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Pricing for a vulnerable bull spread options using a mixed modified fractional Hull-White-Vasicek model

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Abstract

In this paper, in order to serve credit risk management, we introduce a pricing model for a vulnerable Bull Spread options in a Mixed Modified Fractional Hull-White-Vasicek stochastic volatility and stochastic interest rate model. We use Milstein scheme to find the sample paths of asset price and its volatility, and the sample paths of interest rates of asset price movement. We use the double Mellin transform to obtain an analytical vulnerable bull spread call option formula and an analytical vulnerable bull spread put option formula under fractional stochastic volatility and fractional stochastic interest rates.

Keywords: Bull spread option; Hull-White-Vasicek model; Double Mellin transform.

MSC 2010: 60G22, 60G18.

1 Introduction

Credit risk management is crucial for financial institutions (banks, insurance companies, mutual funds and pension funds, etc.), but also for national or international regulatory institutions. Indeed, within the framework of market surveillance and the risk of bankruptcy of financial institutions, the bank for international settlements requires compliance (within the framework of the application of the Basel III or Solvency II systems), that certain prudential rules are respected. Thus, hedging financial risks through the use of financial derivatives is increasingly common. It is enough to see the extension works of the option pricing models proposed by Black and Scholes [3], to understand the growing interest in the valuation of derivative products. With regard to the risks of credit default on the over-the-counter markets, some authors have, under certain assumptions, proposed pricing models of vulnerable options. We must mention the recent occurrence of economic crises (for example the subprime crisis in 2008), with, as a corollary, the increase in the probability of default by certain financial institutions, companies or municipalities. Among the authors who have proposed pricing models for vulnerable options, the holder of which is subject to the risk of default, in addition to the work of Johnson and Stuly [12] where the first to substituted default risk in option pricing and put forward a new definition called the vulnerable option, Klein [22] has derived an analytical pricing formula which not only allows a correlation between the underlying asset of the option and the default risk of the counterparty, but also for the writer of the option to have other liabilities in the capital structure. Ammann [1] developed Klein's credit risk model on the basis of a structural approach. He finally obtained the explicit expression for the vulnerable option under the assumption of interest rate and default intensity obeying a stochastic differential equation.

Other scholars like Chang and Hung [6] have also discussed this problem, while all the above mentioned discussions are in the environment of geometric Brownian motion. Due to the inadequacies of geometric Brownian motion in describing self-similarity and long-term dependence on stock prices, fractional Brownian motion and mixed fractional Brownian motion are widely used in the valuation of assets. Hu and Oksendal [16], Djeutcha et al.[10] developed the structural approach, respectively, provided that the stock prices followed a mixed modified fractional Brownian motion and mixed fractional Brownian motion and they proved that the correspondence with the fractional market and the mixed modified fractional Black-Scholes market had no arbitrage effect for any Hurst parameter greater than 1/2.

For more literature on fractional Brownian motions and mixed modified fractional Brownian motions, among others, we can refer to Oksendal [27], Djetcha and Sadefo Kamdem [11] and Djetcha et al.[10]. But there is another problem that the Black-Scholes fractional market does not have an equivalent martingale measure according to Sottinen and Valkeila [30]. Necula [26] applied the quasi-martingale method to risk-neutral measurement. Huang et al. [17] obtained the explicit expression for the price of the European option under the hypothesis of a Black-Scholes fractional market. Su and Wang [29] and Li and Ma [24] derived the closed form formula of the price of the vulnerable European option by the method of changing measures.

The constant volatility assumption of the Black-Scholes[3] model is unreliable, because in the market volatility is in the form of a smile. Hence the need to introduce in the market stochastic volatility models which give more realistic results in relation with the historical asset returns and interest rate variation, compared to the classical Black-Scholes model. Pioneers in the field are: Hull-White [18], Stein-Stein[28], Heston[15], Njamen et al.[13], Bakshi, Cao-Chen[2] and Schobel-Zhu[31]. Each of these authors model the volatility of the underlying and propose models that present analytical solutions and closed formulas for option pricing.

We therefore appeal to the class of stochastic volatility models and the pioneers in this field are among these stochastic volatility models, the best known and studied is that of Hull-White. But, in the Hull-White model we consider the interest rate to be stochastic and follow the Vasicek model, we can define a new kind of three-factor model. In general, these classes of three-factor models are not studied in the literature, at least there is a lack of method to determine the analytical explicit expression for the option hedging problem.

These competing models modeled the interest rate of the Black and Scholes model and authors such as Vasicek [33], Dothan [8], Cox-Ingersoll-Ross [7], Djetcha et al.[10], Yousouf et al.[34], Mohamadinejad et al.[25] and Black-Derman-Toy [4] have worked on these models.

In this article, we will be using the double Mellin transform method as described in Ji-Hun and Jeong-Hoon[19], so that we can derive the general pricing formula for the vulnerable bull spread option under the stock price assumption, with the stock's volatility obeying the Hull and White diffusion model and the underlying interest rate obeying the Vasicek model which are driven by the mixed modified fractional Brownian motion.

The competition over bull spread options pricing is more intense comparison with vulnerable types. This is due to the uncertainty of the execution time in bull spread option and in result. The vulnerable option is a kind of option with credit risk which refers to a risk that a borrower will default on any type of debt by failing to make the required payments.

The bull spread option is an optimistic option strategy designed to take advantage of a moderate rise in the price of a security or asset. If we replace in the European vulnerable option, the payoff function of the European option with the payoff function of the bull spread option, we get a new option called the vulnerable bull spread option.

Our paper looks at these two families of models (stochastic interest rate) and stochastic volatility). But our motivation comes from the fact that we model the two families of models by assuming that their random parts are described by mixed modified fractional Brownian motions in such a way that our market is without arbitrage. The combination of the stochastic interest rate model (Vasicek[33]) and the stochastic volatility model (Hull-White[18]) allows us to define a three-factors model. This new model has a double advantage in that, it can correctly price any bond unlike the one-factor models mentioned above and can allow any options to be valued. In this paper, we will therefore be talking about simulating of our model using Milstein's discretization approach and proposing a closed formula for the pricing of the particular case of vulnerable options and the bull spread option using the double Mellin's transform method.

Nammed MMFHWV model for "Mixed-Modified-Fractional-Hull-White-Vasiseck ", this new model is a combination of the Hull and White[18] and the Vasicek[33] model. In this model, the volatility process and asset model are not correlated, while the interest rate process and asset model, the volatility process and interest rates process are correlated, with each other and they are controlled by a distinct diffusion process. In MMFHWV model, the existence of the mean reversion process

causes the adjustment of the volatility process and the interest rates behavior in the financial markets and it is a benefit of the MMFHWV model. The rest of this paper is organized as follows. Section 2 gives the preliminaries and basic assumptions. In Section 3, we introduce the MMFHWV model. We simulate the MMFHWV Model based on Milstein discretization. In section 5, we study the pricing of a vulnerable bull spread option under the MMFHWV model and conclusions are presented in the last section.

2 Preliminaries and Basic Assumptions

2.1 Preliminaries

In this subsection, we shall briefly review the definition and some main properties of the mixed modified fractional Brownian motion, the double Mellin transform, we also recall the Hull-White model and the Vasicek model. Fractional Brownian motion or Mellin transform have also been used for option pricing in Djoutcha et al.[9], Thao[32], Sadefo Kamdem [21], Ji-Hun[19], Hull-White[18] and Vasicek[33].

Definition 2.1. *The double Mellin transform is defined by*

$$\mathcal{M}_{xy}(f(x, y), w_1, w_2) = \widehat{f}(w_1, w_2) = \int_0^\infty \int_0^\infty f(x, y)x^{w_1-1}y^{w_2-1}dxdy \quad (1)$$

where $f(x, y)$ is a locally Lebesgue integrable function and w_1 and w_2 are complex numbers. Also, if $a < \text{Re}(w_1)$, $\text{Re}(w_2) < b$ and if c_1 and c_2 are such that $a < c_1 < b$ and $a < c_2 < b$, then the inverse of the double Mellin transform is given by

$$f(x, y) = \mathcal{M}_{xy}^{-1}(\widehat{f}(w_1, w_2)) = \frac{1}{(2\pi)^2} \int_{a_1-i\infty}^{a_1+i\infty} \int_{a_2-i\infty}^{a_2+i\infty} \widehat{f}(w_1, w_2)x^{-w_1}y^{-w_2}dw_1dw_2 \quad (2)$$

We have the following properties for derivative

$$\mathcal{M}_{xy}(x^2 \frac{\partial f(x, y)}{\partial x^2}, w_1, w_2) = w_1(w_1 + 1)\widehat{f}(w_1, w_2) \quad (3)$$

$$\mathcal{M}_{xy}(y^2 \frac{\partial f(x, y)}{\partial y^2}, w_1, w_2) = w_2(w_2 + 1)\widehat{f}(w_1, w_2) \quad (4)$$

Lemma 2.1. Ji-Hun and Jeong-Hoon[19] *Given complex numbers α and β with $\text{Re}(\alpha) \geq 0$, let*

$$f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \widehat{f}(w)x^{-w}dw,$$

where $\widehat{f}(w)x^{-w} = e^{\alpha(w+\beta)^2}$. Then $f(x) = \frac{1}{2} \frac{1}{\sqrt{\pi\alpha}} x^\beta e^{\frac{1}{4\alpha}(\ln x)^2}$ holds.

Lemma 2.2. Ji-Hun and Jeong-Hoon[19] *Let f and g be functions from \mathbb{R}_+^n into \mathbb{C} . If $\widehat{f}(w_1, w_2)$ and $\widehat{g}(w_1, w_2)$ are the double Mellin transform of $f(x, y)$ and $g(x, y)$, respectively, given by*

$$\widehat{\mathbf{f}}(w_1, w_2) = \int_0^\infty \int_0^\infty \mathbf{f}(x, y)x^{w_1-1}y^{w_2-1}dxdy. \quad (5)$$

Then the double Mellin convolution of f and g is given by inverse double Mellin transform of $\widehat{f}(w_1, w_2)\widehat{g}(w_1, w_2)$ as follows

$$\begin{aligned} f(x, y) \star g(x, y) &= \mathcal{M}_{w_1 w_2}^{-1} \left(\widehat{f}(w_1, w_2)\widehat{g}(w_1, w_2); x, y \right) \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty f\left(\frac{x}{u}, \frac{y}{w}\right) g(u, w)u^{-1}w^{-1}dudw \end{aligned} \quad (6)$$

Definition 2.2. *Hull and White[18]* The Hull-White model assumes that the underlying stock price, S_t and variance V_t follow a Black-Scholes-type stochastic process. Hence, the Hull-White model is represented by the bivariate system of stochastic differential equations (SDEs):

$$\begin{cases} dS_t = \mu S_t dt + S_t \sigma_s dB_{t,s}, \\ dV_t = \mu V_t dt + \sigma_v V_t dB_{t,v}. \end{cases} \quad (7)$$

Where $\mathbb{E}^{\mathbb{P}} [dB_{t,s} dB_{t,v}] = \rho_{sv} dt$. $B_{\cdot,s}$ and $B_{\cdot,v}$ are the wiener processes; $\rho_{sv} \in [-1; 1]$ is the correlation between $W_{t,s}$ and $W_{t,v}$.

The parameters of the model are given in Appendix A.1

Definition 2.3. *Vasicek[33]* The Vasicek model is a stochastic interest rate model and the corresponding Stochastic Differential Equation (SDE) as follows,

$$dr_t = \kappa(\theta - r_t)dt + r_t \sigma_r dB_{t,r}, \quad (8)$$

where $B_{\cdot,r}$ is the wiener process.

The parameters of the model are described in Appendix A.1.

Definition 2.4. *Djeutcha et al.[9, 11]* A Mixed Modified Fractional Brownian Motion process whose parameters a, b, ε and H is a linear combination of Brownian Motion B_t and independent semi-martingale process $B_t^{H,\varepsilon}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by:

$$M_t^{H,a,b,\varepsilon} = M_t^{H,\varepsilon} = aB_t + bB_t^{H,\varepsilon}, \quad \forall (a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+, \quad t \in [0, T]. \quad (9)$$

Where $H \in]\frac{1}{2}, 1[$ is the Hurst parameter.

In this paper, our aim is to establish an extension of Hull-White model Eq.(7) and Vasicek model Eq.(8), by exchanging in Eq.(7) and Eq.(8) the Brownian motion $B_{\cdot,s}, B_{\cdot,v}$ and $B_{\cdot,r}$ by the Mixed Modified Fractional Brownian process defined by the following system:

$$\begin{cases} M_{t,s}^{H_1,\varepsilon} = aB_{t,s}(t) + bB_{t,s}^{H_1,\varepsilon} \\ M_{t,v}^{H_2,\varepsilon} = aB_{t,v}(t) + bB_{t,v}^{H_2,\varepsilon} \\ M_{t,r}^{H_3,\varepsilon} = aB_{t,r}(t) + bB_{t,r}^{H_3,\varepsilon}. \end{cases} \quad (10)$$

We suppose that: $B_{\cdot,s} \& B^{H_1}, B_{\cdot,v} \& B^{H_2}$ and $B_{\cdot,r} \& B^{H_3}$ are independents.

