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Projections of martingales in enlargements of filtrations under Jacod's equivalence hypothesis for marked point processes*

Pavel V. Gapeev[†] Monique Jeanblanc[‡] Dongli Wu[§]

Abstract

We consider the initial and progressive enlargements of a filtration generated by a marked point process (called the reference filtration) with a strictly positive random time. We assume Jacod's equivalence hypothesis, that is, the existence of a strictly positive conditional density for the random time with respect to the reference filtration. Then, starting with the predictable integral representation of a martingale in the initially enlarged reference filtration, we derive explicit expressions for the coefficients which appear in the predictable integral representations for the optional projections of the martingale on the progressively enlarged filtration and on the reference filtration. We also provide similar results for the optional projection of a martingale in the progressively enlarged filtration on the reference filtration. [This paper represents the continuation of research contained in our previous paper in EJP \[11\].](#)

1 Introduction

In this paper, we consider the initial (resp. progressive) enlargements of a filtration \mathbb{F} (called hereafter the reference filtration) with a strictly positive random variable τ (called hereafter the random time), denoted by $\mathbb{F}^{(\tau)}$ (resp. \mathbb{G}). We study the case in which \mathbb{F} is generated by a marked point process (MPP). We assume that the law of τ has no atoms and that Jacod's equivalence hypothesis introduced in [3, 12] holds (see Section 3 for details). We prove that these hypotheses imply that the weak predictable representation property holds in the filtration

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$\mathbb{F}^{(\tau)}$ and (adding a pure jump martingale) in the filtration \mathbb{G} . We study the relationship between the representation of martingales in the initially (resp. progressively) enlarged filtration and the various optional projections. The paper is an extension of our previous paper [11] for the case of models driven by marked point processes. We refer the reader to the monograph [1] for results on enlargements of filtrations. Our results can be useful to compare the optimal strategies of investors having different information flows, and to investigate optimal stopping problems in different filtrations.

The reason why we are working with marked point processes is that a marked point process in \mathbb{F} remains a marked point processes, in particular a semi-martingale, in any enlargement of \mathbb{F} with possibly a different compensator.

The paper is organised as follows. In Section 2, we recall standard definitions of projections as well as other results of stochastic analysis that we use in the paper. In Section 3, we give some basic definitions and results related to the initial and progressive enlargements of a filtration \mathbb{F} generated by a marked point process (MPP) with a random time τ , denoted by $\mathbb{F}^{(\tau)}$ and \mathbb{G} , respectively, under Jacod's equivalence hypothesis. In Section 4, we recall that the weak predictable representation property holds in the reference filtration with respect to the compensated random measure and prove that the weak predictable representation property holds with respect to an explicit martingale and a compensated random measure in the enlargements of filtration involved. In Section 5, we consider the optional projections of an $\mathbb{F}^{(\tau)}$ -martingale on the filtrations \mathbb{G} and \mathbb{F} . We derive explicit expressions for the coefficients in the integral representations of these optional projections in terms of the original $\mathbb{F}^{(\tau)}$ -martingale and the components in its representation as a stochastic integral and give analogous results in the case of \mathbb{F} -optional projections of a \mathbb{G} -martingale. In Section 6, we consider the optional projections of a positive $\mathbb{F}^{(\tau)}$ -martingale on \mathbb{G} and \mathbb{F} and the \mathbb{F} -optional projection of a positive \mathbb{G} -martingale. We describe the set of equivalent martingale measures in the associated extension of the exponential model driven by an MPP and enhanced with the random time τ . In particular, we show that the set of equivalent martingale measures in the model with the progressively enlarged filtration \mathbb{G} is essentially larger than the one obtained by means of the optional projections on \mathbb{G} of the Radon-Nikodym densities in the model with the initially enlarged filtration $\mathbb{F}^{(\tau)}$. Some technical proofs are presented in Appendix.

Note that, with any difficulty, one can study a model driven by a MMP and an independent Brownian motion, using the result in [11]. It would simply lead to longer formulae.

2 Preliminary definitions and results

We denote by $\mathcal{B}(\mathbb{R}^*)$ (resp. $\mathcal{B}(\mathbb{R}_+)$) the Borel sets of \mathbb{R}^* (resp. of $\mathbb{R}_+ = [0, \infty)$) and set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

We work on a standard complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$, on which there exists a sequence $(T_n, Z_n)_{n \geq 1}$, where $(T_n)_{n \geq 1}$ is a strictly increasing sequence of finite strictly positive random variables with no accumulation point, and $(Z_n)_{n \geq 1}$ a sequence of real-valued random variables. We shall say that the sequence $N = (T_n, Z_n)_{n \geq 1}$ is a *marked point process* (MPP) on \mathbb{R} (see Def. 1.1.6 in [26], Section 1.2, pages 3-4 in [23], and Chapter VIII in [6]).

We introduce the associated random measure μ on $\Omega \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^*)$ which is defined, for

any set $A \in \mathcal{B}(\mathbb{R})$ and any $t \geq 0$, by

$$\mu(\omega; (0, t], A) = \sum_{n \geq 1} \mathbb{1}_{\{T_n(\omega) \leq t\}} \mathbb{1}_{\{Z_n(\omega) \in A\}} = N_t(A, \omega),$$

which is called the *jump measure* of the marked point process N . Note that the process $N(A) = (N_t(A))_{t \geq 0}$ is a counting process. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the natural filtration of the MPP

$$\mathcal{F}_t = \sigma(\mu((a, b], A), 0 \leq a < b \leq t, A \in \mathcal{B}(\mathbb{R}^*)), \forall t \geq 0,$$

which is a right-continuous filtration (see Proposition 3.39 in [16]). We call \mathbb{F} hereafter the *reference filtration*, and note that T_n , for $n \geq 1$, are \mathbb{F} -stopping times. We define the compensator ν of the jump measure μ with respect to \mathbb{F} as the unique random measure¹ on $\Omega \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^*)$ such that, for any $A \in \mathcal{B}(\mathbb{R}^*)$, the process

$$\nu(\omega; (0, t], A) = \int_0^t \int_A \nu(\omega; ds, dz), \forall t \geq 0, \quad (1)$$

is \mathbb{F} -predictable and the process $\bar{N}(A) = (\bar{N}_t(A))_{t \geq 0}$ given by

$$\bar{N}_t(A) = \mu((0, t], A) - \nu((0, t], A), \forall t \geq 0, \quad (2)$$

is an \mathbb{F} -martingale. We shall say that \bar{N} is the \mathbb{F} -*compensated* martingale of the marked point process N , and, by abuse of language, that ν is the compensator of N . More generally, if \mathbb{K} is a filtration larger than \mathbb{F} , we say that $\nu^{\mathbb{K}}$ is the \mathbb{K} -compensator of N if, for any $A \in \mathcal{B}(\mathbb{R}^*)$, the process

$$\mu((0, t], A) - \nu^{\mathbb{K}}((0, t], A), \forall t \geq 0, \quad (3)$$

is a \mathbb{K} -martingale, and the process $\nu^{\mathbb{K}}((0, \cdot], A)$ is \mathbb{K} -predictable.

We assume, as in Chapter VIII, Definition D5, page 236 of [6] and [28] that the compensator ν admits the representation

$$\nu(\omega; dt, dz) = \eta_t(\omega; dz) dt, \forall t \geq 0, \quad (4)$$

where $\eta(dz)$ is a transition kernel, so that $\eta(A) = (\eta_t(A) = \int_A \eta_t(dz), t \geq 0)$ is the intensity of the counting process $N(A)$.

As usual, $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{F})$) is the predictable (resp. optional) σ -algebra on \mathbb{F} . For a family of processes $\xi(z) = (\xi_t(z))_{t \geq 0}$ parameterized by $z \in \mathbb{R}$, we shall say that ξ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable, if the map $(t, \omega, z) \rightarrow \xi_t(\omega; z)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable, and we define $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable processes in a similar way.

Recall that, if ξ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process such that

$$\int_0^t \int_{\mathbb{R}^*} |\xi_s(z)| \eta_s(dz) ds < \infty, \forall t \geq 0, \quad (5)$$

¹The compensator ν is given, for any $A \in \mathcal{B}(\mathbb{R}^*)$, by

$$\nu(\omega; (0, t], A) = \int_0^t \int_A \sum_{k \geq 1} \mathbb{1}_{\{T_{k-1} < s \leq T_k\}} \frac{\mathbb{P}(T_k \in ds, Z_k \in dz | \mathcal{F}_{T_{k-1}})}{\mathbb{P}(T_k > s | \mathcal{F}_{T_{k-1}})}, \forall t \geq 0,$$

(see, e.g., Section 1.10 in [23], [14] or Chapter 11, and Section 4 in [13]).

the process $Y = (Y_t)_{t \geq 0}$ defined as

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}^*} \xi_s(z) (\mu(ds, dz) - \eta_s(dz) ds), \quad \forall t \geq 0, \quad (6)$$

is an \mathbb{F} -local martingale. Under the stronger assumption

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} |\xi_s(z)| \eta_s(dz) ds \right] < \infty, \quad \forall t \geq 0, \quad (7)$$

the process Y is an \mathbb{F} -martingale (see Chapter VIII, Corollary C4, page 235 in [6]).

Furthermore, any \mathbb{F} -martingale Y admits a representation as in (6) with ξ satisfying (5) (see Chapter VIII, Theorem T8, page 239 in [6] and Theorem 2.2 in [28]). This property is referred to as the *weak predictable representation property* (WPRP) of the marked point process N on the filtration \mathbb{F} with respect to the *compensated jump measure* $\mu - \nu$ (see also Theorem 13.19 in [13], or Th. 1.13.2 in [23], or Theorem 1.1.21 in [26]). Such a representation is essentially unique ($\mathbb{P} \times \eta_t(dz) \times dt$ -a.s.).

Let $X = (X_t)_{t \geq 0}$ be a measurable process and \mathbb{H} be a filtration satisfying the usual hypotheses of completeness and right continuity. We denote by ${}^{p,\mathbb{F}}X = ({}^{p,\mathbb{F}}X_t)_{t \geq 0}$ (resp. ${}^{o,\mathbb{F}}X = ({}^{o,\mathbb{F}}X_t)_{t \geq 0}$) its \mathbb{F} -predictable (resp. optional) projection when they exist (see Chapter V, Th. 5.1 (resp. 5.2) in [13] or Section 1.3.1, page 15 in [1]).

We introduce the *stochastic exponential* of a càdlàg \mathbb{H} -local martingale X which is the \mathbb{H} -local martingale $\mathcal{E}(X) = (\mathcal{E}(X)_t)_{t \geq 0}$ defined by

$$\mathcal{E}(X)_t = e^{X_t - X_0 - \frac{1}{2} \langle X^c, X^c \rangle_t} \prod_{0 < s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}, \quad \forall t \geq 0, \quad (8)$$

where X^c is the continuous \mathbb{H} -martingale part of X , and $\Delta X_t = X_t - X_{t-}$ (see, e.g., Chapter IX, Theorem 9.39, page 248 in [13] or Chapter I, Formula 4.64, page 59 in [17]). The process $\mathcal{E}(X)$, called the Doléans-Dade exponential of X , is the unique solution of the stochastic differential equation (see Chapter I, Formula 4.59, page 59 in [17] or Chapter IX, Theorem 9.39, page 248 in [13])

$$dZ_t = Z_{t-} dX_t, \quad Z_0 = 1.$$

The following proposition is a particular case of the result of Chapter II, Theorem 8.21, page 138 in [17], suitable for our purposes.

