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Absence of evidence for entanglement in a three-photon Greenberger-Horn-Zeilinger type experiment

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Abstract. A famous experiment by Pan *et al.* (Nature **403**, 515 (2000)) on a three-photon Bell state is considered to prove the existence of entanglement without relying on Bell inequalities. We show that the theory this proof is based on contains errors in the calculations that are beyond repair. Indeed, the formalism to describe the internal dynamics of electrons (at rest) and photons in quantum mechanics (QM) is based on the representations of the rotation group. For electrons, who have spin 1/2, the representation is SU(2), for photons, who have spin 1 it is SO(2). Rotations are just a subject of elementary Euclidean geometry, which does not contain any physics, let alone “quantum magic”. Therefore all quantum paradoxes that oppose classical mechanics and QM (when it is purely based on the algebra of SU(2)), can only be due to logical errors and a lack of understanding of the group representation theory. The three-photon Bell state is extended but not non-local and as such does not imply a “spooky action at a distance”. (This version supersedes the previous version which contained an error).

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1 Motivation

In [1] we have been able to derive the Dirac equation from scratch using group representation theory. The proof is mathematically rigorous. It just uses the *Ansatz* that the electron spins. This seems to run contrary to some well-known objections formulated by Lorentz but we have been able to explain why these are not compelling. Quantum mechanics (QM) is reputed to be full of mysteries. It is therefore very surprising that no special stunning assumptions were required to obtain the mathematical proof for this fundamental equation. There is no quantum magic involved at all. The derivation is entirely classical and there is no evidence of any contradiction with the theory of relativity. QM seems to be just a part of relativity expressed in a somewhat counterintuitive language, i.e. the formalism of group representation theory. We have also been able to explain the Stern-Gerlach experiment classically [2]. These results suggest that QM could be just based on classical reasoning expressed in group representation theory. But this idea is firmly contradicted by a vast body of experimental results reporting violations of Bell inequalities designed to test hidden-variable theories [3,4,5,6,7] We have therefore made a study of the Bell inequalities in order to reach a better understanding of the limitations of our approach. We found out that the derivation of the inequalities used by Aspect *et al.* contains an error [8]. A further confirmation of this is that we have been able also to calculate Malus’ law by classical reasoning [9]. This is reassuring because entanglement questions realism and/or locality, which are corner stones of the foundations of the theory of relativity. But herewith the problems are not over yet. Entanglement has also been reported in measurements with more than two particles where the argument does not rely on a violation of Bell inequalities. Famous in this respect is an experiment [10] based on work of Greenberger, Horne and Zeilinger (GHZ) [11]. We have investigated it and are now able to show that the theory underlying this experiment also contains errors such that the experimental results do not constitute a proof for the existence of entanglement.

2 Some preliminaries about SU(2)

First we need some preliminaries about the representation SU(2) of the rotations in \mathbb{R}^3 . More extensive details are given in [1,12] In SU(2) a rotation through an angle φ around an axis specified by a unit vector \mathbf{s} is given by the Rodrigues formula:

$$\mathbf{R}(\mathbf{s}, \varphi) = \cos(\varphi/2)\mathbf{1} - \imath \sin(\varphi/2)[\mathbf{s} \cdot \boldsymbol{\sigma}]. \quad (1)$$

Here $\mathbf{1}$ is the 2×2 unit matrix and it is easily checked that $\det \mathbf{R} = 1$. The notation $[\mathbf{s} \cdot \boldsymbol{\sigma}]$ is not a true scalar product. It is just a shorthand for $s_x \sigma_x + s_y \sigma_y + s_z \sigma_z$, where:

$$\sigma_x = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} & -\imath \\ \imath & \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \quad (2)$$

are the Pauli matrices. In fact, a reflection A with respect to a plane \mathcal{A} defined by the unit vector $\mathbf{a} = (a_x, a_y, a_z) \perp \mathcal{A}$ is in this formalism represented by:

$$\mathbf{A} = [\mathbf{a} \cdot \boldsymbol{\sigma}] = \begin{bmatrix} a_z & a_x - \imath a_y \\ a_x + \imath a_y & -a_z \end{bmatrix}, \quad \det \mathbf{A} = -1. \quad (3)$$

The Rodrigues equation is obtained by calculating the product $\mathbf{R} = \mathbf{BA}$ of two reflections. It is based on the fact that the product of two reflections is a rotation, see e.g. [1], p.10, Fig.1. For a rotation through an angle φ around the z -axis, the Rodrigues equation implies:

$$\mathbf{R}(\varphi, \mathbf{e}_z) = \cos(\varphi/2)\mathbf{1} - \imath \sin(\varphi/2)\sigma_z = \begin{bmatrix} e^{-\imath\varphi/2} & \\ & e^{+\imath\varphi/2} \end{bmatrix}. \quad (4)$$

By putting $\varphi = \omega t$ one obtains this way the description of an object that spins around the z -axis:

$$\mathbf{R}(\omega t, \mathbf{e}_z) = \begin{bmatrix} e^{-\imath\omega t/2} & \\ & e^{+\imath\omega t/2} \end{bmatrix}. \quad (5)$$

It describes the spinning motion in the very same way as a varying position vector $\mathbf{r}(t)$ describes the displacement of an object along its orbit in classical mechanics. It is by using this method to describe a spinning electron that we have been able to derive in [1] the Dirac equation from scratch. One just has to introduce $E = m_0 c^2 = \hbar\omega/2$.

As is easily checked on Eq. 3, $\det \mathbf{A} = -1$. For all group elements g with representation matrix \mathbf{G} of the group generated by the reflections we have therefore $\det \mathbf{G} \in \{-1, +1\}$. The group contains rotations (obtained from an even number of reflections), reversals (obtained from an odd number of reflections) and reflections, which are the special reversals which square to $\mathbf{1}$. An SU(2) rotation matrix has the abstract form:

$$\begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix}, \quad uu^* + vv^* = 1. \quad (6)$$

This can easily be checked on the Rodrigues formula. It is readily verified on Eq. 6 that the first column of an SU(2) matrix contains already the information about the whole matrix. It can therefore be used as a shorthand for the group representation matrices. These shorthands are the spinors. Spinors in SU(2) correspond to the first columns of the representation matrices. They therefore represent group elements. Spinors *are* group elements. They are a very useful and compact tool to write rotations. Like $x^2 + y^2 = R^2$ is the expression for a circle in algebraic geometry, a spinor can express rotational motion in the algebra of representation theory. In both cases this correspondence between the algebra and the geometry permits a very useful dialogue between the geometry and the algebra. The spinors of the dynamical rotation matrix 5 are then:

$$e^{-\imath\omega t/2} \uparrow = e^{-\imath\omega t/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e^{+\imath\omega t/2} \downarrow = e^{+\imath\omega t/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (7)$$

where \uparrow and \downarrow are the electron spin-up and spin-down states of the electron in QM. It may surprise that there are two signs for ω in the algebra. The reason for this is that the formalism describes the same spinning motion both in right-handed and left-handed reference frames (given by the second columns of the SU(2) matrices). The inverse motion introduces two further spinors. As the dynamical rotation matrices describe the spin dynamics of the electron, also the spinors describe these dynamics.

3 Spinors do not form a vector space

The group of rotations in \mathbb{R}^3 form a non-abelian group. For any group (\mathcal{G}, \circ) , where \circ is the composition law defined on the set \mathcal{G} , a linear combination of group elements $\sum_{j=1}^n c_j g_j$, where $c_j \in \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $g_j \in \mathcal{G}$, is not defined by the axioms for a group. In fact, the operations of summing and multiplying by a scalar are not part of the structure of a group. They are defined by the richer and more encompassing structure of a vector space and non-abelian groups are not vector spaces but curved manifolds.