From Thao[32], we also deduce that $B_s^{H_1,\varepsilon}, B_v^{H_2,\varepsilon}$ and $B_r^{H_3,\varepsilon}$ are three local martingale defined as follows:

$$\begin{cases} B_{t,s}^{H_1,\varepsilon} = \int_0^t (t + \varepsilon - s^*)^{H_1 - \frac{1}{2}} dB_{s^*,s} \\ B_{t,v}^{H_2,\varepsilon} = \int_0^t (t + \varepsilon - s^*)^{H_2 - \frac{1}{2}} dB_{s^*,v} \\ B_{t,r}^{H_3,\varepsilon} = \int_0^t (t + \varepsilon - s^*)^{H_3 - \frac{1}{2}} dB_{s^*,r}. \end{cases} \quad (11)$$

From Eq.(10), we define

$$\begin{cases} dM_{t,s}^{H_1,\varepsilon} = a dB_{t,s}(t) + b dB_{t,s}^{H_1,\varepsilon} \\ dM_{t,v}^{H_2,\varepsilon} = a dB_{t,v}(t) + b dB_{t,v}^{H_2,\varepsilon} \\ dM_{t,r}^{H_3,\varepsilon} = a dB_{t,r}(t) + b dB_{t,r}^{H_3,\varepsilon}. \end{cases} \quad (12)$$

We know that

$$\begin{cases} dB_{t,s}^{H_1,\varepsilon} = \phi_{t,s}^\varepsilon dt + \left(a + b\varepsilon^{H_1 - \frac{1}{2}} \right) dB_{t,s} \\ dB_{t,v}^{H_2,\varepsilon} = \phi_{t,v}^\varepsilon dt + \left(a + b\varepsilon^{H_2 - \frac{1}{2}} \right) dB_{t,v} \\ dB_{t,r}^{H_3,\varepsilon} = \phi_{t,r}^\varepsilon dt + \left(a + b\varepsilon^{H_3 - \frac{1}{2}} \right) dB_{t,r}. \end{cases} \quad (13)$$

where

$$\begin{cases} \phi_{t,s}^\varepsilon = \int_0^t (H_1 - \frac{1}{2})(t + \varepsilon - s^*)^{H_1 - \frac{3}{2}} dB_{s^*,s}, \\ \phi_{t,v}^\varepsilon = \int_0^t (H_2 - \frac{1}{2})(t + \varepsilon - s^*)^{H_2 - \frac{3}{2}} dB_{s^*,v}, \\ \phi_{t,r}^\varepsilon = \int_0^t (H_3 - \frac{1}{2})(t + \varepsilon - s^*)^{H_3 - \frac{3}{2}} dB_{s^*,r}. \end{cases} \quad (14)$$

By Letting $\chi^{H_i,\varepsilon} = \left(a + b\varepsilon^{H_i - \frac{1}{2}}\right)^2$, $i = 1, 2, 3$. Eq.(12) helps to define the following expressions:

$$\begin{cases} dM_{t,s}^{H_1,\varepsilon} = \phi_{t,s}^\varepsilon dt + \sqrt{\chi^{H_1,\varepsilon}} dB_{s^*,s}, \\ dM_{t,v}^{H_2,\varepsilon} = \phi_{t,v}^\varepsilon dt + \sqrt{\chi^{H_2,\varepsilon}} dB_{s^*,v}, \\ dM_{t,r}^{H_3,\varepsilon} = \phi_{t,r}^\varepsilon dt + \sqrt{\chi^{H_3,\varepsilon}} dB_{s^*,r}. \end{cases} \quad (15)$$

2.2 Basic assumptions

In this subsection, we introduce the basic assumptions. Some of them are those of Djetcha et al.[9].

- (a) Consider the probability space (Ω, F, \mathbb{P}) where $F = (F_t)_{t \geq 0}$ is a complete and right continuous filtration generated by $F_t = \sigma \left\{ \left(M_{t,s}^{H_1,\varepsilon}; M_{t,s}^{H_2,\varepsilon}; M_{t,s}^{H_3,\varepsilon} \right) \right\}$ and \mathbb{P} is a probability measure on Ω .
- (b) The dynamic of stock price S_t and its volatility V_t are given by a Mixed Modified Fractional Hull-WHITE no-arbitrage model, namely :

$$\begin{cases} dS_t = \mu S_t dt + \sigma_s S_t dM_{t,s}^{H_1,\varepsilon}, \\ dV_t = \mu V_t dt + \sigma_v V_t dM_{t,v}^{H_2,\varepsilon}. \end{cases} \quad (16)$$

where $M_{\cdot,s}^{H_1,\varepsilon}$ and $M_{\cdot,v}^{H_1,\varepsilon}$ are two Mixed Modified Fractional Brownian motions with Hurst parameters $H_1, H_2 \in]\frac{1}{2}, 1[$.

- (c) The evolution of interest rate r_t is given by the Mixed Modified Fractional Vasicek no-arbitrage model, namely :

$$dr_t = \kappa(\theta - r_t)dt + r_t \sigma_r dM_{t,r}^{H_3,\varepsilon}. \quad (17)$$

where $M_{\cdot,r}^{H_3,\varepsilon}$ is the Mixed Modified Fractional Brownian motion with Hurst parameter $H_3 \in]\frac{1}{2}, 1[$.

- (d) The hurst parameters H_1, H_2 and H_3 can be verified $H_3 > H_2 > H_1 > \frac{1}{2}$.
- (e) The correlation of the Mixed Modified Fractional Brownian motions $M_{\cdot,s}^{H_1,\varepsilon}$ and $M_{\cdot,v}^{H_1,\varepsilon}$ is given by the following matrix defined from Eq.(15) as follows:

$$\mathbb{E}^{\mathbb{P}}[dM_{t,s}^{H_i,\varepsilon} dM_{t,s}^{H_j,\varepsilon}] = \begin{cases} \chi^{H_i,\varepsilon} dt, & \text{if } i = j \\ \rho_{sv} \sqrt{\chi^{H_i,\varepsilon} \chi^{H_j,\varepsilon}} dt, & \text{if } i \neq j, i, j = 1, 2. \\ \rho_{sr} \sqrt{\chi^{H_i,\varepsilon} \chi^{H_j,\varepsilon}} dt, & \text{if } i \neq j, i, j = 1, 3. \\ \rho_{vr} \sqrt{\chi^{H_i,\varepsilon} \chi^{H_j,\varepsilon}} dt, & \text{if } i \neq j, i, j = 2, 3. \end{cases} \quad (18)$$

Where

$$\sqrt{\chi^{H_i,\varepsilon} \chi^{H_j,\varepsilon}} = a^2 + ab\varepsilon^{H_i - \frac{1}{2}} + ab\varepsilon^{H_j - \frac{1}{2}} + b^2\varepsilon^{H_i + H_j - 1}, \quad \forall i \neq j. \quad (19)$$

- (f) The correlation coefficients ρ_{sv}, ρ_{sr} and ρ_{vr} are the parameters satisfying

$$\rho_{sv}^2 < 1; \rho_{sr}^2 < 1; \rho_{vr}^2 < 1; \rho_{sv}^2 + \rho_{sr}^2 + \rho_{vr}^2 - 2\rho_{sv}\rho_{sr}\rho_{vr} < 1. \quad (20)$$

3 The MMFHWV Model Framework

In the Mixed Modified Fractional Hull-White model Eq.(16), if we select the parameter μ as a stochastic process(see Mixed Modified Fractional Vasicek model(16)), we obtain the Mixed Modified Fractional Hull-White-Vasicek(MMFHWV) model defined system of trivariate fractional stochastic differential equations (SDEs):

$$\begin{cases} dS_t = r_t S_t dt + \sigma_s S_t dM_{t,s}^{H_1,\varepsilon}, \\ dV_t = r_t V_t dt + \sigma_v V_t dM_{t,v}^{H_2,\varepsilon}, \\ dr_t = \kappa(\theta - r_t)dt + r_t \sigma_r dM_{t,r}^{H_3,\varepsilon}. \end{cases} \quad (21)$$

Where $M_{t,s}^{H_1,\varepsilon}$, $M_{t,v}^{H_2,\varepsilon}$, and $M_{t,r}^{H_3,\varepsilon}$ are three correlated standard Mixed Modified Fractional Brownian motions.

The stock price and variance follow the processes in Eq.(16) and the interest rate follow the processes in Eq.(17) under the historical measure \mathbb{P} also called the physical measure. For pricing purposes, however, we need the processes for (S_t, V_t, r_t) under the risk-neutral measure \mathbb{Q} . In the MMFHWV model Eq.(21), this is done by modifying each SDE in Eq.(16) and Eq.(17) separately by an application of Girsanov's theorem. The risk-neutral process for the MMFHWV model Eq.(21) is defined by

$$\begin{cases} dS_t = r_t S_t dt + \sigma_s S_t d\widetilde{M}_{t,s}^{H_1,\varepsilon}, \\ dV_t = r_t V_t dt + \sigma_v V_t d\widetilde{M}_{t,v}^{H_2,\varepsilon}, \\ dr_t = \kappa^*(\theta^* - r_t)dt + r_t \sigma_r d\widetilde{M}_{t,r}^{H_3,\varepsilon}. \end{cases} \quad (22)$$

With

$$\begin{cases} \widetilde{M}_{t,s}^{H_1,\varepsilon} = M_{t,s}^{H_1,\varepsilon} + \frac{\mu - r}{\sigma_s} t \\ \widetilde{M}_{t,v}^{H_2,\varepsilon} = M_{t,v}^{H_2,\varepsilon} + \frac{\mu - r}{\sigma_v} t \\ \widetilde{M}_{t,r}^{H_3,\varepsilon} = M_{t,r}^{H_3,\varepsilon} + \frac{\mu - r}{\sigma_r \sqrt{\chi^{H_3,\varepsilon}}} t. \end{cases} \quad (23)$$

Where $\widetilde{M}_{t,s}^{H_1,\varepsilon}$, $\widetilde{M}_{t,v}^{H_2,\varepsilon}$, and $\widetilde{M}_{t,r}^{H_3,\varepsilon}$ are three correlated standard Mixed Modified Fractional Brownian motions. Its correlation matrix is defined from Eq.(15) as follows:

$$\mathbb{E}^{\mathbb{Q}}[d\widetilde{M}_{t,s}^{H_i,\varepsilon} d\widetilde{M}_{t,s}^{H_j,\varepsilon}] = \begin{cases} \chi^{H_i,\varepsilon} dt, & \text{if } i = j \\ \rho_{sv} \sqrt{\chi^{H_i,\varepsilon} \chi^{H_j,\varepsilon}} dt, & \text{if } i \neq j, i, j = 1, 2. \\ \rho_{sr} \sqrt{\chi^{H_i,\varepsilon} \chi^{H_j,\varepsilon}} dt, & \text{if } i \neq j, i, j = 1, 3. \\ \rho_{vr} \sqrt{\chi^{H_i,\varepsilon} \chi^{H_j,\varepsilon}} dt, & \text{if } i \neq j, i, j = 2, 3. \end{cases} \quad (24)$$

Where the expression of $\sqrt{\chi^{H_i,\varepsilon} \chi^{H_j,\varepsilon}}$ is given by Eq.(19), and \mathbb{Q} is the risk-neutral measure.

Using Eq.(15), the risk-neutral process for the stock price is

$$dS_t = S_t \left\{ (r + b\sigma_s \phi_{t,s}^\varepsilon) dt + \sigma_s \sqrt{\chi^{H_1,\varepsilon}} d\widetilde{B}_{t,s} \right\}. \quad (25)$$

where

$$\widetilde{B}_{t,s} = B_{t,s} + \frac{\mu - r}{\sigma_s \sqrt{\chi^{H_1,\varepsilon}}} t \quad (26)$$

The risk-neutral process for the variance of stock price is

$$dV_t = V_t \left\{ (r + b\sigma_v \phi_{t,v}^\varepsilon) dt + \sigma_v \sqrt{\chi^{H_2,\varepsilon}} d\widetilde{B}_{t,v} \right\}. \quad (27)$$

where

$$\widetilde{B}_{t,v} = B_{t,v} + \frac{\mu - r}{\sigma_v \sqrt{\chi^{H_2,\varepsilon}}} t \quad (28)$$

The risk-neutral process for the interest rate is

$$dr_t = (\kappa^*(\theta^* - r_t) + \sigma_r \phi_{t,r}^\varepsilon) dt + \sigma_r \sqrt{\chi^{H_3,\varepsilon}} d\tilde{B}_{t,r}. \quad (29)$$

where

$$\tilde{B}_{t,r} = B_{t,r} + \frac{\lambda r_t}{\sigma_r \sqrt{\chi^{H_3,\varepsilon}}} t \quad (30)$$

with $\kappa^* = \kappa + \frac{\lambda}{\sigma_r \sqrt{\chi^{H_3,\varepsilon}}}$ and $\theta^* = \frac{\kappa \theta}{\kappa + \frac{\lambda}{\sigma_r \sqrt{\chi^{H_3,\varepsilon}}}}$, are the risk-neutral parameter and λ the interest rate risk parameter which is given by Breeden[5].