Proposition 2.1 *Let $M = (M_t)_{t \geq 0}$ be an \mathbb{H} -martingale and $X = 1 + (\theta \bullet M)$, where $\theta = (\theta_t)_{t \geq 0}$ is an \mathbb{H} -predictable and locally bounded process. We assume that X and X_- take their values in $(0, \infty)$. Then, the process X is a stochastic exponential martingale, that is, there exists an \mathbb{H} -predictable locally bounded process ψ such that equality $X = \mathcal{E}(\psi \bullet M)$ holds.*

A probability measure \mathbb{Q} is said to be *locally equivalent* to \mathbb{P} on the filtration \mathbb{H} if there exists a strictly positive \mathbb{H} -martingale $L = (L_t)_{t \geq 0}$ such that

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{H}_t} = L_t, \quad \forall t \geq 0.$$

The martingale L is called the *Radon-Nikodym density* of \mathbb{Q} with respect to \mathbb{P} . The "locally" terminology is needed, since, in Section 6 of our paper, as in [3], we cannot define the new probability measure \mathbb{Q} on \mathcal{H}_∞ .

3 Jacod's equivalence hypothesis

In the whole paper, we work on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ which supports a marked point process with a right-continuous and completed natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and a strictly positive random variable τ . Note that the inclusion $\mathcal{F}_\infty \subset \mathcal{G}$ holds and, in general, this inclusion is strict. We recall that any \mathbb{F} -martingale admits a càdlàg modification, see Corollary 2.48 in [13].

Hypothesis 3.1 We assume in the whole paper, as in [3] and [12], that Jacod's equivalence hypothesis holds, that is, the regular conditional distributions of τ given \mathcal{F}_t are equivalent to ρ , the unconditional law of the random variable τ :

$$\mathbb{P}(\tau \in \cdot | \mathcal{F}_t) \sim \mathbb{P}(\tau \in \cdot), \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)}.$$

In our model, this assumption implies (see Lemma 2.3 in [10]) that there exists a family of strictly positive processes $p(u) = (p_t(u))_{t \geq 0}$ such that the function $(\omega, t, u) \mapsto p_t(u; \omega)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable, and, for each $u \geq 0$, the process $p(u)$ is a càdlàg \mathbb{F} -martingale. Moreover, denoting by ρ the law of τ , for any Borel bounded function f , the following equality holds

$$\mathbb{E}[f(\tau) | \mathcal{F}_t] = \int_0^\infty f(u) p_t(u) \rho(du), \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)}. \quad (9)$$

The expression in (9) implies that the following equality holds

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \int_s^\infty p_t(u) \rho(du), \forall t, s \geq 0 \text{ (}\mathbb{P}\text{-a.s.)},$$

so that, from the assumption of strict positivity of τ , the equality

$$\int_0^\infty p_t(u) \rho(du) = 1, \text{ (}\mathbb{P}\text{-a.s.)},$$

is satisfied, and $p_0(u) = 1$, for each $u \geq 0$.

We shall call the family of \mathbb{F} -optional processes $p(u)$, for each $u \geq 0$, the \mathbb{F} -conditional density family with respect to $\rho(du)$ (or the conditional density of τ if there is no ambiguity on the filtration). \square

The following proposition is proved as a consequence of the WPRP in [25, Pro. 2.1].

Proposition 3.2 *For each $u \geq 0$, the strictly positive \mathbb{F} -martingale $p(u)$ satisfies the stochastic differential equation*

$$dp_t(u) = p_{t-}(u) \int_{\mathbb{R}^*} (f_t(u, z) - 1) (\mu(dt, dz) - \eta_t(dz) dt), \quad p_0(u) = 1, \quad (10)$$

or, equivalently, $p(u)$ admits the representation

$$p_t(u) = \exp \left(\int_0^t \int_{\mathbb{R}^*} \ln(f_s(u, z)) \mu(ds, dz) - \int_0^t \int_{\mathbb{R}^*} (f_s(u, z) - 1) \eta_s(dz) ds \right), \forall t \geq 0, \quad (11)$$

that is,

$$p_t(u) = \mathcal{E} \left(\int_{\mathbb{R}^*} (f(u, z) - 1) (\mu(\cdot, dz) - \nu(\cdot, dz)) \right) \forall t \geq 0, \quad (12)$$

for a strictly positive and $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process f satisfying

$$\int_0^t \int_{\mathbb{R}^*} |f_s(u, z) - 1| \eta_s(dz) ds < \infty, \forall t \geq 0. \quad (13)$$

Let us denote by $H = (H_t)_{t \geq 0}$ with $H_t = \mathbb{1}_{\{\tau \leq t\}}$, for all $t \geq 0$, which is called the *indicator default process* in the credit risk theory, where τ denotes the time at which a default occurs. Moreover, since H is a càdlàg process, we can introduce the \mathbb{F} -supermartingale $G = (G_t)_{t \geq 0}$ defined by $G = {}^{\circ}\mathbb{F}(1 - H)$, that is, the \mathbb{F} -optional projection of $1 - H$ satisfying the property

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t), \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)}, \quad (14)$$

which, according to the equality (9), can be represented in the form

$$G_t = \int_t^\infty p_t(u) \rho(du), \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)}. \quad (15)$$

Note that G is strictly positive and that, from the assumption of strict positivity of the random variable τ , one has $G_0 = 1$. The \mathbb{F} -supermartingale G is called the *conditional survival process* or the *Azéma supermartingale* of the random time τ .

Hypothesis 3.3 *We assume that the distribution law ρ of the strictly positive random variable τ is non-atomic.*

Remark 3.4 It is known that, under the assumption that the law ρ of the random variable τ is non-atomic and, under Jacod's equivalence hypothesis, the random time τ avoids all \mathbb{F} -stopping times (see Corollary 2.2 in [9]). This will allow us to obtain a simpler formula for the semimartingale decomposition. More precisely, under the avoidance hypothesis, the dual optional projection of H is continuous and equal to the dual predictable projection of H , denoted by H^p (see Proposition 1.48 (a), page 22 in [1]). Therefore the martingale m which appears in the general formulae of the semimartingale decomposition (see Proposition 5.30, page 116 in [1]) is equal to the martingale part of the Doob-Meyer decomposition of G , that is, one has $G = m - H^p$. In particular, the predictable projection of G is ${}^pG = {}^pm - H^p = m_- - H^p = G_-$. The fact that ρ is non-atomic implies that, for a càdlàg process X , one has

$$\int_0^t X_{s-} \rho(ds) = \int_0^t X_s \rho(ds), \forall t \geq 0.$$

4 Enlargement of filtrations and martingales

We will consider two enlarged filtrations: the initial enlargement of \mathbb{F} obtained by adding the σ -field $\sigma(\tau)$ at time 0 and denoted $\mathbb{F}^{(\tau)}$, and the progressive enlargement of \mathbb{F} obtained by progressively adding information $\sigma(\tau \wedge t)$ at time $t \geq 0$, or, in other terms, the smallest filtration \mathbb{G} containing \mathbb{F} and turning out τ into a stopping time.

The aim of the paper is to explicitly compute the components in the integral representations of the optional projections of the $\mathbb{F}^{(\tau)}$ -martingales and of the \mathbb{G} -martingales. In this section, we recall some well known results. We give the form of the $\mathbb{F}^{(\tau)}$ -semimartingale decomposition and \mathbb{G} -semimartingale decomposition of $N(A)$ defined in (2) as well as the \mathbb{G} -semimartingale decomposition of H . We underline that the martingale part $\overline{N}^{(\tau)}(A)$ of the $\mathbb{F}^{(\tau)}$ -semimartingale decomposition of $N(A)$ enjoys the $\mathbb{F}^{(\tau)}$ -predictable representation property, while the pair $(\overline{N}^{\mathbb{G}}(A), M^{\mathbb{G}})$ of the martingale parts of the \mathbb{G} -semimartingale decompositions of $N(A)$ and H enjoys the \mathbb{G} -predictable representation property, where the integral with respect to the pair is understood componentwise as in (36) below.

4.1 The initially enlarged filtration

As in the introduction, let us denote by $\mathbb{F}^{(\tau)} = (\mathcal{F}_t^{(\tau)})_{t \geq 0} = (\mathcal{F}_t \vee \sigma(\tau))_{t \geq 0}$ the initial enlargement of the filtration \mathbb{F} with the random time τ . We recall that, under Jacod's equivalence hypothesis, any \mathbb{F} -local martingale is an $\mathbb{F}^{(\tau)}$ -special semimartingale (see, e.g., Theorem 2.1 in [15] or Proposition 5.30, page 116 in [1]). Note that, according to Proposition 3.3 in [3], the filtration $\mathbb{F}^{(\tau)}$ is right-continuous.

We further denote $\mathbb{F}^{(\tau)}$ -optional processes with the superscript (τ) as in $Y^{(\tau)}$. We denote \mathbb{F} -adapted processes by capital letters as X , or lower case x , or φ , or even x^0 .

We also recall that, for any $t \geq 0$ fixed, any $\mathcal{F}_t^{(\tau)}$ -measurable random variable $Y_t^{(\tau)}$ is of the form $Y_t(\omega, \tau(\omega))$, for some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable function $(\omega, u) \mapsto Y_t(\omega, u)$ (see, e.g., Proposition 2.7, part (i) in [7]). In particular, any $\mathcal{F}_0^{(\tau)}$ -measurable random variable is a Borel function of τ . Recall that any $\mathbb{F}^{(\tau)}$ -predictable process can be represented in the form $Y_t(\omega, \tau(\omega))$, for all $t \geq 0$, where the mapping $(\omega, t, u) \mapsto Y_t(\omega, u)$ defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+$ and valued in \mathbb{R} is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable. Moreover, under Jacod's equivalence density hypothesis, any $\mathbb{F}^{(\tau)}$ -optional process $Y^{(\tau)} = (Y_t^{(\tau)})_{t \geq 0}$ can be written as $Y_t^{(\tau)} = Y_t(\tau)$, for all $t \geq 0$, where the process Y is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable (see Theorem 6.9 in [29]).

As an immediate consequence of Jacod's equivalence hypothesis, we observe that, for each $t \geq 0$, if the $\mathcal{F}_t^{(\tau)}$ -measurable random variable $Y_t(\tau)$ is integrable, then the following representation holds

$$\mathbb{E}[Y_t(\tau) | \mathcal{F}_t] = \int_0^\infty Y_t(u) p_t(u) \rho(du), \quad \forall t \geq 0 \quad (16)$$

(see, e.g., Proposition 4.18 (b), page 85 in [1]).