Hence when we make linear combinations of group representation matrices $\sum_{j=1}^n c_j \mathbf{D}(g_j)$ this may well be defined algebraically but it remains geometrically meaningless, because $\sum_{j=1}^n c_j g_j$, which would be its counterpart in the isomorphism between g_j and $\mathbf{D}(g_j)$ is just not defined. To attribute a meaning to $\sum_{j=1}^n c_j g_j$, we must embed the group in a vector space \mathcal{V} where the operations become allowed. By the presence of the operation \circ , the vector space $(\mathcal{V}, \cdot, +)$ then actually becomes an algebra $(\mathcal{V}, \cdot, +, \circ)$ on the set \mathcal{V} .

Despite the extension of \mathcal{G} to the embedding vector space \mathcal{V} with the calculus of its algebra, the elements of $\mathcal{V} \setminus \mathcal{G}$ will in general remain meaningless. An example of this is given by general relativity where space-time is a curved manifold which we could embed mathematically in a “flat” vector space \mathbb{R}^5 but where the points of \mathbb{R}^5 that do not belong to space-time are physically meaningless. It is for these reasons that general relativity is formulated intrinsically without any reference to an embedding vector space. The same ideas apply also to the three-dimensional curved manifold of the rotation group. We have shown in [13] that embedding the three-dimensional manifold of the rotation group in a vector space leads to different results in $SU(2)$ and in $SO(3)$. One can calculate $h(\mathbf{r})$ for a vector $\mathbf{r} \in \mathbb{R}^3$, where h is defined as the sum of two group elements $h = g_1 + g_2$. The values one obtains then for $h(\mathbf{r})$ are not the same in $SU(2)$ and $SO(3)$. This is hardly surprising as vectors transform quadratically rather than linearly in $SU(2)$.

4 Probabilities and the Born rule

But we cannot pretend to ignore that physics uses such linear combinations all the time with excellent results. There must therefore exist a justification for this. We have in this respect been able to show that the linear combinations can be given a meaning in terms of sets. This can be deduced from the fact that for finite groups one defines so-called all-commuting or Casimir operators:

$$\sum_{h \in \mathcal{C}} \mathbf{D}(h), \quad \text{for which: } \forall g \in \mathcal{G} : \mathbf{D}(g) \left[\sum_{h \in \mathcal{C}} \mathbf{D}(h) \right] = \left[\sum_{h \in \mathcal{C}} \mathbf{D}(h) \right] \mathbf{D}(g). \quad (8)$$

Here $\mathcal{C} = \{h_1, h_2, \dots, h_k\} \subset \mathcal{G}$ is a so-called normal subgroup and the equation then just translates the definition of a normal subgroup:

$$\forall g \in \mathcal{G} : g \circ \mathcal{C} = \mathcal{C} \circ g, \quad (9)$$

into *a priori* meaningless algebra, by writing down purely formally:

$$g \circ \{h_1, h_2, \dots, h_k\} = \{h_1, h_2, \dots, h_k\} \circ g \quad \text{as: } g \circ (h_1 + h_2 + \dots + h_k) = (h_1 + h_2 + \dots + h_k) \circ g. \quad (10)$$

Based on this observation it becomes natural to introduce the notion that the sum defines a set and can be used to represent a set of group elements. In $SU(2)$, the linear combination $\sum_{j=1}^n c_j \mathbf{D}(g_j)$ describes then a set containing $|c_j|^2$ copies of g_j . These copies could be objects that are in the orientation (or the state) described by g_j , as illustrated by Eq. 7. The reason for using the rule $|c_j|^2$ is that we associate each element of the set with a spinor $\xi = [\xi_0, \xi_1]^\top$, and that spinors satisfy $\xi \xi^\dagger = 1$. When we want to count group elements or the objects they represent, we must count the spinors associated with them. This can be done by counting the spinors in terms of the units $\xi \xi^\dagger = 1$ they intrinsically carry with them in their algebraic structure. Hence, originally we must conceive that $|c_j|^2 \in \mathbb{N} \cup \{0\}$. But by using frequencies instead of counts we can give also meaning to $|c_j|^2 \in \mathbb{R}$. It becomes then a probability and we obtain a justification for the Born rule. The sums that correspond to the sets are not true sums for which carrying out the algebra would be defined. They must be considered as mere juxtapositions, just like elements of sets are occurring by mere juxtaposition in those sets. In other words, this procedure explains the rule for incoherent summing of probabilities. For coherent summing we have not found a group-theoretical justification, such that coherent summing will always call for special caution and treatment. All this shows how crucial it is to be aware of the fact that sums and multiplications by scalars are in principle not defined for group elements. It is an unsuspected eye-opener that unlocks the door to a probability calculus on sets, which otherwise could have remained beyond our conceptual grasps. The sets will be beams or statistical ensembles of particles represented by waves (because they have a periodic internal motion). Hence, when we will describe a Bell state $\frac{1}{\sqrt{2}}(\uparrow\uparrow + \downarrow\downarrow)$ for three photons below, it will not mean that the

three photons are simultaneously $\uparrow\uparrow\uparrow$ and $\downarrow\downarrow\downarrow$ in a mixed superposition state and that the measurement optics bangs them into a pure state. It just describes a beam of triplets of photons, whereby half of them are $\uparrow\uparrow\uparrow$ and half of them are $\downarrow\downarrow\downarrow$. But an even more important lesson to be learned from all this is that probability calculus on group elements can follow different rules from those we are used to with more mundane physical quantities like scalars or vectors. After all a rotation is a function that maps points of \mathbb{R}^3 to points of \mathbb{R}^3 . That is not a usual physical object one can see or grasp with one's hand. The new probability rules are by and large at odds with common sense of traditional textbook teachings, but they are not impenetrable to classical thinking.

5 Sets as eigenvectors of reflection matrices

5.1 Phase factors

To summarize what we explained in the previous sections, it is *a priori* meaningless to use \uparrow and \downarrow as two basis vectors for the set of spinors. But as QM uses this kind of linear algebra all the time with good results, some mathematical justification for these successes had to exist. We have shown that such a justification can be given a meaning in terms of sets. Some other reflexes from linear algebra on vector spaces must be unlearned.

In fact, as we learn in linear algebra that eigenvectors are defined up to a phase factor, these phase factors are often dropped in QM. But that is because linear algebra is defined for vector spaces. The spinors of $SU(2)$ are not vectors. Their phase factors contain very relevant information about the angular frequency of the spinning motion $\varphi = \omega t$ and therefore also about the dynamics. E.g. the rest energy of the electron is $\hbar\omega/2$. That information is entirely locked up inside the phase factor. If we drop that information because we are unable to realize that the calculations on group elements of a non-abelian group (which is a curved manifold) and the calculations on vectors (which belong to a flat vector space) are entirely different things despite the fact that we are using the same matrix tools to make these calculations, we will no longer have access to that information. And that is exactly the trick our eyes and instruments are playing us because what we observe is governed by the Born rule which throws the phase factors out of the window. We should not treat the phase factor with the same contempt but rather acknowledge the interesting information it contains.

5.2 Eigenvectors

Let us now discuss the eigenvectors and eigenvalues of reflection operators \mathbf{A} for which $\det \mathbf{A} = -1$. Reflections do not have group elements for eigenstates, because a reflection turns a rotation into a reversal and a reversal into a rotation. Therefore sets will be needed to render it possible to define eigenvectors. Let us illustrate this with the example of the reflection operator σ_z . We have then $\sigma_z \mathbb{1} = \sigma_z$ and $\sigma_z \sigma_z = \mathbb{1}$, which illustrates that a reflection turns a rotation into a reversal and a reversal into a rotation. Therefore we can only obtain eigenvectors under the form of sets. One such set is $S_+ = \{\mathbb{1}, \sigma_z\}$. For this set we have $\sigma_z \{\mathbb{1}, \sigma_z\} = \{\mathbb{1}, \sigma_z\}$, or $\sigma_z S_+ = S_+$. Expressed algebraically under the form of a sum this becomes: $\sigma_z(\mathbb{1} + \sigma_z) = (\mathbb{1} + \sigma_z)$. The set $S_+ = \{\mathbb{1}, \sigma_z\}$ is this way an eigenvector of σ_z with eigenvalue 1 but its individual members $\mathbb{1}$ and σ_z are not eigenvectors of σ_z .

