$$\lambda_\alpha\partial_x\left(h_\alpha^2\left(\frac{(\delta q)_\alpha}{20}+\frac{\pi_\alpha}{30}\right)\right)+\frac{h_\alpha^2\psi_\alpha}{30}\left((\delta q)_\alpha-\pi_\alpha\right)=\partial_x\left(h_\alpha^2\lambda_\alpha\left(\frac{(\delta q)_\alpha}{20}+\frac{\pi_\alpha}{30}\right)\right)-\frac{h_\alpha^2}{12}\partial_x\lambda_\alpha(\delta q)_\alpha,$$

where we remind that $(\delta q)_\alpha = q_{\alpha+1/2} - q_{\alpha-1/2}$. Therefore, using (12b) again to recover the expression of $w^{\mp}_{\alpha\pm1/2}$ (see (13)), the pressure terms are rewritten as

$$\begin{aligned}
P_{NH,\alpha} &= \partial_x \left(h_\alpha \left(\overline{q}_\alpha \overline{u}_\alpha + h_\alpha \lambda_\alpha \left(\frac{(\delta q)_\alpha}{20} + \frac{\pi_\alpha}{30} \right) \right) \right) \\
&\quad + q_{\alpha+1/2} \left(- \left(\overline{u}_\alpha + \frac{h_\alpha \lambda_\alpha}{2} \right) \partial_x z_{\alpha+1/2} + \overline{w}_\alpha + \frac{h_\alpha \varphi_\alpha}{2} + \frac{h_\alpha^2 \psi_\alpha}{12} \right) \\
&\quad - q_{\alpha-1/2} \left(- \left(\overline{u}_\alpha - \frac{h_\alpha \lambda_\alpha}{2} \right) \partial_x z_{\alpha-1/2} + \overline{w}_\alpha - \frac{h_\alpha \varphi_\alpha}{2} + \frac{h_\alpha^2 \psi_\alpha}{12} \right) \\
&= \partial_x \left(h_\alpha \left(\overline{q}_\alpha \overline{u}_\alpha + h_\alpha \lambda_\alpha \left(\frac{(\delta q)_\alpha}{20} + \frac{\pi_\alpha}{30} \right) \right) \right) \\
&\quad + q_{\alpha+1/2} \left(\partial_t z_{\alpha+1/2} + \Gamma_{\alpha+1/2} \right) - q_{\alpha-1/2} \left(\partial_t z_{\alpha-1/2} + \Gamma_{\alpha-1/2} \right),
\end{aligned}$$

where the definition of the mass transfer terms (16b) has been used in last equality. Notice that the terms on $q_{\alpha\pm1/2}$ vanish when summing up from $\alpha = 1, \ldots, L$, leading to

$$\sum_{\beta=1}^{L} P_{NH,\beta} = \sum_{\beta=1}^{L} \partial_x \left(h_\beta \left(\overline{q}_\beta \overline{u}_\beta + h_\beta \lambda_\beta \left(\frac{(\delta q)_\beta}{20} + \frac{\pi_\beta}{30} \right) \right) \right),$$

which is a conservative term.