Using Eq.(25), Eq.(27) and Eq.(29), we propose the new model appointed Mixed-Modified-Fractional-Hull-White-Vasicek(MMFHWV) model defined by the trivariate system of stochastic differential equations(SDEs) given in the following definition:

Definition 3.1. (MMFHWV model) *In the Mixed Modified Fractional Hull-White model Eq.(25) and Eq.(27), if we select the parameter μ as a stochastic process(see Mixed Modified Fractional Vasicek model Eq.(29)), then we will obtain the new model appointed Mixed-Modified-Fractional-Hull-White-Vasicek (MMFHWV) model and then, under the risk-neutral measure \mathbb{Q} , the dynamics of S_t , V_t and r_t are given by the SDEs:*

$$\begin{cases} dS_t = S_t \left\{ (r + b\sigma_s \phi_{t,s}^\varepsilon) dt + \sigma_s \sqrt{\chi^{H_1,\varepsilon}} d\tilde{B}_{t,s} \right\}, \\ dV_t = V_t \left\{ (r + b\sigma_v \phi_{t,v}^\varepsilon) dt + \sigma_v \sqrt{\chi^{H_2,\varepsilon}} d\tilde{B}_{t,v} \right\}, \\ dr_t = (\kappa^*(\theta^* - r_t) + \sigma_r \phi_{t,r}^\varepsilon) dt + \sigma_r \sqrt{\chi^{H_3,\varepsilon}} d\tilde{B}_{t,r}. \end{cases} \quad (31)$$

where $\tilde{B}_{t,s}$, $\tilde{B}_{t,v}$ and $\tilde{B}_{t,r}$ are three Brownian motions defined by Eq.(26), Eq.(28) and Eq.(30) respectively; with $\kappa^* = \kappa + \frac{\lambda}{\sigma_r \sqrt{\chi^{H_3,\varepsilon}}}$, $\theta^* = \frac{\kappa \theta}{\kappa + \frac{\lambda}{\sigma_r \sqrt{\chi^{H_3,\varepsilon}}}}$, are the risk-neutral parameters and λ the interest rate risk parameter.

4 Simulation of MMFHWV Model

We often face some difficult problems which can't be solved analytically. In these cases, simulation methods are appropriate and really helpful. In order to simulate the asset price S_t at time t , the volatility of asset price V_t at time t and the interest rate denote by r_t at time t , we need sample paths of asset price movement, sample paths of volatility of asset price movement and sample paths of interest rates evolution.

The Milstein discretization can be used to approximate the asset path of the asset, the volatility path of the volatility and the interest path of the interest rate on a discrete time grid, see Glasserman[14], Kloeden and Platen[23] and Kahl and Jackel[20].

Let $\Upsilon = \{t_1, t_2, \dots, t_N\}$ be a partition on time interval $[0, T]$; i.e. $0 = t_0 < t_1 < \dots < t_N = T$.

Then, we have

$$\begin{cases} S_{j+1} = S_j + \left\{ (r_j + b\sigma_s \phi_{j,s}^\varepsilon) S_j \Delta_t + \sigma_s \sqrt{\chi^{H_1,\varepsilon}} \Delta B_{i,j} \right\} \\ V_{j+1} = V_j + \left\{ (r_j + b\sigma_v \phi_{j,v}^\varepsilon) V_j \Delta_t + \sigma_v \sqrt{\chi^{H_2,\varepsilon}} \Delta B_{i,j} \right\} \\ r_{j+1} = r_j + \left\{ (\kappa^*(\theta^* - r_j) + \sigma_r \phi_{j,r}^\varepsilon) \Delta_t + \sigma_r \sqrt{\chi^{H_3,\varepsilon}} \Delta B_{i,j} \right\} \end{cases} \quad (32)$$

where $B_{i,j}$ is a Brownian motion and we have the following formula:

$$B_{i,j} = B_i(t_{j+1}) - B_i(t_j), 0 \leq j \leq N-1 \text{ and } i = 1, 2, 3.$$

Moreover, $\Delta B_{i,j} \sim \mathcal{N}(0, \Delta t)$ and $t_j = j \Delta t$.

According to the Central Limit Theorem (CLT), we have $\Delta B_{i,j} = Z_i \sqrt{\Delta t}$, $Z_i \sim \mathcal{N}(0, 1)$ and we obtain from Eq.(32)

$$\begin{cases} S_{j+1} = S_j + \left(r_t + b\sigma_s \phi_{j,s}^\varepsilon \right) S_j \Delta t + S_j \sigma_s \sqrt{\chi^{H_1, \varepsilon}} \sqrt{\Delta t} Z_1 + \frac{1}{2} \sigma_s^2 \sqrt{\chi^{H_1, \varepsilon}} S_j \Delta t (Z_1^2 - 1) \\ V_{j+1} = V_j + \left(r_j + b\sigma_s \phi_{j,s}^\varepsilon \right) V_j \Delta t + V_j \sigma_s \sqrt{\chi^{H_2, \varepsilon}} \sqrt{\Delta t} \Psi_1 + \frac{1}{2} \sigma_v^2 \sqrt{\chi^{H_2, \varepsilon}} V_j \Delta t (\Psi_1^2 - 1) \\ r_{j+1} = r_j + \left(\kappa^* (\theta^* - r_j) + b\sigma_r \phi_{j,r}^\varepsilon \right) \Delta t + \sigma_r \sqrt{\chi^{H_3, \varepsilon}} \sqrt{\Delta t} \Psi_2. \end{cases} \quad (33)$$

Where $\Psi_1 = \rho_{sv} Z_1 + \sqrt{1 - \rho_{sv}^2} Z_2$ and $\Psi_2 = \rho_{sr} Z_1 + \sqrt{1 - \rho_{sr}^2} Z_3$ are two correlated normal variables generated by the Cholesky decomposition. We have

$$\begin{cases} S_{j+1} = S_j + \left(r_t + b\sigma_s \phi_{j,s}^\varepsilon \right) S_j \Delta t + S_j \sigma_s \sqrt{\chi^{H_1, \varepsilon}} \sqrt{\Delta t} Z_1 + \frac{1}{2} \sigma_s^2 \sqrt{\chi^{H_1, \varepsilon}} S_j \Delta t (Z_1^2 - 1), \\ V_{j+1} = V_j + \left(r_j + b\sigma_s \phi_{j,s}^\varepsilon \right) V_j \Delta t + V_j \sigma_s \sqrt{\chi^{H_2, \varepsilon}} \sqrt{\Delta t} \Psi_1 + \frac{1}{2} \sigma_v^2 \sqrt{\chi^{H_2, \varepsilon}} V_j \Delta t (\Psi_1^2 - 1), \\ r_{j+1} = r_j + \left(\kappa^* (\theta^* - r_j) + b\sigma_r \phi_{j,r}^\varepsilon \right) \Delta t + \sigma_r \sqrt{\chi^{H_3, \varepsilon}} \sqrt{\Delta t} \Psi_2. \end{cases} \quad (34)$$

The simulation of the MMFHWV model is given by the following algorithm.

Algorithm. MMFHWV model simulation.

1. Set $\Delta t = \frac{t}{N}$.
 2. For $i = 1$ to number of simulation.
 3. Generate independent standard normal variables, $Z_j \sim \mathcal{N}(0, 1)$, $j = 1, \dots, N$.
 4. Set $V_{j+1} = V_j + \left(r_j + b\sigma_s \phi_{j,s}^\varepsilon \right) V_j \Delta t + V_j \sigma_s \sqrt{\chi^{H_2, \varepsilon}} \sqrt{\Delta t} \Psi_1 + \frac{1}{2} \sigma_v^2 \sqrt{\chi^{H_2, \varepsilon}} V_j \Delta t (\Psi_1^2 - 1)$.
 5. Set $S_{j+1} = S_j + \left(r_t + b\sigma_s \phi_{j,s}^\varepsilon \right) S_j \Delta t + S_j \sigma_s \sqrt{\chi^{H_1, \varepsilon}} \sqrt{\Delta t} Z_1 + \frac{1}{2} \sigma_s^2 \sqrt{\chi^{H_1, \varepsilon}} S_j \Delta t (Z_1^2 - 1)$.
 6. Set $r_{j+1} = r_j + \left(\kappa^* (\theta^* - r_j) + b\sigma_r \phi_{j,r}^\varepsilon \right) \Delta t + \sigma_r \sqrt{\chi^{H_3, \varepsilon}} \sqrt{\Delta t} \Psi_2$.
 7. End For.
-

In fig.1, 2 and 3, we illustrate changes in the value of stock path, volatility of the stock's path and interest rate path under modified Hurst parameter. The result indicate that:

- By increasing the value of H_1 , (see Fig.1), the value of the stock price is reduced

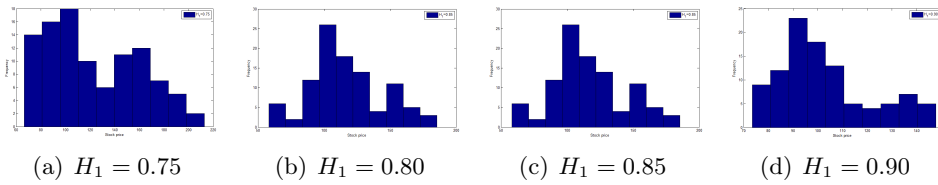


Figure 1: The Stock price paths under MMFHWV model with $S_0 = 115$, $V_0 = 0.01$, $r_0 = 0.9$, $\kappa = 0.2$, $\theta = 0.2$, $\sigma = 0.8$.

- By increasing the value of H_2 , (see Fig.2), the value of the volatility decrease and more, quickly convergence to zero.
- By increasing the value of H_3 , (see Fig.3), the value of the Interest rate increase.

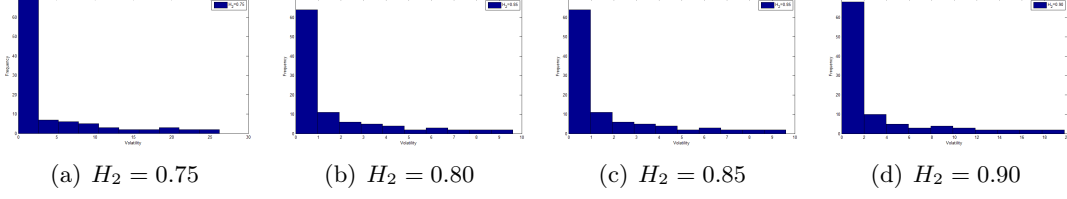


Figure 2: The volatility paths under MMFHWV model with $S_0 = 115, V_0 = 0.01, r_0 = 0.9, \kappa = 0.2, \theta = 0.2, \sigma = 0.8$.

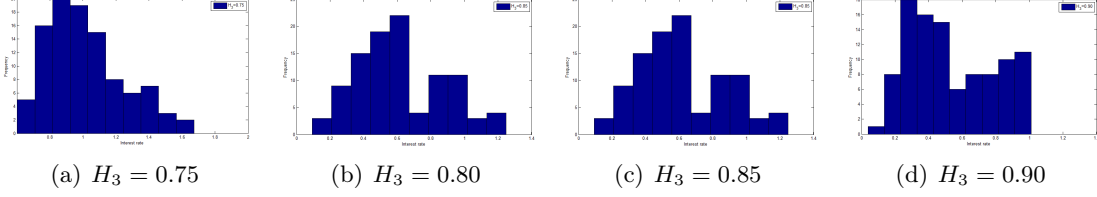


Figure 3: The Interest rate paths under MMFHWV model with $S_0 = 115, V_0 = 0.01, r_0 = 0.9, \kappa = 0.2, \theta = 0.2, \sigma = 0.8$.

5 Value of the call option under the MMFHWV model using the double Mellin transform

5.1 Bond price formula

In this subsection, we first give the pricing formula of a zero coupon bond in the mixed modified fractional Hull-White Vasicek interest rate model.

Let $P(t, r, T)$ denote the price of a default-free zero-coupon bond with the nominal value 1 unit of money at time T . By the Feynman-Kac formula, we have the following pricing formula.