In the following proposition, we give the semimartingale decomposition of $\overline{N}(A)$, defined in (2), in $\mathbb{F}^{(\tau)}$.

Proposition 4.1 *For any Borel set A , the $\mathbb{F}^{(\tau)}$ -semimartingale decomposition of the \mathbb{F} -martingale $\overline{N}(A)$ is given by*

$$\overline{N}_t(A) = \overline{N}_t^{(\tau)}(A) + \int_0^t \int_A (f_s(\tau, z) - 1) \eta_s(dz) ds, \quad \forall t \geq 0, \quad (17)$$

where $\overline{N}^{(\tau)}(A)$ is an $\mathbb{F}^{(\tau)}$ -martingale and f is given in (10). In other terms, the process $N = (T_n, Z_n)_{n \geq 1}$ is a marked point process with $\mathbb{F}^{(\tau)}$ -compensator $\nu^{(\tau)}$, where we have

$$\nu^{(\tau)}(dt, dz) = f_t(\tau, z) \eta_t(dz) dt, \quad \forall t \geq 0, \quad \forall z \in \mathbb{R}^*. \quad (18)$$

Proof: From the results of initial enlargement², for any $A \in \mathcal{B}(\mathbb{R}^*)$, the process $\overline{N}^{(\tau)}(A) = (\overline{N}_t^{(\tau)}(A))_{t \geq 0}$ defined by

$$\overline{N}_t^{(\tau)}(A) = \overline{N}_t(A) - \int_0^t \frac{d\langle \overline{N}(A), p(u) \rangle_s^{\mathbb{F}}}{p_{s-}(u)} \Big|_{u=\tau}, \quad \forall t \geq 0,$$

is an $\mathbb{F}^{(\tau)}$ -martingale. In order to compute the predictable covariation, we start by computing the quadratic covariation of the processes $\overline{N}(A)$ and $p(u)$, for each $u \geq 0$. Obviously, we have

$$[\overline{N}(A), p(u)]_t = \int_0^t \int_A p_{s-}(u) (f_s(u, z) - 1) \mu(ds, dz), \quad \forall t, u \geq 0,$$

and hence, from Theorem 6.28 part 2 in [13],

$$\langle \overline{N}(A), p(u) \rangle_t^{\mathbb{F}} = \int_0^t \int_A p_{s-}(u) (f_s(u, z) - 1) \eta_s(dz) ds, \quad \forall t \geq 0, \forall z \in \mathbb{R}^*.$$

It follows that

$$\overline{N}_t(A) - \int_0^t \int_A (f_s(\tau, z) - 1) \eta_s(dz) ds, \quad \forall t \geq 0,$$

is an $\mathbb{F}^{(\tau)}$ -martingale and the $\mathbb{F}^{(\tau)}$ -compensator of N is

$$\nu^{(\tau)}(dt, dz) = f_t(\tau, z) \eta_t(dz) dt, \quad \forall t \geq 0, \forall z \in \mathbb{R}^*.$$

This completes the proof. □

Note that Jacod's equivalence hypothesis allows us to prove the stability of weak predictable representation property in the enlargement of filtration by means of the following lemma (see [12] or Theorem 4.37, page 94 in [1]).

Lemma 4.2 *Let \mathbb{P}^* be the probability on $\mathbb{F}^{(\tau)}$ defined by means of*

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \Big|_{\mathcal{F}_t^{(\tau)}} = \frac{1}{p_t(\tau)}, \quad \forall t \geq 0.$$

Then, \mathbb{F} and τ are independent under \mathbb{P}^ , as well as $\mathbb{P}^*|_{\mathcal{F}_t} = \mathbb{P}^*|_{\mathcal{F}_t}$, for all $t \geq 0$, and $\mathbb{P}^*|_{\sigma(\tau)} = \mathbb{P}|_{\sigma(\tau)}$. In particular, immersion holds under \mathbb{P}^* , that is, any $(\mathbb{P}^*, \mathbb{F})$ -martingale is a $(\mathbb{P}^*, \mathbb{G})$ -martingale.*

²One applies Theorem 2.1 in [15] which states that, under Jacod's hypothesis, for any \mathbb{F} -martingale $X = (X_t)_{t \geq 0}$, the process $X(\tau) = (X_t(\tau))_{t \geq 0}$ defined by

$$X_t(\tau) = X_t - \int_0^t \frac{d\langle X, p(u) \rangle_s^{\mathbb{F}}}{p_{s-}(u)} \Big|_{u=\tau}, \quad \forall t \geq 0, \tag{19}$$

is an $\mathbb{F}^{(\tau)}$ -martingale.

Proposition 4.3 Each $(\mathbb{P}, \mathbb{F}^{(\tau)})$ -martingale $Y(\tau) = (Y_t(\tau))_{t \geq 0}$ admits a representation of the form

$$Y_t(\tau) = Y_0(\tau) + \int_0^t \int_{\mathbb{R}^*} \psi_s(\tau, z) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)), \forall t \geq 0, \quad (20)$$

for some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R})$ -measurable process ψ satisfying

$$\int_0^t \int_{\mathbb{R}^*} |\psi_s(\tau, z)| \nu^{(\tau)}(ds, dz) < \infty, \forall t \geq 0, \quad (21)$$

where $\nu^{(\tau)}$ is defined in (18).

Proof: Note that, under the probability measure \mathbb{P}^* , the conditional density of the random variable τ is equal to ρ . Hence, we conclude, applying Proposition 2.1 in [7] to the probability \mathbb{P}^* that any $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$ -martingale $Y^{*,(\tau)}$ is of the form $Y_t^{*,(\tau)} = Y_t^*(\tau)$, where $Y^*(u)$, for each $u \geq 0$, is a $(\mathbb{P}^*, \mathbb{F})$ -martingale, hence a (\mathbb{P}, \mathbb{F}) -martingale, which is a stochastic integral with respect to the (\mathbb{P}, \mathbb{F}) (or equivalently $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$)-compensated jump measure $\mu - \nu$. Observe that, for each $u \geq 0$, we have

$$Y_t^*(u) = Y_0^*(u) + \int_0^t \int_{\mathbb{R}^*} \psi_s^*(u, z) (\mu(ds, dz) - \eta_s(dz) ds), \forall t \geq 0,$$

with ψ^* being $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R})$ -measurable and satisfying (21), and thus

$$Y_t^*(\tau) = Y_0^*(\tau) + \int_0^t \int_{\mathbb{R}^*} \psi_s^*(\tau, z) (\mu(ds, dz) - \eta_s(dz) ds), \forall t \geq 0,$$

and WPRP holds for $\mathbb{F}^{(\tau)}$ under \mathbb{P}^* . Since WPRP is stable by equivalent change of probability measures (see, e.g., Chapter 13, Th. 13.22 in [13]), it follows that the weak predictable representation property holds for $\mathbb{F}^{(\tau)}$ under \mathbb{P} with respect to $\mu - \nu^{(\tau)}$. \square

As a particular case, we can represent all strictly positive $\mathbb{F}^{(\tau)}$ -local martingales:

Proposition 4.4 Each strictly positive $\mathbb{F}^{(\tau)}$ -local martingale $L(\tau) = (L_t(\tau))_{t \geq 0}$ can be represented as

$$L_t(\tau) = L_0(\tau) + \int_0^t L_{s-}(\tau) \int_{\mathbb{R}^*} (\Theta_s(\tau, z) - 1) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)), \forall t \geq 0, \quad (22)$$

or, equivalently, $L(\tau)$ is given by

$$L_t(\tau) = L_0(\tau) \exp \left(\int_0^t \int_{\mathbb{R}^*} \ln(\Theta_s(\tau, z)) \mu(ds, dz) - \int_0^t \int_{\mathbb{R}^*} (\Theta_s(\tau, z) - 1) \nu^{(\tau)}(ds, dz) \right), \forall t \geq 0, \quad (23)$$

that is,

$$L_t(\tau) = L_0(\tau) \mathcal{E} \left(\int_{\mathbb{R}^*} (\Theta(\tau, z) - 1) (\mu(\cdot, dz) - \nu^{(\tau)}(\cdot, dz)) \right)_t, \forall t \geq 0, \quad (24)$$

for a strictly positive and $\mathcal{P}(\mathbb{F}^{(\tau)}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process Θ satisfying

$$\int_0^t \int_{\mathbb{R}^*} |\Theta_s(\tau, z) - 1| \nu^{(\tau)}(ds, dz) < \infty, \forall t \geq 0, \quad (25)$$

where $\nu^{(\tau)}$ is defined in (18),

4.2 The progressively enlarged filtration

We denote by $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ the progressive enlargement of \mathbb{F} with τ , that is,

$$\mathcal{G}_t = \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(\tau \wedge s)), \quad \forall t \geq 0. \quad (26)$$

Note that τ is a \mathbb{G} -stopping time and that, according to the hypothesis that the random variable τ is strictly positive, the σ -algebra \mathcal{G}_0 is trivial, so that the initial value of a \mathbb{G} -adapted process is a deterministic one. Observe that, under Jacod's equivalence hypothesis, any \mathbb{F} -martingale is a \mathbb{G} -semimartingale (see, e.g., Proposition 5.30, page 116 in [1] or Theorem 3.1 in [18]), and thus, a special semimartingale according to Chapter VI, Theorem 4, page 367 in [27].

We observe that the completion of the two enlargements \mathbb{G} and $\mathbb{F}^{(\tau)}$ follows from $\mathcal{F}_\infty \subset \mathcal{G}_\infty \subset \mathcal{F}_\infty^{(\tau)} \subset \mathcal{A}$, and we note that $\mathcal{F}_0^{(\tau)} = \sigma(\tau)$.

We further indicate with the superscript \mathbb{G} the processes which are \mathbb{G} -adapted, as $Y^{\mathbb{G}}$, except for the \mathbb{G} -adapted process H .

We recall that, under Jacod's equivalence density hypothesis, any \mathbb{G} -optional process $Y^{\mathbb{G}}$ can be written as

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} Y_t^0 + \mathbb{1}_{\{\tau \leq t\}} Y_t^1(\tau), \quad \forall t \geq 0, \quad (27)$$

where Y^0 is \mathbb{F} -optional and Y^1 is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable (see Theorem 6.9 in [29]). A particular case occurs when $Y^{\mathbb{G}}$ is the optional projection of a process $Y(\tau)$. In that case, for each $u \geq 0$, one has

$$Y_t^0 = \frac{1}{G_t} \int_t^\infty Y_t(u) p_t(u) \rho(du), \quad \forall t \geq 0, \quad \text{and} \quad Y_t^1(u) = Y_t(u), \quad \forall t \geq u, \quad (28)$$

where the process G is defined in (26). Here, Y^0 is called the \mathbb{F} -optional reduction of $Y^{\mathbb{G}}$. We also recall that any \mathbb{G} -predictable process $Y_t^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t \geq 0}$ can be written as

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \geq t\}} Y_t^0 + \mathbb{1}_{\{\tau < t\}} Y_t^1(\tau), \quad \forall t \geq 0,$$

where the process Y^0 is \mathbb{F} -predictable and Y^1 is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable (see, e.g., Proposition 2.11, page 36 in [1]). In this case, Y^0 is called the \mathbb{F} -predictable reduction of $Y^{\mathbb{G}}$. Note that the process $(\mathbb{1}_{\{\tau < t\}})_{t \geq 0}$ is predictable.

