



Algebraicity and smoothness of fixed point stacks

Matthieu Romagny

► To cite this version:

| Matthieu Romagny. Algebraicity and smoothness of fixed point stacks. 2022. hal-03675148v3

HAL Id: hal-03675148

<https://hal.science/hal-03675148v3>

Preprint submitted on 15 Sep 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Algebraicity and smoothness of fixed point stacks

Matthieu Romagny

To the memory of Bas Edixhoven

Abstract. We study algebraicity and smoothness of fixed point stacks for flat group schemes which have a finite composition series whose factors are either reductive or proper, flat, finitely presented, acting on algebraic stacks with affine, finitely presented diagonal. For this, we extend some theorems of [SGA3.2] on functors of homomorphisms $\mathrm{Hom}(G, H)$ and functors of reductive subgroups $\mathrm{Sub}(H)$ for an affine, possibly non-flat group scheme H .

1 Introduction

1.1 Context and motivation. In various situations of algebraic geometry, one needs to consider the fixed points of a flat group scheme acting on an algebraic stack. Currently, probably the biggest provider of such examples is the enumerative industry: Gromov-Witten and Donaldson-Thomas theories provide a wealth of apparitions of fixed points in localization formulas for virtual classes in equivariant cohomology. We refer to Joyce [Jo21] for a recent account. Accordingly, fixed point stacks pervade research articles in the last two decades; with no attempt at exhaustivity, let us mention the works [CLCT09], [Di12], [We11], [Sk13], [KL13], [GJK17], [OS19], [MR19], [KS20], [LS20], [BTN21], [CJW21]. The paper [Ro05] settles the question of algebraicity of fixed point stacks only in the case of actions of *proper* groups, which limitates the scope of applications (and is typically not sufficient in most works cited above). It is the purpose of the present article to extend the results of *loc. cit.*, providing algebraicity and smoothness statements for fixed point stacks in greater generality. A related issue is that of representability of functors of group homomorphisms $\mathrm{Hom}(G, H)$ and functors of subgroups $\mathrm{Sub}(H)$. In [SGA3.2], Exp. X1, § 4 such representability is proved in the case where the target group scheme H is *smooth*. Unfortunately, for the application to the fixed points of a group G acting on an algebraic stack \mathcal{X} , it is the inertia $I_{\mathcal{X}} \rightarrow \mathcal{X}$ which plays the role of H , and this is almost never flat. We also explore these issues, working with a possibly non-flat group H . We answer questions raised in [SGA3.2] by both relaxing the assumptions and strengthening the results.

1.2 Main results. Throughout the paper, we denote by S the base scheme. To get to the heart of the matter we need to recall some terminology. Following [SGA3.2], Exp. XIX, 2.7 and [AOV08], 2.2 or [Alp13], 12.1, we say that a group scheme $G \rightarrow S$ is:

- (i) *reductive* if it is affine and smooth with connected, reductive geometric fibres;

2020 Mathematics Subject Classification: Primary: 14A20, Secondary: 14L15, 14D23

Keywords: fixed points, algebraic stack, Hom functor, reductive group scheme, linearly reductive group scheme

Date: September 15, 2022

(ii) *linearly reductive* if it is flat, separated, of finite presentation, and the functor $\mathrm{QCoh}^G(S) \rightarrow \mathrm{QCoh}(S)$, $\mathcal{F} \mapsto \mathcal{F}^G$ is exact. This includes: group schemes of multiplicative type, finite locally free group schemes of order invertible on S , abelian schemes, reductive group schemes if S is a \mathbb{Q} -scheme, and all extensions of such group schemes (we refer to 4.3.5 and the comments after it).

Here is our result on fixed point stacks; see 4.1.4 and 4.3.6.

1.2.1 Theorem. *Let \mathcal{X} be an S -algebraic stack with affine, finitely presented diagonal. Let G be a flat, finitely presented S -group algebraic space acting on \mathcal{X} .*

- (1) *Assume that G has a finite composition series whose factors are either reductive or proper, flat, finitely presented. Then the fixed point stack \mathcal{X}^G is algebraic, and the morphism $\mathcal{X}^G \rightarrow \mathcal{X}$ is representable by algebraic spaces, separated and locally of finite presentation. If G is reductive, this morphism is even representable by schemes.*
- (2) *Assume that \mathcal{X} is smooth and G is linearly reductive. Then \mathcal{X}^G is smooth.*

The proof necessitates results on functors of group homomorphisms. Before stating them, we bring to the reader's attention the subtle question of the (non-)affineness of $\mathrm{Hom}(G, H)$, when it is representable. Quoting [SGA3.2], Exp. XI, Rem. 4.6 we know that if G is of multiplicative type and H is a closed subgroup of some GL_n , then $\mathrm{Hom}(G, H)$ is a disjoint sum of affine schemes, but ‘on se gardera de croire cependant que les préschémas qui représentent [ces] foncteurs sont toujours des sommes d’une famille de schémas affines sur S ’ (‘the reader should refrain from thinking that the schemes representing these functors are always a sum of a family of affine S -schemes’). In recent work of Brion, the same problem is encountered and certain conditions (FT) and (AFT) are introduced in order to best describe this phenomenon; see [Bri21], § 4.2. The statement in item (3) below is our contribution to this question, in the present generality.

The following result is found in 3.1.4, 3.3.2, 4.4.2.

1.2.2 Theorem. *Let G be an S -group space that has a finite composition series whose factors are either reductive or proper, flat, finitely presented. Let H be an affine, finitely presented S -group scheme.*

- (1) *The functor $\mathrm{Hom}(G, H)$ is representable by an S -algebraic space separated and locally of finite presentation.*
- (2) *The subfunctor of monomorphisms $\mathrm{Mono}(G, H)$ is representable by an open subspace of $\mathrm{Hom}(G, H)$. Moreover, all monomorphisms $G \rightarrow H$ are closed immersions.*
- (3) *If G is reductive, $\mathrm{Hom}(G, H)$ is representable by a scheme with the following property: each subscheme (resp. closed subscheme) which is quasi-compact over S , is quasi-affine (resp. affine) over S .*
- (4) *If G is linearly reductive and H is flat, the algebraic stack $\mathcal{H}\mathrm{om}(BG, BH)$ is smooth. In particular,*
 - (i) *$\mathrm{Hom}(G, H) \rightarrow S$ is flat and locally complete intersection,*
 - (ii) *$\mathrm{Hom}(G, H) \rightarrow S$ is smooth if moreover $H \rightarrow S$ is smooth.*

The assumptions on G and H are close to optimal. Indeed, it is classical that the presence of unipotent factors in G is an obstacle to representability; taking $G = \mathbb{G}_a$ and $H = \mathbb{G}_m$ for simplicity, this is due to the existence of exponentials that do not algebrize, violating effectivity (the axiom called (F_3) in the text). Also an assumption “affine” or at least “quasi-affine” on H is necessary, as shown by the example of an elliptic curve with multiplicative reduction ([SGA3.2], Exposé IX, Rem. 7.4).

Our third and last main result is about functors of reductive subgroups of affine, possibly non-flat group schemes. We refer to 3.2.1 and 4.4.5.

1.2.3 Theorem. *Let H be an affine, finitely presented S -group scheme.*

(1) *The functor $\text{Sub}_{\text{red}}(H)$ of reductive subgroups of H is representable by an algebraic space separated and locally of finite presentation which is a disjoint sum indexed by the types of reductive groups:*

$$\text{Sub}_{\text{red}}(H) = \coprod_{\mathbf{t}} \text{Sub}_{\mathbf{t}}(H).$$

(2) *The summand of subgroups of multiplicative type*

$$\text{Sub}_{\text{mult}}(H) = \coprod_{\mathbf{t}=[(M, M^*, \emptyset, \emptyset)]} \text{Sub}_{\mathbf{t}}(H)$$

is representable by a scheme with the following property: each subscheme (resp. closed subscheme) which is quasi-compact over S , is quasi-affine (resp. affine) over S .

(3) *Assume moreover that $H \rightarrow S$ is flat. Then $\text{Sub}_{\text{mult}}(H) \rightarrow S$ is flat and locally complete intersection, and smooth if $H \rightarrow S$ is. If S is of characteristic 0 then $\text{Sub}_{\text{red}}(H) \rightarrow S$ is smooth.*

1.3 Main ideas of the proofs. The proofs of the results 1.2.2(1)–(3) and 1.2.3(1)–(2) are intertwined in a single line of reasoning. By dévissage for G we treat separately the reductive case and the proper flat case. The key case is when G is the multiplicative group \mathbb{G}_m (the case of proper flat groups is standard using representability of the Hilbert scheme; and one passes from general reductive groups to a maximal torus using a result of [SGA3.2] and then to \mathbb{G}_m). For a suitably chosen prime ℓ , the density theorem shows that $F := \text{Hom}(\mathbb{G}_m, H)$ is a subfunctor of the affine scheme $L := \lim_n \text{Hom}(\mu_{\ell^n}, H)$. To complete the proof, one verifies the eight axioms of Grothendieck’s theorem on unramified functors for the monomorphism $F \rightarrow L$. The axiom which is hardest to check is the final one, and this is proved using a statement of “descent along schematically dominant morphisms” whose proof is inspired from the proof of algebraicity of formal homomorphisms. From these results one deduces 1.2.1(1).

As far as algebraicity statements are concerned, the results for functors of homomorphisms of group schemes imply results for fixed point stacks. For the statements of smoothness, things go in the other direction. That is, we first prove 1.2.1(2) by verifying the infinitesimal criterion of smoothness for $\mathcal{X}^G \rightarrow S$. For this we use the vanishing of cohomology of linearly reductive group schemes to show that a certain stack of liftings, which is a torsor under a certain tangent stack, is trivial. Then we deduce the flatness and smoothness statements 1.2.2(4).

1.4 Comments on related work. The first general results on fixed points and homomorphism functors are of course due to the work of the precursors of [SGA3.2], [SGA3.3], [Ra70]. These provide the foundation for the results presented here.

Fixed point stacks are defined and studied in some generality for actions of proper groups in [Ro05], of which the results of the present text can be seen as a natural continuation. In Subsection 4.5 we take the opportunity to correct a claim made in [Ro05], Rem. 2.4 which turns out to be partially false. Namely, say that G, N are flat, finitely presented group schemes with N a normal subgroup of G . Assume that G acts on a stack \mathcal{X} (we may take \mathcal{X} algebraic and assume that the fixed point and quotient stacks below are algebraic). Then, we provide an example where $(\mathcal{X}^N)^{G/N}$ and \mathcal{X}^G are *not* isomorphic (in fact it is not clear how to let G/N act on \mathcal{X}^N and we discuss this issue). On the other hand we prove that there is always an isomorphism of stacks $(\mathcal{X}/N)/(G/N) \xrightarrow{\sim} \mathcal{X}/G$.

In the paper [AHR20], Alper, Hall and Rydh show that when \mathcal{X} is a Deligne-Mumford stack locally of finite type over a field with an action of $G = \mathbb{G}_m$, then $\mathcal{X}^G \rightarrow \mathcal{X}$ is a closed immersion. This can be easily extended to the case where \mathcal{X} is a Deligne-Mumford stack, G is smooth with connected fibres,

and the base scheme S is arbitrary. The Deligne-Mumford assumption is essential; in Example 4.1.5 we show that \mathbb{G}_m acts on the classifying stack of α_p -torsors (over a base scheme of characteristic $p > 0$) in such a way that $\mathcal{X}^{\mathbb{G}_m} \rightarrow \mathcal{X}$ is not a monomorphism. On the other hand, if \mathcal{X} has finite inertia and the group scheme G is smooth, the valuative criterion for properness holds for $\mathcal{X}^G \rightarrow \mathcal{X}$. In forthcoming work Aranha, Khan, Latyntsev, Park and Ravi use this fact to prove a (virtual) Atiyah-Bott formula under certain hypotheses.

In [SGA3.2], another feature of the scheme $M := \mathrm{Hom}(G, H)$ is studied. Namely, for $h \in H$ let $\mathrm{inn}(h) : H \rightarrow H$ be the inner automorphism $k \mapsto hkh^{-1}$. In [SGA3.2], Exp. XI, § 5 it is shown that if G, H are finitely presented with G of multiplicative type and H affine and smooth, then the morphism

$$H \times M \longrightarrow M \times M, \quad (h, v) \longmapsto (\mathrm{inn}(h) \circ v, v)$$

is smooth. In the case of a base field, this is extended by Brion [Bri21] to the situation where G is linearly reductive and H is locally of finite type (but not necessarily affine). Following the arguments of [SGA3.2], Exp. IX 3.6 and Exp. XI 2.3, with our running assumptions ‘affine of finite presentation’ on H , it should be possible to extend this further to the case where S is arbitrary.

Recent work of Bhatt, Halpern-Leistner, Preygel (see [Bh16] Lemma 2.5, [BHL17], Section 2, [HLP19], Theorem 5.1.1) seems to indicate that it should be possible to extend our results to algebraic stacks with *quasi-affine* diagonal, and to target group schemes H that are *quasi-affine*. We did not explore this possibility.

Moving away from linearly reductive group schemes, general results on the smoothness of fixed points seem difficult to obtain. Recent work of Hamilton [Ha21] provides an interesting attempt in this direction.

Finally we point out that the present text encompasses the results of the preprint [Ro21] which it supersedes.

1.5 Organization of the paper. The table of contents after the acknowledgements describes the plan of the article.

1.6 Acknowledgements. I wish to express warm thanks to Arkadij Bojko, who provided the initial stimulus with the question of algebraicity of fixed point stacks. I thank Michel Brion for conversations on the topic of this article, and for bringing the paper [Bri21] to my attention. For various conversations and feedback, I also thank Dhyan Aranha, Alice Bouillet, Pierre-Emmanuel Chaput, Philippe Gille, Luc Illusie, Marion Jeannin, Bernard Le Stum, Laurent Moret-Bailly, Cédric Pépin, Simon Riche and Angelo Vistoli.

This work was supported by the ANR project CLap-CLap (ANR-18-CE40-0026-01) and by the Centre Henri Lebesgue (ANR-11-LABX-0020-01). I would like to thank the executive and administrative staff of IRMAR and of the Centre Henri Lebesgue for creating an attractive mathematical environment.

Contents

1	Introduction	1
2	Homomorphisms from a diagonalizable group	5
2.1	Statement and first reductions	5
2.2	Descent along schematically dominant morphisms	6
2.3	Representability using Grothendieck’s theorem on unramified functors	10

3	Homomorphisms from a group with reductive and proper composition factors	14
3.1	Homomorphisms from a reductive group	14
3.2	Functor of reductive subgroups	16
3.3	Homomorphisms from a group with reductive and proper composition factors	18
4	Algebraicity and smoothness of fixed points stacks	19
4.1	Algebraicity	19
4.2	Geometric interpretation of group cohomology in degrees 1 and 2	22
4.3	Smoothness	24
4.4	Applications	27
4.5	Failure of transitivity (erratum to [Ro05])	30
A	Some results around purity	33

2 Homomorphisms from a diagonalizable group

The proof of Theorem 1.2.2 builds on the key case where G is a diagonalizable group scheme $D(M)$. In this section, we establish representability in that case. In Subsection 2.1 we state the result we want to prove and we reduce it to the more specific statement 2.1.2. In Subsection 2.2 we prove a crucial descent statement used in Subsection 2.3 to complete the proof of 2.1.2 by verifying the conditions of Grothendieck's theorem on unramified functors.

2.1 Statement and first reductions

2.1.1 Theorem. *Let G, H be finitely presented S -group schemes with G diagonalizable and H affine. Then $\mathrm{Hom}(G, H)$ is representable by an S -scheme separated and locally of finite presentation.*

Proof : The group G is a product $G = N \times \mathbb{G}_m^r$ where N is finite diagonalizable. For a product $G = G_1 \times G_2$, the functor $\mathrm{Hom}(G, H)$ is the subfunctor of $\mathrm{Hom}(G_1, H) \times \mathrm{Hom}(G_2, H)$ composed of pairs of maps that commute. Using [SGA3.2], Exp. VIII, 6.5.b) we see that this is a closed subfunctor. It follows that if the theorem is true for G_1 and G_2 then it is true for G , hence it is enough to consider the factors individually. If $G = N$ is finite, it is classical and recalled in Lemma A.10 that $\mathrm{Hom}(N, H)$ is representable by an affine S -scheme. It remains to handle the case $G = \mathbb{G}_m$, which we now do.

The assumptions and conclusions of the theorem being local for the Zariski topology on S , we can assume that S is affine. Since G and H are of finite presentation, with the usual results on limits ([EGA] IV, § 8) we see that $\mathrm{Hom}(G, H) \rightarrow S$ is locally of finite presentation. Consequently we can further reduce to the case where S is of finite type over $\mathrm{Spec}(\mathbb{Z})$.

For a prime number ℓ let $S_\ell \subset S$ be the open subscheme where ℓ is invertible. Choose two distinct primes ℓ, ℓ' and write $S = S_\ell \cup S_{\ell'}$. Since the question of representability is local on S , it is enough to handle S_ℓ and $S_{\ell'}$ separately. In this way we reduce to the case where $\ell \in \mathcal{O}_S^\times$.

Let $\mu_{\ell^n} \subset \mathbb{G}_m$ be the group scheme of ℓ^n -th roots of unity. By Lemma A.10 again, the functor $\mathrm{Hom}(\mu_{\ell^n}, H)$ is representable by an affine S -scheme. In particular, the morphisms $\mathrm{Hom}(\mu_{\ell^{n+1}}, H) \rightarrow \mathrm{Hom}(\mu_{\ell^n}, H)$ are affine so the limit

$$L := \lim_n \mathrm{Hom}(\mu_{\ell^n}, H)$$

is representable by a scheme which is affine over H , hence over S also. By restricting morphisms to the torsion subschemes, we have a map of functors :

$$\varphi : \mathrm{Hom}(\mathbb{G}_m, H) \longrightarrow L, \quad f \longmapsto \{f|_{\mu_{\ell^n}}\}_{n \geq 0}.$$

It is enough to prove that φ is representable by schemes. For this let T be an S -scheme and let $T \rightarrow L$ be a map, that is, a compatible collection $\{u_n : \mu_{\ell^n, T} \rightarrow H_T\}$ of morphisms of T -group schemes. We want to prove that the fibred product $\mathrm{Hom}(\mathbb{G}_m, H) \times_L T$ is representable. For this we change our notation, rename T as S , reduce to the case where T is affine as before, and the result is exactly Theorem 2.1.2 below. \square

We have thus reduced the proof of 2.1.1 to the following statement, whose proof occupies the rest of the section.

2.1.2 Theorem. *Let ℓ be a prime number, $S = \mathrm{Spec}(R)$ an affine $\mathbb{Z}[1/\ell]$ -scheme of finite type, H a finitely presented affine S -group scheme, and $\{u_n : \mu_{\ell^n} \rightarrow H\}_{n \geq 0}$ a family of morphisms of S -group schemes such that u_{n+1} extends u_n for each n . Let F be the functor defined for all S -schemes T by:*

$$F(T) = \{ \text{morphisms of groups } f : \mathbb{G}_{m, T} \rightarrow H_T \text{ that extend the } u_{n, T}, n \geq 0 \}.$$

Then F is representable by an S -scheme separated and locally of finite presentation.

2.2 Descent along schematically dominant morphisms

We keep all notations as in 2.1.2. The Density Theorem ([SGA3.2], Exp. IX, Théorème 4.7 and Remark 4.10) implies that $F(T)$ contains at most one point; that is, $F \rightarrow S$ is a monomorphism. To prove that F is representable, we will use Grothendieck's theorem on unramified functors. The verification that F fulfills the conditions of the theorem will be based to a large extent on the following fact: the map $F(T) \rightarrow F(T')$ is an isomorphism for all schematically dominant morphisms of schemes $T' \rightarrow T$. This is Lemma 2.2.6 below. Its proof will use a variation on the argument used to show that formal homomorphisms from a group scheme of multiplicative type to an affine group scheme are algebraic, see [SGA3.2] Exp. IX, § 7. It is the purpose of this subsection to settle this.