Theorem 5.1. *In the fixed MMFHWV model, the price of a zero-coupon bond at time t is given by:*

$$P(\tau, r, T) = A(\tau)e^{-B(\tau)\tau}, \quad (35)$$

with $\tau = T - t$. $A(\tau)$ and $B(\tau)$ are given by

$$\begin{cases} A(\tau) = \exp \left\{ \frac{\chi^{H_3, \varepsilon} \sigma_r^2}{2\kappa^2 \left(\tau + \frac{2}{\kappa^*} (e^{-\kappa^* \tau} - 1) - \frac{1}{2\kappa^*} (e^{-\kappa^* \tau} - 1) \right)} \right\} - \int_0^\tau \kappa^* \theta^* (T - z) B(z) dz \\ B(\tau) = \frac{1 - e^{-\kappa^* \tau}}{\kappa^*} \end{cases} \quad (36)$$

Proof of Theorem 5.1. We recall that the no-arbitrage price at time t of the bond with the nominal value 1 is given by

$$P(t, r, T) = \mathbb{E} \left[e^{-\int_t^T r_s ds} | r_t = r \right], \quad t < T. \quad (37)$$

with the final condition $P(T, r, T) = 1$. Then by using the Mixed modified Fractional Feynman-Kac formula, the solution of $P(t, r, T)$ satisfies the following PDE problem

$$\frac{\partial P}{\partial t} + \frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 \frac{\partial^2 P}{\partial r^2} + \kappa^* (\theta^* - r) \frac{\partial P}{\partial r} - rP = 0. \quad (38)$$

The affine term structure solution of Eq.(38) is given by

$$P(\tau, r, T) = A(\tau)e^{-B(\tau)r}, \quad \tau = T - t \quad (39)$$

with If we substitute Eq.(39) into Eq.(38), we derive the following ordinary differential equation:

$$\frac{dA(\tau)}{d\tau} - \kappa^* \theta^* B(\tau) + \frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 B^2(\tau) = 0 \quad (40)$$

and

$$\frac{dB(\tau)}{d\tau} - \kappa^* B(\tau) + 1 = 0 \quad (41)$$

The resolution of Eq.(41) gives

$$B(\tau) = \frac{1 - e^{-\kappa^* \tau}}{\kappa^*} \quad (42)$$

and the resolution of Eq.(40) can be expressed in terms of $B(\tau)$ to give the value of $A(\tau)$

5.2 Pricing for Bull Spread Option under MMFHWV model

We recall that, if we use the traditional probabilistic method such as the Monte Carlo method to solve system given by Eq.(31), for Bull Spread options, that method produce interesting result, but it could requires a long time, depending on the complexity of calculations. However, in this section, using the double Mellin transform, we obtain a closed form formula for the option price, that is credible for both the accuracy and efficiency.

The bull spread option is an optimistic option strategy designed to take advantage of a moderate rise in the price of a security or asset. Its payoff function is defined as follows:

$$l(S_T) = [(S_T - K_1)^+ - (S_T - K_2)^+] \quad (43)$$

The vulnerable option is a kind of option with credit risk which refers to a risk that a borrower will default on any type of debt by failing to make the required payments. Its payoff function is defined as follows:

$$m(S_T) = [(S_T - K)^+] \left[\mathbb{1}_{\{V_T \geq \gamma^*\}} + \mathbb{1}_{\{V_T < \gamma^*\}} \frac{(1 - \alpha)V_T}{\gamma} \right], \quad (44)$$

where

$$\alpha$$

denotes the dead weight costs associated with bankruptcy expressed as a percentage of the value of the assets of the counterparty, the constant parameter D denotes the default boundary. The parameter γ^* may be less than D due to the possibility of a counterparty continuing its operation even while $V_T < \gamma$. In the event of a credit loss, only the proportion $\frac{(1-\alpha)V_T}{\gamma}$ of the nominal claim is paid out by the counterparty.

As aforementioned case for vulnerable bull spread option, expiring at time T , with the strike price K_1 and K_2 , the payoff function is given by

$$h(S_T, V_T) = [(S_T - K_1)^+ - (S_T - K_2)^+] \left[\mathbb{1}_{\{V_T \geq \gamma^*\}} + \mathbb{1}_{\{V_T < \gamma^*\}} \frac{(1 - \alpha)V_T}{\gamma} \right], \quad (45)$$

with γ^* is a critical value such that a credit loss occurs if the value of the option writer's assets and the no-arbitrage price of the option is given by

$$U(t, s, v, r) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} h(S_T, V_T) | S_t = s, V_t = v, r_t = r \right]. \quad (46)$$

By the non-arbitrage assumptions, the corresponding MMFHWV PDE of $U(t, s, v, r)$ according to the Feynman-Kac formula of bull spread option.

In this paper, we consider $U = U(S, V, r, T - t)$ as the common arguments in financial mathematics imply that if the option value function U is sufficiently smooth then under the MMFHWV model, it satisfies the partial differential equation given in the following theorem.

Theorem 5.2. *Under the MMFHWV model, the option value function U satisfies in the domain:*

$$\Omega = \{(S, V, r, t) : 0 < S < +\infty, 0 < V < +\infty, -\infty < r < +\infty, 0 < t < T\}, \quad (47)$$

the following partial differential equation is given by:

$$\begin{aligned} & \frac{\partial U}{\partial t} + \frac{1}{2} \chi^{H_1, \varepsilon} \sigma_s^2 s^2 \frac{\partial^2 U}{\partial s^2} + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 v^2 \frac{\partial^2 U}{\partial v^2} + \frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 \frac{\partial^2 U}{\partial r^2} \\ & + \rho_{sv} \sigma_s \sigma_v s v \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} \frac{\partial^2 U}{\partial s \partial v} + \rho_{sr} \sigma_s \sigma_r s \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \frac{\partial^2 U}{\partial s \partial r} \\ & + \rho_{vr} \sigma_v \sigma_r v \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \frac{\partial^2 U}{\partial v \partial r} + r \left(s \frac{\partial U}{\partial s} + v \frac{\partial U}{\partial v} \right) + \kappa^* (\theta^* - r) \frac{\partial U}{\partial r} - rU = 0 \end{aligned} \quad (48)$$

with the terminal condition $U(T, s, v, r) = h(s, v)$ and $\chi^{H_i, \varepsilon} = (a + b\varepsilon^{H_i - \frac{1}{2}})^2, i=1.2.3$.

Proof of Theorem 5.2. *The proof of this result is found in Appendix A.2*

5.3 Option price formula

In this subsection, we use the approach of Ji-Hun and Jeong-Hoon[19] to derive a price formula of a vulnerable bull Spread option under the Mixed-Modified-Fractional-Hull-White-Vasicek model by using the double mellin transform.

We observe that the payoff function h of a vulnerable bull spread call option is not bounded, we require to modify h by define a sequence of function h_n such that each h_n is bounded and $\lim_{n \rightarrow \infty} h_n(s, v, r) = h(s, v, r)$. They are given by

$$\begin{cases} h_n(s, v, r) = h_n^1(s, v, r)h_n^2(s, v, r) \\ h_n^1(s, v, r) = \begin{cases} (s - K_1) - (s - K_2), \text{ if } K_2 \leq K_1 < s < n \\ 0, \text{ otherwise} \end{cases} \\ h_n^2(s, v, r) = \begin{cases} 1, \text{ if } \gamma^* \leq v < n \\ \frac{(1-\alpha)v}{\gamma}, \text{ otherwise} \end{cases} \end{cases} \quad (49)$$

Setting $U_n(T, s, v, r)$ by

$$U_n(T, s, v, r) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} h(S_T, V_T) | S_t = s, V_t = v, r_t = r \right]. \quad (50)$$

We observe that, $U_n(t, s, v, r)$ satisfies Eq.(48) with the terminal condition $U_n(T, s, v, r) = h_n(s, v)$.

The price $U(t, s, v, r)$ is given by taking the limit $U(t, s, v, r) = \lim_{n \rightarrow \infty} U_n(t, s, v, r)$. Let $\widehat{U}_n(t, w_1, w_2, r)$ be the double Mellin transform of $U_n(t, s, v, r)$. Using Eq.(2), the expression of the inverse double Mellin transform of $U_n(t, s, v, r)$ is given by

$$U_n(t, s, v, r) = \frac{1}{(2\pi)^2} \int_{a_1 - i\infty}^{a_1 + i\infty} \int_{a_2 - i\infty}^{a_2 + i\infty} \widehat{U}_n(t, w_1, w_2, r) s^{-w_1} v^{-w_2} dw_1 dw_2 \quad (51)$$

By applying Eq.(51) into Eq.(48), we obtain

$$\begin{aligned} & -\frac{\partial \widehat{U}_n}{\partial \tau} + \left[\frac{1}{2} \chi^{H_1, \varepsilon} \sigma_s^2 w_1 (w_1 + 1) + \rho_{sv} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} \sigma_v \sigma_s w_1 w_2 + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 w_2 (w_2 + 1) - r(w_1 + w_2 + 1) \right] \widehat{U}_n \\ & + \frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 \frac{\partial^2 \widehat{U}_n}{\partial r^2} + \left(\kappa^* \theta^* - \kappa^* r - \rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r w_1 - \rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r w_2 \right) \frac{\partial \widehat{U}_n}{\partial r} = 0 \end{aligned} \quad (52)$$

with $\tau = T - t$ and $\widehat{U}_n(0, w_1, w_2, r) = \widehat{h}(w_1, w_2)$ which is the Mellin transform of $h_n(s, v)$.

Let

$$\begin{aligned} \widehat{U}_n(\tau, w_1, w_2, r) &= \exp \left\{ \left(\frac{1}{2} \chi^{H_1, \varepsilon} \sigma_s^2 w_1 (w_1 + 1) + \rho_{sv} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} \sigma_v \sigma_s w_1 w_2 + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 w_2 (w_2 + 1) \right) \tau \right\} \\ &\times \widehat{f}_n(\tau, w_1, w_2, r). \end{aligned} \quad (53)$$

Then, Eq.(52) is transformed into the following PDE for \widehat{f}_n .

$$\begin{aligned} & -\frac{\partial \widehat{f}_n}{\partial \tau} + \frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 \frac{\partial^2 \widehat{f}_n}{\partial r^2} + \left(\kappa^* \theta^* - \kappa^* r - \rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r w_1 - \rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r w_2 \right) \frac{\partial \widehat{f}_n}{\partial r} \\ & - r(w_1 + w_2 + 1) \widehat{f}_n = 0 \end{aligned} \quad (54)$$

with $\widehat{f}_n(0, w_1, w_2, r) = \widehat{U}_n(0, w_1, w_2, r) = \widehat{h}(w_1, w_2)$.

To solve Eq.(54), we let

$$\widehat{f}_n(\tau, w_1, w_2, r) = \widehat{h}(w_1, w_2) Q(\tau, w_1, w_2) e - L(\tau)(w_1 + w_2 + 1)r. \quad (55)$$

We have the terminal condition $\widehat{f}_n(0, w_1, w_2, r) = \widehat{h}_n(w_1, w_2)$.

Substituting Eq.(55) into Eq.(54), we obtain the following ODEs:

$$\begin{aligned} & \frac{\partial Q}{\partial \tau} + \left(\left(\kappa \theta (T - \tau) - \rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r w_1 - \rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r w_2 \right) L(\tau)(w_1 + w_2 + 1) \right) Q \\ & - \left(\frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\tau)(w_1 + w_2 + 1)^2 \right) Q = 0 \end{aligned} \quad (56)$$

$$\frac{\partial L}{\partial \tau} + \kappa L - 1 = 0 \quad (57)$$

where $Q(0, w_1, w_2) = 1$ and $L(0) = 0$.

Then, by solving Eq.(56), we have

$$\begin{aligned} Q(\tau, w_1, w_2) &= \exp \left\{ \frac{\chi^{H_3, \varepsilon} \sigma_r^2}{2\kappa^2} \left(\tau + \frac{2}{\kappa^*} (e^{-\kappa^* \tau} - 1) - \frac{1}{2\kappa^*} (e^{-2\kappa^* \tau} - 1) \right) (w_1 + w_2 + 1)^2 \right\} \\ &+ \exp \left\{ \left(\frac{\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r}{\kappa^*} w_1 (w_1 + w_2 + 1) + \frac{\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r}{\kappa^*} w_2 (w_1 + w_2 + 1) \right) (\tau - L(\tau)) \right\} \\ &- \exp \left\{ \int_0^\tau (w_1 + w_2 + 1) \kappa^* \theta^* (T - \nu) L(\nu) d\nu \right\} \end{aligned}$$

and

$$L(\tau) = \frac{1 - e^{-\kappa^* \tau}}{\kappa^*}. \quad (58)$$

Then, from Eq.(53), we obtain

$$\begin{aligned} \widehat{U}_n(\tau, w_1, w_2, r) &= \widehat{h}_n(w_1, w_2) \exp \{ (N(\tau)(w_1 + w_2 + 1)^2) \} \\ &+ \exp \left\{ \left(\frac{1}{2} \chi^{H_1, \varepsilon} \sigma_s^2 w_1 (w_1 + 1) + \rho_{sv} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 w_2 (w_2 + 1) \right) \right\} \\ &+ \exp \left\{ \left(\frac{\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r}{\kappa^*} w_1 (w_1 + w_2 + 1) + \frac{\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r}{\kappa^*} w_2 (w_1 + w_2 + 1) \right) (\tau - L(\tau)) \right\} \\ &- \exp \{ M(\tau) + L(\tau)r(w_1 + w_2 + 1) \} \end{aligned} \quad (59)$$

with $N(\tau)$ and $M(\tau)$ are given by

$$\begin{cases} N(\tau) = \frac{\chi^{H_3, \varepsilon} \sigma_r^2}{2(\kappa^*)^2} \left(\tau + \frac{2}{\kappa^*} (e^{-\kappa^* \tau} - 1) - \frac{1}{2\kappa^*} (e^{-2\kappa^* \tau} - 1) \right) \\ M(\tau) = \int_0^\tau \kappa \theta (T - \nu) L(\nu) d\nu \end{cases} \quad (60)$$