Concerning the mass transfer terms, they are

$$\begin{aligned}
MT_\alpha &= g \left(z_b + H \right) \left(\Gamma_{\alpha-1/2} - \Gamma_{\alpha+1/2} \right) \\
&\quad + \frac{\Gamma_{\alpha+1/2}}{2} \left[\left(\overline{u}_\alpha^2 + \left(\frac{h_\alpha \lambda_\alpha}{2} \right)^2 + \overline{u}_\alpha h_\alpha \lambda_\alpha \right) - 2 \widetilde{u}_{\alpha+1/2} \left(\overline{u}_\alpha + \frac{h_\alpha \lambda_\alpha}{2} \right) \right] \\
&\quad - \frac{\Gamma_{\alpha-1/2}}{2} \left[\left(\overline{u}_\alpha^2 + \left(\frac{h_\alpha \lambda_\alpha}{2} \right)^2 - \overline{u}_\alpha h_\alpha \lambda_\alpha \right) - 2 \widetilde{u}_{\alpha-1/2} \left(\overline{u}_\alpha - \frac{h_\alpha \lambda_\alpha}{2} \right) \right] \\
&\quad + \frac{\Gamma_{\alpha+1/2}}{2} \left[\overline{w}_\alpha^2 + \left(\frac{h_\alpha \varphi_\alpha}{2} \right)^2 + \overline{w}_\alpha h_\alpha \varphi_\alpha + \frac{h_\alpha^2 \psi_\alpha}{6} \left(\overline{w}_\alpha + \frac{h_\alpha \varphi_\alpha}{2} \right) + \left(\frac{h_\alpha^2 \psi}{12} \right)^2 \right. \\
&\qquad \left. - 2 \widetilde{w}_{\alpha+1/2} \left(\overline{w}_\alpha + \frac{h_\alpha \varphi_\alpha}{2} + \frac{h_\alpha^2 \psi_\alpha}{12} \right) \right] \\
&\quad - \frac{\Gamma_{\alpha-1/2}}{2} \left[\overline{w}_\alpha^2 + \left(\frac{h_\alpha \varphi_\alpha}{2} \right)^2 - \overline{w}_\alpha h_\alpha \varphi_\alpha + \frac{h_\alpha^2 \psi_\alpha}{6} \left(\overline{w}_\alpha - \frac{h_\alpha \varphi_\alpha}{2} \right) + \left(\frac{h_\alpha^2 \psi}{12} \right)^2 \right. \\
&\qquad \left. - 2 \widetilde{w}_{\alpha-1/2} \left(\overline{w}_\alpha - \frac{h_\alpha \varphi_\alpha}{2} + \frac{h_\alpha^2 \psi_\alpha}{12} \right) \right].
\end{aligned}$$

Looking at the velocities at the interfaces $u^{\mp}_{\alpha\pm1/2}, w^{\mp}_{\alpha\pm1/2}$ (see (7) and (13)), the latter equation yields

$$
\begin{aligned}
MT_\alpha = {} & g\left(z_b+H\right)\left(\Gamma_{\alpha-1/2}-\Gamma_{\alpha+1/2}\right) \\
& + \frac{\Gamma_{\alpha+1/2}}{2}\left(\left(u^-_{\alpha+1/2}\right)^2 - 2\widetilde{u}_{\alpha+1/2}u^-_{\alpha+1/2} \;+\; \left(w^-_{\alpha+1/2}\right)^2 - 2\widetilde{w}_{\alpha+1/2}w^-_{\alpha+1/2}\right) \\
& - \frac{\Gamma_{\alpha-1/2}}{2}\left(\left(u^+_{\alpha-1/2}\right)^2 - 2\widetilde{u}_{\alpha-1/2}u^+_{\alpha-1/2} \;+\; \left(w^+_{\alpha-1/2}\right)^2 - 2\widetilde{w}_{\alpha-1/2}w^+_{\alpha-1/2}\right) \\
= {} & g\left(z_b+H\right)\left(\Gamma_{\alpha-1/2}-\Gamma_{\alpha+1/2}\right) \;+\; \frac{\Gamma_{\alpha-1/2}}{2}\left(u^-_{\alpha-1/2}u^+_{\alpha-1/2} \;+\; w^-_{\alpha-1/2}w^+_{\alpha-1/2}\right) \\
& \qquad - \frac{\Gamma_{\alpha+1/2}}{2}\left(u^-_{\alpha+1/2}u^+_{\alpha+1/2} \;+\; w^-_{\alpha+1/2}w^+_{\alpha+1/2}\right).
\end{aligned}
$$

It is easy to see that, since we have set $\Gamma_{1/2} = \Gamma_{N+1/2} = 0$, it holds

$$
\sum_{\beta=1}^{L} MT_\beta = 0, \qquad \text{and} \qquad \frac{g}{2}\sum_{\beta=1}^{L}\left(h_\alpha\partial_t H - H\partial_t h_\alpha\right) = 0,
$$

and thus, summing up (71) in all the layers, the proof is finished. □

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