We work over a $\mathbb{Z}[1/\ell]$ -algebra A .

2.2.1 ℓ -power roots of unity. We consider the scheme of ℓ -power roots of unity:

$$\mu_{\ell^\infty} = \mathrm{colim} \mu_{\ell^n}.$$

This is the disjoint sum of the schemes of *primitive* roots of unity:

$$\mu_{\ell^\infty} = \coprod_{n \geq 0} \mu_{\ell^n}^*.$$

If Φ_n denotes the ℓ^n -th cyclotomic polynomial, we have $A[\mu_{\ell^n}^*] = A[z]/(\Phi_n)$ and

$$A[\mu_{\ell^\infty}] = \prod_{n \geq 0} A[z]/(\Phi_n).$$

The restriction of functions is a canonical injective morphism:

$$c : A[\mathbb{G}_m] \hookrightarrow A[\mu_{\ell^\infty}]$$

which we describe further below.

2.2.2 Cyclotomic expansion of Laurent polynomials. For relative integers $i \leq j$ let

$$A(i; j) = \left\{ P = \sum_{i \leq s \leq j} a_s z^s \in A[z^{\pm 1}] \right\}$$

be the module of Laurent polynomials whose monomials have degree in the range $\{i, \dots, j\}$.

2.2.3 Lemma. *Each nonzero Laurent polynomial $P \in A[z^{\pm 1}]$ has a unique expression*

$$P = r_0 + r_1(z - 1) + r_2(z^\ell - 1) + \dots + r_n(z^{\ell^{n-1}} - 1)$$

with $r_i \in A(-\lfloor \varphi(\ell^i)/2 \rfloor; \varphi(\ell^i) - \lfloor \varphi(\ell^i)/2 \rfloor - 1)$ and $r_n \neq 0$. In other words we have a decomposition

$$A[z^{\pm 1}] = \bigoplus_{n \geq 0} D_n$$

into sub- A -modules $D_n := A(-\lfloor \varphi(\ell^n)/2 \rfloor; \varphi(\ell^n) - \lfloor \varphi(\ell^n)/2 \rfloor - 1) \cdot (z^{\ell^{n-1}} - 1)$.

Proof : Let \deg be the degree and val the valuation. If P is constant, the result is clear. Otherwise, there is $n \geq 1$ minimal with the property that

$$-\lfloor \varphi(\ell^n)/2 \rfloor \leq \text{val}(P) \leq \deg(P) < \ell^n - \lfloor \varphi(\ell^n)/2 \rfloor.$$

Let $Q_0 = z^{\lfloor \varphi(\ell^n)/2 \rfloor} P$, so we have:

$$0 \leq \text{val}(Q_0) \leq \deg(Q_0) < \ell^n.$$

Let $B_n = A(-\lfloor \varphi(\ell^n)/2 \rfloor; \varphi(\ell^n) - \lfloor \varphi(\ell^n)/2 \rfloor - 1) \cdot z^{\lfloor \varphi(\ell^n)/2 \rfloor}$ be the $z^{\lfloor \varphi(\ell^n)/2 \rfloor}$ -translate of the A -module in the statement. For each i the module B_i is finite free of rank $\varphi(\ell^i) = \deg(\Phi_i)$, hence

$$\rho_i : B_i \hookrightarrow A[z] \longrightarrow A[z]/(\Phi_i)$$

is an isomorphism and B_i can serve as a module of representatives of residue classes for Euclidean division modulo Φ_i . We define a sequence of polynomials $\{Q_i\}$ by running the division algorithm:

- we set $s_0 = \rho_0^{-1}(Q_0 \bmod \Phi_0)$ and get a division $Q_0 = s_0 + \Phi_0 Q_1$;
- inductively, while $Q_i \neq 0$ we set $s_i = \rho_i^{-1}(Q_i \bmod \Phi_i)$ and get a division $Q_i = s_i + \Phi_i Q_{i+1}$.

Since the sequence $\{\deg(Q_i)\}_{i \geq 0}$ is strictly decreasing, the process eventually stops. We obtain the desired expression for P by setting $r_i = z^{-\lfloor \varphi(\ell^i)/2 \rfloor} s_i$. \square

Like in the proof of Lemma 2.2.3, since $\text{rank}(D_i) = \varphi(\ell^i)$ the map $D_n \hookrightarrow A[z^{\pm 1}] \rightarrow A[z^{\pm 1}]/(\Phi_n)$ is an isomorphism.

2.2.4 Lemma. *Let $c : A[\mathbb{G}_m] \rightarrow A[\mu_{\ell^\infty}] = \prod_{n \geq 0} A[z]/(\Phi_n)$, $P \mapsto (P \bmod \Phi_0, P \bmod \Phi_1, P \bmod \Phi_2, \dots)$ be the map of §2.2.1.*

(1) *For an element $q_i \in A[z]/(\Phi_i)$, write P_i the unique Laurent polynomial in D_i with $q_i = P_i \bmod \Phi_i$. The image of c is the set of families (q_0, q_1, q_2, \dots) such that there is N such that $q_n = P_0 + P_1 + \dots + P_N \bmod \Phi_n$ for all $n \geq N$. In this case, we have $(q_0, q_1, q_2, \dots) = c(P)$ with $P = P_0 + P_1 + \dots + P_N$.*

(2) If $A \rightarrow B$ is an injective ring homomorphism, the commutative square of inclusions

$$\begin{array}{ccc} A[\mathbb{G}_m] & \hookrightarrow & A[\mu_{\ell^\infty}] \\ \downarrow & & \downarrow \\ B[\mathbb{G}_m] & \hookrightarrow & B[\mu_{\ell^\infty}]. \end{array}$$

is cartesian.

Proof : (1) If $(q_0, q_1, q_2, \dots) = c(P)$, we can write $P = P_0 + P_1 + \dots + P_N$ with $P_i \in D_i$. Since $D_i \subset (z^{\ell^i} - 1)A[z^{\pm 1}]$ and Φ_i divides $z^{\ell^i} - 1$, we have $P_i = 0 \pmod{\Phi_j}$ for all $j \geq i$. Therefore for all $n \geq N$ we have:

$$q_n = P \pmod{\Phi_n} = P_0 + P_1 + \dots + P_N \pmod{\Phi_n}.$$

Conversely if $q_n = P_0 + P_1 + \dots + P_N \pmod{\Phi_n}$ for all $n \geq N$ where $P_i \in D_i$ reduces to q_i modulo Φ_i , then obviously $(q_0, q_1, q_2, \dots) = c(P)$ with $P = P_0 + P_1 + \dots + P_N$.

(2) This follows from the description in (1). \square

We can now prove that the objects of the functor F descend along schematically dominant morphisms. For the latter notion, we refer the reader to [EGA] IV₃.11.10. The next two lemmas are two variants of this descent statement.

2.2.5 Lemma. *Let $T' \rightarrow T$ be a morphism of S -schemes such that $T = \text{Spec}(A)$ is affine and $T' = \coprod_i \text{Spec}(A_i)$ is a disjoint sum of affines, with $A \rightarrow \prod_i A_i$ injective. Then the map $F(T) \rightarrow F(T')$ is bijective.*

The result is easier when $T' \rightarrow T$ is quasi-compact, but the general case will be crucial for us.

Proof : Since $F(T)$ has at most one point, the map $F(T) \rightarrow F(T')$ is injective and it is enough to prove that it is surjective. We start with an element of $F(T')$, i.e. a family of morphisms of A_i -group schemes $f_i : \mathbb{G}_{m, A_i} \rightarrow H_{A_i}$ each of which extends the morphisms $u_n : \mu_{n, A_i} \rightarrow H_{A_i}$, $n \geq 0$. For simplicity, in the sequel we write again $f_i : \mathcal{O}_H \otimes A_i \rightarrow A_i[z^{\pm 1}]$ and $u_n : \mathcal{O}_H \rightarrow R[z]/(z^{\ell^n} - 1)$ the corresponding comorphisms of Hopf algebras; this should not cause confusion. For each R -algebra A we also write $u_{\infty, A} : \mathcal{O}_H \otimes A \rightarrow A[\mu_{\ell^\infty}]$ for the product of the $u_{n, A}$. These fit in a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_H \otimes A_i & & \\ f_i \downarrow & \searrow u_{\infty, A_i} & \\ A_i[z^{\pm 1}] & \xrightarrow{c_{A_i}} & A_i[\mu_{\ell^\infty}]. \end{array}$$

We now reduce to the case where A and the A_i are noetherian. For this let L resp. L_i be the image of $R \rightarrow A$, resp. of $R \rightarrow A_i$. Being quotients of R , the rings L and L_i are noetherian. Moreover, since $A \rightarrow \prod A_i$ is injective then so is $L \rightarrow \prod L_i$. Since the u_n are defined over R hence over L , we have a

commutative diagram:

$$\begin{array}{ccccc}
\mathcal{O}_H \otimes L_i & & & & \\
\downarrow & \searrow^{f_i, L_i} & & \searrow^{u_{\infty, L_i}} & \\
\mathcal{O}_H \otimes L_i & & L_i[z^{\pm 1}] & \xrightarrow{c_{L_i}} & L_i[\mu_{\ell^\infty}] \\
& \searrow^{f_i} & \downarrow & & \downarrow \\
& & A_i[z^{\pm 1}] & \xrightarrow{c_{A_i}} & A_i[\mu_{\ell^\infty}].
\end{array}$$

Since the lower right square is cartesian, there is an induced dotted arrow. In this way we see that f_i is actually defined over L_i . So replacing A (resp. A_i) by L (resp. L_i), we obtain the desired reduction.

Let $\hat{A} := \prod_i A_i$. Taking products over i , we build a commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}_H \otimes A & \xrightarrow{u_{\infty, A}} & A[\mu_{\ell^\infty}] \\
\downarrow & \searrow^{\Phi_0} & \downarrow \\
\mathcal{O}_H \otimes \hat{A} & & \hat{A}[\mu_{\ell^\infty}] \\
\downarrow & & \downarrow \\
\prod_i (\mathcal{O}_H \otimes A_i) & \xrightarrow{\prod f_i} \prod_i (A_i[z^{\pm 1}]) \xrightarrow{\prod c_{A_i}} & \hat{A}[\mu_{\ell^\infty}]
\end{array}$$

Henceforth we set $C := \mathcal{O}_H \otimes A$ and we write Φ_0 the dotted composition in the diagram above. What the diagram shows is that

$$C \xrightarrow{\Phi_0} \prod_i (A_i[z^{\pm 1}]) \xrightarrow{\prod c_{A_i}} \hat{A}[\mu_{\ell^\infty}]$$

factors through $A[\mu_{\ell^\infty}]$. According to Lemma 2.2.4(2) applied with $B = \hat{A}$, this implies that

$$C \xrightarrow{\Phi_0} \prod_i (A_i[z^{\pm 1}]) \xrightarrow{\prod \text{can}_{A_i}} \hat{A}^{\mathbb{Z}}$$

factors through $A^{\mathbb{Z}}$, providing a map $\Phi : C \rightarrow A^{\mathbb{Z}}$. From the diagrams expressing the fact that the f_i respect the comultiplications, taking products over i , we obtain a commutative diagram:

$$\begin{array}{ccc}
C & \xrightarrow{\Phi} & A^{\mathbb{Z}} \\
\downarrow & & \downarrow \\
C \otimes_A C & \xrightarrow{\Phi \otimes \Phi} & A^{\mathbb{Z}} \otimes_A A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z} \times \mathbb{Z}}.
\end{array}$$

Let $g \in C$ and write $\Phi(g) = (a_m)_{m \in \mathbb{Z}}$. Since A is noetherian, Lemme 7.2 of [SGA3.2], Exp. IX is applicable and shows that only finitely many of the a_m are nonzero, that is $\Phi(g) \in A[z^{\pm 1}]$. Therefore Φ gives rise to a map $f : C \rightarrow A[z^{\pm 1}]$. The fact that f respects the comultiplication of the Hopf algebras follows immediately by embedding $A[z^{\pm 1}] \otimes A[z^{\pm 1}]$ into $\hat{A}[z^{\pm 1}] \otimes \hat{A}[z^{\pm 1}]$ where the required commutativity holds by assumption. The fact that f respects the counits is equally clear. \square

2.2.6 Lemma. *Let $T' \rightarrow T$ be a morphism of S -schemes which is schematically dominant. Then the map $F(T) \rightarrow F(T')$ is bijective.*

Proof : Since $F(T)$ has at most one point, the map $F(T) \rightarrow F(T')$ is injective and it is enough to prove that it is surjective. By fpqc descent of morphisms, this holds when $T' \rightarrow T$ is a covering for the fpqc topology. Applying this remark with chosen Zariski covers $\coprod T_i \rightarrow T$ and $\coprod_{i,j} T'_{ij} \rightarrow \coprod_i T_i \times_T T' \rightarrow T'$, we see that the vertical maps in the following commutative square are bijective:

$$\begin{array}{ccc} F(T) & \longrightarrow & F(T') \\ \downarrow \wr & & \downarrow \wr \\ \prod F(T_i) & \longrightarrow & \prod F(T'_{ij}). \end{array}$$

Therefore it is enough to prove that $F(T_i) \rightarrow \prod_j F(T'_{ij})$ is bijective, for each i . Choosing $T_i = \text{Spec}(A_i)$ and $T'_{ij} = \text{Spec}(A'_{ij})$ affine, the assumption that $T' \rightarrow T$ is schematically dominant implies that $A_i \rightarrow \prod A'_{ij}$ is injective. In this way we are reduced to the statement of Lemma 2.2.5. \square

2.3 Representability using Grothendieck's theorem on unramified functors

Recall the statement of Grothendieck's theorem on representation of unramified functors; all affine schemes $\text{Spec}(A)$ appearing are assumed to be S -schemes, and we write $F(A)$ instead of $F(\text{Spec}(A))$.

2.3.1 Theorem (Grothendieck [Mu65]). *Let S be a locally noetherian scheme and F a set-valued contravariant functor on the category of S -schemes. Then F is representable by an S -scheme which is locally of finite type, unramified and separated if and only if Conditions (F_1) to (F_8) below hold.*

- (F_1) *The functor F is a sheaf for the fpqc topology.*
- (F_2) *The functor F is locally of finite presentation; that is, for all filtering colimits of rings $A = \text{colim } A_\alpha$, the map $\text{colim } F(A_\alpha) \rightarrow F(A)$ is bijective.*
- (F_3) *The functor F is effective; that is, for all noetherian complete local rings (A, m) , the map $F(A) \rightarrow \varprojlim F(A/m^k)$ is bijective.*
- (F_4) *The functor F is homogeneous; more precisely, for all exact sequences of rings $A \rightarrow A' \rightrightarrows A' \otimes_A A'$ with A local artinian, $\text{length}_A(A'/A) = 1$ and trivial residue field extension $k_A = k_{A'}$, the diagram $F(A) \rightarrow F(A') \rightrightarrows F(A' \otimes_A A')$ is exact.*
- (F_5) *The functor F is formally unramified.*
- (F_6) *The functor F is separated; that is, it satisfies the valuative criterion of separation.*

For the last two conditions we let A be a noetherian ring, N its nilradical, I a nilpotent ideal such that $IN = 0$, $T = \text{Spec}(A)$, $T' = \text{Spec}(A/I)$. We assume that T is irreducible and we call t its generic point.

- (F_7) *Assume moreover that A is complete one-dimensional local with a unique associated prime. Then any point $\xi' : \text{Spec}(A/I) \rightarrow F$ such that*

$$\xi'_t : \text{Spec}((A/I)_t) \longrightarrow \text{Spec}(A/I) \longrightarrow F$$

can be lifted to a point $\xi^ : \text{Spec}(A_t) \rightarrow F$, can be lifted to a point $\xi : \text{Spec}(A) \rightarrow F$.*

(F₈) Assume that $\xi' : \operatorname{Spec}(A/I) \rightarrow F$ is such that

$$\xi'_t : \operatorname{Spec}((A/I)_t) \longrightarrow \operatorname{Spec}(A/I) \longrightarrow F$$

can not be lifted to any subscheme of $\operatorname{Spec}(A_t)$ which is strictly larger than $\operatorname{Spec}((A/I)_t)$. Then there exists a nonempty open set $W \subset T$ such that for all open subschemes $W_1 \subset T$ contained in W , the restriction $\xi'_{|W'_1} : W'_1 \rightarrow F$ (with $W'_1 = W_1 \times_T T'$) can not be lifted to any subscheme of W_1 which is strictly larger than W'_1 .

In what follows we apply Grothendieck's theorem to prove 2.1.2, whose notation we use. Recall in particular that

$$F(T) = \{\text{morphisms } f : \mathbb{G}_{m,T} \rightarrow H_T \text{ that extend the } u_{n,T}, n \geq 0\}.$$

We verify the conditions one by one for this functor.

2.3.2 Conditions (F₁), (F₄), (F₅), (F₆), (F₇). We begin by checking the easiest conditions.

(F₁) This follows from fpqc descent, see e.g. [SGA1], Exp. VIII, Th. 5.2.

(F₄) Since $A \rightarrow A'$ is injective, by Lemma 2.2.5 the map $F(A) \rightarrow F(A')$ is a bijection. This gives a statement which is much stronger than the (F₄) in the theorem.

(F₅) Since $F \rightarrow S$ is a monomorphism, it is formally unramified.

(F₆) Since $F \rightarrow S$ is a monomorphism, it is separated.

(F₇) Since A has a unique associated prime, the map $\operatorname{Spec}(A_t) \rightarrow \operatorname{Spec}(A)$ is schematically dominant. Hence by Lemma 2.2.5, the point $\xi^* : \operatorname{Spec}(A_t) \rightarrow F$ automatically extends to $\operatorname{Spec}(A)$.

2.3.3 Condition (F₂). Let $\mathcal{F} := \operatorname{Hom}(\mathbb{G}_m, H)$ be the functor of *all* morphisms of group schemes $\mathbb{G}_m \rightarrow H$, that is, not just those that extend the collection u_n . It is standard that \mathcal{F} is locally of finite presentation, see [EGA] IV₃.8.8.3. Should the affine scheme $\lim_n \operatorname{Hom}(\mu_{\ell^n}, H)$ be locally of finite type over S , it would follow that $F \rightarrow S$ is locally of finite presentation ([EGA] IV₁.1.4.3(v)). However this is not the case in general, and the verification of (F₂) needs more work.

So let $A = \operatorname{colim} A_\alpha$ be a filtering colimits of rings. We want to prove that $\operatorname{colim} F(A_\alpha) \rightarrow F(A)$ is bijective. We look at the diagram

$$\begin{array}{ccc} \operatorname{colim} F(A_\alpha) & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ \operatorname{colim} \mathcal{F}(A_\alpha) & \xrightarrow{\sim} & \mathcal{F}(A). \end{array}$$

Since \mathcal{F} is locally of finite presentation, the bottom row is an isomorphism. We deduce that the upper row is injective. We shall now prove that the upper row is surjective, and in fact that the diagram is cartesian.

2.3.4 Lemma. *Let $f : \mathbb{G}_{m,A} \rightarrow H_A$ be a morphism extending $u_{n,A} : \mu_{\ell^n,A} \rightarrow H_A$ for all $n \geq 0$. Then there exists an index α such that f descends to a map $f_\alpha : \mathbb{G}_{m,A_\alpha} \rightarrow H_{A_\alpha}$ extending u_{n,A_α} for all $n \geq 0$.*

Proof : Since G and H are finitely presented, the morphism f is defined at finite level, that is there exists an index α and a morphism of A_α -group schemes $g : \mathbb{G}_{m,A_\alpha} \rightarrow H_{A_\alpha}$ whose pullback along $\operatorname{Spec}(A) \rightarrow \operatorname{Spec}(A_\alpha)$ is f . Since the groups are affine, the morphism g is given by a map of rings $g^\# : \mathcal{O}_H \otimes A_\alpha \rightarrow A_\alpha[z^{\pm 1}]$. Fix a presentation $\mathcal{O}_H = R[x_1, \dots, x_s]/(P_1, \dots, P_t)$. Then :

- u_n^\sharp is determined by the elements $z_{n,j} := u_n^\sharp(x_j) \in R[z]/(z^{\ell^n} - 1)$ satisfying $P_k(z_{n,1}, \dots, z_{n,s}) = 0$ for $k = 1, \dots, t$,
- g^\sharp is determined by the elements $y_j = g^\sharp(x_j) \in A_\alpha[z^{\pm 1}]$ satisfying $P_k(y_1, \dots, y_s) = 0$ for $k = 1, \dots, t$.