The other eigenvector is the set $S_- = \{\mathbb{1}, -\sigma_z\}$. We have then $\sigma_z \{\mathbb{1}, -\sigma_z\} = -\{\mathbb{1}, \sigma_z\}$. We see that the set $S_- = \{\mathbb{1}, -\sigma_z\}$ is an eigenvector of σ_z with eigenvalue -1 , i.e. $\sigma_z S_- = -S_-$. We can write this algebraically as $\sigma_z(\mathbb{1} - \sigma_z) = -(\mathbb{1} - \sigma_z)$. These eigenvectors may conceptually look somewhat hilarious. When we are talking about eigenvectors, what we have in mind are 2×1 matrices, not sets. But first of all we can write σ_z and $\mathbb{1}$ down under the form of 2×2 matrices and write the sets algebraically using the notation based on summing. Next, we have to remind that spinors are 2×1 matrices corresponding to the first columns of the 2×2 matrices. This is the way to write the sets as 2×1 column matrices.

Before we do this let us first write things out in full. This tells us that using $\mathbb{1} + \sigma_z$ as a notation for $S_+ = \{\mathbb{1}, \sigma_z\}$ corresponds to:

$$\mathbb{1} + \sigma_z = 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where now: } \sigma_z \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (11)$$

such that this indeed leads to an equivalence between $\sigma_z S_+ = S_+$ and $\sigma_z(\mathbb{1} + \sigma_z)$. Furthermore, we can see that the first column is indeed our up-state \uparrow . But what the usual method to calculate the eigenvectors is hiding us is that this up-state \uparrow is a superposition state. This goes unnoticed because both $\mathbb{1}$ and σ_z have the up-state as their first column. But it is the simultaneous presence of these two operators which ensures that the second column is completely padded with zeros. When you apply the same logic of sets to σ_x , whose eigenvectors are $\mathbb{1} + \sigma_x$ and $\mathbb{1} - \sigma_x$, the degeneracy in the notations will be lifted. The two contributions, $\mathbb{1}$ and σ_x , will no longer be stacked one on top of the other, such that they can be spotted individually. The matrix in Eq. 11 is actually one of the four basis vectors for the linear space

of 2×2 matrices. These four basis vectors correspond to $\frac{1}{\sqrt{2}}(\mathbb{1} + \sigma_z)$, $\frac{1}{\sqrt{2}}(\mathbb{1} - \sigma_z)$, $\frac{1}{\sqrt{2}}(\sigma_x + i\sigma_y)$ and $\frac{1}{\sqrt{2}}(\sigma_x - i\sigma_y)$. It is conceptually very important to be aware of the fact that the eigenvectors \uparrow and \downarrow are mixed states. The Bell state $\frac{1}{\sqrt{2}}(\uparrow\uparrow + \downarrow\downarrow)$, which is an eigenvector of $\sigma_z \otimes \sigma_z$, represents a set with eigenvalue 1. Its pendant with eigenvalue -1 is the other Bell state $\frac{1}{\sqrt{2}}(\uparrow\uparrow - \downarrow\downarrow)$.

We can generalize what we have said here to any reflection operator $\mathbf{A} = [\mathbf{a} \cdot \boldsymbol{\sigma}]$, for which $\det \mathbf{A} = -1$. Reflections do not have group elements for eigenstates, because a reflection turns a rotation into a reversal and a reversal into a rotation. The reflection operator \mathbf{A} turns $\mathbb{1}$ into \mathbf{A} and \mathbf{A} into $\mathbb{1}$. The eigenvectors of \mathbf{A} must therefore be sets. The first set is $S_+ = \{\mathbb{1}, \mathbf{A}\}$. For this set, $\mathbf{A}S_+ = S_+$, such that S_+ is an eigenvector of \mathbf{A} corresponding to its eigenvalue $\lambda = 1$. As a matter of fact all reflections go by the rule $\mathbf{A}^2 = \mathbb{1}$. That is just part of the definition of a reflection. We can write the set S_+ in terms of sums as: $\mathbf{A}(\mathbb{1} + \mathbf{A}) = (\mathbb{1} + \mathbf{A})$. Similarly $S_- = \{-\mathbb{1}, +\mathbf{A}\}$. We have now $\mathbf{A}S_- = -S_-$, such that S_- is an eigenvector of \mathbf{A} corresponding to its eigenvalue $\lambda = -1$. We can write this as $\mathbf{A}(\mathbf{A} - \mathbb{1}) = -(\mathbf{A} - \mathbb{1})$. We can now take the first columns of these eigenvectors, the spinors, as eigenvectors under the form of column matrices, thereby using the expression of \mathbf{A} given in Eq. 3. In other words, there is no need to make calculations. One can write the eigenvectors down immediately. One can use both rotation matrices \mathbf{R} and reflection matrices \mathbf{V} to perform the diagonalization. By this we mean that we can write the diagonalization of \mathbf{A} both under the forms $\mathbf{R}\mathbf{A}\mathbf{R}^{-1}$ and $\mathbf{V}\mathbf{A}\mathbf{V}^{-1}$. We can choose between \mathbf{R} and \mathbf{V} easily by playing with the signs of the eigenvectors to obtain $\det \mathbf{R} = 1$ or $\mathbf{V} = -1$.

5.3 Caveats

Remember that this whole discussion all started from the observation that the axioms of a group do not define linear combinations of group elements! The set S_+ contains two group elements of a different nature, a rotation (the identity element $\mathbb{1}$ with determinant 1) and a reversal (the reflection with respect to the Oxy -plane with determinant -1). We can attribute a $+$ sign to the set S_+ because its eigenvalue is $+1$. But we cannot attribute this $+$ sign to its individual members which according to their determinant have the sign $+$ for the rotations and $-$ for the reversals. When we are attributing signs to the eigenvectors of the reflection operators we are therefore not tagging the individual particles, but sets of particles, because S_+ can equally well describe a beam of photons wherein half of them are in a $\det = +1$ state and half of them in a $\det = -1$ state.

In the identity $\sigma_z \uparrow = \uparrow$, \uparrow does therefore not represent a group element, but actually the infinite set $\{e^{i\varphi/2}\mathbb{1}, e^{i\varphi/2}\sigma_z \mid \varphi \in \mathbb{R}\}$, which must be replaced by $\{e^{i(\omega t + \varphi)/2}\mathbb{1}, e^{i(\omega t + \varphi)/2}\sigma_z \mid \varphi \in \mathbb{R}\} \cup \{e^{-i(\omega t + \varphi)/2}\mathbb{1}, e^{-i(\omega t + \varphi)/2}\sigma_z \mid \varphi \in \mathbb{R}\}$, where now the phase factor φ can be dropped because it is only $\pm\omega$ which is specifying the dynamical state. Of course $e^{i(\omega t + \varphi)/2}\mathbb{1}$ is hard to understand. Hence we should replace $\mathbb{1}$ by $\mathbf{R}(0)$ and σ_z by $\sigma_z \mathbf{R}(0)$. As the group element in Eq. 5 is used to represent an electron, the set $e^{\pm i(\omega t + \varphi)/2}S_+$ represents an infinity of orientations of electrons. If the electrons could be tagged such that we could see their phase φ , then the group theory could account for it.

6 SU(2) is only a part of Euclidean geometry

6.1 Preamble

Depending on their contents, photon beams can be circularly or linearly polarized. As photons have spin, individual photons must be circularly polarized [14]. Linearly polarized beams would contain then equal amounts of photons of either type. And if the balance is not exact, the beam would be elliptically polarized. Therefore using the eigenvector equation $\sigma_z \uparrow = \uparrow$ to draw conclusions about single particles is according to Subsection 5.2 wrong because it represents $\sigma S_+ = S_+$, but in [9] we have described how the group theory can account for particles that can be both circularly and linearly polarized, because the spin-1 representation is obtained from the tensor product of two spin 1/2 representations (see Appendix). The same applies *mutatis mutandis* for the eigenvector equation $\sigma_z \downarrow = -\downarrow$.