We have

$$U_n(\tau, s, v, r) = \frac{1}{(2\pi)^2} \int_{a_1 - i\infty}^{a_1 + i\infty} \int_{a_2 - i\infty}^{a_2 + i\infty} \widehat{h}_n(w_1, w_2) e^{\widehat{J}(\tau, w_1, w_2, r)} s^{-w_1} v^{-w_2} dw_1 dw_2 \quad (61)$$

where $\widehat{J}(\tau, w_1, w_2, r)$ is given by

$$\begin{aligned} \widehat{J}(\tau, w_1, w_2, r) &= N(\tau)(w_1 + w_2 + 1)^2 + \left(\frac{1}{2} \chi^{H_2, \varepsilon} \sigma_s^2 w_1 (w_1 + 1) + \rho_{sv} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 w_2 (w_2 + 1) \right) \tau \\ &+ \left(\frac{\rho_{sr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r}{\kappa^*} w_1 (w_1 + w_2 + 1) + \frac{\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r}{\kappa^*} w_2 (w_1 + w_2 + 1) \right) (\tau - L(\tau)) \\ &- (M(\tau) + L(\tau)r)(w_1 + w_2 + 1). \end{aligned} \quad (62)$$

We define the following expression to use to compute $U_n(\tau, s, v, r)$

$$\Lambda(\tau, s, v, r) = \frac{1}{(2\pi)^2} \int_{a_1 - i\infty}^{a_1 + i\infty} e^{\overline{C}(\tau, w_2, r)} \left[\int_{a_2 - i\infty}^{a_2 + i\infty} \widehat{h}_n(w_1, w_2) e^{\{\overline{A}(\tau)\xi^2(\tau)\}} s^{-w_1} dw_1 \right] v^{-w_2} dw_2. \quad (63)$$

Now, the term $\widehat{J}(\tau, w_1, w_2, r)$ have the following quadratic form of w_1 and we obtain

$$\Lambda(\tau, s, v, r) = \frac{1}{(2\pi)^2} \int_{a_1 - i\infty}^{a_1 + i\infty} e^{\overline{C}(\tau, w_2, r)} \left[\int_{a_2 - i\infty}^{a_2 + i\infty} \widehat{h}_n(w_1, w_2) e^{\{\overline{A}(\tau)\xi^2(\tau)\}} s^{-w_1} dw_1 \right] v^{-w_2} dw_2 \quad (64)$$

with $\xi(\tau) = w_1 + \frac{\overline{A}(\tau) + \overline{B}(\tau) + \overline{B}(\tau)w_2}{2\overline{A}(\tau)}$, where

$$\begin{cases} \overline{A}(\tau) = N(\tau) + \frac{1}{2} \chi^{H_1, \varepsilon} \sigma_s^2 \tau + \frac{\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r}{\kappa^*} (\tau - L(\tau)) \\ \overline{B}(\tau) = 2N(\tau) + \rho_{sv} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} + \left(\frac{\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r}{\kappa^*} + \frac{\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r}{\kappa^*} \right) (\tau - L(\tau)) \\ \overline{\overline{B}}(\tau) = N(\tau) - M(\tau) - L(\tau)r. \end{cases}$$

and

$$\begin{aligned}\bar{C}(\tau, w_2, r) &= -\frac{1}{4} \frac{\left(\bar{A}(\tau) + \bar{B}(\tau) + \bar{B}(\tau)w_2\right)^2}{\bar{A}(\tau)} + N(\tau)(w_2 + 1)^2 \\ &+ \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 (w_2^2 + w_2) \tau + \frac{\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon} \sigma_v \sigma_r}}{\kappa^*} (\tau - L(\tau)) \\ &- M(\tau)(w_2 + 1) - L(\tau)(w_2 + 1)r.\end{aligned}$$

We observe that $\bar{A}(\tau) \geq \frac{1}{2} \int_0^\tau (\sigma_s - \sigma_r L(\nu))^2 d\nu$, we deduce that $\bar{A}(\tau) > 0$, for all $\tau > 0$, according Lemma 2.1, we have

$$\frac{1}{2\pi i} \left[\int_{a_2 - i\infty}^{a_2 + i\infty} e^{\{\bar{A}(\tau)\xi^2(\tau)\}} s^{-w_1} dw_1 \right] = \frac{1}{\sqrt{\pi \bar{A}(\tau)}} s^{\frac{\bar{A}(\tau) + \bar{B}(\tau) + \bar{B}(\tau)w_2}{2\bar{A}(\tau)}} e^{-\frac{1}{4\bar{A}(\tau)} (\ln s)^2}.$$

Therefore, Eq.(64) becomes

$$\Lambda(\tau, s, v, r) = \frac{e^{-\frac{1}{4\bar{A}(\tau)} (\ln s)^2}}{2\sqrt{\pi \bar{A}(\tau)}} s^{\frac{\bar{A}(\tau) + \bar{B}(\tau)}{2\bar{A}(\tau)}} \frac{1}{2\pi i} \int_{a_1 - i\infty}^{a_1 + i\infty} e^{\bar{C}(\tau, w_2, r)} s^{\frac{\bar{B}(\tau)w_2}{2\bar{A}(\tau)}} v^{-w_2} dw_2. \quad (65)$$

In the Eq.(65), the integrand have a quadratic form of w_2 as follows

$$\begin{aligned}& \frac{1}{2\pi i} \int_{a_1 - i\infty}^{a_1 + i\infty} e^{\bar{C}(\tau, w_2, r)} s^{\frac{\bar{B}(\tau)w_2}{2\bar{A}(\tau)}} v^{-w_2} dw_2 \\ &= \exp \left\{ -\frac{\left(B_3(\tau) + \frac{\bar{B}(\tau)}{2\bar{A}(\tau)} \ln s\right)^2}{4B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}}} - \frac{(\bar{A}(\tau) - \bar{B}_2)^2}{4\bar{A}(\tau)} \right\} \\ &\times \frac{1}{2\pi i} \int_{a_1 - i\infty}^{a_1 + i\infty} \exp \left\{ \left(B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}\right) \left(w_2^2 + \frac{B_3(\tau) + \frac{\bar{B}(\tau)}{2\bar{A}(\tau)} \ln s}{2B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}\right)^2 \right\} v^{-w_2} dw_2,\end{aligned} \quad (66)$$

where

$$\begin{cases} B_1(\tau) = N(\tau) + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 \tau + \frac{\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon} \sigma_v \sigma_r}}{\kappa^*} (\tau - L(\tau)) \\ B_3(\tau) = -\frac{\bar{B}(\tau)}{2\bar{A}(\tau)} (\bar{A}(\tau) + B_2(\tau)) + B_1(\tau) + B_2(\tau).\end{cases}$$

By setting

$$\begin{cases} \hat{\sigma}_s^2(t) = \chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon} \sigma_s \sigma_r} L(t) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(t) \\ \hat{\sigma}_v^2(t) = \chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon} \sigma_v \sigma_r} L(t) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(t).\end{cases} \quad (67)$$

We have

$$\begin{cases} \bar{A}(\tau) = \frac{1}{2} \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \\ B_1(\tau) = \frac{1}{2} \int_0^\tau \hat{\sigma}_v^2(\nu) d\nu \\ \bar{B}(\tau) = \int_0^\tau \left[\rho_{sv} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon} \sigma_s \sigma_v} + \sigma_r L(\nu) (\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon} \sigma_s} + \rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon} \sigma_r}) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right] d\nu.\end{cases} \quad (68)$$

While the correlation coefficients ρ_{sr} and ρ_{vr} are constants satisfying Eq.(20), then $B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} > 0$ is satisfied for $\tau > 0$. From Lemma 2.1 applied to Eq.(66), we have

$$\begin{aligned}& \frac{1}{2\pi i} \int_{a_1 - i\infty}^{a_1 + i\infty} \exp \left\{ \left(B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}\right) \left(w_2^2 + \frac{B_3(\tau) + \frac{\bar{B}(\tau)}{2\bar{A}(\tau)} \ln s}{2B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}\right)^2 \right\} v^{-w_2} dw_2 \\ &= \frac{1}{2\sqrt{\pi \left(B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}\right)}} \exp \left\{ -\frac{1}{4 \left(B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}\right)} (\ln v)^2 \right\} v^{\frac{B_3(\tau) + \frac{\bar{B}(\tau)}{2\bar{A}(\tau)} \ln s}{2B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}}\end{aligned} \quad (69)$$

Therefore, using Eq.(65), Eq.(66) and Eq.(69), we obtain the following explicit of $\Lambda(\tau, s, v, r)$ which is given by

$$\begin{aligned} \Lambda(\tau, s, v, r) &= \exp \left\{ -\frac{\left(B_3(\tau) + \frac{\overline{B}(\tau)}{2\overline{A}(\tau)} \ln s \right)^2}{4 \left(B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)} \right)} - \frac{\left(\overline{A}(\tau) - \overline{B}(\tau) \right)^2}{4\overline{A}(\tau)} \right\} \\ &\times s^{\frac{\overline{A}(\tau) + \overline{B}(\tau) + \overline{B}(\tau) w_2}{2\overline{A}(\tau)}} v^{\frac{B_3(\tau) + \frac{\overline{B}(\tau)}{2\overline{A}(\tau)} \ln s}{2 \left(B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)} \right)}} \\ &\times \frac{e^{-\frac{1}{4\overline{A}(\tau)} (\ln s)^2}}{2\sqrt{\pi\overline{A}(\tau)}} e^{-\frac{1}{4 \left(B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)} \right)} (\ln v)^2}}{2\sqrt{\pi \left(B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)} \right)}}. \end{aligned} \quad (70)$$

Now the expression of $U_n(\tau, s, v, r)$ is obtain by using the Mellin convolution result given by Lemma 2.2, since $\widehat{h}(w_1, w_2)$ and $e^{\widehat{J}(\tau, s, v, r)}$ are the double Mellin transforms of $h_n(s, v)$ and $\Lambda(\tau, s, v, r)$, respectively.

By using Eq.(45), Eq.(61) becomes

$$\begin{aligned} U_n(\tau, s, v, r) &= \int_0^\infty \int_0^\infty h_n(u, w) \Lambda\left(\tau, \frac{s}{u}, \frac{v}{w}, r\right) u^{-1} w^{-1} dudw \\ &= e^{\widehat{k}} \int_{\gamma^*}^n \int_{K_1}^n e^{\widehat{\phi}_1(u)} (u - K_1) \widehat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw - e^{\widehat{k}} \int_{\gamma^*}^n \int_{K_2}^n e^{\widehat{\phi}_1(u)} (u - K_2) \widehat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw \\ &+ e^{\widehat{k}} \widehat{\delta} \int_0^{\gamma^*} \int_{K_1}^\infty e^{\widehat{\phi}_1(u)} w (u - K_1) \widehat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw - e^{\widehat{k}} \widehat{\delta} \int_0^{\gamma^*} \int_{K_2}^\infty e^{\widehat{\phi}_1(u)} w (u - K_2) \widehat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw. \end{aligned}$$

with

$$\widehat{k} = -\frac{1}{4} \frac{\left(\overline{A}(\tau) - \overline{B}(\tau) \right)^2}{\overline{A}(\tau)},$$

where

$$\begin{cases} \widehat{\delta} = \frac{1-\alpha}{\gamma} \\ \widehat{\phi}_1(u) = -\frac{1}{4} \frac{\left(B_3(\tau) + \frac{\overline{B}(\tau)}{2\overline{A}(\tau)} \ln s \right)^2}{B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)}}, \end{cases}$$

and

$$\begin{aligned} \widehat{\phi}_2(u, w) &= \left(\frac{s}{u} \right)^{\frac{\overline{A}(\tau) + \overline{B}(\tau)}{2\overline{A}(\tau)}} \left(\frac{v}{w} \right)^{\frac{B_3(\tau) + \frac{\overline{B}(\tau)}{2\overline{A}(\tau)} \ln \left(\frac{s}{u} \right)}{B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)}}} \\ &\times \frac{e^{-\frac{1}{4\overline{A}(\tau)} (\ln \left(\frac{s}{u} \right))^2}}{2\sqrt{\pi\overline{A}(\tau)}} e^{-\frac{1}{4 \left(B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)} \right)} (\ln \left(\frac{v}{w} \right))^2}}{2\sqrt{\pi \left(B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)} \right)}}. \end{aligned}$$