As it follows from the Doob-Meyer decomposition of the supermartingale H and the fact that any \mathbb{G} -predictable process is equal, on the set $\{\tau \geq t\}$ to an \mathbb{F} -predictable process, there exists an \mathbb{F} -predictable increasing process $\Lambda = (\Lambda_t)_{t \geq 0}$ such that the process $M^{\mathbb{G}} = (M_t^{\mathbb{G}})_{t \geq 0}$ defined by

$$M_t^{\mathbb{G}} = H_t - \Lambda_{t \wedge \tau}, \quad \forall t \geq 0, \quad (29)$$

is a \mathbb{G} -martingale. It is known that, under Jacod's equivalence hypothesis, the process Λ admits the representation (we also use the fact that ρ has no atoms)

$$\Lambda_t = \int_0^t \frac{p_s(s)}{G_s} \rho(ds) = \int_0^t \frac{p_{s-}(s)}{G_{s-}} \rho(ds), \quad \forall t \geq 0, \quad (30)$$

(see Proposition 4.4 in [9] or Corollary 5.27 (b), page 114 in [1]). In this respect, the process $\lambda = (\lambda_t)_{t \geq 0}$ defined by $\lambda_t = p_{t-}(t)/G_{t-}$, for $t \geq 0$, is the intensity rate of τ with respect to the measure ρ (see Proposition 2.15, page 37 in [1]).

The Doob-Meyer decomposition of the Azéma supermartingale can be given explicitly and its multiplicative decomposition is as follows.

Proposition 4.5 *Suppose that Jacod's equivalence hypothesis holds. The Doob-Meyer decomposition of the Azéma supermartingale G is*

$$G_t = 1 - \int_0^t G_s \lambda_s \rho(ds) + \int_0^t G_{s-} \int_{\mathbb{R}^*} (\varphi_s(z) - 1) (\mu(ds, dz) - \eta_s(dz) ds), \quad \forall t \geq 0, \quad (31)$$

where the function φ defined by

$$\varphi_t(z) = \frac{1}{G_{t-}} \int_t^\infty p_{t-}(u) f_t(u, z) \rho(du), \quad \forall t \geq 0, \quad \forall z \in \mathbb{R}^*, \quad (32)$$

is strictly positive and $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable.

The multiplicative decomposition of the Azéma supermartingale G has the form

$$G_t = e^{-\Lambda t} \exp \left(\int_0^t \int_{\mathbb{R}^*} \ln(\varphi_s(z)) \mu(ds, dz) - \int_0^t \int_{\mathbb{R}^*} (\varphi_s(z) - 1) \eta_s(dz) ds \right), \quad \forall t \geq 0, \quad (33)$$

where Λ is given by (30).

Proof: The Doob-Meyer decomposition of G is obtained using Itô-Ventzell formula as developed in Theorem 3.1 in [24] to the process

$$G_t(s) = \mathbb{P}(\tau > s | \mathcal{F}_t) = \int_s^\infty p_t(u) \rho(du), \quad \forall t, s \geq 0,$$

with parameter s , where the forward integral (with respect to the compensated measure) in [24] is the usual stochastic integral in our setting since we integrate predictable processes. \square

In the following proposition, we give the semimartingale decomposition of the process $\overline{N}(A)$ defined in (2) in the filtration \mathbb{G} .

Proposition 4.6 *For any $A \in \mathcal{B}(\mathbb{R}^*)$, the \mathbb{G} -semimartingale decomposition of the \mathbb{F} -martingale $\overline{N}(A)$ is given by*

$$\overline{N}_t(A) = \overline{N}_t^{\mathbb{G}}(A) + \int_0^{t \wedge \tau} \int_A (\varphi_s(z) - 1) \eta_s(dz) ds + \int_{t \wedge \tau}^t \int_A (f_s(\tau, z) - 1) \eta_s(dz) ds, \quad \forall t \geq 0, \quad (34)$$

where $\overline{N}^{\mathbb{G}}(A)$ is a \mathbb{G} -martingale, φ is defined in (32), and f is defined in (10). The predictable random measure

$$\nu^{\mathbb{G}}(dt, dz) = (\mathbb{1}_{\{\tau \geq t\}} \varphi_t(z) + \mathbb{1}_{\{\tau < t\}} f_t(\tau, z)) \eta_t(dz) dt, \quad \forall t \geq 0, \quad \forall z \in \mathbb{R}^*, \quad (35)$$

is the \mathbb{G} -compensator of the random jump measure μ of the marked point process N .

Proof: Recall that the process G admits a Doob-Meyer decomposition as $G = m - H^p$ (see Remark 3.4). The \mathbb{G} -semimartingale decomposition³ of the \mathbb{F} -martingale $\bar{N}(A)$ is given by

$$\begin{aligned}\bar{N}_t(A) &= \bar{N}_t^{\mathbb{G}}(A) + \int_0^{t \wedge \tau} \frac{d\langle \bar{N}(A), m \rangle_s^{\mathbb{F}}}{G_{s-}} + \int_{t \wedge \tau}^t \frac{d\langle \bar{N}(A), p(u) \rangle_s^{\mathbb{F}}}{p_{s-}(u)} \Big|_{u=\tau} \\ &= \bar{N}_t^{\mathbb{G}}(A) + \int_0^{t \wedge \tau} \int_A \frac{(\varphi_s(z) - 1)G_{s-}}{G_{s-}} \eta_s(dz) ds + \int_{t \wedge \tau}^t \int_A (f_s(\tau, z) - 1) \eta_s(dz) ds \\ &= \bar{N}_t^{\mathbb{G}}(A) + \int_0^{t \wedge \tau} \int_A (\varphi_s(z) - 1) \eta_s(dz) ds + \int_{t \wedge \tau}^t \int_A (f_s(\tau, z) - 1) \eta_s(dz) ds, \quad \forall t \geq 0,\end{aligned}$$

where $\bar{N}^{\mathbb{G}}(A) = (\bar{N}_t^{\mathbb{G}}(A))_{t \geq 0}$ is a \mathbb{G} -martingale. It thus follows that the \mathbb{G} -compensator of μ is given by (35). \square

Proposition 4.7 *Each (\mathbb{P}, \mathbb{G}) -martingale $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t \geq 0}$ can be represented as*

$$Y_t^{\mathbb{G}} = Y_0^{\mathbb{G}} + \int_0^t \int_{\mathbb{R}^*} \alpha_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) + \int_0^t \beta_s^0 dM_s^{\mathbb{G}}, \quad \forall t \geq 0, \quad (36)$$

for some $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ -measurable process $\alpha^{\mathbb{G}}$ satisfying

$$\int_0^t \int_{\mathbb{R}^*} |\alpha_s^{\mathbb{G}}(z)| \nu^{\mathbb{G}}(ds, dz) < \infty, \quad \forall t \geq 0, \quad (37)$$

where $\nu^{\mathbb{G}}$ is defined in (35). Here, the process $\alpha^{\mathbb{G}}$ is of the form

$$\alpha_t^{\mathbb{G}}(z) = \mathbb{1}_{\{\tau \geq t\}} \alpha_t^0(z) + \mathbb{1}_{\{\tau < t\}} \alpha_t(\tau, z), \quad \forall t \geq 0, \quad \forall z \in \mathbb{R}^*, \quad (38)$$

where α^0 is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable process, α is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R})$ -measurable process, while β^0 is an \mathbb{F} -predictable process.

Proof: The weak predictable representation property holds for the filtration \mathbb{G} under the probability measure \mathbb{P}^* , due to the independence between \mathbb{F} and $\sigma(\tau)$ under \mathbb{P}^* . This property means that any $(\mathbb{P}^*, \mathbb{G})$ -martingale $Y^{*, \mathbb{G}} = (Y_t^{*, \mathbb{G}})_{t \geq 0}$ admits the representation

$$Y_t^{*, \mathbb{G}} = Y_0^{*, \mathbb{G}} + \int_0^t \int_{\mathbb{R}^*} \alpha_s^{\mathbb{G}}(z) (\mu(ds, dz) - \eta_s(dz) ds) + \int_0^t \beta_s^0 dM_s^{*, \mathbb{G}}, \quad \forall t \geq 0, \quad (39)$$

with some $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ -measurable process $\alpha^{\mathbb{G}}$ satisfying (37) and being of the form (38), for some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable process α^0 , some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R})$ -measurable process α , and

³One can use Remark 3.4 and Theorem 5.30, page 116 in [1] to deduce that, for any \mathbb{F} -martingale X , the process $X^{\mathbb{G}} = (X_t^{\mathbb{G}})_{t \geq 0}$ defined by

$$X_t^{\mathbb{G}} = X_t - \int_0^{t \wedge \tau} \frac{d\langle X, m \rangle_s^{\mathbb{F}}}{G_{s-}} - \int_{t \wedge \tau}^t \frac{d\langle X, p(u) \rangle_s^{\mathbb{F}}}{p_{s-}(u)} \Big|_{u=\tau}, \quad \forall t \geq 0,$$

is a \mathbb{G} -martingale.

some \mathbb{F} -predictable process β^0 . Note that, due to immersion property the $(\mathbb{P}^*, \mathbb{G})$ -compensator of μ is the measure $\eta_t(dz)dt$. Here the process $M^{*,\mathbb{G}} = (M_t^{*,\mathbb{G}})_{t \geq 0}$ is the $(\mathbb{P}^*, \mathbb{G})$ -compensated martingale associated with H defined similar to $M^{\mathbb{G}}$ in (29), but under the probability measure \mathbb{P}^* . Since the weak predictable representation property (WPRP) is stable under an equivalent change of probability measure (see Th. 13.22 in [13]), the result follows. \square

Remark 4.8 Note that, if the process $\beta^{\mathbb{G}}$ admits the representation

$$\beta_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \geq t\}} \beta_t^0 + \mathbb{1}_{\{\tau < t\}} \beta_t^1(\tau), \quad \forall t \geq 0,$$

then the equality

$$\int_0^t \beta_s^{\mathbb{G}} dM_s^{\mathbb{G}} = \int_0^t \beta_s^0 dM_s^{\mathbb{G}}, \quad \forall t \geq 0, \quad (40)$$

holds, for any choice of the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ -measurable process β^1 , since $M^{\mathbb{G}}$ is flat after τ (i.e., $M_t^{\mathbb{G}} = M_{t \wedge \tau}^{\mathbb{G}}$, for all $t \geq 0$).