Let us now describe how this group formalism is used to describe the Stern-Gerlach experiment. We start with electrons of two types, right-handed ones ($e^{-i\omega t/2} \uparrow$) and left-handed ones ($e^{+i\omega t/2} \downarrow$). In the calculations I only considered one sense of spinning. I should perhaps have considered the other sense as well, but it would only have duplicated the calculations. The electrons are entering a magnetic field with a gradient. This magnetic field behaves exactly as a dynamical rotation. This rotation can again be right-handed ($e^{-i\Omega t/2} \uparrow$) or left-handed ($e^{+i\Omega t/2} \downarrow$). This leads to four combinations, but only to two types of beams in terms of their energies $|\hbar(\omega \pm \Omega)/2|$. The combinations describe spinning electrons that are precessing in a magnetic field. We end up with two different energies in the beam which is split into two beams according to these energies by the gradient of the magnetic field. The resulting beams are not in spin-up and spin-down but precession-up and precession-down states, which makes it far less mysterious than the traditional spin-up and spin-down scenario we have been taught. Each of the two beams contains two algebraic contributions. In the calculus, each group element comes with two signs and we just must allow for all possible

combinations of the signs. It would be the same as considering the interaction of two billiard balls which both can spin clockwise or counterclockwise. That leads also to four possibilities. One may object that one does not believe in the existence of two types of action by magnetic fields, but all this is only about the same rotation described in right-handed and left-handed frames as may transpire from Eq. 5 which only describes only one rotation. In any case, the worst thing that can happen is that we are calculating the same things twice due to ignorance, without any effect on the final conclusions.

The reflection operators of the rotation group exist also with two signs depending on the handedness of their frames. We cannot see all this happening on the microscopic level with our own eyes, but this philosophy stands the test of the comparison with the experimental data. All we have done in such calculations is applying the group theory to describe spinning motions. We call it QM and learn that its predictions are in agreement with experimental data.

6.2 A tale of two triplets of spinning tops

Let us consider now a *Gedankenexperiment* with spinning tops. We will imagine six perfectly identical tops. They are all white. We imagine that we can make them spin in a perfectly vertical position and that the tops are made so perfectly regular and smooth that you would not be able to detect it visually when they are spinning. To become aware of their spinning motion one would have to touch them.

We imagine that our tops would be able to spin without any friction such that once one has set them in motion, they could happily spin until the end of times. To render it possible to detect the spinning motion of the tops visually we will now imagine a right-handed reference frame $Oxyz$ where O coincides with the centre of gravity of a top and the top is in the vertical position. Where the three axes intersect the surface of the top we will put spots on the surface of the top. We could give them different colours. A red one for the x -axis, a blue one for the y -axis and a black one for the z -axis. The idea is only that they will visualize the reference frame $Oxyz$ unambiguously. When the tops spin, the set of three moving spots will represent a rotating reference frame visualizing the spinning motion. We proceed this way for three of the six tops. We will then be able to follow the motion of the tops by watching the dots. Perhaps we will need to shoot a movie of it and play the movie in slow motion, but in principle the dots should do the job.

For the three other tops we proceed somewhat differently. Instead of a right-handed frame $Oxyz$ we use a left-handed one. We will in this respect refer to the two possibilities in terms of right-handed and left-handed tops. We define the initial conditions, by aligning the x and y axes of all six tops along mutually parallel directions. For the right-handed tops the direction of the z -axis will then be given by the right-hand rule and point upwards. For the left-handed tops the direction of the z -axis will be given by the left-hand rule and point downwards. Now we imagine that we can put the six tops into motion simultaneously with the same angular frequency ω . For all six tops we can imagine that the motion is counterclockwise. The spin vectors used to qualify the motions of the right-handed tops will then be parallel with the z -axis and point upwards. However, the spin vectors for the left-handed tops will be parallel with the z -axis and point downwards. That is because in a left-handed frame the vector product \times must be calculated with the left-hand rule, while $SU(2)$ uses the product \wedge . $SU(2)$ attributes therefore opposite algebraic signs $\pm\omega$ to the angular frequencies ω . It takes this way into account the handedness of the frames and the tops. In $SU(2)$, the motion of the right-handed top is therefore described by $\uparrow e^{-i\omega t/2}$. The motion of the left-handed top is described by $\downarrow e^{+i\omega t/2}$. The six tops are then described by $\frac{1}{\sqrt{2}}(\uparrow e^{-i\omega t/2} \otimes \uparrow e^{-i\omega t/2} \otimes \uparrow e^{-i\omega t/2} + \downarrow e^{+i\omega t/2} \otimes \downarrow e^{+i\omega t/2} \otimes \downarrow e^{+i\omega t/2})$, which is the Bell state, we noted by the shorthand $\frac{1}{\sqrt{2}}(\uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow)$. That $\downarrow e^{+i\omega t/2}$ and $\uparrow e^{-i\omega t/2}$ are really turning both counterclockwise follows from the fact that they both occur in the matrix in Eq. 5 which describes a counterclockwise spinning motion.

Now we can put the two triplets in simultaneous motion. We can put them e.g. in a dark room and have light beams whereby we can arrange things in such a way that either all three right-handed tops are lit simultaneously or alternatively all three left-handed tops are lit simultaneously. Each time, we switch only one of the two light beams on at a time, and do it in a random sequence. But we make sure that the statistical averages for both choices are $1/2$. That would visualize successive observations of the states of triplets of photons. We will have this way visualized a statistical ensemble of Bell states, because the algebraic description of the simultaneous motion of these two triplets of tops in $SU(2)$ is the Bell state.

We can identify the three tops in each triplet by naming them 1, 2 or 3. Next to both tops called “1” we put now a vertical mirror whose normal points in the x -direction, next to both tops “2” we put a vertical mirror whose normal points in the y -direction, and next to both tops “3” we put also a vertical mirror whose normal points in the y -direction. That will be the scenario P described by the operator \mathbf{P} (defined below, see Eqs. 22 and 23-26) in $SU(2)$ because the operators σ_x and σ_y are reflections and model therefore the actions of the mirrors. We can similarly visualize the scenarios Q , R and S defined below in Eqs. 22 and 23-26. It would require displacing the mirrors to change the configuration from P to Q , etc. . . . But we could actually have four triplets of right-handed tops and four triplets of left-handed tops such that we could even visualize all experiments P , Q , R , and S simultaneously. For the

moment, such measurements are not carried out simultaneously but I am quite sure that in the future experimental physicists with golden fingers and unfaltering patience will manage to prepare 24 photons in the appropriate state.

We would need 24 tops and 24 mirrors to build the macroscopic model and we should be able to switch between eight light beams rather than two. All we have to do now is to play with the light switch such that there are two possibilities. (1) We see the four measurements P , Q , R , and S simultaneously working on the $\uparrow\uparrow\uparrow$ states (2) We see the four measurements P , Q , R , and S simultaneously working on the $\downarrow\downarrow\downarrow$ states. As pointed out, it is actually not necessary to carry out the measurements P , Q , R , and S simultaneously. But we have now translated exactly the mathematics of the GHZ experiment into a macroscopic realization. If we wanted to describe this macroscopic *Gedankenexperiment* with the group representation theory of $SU(2)$ the algebra would be rigorously identical to that of the GHZ experiment.

But if this is so, here is then the question: where are the quantum effects? Where is the non-locality? Where is the entanglement? If you announced to the audience that you were going to show them the stunning magic of QM and entanglement caught in the act, they might want their money back! We will see that the inconsistency supposed to be a proof for the existence of entanglement in [10] is just due to errors in using the group representation theory.

6.3 “What did you expect?”

We can observe on the basis of the *Gedankenexperiment* that the whole algebra used to calculate the results of a GHZ experiment by QM, if done correctly, just corresponds to the group representation theory of $SU(2)$. Rotations are part of Euclidean geometry, the stuff kids learn at school when they are 13-14 years old. It actually does not contain any physics, let alone that it would contain “quantum magic” ready to pop up in plain sight in a dazzling dance before our perplexed gazes. It is just mere elementary Euclidean geometry. That should demystify it quite a bit. Is it not somewhat hard going to claim that rotations do not have physical reality? Contrary to what has been claimed on the basis of the theory of the GHZ experiment, it will be just impossible to create internal contradictions between QM and classical physics within $SU(2)$, because the QM formalism is pure geometry. The group theory is contradiction-free and its calculations, which we are dubbing quantum theory, confirms it by testing them against experimental data. And if some fancy calculation one may want to apply to these data does not work out, it just means that the procedures used are lacking internal consistence, not that classical mechanics is not able to cope with the physics.