Moreover, to say that f extends $u_{n,A}$ means that we have the equality $z_{n,j} = \pi_n(y_j)$ in $A[z]/(z^{\ell^n} - 1)$, for all j , where

$$\pi_n : A_\alpha[z^{\pm 1}] \rightarrow A[z]/(z^{\ell^n} - 1)$$

is the projection. So we have to prove that we may enlarge the index α in such a way that g extends u_{n,A_α} for all $n \geq 0$.

For $n_0 \geq 0$ an integer, consider the finite free R -module $E_0 = R(-\lfloor \ell^{n_0}/2 \rfloor; \ell^{n_0} - \lfloor \ell^{n_0}/2 \rfloor - 1)$ in the notation of 2.2.2. Then $\pi_{n_0|E_0}$ is an isomorphism and for all $n \geq n_0$ we can define

$$\chi_n = \pi_n \circ (\pi_{n_0|E_0})^{-1} : R[z]/(z^{\ell^{n_0}} - 1) \longrightarrow R[z]/(z^{\ell^n} - 1).$$

By base change, these objects are defined over any R -algebra. We choose n_0 large enough so that $E_0 \otimes_R A_\alpha$ contains the Laurent polynomials y_1, \dots, y_s .

In the present context, the condition that f extends all the maps $u_{n,A} : G_{n,A} \rightarrow H_A$ is a finiteness constraint imposed by f on $\{u_n\}$ (whereas in other places of our arguments it is best seen as a condition imposed by $\{u_n\}$ on f). Indeed, from the relations $z_{n,j} = \pi_n(y_j)$ in A , we deduce that

$$z_{n,j} = \chi_n(z_{n_0,j}) \quad \text{in } A[z]/(z^{\ell^n} - 1) \quad \text{for all } n \geq n_0,$$

namely $\chi_n(z_{n_0,j}) = (\pi_n \circ (\pi_{n_0|E_0})^{-1})(\pi_{n_0}(y_j)) = \pi_n(y_j) = z_{n,j}$. We claim that we may increase α to achieve that these equalities hold in $A_\alpha[z]/(z^{\ell^n} - 1)$, for all $n \geq n_0$ and all j . In order to see this, note that the elements $\delta_{n,j} := z_{n,j} - \chi_n(z_{n_0,j})$ are defined over R , and as we have just proved, they belong to the kernel of the morphism $R[z]/(z^{\ell^n} - 1) \rightarrow A[z]/(z^{\ell^n} - 1)$. Let $I \subset R$ be the ideal generated by the coefficients of the expressions of $\delta_{n,j}$ on the monomial basis, for varying $n \geq n_0$ and j . Since R is noetherian, I is generated by finitely many elements. These elements vanish in A , hence they vanish in A_α provided we increase α a little, whence our claim.

The relations $z_{n,j} = \pi_n(y_j)$ in $A[z]/(z^{\ell^n} - 1)$ with $j = 1, \dots, s$ and $n \leq n_0$ being finite in number, we may increase α so as to ensure that all of them hold in $A_\alpha[z]/(z^{\ell^n} - 1)$. Then for $n \geq n_0$ we have

$$z_{n,j} = \chi_n(z_{n_0,j}) = \chi_n(\pi_{n_0}(y_j)) = \pi_n(y_j) \quad \text{in } A_\alpha[z]/(z^{\ell^n} - 1)$$

again. That is, g extends the maps u_{n,A_α} for all $n \geq 0$. □

2.3.5 Condition (F_3). Let (A, m) be a noetherian complete local ring. We want to prove that the map $F(A) \rightarrow \lim F(A/m^k)$ is bijective. We write again $\mathcal{F} := \text{Hom}(\mathbb{G}_m, H)$. We look at the diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & \lim F(A/m^k) \\ \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{\sim} & \lim \mathcal{F}(A/m^k). \end{array}$$

From [SGA3.2], Exp. IX, Th. 7.1 we know that \mathcal{F} is effective, that is the bottom arrow is bijective. We deduce that the upper row is injective. We shall now prove that the upper row is surjective, and in fact that the diagram is cartesian. So let $f_k : \mathbb{G}_{m,A/m^k} \rightarrow H_{A/m^k}$ be a collection of A/m^k -morphisms such

that f_k extends $u_{n,A/m^k} : \mu_{\ell^n,A/m^k} \rightarrow H_{A/m^k}$ for all $n \geq 0$, and let $f : \mathbb{G}_{m,A} \rightarrow H_A$ be a morphism that algebraizes the f_k . We must prove that f extends $u_{n,A}$, for each n . For this let $i_n : \mu_{\ell^n,A} \rightarrow \mathbb{G}_{m,A}$ be the closed immersion. The two maps $f \circ i_n$ and u_n coincide modulo m^k for each $k \geq 1$, hence so do the morphisms of Hopf algebras

$$(f \circ i_n)^\sharp, u_n^\sharp : \mathcal{O}_H \otimes A \rightarrow A[z]/(z^{\ell^n} - 1).$$

Since $A[z]/(z^{\ell^n} - 1)$ is separated for the m -adic topology, we deduce that $(f \circ i_n)^\sharp = u_n^\sharp$ and hence $f \circ i_n = u_n$. This concludes the argument.

2.3.6 Remark. We could also appeal to the following more general result extending the injectivity part of [EGA] III₁.5.4.1: *let (A, m) be a noetherian complete local ring, and $S = \text{Spec}(A)$. Let X, Y be S -schemes of finite type with X pure and Y separated. Let $f, g : X \rightarrow Y$ be S -morphisms. If we have the equality of completions $\hat{f} = \hat{g}$, then $f = g$.* For the notion of a pure morphism of schemes we refer to Appendix A. For the proof of the italicized statement, by [EGA1_{new}], 10.9.4 the morphisms f and g agree in an open neighbourhood of $\text{Spec}(A/m)$. Then the arguments in the proof of [Ro12], Lemma 2.1.9 apply verbatim.

2.3.7 Condition (F_8) . This condition will be verified with the help of the following lemma.

2.3.8 Lemma. *Let T be a scheme and T' a closed subscheme. Let $\xi' : T' \rightarrow F$ be a point. Then there is a largest closed subscheme $\mathcal{Z}_T \subset T$ such that ξ' extends to \mathcal{Z}_T . Moreover, its formation is Zariski local: if $U \subset T$ is an open subscheme and $U' = U \cap T'$, we have $\mathcal{Z}_T \cap U = \mathcal{Z}_U$.*

Proof : Throughout, for all open subschemes $U \subset T$ we write $U' = U \cap T'$ and all closed subschemes Z of U such that $\xi'_{|U'}$ extends to Z are implicitly assumed to contain U' . We proceed by steps.

Let $U = \text{Spec}(A)$ be an affine open subscheme of T . Consider the family of all closed subschemes $Z_\alpha = V(I_\alpha) \subset U$ to which $\xi'_{|U'}$ extends. Consider the ideal $I = \bigcap I_\alpha$ and define $\mathcal{Z}_U = V(I)$. Since the map $A/I \rightarrow \prod A/I_\alpha$ is injective, applying Lemma 2.2.5, we see that $\xi'_{|U'}$ extends to \mathcal{Z}_U . By its very definition the closed subscheme \mathcal{Z}_U is largest.

Let U, V be two affine opens of T with $U \subset V$. We claim that $\mathcal{Z}_V \cap U = \mathcal{Z}_U$. Indeed, since $\xi'_{|V'}$ extends to \mathcal{Z}_V then $\xi'_{|U'}$ extends to $\mathcal{Z}_V \cap U$, hence $\mathcal{Z}_V \cap U \subset \mathcal{Z}_U$. Conversely, let Z be the schematic image of $\mathcal{Z}_U \rightarrow U \rightarrow V$. The latter map being quasi-compact, the map $\mathcal{Z}_U \rightarrow Z$ is schematically dominant. By Lemma 2.2.6 it follows that $\xi'_{|Z}$ extends to Z . By maximality this forces $Z \subset \mathcal{Z}_V$, hence $\mathcal{Z}_U \subset \mathcal{Z}_V \cap U$.

Let U, V be arbitrary affine opens of T . We claim that $\mathcal{Z}_U \cap V = \mathcal{Z}_V \cap U$. Indeed, by the previous step, for all affine opens $W \subset U \cap V$ we have $\mathcal{Z}_U \cap V \cap W = \mathcal{Z}_W = \mathcal{Z}_V \cap U \cap W$.

Let \mathcal{Z}_T be the closed subscheme of T obtained by gluing the \mathcal{Z}_U when U varies over all affine opens; thus $\mathcal{Z}_T \cap U = \mathcal{Z}_U$ by construction. Now ξ' extends to \mathcal{Z}_T , because $\xi'_{|U'}$ extends to \mathcal{Z}_U for each U , and we can glue these extensions. Moreover \mathcal{Z}_T is maximal with this property, because if ξ' extends to some closed subscheme $Z \subset T$ then for each affine U the element $\xi'_{|U'}$ extends to $Z \cap U$, hence $Z \cap U \subset \mathcal{Z}_U = \mathcal{Z}_T \cap U$, hence $Z \subset \mathcal{Z}_T$.

The fact that $\mathcal{Z}_T \cap U = \mathcal{Z}_U$ for all open subschemes U follows by restricting to affine opens. \square

In order to verify Condition (F_8) , we write $T = \text{Spec}(A)$ and $T' = \text{Spec}(A/I)$. The assumption that $\xi'_t : \text{Spec}((A/I)_t) \rightarrow \text{Spec}(A/I) \rightarrow F$ does not lift to any subscheme of $\text{Spec}(A_t)$ which is strictly larger than $\text{Spec}((A/I)_t)$ means that the inclusion $T' \subset \mathcal{Z}_T$ is an equality at the generic point. It follows that $T' \cap W = \mathcal{Z}_T \cap W = \mathcal{Z}_W$ for some open W . Applying Lemma 2.3.8 to variable opens $W_1 \subset W$, we obtain $T' \cap W_1 = \mathcal{Z}_T \cap W_1 = \mathcal{Z}_{W_1}$, which shows that W fulfills the required condition and we are done. \square

This concludes the proof of Theorem 2.1.2 and thus also of Theorem 2.1.1.

3 Homomorphisms from a group with reductive and proper composition factors

In this section we build on Theorem 2.1.1 to prove our main results on the functors of homomorphisms of group schemes and functors of reductive subgroups: to wit, Theorems 1.2.2 and 1.2.3 from the Introduction.

3.1 Homomorphisms from a reductive group

In this section G and H are S -group schemes.

3.1.1 Lemma. *Assume that $G \rightarrow S$ is reductive and $H \rightarrow S$ is separated and of finite presentation. Let $\mathrm{Hom}(G, H)$ be the functor of morphisms of group schemes, and $\mathrm{Mono}(G, H)$ the subfunctor of monomorphisms. Then the following hold.*

- (1) *The inclusion $\mathrm{Mono}(G, H) \subset \mathrm{Hom}(G, H)$ is representable by open immersions.*
- (2) *If $G \rightarrow S$ has no geometric fibre of characteristic 2 containing a direct factor isomorphic to SO_{2n+1} for some $n \geq 1$, then for each maximal torus $T \subset G$ the following commutative diagram is cartesian:*

$$\begin{array}{ccc} \mathrm{Mono}(G, H) & \hookrightarrow & \mathrm{Hom}(G, H) \\ \downarrow & & \downarrow \\ \mathrm{Mono}(T, H) & \hookrightarrow & \mathrm{Hom}(T, H). \end{array}$$

The refined statement (2) will not be needed in the paper, but we find it worth reporting.

Proof : We handle both cases (1) and (2) simultaneously, with only a little variation in the end.

The question of representability of $\mathrm{Mono}(G, H)$ by an open subscheme of $\mathrm{Hom}(G, H)$ is étale-local over S so we may assume that S is affine and that there exists a maximal torus $T \subset G$ ([SGA3.2], Exp. XII, Th. 1.7). Also since G and H are finitely presented, we may assume that S is noetherian.

Let $f : G \rightarrow H$ and $K := \ker(f)$. Restricting to the open locus where $f|_T$ is a monomorphism ([SGA3.2], Exp. IX, Cor. 6.6) we may also assume that $K \cap T = 1$.

Let $S_0 \subset S$ be the locus of points s such that K_s is trivial. We claim it is enough to prove that S_0 is open. Indeed, in this case $K \times_S S_0 \rightarrow S_0$ is finite and the augmentation ideal of its structure sheaf is zero in each fibre, hence zero. It follows that $K \times_S S_0$ is trivial and S_0 represents the subfunctor of S defined by the condition that f is a monomorphism.

We proceed to prove that S_0 is open in S . Since it is constructible and S is noetherian, it is enough to prove that it is stable by generization, see [EGA] IV₃.9.6.1. Let $s \rightsquigarrow s_0$ be a specialization with $s_0 \in S_0$ and let $S' \rightarrow S$ be a morphism from a trait (spectrum of a discrete valuation ring) whose image witnesses this specialization. Let $S'' \rightarrow S'$ be a ramified extension whose residue field contains the field of definition of the geometric nilpotent ideal of $K_{s''}$, so in particular $(K_{s''})_{\mathrm{red}}$ is a smooth subgroup scheme of the generic fibre $K_{s''}$. Replacing S by S'' we can assume that S is a trait and $(K_s)_{\mathrm{red}}$ is a smooth subgroup. We have to prove that K_s is trivial, assuming that $K_{s_0} = 1$ in case (1) and that $K_{s_0} \cap T_{s_0} = 1$ in case (2).

Let $(K_s)_{\mathrm{red}}^\circ$ be the reduced identity component. By the assumption $K \cap T = 1$ together with conjugacy of maximal tori, the normal, smooth connected subgroup $(K_s)_{\mathrm{red}}^\circ$ contains no torus, hence is unipotent. Since G_s is reductive, we find $(K_s)_{\mathrm{red}}^\circ = 1$. This shows that K_s is finite. If $L \subset K_s$ is a subgroup of multiplicative type which is either étale or infinitesimal, there is a maximal torus of G_s containing L : in

the étale case this is standard, and in the infinitesimal case this is Th. 1.1 of Geiss and Voigt [GV04]. After a further étale extension S'/S if needed, a suitable conjugate of L lies in $K \cap T$, hence $L = 1$. It follows that K_s is finite unipotent.

The neutral component K_s° is infinitesimal unipotent. When $G \rightarrow S$ has no characteristic 2 geometric fibre containing a direct factor isomorphic to SO_{2n+1} , the main result of Vasiu [Va05] says that G_s has no infinitesimal unipotent group scheme, hence $K_s^\circ = 1$. In the general case, letting $G_d = \ker(F^d : G \rightarrow G^{(d)})$ be the d -th Frobenius kernel of $G \rightarrow S$, we have $K_s^\circ \subset (G_d)_s$ for large enough d . Since G_d is finite flat over S , the scheme-theoretic closure $C \subset G_d$ of K_s° is finite flat also. Since H is separated we have $C \subset K$ and from the assumption $K_{s_0} = 1$ we deduce that the rank of $C \rightarrow S$ is 1, that is $C = 1$. Thus $K_s^\circ = C_s = 1$ and the upshot is that K_s is étale (and finite unipotent).

Then the automorphism scheme $\mathrm{Aut}(K_s)$ is étale, so by connectedness of G_s the conjugation action $G_s \rightarrow \mathrm{Aut}(K_s)$ is trivial. Therefore K_s is central, and since the center of G_s is a torus, it is trivial. \square

3.1.2 Remark. Let S be the spectrum of a field k of characteristic 2 and $V = ke_0 \oplus \cdots \oplus ke_{2n}$ the standard vector space of dimension $2n + 1$. Let $G = \mathrm{SO}(q)$ be the orthogonal group of the quadratic form $q(x_0, \dots, x_{2n}) = x_0^2 + x_1x_{n+1} + \cdots + x_nx_{2n}$. The polarization $\psi(x, y) := q(x + y) - q(x) - q(y)$ is alternating with kernel $V_0 = ke_0$, inducing a nondegenerate alternating form ψ_0 on $W_0 = V/V_0$. This gives rise to an isogeny $f : \mathrm{SO}_{2n+1} = \mathrm{SO}(q) \rightarrow \mathrm{Sp}_{2n} = \mathrm{Sp}(\psi_0)$ whose kernel is isomorphic to α_2^{2n} , see [Va05], § 2.1. Let $T \subset \mathrm{SO}_{2n+1}$ be a maximal torus. The restriction of f to T is a monomorphism; in other words, the commutative square of Lemma 3.1.1 is not cartesian.

3.1.3 Lemma. *Assume that $G \rightarrow S$ is reductive and $H \rightarrow S$ is separated, of finite presentation, flat and pure. Let $\mathrm{Isom}(G, H)$ be the functors of isomorphisms of group schemes.*

- (1) *The inclusion $\mathrm{Isom}(G, H) \subset \mathrm{Mono}(G, H)$ is representable by closed immersions of finite presentation,*
- (2) *The inclusion $\mathrm{Isom}(G, H) \subset \mathrm{Hom}(G, H)$ is representable by immersions locally of finite presentation.*

Proof : (1) By [SGA3.2], Exp. XVI, Cor. 1.5.a) any monomorphism $f : G \rightarrow H$ is a closed immersion. Thus $\mathrm{Isom}(G, H) \subset \mathrm{Mono}(G, H)$ is the subfunctor defined by the condition that the surjective map of sheaves $\mathcal{O}_H \rightarrow f_*\mathcal{O}_G$ is an isomorphism. It follows from [RG71], Première partie, Th. (4.1.1) that this condition is representable by closed immersions of finite presentation.

(2) Follows from (1) and Lemma 3.1.1. \square

3.1.4 Theorem. *Assume that $G \rightarrow S$ is reductive and $H \rightarrow S$ is affine and of finite presentation. Then, the following hold.*

- (1) *$\mathrm{Hom}(G, H)$ is representable by an S -scheme separated and locally of finite presentation.*
- (2) *Each subscheme (resp. closed subscheme) of $\mathrm{Hom}(G, H)$ which is quasi-compact over S is quasi-affine (resp. affine) over S .*

Proof : The fact that $\mathrm{Hom}(G, H) \rightarrow S$ is locally of finite presentation is checked using the usual results on limits from [EGA] IV, § 8. Hence it will be enough to complete the following three steps:

- (i) prove that $\mathrm{Hom}(G, H)$ is a separated S -algebraic space;
- (ii) prove (2) (with “subspace” replacing “subscheme”);
- (iii) deduce that $\mathrm{Hom}(G, H)$ is a scheme.

(i) This question is étale-local over S so by [SGA3.3], Exp. XIX, Th. 2.5 we may assume that there exists a split maximal torus $T \subset G$. Then it follows from [SGA3.3], Exp. XXIV, Cor. 7.1.9 that the restriction map $\mathrm{Hom}(G, H) \rightarrow \mathrm{Hom}(T, H)$ is representable and affine. Since $\mathrm{Hom}(T, H) \rightarrow S$ is representable and separated by Theorem 2.1.1, we see that $\mathrm{Hom}(G, H) \rightarrow S$ is representable by a separated S -algebraic space.