As a particular case of Proposition 4.7, we obtain:

Proposition 4.9 *Each strictly positive \mathbb{G} -local martingale $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$ can be represented as*

$$L_t^{\mathbb{G}} = L_0^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}} \int_{\mathbb{R}^*} (\varkappa_s^{\mathbb{G}}(z) - 1) (\mu(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) + \int_0^t L_{s-}^{\mathbb{G}} \xi_s^0 dM_s^{\mathbb{G}}, \quad \forall t \geq 0, \quad (41)$$

or, equivalently, $L^{\mathbb{G}}$ is given by

$$\begin{aligned} L_t^{\mathbb{G}} &= L_0^{\mathbb{G}} \exp \left(\int_0^t \int_{\mathbb{R}^*} \ln(\varkappa_s^{\mathbb{G}}(z)) \mu(ds, dz) - \int_0^t \int_{\mathbb{R}^*} (\varkappa_s^{\mathbb{G}}(z) - 1) \nu^{\mathbb{G}}(ds, dz) \right) \\ &\quad \times (1 + \xi_{\tau}^0)^{H_t^{\mathbb{G}}} \exp \left(- \int_0^{t \wedge \tau} \xi_s^0 \lambda_s ds \right), \quad \forall t \geq 0, \end{aligned} \quad (42)$$

that is,

$$L_t^{\mathbb{G}} = L_0^{\mathbb{G}} \mathcal{E} \left(\int_{\mathbb{R}^*} (\varkappa^{\mathbb{G}}(z) - 1) \cdot (\mu(\cdot, dz) - \nu^{\mathbb{G}}(\cdot, dz)) \right)_t \mathcal{E}(\xi^0 \cdot M^{\mathbb{G}})_t, \quad \forall t \geq 0, \quad (43)$$

for a strictly positive and $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ -measurable process $\varkappa^{\mathbb{G}}$ satisfying

$$\int_0^t \int_{\mathbb{R}^*} |\varkappa_s^{\mathbb{G}}(z) - 1| \nu^{\mathbb{G}}(ds, dz) < \infty, \quad \forall t \geq 0,$$

where $\nu^{\mathbb{G}}$ is defined in (35), and $H_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \leq t\}}$, for all $t \geq 0$. Here, the process $\varkappa^{\mathbb{G}}$ is of the form

$$\varkappa_t^{\mathbb{G}}(z) = \mathbb{1}_{\{\tau \geq t\}} \varkappa_t^0(z) + \mathbb{1}_{\{\tau < t\}} \varkappa_t(\tau, z), \quad \forall t \geq 0, \quad \forall z \in \mathbb{R}^*,$$

where \varkappa^0 is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process, \varkappa is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process, while $\xi^0 > -1$ is an \mathbb{F} -predictable process.

5 Optional projections of martingales

Let $Y(\tau)$ be an $\mathbb{F}^{(\tau)}$ -martingale. Then, $Y(\tau)$ admits the integral representation given by (20). We study the \mathbb{G} -optional projection $Y^{\mathbb{G}}$ of the process $Y(\tau)$ and the \mathbb{F} -optional projection Y of $Y(\tau)$. Note that $Y^{\mathbb{G}}$ is a \mathbb{G} -martingale and Y is an \mathbb{F} -martingale. The \mathbb{G} -martingale $Y^{\mathbb{G}}$ admits the integral representation given by (36), with some processes $\alpha^{\mathbb{G}}$ and β^0 that can be represented in the form of (38), where α^0 is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable, α is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ -measurable, and β^0 is an \mathbb{F} -predictable process.

Observe that any *square integrable \mathbb{F} -martingale* Y admits the representation (6) with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable process ξ satisfying

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \xi_s^2(z) \eta_s(dz) ds \right] < \infty, \forall t \geq 0, \quad (44)$$

(see Chapter VIII, Theorem T8, page 239 in [6]).

Similarly, we observe that any *square integrable $\mathbb{F}^{(\tau)}$ -martingale* $Y(\tau)$ admits the representation (20) with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process ψ satisfying

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \psi_s^2(\tau, z) \nu^{(\tau)}(ds, dz) \right] < \infty, \forall t \geq 0, \quad (45)$$

where $\nu^{(\tau)}$ is defined in (18) (see Chapter VIII, Theorem T8, page 239 in [6]).

Finally, we observe that any *square integrable \mathbb{G} -martingale* $Y^{\mathbb{G}}$ admits the representation (36) with some $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process $\alpha^{\mathbb{G}}$ satisfying

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} (\alpha_s^{\mathbb{G}}(z))^2 \nu^{\mathbb{G}}(ds, dz) \right] < \infty, \forall t \geq 0, \quad (46)$$

and \mathbb{F} -predictable process β^0 , where $\nu^{\mathbb{G}}$ is defined in (35) (see Chapter VIII, Theorem T8, page 239 in [6]).

5.1 The projections of $\mathbb{F}^{(\tau)}$ -martingales on \mathbb{G}

Proposition 5.1 *Let $Y^{(\tau)} = Y(\tau)$ (with Y being $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable) be an $\mathbb{F}^{(\tau)}$ -martingale with the representation (20), where ψ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable. Then, its \mathbb{G} -optional projection $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t \geq 0}$ is the \mathbb{G} -martingale with representation given by the equation (36) with $Y_0 = \mathbb{E}[Y_t(\tau)]$, where $\alpha^{\mathbb{G}}$ admits the decomposition (38). The $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process α^0 , the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process α and the \mathbb{F} -predictable process β^0 are of the form*

$$\begin{aligned} \alpha_t^0(z) &= \frac{1}{\varphi_t(z) G_{t-}} \int_t^\infty \left((\psi_t(u, z) + Y_{t-}(u)) f_t(u, z) - Y_{t-}(u) \varphi_t(z) \right) p_{t-}(u) \rho(du), \forall t \geq 0, \forall z \in \mathbb{R}^*, \\ \alpha_t(u, z) &= \psi_t(u, z), \forall u \geq t \geq 0, \forall z \in \mathbb{R}^*, \\ \beta_t^0 &= {}^{p, \mathbb{F}}(Y_{t-}(t)) - Y_{t-}^0, \forall t \geq 0, \end{aligned}$$

with

$$Y_{t-}^0 = \frac{1}{G_{t-}} \int_t^\infty Y_{t-}(u) p_{t-}(u) \rho(du), \forall t \geq 0,$$

where $Y^0 = (Y_t^0)_{t \geq 0}$ is the \mathbb{F} -predictable reduction of $Y^{\mathbb{G}}$ and Y_-^0 its left limit.

Proof: In the first part of the proof (the first and the second step), we assume that the $\mathbb{F}^{(\tau)}$ -martingale $Y(\tau)$ is square integrable, so that the \mathbb{G} -martingale $Y^{\mathbb{G}}$ is square integrable too. In the first step, we determine $\alpha^{\mathbb{G}}(z)$, for each $z \in \mathbb{R}^*$, and, in the second step, we determine β^0 . We generalize the result to any $\mathbb{F}^{(\tau)}$ -martingale by localisation in the second part of the proof (third step).

We introduce the sign $\stackrel{\text{TP}}{=}$ to indicate that the tower property for conditional expectations is applied.

First step: We assume that the $\mathbb{F}^{(\tau)}$ -martingale $Y(\tau)$ is square integrable, so that the \mathbb{G} -martingale $Y^{\mathbb{G}}$ is square integrable too. In particular, $Y_0(\tau)$ is square integrable and the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process ψ satisfies (45) as well as the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process $\alpha^{\mathbb{G}}$ satisfies (46). Then, consider a bounded $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process $\gamma^{\mathbb{G}}$ such that $\gamma^{\mathbb{G}}(z) = (\gamma_t^{\mathbb{G}}(z))_{t \geq 0}$, for each $z \in \mathbb{R}^*$, as well as a bounded \mathbb{F} -predictable process $\theta^0 = (\theta_t^0)_{t \geq 0}$, and define the process $K^{\mathbb{G}} = (K_t^{\mathbb{G}})_{t \geq 0}$ by

$$K_t^{\mathbb{G}} = K_0^{\mathbb{G}} + \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) + \int_0^t \theta_s^0 dM_s^{\mathbb{G}}, \quad \forall t \geq 0, \quad (47)$$

where $\nu^{\mathbb{G}}$ is defined in (35). It is seen that the process $K^{\mathbb{G}}$ is a square integrable \mathbb{G} -martingale, since $\gamma^{\mathbb{G}}$ satisfies the condition

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} (\gamma_s^{\mathbb{G}}(z))^2 \nu^{\mathbb{G}}(ds, dz) \right] < \infty, \quad \forall t \geq 0, \quad (48)$$

and the process θ^0 is \mathbb{F} -predictable and bounded. In this case, the square integrable random variable $Y_t^{\mathbb{G}} = \mathbb{E}[Y_t(\tau) | \mathcal{G}_t]$ is the only \mathcal{G}_t -measurable random variable such that

$$\mathbb{E}[Y_t(\tau) K_t^{\mathbb{G}}] = \mathbb{E}[Y_t^{\mathbb{G}} K_t^{\mathbb{G}}], \quad \forall t \geq 0, \quad (49)$$

holds. Thus, since one has

$$\mathbb{E}[Y_t(\tau) K_0^{\mathbb{G}}] = \mathbb{E}[Y_t^{\mathbb{G}} K_0^{\mathbb{G}}], \quad \forall t \geq 0,$$

the equality (49) is equivalent to the system of two following equalities

$$\begin{aligned} & \mathbb{E} \left[Y_t(\tau) \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) \right] \\ &= \mathbb{E} \left[Y_t^{\mathbb{G}} \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) \right], \quad \forall t \geq 0, \end{aligned} \quad (50)$$

and

$$\mathbb{E} \left[Y_t(\tau) \int_0^t \theta_s^0 dM_s^{\mathbb{G}} \right] = \mathbb{E} \left[Y_t^{\mathbb{G}} \int_0^t \theta_s^0 dM_s^{\mathbb{G}} \right], \quad \forall t \geq 0. \quad (51)$$

We now determine the processes α^0 and α from the equality (50). On the one hand, one has

$$\begin{aligned} & \mathbb{E} \left[Y_t(\tau) \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) \right] \\ &= \mathbb{E} \left[Y_t(\tau) \left(\int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)) \right. \right. \\ & \quad \left. \left. + \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\nu^{(\tau)}(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) \right) \right], \quad \forall t \geq 0, \end{aligned}$$

where $\nu^{(\tau)}$ is defined in (18). Integrating by parts on the time interval $[0, t]$ the product the two $\mathbb{F}^{(\tau)}$ -martingales $Y(\tau)$ and $\Upsilon = (\Upsilon_t)_{t \geq 0}$ defined by

$$\Upsilon_t = \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)), \forall t \geq 0,$$

and taking into account the fact that $\Upsilon_{t-} dY_t(\tau)$ and $Y_{t-}(\tau) d\Upsilon_t$ correspond to true martingales, as we shall prove in Appendix below, one has

$$\begin{aligned} & \mathbb{E} \left[Y_t(\tau) \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)) \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) \psi_s(\tau, z) \nu^{(\tau)}(ds, dz) \right], \forall t \geq 0. \end{aligned}$$