We have two methods to assign parities to objects, viz. $+1$ and -1 for the eigenvectors based on their corresponding eigenvalues, and $+1$ and -1 for individual particles depending on their handedness betrayed by the determinant of the matrix that presents the group elements which corresponds to their state of motion. The eigenvectors represent sets, i.e. particle beams. If we stick to these two choices in a consistent way, no paradox can emerge. And even if we are not able to keep the books about the tagging in the experiments because the signs keep mixing up, like the four states that merge into only two observable beams in the Stern-Gerlach experiment, this is not good enough a reason to claim that there is no classical physical reality to rotations.

But people who are unwittingly making errors or using double standards in the assignments may indeed produce quantum magic. It would be like having a boy with a yellow stocking on his left foot and a red stocking on his right foot. Turn him through an angle of 45 degrees in space and oops, the colours have swapped. That would be quantum magic! It would not happen for real but only purely mentally in your mind because you have not been consistent in your colour assignments. As $SU(2)$ is only about rotations, we do not have to buy the narrative about entanglement.

7 Where did it go wrong?

7.1 Diagonalization and transformation of the reflection matrices

It is easy to check on the simple examples of Eqs. 3 and 5 that rotations do not square to unity while reflections do. A rotation $\mathbf{R}(\mathbf{s}, \varphi)$ squares to $\mathbf{R}(\mathbf{s}, 2\varphi)$. In fact Eq. 6 implies $\mathbf{R}\mathbf{R}^\dagger = 1$ and $\mathbf{R}^\dagger \neq \mathbf{R}$. In general group elements \mathbf{H} transform under the action of other group elements \mathbf{G} according to $\mathbf{H} \rightarrow \mathbf{G}\mathbf{H}$. But a group element will in general not transform a reflection into another reflection. In general it transforms a reflection into a reversal (a group element with determinant -1 that does not square to unity) or into a rotation with determinant $+1$. To transform a reflection \mathbf{A} into another reflection by a group element \mathbf{G} we must use a similarity transformation which preserves its geometrical properties. In other words, a reflection \mathbf{A} is transformed into another reflection by: $\mathbf{A} \rightarrow \mathbf{G}\mathbf{A}\mathbf{G}^{-1}$. Here $\mathbf{G}^{-1} = \mathbf{G}^\dagger$, and \mathbf{G} can be both a reflection \mathbf{V} or a rotation \mathbf{R} , as we will illustrate below in Eqs. 12, 14, 15.

In fact, the effect of an in-plane reflection in $SO(2)$ (e.g. in the plane Oxy) can also be obtained by a rotation in $SO(3)$. We have learned this at school. We can consider a triangle $\triangle ABC$ in the Oxy plane, with $AB \neq BC \neq CA$, and another triangle $\triangle A'B'C'$ obtained from it by a reflection in the plane. Despite the opposite parities in \mathbb{R}^2 the triangles are congruent in \mathbb{R}^3 . The proof requires a rather trivial form of thinking out of the box. It boils down to breaking away from the straightjacket of \mathbb{R}^2 and turning the triangles in \mathbb{R}^3 . In an analogous way, transforming the

Oyz -plane into the Oxz -plane can be done in several ways. It can be done by a reflection with respect to the plane $x = y$. The transformation can also be obtained by a rotation around the z -axis through an angle of $\pi/2$ or by a rotation through an angle of π around the axis defined by $z = 0$ and $x = y$. This can be translated *mutatis mutandis* into the transformations between the Oxy plane (defining σ_z) and the Oyz plane (defining σ_x) or the Oxz plane (defining σ_y).

Using this idea, we will now show that we can make a similarity transformation between two Pauli matrices in several ways, by using reflection or rotation matrices for the transfer matrix. These different similarity transformations correspond conceptually to the different ways we can relate two congruent triangles or two planes in the examples, because Pauli matrices correspond to reflections defined by planes.

The Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ represent reflections with respect to planes defined by the normals $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. They square to unity. The matrices $\sigma_x, \sigma_y, \sigma_z$ can be diagonalized as follows:

$$\begin{aligned}\sigma_z &= \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} = \begin{bmatrix} +1 & \\ & +1 \end{bmatrix} \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} \begin{bmatrix} +1 & \\ & +1 \end{bmatrix} \\ \sigma_x &= \begin{bmatrix} & +1 \\ +1 & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix} \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} \begin{bmatrix} +1 & +1 \\ -1 & +1 \end{bmatrix} \frac{1}{\sqrt{2}} \\ \sigma_y &= \begin{bmatrix} & -i \\ +i & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 & +i \\ +i & +1 \end{bmatrix} \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} \begin{bmatrix} +1 & -i \\ -i & +1 \end{bmatrix} \frac{1}{\sqrt{2}}.\end{aligned}\quad (12)$$

We have thus $\sigma_z = \mathbf{Z}\mathbf{\Lambda}\mathbf{Z}^{-1}$, $\sigma_x = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$ and $\sigma_y = \mathbf{Y}\mathbf{\Lambda}\mathbf{Y}^{-1}$. The diagonalization of σ_z is trivial because it is already diagonal. We note that $\det \mathbf{X} = \det \mathbf{Y} = 1$ such that they are rotation matrices. For \mathbf{X} this is a rotation about the y -axis through an angle $\pi/2$. For \mathbf{Y} it is a rotation about the x -axis through an angle $-\pi/2$. One may note that for all three Pauli matrices the diagonal matrix $\mathbf{\Lambda}$ is σ_z itself. We see thus that we can transform between them in the following way:

$$\begin{aligned}\sigma_z &\rightarrow \sigma_x \quad \text{by:} \quad \sigma_z \rightarrow \sigma_x = \mathbf{X}\sigma_z\mathbf{X}^{-1} \\ \sigma_z &\rightarrow \sigma_y \quad \text{by:} \quad \sigma_z \rightarrow \sigma_y = \mathbf{Y}\sigma_z\mathbf{Y}^{-1} \\ \sigma_x &\rightarrow \sigma_z \quad \text{by:} \quad \sigma_x \rightarrow \sigma_z = \mathbf{X}^{-1}\sigma_x\mathbf{X} \\ \sigma_y &\rightarrow \sigma_z \quad \text{by:} \quad \sigma_y \rightarrow \sigma_z = \mathbf{Y}^{-1}\sigma_y\mathbf{Y}\end{aligned}\quad (13)$$

As we mentioned, we can also make the transformation (and therefore the diagonalization) using reflections. It looks then this way:

$$\begin{aligned}\sigma_x &= \begin{bmatrix} & +1 \\ +1 & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} \begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \\ \sigma_y &= \begin{bmatrix} & -i \\ +i & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} +1 & +i \\ -i & -1 \end{bmatrix} \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} \begin{bmatrix} +1 & +i \\ -i & -1 \end{bmatrix} \frac{1}{\sqrt{2}}.\end{aligned}\quad (14)$$

The transfer matrices \mathbf{X}' for σ_x and \mathbf{Y}' for σ_y are now reflection matrices. They have a determinant equal to -1 . They are truly reflections and not reversals because they square to $\mathbf{1}$. The reflection matrix \mathbf{X}' that diagonalizes σ_x is the reflection $\frac{1}{\sqrt{2}}(\sigma_z - \sigma_x)$, i.e. the reflection with normal $\frac{1}{\sqrt{2}}(-1, 0, 1)$. The matrix \mathbf{Y}' that diagonalizes σ_y is the reflection $\frac{1}{\sqrt{2}}(\sigma_z - \sigma_y)$, i.e. the reflection with normal $\frac{1}{\sqrt{2}}(0, -1, 1)$. The two types of transfer matrices correspond to the two possible types of operations relating the two triangles we mentioned above: a reflection or a rotation (here through an angle of $\pi/2$). It is the selection of the transfer matrix in the Eqs. 12 and 14 which determines then if we obtain the in-plane reflection by a true reflection or by a rotation through an angle $\pi/2$. We can easily flip the sign of the determinant of the transfer matrix by fiddling with the signs of the eigenvectors. That is the reason why we can select both a rotation and a reflection matrix to play the rôle of the transfer matrix.