(ii) Let $Y \subset \mathrm{Hom}(G, H)$ be a subspace which is quasi-compact over S . In order to show that Y is quasi-affine over S we can afford an étale base change $S' \rightarrow S$, so we can assume that there exists a split maximal torus $T \subset G$. Let $T_n = \ker(n : T \rightarrow T)$ be the finite flat torsion subschemes. The limit

$$L = \lim \mathrm{Hom}(T_n, H)$$

is an affine S -scheme. As in [SGA3.2] 7.1.1 we choose a pinning of G with a set of simple roots Δ of cardinality n , and associated elements $u_\alpha \in U_\alpha^\times(S)$, $w_\alpha \in \mathrm{Norm}_G(T)(S)$ for $\alpha \in \Delta$ (see *loc. cit.* for the precise definition of these elements). It follows from [SGA3.2], Proposition 7.1.2 that the morphism

$$\begin{aligned} \mathrm{Hom}(G, H) &\longrightarrow \mathrm{Hom}(T, H) \times H^{2n} \\ f &\longmapsto (f|_T, (f(u_\alpha), f(w_\alpha))_{\alpha \in \Delta}) \end{aligned}$$

is a closed immersion. Using the Density Theorem we have a sequence of monomorphisms

$$F := \mathrm{Hom}(G, H) \hookrightarrow \mathrm{Hom}(T, H) \times H^{2n} \hookrightarrow L' := L \times H^{2n}.$$

Let $Z \subset L'$ be the schematic image of $Y \rightarrow L'$; this is a closed subscheme, hence affine over S . Since Y is quasi-compact and L' is separated, then $Y \rightarrow L'$ is quasi-compact. Thus the morphism $Y \rightarrow Z$ is schematically dominant ([SP22], Tag 01R8). It then follows from Lemma 2.2.6 that $Y \rightarrow \mathrm{Hom}(T, H)$ factors uniquely through a map $Z \rightarrow \mathrm{Hom}(T, H)$. Thus the closed immersion $Z \rightarrow L'$ factors through a closed immersion $Z \rightarrow \mathrm{Hom}(T, H) \times H^{2n}$. Then $Y \rightarrow Z$ is an immersion; in particular it is quasi-affine so Y is quasi-affine over S . If at the start it is assumed that $Y \subset \mathrm{Hom}(G, H)$ is an S -quasi-compact *closed* subspace, then $Y \rightarrow Z$ is a closed immersion; being also schematically dominant, it is an isomorphism, hence Y is affine over S in this case.

(iii) Let $\mathrm{Hom}(G, H) = \cup Z_i$ be a covering by quasi-compact open subspaces. According to (ii) the Z_i are quasi-affine over S . In particular they are schemes, hence $\mathrm{Hom}(G, H)$ is a scheme. \square

3.2 Functor of reductive subgroups

Recall that a *type* is by definition an isomorphism class $\mathbf{t} = [\mathcal{R}]$ of *root datum* ([SGA3.2], XXII.2.6.1), the latter being a quadruple $\mathcal{R} = (M, M^*, R, R^*)$ composed of finite type free \mathbb{Z} -modules in duality M, M^* and finite subsets $R \subset M$, $R^* \subset M^*$ in duality called *root system* and *coroot system* ([SGA3.2], XXI.1.1.1). For example, the type of a diagonalizable group $T = D(M)$ is $\mathbf{t} = [(M, M^*, \emptyset, \emptyset)]$.

3.2.1 Theorem. *Assume that $H \rightarrow S$ is an affine group scheme of finite presentation.*

(1) *The functor $\mathrm{Sub}_{\mathrm{red}}(H)$ of reductive subgroups of H is representable by an algebraic space separated and locally of finite presentation which is a disjoint sum*

$$\mathrm{Sub}_{\mathrm{red}}(H) = \coprod_{\mathbf{t}} \mathrm{Sub}_{\mathbf{t}}(H)$$

indexed by the types of reductive groups.

(2) *The summand of subgroups of multiplicative type*

$$\mathrm{Sub}_{\mathrm{mult}}(H) = \coprod_{\mathbf{t}=[(M, M^*, \emptyset, \emptyset)]} \mathrm{Sub}_{\mathbf{t}}(H)$$

is representable by a scheme with the following property: each subscheme (resp. closed subscheme) of $\mathrm{Sub}_{\mathrm{mult}}(H)$ which is quasi-compact over S , is quasi-affine (resp. affine) over S .

Proof : (1) The disjoint sum decomposition reflects the fact that the type of a reductive group is locally constant on the base ([SGA3.2], XXII.2.8). Thus it is enough to establish that $\mathrm{Sub}_{\mathbf{t}}(H)$ is representable.

Let $G(\mathbf{t})$ be the split reductive group scheme of type \mathbf{t} as in [SGA3.3], XXV.1.1. By Theorem 3.1.4, the functor $\mathrm{Hom}(G(\mathbf{t}), H)$ is representable by a scheme. It follows from Lemma 3.1.1, item (1) that the subfunctor

$$\mathrm{Mono}(G(\mathbf{t}), H) \subset \mathrm{Hom}(G(\mathbf{t}), H)$$

of monomorphisms of group schemes is an open subscheme. Moreover, since $G(\mathbf{t})$ is reductive and H is finitely presented and separated, by [SGA3.2], XVI.1.5.a) any monomorphism $f : G(\mathbf{t}) \rightarrow H$ is a closed immersion, inducing an isomorphism between $G(\mathbf{t})$ and a closed subgroup scheme $K \hookrightarrow H$. By taking a monomorphism f to its image K , we obtain a morphism of functors:

$$\pi : \mathrm{Mono}(G(\mathbf{t}), H) \longrightarrow \mathrm{Sub}_{\mathbf{t}}(H).$$

Let $A = \mathrm{Aut}(G(\mathbf{t}))$ be the functor of automorphisms of $G(\mathbf{t})$; this is a smooth, separated S -group scheme by [SGA3.3], XXIV.1.3. It acts freely on $\mathrm{Mono}(G(\mathbf{t}), H)$ by the rule $af = f \circ a^{-1}$ for $a \in \mathrm{Aut}(G(\mathbf{t}))$ and $f \in \mathrm{Mono}(G(\mathbf{t}), H)$. Let $\mathrm{Mono}(G(\mathbf{t}), H)/A$ be the quotient sheaf. Since the morphism π is A -equivariant, it induces a morphism

$$i : \mathrm{Mono}(G(\mathbf{t}), H)/A \longrightarrow \mathrm{Sub}_{\mathbf{t}}(H).$$

We claim that i is an isomorphism. It is enough to prove that it is an isomorphism of fppf sheaves:

- surjectivity: let $K \subset H$ be a reductive subgroup scheme. Around each point $s \in S$, after étale localization there is a maximal torus T ([SGA3.3], XIX.2.5) and after further Zariski localization there is a root system for T providing a splitting for K ([SGA3.3], XXII.2.1). So we can assume that K is split and isomorphic to $G(\mathbf{t})$. We obtain a monomorphism $G(\mathbf{t}) \simeq K \subset H$ which provides a point of $\mathrm{Mono}(G(\mathbf{t}), H)$ lifting K .
- injectivity: if $f_i : G(\mathbf{t}) \rightarrow H$ are two monomorphisms with the same image K , then $f_2^{-1} \circ f_1 : G(\mathbf{t}) \rightarrow K \rightarrow G(\mathbf{t})$ is an automorphism of $G(\mathbf{t})$.

Now by Artin's Theorem (see [SP22], Tag 04S5) the quotient $\mathrm{Mono}(G(\mathbf{t}), H)/A$ is an algebraic space locally of finite presentation, hence so is $\mathrm{Sub}_{\mathbf{t}}(H)$. Moreover, using that monomorphisms $G(\mathbf{t}) \rightarrow H$ are closed immersions we see that $\mathrm{Sub}_{\mathbf{t}}(H)$ is separated. This concludes the proof of (1).

(2) Let us write $\mathrm{Sub}_M(H) := \mathrm{Sub}_{\mathbf{t}}(H)$ when $\mathbf{t} = [(M, M^*, \emptyset, \emptyset)]$. To prove that $\mathrm{Sub}_M(H)$ is representable by a scheme, let $L := \lim \mathrm{Sub}_{M/nM}(H)$ be the limit of the functors of finite flat multiplicative type subgroups of type M/nM . Since $\mathrm{Sub}_{M/nM}(H)$ is representable and affine (Lemma A.10), the functor L is an affine scheme. By mapping any subgroup $G \hookrightarrow H$ to the collection of subgroups $G_n \hookrightarrow H$ where $G_n = \ker(n : G \rightarrow G)$, we define a morphism of functors $u : \mathrm{Sub}_M(H) \rightarrow L$. By the Density Theorem, this is a monomorphism. As $\mathrm{Sub}_M(H) \rightarrow S$ is locally of finite type, so is u . In particular u is a separated, locally quasi-finite morphism. By [SP22], Tag 0418 all such morphisms are representable by schemes, hence $\mathrm{Sub}_M(H)$ is a scheme. Finally, in order to prove that each subscheme (resp. closed subscheme) which is quasi-compact over S is quasi-affine (resp. affine) over S , we proceed as in the proof of 3.1.4. \square

3.3 Homomorphisms from a group with reductive and proper composition factors

3.3.1 Lemma. *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence of flat, finitely presented S -group schemes with $N \rightarrow S$ pure. Let H be an affine, finitely presented S -group scheme and $f_0 : N \rightarrow H$ a morphism of group schemes. Assume that*

- (i) Q is reductive, or
- (ii) Q is proper.

Then the functor $\mathrm{Hom}^{f_0}(G, H)$ of morphisms of group schemes $f : G \rightarrow H$ extending f_0 is representable by a locally finitely presented, separated S -algebraic space.

Proof : Let $\Gamma_0 \subset N \times H \subset G \times H$ be the graph of f_0 . Since $\Gamma_0 \simeq N$ is pure, by Corollary A.8 its normalizer $\mathrm{Norm}(\Gamma_0) \subset G \times H$ is a closed, finitely presented subgroup scheme of $G \times H$. Let $\pi : G \times H \rightarrow G$ be the projection and $\pi' : \mathrm{Norm}(\Gamma_0)/\Gamma_0 \rightarrow G/N = Q$ the morphism it induces.

Step 1: the map $\pi' : \mathrm{Norm}(\Gamma_0)/\Gamma_0 \rightarrow Q$ is affine. Since the closed immersion $\mathrm{Norm}(\Gamma_0) \hookrightarrow G \times H$ induces a closed immersion of algebraic spaces $\mathrm{Norm}(\Gamma_0)/\Gamma_0 \hookrightarrow (G \times H)/\Gamma_0$ (beware that the target does not a priori carry a group structure), it is enough to prove that $(G \times H)/\Gamma_0 \rightarrow Q$ is affine. It is enough to check this after the fppf base change $G \rightarrow Q$. For this, we consider the morphism:

$$\begin{aligned} (G \times H) \times_Q G &\xrightarrow{\alpha} H \times G \\ ((g_1, h), g_2) &\longmapsto (f_0(g_2 g_1^{-1})h, g_2) \end{aligned}$$

(note that $g_2 g_1^{-1}$ is a section of N). This is invariant by the action of Γ_0 on $(G \times H) \times_Q G$ by translation on the first factor. By commutation of the quotient $G \times H \rightarrow (G \times H)/\Gamma_0$ with the flat base change $G \rightarrow Q$, from α we deduce a morphism

$$((G \times H)/\Gamma_0) \times_Q G \xrightarrow{\beta} H \times G.$$

It is easy to see that the map $(h, g) \mapsto (g, h, g)$ provides an inverse to β which therefore is an isomorphism. Since the right-hand side is affine over G , this concludes Step 1.

Step 2: conclusion. Attaching to a morphism $f : G \rightarrow H$ its graph Γ yields a correspondence between the functor $\mathrm{Hom}^{f_0}(G, H)$ on one side, and the functor of subgroups $\Gamma \subset G \times H$ containing Γ_0 and such that $\pi|_{\Gamma} : \Gamma \rightarrow G$ is an isomorphism, on the other side. Note that $\Gamma \subset \mathrm{Norm}(\Gamma_0)$ because N is normal in G ; hence the latter functor is in correspondence with the functor of subgroups Γ' of $\mathrm{Norm}(\Gamma_0)/\Gamma_0$ such that $\pi'_{|\Gamma'} : \Gamma' \rightarrow Q$ is an isomorphism. It remains to prove that the latter is representable.

In case (i), by Step 1 the map $\mathrm{Norm}(\Gamma_0)/\Gamma_0 \rightarrow Q \rightarrow S$ is affine. By Theorem 3.2.1 the functor of reductive subgroups of $\mathrm{Norm}(\Gamma_0)/\Gamma_0$ is representable. The subfunctor of those subgroups Γ' for which $\pi'_{|\Gamma'} : \Gamma' \rightarrow Q$ is an isomorphism is representable by a locally finitely presented subscheme by Lemma 3.1.3.

In case (ii) recall from Lemma A.11 that the functor of proper, flat, finitely presented subgroups of $\mathrm{Norm}(\Gamma_0)/\Gamma_0$ is representable. According to Olsson [Ol06], Lemma 5.2, the subfunctor of those subgroups Γ' for which $\pi'_{|\Gamma'} : \Gamma' \rightarrow Q$ is an isomorphism is representable by an open subscheme. \square

3.3.2 Theorem. *Assume that $G \rightarrow S$ has a finite composition series whose factors are either reductive or proper, flat, finitely presented, and that $H \rightarrow S$ is affine and of finite presentation.*

- (1) $\mathrm{Hom}(G, H)$ is representable by an S -algebraic space separated and locally of finite presentation.
- (2) $\mathrm{Mono}(G, H)$ is representable by an open subspace of $\mathrm{Hom}(G, H)$. Moreover, all monomorphisms $G \rightarrow H$ are closed immersions.

Proof : According to Corollaries A.3 and A.5, all group schemes having a finite composition series as indicated are pure. We prove (1) and (2) for group schemes admitting a composition series of length n , by induction on n . If $n = 0$ we have $G = 1$ and both statements are clear, so we now assume that G admits a composition series of length $n \geq 1$. Thus there is an exact sequence $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ where Q is reductive or proper and N admits a composition series of length $n - 1$.

(1) By induction the functor $\mathrm{Hom}(N, H)$ is representable by an S -algebraic space separated and locally of finite presentation, so it is enough to prove that the restriction homomorphism $\mathrm{Hom}(G, H) \rightarrow \mathrm{Hom}(N, H)$ is representable, separated and locally of finite presentation. This is exactly what Lemma 3.3.1 says.

(2) By induction we have an open immersion $\mathrm{Mono}(N, H) \subset \mathrm{Hom}(N, H)$ and it is enough to prove that $\mathrm{Mono}(G, H)$ is open in the space $T = \mathrm{Hom}(G, H) \times_{\mathrm{Hom}(N, H)} \mathrm{Mono}(N, H)$ of morphisms whose restriction to N is monomorphic. Let $f : G \rightarrow H$ be the universal morphism over T and let $N' = f(N) \simeq N$. The normalizer $\mathrm{Norm}(N') \subset H$ is a closed, finitely presented subgroup scheme of H thanks to Corollary A.8. Moreover f maps into $\mathrm{Norm}(N')$ and induces a morphism $f' : Q = G/N \rightarrow \mathrm{Norm}(N')/N'$. Now $\mathrm{Mono}(G, H)$ is the subfunctor of T where f' is a monomorphism, which is an open subscheme by Lemma 3.1.1 when Q is reductive and by Lemma A.10 when Q is proper.

Finally the fact that all monomorphisms $G \rightarrow H$ are closed immersions follows directly from the same statements for reductive and proper groups. For reductive groups this is [SGA3.2], Exp. XVI, Cor. 1.5.a) and for proper groups this is because proper monomorphisms are closed immersions. \square

4 Algebraicity and smoothness of fixed points stacks

Let $\mathcal{X} \rightarrow S$ be an algebraic stack and $G \rightarrow S$ a group algebraic space acting on it.

4.1 Algebraicity

The stack of fixed points \mathcal{X}^G has for sections over a scheme $T \rightarrow S$ the pairs $(x, \{\alpha_g\}_{g \in G(T)})$ composed of an object $x \in \mathcal{X}(T)$ and a collection of isomorphisms $\alpha_g : gx \rightarrow x$ satisfying the cocycle condition $\alpha_{gh} = \alpha_g \circ g\alpha_h$ (see [Ro05], Prop. 2.5), pictured by a commutative triangle:

$$\begin{array}{ccc} & gx & \\ g\alpha_h \nearrow & & \searrow \alpha_g \\ (gh)x & \xrightarrow{\alpha_{gh}} & x. \end{array}$$

An interesting viewpoint on \mathcal{X}^G is that it can be expressed as a certain Weil restriction of the *universal stabilizer* $\mathrm{St}_{\mathcal{X}, G}$ of the action of G on \mathcal{X} . The latter is the 2-fibred product:

$$\begin{array}{ccc} \mathrm{St}_{\mathcal{X}, G} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \Delta \\ G \times \mathcal{X} & \xrightarrow{\mathrm{act} \times \mathrm{pr}_2} & \mathcal{X} \times \mathcal{X}. \end{array}$$

In particular $\mathrm{St}_{\mathcal{X},G} \rightarrow \mathcal{X}$ is representable by algebraic spaces. The top horizontal map of the diagram makes $\mathrm{St}_{\mathcal{X},G}$ an \mathcal{X} -group functor: for each $x : T \rightarrow \mathcal{X}$ we have:

$$\mathrm{St}_{\mathcal{X},G}(T) = \{(g, \alpha); g \in G(T) \text{ and } \alpha : gx \xrightarrow{\sim} x \text{ an isomorphism}\},$$

with law of multiplication $(g, \alpha) \cdot (h, \beta) := (gh, \alpha \circ g\beta)$ and neutral element $(g, \alpha) = (1, \mathrm{id}_x)$. The left vertical map is the map $\mathrm{St}_{\mathcal{X},G} \rightarrow G_{\mathcal{X}}$, $(g, \alpha) \mapsto g$. It is a morphism of \mathcal{X} -group spaces with kernel equal to the inertia stack $I_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{X}} \mathcal{X}} \mathcal{X}$, whence an exact sequence of \mathcal{X} -group functors:

$$1 \longrightarrow I_{\mathcal{X}} \longrightarrow \mathrm{St}_{\mathcal{X},G} \longrightarrow G_{\mathcal{X}}.$$

If the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is affine and finitely presented, then so is $\mathrm{St}_{\mathcal{X},G} \rightarrow G_{\mathcal{X}}$.

4.1.1 Definition. In general, if $\Sigma \rightarrow G$ is a morphism of S -group spaces we write

$$(\mathrm{GRes}_{G/S} \Sigma)(T) = \{\text{group-theoretic sections of } \Sigma_T \rightarrow G_T\}.$$

We call $\mathrm{GRes}_{G/S} \Sigma$ the *group-theoretic Weil restriction of Σ along $G \rightarrow S$* .

4.1.2 Lemma. *We have an \mathcal{X} -isomorphism $\mathcal{X}^G \xrightarrow{\sim} \mathrm{GRes}_{G_{\mathcal{X}}/\mathcal{X}} \mathrm{St}_{\mathcal{X},G}$.*

Proof : A section of \mathcal{X}^G over $x : T \rightarrow \mathcal{X}$ is a collection $(\{\alpha_g\}_{g \in G(T)})$ satisfying the cocycle condition $\alpha_{gh} = \alpha_g \circ g\alpha_h$. This is exactly a group-theoretic section of $\mathrm{St}_{\mathcal{X},G} \times_{\mathcal{X}} T \rightarrow G_{\mathcal{X}} \times_{\mathcal{X}} T$. \square

For the proof of Theorem 4.1.4 below we need a variant of Lemma 3.3.1.

4.1.3 Lemma. *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence of flat, finitely presented S -group schemes with $N \rightarrow S$ pure. Let E be an S -group scheme and $\pi : E \rightarrow G$ a morphism of S -group schemes which is affine. Let $f_0 : N \rightarrow E$ be a morphism of group schemes which is a section of $\pi|_N : E \times_G N \rightarrow N$. Assume that*

- (i) Q is reductive, or
- (ii) Q is proper.