Now, integrating by parts on the time interval $[0, t]$ the product of the martingale $Y(\tau)$ and the bounded variation process $\Gamma(\tau) = (\Gamma_t(\tau))_{t \geq 0}$ defined by

$$\Gamma_t(\tau) = \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\nu^{(\tau)}(ds, dz) - \nu^{\mathbb{G}}(ds, dz)), \forall t \geq 0,$$

one obtains

$$\mathbb{E}[Y_t(\tau) \Gamma_t(\tau)] = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} Y_{s-}(\tau) \gamma_s^{\mathbb{G}}(z) (\nu^{(\tau)}(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) \right], \forall t \geq 0.$$

On the other hand, one has by integration by parts

$$\mathbb{E} \left[Y_t^{\mathbb{G}} \int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) \right] = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) \alpha_s^{\mathbb{G}}(z) \nu^{\mathbb{G}}(ds, dz) \right], \forall t \geq 0.$$

Finally, (50) is equivalent to, for any $\gamma^{\mathbb{G}}$ satisfying (48), we have

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) \left(\psi_s(\tau, z) \nu^{(\tau)}(ds, dz) + Y_{s-}(\tau) (\nu^{(\tau)}(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) \right) \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^{\mathbb{G}}(z) \alpha_s^{\mathbb{G}}(z) \nu^{\mathbb{G}}(ds, dz) \right], \forall t \geq 0. \end{aligned} \tag{52}$$

For $\gamma^{\mathbb{G}}(z)$ such that $\gamma_s^{\mathbb{G}}(z) = \mathbb{1}_{\{\tau \geq s\}} \gamma_s^0(z)$, for all $s > 0$, where γ^0 is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable, using the identities (18) and (35), we have

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^0(z) \mathbb{1}_{\{\tau \geq s\}} \left((\psi_s(\tau, z) + Y_{s-}(\tau)) f_s(\tau, z) - Y_{s-}(\tau) \varphi_s(z) \right) \eta_s(dz) ds \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^0(z) \mathbb{1}_{\{\tau \geq s\}} \alpha_s^0(z) \varphi_s(z) \eta_s(dz) ds \right], \forall t \geq 0, \end{aligned} \tag{53}$$

and, introducing by tower property a conditioning with respect to \mathcal{F}_s and using the existence of the conditional density, the left hand side of (53) is

$$\begin{aligned}
& \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^0(z) \mathbb{1}_{\{\tau \geq s\}} \left((\psi_s(\tau, z) + Y_{s-}(\tau)) f_s(\tau, z) - Y_{s-}(\tau) \varphi_s(z) \right) \eta_s(dz) ds \right] \\
& \stackrel{\text{TP}}{=} \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^0(z) \left(\int_s^\infty \left((\psi_s(u, z) + Y_{s-}(u)) f_s(u, z) - Y_{s-}(u) \varphi_s(z) \right) p_s(u) \rho(du) \right) \eta_s(dz) ds \right] \\
& = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^0(z) \left(\int_s^\infty \left((\psi_s(u, z) + Y_{s-}(u)) f_s(u, y) - Y_{s-}(u) \varphi_s(z) \right) p_{s-}(u) \rho(du) \right) \eta_s(dz) ds \right], \forall t \geq 0,
\end{aligned} \tag{54}$$

where, in the last equality, we have used the fact that the \mathbb{F} -predictable projection of $p(u)$ is $p_-(u)$, the process $p(u)$ being a martingale, for each $u \geq 0$.

We note also that, using the fact that G_- is the \mathbb{F} -predictable projection of G (see Remark 3.4), the right-hand side of (53) is

$$\begin{aligned}
& \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^0(z) \mathbb{1}_{\{\tau \geq s\}} \alpha_s^0(z) \varphi_s(z) \eta_s(dz) ds \right] \\
& = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^0(z) G_s \alpha_s^0(z) \varphi_s(z) \eta_s(dz) ds \right] \\
& = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s^0(z) G_{s-} \alpha_s^0(z) \varphi_s(z) \eta_s(dz) ds \right], \forall t \geq 0.
\end{aligned} \tag{55}$$

It follows from (53) that the right-hand sides of (54) and (55) are equal for any γ^0 , hence

$$\alpha_s^0(z) = \frac{1}{\varphi_s(z) G_{s-}} \int_s^\infty \left((\psi_s(u, z) + Y_{s-}(u)) f_s(u, z) - Y_{s-}(u) \varphi_s(z) \right) p_{s-}(u) \rho(du), \forall s \geq 0.$$

Using the identities (18) and (35), for $\gamma^{\mathbb{G}}$ of the form $\gamma_s^{\mathbb{G}} = \gamma_s(\tau, z) \mathbb{1}_{\{\tau < s\}}$, for all $s > 0$, for $\gamma \in \mathbb{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^*)$, equality (52) leads to

$$\begin{aligned}
& \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s(\tau, z) \mathbb{1}_{\{\tau < s\}} \psi_s(\tau, z) f_s(\tau, z) \eta_s(dz) ds \right] \\
& = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \gamma_s(\tau, z) \mathbb{1}_{\{\tau < s\}} \alpha_s(\tau, z) f_s(\tau, z) \eta_s(dz) ds \right], \forall t \geq 0,
\end{aligned}$$

and we can choose $\alpha = \psi$.

Second step: In the second step, we compute the value of β^0 , from (51). It is straightforward to see that

$$\mathbb{E} \left[Y_t^{\mathbb{G}} \int_0^t \theta_s^0 dM_s^{\mathbb{G}} \right] = \mathbb{E} \left[\int_0^t \beta_s^0 \theta_s^0 \lambda_s \mathbb{1}_{\{\tau > s\}} \rho(ds) \right] \stackrel{\text{TP}}{=} \mathbb{E} \left[\int_0^t \beta_s^0 \theta_s^0 \lambda_s G_s \rho(ds) \right], \forall t \geq 0.$$

From the definition of $M^{\mathbb{G}}$, it follows that

$$\begin{aligned}
\mathbb{E} \left[Y_t(\tau) \int_0^t \theta_s^0 dM_s^{\mathbb{G}} \right] &= \mathbb{E} \left[Y_t(\tau) \left(\mathbb{1}_{\{\tau \leq t\}} \theta_\tau^0 - \int_0^t \mathbb{1}_{\{\tau > s\}} \theta_s^0 \lambda_s \rho(ds) \right) \right] \\
&\stackrel{\text{TP}}{=} \mathbb{E} \left[\int_0^t Y_t(s) \theta_s^0 p_t(s) \rho(ds) - \int_0^t \theta_s^0 \lambda_s \mathbb{E}[Y_s^{(\tau)} \mathbb{1}_{\{\tau > s\}} | \mathcal{F}_s] \rho(ds) \right] \\
&= \mathbb{E} \left[\int_0^t Y_{s-}(s) p_{s-}(s) \theta_s^0 \rho(ds) - \int_0^t \theta_s^0 \lambda_s \left(\int_s^\infty Y_{s-}(u) p_{s-}(u) \rho(du) \right) \rho(ds) \right] \\
&= \mathbb{E} \left[\int_0^t {}^{p, \mathbb{F}} \Sigma_s p_{s-}(s) \theta_s^0 \rho(ds) - \int_0^t \theta_s^0 \lambda_s \left(\int_s^\infty Y_{s-}(u) p_{s-}(u) \rho(du) \right) \rho(ds) \right], \forall t \geq 0,
\end{aligned}$$

where we have used in the third equality that $Y(u)p(u)$ is an \mathbb{F} -martingale with predictable projection $Y_{-}(u)p_{-}(u)$, for each $u \geq 0$, and defined $\Sigma = (\Sigma_t)_{t \geq 0}$ by $\Sigma_t = Y_{t-}(t)$, for all $t \geq 0$. We are not able to give conditions so that Σ is predictable, since we do not have regularity of the process $Y_{t-}(u)$ with respect to u , for each $u \geq 0$, this is why we have to take its predictable projection.

It follows that

$$\begin{aligned}
\beta_t^0 &= \frac{1}{\lambda_t G_{t-}} \left({}^{p, \mathbb{F}} \Sigma_t p_{t-}(t) - \lambda_t \int_t^\infty Y_{t-}(u) p_{t-}(u) \rho(du) \right) \\
&= {}^{p, \mathbb{F}} \Sigma_t - \frac{1}{G_{t-}} \int_t^\infty Y_{t-}(u) p_{t-}(u) \rho(du), \forall t \geq 0,
\end{aligned}$$

where we have used the fact that $\lambda_t = p_{t-}(t)/G_{t-}$, for $t \geq 0$.

Third step: The extension to $\mathbb{F}^{(\tau)}$ -martingales is done using usual localisation procedure (see Third step of Proof of Proposition 5.1 in [11]). \square

5.2 The projections of $\mathbb{F}^{(\tau)}$ -martingales on \mathbb{F}

Proposition 5.2 *Let $Y(\tau)$ be an $\mathbb{F}^{(\tau)}$ -martingale with the representation given by equality (20). Then, its \mathbb{F} -optional projection $Y = (Y_t)_{t \geq 0}$ admits the representation (6), with $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process ξ , given by*

$$\xi_t(z) = \int_0^\infty \left((\psi_t(u, z) + Y_{t-}(u)) f_t(u, z) - Y_{t-}(u) \right) p_{t-}(u) \rho(du), \forall t \geq 0. \quad (56)$$

Proof: As before, we assume that $Y(\tau)$ is square integrable. Then, consider a bounded $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process ζ such that $\zeta(z) = (\zeta_t(z))_{t \geq 0}$, for each $z \in \mathbb{R}^*$, and define the process $K = (K_t)_{t \geq 0}$ by

$$K_t = K_0 + \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\mu(ds, dz) - \eta_s(dz) ds), \forall t \geq 0.$$

It is seen that the process K is a square integrable \mathbb{G} -martingale, since the process ζ satisfies the condition

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \zeta_s^2(z) \eta_s(dz) ds \right] < \infty, \forall t \geq 0.$$

In this case, the square integrable random variable $Y_t = \mathbb{E}[Y_t(\tau) | \mathcal{F}_t]$ is the only \mathcal{F}_t -measurable random variable such that

$$\mathbb{E}[Y_t(\tau) K_t] = \mathbb{E}[Y_t K_t], \forall t \geq 0, \quad (57)$$

holds. Thus, the equality (57) is equivalent to the following equality

$$\begin{aligned} & \mathbb{E} \left[Y_t(\tau) \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\mu(ds, dz) - \eta_s(dz) ds) \right] \\ &= \mathbb{E} \left[Y_t \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\mu(ds, dz) - \eta_s(dz) ds) \right], \forall t \geq 0. \end{aligned} \quad (58)$$