Finally, we mentioned that we can also relate the triangles by rotations through an angle π . This gives rise to a third way to diagonalize σ_x and σ_y . The transformations are as then follows:

$$\begin{aligned}\sigma_x &= \begin{bmatrix} & +1 \\ +1 & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -i \\ -i & +i \end{bmatrix} \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} \begin{bmatrix} +i & +i \\ +i & -i \end{bmatrix} \frac{1}{\sqrt{2}} \\ \sigma_y &= \begin{bmatrix} & -i \\ +i & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 \\ +1 & +i \end{bmatrix} \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} \begin{bmatrix} +i & +1 \\ -1 & -i \end{bmatrix} \frac{1}{\sqrt{2}}.\end{aligned}\quad (15)$$

For σ_x the rotation is around $\frac{1}{\sqrt{2}}(1, 0, 1)$, for σ_y the rotation is around $\frac{1}{\sqrt{2}}(0, 1, 1)$.

The Eqs. 12, 14, 15 are all telling a geometrical truth. But in general people are not aware of this and they diagonalize casually. But what is at stake here are not two independent isolated diagonalization procedures but a simultaneous diagonalization of the two matrices. What you are address in the GHZ experiment is a configuration which is extended in space. You are treating the configuration, not the isolated elements. You want to carry out a global consistency check for the possible hidden variables if they exist. It must be clear that you must carry out all your calculations in a self-consistent way. You cannot claim simultaneously that a given element of the optics is turning the linear polarization and another identical element reflecting the linear polarization. That is inconsistent, it is forcibly twisting the truth and you will be confronted with your doublespeak when you are checking the global consistency in the end and discover the internal contradiction you introduced yourself in a different guise. You may scream blue murder, but for one thing, classical mechanics is not to blame for what you will discover, you are! You are only looking into a mirror and all you might see is about you, not about classical mechanics.

7.2 Two mathematical errors

If one does the diagonalization carelessly we can run into problems and make calculations that are geometrically meaningless. Of course, it seems as though errors in the diagonalization procedures could happen almost subliminally because there are no known red flags associated with diagonalization. People are not forcedly aware of the fact that the choice of the transfer matrices has geometrical implications. In [10], the following matrices are used to diagonalize σ_x and σ_y :

$$\mathbf{T}_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & +1 \\ 1 & -1 \end{bmatrix}, \det \mathbf{T}_x = -1, \quad \text{and:} \quad \mathbf{T}_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & +i \\ 1 & -i \end{bmatrix}, \det \mathbf{T}_y = -i. \quad (16)$$

Two inconsistencies have been introduced into this algebraic formulation whilst no alarm bells were ringing. The matrix \mathbf{T}_x corresponds to a reflection with normal $\frac{1}{\sqrt{2}}(1, 0, 1)$. But with its determinant -1 the matrix \mathbf{T}_y does not correspond to a group element, and is therefore totally meaningless. It is further totally illogical to diagonalize σ_x and σ_y , which are geometrical operators of the same type, with transfer matrices of entirely different types (as revealed by their determinants), especially in a context where the whole issue at stake is the consistency of a global scheme.¹ Let us develop this latter point first. The sting is here that we should not address an extended global problem with interconnections by using a tool kit of purely disconnected local techniques. Instead of several independent *individual diagonalizations* we must at least apply one single *global diagonalization scheme*, by applying the same geometrical type of transfer matrix, i.e. with the same determinant. We could qualify this principle as not readily suspected.

But let us now discuss the issue of what happened with $\det \mathbf{T}_y = -i$, which should be in everyone's cross hairs. The matrix \mathbf{T}_y is supposed to contain the eigenvectors of σ_y . But σ_y has the following action on these eigenvectors:

$$\begin{bmatrix} & -i \\ +i & \end{bmatrix} \begin{bmatrix} 1 & +i \\ 1 & -i \end{bmatrix} = \begin{bmatrix} -i & -1 \\ +i & -1 \end{bmatrix}, \quad \text{that is:} \quad \begin{cases} \mathbf{w}_1 & \rightarrow & -\mathbf{w}_2 \\ \mathbf{w}_2 & \rightarrow & -\mathbf{w}_1 \end{cases}, \quad (17)$$

instead of:

$$\begin{bmatrix} & -i \\ +i & \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ i & +i \end{bmatrix}, \quad \text{that is:} \quad \begin{cases} \mathbf{v}_1 & \rightarrow & +\mathbf{v}_1 \\ \mathbf{v}_2 & \rightarrow & -\mathbf{v}_2 \end{cases}. \quad (18)$$

Cetera desint.

7.3 The correct calculation

We will show that the calculations in [10] are wrong, due to these inconsistent choices for the matrices used in the diagonalization. We will use the following order in the calculations of the tensor products:

$$\boldsymbol{\xi} \otimes \boldsymbol{\eta} = \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \otimes \begin{bmatrix} \eta_0 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} \xi_0 \boldsymbol{\eta} \\ \xi_1 \boldsymbol{\eta} \end{bmatrix} \quad (19)$$

¹ One of the global aspects that cannot be noticed locally is the topology of the group manifold. For any element $\mathbf{G} \in \text{SU}(2)$, also $-\mathbf{G} \in \text{SU}(2)$, where \mathbf{G} and $-\mathbf{G}$ correspond to the same rotation of $\text{SO}(3)$. It is therefore that one must rotate a wave function through an angle of 4π , rather than 2π to get back to its initial value. Such a double covering is perhaps an astounding mathematical truth but it can be perfectly explained, see e.g. [1], p. 12, Remark 10. The explanation given for it in [15], p. 1146, Fig. 41.6, gives the misleading impression that this inversion of the sign after a rotation through an angle 2π is related to a kind of entanglement. It seems to link a state by wires to the rest of the Universe. But the explanation given in [1] shows that the truth is much more mundane and that there are no such concepts at stake. The change of sign is just like the change of sign of a normal vector after making a round trip on a Moebius ring (see e.g. [16], p. 84, Fig. 3.5).

We will use the same order for the 2×2 SU(2) matrices. The tensor products of column matrices will therefore be of the following form:

$$\begin{bmatrix} a_+ \\ a_- \end{bmatrix} \otimes \begin{bmatrix} b_+ \\ b_- \end{bmatrix} \otimes \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} a_+b_+c_+ \\ a_+b_+c_- \\ a_+b_-c_+ \\ a_+b_-c_- \\ a_-b_+c_+ \\ a_-b_+c_- \\ a_-b_-c_+ \\ a_-b_-c_- \end{bmatrix} \quad (20)$$

We can then substitute in this identity (a, b, c) by (x, x, x) , (x, y, y) , (z, z, z) , etc... to figure out the indices for the entries of a column matrix. Here $[z_+, z_-]^\top$ stands then for $[\uparrow, \downarrow]^\top$. We use the transfer matrices from Eq. 12. to calculate the components of tensor products of $[x_+, x_-]^\top$ and $[y_+, y_-]^\top$ in their basis after the action of the operators on the Bell state $\frac{1}{\sqrt{2}}(\uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow)$, e.g.:

$$\begin{bmatrix} x_+y_+y_+ \\ x_+y_+y_- \\ x_+y_-y_+ \\ x_+y_-y_- \\ x_-y_+y_+ \\ x_-y_+y_- \\ x_-y_-y_+ \\ x_-y_-y_- \end{bmatrix} = \mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Y} \begin{bmatrix} z_+z_+z_+ \\ z_+z_+z_- \\ z_+z_-y_+ \\ z_+z_-z_- \\ z_-z_+z_+ \\ z_-z_+z_- \\ z_-z_-z_+ \\ z_-z_-z_- \end{bmatrix}, \quad \text{where:} \quad \begin{bmatrix} z_+z_+z_+ \\ z_+z_+z_- \\ z_+z_-y_+ \\ z_+z_-z_- \\ z_-z_+z_+ \\ z_-z_+z_- \\ z_-z_-z_+ \\ z_-z_-z_- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (21)$$