By setting $U(\tau, s, v, r) = \lim_{n \rightarrow \infty} U_n(\tau, s, v, r)$, it follow that

$$\begin{aligned} U(\tau, s, v, r) &= \int_0^\infty \int_0^\infty h_n(u, w) \Lambda\left(\tau, \frac{s}{u}, \frac{v}{w}, r\right) u^{-1} w^{-1} dudw \\ &= e^{\widehat{k}} \int_{\gamma^*}^n \int_{K_1}^n e^{\widehat{\phi}_1(u)} (u - K_1) \widehat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw - e^{\widehat{k}} \int_{\gamma^*}^n \int_{K_2}^n e^{\widehat{\phi}_1(u)} (u - K_2) \widehat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw \\ &+ e^{\widehat{k}} \widehat{\delta} \int_0^{\gamma^*} \int_{K_1}^\infty e^{\widehat{\phi}_1(u)} w (u - K_1) \widehat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw - e^{\widehat{k}} \widehat{\delta} \int_0^{\gamma^*} \int_{K_2}^\infty e^{\widehat{\phi}_1(u)} w (u - K_2) \widehat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw \\ &= U^1(\tau, s, v, r) + V^1(\tau, s, v, r). \end{aligned} \quad (71)$$

We deduce that

$$U(\tau, s, v, r) = U^1(\tau, s, v, r) + V^1(\tau, s, v, r), \quad (72)$$

with

$$U^1(\tau, s, v, r) = \tilde{U}^1(\tau, s, v, r) - \bar{U}^1(\tau, s, v, r) \quad (73)$$

$$V^1(\tau, s, v, r) = \tilde{V}^1(\tau, s, v, r) - \bar{V}^1(\tau, s, v, r), \quad (74)$$

where

$$\tilde{U}^1(\tau, s, v, r) = e^{\hat{k}} \int_{\gamma^*}^n \int_{K_1}^n e^{\hat{\phi}_1(u)} (u - K_1) \hat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw \quad (75)$$

$$\bar{U}^1(\tau, s, v, r) = e^{\hat{k}} \int_{\gamma^*}^n \int_{K_2}^n e^{\hat{\phi}_1(u)} (u - K_2) \hat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw. \quad (76)$$

and

$$\tilde{V}^1(\tau, s, v, r) = e^{\hat{k}} \hat{\delta} \int_0^{\gamma^*} \int_{K_1}^{\infty} e^{\hat{\phi}_1(u)} w (u - K_1) \hat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw \quad (77)$$

$$\bar{V}^1(\tau, s, v, r) = e^{\hat{k}} \hat{\delta} \int_0^{\gamma^*} \int_{K_2}^{\infty} e^{\hat{\phi}_1(u)} w (u - K_2) \hat{\phi}_2(u, w) \frac{1}{u} \frac{1}{w} dudw. \quad (78)$$

The explicit calculation of $U^1(\tau, s, v, r)$ and $V^1(\tau, s, v, r)$ of Eq.(72) leads to the following analytic closed form formula of the price of vulnerable bull spread call option.

By letting $U(t, s, v, r) = CallBullSpread(t, s, v, r)$, we have:

Theorem 5.3. *The price of Bull Spread vulnerable call option denote by $CallBullSpread(t, s, v, r)$ with the Mixed-Modified-Fractional-Hull-White-Vasicek model, defined by Eq.(46), is explicitly given by*

$$CallBullSpread(t, s, v, r) = s\mathcal{N}(\hat{a}_1, \hat{a}_2, \xi) - K_1 P(t, r, T) \mathcal{N}(\hat{b}_1, \hat{b}_2, \xi) \quad (79)$$

$$- s\mathcal{N}(\hat{a}_1, \hat{a}_2, \xi) + K_2 B(t, r, T) \mathcal{N}(\hat{b}_1, \hat{b}_2, \xi) \\ + \hat{\delta} v \left(s \frac{p(t, T)}{P(t, r, T)} \mathcal{N}(\hat{c}_1, \hat{c}_2, -\xi) - K_1 \mathcal{N}(\hat{d}_1, \hat{d}_2, -\xi) \right) \quad (80)$$

$$- \hat{\delta} v \left(s \frac{p(t, T)}{P(t, r, T)} \mathcal{N}(\hat{c}_1, \hat{c}_2, -\xi) + K_2 \mathcal{N}(\hat{d}_1, \hat{d}_2, -\xi) \right) \quad (81)$$

with $\xi = \frac{\bar{B}(\tau)}{2\sqrt{A(\tau)B_1(\tau)}}$ where

$$\hat{a}_1 = \frac{\ln\left(\frac{s}{K_1}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^T \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^T \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \quad (82)$$

$$\hat{b}_1 = \frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^T \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^T \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \\ + \frac{2\xi \int_0^T \hat{\sigma}_s^2(\nu) d\nu \int_0^T \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^T \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \quad (83)$$

$$\hat{a}_2 = \frac{\ln\left(\frac{s}{K_1}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^T \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^T \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \quad (84)$$

$$\hat{b}_2 = \frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^T \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^T \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \quad (85)$$

$$\begin{aligned} \widehat{c}_1 = & \frac{\ln\left(\frac{s}{K_2}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\ & + \frac{2\xi \int_0^\tau \widehat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \end{aligned} \quad (94)$$

$$\begin{aligned} \widehat{c}_2 = & - \frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\ & - \frac{2\xi \int_0^\tau \widehat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \end{aligned} \quad (95)$$

$$\begin{aligned} \widehat{d}_1 = & \frac{\ln\left(\frac{s}{K_2}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\ & + \frac{2\xi \int_0^\tau \widehat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \end{aligned} \quad (96)$$

$$\widehat{d}_2 = - \frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \quad (97)$$

Proof of Theorem 5.3. *The proof of this result is found in Appendix A.3*

In the case of a vulnerable bull spread put option with Hull-White-Vasicek volatility and interest rates model using Mixed Modified Fractional Brownian Motion, the option pricing formula is given by the following result:

Theorem 5.4. *The price of Bull Spread vulnerable put option denote by $PutBullSpread(t, s, v, r)$, with the Mixed-Modified-Fractional-Hull-White-Vasicek volatility and interest rates model, defined by Eq.(46), is explicitly given by*

$$PutBullSpread(t, s, v, r) = -s\mathcal{N}(-\widehat{a}_1, \widehat{a}_2, \xi) + K_1 P(t, r, T) \mathcal{N}(-\widehat{b}_1, \widehat{b}_2, \xi) \quad (98)$$

$$+ s\mathcal{N}(-\widehat{a}_1, \widehat{a}_2, \xi) - K_2 P(t, r, T) \mathcal{N}(-\widehat{b}_1, \widehat{b}_2, \xi) \\ + \widehat{\delta}v \left(-s \frac{p(t, T)}{P(t, r, T)} \mathcal{N}(-\widehat{c}_1, \widehat{c}_2, -\xi) + K_1 \mathcal{N}(-\widehat{d}_1, \widehat{d}_2, -\xi) \right) \quad (99)$$

$$- \widehat{\delta}v \left(-s \frac{p(t, T)}{P(t, r, T)} \mathcal{N}(-\widehat{c}_1, \widehat{c}_2, -\xi) - K_2 \mathcal{N}(-\widehat{d}_1, \widehat{d}_2, -\xi) \right) \quad (100)$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2, \widehat{c}_1, \widehat{c}_2, \widehat{d}_1, \widehat{d}_2, \widehat{c}_1, \widehat{c}_2, \widehat{d}_1, \widehat{d}_2, \xi$ and $p(t, T)$ are given by Theorem 5.3.

We deduce that the bull Call spread reduces the cost of the call option, but it comes with a trade-off. The gains in the stock's price are also capped, creating a limited range where the investor can make a profit. Traders will use the bull call spread if they believe an asset will moderately rise in value.

6 Conclusion

As an extension of Hull and White [18] and Vasicek[33] models, this paper proposes a new three-factor pricing model for vulnerable European options incorporating spreads and interest rates whose movements

are governed by mixed modified fractional stochastic differential equations. The dynamics of the short-term interest rate is represented implicitly by a fractional stochastic process of the price of zero-coupon bond. Under our original assumption, we have obtain the bond price closed form formula. We have discretize the stochastic process with Milstein discretization scheme and we have showed the sample paths of asset price movement, the sample paths of volatility of asset price movement and the sample paths of interest rates of asset price movement. Furthermore, we exploit Mellin's transform techniques to derive a closed-form solution for vulnerable Bull Spread options model, which is simply calculated using the standard normal cumulative distribution function so that pricing and coverage of vulnerable European options can be calculated accurately and quickly.

The applications of our results are numerous, in particular to for a realistic credit risk measurement (see Johnson and Stuly [12]) and Klein[22]) and Amman[1].

A Appendix

A.1 Notations

- μ the drift of the process for the stock and for the variance;
- σ_s is the volatility of the stock price;
- σ_v is the volatility of the variance;
- κ the mean reversion speed for the interest rate;
- θ the mean reversion level for the interest rate;
- σ_r the volatility of the interest rate;
- μ the drift of the process for the stock and for the variance;
- $S_0 > 0$ is the spot asset price;
- $V_0 > 0$ is the initial (time zero)level of the variance;
- $r_0 > 0$ is the initial (time zero)level of the interest rates;
- \mathbb{Q} is a Risk neutral probability measure and \mathbb{P} is a probability measure;
- S_t is the dynamic stock price, V_t is its volatility and r_t is its interest rate;
- $P(t, r, T)$ is the default price of a zero-coupon bond maturing at time T with the nominal value 1;
- U is the price of the bull spread vulnerable option at time t ;
- \mathcal{N} is the central distribution function of the normal distribution, $\mathcal{N}(0, 1)$;
- $M_{.,s}^{H_1,\varepsilon}$ and $M_{.,v}^{H_2,\varepsilon}, M_{.,r}^{H_3,\varepsilon}$ are three Mixed Modified Fractional Brownian motions with Hurst parameters $H_1, H_2, H_3 \in [\frac{1}{2}, 1[$;
- ε is the adjustment coefficient between the times of cotations on the financial market;
- γ^* is a critical value such that a credit loss occurs if the value of the option writer's assets;
- K_1 and K_2 are the strike of the bull spread option;
- $p(t, T)$ is the price at time $0 < t < T$ of a zero coupon with maturity T .

A.2 Proof of Theorem 5.2

Let $U = U(s, v, r, t)$ be the price of a bull spread option and Π denotes the value of the portfolio at time t . We have

$$\Pi = U - n_1 s - n_2 v - n_3 P \quad (101)$$

The variation of the portfolio is defined by

$$d\Pi = dU - n_1 ds - n_2 dv - n_3 dP \quad (102)$$

By applying the fractional Ito's lemma to Eq.(102) as well as choosing

$$n_1 = \frac{\partial U}{\partial s}; n_2 = \frac{\partial U}{\partial v}; n_3 = \frac{\partial U}{\partial r}, \quad (103)$$

to eliminate the stochastic noise contained in the dS , dV and dr respectively. We obtain

$$d\Pi = \frac{\partial U}{\partial t} dt + \frac{1}{2} \frac{\partial^2 U}{\partial s^2} (ds)^2 + \frac{1}{2} \frac{\partial^2 U}{\partial v^2} (dV)^2 + \frac{\partial^2 U}{\partial s \partial v} (dsdV) + \frac{\partial^2 U}{\partial s \partial r} (dsdr) + \frac{\partial^2 U}{\partial v \partial r} (dVdr) - n_3 \left[\frac{\partial U}{\partial t} dt + \frac{1}{2} \frac{\partial^2 U}{\partial r^2} (dr)^2 \right] \quad (104)$$

with $(dS_t)^2$, $(dV_t)^2$, $(dr)^2$, $dS_t dV_t$, $dS_t dr$, $dV_t dr_t$ are given respectively by

$$\begin{cases} (dS_t)^2 = \chi^{H_1, \varepsilon} \sigma_s^2 s^2 dt \\ (dV_t)^2 = \chi^{H_2, \varepsilon} \sigma_v^2 v^2 dt \\ (dr_t)^2 = \chi^{H_3, \varepsilon} \sigma_r^2 dt \\ dS_t dV_t = \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} \sigma_s \sigma_v s v \rho_{sv} dt \\ dS_t dr_t = \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r s \rho_{sr} dt \\ dV_t dr_t = \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r v \rho_{vr} dt. \end{cases} \quad (105)$$

Substituting Eq.(105) into Eq.(104), we have

$$\begin{cases} d\Pi = \left\{ \frac{\partial U}{\partial t} + \left[\frac{1}{2} \chi^{H_1, \varepsilon} \sigma_s^2 s^2 \frac{\partial^2 U}{\partial s^2} + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 v^2 \frac{\partial^2 U}{\partial v^2} \right] \right\} dt \\ + \left\{ \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} \sigma_s \sigma_v s v \rho_{sv} \frac{\partial^2 U}{\partial s \partial v} + \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r s \rho_{sr} \frac{\partial^2 U}{\partial s \partial r} + \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r v \rho_{vr} \frac{\partial^2 U}{\partial v \partial r} \right\} dt \\ - n_3 \left[\frac{\partial U}{\partial t} + \frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 \frac{\partial^2 U}{\partial r^2} \right]. \end{cases} \quad (106)$$