Then the functor $\mathrm{GRes}_{G/S}^{f_0}(E)$ of group-theoretic sections of π extending f_0 is representable by a locally finitely presented, separated S -algebraic space.

Proof : The section f_0 induces an isomorphism between N and the image subgroup $\Sigma_0 := f_0(N)$. Let $\mathrm{Norm}(\Sigma_0) \subset E$ be its normalizer. Since $\Sigma_0 \simeq N$ is pure, by Corollary A.8 this is a closed, finitely presented subgroup scheme of E . Let $\pi' : \mathrm{Norm}(\Sigma_0)/\Sigma_0 \rightarrow G/N = Q$ be the morphism induced by $\pi : E \rightarrow G$.

Step 1: the map $\pi' : \mathrm{Norm}(\Sigma_0)/\Sigma_0 \rightarrow Q$ is affine. Since $\mathrm{Norm}(\Sigma_0)/\Sigma_0 \hookrightarrow E/\Sigma_0$ is a closed immersion, it is enough to prove that $E/\Sigma_0 \rightarrow Q$ is affine. In turn, it is enough to prove this after the flat base change $G \rightarrow Q$. The morphism

$$\alpha : E \times_Q G \longrightarrow E, \quad (e, g) \longmapsto (f_0(g\pi(e)^{-1})e$$

is invariant by the action of Σ_0 on $E \times_Q G$ by translation on the first factor, hence induces

$$(E/\Sigma_0) \times_Q G \xrightarrow{\beta} E.$$

The map $e \mapsto (e, \pi(e))$ is an inverse to β which therefore is an isomorphism. Since the right-hand side is affine over G , our claim follows.

Step 2: conclusion. We have an isomorphism of functors between $\mathrm{GRes}_{G/S}^{f_0}(E)$ and the functor of subgroups $\Sigma \subset E$ containing Σ_0 such that $\pi|_{\Sigma}$ is an isomorphism. That functor is isomorphic to the functor of subgroups Σ' of $\mathrm{Norm}(\Sigma_0)/\Sigma_0$ such that $\pi'_{\Sigma'} : \Sigma' \rightarrow Q$ is an isomorphism. It remains to prove that the latter is representable.

In case (i), by Step 1 the composition $\mathrm{Norm}(\Sigma_0)/\Sigma_0 \rightarrow Q \rightarrow S$ is affine. By Theorem 3.2.1 the functor of reductive subgroups of $\mathrm{Norm}(\Sigma_0)/\Sigma_0$ is representable. The subfunctor of those subgroups Σ' for which $\pi'_{\Sigma'}$ is an isomorphism is representable by a locally finitely presented subscheme by Lemma 3.1.3.

In case (ii) recall from Lemma A.11 that the functor of proper, flat, finitely presented subgroups of $\mathrm{Norm}(\Sigma_0)/\Sigma_0$ is representable. According to Olsson [Ol06], Lemma 5.2, the subfunctor of those subgroups Σ' for which $\pi'_{\Sigma'}$ is an isomorphism is representable by an open subspace. \square

4.1.4 Theorem. *Let $\mathcal{X} \rightarrow S$ be an algebraic stack with affine, finitely presented diagonal and let $G \rightarrow S$ be a group space acting on \mathcal{X} . Assume that $G \rightarrow S$ has a finite composition series whose factors are either reductive or proper, flat, finitely presented. Then the fixed point stack $\mathcal{X}^G \rightarrow S$ is algebraic, and the morphism $\mathcal{X}^G \rightarrow \mathcal{X}$ is representable by algebraic spaces, separated and locally of finite presentation. If $G \rightarrow S$ is reductive, this morphism is even representable by schemes.*

Proof : According to Corollaries A.3 and A.5, all group schemes having a finite composition series as indicated are pure. Therefore the assumption implies that there is an exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

of flat, finitely presented S -group schemes with $N \rightarrow S$ pure and $Q \rightarrow S$ either reductive or proper. By induction on the number of factors in a composition series, it is enough to prove that the map $\mathcal{X}^G \rightarrow \mathcal{X}^N$ is representable by algebraic spaces separated and locally of finite presentation. Let $\mathrm{St} := \mathrm{St}_{\mathcal{X}, G}$ be the universal stabilizer of G acting on \mathcal{X} . Let $x : T \rightarrow \mathcal{X}^N$ be a point from an S -scheme; this corresponds to a group-theoretic section $f_0 : N \rightarrow \mathrm{St}$ of $\mathrm{St} \times_G N \rightarrow N$. The functor $\mathcal{X}^G \times_{\mathcal{X}^N} T$ classifies the group-theoretic sections of $\mathrm{St} \rightarrow G$ that extend f_0 . By Lemma 4.1.3, this is representable by an algebraic space enjoying the announced properties. \square

Alper, Hall and Rydh proved this result in [AHR20], Theorem 5.16 when \mathcal{X} is a Deligne-Mumford stack locally of finite type over a field, and $G = \mathbb{G}_m$. They further showed that in this situation $\mathcal{X}^G \rightarrow \mathcal{X}$ is a monomorphism; this can be easily extended to the case where \mathcal{X} is a Deligne-Mumford stack, G is smooth with connected fibres, and the base scheme S is arbitrary. In the following example, we show that the Deligne-Mumford assumption is essential.

4.1.5 Example. *(An algebraic stack \mathcal{X} with finite inertia with action of a torus T such that $\mathcal{X}^T \rightarrow \mathcal{X}$ is not a monomorphism.)* Let S be a scheme of characteristic $p > 0$. Let $\mathcal{X} = B\alpha_p$ be the classifying stack of α_p . Consider the exact sequence of commutative S -group schemes:

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \longrightarrow 0.$$

The group $T := \mathbb{G}_m = \mathrm{Aut}(\mathbb{G}_a)$ acts on the first and second term naturally, and on the third term via Frobenius, that is $\lambda \cdot x := \lambda^p x$. In this way the sequence is an exact sequence of T -modules. There is an induced, T -equivariant exact sequence of Picard categories:

$$0 \longrightarrow \alpha_p(S) \longrightarrow \mathbb{G}_a(S) \longrightarrow \mathbb{G}_a(S) \longrightarrow (B\alpha_p)(S) \longrightarrow (B\mathbb{G}_a)(S) \longrightarrow (B\mathbb{G}_a)(S)$$

(see Giraud [Gi71], Chap. III, § 3, Prop. 3.2.1).

We claim that $\mathcal{X}^T \rightarrow \mathcal{X}$ is not a monomorphism. Indeed, if $S = \operatorname{Spec}(R)$ and $\lambda \in T(R) = R^\times$ then any α_p -torsor over S is of the form $P = \operatorname{Spec}(R[x]/(x^p - r))$ with α_p -action given by $a \cdot x = a + x$, for some $r \in R$. Moreover, the torsor $\lambda \cdot P$ is $\operatorname{Spec}(R[x]/(x^p - \lambda^p r))$ with action $a \cdot x = \lambda a + x$. Let us fix such a torsor $P \rightarrow S$, that is, a map $S \rightarrow \mathcal{X}$. The fibre product $\mathcal{X}^T \times_{\mathcal{X}} S$ is the functor of T -linearizations of P . This is identified with the functor of group-theoretic sections of the extension

$$1 \longrightarrow \alpha_p = \operatorname{Aut}(P) \longrightarrow E \longrightarrow \mathbb{G}_m \longrightarrow 1 \quad (1)$$

where $E = \{(\lambda, \varphi); \lambda \in T \text{ and } \varphi : \lambda \cdot P \rightarrow P \text{ an isomorphism}\}$. One computes that all the isomorphisms $\varphi : \lambda \cdot P \rightarrow P$ are described by a map of algebras $R[x]/(x^p - r) \rightarrow R[x]/(x^p - \lambda^p r)$, $x \mapsto u + \lambda^{-1}x$ for some $u \in \alpha_p(R)$. Thus we see that E is the group scheme whose points are pairs $(\lambda, u) \in T \times \alpha_p$ with law of multiplication

$$(\lambda, u) \cdot (\mu, v) = (\lambda\mu, u + \lambda^{-1}v).$$

The sections of the extension (1) are the maps $\mathbb{G}_m \rightarrow E$, $\lambda \mapsto (\lambda, u(\lambda))$ where $\lambda \mapsto u(\lambda)$ is a crossed homomorphism, that is $u(\lambda\mu) = u(\lambda) + \lambda^{-1}u(\mu)$. Those are all of the form $u(\lambda) = s(1 - \lambda^{-1})$ for some $s \in \alpha_p(R)$. In conclusion the functor of T -linearizations of P is isomorphic to α_p and is not trivial, proving that $\mathcal{X}^T \rightarrow \mathcal{X}$ is not a monomorphism. For more material on extensions with quotient of multiplicative type, we refer to Demazure and Gabriel [DG70], Chap. III, § 6, n° 6.

4.2 Geometric interpretation of group cohomology in degrees 1 and 2

In this section we consider a sheaf of groups G and a sheaf of G -modules A over S (that is, an abelian sheaf endowed with an additive action of G) and we give the interpretation of $H^1(G, A)$ and $H^2(G, A)$ in terms of equivariant torsors and gerbes. Since this basically amounts to reviewing the classical correspondences between geometric objects and cohomological classes and proving that they are G -equivariant, we sometimes omit some details.

We denote by A° the underlying abelian sheaf of A , devoid of G -action.

4.2.1 Definition. An A -torsor is an A° -torsor P endowed with an action of G such that the action morphism $A^\circ \times P \rightarrow P$ is G -equivariant.

4.2.2 Lemma. *There is a canonical bijection between the set of isomorphism classes of A -torsors over S and the cohomology group $H^1(G, A)$.*

In the proof below, starting from an A -torsor P we will provide a construction of a canonical extension $1 \rightarrow A \rightarrow E \rightarrow \mathbb{Z} \rightarrow 1$ such that P is the preimage of $E \rightarrow \mathbb{Z}$ at $1 \in \mathbb{Z}$. This is classical when $G = 1$ and the novelty here is to make sure that the procedure is G -equivariant. Our construction is different from those of [Ol16], 12.1.4 or [SP22, Tag 02FQ], having a more geometric flavour. Moreover, if A is a sheaf of possibly noncommutative groups, the construction works equally well by working with (A, A) -bitorsors instead of A -torsors, providing an extension of the result to that case.

Proof : For an A -torsor P , recall that the opposite A° -torsor P^{-1} is P with the opposite A° -action, that is the action given by $a \cdot p = a^{-1}p$. The action of G on P commutes with the opposite A° -action, turning P^{-1} into an A -torsor. Iterating the contracted product of torsors denoted by a wedge, we define the *contracted powers* of P as follows:

$$P^{\wedge 0} := A; \quad P^{\wedge n+1} := P^{\wedge n} \wedge P \text{ for all } n \geq 0; \quad P^{\wedge n-1} := P^{\wedge n} \wedge P^{-1} \text{ for all } n \leq 0.$$

When endowed with the diagonal action $g \cdot (p_1 \wedge \cdots \wedge p_n) = gp_1 \wedge \cdots \wedge gp_n$, where the p_i are local sections of P or P^{-1} , the A° -torsors $P^{\wedge n}$ become A -torsors for all $n \in \mathbb{Z}$. We define:

$$E := \coprod_{n \in \mathbb{Z}} P^{\wedge n}.$$

The contracted product induces maps $P^{\wedge m} \times P^{\wedge n} \rightarrow P^{\wedge m+n}$ (for all $m, n \in \mathbb{Z}$) which assembled together endow E with a group law such that the map $f : E \rightarrow \mathbb{Z}$ mapping $P^{\wedge n}$ into n is a G -equivariant group homomorphism. Of course $\ker(f) = A$ and $f^{-1}(1) = P$. The class of the extension

$$1 \longrightarrow A \longrightarrow E \longrightarrow \mathbb{Z} \longrightarrow 1$$

defines the element of $\text{Ext}_G^1(\mathbb{Z}, A) \simeq H^1(G, A)$ that completes the definition of the desired bijection. The fact that the map is indeed bijective is easy and left to the reader. \square

For a reminder of the definition of a gerbe banded by an abelian group, one can consult [Ol16], 12.2.2.

4.2.3 Definition. An A -gerbe is an A° -gerbe \mathcal{G} endowed with an action of G such that for all sections $x \in \mathcal{G}(T)$ over some S -scheme T , the given isomorphism $A_T^\circ \xrightarrow{\sim} \text{Aut}_T(x)$ is G_T -equivariant.

4.2.4 Lemma. *There is a canonical bijection between the set of isomorphism classes of A -gerbes over S and the cohomology group $H^2(G, A)$.*

Similarly as before, starting from an A -gerbe \mathcal{G} we construct a length two extension $1 \rightarrow A \rightarrow E \rightarrow F \rightarrow \mathbb{Z} \rightarrow 1$. It would be very interesting to produce a *canonical* extension. Since we do not know how (and fortunately we do not need) to do this, we merely adapt the proof of [Ol16], 12.2.8.

Proof : Let \mathcal{G} be an A -gerbe. Choose an injection $i : A \rightarrow I$ into an injective sheaf of G -modules and let $K = I/A$ be the quotient. Since I is injective, the gerbe $i_*\mathcal{G}$ is neutral (see [Ol16], 12.2.9) hence there exists a section $\eta : S \rightarrow \mathcal{G}$. Let \mathcal{P} be the K° -torsor of sections of \mathcal{G} that induce η , defined as in the proof of [Ol16], 12.2.8. Since the previous constructions are G -equivariant, the torsor \mathcal{P} acquires a G -action making it a K -torsor. Let $1 \rightarrow K \rightarrow E \rightarrow \mathbb{Z} \rightarrow 1$ be the extension attached to this torsor like in the proof of 4.2.2. We obtain a length 2 extension of G -modules

$$1 \longrightarrow A \longrightarrow I \longrightarrow E \longrightarrow \mathbb{Z} \longrightarrow 1$$

whose class in $\text{Ext}_G^2(\mathbb{Z}, A) \simeq H^2(G, A)$ defines the desired bijection. Again, the verification that the extension class does not depend on the choices of $i : A \rightarrow I$ and η , and that the resulting map is bijective, being classical, are left to the reader. \square

4.2.5 Definition. A G -Picard stack is a Picard stack \mathcal{P} over S endowed with a G -action such that the addition morphism $+: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ is G -equivariant. We denote by \mathcal{P}° the underlying Picard stack, devoid of G -action.

4.2.6 Definition. Let \mathcal{P} be a G -Picard stack. A \mathcal{P} -torsor is an S -stack \mathcal{Q} endowed with an action $\mu : \mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{Q}$ of \mathcal{P} such that $(\mu, \text{pr}_2) : \mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{Q} \times \mathcal{Q}$ is an isomorphism (that is, the action is free and locally transitive).

If \mathcal{P} is a G -Picard stack, the sheaf of isomorphism classes A and the sheaf of automorphisms of the neutral object $e \in \mathcal{P}(S)$ are sheaves of G -modules.

Endowed with the contracted product, the set of isomorphism classes of \mathcal{P}° -torsors is a group denoted $H^1(S, \mathcal{P}^\circ)$, see [Bre90], Prop. 6.2. Similarly we can define a group of isomorphism classes of \mathcal{P} -torsors which we denote $H^1(G, \mathcal{P})$. It is proved in [Bro21], Prop. 5.11 that $H^2(S, A^\circ) = H^1(S, BA^\circ)$ and this group classifies A° -gerbes or BA° -torsors (clearly $(BA)^\circ = B(A^\circ)$). As we did before, one can follow the constructions of the proof of *loc. cit.* and notice that they are G -equivariant, thereby enhancing the previous isomorphism to an isomorphism $H^2(G, A) = H^1(G, BA)$, both groups classifying A -gerbes or BA -torsors. Since we will not need this, we omit the details.

By assembling together torsors and gerbes, we can prove the following triviality result for torsors under certain Picard stacks which will be the key to the proof of Theorem 4.3.6 below.

4.2.7 Lemma. *Let \mathcal{P} be a G -Picard stack over S . Let P be the sheaf of isomorphism classes and A the sheaf of automorphisms of the neutral object $e \in \mathcal{P}(S)$. If $H^1(G, P) = H^2(G, A) = 0$ then $H^1(G, \mathcal{P}) = 0$, that is, all \mathcal{P} -torsors over S are trivial.*

Proof : Let $\mathcal{Q} \rightarrow S$ be a \mathcal{P} -torsor, so we have an isomorphism:

$$\mathcal{P} \times \mathcal{Q} \xrightarrow{\sim} \mathcal{Q} \times \mathcal{Q}.$$

Passing to sheaves of isomorphism classes, we obtain:

$$P \times Q \xrightarrow{\sim} Q \times Q,$$

that is $Q \rightarrow S$ is a P -torsor. Since $H^1(G, P) = 0$, by Lemma 4.2.2 this torsor has a section $q : S \rightarrow Q$. Let $\mathcal{G} = q^* \mathcal{Q}$, a gerbe over S . The isomorphism $\mathcal{P} \times \mathcal{Q} \xrightarrow{\sim} \mathcal{Q} \times \mathcal{Q}$ sends $(0, q)$ to (q, q) ; passing to inertia stacks in this isomorphism we obtain

$$BA \times \mathcal{G} \xrightarrow{\sim} \mathcal{G} \times \mathcal{G},$$

that is $\mathcal{G} \rightarrow S$ is a BA -torsor. This means that \mathcal{G} is an A -gerbe; let us provide the easy verifications of this. Let T be an S -scheme and $x \in \mathcal{G}(T)$ a section. We thus have an isomorphism:

$$f : A \times \text{Aut}(x) \xrightarrow{\sim} \text{Aut}(x) \times \text{Aut}(x), \quad f(a, u) = (u, a \cdot u).$$

By computing the images of $(ab, \text{id}_x) = (a, \text{id}_x)(b, \text{id}_x)$ in two different ways, one finds that the map $\iota_x : A \rightarrow \text{Aut}(x)$, $a \mapsto a \cdot \text{id}_x$ is a morphism of groups. By using that f is bijective as a sheaf map, we find that the same is true for ι_x . Finally, by using that f is G -equivariant we obtain the same conclusion for ι_x . The collection of isomorphisms $\{\iota_x\}$ shows that \mathcal{G} is an A -gerbe. Since $H^2(G, A) = 0$, by Lemma 4.2.4 this gerbe has a section $\alpha : S \rightarrow \mathcal{G}$. Using homogeneity we have $\mathcal{Q} \xrightarrow{\sim} Q \times \mathcal{G}$ which the section $(q, \alpha) : S \rightarrow Q \times \mathcal{G}$ trivializes and finally \mathcal{Q} is trivial. \square

4.3 Smoothness

In this subsection, we study the smoothness of fixed point stacks. For fixed point schemes, a useful reference is [SGA3.2], Exposé XII, § 9 (the reader should be careful however that in Prop. 9.2 of *loc. cit.* the assumption that X is separated over S is missing).

4.3.1 The equivariant cotangent complex. Let $\mathcal{X} \rightarrow S$ be a smooth algebraic stack. We want to recall the elementary description of the cotangent complex in this context; since we will have to handle stacks endowed with a group action, it is appropriate to work with sheaves on the equivariant site. The reader is assumed to be familiar with basics on equivariant quasi-coherent sheaves on schemes, like for instance in [AOV08], § 2.1.

Let $G \rightarrow S$ be a flat, locally finitely presented group algebraic space acting on $\mathcal{X} \rightarrow S$. The *equivariant lisse-étale site* is the site $\text{Lis-}\acute{\text{Et}}^G(\mathcal{X})$ whose underlying category is the category of smooth G -schemes $U \rightarrow \mathcal{X}$ (that is G -schemes $U \rightarrow S$ with a smooth equivariant morphism $U \rightarrow \mathcal{X}$), and whose covering families $\{U_i \rightarrow U\}_{i \in I}$ are families of étale G -equivariant morphisms such that $\coprod_{i \in I} U_i \rightarrow U$ is surjective. Particular objects of this site can be obtained as pullbacks $U := V \times_{\mathcal{X}/G} \mathcal{X}$ of objects $V \rightarrow \mathcal{X}/G$ of the ordinary, non-equivariant lisse-étale site of the quotient stack. In particular, we see that \mathcal{X} has equivariant smooth atlases.