We have four measurement protocols:

$$\begin{aligned} \mathbf{P} &= \sigma_x \otimes \sigma_y \otimes \sigma_y, \\ \mathbf{Q} &= \sigma_y \otimes \sigma_x \otimes \sigma_y, \\ \mathbf{R} &= \sigma_y \otimes \sigma_y \otimes \sigma_x, \\ \mathbf{S} &= \sigma_x \otimes \sigma_x \otimes \sigma_x. \end{aligned} \quad (22)$$

The “entangled” state $\frac{1}{\sqrt{2}}(\uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow)$ consists of three photons which are emitted simultaneously and have been prepared to be all in exactly the same spin state. In other words, it has been made sure that there are only two possibilities: either they are simultaneously all up, or they are simultaneously all down, a clever experimental *tour de force*. The indices only serve to explain the different operations the three photons are subjected to in the optics. Some people would note e.g. \mathbf{P} as $\sigma_x \sigma_y \sigma_y$ but this is somewhat confusing because one could read it as three operations on one particle in succession. What really happens is three distinct measurements taking place simultaneously on the three photons who are in the same state. The result obtained is then used to derive the contradiction reported in [10]. Using the diagonalization scheme of Eq. 12, the four matrices which express the outcome of the experiments are then:

$$\text{set-up } \mathbf{P}: \quad \mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Y} = \frac{1}{2\sqrt{2}} \begin{bmatrix} +1 & +i & +i & -1 & -1 & -i & -i & +1 \\ +i & +1 & -1 & +i & -i & -1 & +1 & -i \\ +i & -1 & +1 & +i & -i & +1 & -1 & -i \\ -1 & +i & +i & +1 & +1 & -i & -i & -1 \\ +1 & +i & +i & -1 & +1 & +i & +i & -1 \\ +i & +1 & -1 & +i & +i & +1 & -1 & +i \\ +i & -1 & +1 & +i & +i & -1 & +1 & +i \\ -1 & +i & +i & +1 & -1 & +i & +i & +1 \end{bmatrix} \quad (23)$$

$$\text{set-up } \mathbf{Q}: \quad \mathbf{Y} \otimes \mathbf{X} \otimes \mathbf{Y} = \frac{1}{2\sqrt{2}} \begin{bmatrix} +1 & +i & -1 & -i & +i & -1 & -i & +1 \\ +i & +1 & -i & -1 & -1 & +i & +1 & -i \\ +1 & +i & +1 & +i & +i & -1 & +i & -1 \\ +i & +1 & +i & +1 & -1 & +i & -1 & +i \\ +i & -1 & -i & +1 & +1 & +i & -1 & -i \\ -1 & +i & +1 & -i & +i & +1 & -i & -1 \\ +i & -1 & +i & -1 & +1 & +i & +1 & +i \\ -1 & +i & -1 & +i & +i & +1 & +i & +1 \end{bmatrix} \quad (24)$$

$$\text{set-up } \mathbf{R}: \quad \mathbf{Y} \otimes \mathbf{Y} \otimes \mathbf{X} = \frac{1}{2\sqrt{2}} \begin{bmatrix} +1 & -1 & +i & -i & +i & -i & -1 & +1 \\ +1 & +1 & +i & +i & +i & +i & -1 & -1 \\ +i & -i & +1 & -1 & -1 & +1 & +i & -i \\ +i & +i & +1 & +1 & -1 & -1 & +i & +i \\ +i & -i & -1 & +1 & +1 & -1 & +i & -i \\ +i & +i & -1 & -1 & +1 & +1 & +i & +i \\ -1 & +1 & +i & -i & +i & -i & +1 & -1 \\ -1 & -1 & +i & +i & +i & +i & +1 & +1 \end{bmatrix}. \quad (25)$$

$$\text{set-up } \mathbf{S}: \quad \mathbf{X} \otimes \mathbf{X} \otimes \mathbf{X} = \frac{1}{2\sqrt{2}} \begin{bmatrix} +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\ +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \end{bmatrix}. \quad (26)$$

These four matrices describe the action on any state expressed in the basis $\sigma_z \otimes \sigma_z \otimes \sigma_z$. The Bell state has only two components in this basis, such that we obtain:

$$\begin{aligned} [\mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Y}][\uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow] &= \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ i \\ i \\ 0 \end{bmatrix}, & [\mathbf{Y} \otimes \mathbf{X} \otimes \mathbf{Y}][\uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow] &= \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ i \\ 0 \\ -1 \\ i \\ 0 \end{bmatrix}, \\ [\mathbf{Y} \otimes \mathbf{Y} \otimes \mathbf{X}][\uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow] &= \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ i \\ 0 \\ i \\ -1 \\ 0 \end{bmatrix}, & [\mathbf{X} \otimes \mathbf{X} \otimes \mathbf{X}][\uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow] &= \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned} \quad (27)$$

The three columns corresponding to the tensor products $\mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Y}$, $\mathbf{Y} \otimes \mathbf{X} \otimes \mathbf{Y}$, and $\mathbf{Y} \otimes \mathbf{Y} \otimes \mathbf{X}$ are representing just the same states after a relabelling induced by changing the order of the terms in the calculation of the tensor products. Physically, they represent the three possible permutations of the combined positions of the two mirrors represented by σ_y , or just relabelling the tops. We see that these columns do indeed contain the same coefficients in a different order. Following the strategy of [10] we can calculate the products of the components inside one of these single columns, because they are a global indicator of the coefficients that occur in a product. Let us note these products as $(XY Y)$, (YXY) , (YYX) , (XXX) . The products obtained for the three columns of the type $XY Y$ all yield 1. And the global product $(XY Y)(YXY)(YYX)$ of these three products yields the same result as (XXX) . Both $(XY Y)(YXY)(YYX)$ and (XXX) contain the same number of X terms. We therefore do not obtain an inconsistency of the type signalled in [10]. In fact, it is all geometry, such that there is nothing that can go wrong as long as we do things correctly. A hypothetical inconsistency would not oppose QM to classical mechanics, but classical mechanics to classical mechanics. We have performed this consistency check for the three diagonalization schemes Eq. 12, 14 and 15.

We may further note that the Pauli matrices are reflection operators. Therefore we must transform them according to a scheme of the type:

$$\sigma_z \otimes \sigma_z \otimes \sigma_z \rightarrow [\mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Y}][\sigma_z \otimes \sigma_z \otimes \sigma_z][\mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Y}]^{-1}, \quad (28)$$

if we want that the end result is again a reflection operator. Such operations cannot be carried out on spinors which can only be acted on from the left. To make the similarity transformation one must then use the procedure explained

in the diagram of [1], p.38, Eq. 84, i.e. reconstructing the square matrix, perform the similarity transformation and after that writing the result in spinor form again. When we make calculations:

$$\sigma_z \otimes \sigma_z \otimes \sigma_z \rightarrow [\mathbf{X} \otimes \mathbf{Y} \otimes \mathbf{Y}][\sigma_z \otimes \sigma_z \otimes \sigma_z], \quad (29)$$

we transform the reflections into reversals when we choose \mathbf{X} and \mathbf{Y} to be rotations, and into rotations when we choose \mathbf{X} and \mathbf{Y} to be reflections. All this is geometrically meaningful, but it may not correspond to the physical reality of what happens to the photons. The physical reality corresponds to Eq. 28 rather than Eq. 29, due to the fact that reflections are transformed “quadratically”, according to $\mathbf{A} \rightarrow \mathbf{GAG}^{-1}$ rather than linearly according to $\mathbf{A} \rightarrow \mathbf{GA}$. Eq. 28 is just part of the calculation needed to express how the eigenvector equation $[\sigma_z \otimes \sigma_z \otimes \sigma_z][\frac{1}{\sqrt{2}}(\uparrow \otimes \uparrow \otimes \uparrow + \downarrow \otimes \downarrow \otimes \downarrow)] = [\frac{1}{\sqrt{2}}(\uparrow \otimes \uparrow \otimes \uparrow + \downarrow \otimes \downarrow \otimes \downarrow)]$ for the Bell state will be transformed under changes of basis from σ_z to σ_x or σ_y .