On the other hand, it is known that the return for any risk-free portfolio should be r . Hence, the variation value of the portfolio is

$$d\Pi_t = r \left(U - s \frac{\partial U}{\partial s} - v \frac{\partial U}{\partial v} - n_3 P \right) dt \quad (107)$$

By substituting Eq.(106) and Eq.(38) into Eq.(107) gives

$$\begin{cases} \left\{ \frac{\partial U}{\partial t} + \left[\frac{1}{2} \chi^{H_1, \varepsilon} \sigma_s^2 s^2 \frac{\partial^2 U}{\partial s^2} + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 v^2 \frac{\partial^2 U}{\partial v^2} \right] \right\} dt \\ + \left\{ \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} \sigma_s \sigma_v s v \rho_{sv} \frac{\partial^2 U}{\partial s \partial v} + \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r s \rho_{sr} \frac{\partial^2 U}{\partial s \partial r} + \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r v \rho_{vr} \frac{\partial^2 U}{\partial v \partial r} \right\} dt \\ - n_3 \left[\frac{\partial U}{\partial t} + \frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 \frac{\partial^2 U}{\partial r^2} \right] = (rU - rs \frac{\partial U}{\partial s} - rv \frac{\partial U}{\partial v} - rn_3 P). \end{cases} \quad (108)$$

In the above statement Eq.(109), we use Eq.(38) and the value of n_3 given by Eq.(103) to obtain

$$\begin{aligned} & \frac{\partial U}{\partial t} + \frac{1}{2} \chi^{H_1, \varepsilon} \sigma_s^2 s^2 \frac{\partial^2 U}{\partial s^2} + \frac{1}{2} \chi^{H_2, \varepsilon} \sigma_v^2 v^2 \frac{\partial^2 U}{\partial v^2} + \frac{1}{2} \chi^{H_3, \varepsilon} \sigma_r^2 \frac{\partial^2 U}{\partial r^2} \\ & + \rho_{sv} \sigma_s \sigma_v s v \sqrt{\chi^{H_1, \varepsilon} \chi^{H_2, \varepsilon}} \frac{\partial^2 U}{\partial s \partial v} + \rho_{sr} \sigma_s \sigma_r s \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \frac{\partial^2 U}{\partial s \partial r} \\ & + \rho_{vr} \sigma_v \sigma_r v \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \frac{\partial^2 U}{\partial v \partial r} + r \left(s \frac{\partial U}{\partial s} + v \frac{\partial U}{\partial v} \right) + \kappa^* (\theta^* - r) \frac{\partial U}{\partial r} - rU = 0 \end{aligned} \quad (109)$$

A.3 Proof of Theorem 5.3

A.3.1 Expression of $U^1(\tau, s, v, r)$

- **Expression of $\tilde{U}^1(\tau, s, v, r)$.** We begin to take changes of variables

$$\begin{cases} x = \frac{\ln(\frac{s}{w})}{\sqrt{2A(\tau)}} \\ y = \frac{\ln(\frac{v}{w})}{\sqrt{2B(\tau)}}. \end{cases} \quad (110)$$

and

$$\begin{cases} \eta_1(x) = \left(e^{-x\sigma_s\sqrt{\tau}} - K_1 \right) \\ \Sigma(\tau) = \frac{B_3^2(\tau)}{4[B_1(\tau) - \frac{B^2(\tau)}{4A(\tau)}]}. \end{cases} \quad (111)$$

We have:

$$\begin{aligned}
& \tilde{U}^1(\tau, s, v, r) \tag{112} \\
&= \frac{1}{2\pi} \sqrt{\frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \int_{\frac{\ln(\frac{v}{\gamma^*})}{\sqrt{2B_1(\tau)}}}^{\infty} \int_{\frac{\ln(\frac{s}{K_1})}{\sqrt{2\bar{A}(\tau)}}}^{\infty} e^{\hat{k}} \exp \left\{ -\frac{1}{4} \frac{\left(B_3(\tau) + \frac{\bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} x \right)^2}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}} \right\} \eta_1(x) \\
&\times \exp \left\{ \frac{\bar{A}(\tau) + \bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} x + \frac{y\sqrt{B_1(\tau)} \left[B_3(\tau) + \frac{\bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} x \right]}{\sqrt{2} \left[B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \right]} \right\} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2} \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}} y^2} dx dy \\
&= \frac{s}{2\pi} \sqrt{\frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \int_{\frac{\ln(\frac{v}{\gamma^*})}{\sqrt{2B_1(\tau)}}}^{\infty} \int_{\frac{\ln(\frac{s}{K_1})}{\sqrt{2\bar{A}(\tau)}}}^{\infty} \exp \left\{ \hat{k} - \frac{1}{2} \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}} (x^2 + y^2) \right\} \\
&\times \exp \left\{ \left(-\sqrt{2\bar{A}(\tau)} + \frac{\bar{A}(\tau) + \bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} - \frac{1}{2} \frac{\left[B_3(\tau) + \frac{\bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} \right]}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}} \right) x - \Sigma(\tau) \right\} \\
&\times \exp \left\{ \frac{y\sqrt{B_1(\tau)}B_3(\tau)}{\sqrt{2} \left[B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \right] + \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \times B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \times \frac{\bar{B}(\tau)}{2\sqrt{\bar{A}(\tau)}B_1(\tau)} xy \right\} dx dy \\
&\times \frac{K_1}{2\pi} \sqrt{\frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \int_{\frac{\ln(\frac{v}{\gamma^*})}{\sqrt{2B_1(\tau)}}}^{\infty} \int_{\frac{\ln(\frac{s}{K_1})}{\sqrt{2\bar{A}(\tau)}}}^{\infty} \exp \left\{ \hat{k} - \frac{1}{2} \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}} (x^2 + y^2) \right\} \\
&\times \exp \left\{ \left(-\sqrt{2\bar{A}(\tau)} + \frac{\bar{A}(\tau) + \bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} - \frac{1}{2} \frac{\left[B_3(\tau) + \frac{\bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} \right]}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}} \right) x - \Sigma(\tau) \right\} \\
&\times \exp \left\{ \frac{y\sqrt{B_1(\tau)}B_3(\tau)}{\sqrt{2} \left[B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \right] + \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \times B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \times \frac{\bar{B}(\tau)}{2\sqrt{\bar{A}(\tau)}B_1(\tau)} xy \right\} dx dy. \tag{113}
\end{aligned}$$

According to Lemma 2.2, From Eq.(72), we let

$$\begin{aligned}
\tilde{U}^1(\tau, s, v, r) &= \frac{s}{2\pi} \sqrt{\frac{B_1(\tau)}{B_1(\tau) - \frac{\overline{B}^2(\tau)}{2\overline{A}(\tau)}}} \int_{\frac{\ln(\frac{v}{\gamma^*})}{\sqrt{2B_1(\tau)}}}^{\infty} \int_{\frac{\ln(\frac{s}{K_1})}{\sqrt{2\overline{A}(\tau)}}}^{\infty} \exp \left\{ \widehat{k} - \frac{1}{2} \frac{B_1(\tau)}{B_1(\tau) - \frac{\overline{B}^2(\tau)}{2\overline{A}(\tau)}} (x^2 + y^2) \right\} \\
&\times \exp \left\{ \left(-\sqrt{2\overline{A}(\tau)} + \frac{\overline{A}(\tau) + \overline{B}(\tau)}{\sqrt{2\overline{A}(\tau)}} - \frac{1}{2} \frac{B_3(\tau) + \frac{\overline{B}(\tau)}{\sqrt{2\overline{A}(\tau)}}}{B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)}} \right) x - \Sigma(\tau) \right\} \\
&\times \exp \left\{ \frac{y\sqrt{B_1(\tau)}B_3(\tau)}{\sqrt{2} \left[B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)} \right] + \frac{B_1(\tau)}{B_1(\tau) - \frac{\overline{B}^2(\tau)}{2\overline{A}(\tau)}}} \times B_1(\tau) - \frac{\overline{B}^2(\tau)}{4\overline{A}(\tau)} \times \frac{\overline{B}(\tau)}{2\sqrt{\overline{A}(\tau)}B_1(\tau)} xy \right\} dx dy \\
&= s \left[\frac{1}{2\pi\sqrt{\Sigma^*}} \int_{\frac{\ln(\frac{v}{\gamma^*})}{\sqrt{2B_1(\tau)}}}^{\infty} \int_{\frac{\ln(\frac{s}{K_1})}{\sqrt{2\overline{A}(\tau)}}}^{\infty} \exp \left\{ -\frac{1}{2\Sigma^*} [(x + \bar{a})^2 + (y + \bar{b})^2] + \frac{\xi}{2\Sigma^*} [(x + \bar{a})(y + \bar{b})] \right\} \right].
\end{aligned}$$

where

$$\begin{cases} \Sigma^* = 1 - \xi^2, \\ \bar{a} = \frac{\overline{A}(\tau) - \overline{B}(\tau)}{\sqrt{2\overline{A}(\tau)}}, \\ \bar{b} = \frac{\overline{B}(\tau) - B_1(\tau) - \overline{B}(\tau)}{\sqrt{2B_1(\tau)}}, \\ -1 < \xi < 1. \end{cases}$$

Using the following transformation of variables,

$$\begin{cases} x_1 = x + \frac{\overline{A}(\tau) - \overline{B}(\tau)}{\sqrt{2\overline{A}(\tau)}}, \\ y_1 = y + \frac{\overline{B}(\tau) - B_1(\tau) - \overline{B}(\tau)}{\sqrt{2B_1(\tau)}}, \\ x_2 = x - \frac{\overline{A}(\tau) + \overline{B}(\tau)}{\sqrt{2\overline{A}(\tau)}}, \\ y_2 = y - \frac{B_1(\tau) + \overline{B}(\tau)}{\sqrt{2B_1(\tau)}}. \end{cases} \quad (114)$$

We obtain

$$\tilde{U}^1(\tau, s, v, r) = s \left[\frac{1}{2\pi\sqrt{1 - \xi^2}} \int_{-\infty}^{\hat{a}_1} \int_{-\infty}^{\hat{a}_2} e^{\frac{1}{2(1-\xi^2)}(x_1^2 - 2\xi x_1 y_1 + y_1^2)} dx_1 dy_1 \right], \quad (115)$$

where

$$\begin{aligned}
\hat{a}_1 &= \frac{\ln\left(\frac{s}{K_1}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\
\hat{a}_2 &= \frac{\ln\left(\frac{s}{K_1}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}}
\end{aligned}$$

and according always to Lemma 2.2, we also let

$$\begin{aligned}
\tilde{U}^2(\tau, s, v, r) &= \frac{K_1}{2\pi} \sqrt{\frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2A(\tau)}}} \int_{\frac{\ln(\frac{v}{\gamma^*})}{\sqrt{2B_1(\tau)}}}^{\infty} \int_{\frac{\ln(\frac{s}{K_1})}{\sqrt{2A(\tau)}}}^{\infty} \exp \left\{ \hat{k} - \frac{1}{2} \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2A(\tau)}} (x^2 + y^2) \right\} \\
&\times \exp \left\{ \left(-\sqrt{2A(\tau)} + \frac{\bar{A}(\tau) + \bar{B}(\tau)}{\sqrt{2A(\tau)}} - \frac{1}{2} \frac{\left[B_3(\tau) + \frac{\bar{B}(\tau)}{\sqrt{2A(\tau)}} \right]}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{4A(\tau)}} \right) x - \Sigma(\tau) \right\} \\
&\times \exp \left\{ \frac{y\sqrt{B_1(\tau)}B_3(\tau)}{\sqrt{2} \left[B_1(\tau) - \frac{\bar{B}^2(\tau)}{4A(\tau)} \right] + \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2A(\tau)}}} \times B_1(\tau) - \frac{\bar{B}^2(\tau)}{4A(\tau)} \times \frac{\bar{B}(\tau)}{2\sqrt{A(\tau)B_1(\tau)}} xy \right\} dx dy \\
&= \frac{K_1 P(t, r, T)}{2\pi\sqrt{\Sigma^*}} \left[\int_{\frac{\ln(\frac{v}{\gamma^*})}{\sqrt{2B_1(\tau)}}}^{\infty} \int_{\frac{\ln(\frac{s}{K_1})}{\sqrt{2A(\tau)}}}^{\infty} \exp \left\{ P(t, r, T) - \frac{1}{2\Sigma^*} [(x + \bar{c})^2 + (y + \bar{d})^2] + q(\tau) \right\} \right].
\end{aligned}$$

where

$$\begin{cases} q(\tau) = \frac{\xi}{2\Sigma^*} [(x + \bar{c})(y + \bar{d})], \\ \bar{c} = -\frac{\bar{A}(\tau) + \bar{B}(\tau)}{\sqrt{2A(\tau)}}, \\ \bar{d} = -\frac{B_1(\tau) + \bar{B}(\tau)}{\sqrt{2B_1(\tau)}}. \end{cases}$$