We define the *equivariant cotangent complex* $\mathbb{L}_{\mathcal{X}/S}$ of (G, \mathcal{X}) as an object of the derived category of bounded complexes of G -quasi-coherent modules. Let $f : U \rightarrow \mathcal{X}$ be an object in $\text{Lis-}\acute{\text{Et}}^G(\mathcal{X})$. Choose a G -atlas $V \rightarrow \mathcal{X}$ and write $f' : U \times_{\mathcal{X}} V \rightarrow V$ the pullback of f . Then the sheaf $\Omega_{U/\mathcal{X}}^1|_{U \times_{\mathcal{X}} V} := \Omega_{U \times_{\mathcal{X}} V/V}^1$ descends along $U \times_{\mathcal{X}} V \rightarrow U$ to a G -quasi-coherent \mathcal{O}_U -module which we denote $\Omega_{U/\mathcal{X}}^1$. For each object $f : U \rightarrow \mathcal{X}$ in $\text{Lis-}\acute{\text{Et}}^G(\mathcal{X})$, we define a length two complex with sheaves placed in degrees 0 and 1:

$$\mathbb{L}_{\mathcal{X}/S|U} := \left[\Omega_{U/S}^1 \longrightarrow \Omega_{U/\mathcal{X}}^1 \right].$$

If $f : V \rightarrow U$ is a morphism in $\text{Lis-}\acute{\text{Et}}^G(\mathcal{X})$, there is a commutative diagram

$$\begin{array}{ccc} f^* \Omega_{U/S}^1 & \longrightarrow & \Omega_{V/S}^1 \\ \downarrow & & \downarrow \\ f^* \Omega_{U/\mathcal{X}}^1 & \longrightarrow & \Omega_{V/\mathcal{X}}^1 \end{array}$$

which induces a quasi-isomorphism

$$\theta_f : \left[f^* \Omega_{U/S}^1 \rightarrow f^* \Omega_{U/\mathcal{X}}^1 \right] \longrightarrow \left[\Omega_{V/S}^1 \rightarrow \Omega_{V/\mathcal{X}}^1 \right].$$

Moreover, for $W/V/U$ in $G\text{-Lis-}\acute{\text{Et}}(\mathcal{X})$ these quasi-isomorphisms satisfy the cocycle condition. The equivariant cotangent complex is the complex defined by the data $(\mathbb{L}_{\mathcal{X}/S|U}, \theta_f)$.

4.3.2 Deformations of sections of $\mathcal{X} \rightarrow S$. In what follows we work on both the small étale site $S_{\acute{\text{et}}}$ and the big fppf site S_{fppf} . We denote by $\varepsilon : S_{\text{fppf}} \rightarrow S_{\acute{\text{et}}}$ the canonical morphism. For basics on Picard stacks and their torsors, we refer to Deligne [SGA4.3], Exp. XVIII, § 1.4, Breen [Bre90], section 6 and Brochard [Bro21], sections 2 and 5.

Let I be a quasi-coherent \mathcal{O}_S -module. Let $\text{Thick}(S, I)$ be the category of *thickenings* of S by I , which by definition are pairs $(S \hookrightarrow S', u)$ composed of a closed immersion of schemes defined by a square-zero ideal, and an isomorphism $I \simeq \ker(\mathcal{O}_{S'} \rightarrow \mathcal{O}_S)$ which most often is omitted from the notation. There is a stack $\mathcal{Thick}(S, I)$ on $S_{\acute{\text{et}}}$, whose fibre category over $U \rightarrow S$ is $\text{Thick}(U, I|_U)$. (Here we are forced to work on the small étale site in order to guarantee existence of pullbacks: only for étale $T \rightarrow S$ does the thickening $S \hookrightarrow S'$ lift uniquely to a thickening $T \hookrightarrow T'$.) This is endowed with the structure of a Picard stack whose neutral object is the thickening $S \hookrightarrow S[I]$ where $S[I] = \text{Spec}(\mathcal{O}_S \oplus I)$ with $I^2 = 0$.

The tangent stack of \mathcal{X} relative to I is the stack $\mathcal{T}_{\mathcal{X}/S}(I) := \mathcal{H}om(S[I], \mathcal{X})$ on S_{fppf} whose points are the morphisms $S[I] \rightarrow \mathcal{X}$. It comes endowed with a morphism $\mathcal{T}_{\mathcal{X}/S}(I) \rightarrow \mathcal{X}$ induced by the

immersion $S \hookrightarrow S[I]$. If $x : S \rightarrow \mathcal{X}$ is a section, the pullback $x^* \mathcal{T}_{\mathcal{X}/S}(I)$ is the stack of morphisms extending x . Since $\mathcal{X} \rightarrow S$ is smooth, the usual computation shows that $x^* \mathcal{T}_{\mathcal{X}/S}(I)$ is canonically and equivariantly isomorphic to the stack associated as in Deligne's exposé [SGA4.3], Exp. XVIII, § 1.4 to the length two complex $\tau_{\leq 0} R\mathrm{Hom}(x^* \mathbb{L}_{\mathcal{X}/S}, I)$.

Let $\mathrm{Exal}_{\mathcal{X}}(S, I)$ be the category whose objects are the pairs composed of a thickening $S \hookrightarrow S'$ of S by I , and a morphism $x' : S' \rightarrow \mathcal{X}$ extending x (this includes the datum of a 2-isomorphism $u : x'|_S \simeq x$). There is a Picard stack $\mathcal{E}xal_{\mathcal{X}}(S, I)$ on $S_{\mathrm{\acute{e}t}}$, whose fibre category over $U \rightarrow S$ is $\mathrm{Exal}_{\mathcal{X}}(U, I|_U)$. Moreover, this sits in an exact sequence of Picard stacks:

$$0 \longrightarrow x^* \mathcal{T}_{\mathcal{X}/S}(I) \longrightarrow \mathcal{E}xal_{\mathcal{X}}(S, I) \longrightarrow \mathcal{T}hick(S, I) \longrightarrow 0. \quad (2)$$

Here, exactness on the right is guaranteed by the smoothness of $\mathcal{X} \rightarrow S$. It follows that the fibre of $\mathcal{E}xal_{\mathcal{X}}(S, I) \rightarrow \mathcal{T}hick(S, I)$ above a given thickening $S \hookrightarrow S'$ is a torsor under $x^* \mathcal{T}_{\mathcal{X}/S}(I)$.

4.3.3 Group action. Now assume that $x : S \rightarrow \mathcal{X}$ is fixed by G , which means that it is given with a collection of isomorphisms $\{\alpha_g : gx \simeq x\}_{g \in G}$. In this case G acts on $\mathcal{E}xal_{\mathcal{X}}(S, I)$ as follows:

$$g \cdot (x', u) = (g \circ x', \alpha_g \circ gu).$$

The group G also acts on $x^* \mathcal{T}_{\mathcal{X}/S}(I)$ by the same formula; it acts trivially on $\mathcal{T}hick(S, I)$ and the exact sequence (2) is equivariant.

From this, it is natural to approach Theorem 4.3.6 by descending the sequence (2) to an exact sequence on the small étale site of BG and proving smoothness via triviality of a certain $x^* \mathcal{T}_{\mathcal{X}/S}(I)$ -torsor on BG . This indeed works well; however, since proper foundations for Picard stacks over a stacky site such as $(BG)_{\mathrm{\acute{e}t}}$ are lacking, we prefer to work with equivariant objects using the material developed in Subsection 4.2.

4.3.4 Linearly reductive group schemes. We can now introduce linearly reductive group schemes and prove the statement of smoothness in Theorem 1.2.2. We use the notion of linear reductivity given by Alper [Alp13], Def. 12.1; see also Brion [Bri21]. For recent results concerning *affine* linearly reductive group schemes the reader may look at Alper, Hall and Rydh's article [AHR21], Section 19.

4.3.5 Definition. A flat, finitely presented, separated group scheme $G \rightarrow S$ is called *linearly reductive* if the functor $\mathrm{QCoh}^G(S) \rightarrow \mathrm{QCoh}(S)$, $\mathcal{F} \mapsto \mathcal{F}^G$ is exact.

Our interest for linearly reductive group schemes is that if S is affine, the higher Hochschild cohomology of quasi-coherent G - \mathcal{O}_S -modules vanishes, as follows from the definition:

$$H^i(G, \mathcal{F}) = 0 \text{ for all } \mathcal{F} \in \mathrm{QCoh}^G(S) \text{ and } i \geq 1.$$

The class of linearly reductive group schemes is stable by base change, faithfully flat descent ([Alp13], Prop. 12.8), subgroups with affine quotient (Matsushima's Theorem, [Alp13], Th. 12.15), and extensions ([Alp13], Prop. 12.17). It contains:

- (1) group schemes of multiplicative type, by [SGA3.2], Exp. IX, Th. 3.1;
- (2) finite locally free group schemes of order invertible on S , by the existence of explicit averaging operators;
- (3) abelian schemes, by [Alp13], Ex. 12.4;

(4) reductive group schemes, if S is a \mathbb{Q} -scheme, by the following results of [Alp14]: such a group scheme is geometrically reductive (Th. 9.7.5), hence $BG \rightarrow S$ is adequately affine (Def. 9.1.1) and cohomologically affine (Lem. 4.1.6) which is the definition of linearly reductive (Rem. 9.1.3).

4.3.6 Theorem. *Let S be a scheme and $\mathcal{X} \rightarrow S$ an algebraic stack with affine, finitely presented diagonal. Let $G \rightarrow S$ be a linearly reductive group scheme. If $\mathcal{X} \rightarrow S$ is smooth, the fixed point stack $\mathcal{X}^G \rightarrow S$ is smooth.*

Proof : By Theorem 4.1.4 the stack $\mathcal{X}^G \rightarrow S$ is algebraic and locally of finite presentation, hence it is enough to prove that it is formally smooth. Let $x : T \rightarrow \mathcal{X}^G$ be a point with values in some S -scheme T which is affine. After base-changing \mathcal{X} along $T \rightarrow S$ and renaming, we can assume that $T = S$ in what follows. We have the sequence of Picard stacks over S :

$$0 \longrightarrow x^* \mathcal{T}_{\mathcal{X}/S}(I) \longrightarrow \mathcal{E}xal_{\mathcal{X}}(S, I) \xrightarrow{\pi} \mathcal{T}hick(S, I) \longrightarrow 0.$$

Since $\mathcal{X} \rightarrow S$ is smooth, this sequence is exact. Moreover, as explained before, these stacks are naturally endowed with G -actions (the G -action on the stack of thickenings is trivial) and the sequence is equivariant. Let $\iota : S \hookrightarrow S'$ be a thickening; we have to prove that x has a lifting $x' : S' \rightarrow \mathcal{X}^G$. The category of liftings of x along ι is the fibre category $\pi^{-1}(\iota) \subset \mathcal{E}xal_{\mathcal{X}}(S, I)$. This is a torsor under the G -Picard stack $x^* \mathcal{T}_{\mathcal{X}/S}(I)$, whose sheaves of isomorphism classes P and neutral automorphisms A are quasi-coherent. Because $G \rightarrow S$ is linearly reductive, we have $H^1(G, P) = H^2(G, A) = 0$. By lemma 4.2.7, this implies that this torsor is trivial. In other words, it has a section $x' : S \rightarrow \pi^{-1}(\iota)$ which gives the desired lifting for x . \square

4.4 Applications

We conclude by giving several applications of Theorem 4.3.6. The first application is to the flatness or smoothness properties of the space of homomorphisms $\text{Hom}(G, H)$. It relies on the following well-known fact, a proof of which we provide for the convenience of the reader.

4.4.1 Lemma. *Let G, H be sheaves of groups over a base scheme S (for some topology). Then there is an isomorphism of stacks*

$$[\text{Hom}(G, H)/H] \xrightarrow{\sim} \mathcal{H}om(BG, BH)$$

where the quotient is taken for the action of H on $\text{Hom}(G, H)$ by conjugation on the target.

Proof : We define maps in both directions. A section of the stack $[\text{Hom}(G, H)/H]$ is a pair composed of an H -torsor $S' \rightarrow S$ and an H -equivariant map $f : S' \rightarrow \text{Hom}(G, H)$. With these data we define a map $\Phi : BG \rightarrow BH$ as follows. Let $\varphi : G_{S'} \rightarrow H_{S'}$ be the group homomorphism determined by f . To a G -torsor E we attach the $H_{S'}$ -torsor $F_{S'} := E_{S'} \wedge^{\varphi} H_{S'}$. To say that f is equivariant is to say that for all local sections $h \in H$, the pullback of φ along $h : S' \rightarrow S'$ is equal to $h^* \varphi = c_h \circ \varphi$ where $c_h : H \rightarrow H$ is conjugation. Therefore, for all local sections $h \in H$ we have, canonically:

$$h^* F_{S'} = E_{S'} \wedge^{c_h \varphi} H_{S'}.$$

This implies that the isomorphism of $H_{S'}$ -torsors

$$(\text{id}_{E_{S'}}, c_h) : E_{S'} \wedge^{\varphi} H_{S'} \longrightarrow E_{S'} \wedge^{c_h \varphi} H_{S'}$$

is an isomorphism $F_{S'} \xrightarrow{\sim} h^*F_{S'}$. This gives descent data for $F_{S'}$ with respect to $S' \rightarrow S$, and we call $\Phi(E)$ the descent $F \rightarrow S$. Conversely let $\Phi : BG \rightarrow BH$ be a morphism of stacks and let $S' \in BH$ be the image of the trivial torsor G by Φ . After the pullback $S' \rightarrow S$ the torsor F becomes trivial and the map $\text{Aut}_S(G) \rightarrow \text{Aut}_S(\Phi(G))$ becomes a morphism of groups:

$$\varphi' : G_{S'} = \text{Aut}_{S'}(G_{S'}) \longrightarrow \text{Aut}_{S'}(F_{S'}) = \text{Aut}_{S'}(H_{S'}) = H_{S'}.$$

This amounts to a morphism $u : S' \rightarrow \text{Hom}(G, H)$. For each local section $h \in H$, we have S' -group schemes $a : G_{S'} \rightarrow S'$ and $b : H_{S'} \rightarrow S'$ and the pullbacks $h^*G_{S'}$, $h^*H_{S'}$ are isomorphic to $G_{S'}$, $H_{S'}$ with structure maps $h^{-1} \circ a$ and $h^{-1} \circ b$. This shows that $h^*\varphi'$ is $c_h \circ \varphi'$, hence u is H -equivariant. It provides a point of the quotient $[\text{Hom}(G, H)/H]$. The two maps so described are inverse to each other. \square

4.4.2 Corollary. *Let $G \rightarrow S$ be a linearly reductive S -group scheme. Let H be a flat, affine, finitely presented S -group scheme. Then the stack $\mathcal{H}om(BG, BH) \rightarrow S$ is algebraic and smooth. In particular,*

- (i) $\text{Hom}(G, H) \rightarrow S$ is flat and locally complete intersection,
- (ii) $\text{Hom}(G, H) \rightarrow S$ is smooth if moreover $H \rightarrow S$ is smooth.

Proof : The stack $\mathcal{X} = BH$ has affine, finitely presented diagonal and it is smooth because its natural atlas $S \rightarrow BH$ has smooth source (see [SP22, Tag 0DLS]). Letting G act trivially on it, we obtain $\mathcal{X}^G = \mathcal{H}om(S, BH)^G = \mathcal{H}om(BG, BH)$ which is smooth by Theorem 4.3.6. The group scheme $H \rightarrow S$ is locally complete intersection in case (i) and smooth in case (ii); since $\text{Hom}(G, H) \rightarrow \mathcal{H}om(BG, BH)$ is an H -torsor by Lemma 4.4.1, the announced properties are deduced. \square

The reader can find other examples of $\mathcal{H}om$ stacks in 4.5.2(1) below.

4.4.3 Remark. (*Smoothness in case (ii) by deformation theory.*) Because $\text{Hom}(G, H) \rightarrow S$ is locally of finite presentation, in order to prove that it is smooth when H is smooth, it is enough to verify the infinitesimal lifting criterion. Given a nilimmersion of affine schemes $T \hookrightarrow T'$ over S , and a T -morphism $f : G_T \rightarrow H_T$, we seek to lift it to a T' -morphism $f' : G_{T'} \rightarrow H_{T'}$. Changing notation, we can assume that $T = S$, $T' = S'$ which are affine schemes. Let $I = \ker(\mathcal{O}_{S'} \rightarrow \mathcal{O}_S)$ be the square-zero kernel. By Illusie [Ill72], Chap. VII, Th. 3.3.1 the obstruction to lifting f lives in $\mathbb{H}^2(BG/S, f^*\underline{\mathcal{L}}_H^\vee \otimes I)$ where $\underline{\mathcal{L}}_H \in D(BH)$ is the equivariant co-Lie complex of $H \rightarrow S$; here the operations on complexes f^* , $(-)^*$, \otimes are understood in the derived sense. Therefore it is enough to prove that this cohomology group is zero. Let us write once for all

$$K := f^*\underline{\mathcal{L}}_H^\vee \otimes I.$$

Note that $\mathbb{H}^i(BG/S, K)$ denotes *relative* cohomology with respect to the map $S \rightarrow BG$, as in [Ill71], Chap. III, § 4. This is related to ordinary cohomology $\mathbb{H}^i(BG, K)$ via a long exact sequence of which we write the part which is useful for our calculation:

$$\dots \longrightarrow \mathbb{H}^1(S, K) \longrightarrow \mathbb{H}^2(BG/S, K) \longrightarrow \mathbb{H}^2(BG, K) \longrightarrow \mathbb{H}^2(S, K) \longrightarrow \dots$$

We claim that $\mathbb{H}^2(BG, K) = 0$. To compute this group, we use the second hypercohomology spectral sequence:

$$\mathbb{H}_{E_2}^{i,j} = H^i(BG, \mathcal{H}^j(K)) \implies \mathbb{H}^{i+j}(BG, K).$$

We know that quasi-coherent cohomology on BG coincides with group cohomology of G . Hence, by the assumption on G we have $H^i(BG, \mathcal{F}) = H^i(G, \mathcal{F}) = 0$ for $i \geq 1$ for all quasi-coherent sheaves \mathcal{F} on BG . It follows that the spectral sequence collapses at E_2 , giving isomorphisms

$$\mathbb{H}^n(BG, K) \simeq H^0(BG, \mathcal{H}^n(K)) = \Gamma^G(\mathcal{H}^n(K))$$

for all n , where $\Gamma^G(-)$ denotes G -invariant global sections. We claim that the sheaf $\mathcal{H}^n(K)$ vanishes for all $n \geq 2$. The complex $f^* \underline{\mathcal{L}}_H^\vee$ has perfect amplitude in $[0, 1]$ ([Il72], Chap. VII, § 3.1); thus replacing I by a flat resolution $\cdots \rightarrow I_{-2} \rightarrow I_{-1} \rightarrow I_0$ and computing the total complex of $f^* \underline{\mathcal{L}}_H^\vee \otimes I$ we find that it has no cohomology in degrees ≥ 2 , as claimed.

To compute $\mathbb{H}^1(S, K)$ we can proceed similarly. Since S is affine, the quasi-coherent sheaves $\mathcal{H}^j(K)$ have no higher cohomology. It follows that the hypercohomology spectral sequence degenerates at E_2 , giving isomorphisms $\mathbb{H}^n(S, K) \simeq H^0(S, \mathcal{H}^n(K))$. In case 4.4.2(i) the complex $f^* \underline{\mathcal{L}}_H^\vee$ is locally represented by a two-term complex of quasi-coherent sheaves $[F_0 \rightarrow F_1]$; then the total complex has a term $F_1 \otimes I_0$ in degree 1 and we can not conclude to the vanishing of $\mathbb{H}^1(S, K)$. In case 4.4.2(ii) however, the complex $\underline{\mathcal{L}}_H$ is quasi-isomorphic to the sheaf of invariant differentials $\omega_{H/S}^1$ viewed as a complex concentrated in degree 0 and $\mathbb{H}^1(S, K)$ vanishes. In this case $\mathbb{H}^2(BG/S, K) = 0$ and we are done.