Using the linear, “halved” formalism $\mathbf{A} \rightarrow \mathbf{GA}$, thereby reasoning on reflections as on rotations, introduces reversals (when \mathbf{G} is a rotation) or rotations (when \mathbf{G} is a reflection). These group elements have no bearing on the final result \mathbf{GAG}^{-1} . As explained, different diagonalizations will lead to different “intermediate states” \mathbf{GA} in the “halved” formalism. In the analogy, these “intermediate states” reveal, which way the triangle traveled before it landed on its mirror image to enact the identification. That is, they tell if we mapped the triangles one onto another by a reflection or by a rotation. It is the final result \mathbf{GAG}^{-1} , expressing that the triangles are congruent, which counts, not the way the triangle traveled.

One must actually select the operations that correspond to the physical reality of what really happens to $\frac{1}{\sqrt{2}}(\uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow)$ under the transformation. But the physical reality corresponds to Eq. 28 instead of 29. In any case, using operators \mathbf{X} and \mathbf{Y} that are not of the same type is with certainty in blatant conflict with any physical reality in the “halved” formalism because σ_x and σ_y are supposed to model the workings of two optical elements of exactly the same type in mathematical language. And using transfer matrices that are not group elements has no geometrical reality at all, let alone that it would correspond to a physical reality.

The Bell state is an eigenvector of $\sigma_z \otimes \sigma_z \otimes \sigma_z$ and as such is a set, which also transpires from its algebraic expression. The scenario of a set might at face value be contradicted by what we observe for a single photon, which is why we point out in the Appendix that a photon might have the symmetry of a set in its own right.

It cannot be stressed enough that Eqs. 23-27 are not the bedrock of our argument. The crux of our argument is that all results of calculations of the type shown in Eqs. 23-27 are mere properties of the rotation group. As such they are valid independent of their application to physics. It is therefore totally wrong to consider them at any stage as physics. They are just geometry. E.g. we can apply the geometrical transformations on the spinors $\zeta_1, \zeta_2, \zeta_3$ which are used to describe the three photons, all at the origin O of the reference frame or in three separate places A, B, C that are far apart like is done in the experiment. The result will of course be the same. Therefore, the fact that the state is extended cannot play the rôle it is given in [10]. It is consequently just plain wrong to attribute any result of such calculations to some magical quantum effect of entanglement defeating classical realism. In other words, if the calculation in [10] had been completely error-free, the surprising result obtained (if possible) would still not have been the signature of some spooky action at a distance in physics. It would have been a not readily suspected property of the rotation group. It would have required perhaps some further investigation to figure out what it meant geometrically. We already pointed out in Subsection 6.3 that this is the core of our argument. The only aim of the *Gedankenexperiment* was exactly to get this message across, by providing the reader with the geometrical meaning of the operators. E.g. also the fact that the wave function changes sign under a rotation through an angle of 2π is in this respect not some stunning physics but just geometry. As explained in Footnote 1 it is related to the topology of the manifold of the rotation group. We were therefore *a priori* unfazed by the *reductio ad absurdum* proposed by the authors of [10].

Needless to say that this was very difficult to figure out by reasoning logically. There were too many details about the calculations lacking. Also some coincidences, which occur in [10] due to the choice of the meaningless matrix \mathbf{T}_y , disappear in a correct calculation. It is only on consulting [17] that the penny dropped. Of course, the smoking guns were $\det \mathbf{T}_y = -i$, the use of the halved formalism in Eq. 29 and the obvious analogy with the *Gedankenexperiment*.

8 Conclusion

As we anticipated, the GHZ argument had to be wrong because the whole formalism of $SU(2)$ which is being used is mere Euclidean geometry. Whatever riddle or paradox we may ever run into some day in using $SU(2)$, it will always be only geometry and not physics, and therefore will never have anything to do with entanglement. Just like now, the argument will then not be a proof for the existence of non-locality.

Our analysis shows that the calculation used in [10] in order to claim that the experiment would constitute a proof for the existence of entanglement is fatally flawed. The Bell state is extended, like Euclidean geometry is, but does not imply any form of “spooky action at the distance”. The contradiction obtained only pinpoints that an inappropriate

use has been made of the spinor algebra, which requires a completely different way of thinking than we have become used to in vector algebra.

With the results we analyzed up to now the theory of relativity remains unscathed and QM appears to be just a part of it. This is a relief because it is hard to see by what the theory of relativity should be replaced if it were wrong. It has been ever so successful. Even Einstein's gravitational-wave predictions were confirmed by experiment a hundred years later.

We think that the fact that our results are in agreement with the tenet that QM is pure group theory is very important because it permits to understand at least a part of QM and check the calculations in a dialogue between the algebra and the geometry like in algebraic geometry. It permits to spot errors in an argument as the present example clearly illustrates. It solves also a lot of conceptual problems that were reputed unsolvable and therefore were dubbed quantum mysteries. These conceptual problems were generated by replacing the natural geometrical meaning of the algebra by *ad hoc* parallel interpretations, most of the time as a result of misreading the algebraic calculations on a manifold as a calculus on a vector space.

Appendix - Group theory for photons

The polarization dynamics of the photon must be described by $SU(2) \otimes SU(2)$. In fact, the representation $SU(2)$ can be used to describe the dynamics of an electron with spin $1/2$, but photons have spin 1. The representations of the rotation group corresponding to spin $n \in \mathbb{N}$ are obtained from rank- $2n$ tensor products of 2×2 rotation matrices $\mathbf{R} \in SU(2)$ and 2×1 spinor matrices ξ according to:

$$\bigotimes_{j=1}^{2n} \mathbf{R}_j, \quad \text{where: } (\forall j \in [1, 2n] \cap \mathbb{N}) (\mathbf{R}_j = \mathbf{R}), \quad \text{and: } \bigotimes_{j=1}^{2n} \xi_j, \quad \text{where: } (\forall j \in [1, 2n] \cap \mathbb{N}) (\xi_j = \xi). \quad (30)$$

They are tensor products of $2n$ identical copies. Hence for spin 1 we need $SU(2) \otimes SU(2)$. It is the rank-2 tensor product which doubles the spin from $1/2$ to 1. The tensor product $SU(2) \otimes SU(2)$ yields a 4×4 matrix formalism. But one line and one column in this formalism are redundant duplicates due to the fact that we are constructing tensor products of identical copies. After eliminating this line and this column one obtains the representation $SO(3)$. To describe photons we only need the subgroup $SO(2) \subset SO(3)$. That is because photons travel at the speed of light with the effect that the polarization dynamics are restricted to a plane perpendicular to the direction of the motion. Of course that plane changes direction when the photon is reflected. Despite its spin 1 a photon has only two substates rather than three and this can be understood in terms of the restriction of $SO(3)$ to the two-dimensional subgroup $SO(2)$. In the product representation $SO(2)$ all angles are doubled with respect to those that occur in $SU(2)$, just like in $SO(3)$ all angles are doubled with respect to $SU(2)$. Three-dimensional rotation matrices are expressed in the rotation angle φ , not in $\varphi/2$ as in the Rodrigues formula. The resulting formalism is isomorphic to the restriction of $SU(2)$ to the Oxy plane, but with doubled angles, such that the group $SU(2)$ can be used to describe the spin dynamics in the polarization plane. The group theory describes this way the photon as a set of two spin- $1/2$ particles. We do not need to take this literally, it just describes some symmetry, but it is a convenient language to describe that symmetry. We have stated that only a set of two circularly polarized particles could be linearly polarized but in our construction and in the language used, the two particles are the two spin- $1/2$ particles which make up a photon. In this language, the photon is in its interiors itself a mini Bell state of two $1/2$ particles.

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