On the one hand, one has

$$\begin{aligned} & \mathbb{E} \left[Y_t(\tau) \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\mu(ds, dz) - \eta_s(dz) ds) \right] \\ &= \mathbb{E} \left[Y_t(\tau) \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)) \right. \\ & \quad \left. + Y_t(\tau) \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\nu^{(\tau)}(ds, dz) - \eta_s(dz) ds) \right], \forall t \geq 0, \end{aligned}$$

where $\nu^{(\tau)}$ is defined in (18). Integrating by parts on the time interval $[0, t]$ the product of the two $\mathbb{F}^{(\tau)}$ -martingales $Y(\tau)$ and $\Phi(\tau) = (\Phi_t(\tau))_{t \geq 0}$ defined by

$$\Phi_t(\tau) = \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)), \forall t \geq 0,$$

one has, using the square integrability assumption, that

$$\mathbb{E}[Y_t(\tau) \Phi_t(\tau)] = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \zeta_s(z) \psi_s(\tau, z) \nu^{(\tau)}(ds, dz) \right], \forall t \geq 0.$$

By integrating by parts the product of $Y(\tau)$ and the process $\Delta(\tau) = (\Delta_t(\tau))_{t \geq 0}$ of bounded variation defined by

$$\Delta_t(\tau) = \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\nu^{(\tau)}(ds, dz) - \eta_s(dz) ds), \forall t \geq 0,$$

one obtains, using the equality (18), that

$$\mathbb{E}[Y_t(\tau) \Delta_t(\tau)] = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \zeta_s(z) Y_{s-}(\tau) (f_s(\tau, z) - 1) \eta_s(dz) ds \right], \forall t \geq 0.$$

On the other hand, one has

$$\mathbb{E} \left[Y_t \int_0^t \int_{\mathbb{R}^*} \zeta_s(z) (\mu(ds, dz) - \eta_s(dz) ds) \right] = \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \zeta_s(z) \xi_s(z) \eta_s(dz) ds \right], \forall t \geq 0.$$

Finally, (57) implies

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \zeta_s(z) \int_0^\infty \left(\zeta_s(u, z) f_s(u, z) + Y_{s-}(u) (f_s(u, z) - 1) \right) p_{s-}(u) \rho(du) \eta_s(dz) ds \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \zeta_s(z) \xi_s(z) \eta_s(dz) ds \right], \forall t \geq 0, \end{aligned}$$

then, we obtain the expression (56). \square

5.3 The projections of \mathbb{G} -martingales on \mathbb{F}

Proposition 5.3 *Let $Y^\mathbb{G}$ be a \mathbb{G} -martingale with the representation given by equality (36) and decomposition (27). Then, its \mathbb{F} -optional projection Y is given by (6) above, where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process ξ is given by*

$$\begin{aligned} \xi_t(z) &= \alpha_t^0(z) \varphi_t(z) + Y_{t-}^0 (\varphi_t(z) - 1) G_{t-} \\ &\quad + \int_0^t \left(\alpha_t(u, z) f_t(u, z) + Y_{t-}(u) (f_t(u, z) - 1) p_{t-}(u) \right) \rho(du), \forall t \geq 0, \forall z \in \mathbb{R}^*, \end{aligned}$$

with the supermartingale G given by the equality (33).

Proof: As before, for any \mathbb{G} -adapted bounded process $\theta^\mathbb{G}$, we consider the equality satisfied by Y such that

$$\mathbb{E} \left[Y_t \int_0^t \int_{\mathbb{R}^*} \theta_s^\mathbb{G} (\mu(ds, dz) - \eta_s(dz) ds) \right] = \mathbb{E} \left[Y_t^\mathbb{G} \int_0^t \int_{\mathbb{R}^*} \theta_s^\mathbb{G} (\mu(ds, dz) - \eta_s(dz) ds) \right], \forall t \geq 0.$$

The left-hand side is equal to

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \xi_s(z) \theta_s^\mathbb{G} \eta_s(dz) ds \right], \forall t \geq 0.$$

The right-hand side is

$$\begin{aligned} & \mathbb{E} \left[Y_t^\mathbb{G} \int_0^t \int_{\mathbb{R}^*} \theta_s^\mathbb{G} \left((\mu(ds, dz) - \nu_s^\mathbb{G}(ds, dz)) + (\nu_s^\mathbb{G}(ds, dz) - \eta_s(dz) ds) \right) \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \alpha_s^\mathbb{G} \theta_s^\mathbb{G} \nu_s^\mathbb{G}(ds, dz) + \int_0^t \int_{\mathbb{R}^*} \theta_s^\mathbb{G} Y_{s-}^\mathbb{G} (\nu_s^\mathbb{G}(ds, dz) - \eta_s(dz) ds) \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \theta_s^\mathbb{G} \left(\alpha_s^0(z) \varphi_s(z) + Y_{s-}^0 (\varphi_s(z) - 1) \right) G_{s-} \eta_s(dz) ds \right] \\ &+ \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^*} \theta_s^\mathbb{G} \left(\int_0^s \left(\alpha_s(u, z) f_s(u, z) + Y_{s-}(u) (f_s(u, z) - 1) \right) p_{s-}(u) \rho(du) \right) \eta_s(dz) ds \right], \forall t \geq 0, \end{aligned}$$

where $\nu^\mathbb{G}$ is defined in (35). Hence, the proof is complete. \square

6 Changes of probability measures and applications

In this section, as an example of application of the results from the previous section, we consider the relationships between strictly positive $\mathbb{F}^{(\tau)}$ -martingales (or \mathbb{G} -martingales) and their optional projections. We then apply the results in a financial market framework to study the set of equivalent martingale measures in different filtrations.

6.1 Exponential martingales and their projections

6.1.1 The projections of strictly positive $\mathbb{F}^{(\tau)}$ -martingales on \mathbb{G}

Let $L(\tau)$ be a strictly positive $\mathbb{F}^{(\tau)}$ -martingale. Then, in particular, we have $L_0(\tau) > 0$ (\mathbb{P} -a.s.), and we can write $L(\tau)$ in the form of (22) and (23) with a strictly positive and $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process Θ satisfying

$$\int_0^t \int_{\mathbb{R}^*} |\Theta_s(\tau, z) - 1| \nu^{(\tau)}(ds, dz) < \infty, \forall t \geq 0. \quad (59)$$

Note that, if $\mathbb{E}[L_0(\tau)] = 1$, then we can associate to the strictly positive $\mathbb{F}^{(\tau)}$ -martingale $L(\tau)$ the probability measure $\tilde{\mathbb{P}}$ locally equivalent to \mathbb{P} on the filtration $\mathbb{F}^{(\tau)}$ defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t^{(\tau)}} = L_t(\tau), \forall t \geq 0. \quad (60)$$

Remark 6.1 As in Remark 6.1 in [11], the particular choice of $L_0(\tau) = 1$ (\mathbb{P} -a.s.) is equivalent to the property $\tilde{\mathbb{P}}(\tau > u) = \mathbb{P}(\tau > u)$, for each $u \geq 0$.

We now consider the \mathbb{G} -optional projection $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$ of the strictly positive martingale $L(\tau)$. In this case, it follows from the arguments above that $L^{\mathbb{G}}$ admits the representations of (41) and (43) with a strictly positive and $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process $\varkappa^{\mathbb{G}}$ satisfying

$$\int_0^t \int_{\mathbb{R}^*} |\varkappa_s^{\mathbb{G}}(z) - 1| \nu^{\mathbb{G}}(ds, dz) < \infty, \forall t \geq 0,$$

and an \mathbb{F} -predictable process $\xi^0 > -1$. The strict positivity of the process $L^{\mathbb{G}}$ implies the strict positivity of the stochastic exponentials and the property $\xi_\tau^0 > -1$.

Proposition 6.2 *Let $L(\tau) = (L_t(\tau))_{t \geq 0}$ be a strictly positive martingale of the form (22) and (23). Then, its \mathbb{G} -optional projection $L^{\mathbb{G}}$ satisfies (41) and (42) with the \mathbb{G} -predictable processes $\varkappa^{\mathbb{G}}$ and the \mathbb{F} -predictable process ξ^0 given by*

$$\begin{aligned} \varkappa_t^{\mathbb{G}}(z) - 1 &= \mathbb{1}_{\{\tau \geq t\}} \frac{1}{L_{t-}^0 \varphi_t(z) G_{t-}} \int_t^\infty L_{t-}(u) (\Theta_t(u, z) f_t(u, z) - \varphi_t(z)) p_{t-}(u) \rho(du) \\ &\quad + \mathbb{1}_{\{\tau < t\}} (\Theta_t(\tau, z) - 1), \forall t \geq 0, \forall z \in \mathbb{R}^*, \end{aligned} \quad (61)$$

$$\xi_t^0 = \frac{p_{t-}^{\mathbb{F}}(L_{t-}(t))}{L_{t-}^0} - 1, \forall t \geq 0, \quad (62)$$

$$L_{t-}^0 = \frac{1}{G_{t-}} \int_t^\infty L_{t-}(u) p_{t-}(u) \rho(du), \forall t \geq 0, \quad (63)$$

where $L^0 = (L_t^0)_{t \geq 0}$ is the \mathbb{F} -predictable reduction of $L^{\mathbb{G}}$ and L_-^0 is its left limit.

Proof: Consider the $\mathbb{F}^{(\tau)}$ -martingale $L(\tau)$ given by equality (22) and (23). In this case, its \mathbb{G} -optional projection $L^{\mathbb{G}}$ has the form of (41) or (42). Then, Proposition 5.1 applies with $Y(\tau) = L(\tau)$ and $\psi(u, z) = L_-(u)(\Theta(u, z) - 1)$, for all $u \geq 0$ and $z \in \mathbb{R}^*$, and therefore, equalities $L_t^{\mathbb{G}}(\mathcal{X}_t^{\mathbb{G}}(z) - 1) = \alpha_t^{\mathbb{G}}(z)$ (that is, $L_{t-}^0(\mathcal{X}_t^0(z) - 1) = \alpha_t^0(z)$ and $L_{t-}(u)(\mathcal{X}_t(u, z) - 1) = \alpha_t(u, z)$, for all $u \geq 0$ and $z \in \mathbb{R}^*$) and $\xi_t^0 L_{t-}^0 = \beta_t^0$ hold, for all $t \geq 0$ and $z \in \mathbb{R}^*$. \square

Example 6.3 Assume that the $\mathbb{F}^{(\tau)}$ -martingale $L(\tau)$ is given by

$$\begin{aligned} L_t(\tau) &= \ell(\tau) \\ &\times \exp \left(\int_0^t \int_{\mathbb{R}^*} \ln \left(\frac{1}{f_s(\tau, z)} \right) \mu(ds, dz) - \int_0^t \int_{\mathbb{R}^*} \left(\frac{1}{f_s(\tau, z)} - 1 \right) f_s(\tau, z) \eta_s(dz) ds \right) \\ &= \frac{1}{p_t(\tau)}, \forall t \geq 0, \end{aligned}$$

for some strictly positive Borel function $\ell(u)$, for $u \geq 0$. Note that the second equality is an easy consequence of (10). Indeed, Itô's formula and equation (19) lead to

$$d \left(\frac{1}{p_t(\tau)} \right) = \frac{1}{p_{t-}(\tau)} \int_{\mathbb{R}^*} \left(\frac{1}{f_t(\tau, z)} - 1 \right) (\mu(dt, dz) - f_t(\tau, z) \eta_t(dz) dt), \quad \frac{1}{p_0(\tau)} = \ell(\tau).$$