By also using the transformation of variables given by Eq.(114), we have

$$\tilde{U}^2(\tau, s, v, r) = K_1 P(t, r, T) \left[\frac{1}{2\pi\sqrt{1 - \xi^2}} \int_{-\infty}^{\hat{b}_1} \int_{-\infty}^{\hat{b}_2} e^{\frac{1}{2(1 - \xi^2)}(x_2^2 - 2\xi x_2 y_2 + y_2^2)} dx_2 dy_2 \right] \quad (116)$$

where

$$\begin{aligned}
\hat{b}_1 &= \frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\
&+ \frac{2\xi \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\
\hat{b}_2 &= \frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}}
\end{aligned}$$

Combining, Eq.(115) and Eq.(116), the value of $\tilde{U}^1(t, s, v, r)$ given by Eq.(75) becomes

$$\begin{aligned}
\tilde{U}^1(t, s, v, r) &= \tilde{U}^1(t, s, v, r) + \tilde{U}^2(t, s, v, r) \\
&= s\mathcal{N}(\hat{a}_1, \hat{a}_2, \xi) - K_1 P(t, r, T) \mathcal{N}(\hat{b}_1, \hat{b}_2, \xi).
\end{aligned}$$

Therefore, we have:

$$\tilde{U}^1(t, s, v, r) = s\mathcal{N}(\hat{a}_1, \hat{a}_2, \xi) - K_1 P(t, r, T) \mathcal{N}(\hat{b}_1, \hat{b}_2, \xi) \quad (117)$$

- **Expression of $\bar{U}^1(t, s, v, r)$.** We make the same transformation used to have the value of $\tilde{U}^1(t, s, v, r)$ and then, we changes the value of $\eta_1(x)$ by $\eta_2(x) = \left(e^{-x\sigma_s\sqrt{\tau}} - K_2 \right)$. We deduce that, the expression of second term $\bar{U}^1(t, s, v, r)$ given by Eq.(76) is defined as follows:

$$\bar{U}^1(t, s, v, r) = s\mathcal{N}(\hat{\hat{a}}_1, \hat{\hat{a}}_2, \xi) - K_2 P(t, r, T) \mathcal{N}(\hat{\hat{b}}_1, \hat{\hat{b}}_2, \xi) \quad (118)$$

with

$$\hat{a}_1 = \frac{\ln\left(\frac{s}{K_2}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \quad (119)$$

$$\begin{aligned} \hat{b}_1 &= \frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\ &+ \frac{2\xi \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \end{aligned} \quad (120)$$

$$\hat{a}_2 = \frac{\ln\left(\frac{s}{K_2}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \quad (121)$$

$$\hat{b}_2 = \frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \quad (122)$$

Therefore, combining Eq.(117) and Eq.(118), Eq.(73) gives:

$$U^1(t, s, v, r) = s\mathcal{N}(\hat{a}_1, \hat{a}_2, \xi) - K_2 P(t, r, T) \mathcal{N}(\hat{b}_1, \hat{b}_2, \xi) - s\mathcal{N}(\hat{a}_1, \hat{a}_2, \xi) - K_2 P(t, r, T) \mathcal{N}(\hat{b}_1, \hat{b}_2, \xi) \quad (123)$$

A.3.2 Expression of $V^1(\tau, s, v, r)$

- **Expression of $\tilde{V}^1(\tau, s, v, r)$** The expression of $\tilde{V}^1(\tau, s, v, r)$ of Eq.(77) becomes

$$\begin{aligned} \tilde{V}^1(\tau, s, v, r) &= \frac{\hat{\delta} v s}{2\pi} \sqrt{\frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \int_{\frac{\ln\left(\frac{s}{K_1}\right)}{\sqrt{2\bar{A}(\tau)}}}^{\frac{\ln\left(\frac{v}{\gamma^*}\right)}{\sqrt{2B_1(\tau)}}} \int_{\infty}^{\infty} \exp\left\{ \hat{k} - \frac{1}{2} \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}} (x^2 + y^2) \right\} \\ &\times \exp\left\{ \left(-\sqrt{2\bar{A}(\tau)} + \frac{\bar{A}(\tau) + \bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} - \frac{1}{2} \frac{\left[B_3(\tau) + \frac{\bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} \right]}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}} \right) x - \Sigma(\tau) \right\} \\ &\times \exp\left\{ \frac{y\sqrt{B_1(\tau)}B_3(\tau)}{\sqrt{2}\left[B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \right] + \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \times B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \times \frac{\bar{B}(\tau)}{2\sqrt{\bar{A}(\tau)}B_1(\tau)} xy \right\} dx dy \\ &\times \frac{\hat{\delta} v K_1}{2\pi} \sqrt{\frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \int_{\frac{\ln\left(\frac{s}{K_1}\right)}{\sqrt{2\bar{A}(\tau)}}}^{\frac{\ln\left(\frac{v}{\gamma^*}\right)}{\sqrt{2B_1(\tau)}}} \int_{\infty}^{\infty} \exp\left\{ \hat{k} - \frac{1}{2} \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}} (x^2 + y^2) \right\} \\ &\times \exp\left\{ \left(-\sqrt{2\bar{A}(\tau)} + \frac{\bar{A}(\tau) + \bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} - \frac{1}{2} \frac{\left[B_3(\tau) + \frac{\bar{B}(\tau)}{\sqrt{2\bar{A}(\tau)}} \right]}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)}} \right) x - \Sigma(\tau) \right\} \\ &\times \exp\left\{ \frac{y\sqrt{B_1(\tau)}B_3(\tau)}{\sqrt{2}\left[B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \right] + \frac{B_1(\tau)}{B_1(\tau) - \frac{\bar{B}^2(\tau)}{2\bar{A}(\tau)}}} \times B_1(\tau) - \frac{\bar{B}^2(\tau)}{4\bar{A}(\tau)} \times \frac{\bar{B}(\tau)}{2\sqrt{\bar{A}(\tau)}B_1(\tau)} xy \right\} dx dy. \end{aligned} \quad (124)$$

Then, following the same procedure as for $\tilde{U}^1(\tau, s, v, r)$, and using the following transformation defined by:

$$\begin{cases} \tilde{x}_3 = x + \frac{-\bar{B}(\tau) + \bar{A} + \bar{B}}{\sqrt{2\bar{A}}} \\ \tilde{y}_3 = y + \frac{-\bar{B}(\tau) + B_1(\tau) + \bar{B}}{\sqrt{2B_1(\tau)}} \\ \tilde{x}_4 = x + \frac{-\bar{B}(\tau) - \bar{A} + \bar{B}}{\sqrt{2\bar{A}}} \\ \tilde{y}_4 = y + \frac{-\bar{B}(\tau) + B_1(\tau)}{\sqrt{2B_1(\tau)}}, \end{cases} \quad (125)$$

the expression of $\tilde{V}^1(\tau, s, v, r)$ is given by:

$$\begin{aligned} \tilde{V}^1(\tau, s, v, r) &= \hat{\delta} v s e^{-\bar{B}(\tau) + B(\tau)} \left(\frac{1}{2\pi\sqrt{1-\xi^2}} \int_{\hat{c}_2}^{\hat{c}_1} \int_{\infty}^{\infty} e^{-\frac{1}{1-\xi^2}(\tilde{x}_3^2 - 2\xi\tilde{x}_3\tilde{y}_3 + \tilde{y}_3^2)} d\tilde{x}_3 d\tilde{y}_3 \right) \\ &\quad - \hat{\delta} v s K_1 e^{-\bar{B}(\tau) + B(\tau)} \left(\frac{1}{2\pi\sqrt{1-\xi^2}} \int_{\hat{d}_2}^{\hat{d}_1} \int_{\infty}^{\infty} e^{-\frac{1}{1-\xi^2}(\tilde{x}_4^2 - 2\xi\tilde{x}_4\tilde{y}_4 + \tilde{y}_4^2)} d\tilde{x}_4 d\tilde{y}_4 \right) \\ &= \hat{\delta} v \left(\frac{p(t, T)}{P(t, r, T)} \mathcal{N}(\hat{c}_1, -\hat{c}_2, \xi) - K_1 \mathcal{N}(\hat{d}_1, -\hat{d}_2, -\xi) \right). \end{aligned} \quad (126)$$

where

$$\begin{aligned} \hat{c}_1 &= \frac{\ln\left(\frac{s}{K_1}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\ &\quad + \frac{2\xi \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \end{aligned} \quad (127)$$

$$\begin{aligned} \hat{c}_2 &= -\frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\ &\quad - \frac{2\xi \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \end{aligned} \quad (128)$$

and

$$\begin{aligned} \hat{d}_1 &= \frac{\ln\left(\frac{s}{K_1}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \\ &\quad + \frac{2\xi \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \end{aligned} \quad (129)$$

$$\begin{aligned} \hat{d}_2 &= -\frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_2, \varepsilon} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_2, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_v \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_1, \varepsilon} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_1, \varepsilon} \chi^{H_3, \varepsilon}} \sigma_s \sigma_r L(\nu) + \chi^{H_3, \varepsilon} \sigma_r^2 L^2(\nu) \right) d\nu}} \end{aligned} \quad (130)$$

- **Expression of $\bar{V}^1(\tau, s, v, r)$.** The expression of $\bar{V}^1(\tau, s, v, r)$ of Eq.(78) is obtained by following the same procedure as for $\tilde{V}^1(\tau, s, v, r)$. Then, the expression of $\bar{V}^1(\tau, s, v, r)$ is given by:

$$\begin{aligned}
\bar{V}^1(\tau, s, v, r) &= -\hat{\delta}v s e^{-\bar{B}(\tau)+B(\tau)} \left(\frac{1}{2\pi\sqrt{1-\xi^2}} \int_{\infty}^{\hat{c}_1} \int_{\infty}^{\hat{c}_2} e^{-\frac{1}{1-\xi^2}(\tilde{x}_3^2-2\xi\tilde{x}_3\tilde{y}_3+\tilde{y}_3^2)} d\tilde{x}_3 d\tilde{y}_3 \right) \\
&+ \hat{\delta}v s K_2 e^{-\bar{B}(\tau)+B(\tau)} \left(\frac{1}{2\pi\sqrt{1-\xi^2}} \int_{\infty}^{\hat{d}_1} \int_{\infty}^{\hat{d}_2} e^{-\frac{1}{1-\xi^2}(\tilde{x}_4^2-2\xi\tilde{x}_4\tilde{y}_4+\tilde{y}_4^2)} d\tilde{x}_4 d\tilde{y}_4 \right) \\
&= -\hat{\delta}v \left(\frac{p(t, T)}{P(t, r, T)} \mathcal{N}(\hat{c}_1, -\hat{c}_2, \xi) - K_2 \mathcal{N}(\hat{d}_1, -\hat{d}_2, -\xi) \right). \tag{131}
\end{aligned}$$

Where

$$\begin{aligned}
\hat{c}_1 &= \frac{\ln\left(\frac{s}{K_2}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \\
&+ \frac{2\xi \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \tag{132}
\end{aligned}$$

$$\begin{aligned}
\hat{c}_2 &= -\frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \\
&- \frac{2\xi \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \tag{133}
\end{aligned}$$

and

$$\begin{aligned}
\hat{d}_1 &= \frac{\ln\left(\frac{s}{K_2}\right) - \ln P(t, r, T) - \frac{1}{2} \int_0^\tau \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \\
&+ \frac{2\xi \int_0^\tau \hat{\sigma}_s^2(\nu) d\nu \int_0^\tau \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \tag{134}
\end{aligned}$$

$$\begin{aligned}
\hat{d}_2 &= -\frac{\ln\left(\frac{v}{\gamma^*}\right) - \ln P(t, r, T) + \frac{1}{2} \int_0^\tau \left(\chi^{H_{2,\varepsilon}} \sigma_v^2 + 2\rho_{vr} \sqrt{\chi^{H_{2,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_v \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}{\sqrt{\int_0^\tau \left(\chi^{H_{1,\varepsilon}} \sigma_s^2 + 2\rho_{sr} \sqrt{\chi^{H_{1,\varepsilon}} \chi^{H_{3,\varepsilon}}} \sigma_s \sigma_r L(\nu) + \chi^{H_{3,\varepsilon}} \sigma_r^2 L^2(\nu) \right) d\nu}} \tag{135}
\end{aligned}$$

By combining Eq.(126) and Eq.(131), Eq.(74) gives:

$$\begin{aligned}
V^1(\tau, s, v, r) &= \hat{\delta}v \left(\frac{p(t, T)}{P(t, r, T)} \mathcal{N}(\hat{c}_1, -\hat{c}_2, \xi) - K_1 \mathcal{N}(\hat{d}_1, -\hat{d}_2, -\xi) \right) \\
&- \hat{\delta}v \left(\frac{p(t, T)}{P(t, r, T)} \mathcal{N}(\hat{c}_1, -\hat{c}_2, \xi) - K_2 \mathcal{N}(\hat{d}_1, -\hat{d}_2, -\xi) \right). \tag{136}
\end{aligned}$$

By using Eq.(123) and Eq.(136), Eq.(72) gives the expression of $U(\tau, s, v, r)$.

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