Should one want to study potential cases of smoothness of $\text{Hom}(G, H)$ in case (i), a natural approach would be to undertake a more detailed analysis of $\mathbb{H}^1(S, K)$.

4.4.4 Examples. (*Non-smooth examples.*) Here are various examples of schemes $\text{Hom}(G, H)$; the cases in (2) and (3) were communicated by Michel Brion and Angelo Vistoli.

(1) When H is not flat the scheme $\text{Hom}(G, H)$ can exhibit all kind of behaviour: it can be smooth, or on the contrary not even flat. To give examples, let R be a discrete valuation ring with residue field k . If M is a k -group scheme with affine identity component, we let M^\natural be the scheme obtained by gluing the trivial R -group scheme $\{1\}_R$ with M along the unit section $\{1\}_k$ of the special fibre. This is an R -group scheme with affine identity component, which is non-flat when $M \neq \{1\}_k$. Then, one has:

- (i) $\text{Hom}(\mathbb{G}_{m,R}, (\mathbb{G}_{a,k})^\natural) \hookrightarrow \text{Hom}(\mathbb{G}_{m,R}, \mathbb{G}_{a,R}) = 1$ hence $\text{Hom}(\mathbb{G}_{m,R}, (\mathbb{G}_{a,k})^\natural) = 1$ which is smooth;
- (ii) $\text{Hom}(\mathbb{G}_{m,R}, (\mathbb{G}_{m,k})^\natural) = (\mathbb{Z}_k)^\natural$ by a direct computation; this is not flat.

(2) Let $S = \text{Spec}(k)$ where k is a separably closed, non algebraically closed field of characteristic p . Take $G = \mathbb{G}_m$ and H a nontrivial extension of \mathbb{G}_m by α_p ; such extensions are described in [SGA3.2], Exp. XVII, Exemple 5.9.c). If $\text{Hom}(G, H)$ is representable by a smooth scheme T , then $T(k)$ is a point because each homomorphism $G \rightarrow H$ is trivial (for otherwise its image would split the extension). Since the extension is split on the algebraic closure \bar{k} , we have $T(\bar{k}) = \text{Hom}(\mathbb{G}_{m,\bar{k}}, \mathbb{G}_{m,\bar{k}}) = \mathbb{Z}$. But since T is smooth $T(k)$ is dense in T , a contradiction.

(3) Here is an example where G and H are both finite and linearly reductive, and the field k is algebraically closed of characteristic p . Suppose that Δ is a finite connected diagonalizable k -group scheme, H is a nontrivial group of automorphisms of Δ , and consider the semidirect product $G := \Delta \rtimes H$. Then $\text{Hom}(G, G)$ is representable and finite, and contains $\text{Aut}(G)$ as an open subscheme. It is shown in [AOV08], Lemma 2.19 that the connected component of the identity in $\text{Aut}(G)$ is Δ/Δ^H , which is not reduced, so $\text{Hom}(G, G)$ can not be smooth.

We now give an application to functors of subgroups.

4.4.5 Corollary. *Let $H \rightarrow S$ be an affine, flat group scheme of finite presentation. Then:*

- (1) *The functor of subgroups of multiplicative type $\text{Sub}_{\text{mult}}(H)$ is flat and locally complete intersection over S , and it is smooth if H is.*
- (2) *If S is of characteristic 0 then $\text{Sub}_{\text{red}}(H)$ is smooth over S .*

Proof : (1) Recall that $\text{Sub}_{\text{mult}}(H) = \coprod_{\mathbf{t}} \text{Sub}_{\mathbf{t}}(H)$ is a sum indexed by the types $\mathbf{t} = [(M, M^*, \emptyset, \emptyset)]$. Since $G(\mathbf{t})$ is of multiplicative type, it is linearly reductive hence $\text{Hom}(G(\mathbf{t}), H)$ is flat and locally complete intersection (resp. smooth if H is) by 4.4.2. Since $\text{Mono}(G(\mathbf{t}), H)$ is open in the latter by Lemma 3.1.1, it is flat and locally complete intersection also. Finally, remember that in the proof of Theorem 3.2.1 we expressed $\text{Sub}_{\mathbf{t}}(H)$ as the quotient of $\text{Mono}(G(\mathbf{t}), H)$ by the smooth group scheme $\text{Aut}(G(\mathbf{t}))$ acting freely; the result follows.

(2) If S is of characteristic 0, for an arbitrary type \mathbf{t} the reductive group schemes $G(\mathbf{t})$ are linearly reductive and the flat group scheme H is smooth. Therefore the same arguments apply. \square

The final application extends 4.4.2 as well as [Ro05], Cor. 3.11.

4.4.6 Corollary. (Stacks of equivariant objects.) *Let $\mathcal{X} \rightarrow S$ be an algebraic stack with affine, finitely presented diagonal. Let $G \rightarrow S$ be a group space admitting a finite composition series whose factors are either reductive or proper, flat, finitely presented.*

- (1) *The stack $\mathcal{X}[G]$ of pairs (x, α) comprising an object of \mathcal{X} and an action $\alpha : G \rightarrow \text{Aut}(x)$ is algebraic. Moreover $\mathcal{X}^G \rightarrow \mathcal{X}$ is representable by algebraic spaces, separated and locally of finite presentation.*
- (2) *The substack $\mathcal{X}\{G\}$ composed of pairs such that the action α is faithful is open.*
- (3) *If $\mathcal{X} \rightarrow S$ is smooth and G is linearly reductive, then $\mathcal{X}[G] \rightarrow S$ and $\mathcal{X}\{G\} \rightarrow S$ are smooth.*

Proof : Letting G act trivially on \mathcal{X} , we see that the fixed point stack is exactly $\mathcal{X}[G]$. By Theorem 4.1.4, this is algebraic and the substack $\mathcal{X}\{G\}$ is open. Finally the smoothness in (3) follows from Theorem 4.3.6. \square

4.5 Failure of transitivity (erratum to [Ro05])

In [Ro05], Rem. 2.4 it is asserted that if $N \subset G$ is a normal subgroup scheme, then:

- (i) the group G/N acts on \mathcal{X}^N and we have an isomorphism of stacks $\mathcal{X}^G \xrightarrow{\sim} (\mathcal{X}^N)^{G/N}$;
- (ii) the group G/N acts on \mathcal{X}/N and we have an isomorphism of stacks $(\mathcal{X}/N)/(G/N) \xrightarrow{\sim} \mathcal{X}/G$.

In this subsection we wish to correct this statement: in fact, surprisingly point (i) is *incorrect* (Lemma 4.5.1) while point (ii) is *correct* (Proposition 4.5.3). To the author's knowledge the erroneous statement (i) is not used anywhere, but (ii) is used in the papers [LMM14], [Sch17], [Sch18], [AI19].

To understand what happens, recall that an action $\mu : G \times \mathcal{X} \rightarrow \mathcal{X}$ is called *strictly trivial* if μ is equal to the second projection, which we write $\mu = \text{triv}$. The action is called *weakly trivial* if there exists a G -isomorphism $(f, \sigma) : (\mathcal{X}, \mu) \xrightarrow{\sim} (\mathcal{X}, \text{triv})$ with $f = \text{id}_{\mathcal{X}}$; here σ_g^x is an isomorphism $g.f(x) \xrightarrow{\sim} f(g.x)$, as in [Ro05], Def. 2.1. By the 2-universal properties of quotients and fixed points, the stacks \mathcal{X}^N and \mathcal{X}/N come equipped with actions of G , the restriction to N of which are *weakly trivial*. However, in order to induce an action of G/N one needs to find G -equivariant models of \mathcal{X}^N and \mathcal{X}/N on which the action of N is *strictly trivial*, and it is not clear if this is possible at all.

We first provide a counterexample to (i). The example highlights the fact that G -fixed point stacks retain information on the extension structure of G which is not captured by $(\mathcal{X}^N)^{G/N}$.

4.5.1 Lemma. *Let G, N, A be S -group schemes of multiplicative type with $G \rightarrow S$ fibrewise connected and $N \subset G$ a normal subgroup. Let $\mathcal{X} = BA$ be the classifying stack of A , endowed with the trivial action of G . Then:*

- (1) *We have an isomorphism of stacks $\mathcal{X}^N = BA \times \mathrm{Hom}(N, A)$ such that the canonical map $\mathcal{X}^N \rightarrow \mathcal{X}$ is the first projection.*
- (2) *Each action of G on \mathcal{X}^N making $\mathcal{X}^N \rightarrow \mathcal{X}$ equivariant is isomorphic to the trivial action.*
- (3) *Letting G/N act trivially on \mathcal{X}^N , we have an isomorphism of stacks*

$$(\mathcal{X}^N)^{G/N} = BA \times \mathrm{Hom}(G/N, A) \times \mathrm{Hom}(N, A)$$

and the canonical map $\mathcal{X}^G \rightarrow (\mathcal{X}^N)^{G/N}$ is given by $(E, \alpha) \mapsto (E, 0, \alpha|_N)$. In particular, this is not an isomorphism if $\mathrm{Hom}(G/N, A) \neq 0$.

Proof : (1) Since N acts trivially, a section of $(BA)^N$ over S is a pair $(E, \{\alpha_n\}_{n \in N(S)})$ composed of an A -torsor $E \rightarrow S$ and a collection of isomorphisms $\alpha_n : E \rightarrow E$ satisfying the cocycle condition $\alpha_{nm} = \alpha_n \circ \alpha_m$ for all $n, m \in N(S)$. Since $\mathrm{Aut}(E) = A$, this boils down to a pair (E, α) where $\alpha : N \rightarrow A$ is a morphism of groups.

(2) Write $f : \mathcal{X}^N \rightarrow \mathcal{X}$ for the map $(E, \alpha) \mapsto E$. Assume given a G -action (\mathcal{X}^N, μ) such that f extends to a G -equivariant morphism $(f, \sigma) : \mathcal{X}^N \rightarrow \mathcal{X}$. For each S -scheme T and points $(E, \alpha) \in \mathcal{X}^N(T)$, $g \in G(T)$ write the image $g \cdot (E, \alpha)$ as $(\mathfrak{E}(g, E, \alpha), \mathfrak{a}(g, E, \alpha))$ where $\mathfrak{E}(g, E, \alpha)$ is an A -torsor and $\mathfrak{a}(g, E, \alpha)$ is an N -linearization. Now let (E, α) and g be fixed and write $\mathfrak{E} = \mathfrak{E}(g, E, \alpha)$, $\mathfrak{a} = \mathfrak{a}(g, E, \alpha)$ for brevity. We have an isomorphism $\sigma := \sigma_g^{E, \alpha} : E \rightarrow \mathfrak{E}$. Define α' by $\alpha'(n) : E \rightarrow E$, $\alpha'(n) = \sigma^{-1} \circ \mathfrak{a}(n) \circ \sigma$. By setting $g \star (E, \alpha) = (E, \alpha')$ we define a new G -stack (\mathcal{X}^N, μ') for which f is strictly (and not just weakly) invariant, together with a G -isomorphism $(\mathcal{X}^N, \mu') \xrightarrow{\sim} (\mathcal{X}^N, \mu)$ provided by the $\sigma_g^{E, \alpha}$. Moreover the dependence in g for $\alpha' = \alpha'(g, E, \alpha)$ is morphic, that is $g \mapsto \alpha'(g, E, \alpha)$ is an action of G on $\mathrm{Hom}(N, A)$. Since G is connected and $\mathrm{Hom}(N, A)$ is étale, this action is trivial.

(3) Reasoning as in (1) we find that $(\mathcal{X}^N)^{G/N} = BA \times \mathrm{Hom}(G/N, A) \times \mathrm{Hom}(N, A)$. The expression for the map $\mathcal{X}^G \rightarrow (\mathcal{X}^N)^{G/N}$ is dictated by the universal properties. \square

4.5.2 Examples. Here are examples where the map $\mathcal{X}^G \rightarrow (\mathcal{X}^N)^{G/N}$ is an isomorphism.

(1) (*Hom stacks*) Assume that $\mathcal{X} = \mathcal{H}om(Y, Z)$ where Y, Z are algebraic stacks, and we are given an action of G on Y , inducing an action on \mathcal{X} . By taking for Z the stack of vector bundles (or coherent modules, or curves, etc) we obtain for \mathcal{X}^G the stack of equivariant bundles (or coherent modules, etc) on Y . We claim that if $N \subset G$ is a normal subgroup then the map $\mathcal{X}^G \rightarrow (\mathcal{X}^N)^{G/N}$ is an isomorphism. Indeed, it follows from the 2-universal property of quotient stacks and Proposition 4.5.3 below that we have canonical isomorphisms:

$$\mathcal{X}^G = \mathcal{H}om(Y, Z)^G \simeq \mathcal{H}om(Y/G, Z) \simeq \mathcal{H}om((Y/N)/(G/N), Z) \simeq \mathcal{H}om(Y/N, Z)^{G/N} \simeq (\mathcal{X}^N)^{G/N}.$$

(2) (*Product groups*) Assume that $G = H \times K$ is a product group acting on \mathcal{X} . Identify G/H with K . Then the canonical map $\mathcal{X}^G \xrightarrow{\sim} (\mathcal{X}^H)^K$ is an isomorphism. The objects of the stacks on both sides can be identified with collections $(x, \{\alpha_h\}_{h \in H}, \{\beta_k\}_{k \in K})$ where the isomorphisms $\alpha_h : x \rightarrow h^{-1}x$ and $\beta_k : x \rightarrow k^{-1}x$ commute with each other.

In view of this, point (ii) may now seem surprising. We now give the proof. The main idea is to strictify the action by systematically embedding an N -torsor E into the induced G -torsor $\mathrm{Ind}_N^G(E)$.

4.5.3 Proposition. *Let $G \rightarrow S$ be a group scheme and $N \subset G$ a normal subgroup scheme, both flat and locally of finite presentation. Let $\mathcal{X} \rightarrow S$ be an algebraic stack with an action of G . Then there is an action of G/N on the quotient stack \mathcal{X}/N such that the morphism $\mathcal{X}/N \rightarrow \mathcal{X}/G$ is invariant and induces an isomorphism $(\mathcal{X}/N)/(G/N) \xrightarrow{\sim} \mathcal{X}/G$.*

Note that the algebraicity statement [Ro05], Th. 4.1 assumes too strong assumptions on G and N (namely they are required to be separated and of finite presentation) than is necessary for the proof of *loc. cit.* to go through.

Proof : Our main task is to find a G -equivariant model $\mathcal{X}/N \simeq Y$ such that the N -action on Y is strictly trivial, so that there is an induced action of G/N . To this aim, recall that the points of \mathcal{X}/N with values in an S -scheme T are the pairs composed of an N -torsor $E \rightarrow T$ and an N -equivariant map $a : E \rightarrow \mathcal{X}$. Let Y be the S -stack whose T -points are the triples $(F \rightarrow T, E' \subset F, b : F \rightarrow \mathcal{X})$ with

- a G -torsor $F \rightarrow T$;
- a subspace $E' \subset F$ which is N -stable and an N -torsor over S ;
- a G -equivariant map $b : F \rightarrow \mathcal{X}$.

There is a morphism $\lambda : \mathcal{X}/N \rightarrow Y$ which sends $(E \rightarrow T, E \rightarrow \mathcal{X})$ to the triple given by

- $F = \text{Ind}_N^G(E) = G \times^N E$ is the induction of the N -torsor E to G , that is $F = (G \times E)/N$ where N acts by $n(g, e) = (gn^{-1}, ne)$;
- E' is the image of the monomorphism $E \rightarrow F$, $e \mapsto (1, e)$; this may alternatively be seen as the preimage of 1 under the map $F \rightarrow G/N$, $(g, e) \mapsto (g \bmod N)$;
- $b : F \rightarrow \mathcal{X}$ is induced by the map $G \times E \rightarrow \mathcal{X}$, $(g, e) \mapsto g \cdot a(e)$.

There is a morphism $\mu : Y \rightarrow \mathcal{X}/N$ that sends the triple $(F \rightarrow T, E' \subset F, b : F \rightarrow \mathcal{X})$ to $(E' \rightarrow T, a = b|_{E'} : E' \rightarrow \mathcal{X})$. We see that $\mu \circ \lambda \simeq \text{id}_{\mathcal{X}/N}$ by applying the definitions of the objects. We can see also that $\lambda \circ \mu \simeq \text{id}_Y$ because if $F \rightarrow T$ is a G -torsor and $E' \subset F$ is N -stable and an N -torsor over S , then the morphism $G \times^N E' \rightarrow F$ induced by the inclusion $E' \subset F$ is a morphism of G -torsors, hence automatically an isomorphism. Hence $\mathcal{X}/N \simeq Y$. Henceforth we write $(\mathcal{X}/N)^{\text{str}} = Y$.

The stack $(\mathcal{X}/N)^{\text{str}}$ is endowed with a natural G -action by $g \cdot (F, E', b) = (F, g(E'), b)$ where $g(E') \subset F$ is the image of E' by $g : F \rightarrow F$. As a consequence of the fact that N is normal in G , the subspace $g(E') \subset F$ is N -stable, which ensures that it is an N -torsor and $g \cdot (F, E', b)$ is a well-defined point of $(\mathcal{X}/N)^{\text{str}}$. We now prove that $\mu : (\mathcal{X}/N)^{\text{str}} \rightarrow \mathcal{X}/N$ is G -equivariant. For this, recall that the G -action on \mathcal{X}/N is described by $g \cdot (E, a) = ({}^g E, g \circ a)$ where ${}^g E$ is the N -torsor with underlying space E and action defined by $h \star e := g^{-1} h g e$. Now pick sections $(F, E', b) \in (\mathcal{X}/N)^{\text{str}}(T)$ and $g \in G(T)$. From the equivariance property of b we see that g provides an isomorphism of N -torsors:

$$\begin{array}{ccc} {}^g E' & \xrightarrow{g} & g(E') \\ & \searrow \scriptstyle g \circ b & \swarrow \scriptstyle b \\ & \mathcal{X} & \end{array}$$

This shows that μ is G -equivariant. Of course, the restricted action of N on $(\mathcal{X}/N)^{\text{str}}$ is strictly trivial, hence there is an action of G/N . We leave it to the reader to verify that the map $(\mathcal{X}/N)^{\text{str}} \rightarrow \mathcal{X}/N \rightarrow \mathcal{X}/G$ induces an isomorphism $(\mathcal{X}/N)^{\text{str}}/(G/N) \xrightarrow{\sim} \mathcal{X}/G$. \square

A Some results around purity

We shall make use of the results of [SGA3.2], Exp. VIII, § 6 on the representability of Weil restriction of closed subschemes and its consequences for fixed points, transporters, normalizers, etc. It is convenient to state a version of these results that is flexible enough to cover our needs, namely Theorem A.6 and its corollaries. This is best achieved using the notion of *pure morphism*. This is defined in [RG71] for schemes and extends without difficulty to algebraic spaces (or even stacks but we have no need for this); see for instance [Ro11], Appendix B. We recall the definition and the main facts we shall use.

Most of this material is also covered in [SGA3.1], Exp. VI_B, § 6 and [SGA3.2], Exp. XII, § 9.

A.1 Definition. A morphism of schemes or algebraic spaces $X \rightarrow S$ locally of finite type is called *pure* if for each point $s \in S$ with henselization $(\tilde{S}, \tilde{s}) \rightarrow (S, s)$, and each point $\tilde{x} \in \tilde{X} := X \times_S \tilde{S}$ which is an associated point in its fibre, the closure of \tilde{x} in \tilde{X} meets the special fibre $X \otimes k(\tilde{s})$.

For example, if $X \rightarrow S$ is proper then it is pure, because the image in \tilde{S} of the closure of \tilde{x} is closed and nonempty, hence contains \tilde{s} . Another important example is given in [RG71], Première partie, Ex. (3.3.4)(iii). For the convenience of the reader, we provide this example with detailed explanation.

A.2 Lemma. *Let $X \rightarrow S$ be a morphism which is flat, locally of finite presentation, with geometrically irreducible fibres without embedded components. Then X is S -pure.*

Note that irreducible implies nonempty by definition.

Proof : By definition, replacing S by its henselization at an arbitrary point s , we may assume that S is local henselian and we have to prove that the closure of a point $x' \in X$ which is associated in its fibre $X_{s'}$, $s' = f(x')$, meets the special fibre X_s . Let Z be the closure of s' in S . Since S is local, Z meets s and hence we may replace S by Z and assume that S is irreducible with generic point s' . Since $X \rightarrow S$ is open with irreducible fibres, it follows that X is irreducible, see [SP22, Tag 004Z]. Now the fibre $X_{s'}$ is irreducible without embedded component, hence the assassin $\text{Ass}(X_{s'})$ is a single point, that is $\text{Ass}(X_{s'}) = \{x'\}$. This means that x' is the generic point of the generic fibre, hence the generic point of X . It follows that its closure is equal to X , and meets the special fibre. \square

A.3 Corollary. *Let $G \rightarrow S$ be a group algebraic space which is flat, locally of finite presentation, with connected fibres. Then G is S -pure.*

Proof : A connected group space over a field is a scheme and is geometrically irreducible ([SGA3.1], Exp. VI_A, Thm. 2.6.5). Moreover any group scheme over a field is locally complete intersection ([SGA3.1], Exp. VII_B, Cor. 5.5.1) hence without embedded points. It follows from the above lemma that $G \rightarrow S$ is pure. \square

A.4 Lemma. *Let $X \rightarrow Y \rightarrow S$ be morphisms locally of finite type. Assume that $X \rightarrow Y$ is flat and pure, and $Y \rightarrow S$ is pure. Then the composition $X \rightarrow S$ is pure.*

Proof : We may assume that S is henselian with closed point s_0 . Let $x \in X$ be a point and $y \in Y$ its image. Assume that $x \in \text{Ass}(X/S)$. By flatness of $X \rightarrow S$, this means that $x \in \text{Ass}(X/Y)$ and $y \in \text{Ass}(Y/S)$, see [RG71] (3.2.4). By purity of $Y \rightarrow S$, the closure of y in Y meets the special fibre in a point y_0 . Then the henselization $(\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$ hits the point y ; let $\tilde{y} \in \tilde{Y}$ be a lift of y .

Let $\tilde{x} \in \tilde{X} := X \times_Y \tilde{Y}$ be a lift of x . By invariance of the assassin under étale localization ([RG71] Lemma 3.4.5) and stability of purity by base change ([RG71] Corollaire 3.3.7), upon replacing $\{Y, y, X, x\}$ by $\{\tilde{Y}, \tilde{y}, \tilde{X}, \tilde{x}\}$ we may assume that Y is henselian. Since $X \rightarrow Y$ is pure, the closure of x in X meets the special fibre X_{y_0} . Since $X_{y_0} \subset X_{s_0}$ we are done. \square

A.5 Corollary. *Let $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ be an extension of flat group spaces of finite presentation. If $G' \rightarrow S$ and $G'' \rightarrow S$ are pure, then $G \rightarrow S$ is pure.*

Proof : The morphism $G \rightarrow G''$ is a torsor under the flat, pure, finitely presented G'' -group space $G'_{G''}$. By flat descent of purity ([RG71] Corollaire 3.3.7), it follows that $G \rightarrow G''$ is pure. Then the result follows from Lemma A.4. \square

Corollaries A.3 and A.5 show that if an S -group algebraic space has a finite composition series whose factors are reductive or proper, flat, finitely presented, then it is pure. This includes finitely presented group schemes of multiplicative type, because they are canonically an extension of a finite, flat group scheme of multiplicative type by a torus.

A.6 Theorem. *Let $X \rightarrow S$ be a morphism of finite presentation, flat and pure, and let $Z \rightarrow X$ be a closed immersion. Then the Weil restriction $\text{Res}_{X/S} Z$ is representable by a closed subscheme of S . If moreover $Z \rightarrow X$ is of finite presentation, then $\text{Res}_{X/S} Z \rightarrow S$ also.*

Proof : For an arbitrary immersion $Z \rightarrow X$, see [AR12], Prop. B.3. The complement when $Z \rightarrow X$ is of finite presentation, is standard; see for instance [LMB00], Proposition 4.18. \square

The next two corollaries appear in [SGA3.1], Exp. VI, § 6.9.

A.7 Corollary. *Let X, Y be S -schemes with $X \rightarrow S$ flat, pure, finitely presented and $Y \rightarrow S$ is separated. Then the functor $\text{Hom}(X, Y)$ is separated over S : for any two S -morphisms $f, g : X \rightarrow Y$ the equalizer $\text{Eq}(f, g) \subset S$ defined by the condition $f = g$ is representable by a closed subscheme of S . If moreover the diagonal of Y is of finite presentation, then $\text{Eq}(f, g) \hookrightarrow S$ also.*

Proof : Apply Theorem A.6 to the closed immersion $(f, g)^{-1}(\Delta_Y) \hookrightarrow X$ where the source is the preimage of the diagonal $\Delta_Y \hookrightarrow Y \times Y$ by $(f, g) : X \rightarrow Y \times Y$. \square

A.8 Corollary. *Let $G \rightarrow S$ be a group scheme and $H \hookrightarrow G$ a closed subgroup scheme which is flat, pure, of finite presentation over S . Then the functor $\text{Norm}_G(H)$ defined as the normalizer of H in G is representable by a closed subgroup scheme of G . If moreover $H \hookrightarrow G$ is of finite presentation, then $\text{Norm}_G(H) \hookrightarrow G$ also.*

Proof : Apply Theorem A.6 to the Weil restriction along the projection $H \times G \rightarrow G$ of the closed immersion $c^{-1}(H) \hookrightarrow H \times G$ where $c : H \times G \rightarrow G$, $(h, g) \mapsto ghg^{-1}$ is the conjugation map. \square

A.9 Corollary. *Let $H \rightarrow S$ be a group scheme which is separated and of finite presentation. Let $L \subset H$ be a proper, flat, finitely presented closed subscheme. Then the subfunctor of S defined by the condition that L is a subgroup scheme is representable by a closed, finitely presented subscheme of S .*

Proof : Let $e : S \rightarrow H$ be the neutral section of H . Let $a = m_{|L \times L} : L \times L \rightarrow H \times H \rightarrow H$ be the restriction of the multiplication of H . Let $b = i_{|L} : L \rightarrow H \rightarrow H$ be the restriction of the inversion. To say that L is a subgroup scheme is to say that the closed immersions $e^{-1}(L) \hookrightarrow S$, $a^{-1}(L) \hookrightarrow L \times L$ and $b^{-1}(L) \hookrightarrow L$ are isomorphisms. By three applications of Theorem A.6 we see that these conditions are represented by a closed subscheme of S . \square

A.10 Lemma. *Let $G \rightarrow S$ be a proper, flat, finitely presented group scheme. Let $H \rightarrow S$ be a group scheme which is separated and of finite presentation.*

- (1) *The functor $\mathrm{Hom}(G, H)$ is representable by an S -algebraic space separated and locally of finite presentation.*
- (2) *If moreover G is finite and H is affine, then $\mathrm{Hom}(G, H)$ is affine and of finite presentation.*
- (3) *The inclusion $\mathrm{Mono}(G, H) \subset \mathrm{Hom}(G, H)$ is representable by open immersions.*

Proof : (1) Let $T = \mathrm{Hom}_{\mathrm{Sch}}(G, H)$ be the functor of morphisms of schemes. This is representable by an S -algebraic space separated and locally of finite presentation, as follows from e.g. [SP22, Tag 0D1C]. Let $f : G_T \rightarrow H_T$ be the universal point. Then $\mathrm{Hom}(G, H)$ is the subfunctor of T that equalizes the two maps $f \circ m_G$ and $m_H \circ (f \times f)$. This is representable by a closed subscheme by Corollary A.7.

(2) This is [SGA3.2], Exp. XI, Prop. 3.12.b) whose statement requires G to be of multiplicative type but whose proof does not use this assumption – and explicitly points it out.

(3) Let $T = \mathrm{Hom}(G, H)$ and let $f : G \rightarrow H$ be the universal homomorphism over T . The kernel $N = \ker(f)$ is a closed subgroup scheme of G of finite presentation, and the functor $\mathrm{Mono}(G, H)$ is the subfunctor of T which renders the neutral section $e : T \rightarrow N$ an isomorphism. The latter condition is equivalent to $N \rightarrow T$ being a closed immersion; since $N \rightarrow T$ is proper, it follows from [SP22, Tag 05XA] that the subfunctor of interest is representable by an open subscheme of T . \square

A.11 Lemma. *Let $H \rightarrow S$ be a group scheme which is separated and of finite presentation. Then the S -functor of subgroup schemes $L \subset H$ which are proper, flat, finitely presented is representable by an algebraic space separated and locally of finite presentation.*

Proof : Let $T = \mathrm{Hilb}$ be the Hilbert scheme of proper, flat, finitely presented closed subschemes of H . This is representable by an algebraic space separated and locally of finite presentation, see [SP22, Tag 0D01]. It follows from Corollary A.9 that the functor of subgroup schemes is representable by a closed subscheme of T . \square

References

- [AHR20] JAROD ALPER, JACK HALL, DAVID RYDH, *A Luna étale slice theorem for algebraic stacks*, Ann. Math. (2) 191, No. 3, 675–738 (2020).
- [AHR21] JAROD ALPER, JACK HALL, DAVID RYDH, *The étale local structure of algebraic stacks*, <https://arxiv.org/abs/1912.06162>, 2021.
- [AI19] SHAMIL ASGARLI, GIOVANNI INCHIOSTRO, *The Picard group of the moduli of smooth complete intersections of two quadrics*, Trans. Amer. Math. Soc. 372 (2019), no. 5, 3319–3346.

- [Alp13] JAROD ALPER, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) 63 (2013), no. 6, 2349–2402.
- [Alp14] JAROD ALPER, *Adequate moduli spaces and geometrically reductive group schemes*, Algebr. Geom. 1 (2014), no. 4, 489–531.
- [AOV08] DAN ABRAMOVICH, MARTIN OLSSON, ANGELO VISTOLI, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 4, 1057–1091. Corrigendum *ibid.* 64, No. 3, 945–946 (2014).
- [AR12] DAN ABRAMOVICH, MATTHIEU ROMAGNY, *Moduli of Galois p -covers in mixed characteristics*, Algebra Number Theory 6 (2012), no. 4, 757–780.
- [Bh16] BHARGAV BHATT, *Algebraization and Tannaka duality*, Camb. J. Math. 4, No. 4, 403–461 (2016).
- [BHL17] BHARGAV BHATT, DANIEL HALPERN-LEISTNER, *Tannaka duality revisited*, Adv. Math. 316, 576–612 (2017).
- [Bre90] LAWRENCE BREEN, *Bitorseurs et cohomologie non abélienne*, The Grothendieck Festschrift, in honor of the 60th Birthday of A. Grothendieck. Vol. I, Prog. Math. 86, 401–476 (1990).
- [Bri21] MICHEL BRION, *Homomorphisms of algebraic groups: representability and rigidity*, <https://arxiv.org/abs/2101.12460>, 2021.
- [Bro21] SYLVAIN BROCHARD, *Duality for commutative group stacks*, Int. Math. Res. Not. IMRN 2021, no. 3, 2321–2388.
- [BTN21] LUIS ALEJANDRO BARBOSA-TORRES, FRANK NEUMANN, *Equivariant cohomology for differentiable stacks*, J. Geom. Phys. 160 (2021).
- [CJW21] QILE CHEN, FELIX JANDA, RACHEL WEBB, *Virtual cycles of stable (quasi-)maps with fields*, Adv. Math. 385 (2021).
- [CLCT09] TOM COATES, YUAN-PIN LEE, ALESSIO CORTI, HSIAN-HUA TSENG, *The quantum orbifold cohomology of weighted projective spaces*, Acta Math. 202 (2009), no. 2, 139–193.
- [DG70] MICHEL DEMAZURE, PIERRE GABRIEL, *Groupes algébriques*, Masson et Cie, North-Holland Publishing Company. xxvi, 700 p. (1970).
- [Di12] DUILIU-EMANUEL DIACONESCU, *Chamber structure and wallcrossing in the ADHM theory of curves, I*, J. Geom. Phys. 62 (2012), no. 2, 523–547.
- [EGA1_{new}] ALEXANDRE GROTHENDIECK (WITH JEAN DIEUDONNÉ), *Éléments de Géométrie Algébrique I*, Grundlehren der Mathematischen Wissenschaften 166, Springer-Verlag, 1971.
- [EGA] ALEXANDRE GROTHENDIECK (WITH JEAN DIEUDONNÉ), *Éléments de Géométrie Algébrique*, Publ. Math. IHÉS 4 (Chapter 0, 1-7, and I, 1-10), 8 (II, 1-8), 11 (Chapter 0, 8-13, and III, 1-5), 17 (III, 6-7), 20 (Chapter 0, 14- 23, and IV, 1), 24 (IV, 2-7), 28 (IV, 8-15), and 32 (IV, 16-21), 1960-1967.
- [Gi71] JEAN GIRAUD, *Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften. Band 179. Springer-Verlag, ix, 467 p (1971).

- [GJK17] AMIN GHOLAMPOUR, YUNFENG JIANG, MARTIJN KOOL, *Sheaves on weighted projective planes and modular forms*, Adv. Theor. Math. Phys. 21 (2017), no. 6, 1455–1524.
- [GV04] CHRISTOF GEISS, DETLEF VOIGT, *Non-reduced automorphism schemes*, J. Pure Appl. Algebra 193 (2004), no. 1-3, 123–127.
- [Ha21] ELOISE HAMILTON, *Smoothness of non-reductive fixed point sets and cohomology of non-reductive GIT quotients*, <https://arxiv.org/abs/2105.03449>, 2021.
- [HLP19] DANIEL HALPERN-LEISTNER, ANATOLY PREYGEL, *Mapping stacks and categorical notions of properness*, <https://arxiv.org/abs/1402.3204>, 2019.
- [Il71] LUC ILLUSIE, *Complexe cotangent et déformations I*, Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, 1971.
- [Il72] LUC ILLUSIE, *Complexe cotangent et déformations II*, Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, 1972.
- [Jo21] DOMINIC JOYCE, *Enumerative invariants and wall-crossing formulae in abelian categories*, <https://arxiv.org/abs/2111.04694>, 2021.
- [KL13] YOUNG-HOON KIEM, JUN LI, *A wall crossing formula of Donaldson-Thomas invariants without Chern-Simons functional*, Asian J. Math. 17 (2013), no. 1, 63–94.
- [KS20] YOUNG-HOON KIEM, MICHAEL SAVVAS, *Localizing virtual structure sheaves for almost perfect obstruction theories*, Forum Math. Sigma 8 (2020).
- [LMB00] GÉRARD LAUMON, LAURENT MORET-BAILLY, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 39, Springer-Verlag, 2000.
- [LMM14] SHISEN LUO, TOMOO MATSUMURA, W. FRANK MOORE, *Moment angle complexes and big Cohen-Macaulayness*, Algebr. Geom. Topol. 14 (2014), no. 1, 379–406.
- [LS20] CHIU-CHU MELISSA LIU, ARTAN SHESHMANI, *Stacky GKM graphs and orbifold Gromov-Witten theory*, Asian J. Math. 24 (2020), no. 5, 855–902.
- [MR19] SAMOUIL MOLCHO, EVANGELOS ROUTIS, *Localization for logarithmic stable maps*, Trans. Amer. Math. Soc. Ser. B 6 (2019), 80–113.
- [Mu65] JACOB MURRE, *Representation of unramified functors. Applications (according to unpublished results of A. Grothendieck)*, Séminaire Bourbaki, Vol. 9, Exp. No. 294, 243–261, Soc. Math. France, Paris, 1995.
- [Ol06] MARTIN OLSSON, *Hom-stacks and restriction of scalars*, Duke Math. J. 134 (2006), no. 1, 139–164.
- [Ol16] MARTIN OLSSON, *Algebraic spaces and stacks*, American Mathematical Society Colloquium Publications 62, 2016.
- [OS19] GEORG OBERDIECK, JUNLIANG SHEN, *Reduced Donaldson-Thomas invariants and the ring of dual numbers*, Proc. Lond. Math. Soc. (3) 118 (2019), no. 1, 191–220.
- [Ra70] MICHEL RAYNAUD, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lecture Notes in Mathematics 119, Springer-Verlag, 1970.

- [RG71] MICHEL RAYNAUD, LAURENT GRUSON, *Critères de platitude et de projectivité. Techniques de “platification” d’un module*, Invent. Math. 13 (1971), 1–89.
- [Ro05] MATTHIEU ROMAGNY, *Group actions on stacks and applications*, Michigan Math. J. 53 (2005), no. 1, 209–236.
- [Ro11] MATTHIEU ROMAGNY, *Composantes connexes et irréductibles en familles*, Manuscripta Math. 136 (2011), no. 1-2, 1–32.
- [Ro12] MATTHIEU ROMAGNY, *Effective models of group schemes*, J. Algebraic Geom. 21 (2012), no. 4, 643–682.
- [Ro21] MATTHIEU ROMAGNY, *Fixed point stacks under groups of multiplicative type*, <https://arxiv.org/abs/2101.02450>, 2021.
- [Sch17] JOHANNES SCHMITT, *A compactification of the moduli space of self-maps of \mathbb{CP}^1 via stable maps*, Conform. Geom. Dyn. 21 (2017), 273–318.
- [Sch18] JOHANNES SCHMITT, *A correspondence of good G -sets under partial geometric quotients*, Beitr. Algebra Geom. 59 (2018), no. 2, 343–360.
- [SGA1] ALEXANDRE GROTHENDIECK, *Revêtements étales et groupe fondamental*, Séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck. With two papers by M. Raynaud. Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer]. Documents Mathématiques 3, Société Mathématique de France, 2003.
- [SGA3.1] *Schémas en groupes I. Propriétés générales des schémas en groupes*, Séminaire de Géométrie Algébrique du Bois Marie 1962–64. A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J-P. Serre. Revised and annotated edition of the 1970 French original. Edited by Philippe Gille and Patrick Polo. Documents Mathématiques 7, Société Mathématique de France, 2011.
- [SGA3.2] *Schémas en groupes II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*. Directed by M. Demazure and A. Grothendieck. With the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud, J.-P. Serre. Lecture Notes in Mathematics 152, Springer-Verlag, 1970. New edition available at <https://webusers.imj-prg.fr/~patrick.polo/SGA3/>.
- [SGA3.3] *Schémas en groupes III. Structure des schémas en groupes réductifs*. Directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J-P. Serre. Revised and annotated edition of the 1970 French original. Edited by Philippe Gille and Patrick Polo. Documents Mathématiques 8, Société Mathématique de France, 2011.
- [SGA4.3] *Théorie des topos et cohomologie étale des schémas*, Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964. Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics 305, Springer-Verlag, 1973.
- [Sk13] JONATHAN SKOWERA, *Białynicki-Birula decomposition of Deligne-Mumford stacks*, Proc. Amer. Math. Soc. 141 (2013), no. 6, 1933–1937.
- [SP22] THE STACKS PROJECT AUTHORS, *Stacks Project*, located at http://www.math.columbia.edu/algebraic_geometry/stacks-git.

- [Va05] ADRIAN VASIU, *Normal, unipotent subgroup schemes of reductive groups*, C. R. Math. Acad. Sci. Paris 341 (2005), no. 2, 79–84.
- [We11] THORSTEN WEIST, *Torus fixed points of moduli spaces of stable bundles of rank three*, J. Pure Appl. Algebra 215 (2011), no. 10, 2406–2422.

Matthieu ROMAGNY, UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE
Email address: `matthieu.romagny@univ-rennes1.fr`