In this case, its \mathbb{G} -optional projection $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$ is given by

$$\begin{aligned} L_t^{\mathbb{G}} &= L_0^{\mathbb{G}} \exp \left(\int_0^t \int_{\mathbb{R}^*} \ln (\mathcal{X}_s^{\mathbb{G}}(z)) \mu(ds, dz) - \int_0^t \int_{\mathbb{R}^*} (\mathcal{X}_s^{\mathbb{G}}(z) - 1) \nu^{\mathbb{G}}(ds, dz) \right) \\ &\times (1 + \xi_{\tau}^0)^{H_t^{\mathbb{G}}} \exp \left(- \int_0^{t \wedge \tau} \xi_s^0 \lambda_s ds \right), \forall t \geq 0, \end{aligned} \quad (64)$$

with

$$\begin{aligned} \mathcal{X}_t^{\mathbb{G}}(z) - 1 &= -\mathbb{1}_{\{\tau \geq t\}} \frac{(\varphi_t(z) - 1)(1 - F(t))}{\varphi_t(z)G_{t-}} + \mathbb{1}_{\{\tau < t\}} \left(\frac{1}{f_t(\tau, z)} - 1 \right), \forall t \geq 0, \forall z \in \mathbb{R}^*, \\ \xi_t^0 &= \frac{G_{t-}}{p_{t-}(t)(1 - F(t))} - 1, \forall t \geq 0, \end{aligned} \quad (65)$$

where we set $F(t) = \mathbb{P}(\tau \leq t)$, for all $t \geq 0$.

Observe that the probability measure defined through (60) with this choice of $L(\tau)$ (which is a strictly positive martingale with expectation being equal to one) is a preserving and decoupling measure (see [3] and [12] for a discussion of an important role of this strictly positive $\mathbb{F}^{(\tau)}$ -martingale $L(\tau)$).

6.1.2 The projections of strictly positive $\mathbb{F}^{(\tau)}$ -martingales on \mathbb{F}

Let $L(\tau)$ be a strictly positive $\mathbb{F}^{(\tau)}$ -martingale of the form (23). Then, its \mathbb{F} -optional projection admits the integral representation

$$L_t = \mathbb{E}[L_0(\tau)] + \int_0^t L_{s-} \int_{\mathbb{R}^*} \chi_s(z) (\mu(ds, dz) - \eta_s(dz) ds), \forall t \geq 0, \quad (66)$$

where the \mathbb{F} -predictable process $\chi(z) = (\chi_t(z))_{t \geq 0}$, for $z \in \mathbb{R}^*$, can be derived by applying Proposition 5.2 with $Y(\tau) = L(\tau)$ (so that $Y = L$ and $L_- \chi = \xi$),

$$\chi_t(z) = \frac{1}{L_{t-}} \int_0^\infty L_{t-}(u) \left(\Theta_t(u, z) f_t(u, z) - 1 \right) p_{t-}(u) \rho(du), \quad \forall t \geq 0, \forall z \in \mathbb{R}^*. \quad (67)$$

6.1.3 The projections of strictly positive \mathbb{G} -martingales on \mathbb{F}

From Proposition 5.3, any strictly positive \mathbb{G} -martingale $L^\mathbb{G} = (L_t^\mathbb{G})_{t \geq 0}$ admits the equivalent representation (42) and, being a \mathbb{G} -optional process, it admits the decomposition

$$L_t^\mathbb{G} = \mathbb{1}_{\{\tau > t\}} L_t^0 + \mathbb{1}_{\{\tau \leq t\}} L_t^1(\tau), \quad \forall t \geq 0,$$

where the process L^0 is \mathbb{F} -optional and the process L^1 is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable. By similar arguments, it follows that its \mathbb{F} -optional projection $L = (L_t)_{t \geq 0}$ admits the integral representation

$$L_t = L_0^\mathbb{G} + \int_0^t L_{s-} \int_{\mathbb{R}^*} \sigma_s(z) (\mu(ds, dz) - \eta_s(dz) ds), \quad \forall t \geq 0, \quad (68)$$

where $\sigma(z) = (\sigma_t(z))_{t \geq 0}$ is an \mathbb{F} -predictable process. In order to derive σ , it suffices to apply Proposition 5.3 with $Y^\mathbb{G} = L^\mathbb{G}$, $(L_-(\varkappa(z) - 1))^0 = \alpha^0(z)$, $L_-(\varkappa(z) - 1) = \alpha(z)$ and $L_- \sigma(z) = \xi(z)$, for all $z \in \mathbb{R}^*$, so that $Y^0 = L^0$ and $Y = L$. The equality $(L_-(\varkappa(z) - 1))^0 = L_-^0(\varkappa^0(z) - 1)$, for all $z \in \mathbb{R}^*$, follows from the definition of predictable reduction. Therefore, we conclude that

$$\begin{aligned} \sigma_t(z) &= \frac{1}{L_{t-}} \left(L_{t-}^0 (\varkappa_t^0(z) - 1) \varphi_t(z) + L_{t-}^0 (\varphi_t(z) - 1) G_t - \right. \\ &\quad \left. + \int_0^t \left(L_{t-}^0(u) (\varkappa_t(u, z) - 1) f_t(u, z) + L_{t-}^1(u) (f_t(u, z) - 1) \right) p_{t-}(u) \rho(du) \right), \quad \forall t \geq 0, \forall z \in \mathbb{R}^*. \end{aligned} \quad (69)$$

6.2 The equivalent martingale measures

Let us now consider a model of a financial market in which the risky asset price process $S = (S_t)_{t \geq 0}$ follows the stochastic differential equation

$$dS_t = S_t \delta_t dt + S_{t-} \int_{\mathbb{R}^*} h_t(z) (\mu(dt, dz) - \nu(dt, dz)),$$

where h belongs to $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ is greater than -1 (to satisfy the positivity of S).

In the filtration \mathbb{G} , one has, using the equality (35),

$$\begin{aligned} dS_t &= S_t \delta_t dt + S_{t-} \int_{\mathbb{R}^*} h_t(z) (\nu^\mathbb{G}(dt, dz) - \nu(dt, dz)) + S_{t-} \int_{\mathbb{R}^*} h_t(z) (\mu(dt, dz) - \nu^\mathbb{G}(dt, dz)) \\ &= S_{t-} \left(\delta_t + \int_{\mathbb{R}^*} h_t(z) (\mathbb{1}_{\{\tau \geq t\}} \varphi_t(z) + \mathbb{1}_{\{\tau < t\}} f_t(\tau, z) - 1) \eta_t(dz) \right) dt \\ &\quad + S_{t-} \int_{\mathbb{R}^*} h_t(z) (\mu(dt, dz) - \nu^\mathbb{G}(dt, dz)) \\ &= S_{t-} \theta_t dt + S_{t-} \int_{\mathbb{R}^*} h_t(z) (\mu(dt, dz) - \nu^\mathbb{G}(dt, dz)), \end{aligned}$$

where

$$\theta_t = \delta_t + \int_{\mathbb{R}^*} h_t(z) \left(\mathbb{1}_{\{\tau \geq t\}} \varphi_t(z) + \mathbb{1}_{\{\tau < t\}} (f_t(\tau, z) - 1) \right) \eta_t(dz) \quad (70)$$

and the last term is an \mathbb{G} -martingale.

In the filtration $\mathbb{F}^{(\tau)}$, using (18)

$$\begin{aligned} dS_t &= S_t \delta_t dt + S_{t-} \int_{\mathbb{R}^*} h_t(z) (\nu^{(\tau)}(dt, dz) - \nu(dt, dz)) + S_{t-} \int_{\mathbb{R}^*} h_t(z) (\mu(dt, dz) - \nu^{(\tau)}(dt, dz)) \\ &= S_{t-} \left(\delta_t + \int_{\mathbb{R}^*} h_t(z) (f_t(\tau, z) - 1) \eta_t(dz) \right) dt + S_{t-} \int_{\mathbb{R}^*} h_t(z) (\mu(dt, dz) - \nu^{(\tau)}(dt, dz)), \end{aligned}$$

and the last term is an $\mathbb{F}^{(\tau)}$ -martingale.

We assume that the riskless asset has a zero interest rate.

A change of probability in \mathbb{F} has the Radon-Nikodym density process $L = (L_t)_{t \geq 0}$ satisfying the stochastic differential equation given from WRPT [25, Pro. 2.1].

$$dL_t = L_{t-} \int_{\mathbb{R}^*} (\alpha_t(z) - 1) (\mu(dt, dz) - \nu(dt, dz)), \quad L_0 = 1,$$

for strictly a positive $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process α . This Radon-Nikodym density process will correspond to an equivalent martingale measure if SL is an \mathbb{F} -martingale. From the integration-by-parts formula

$$S_t L_t = S_0 + \int_0^t L_{s-} dS_s + \int_0^t S_{s-} dL_s + [S, L]_t, \quad \forall t \geq 0,$$

where the quadratic variation⁴ is given by

$$\begin{aligned} [S, L]_t &= \sum_{T_n \leq t} S_{T_n-} L_{T_n-} h_{T_n}(Z_n) (\alpha_{T_n}(Z_n) - 1) \\ &= \int_0^t S_{s-} L_{s-} \int_{\mathbb{R}^*} h_s(z) (\alpha_s(z) - 1) (\mu(ds, dz) - \nu(ds, dz)) \\ &\quad + \int_0^t S_{s-} L_{s-} \int_{\mathbb{R}^*} h_s(z) (\alpha_s(z) - 1) \eta_s(dz) ds, \quad \forall t \geq 0, \end{aligned}$$

and the first term being an \mathbb{F} -martingale, we obtain that the process SL is an \mathbb{F} -martingale if and only if α satisfies the equality

$$\delta_t + \int_{\mathbb{R}^*} h_t(z) (\alpha_t(z) - 1) \eta_t(dz) = 0, \quad \forall t \geq 0$$

⁴For any two semimartingales $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$, one has

$$[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{0 \leq s \leq t} \Delta X_s \Delta Y_s, \quad \forall t \geq 0,$$

where $\Delta X_t = X_t - X_{t-}$ and $\Delta Y_t = Y_t - Y_{t-}$ (see Chapter VIII, Definition 8.2, page 209 in [13]).

holds. See Prigent [25, Section 2.1] for similar results.

Using the fact that any positive \mathbb{G} -martingale has the form

$$dL_t^{\mathbb{G}} = L_{t-} \int_{\mathbb{R}} (\varkappa_t^{\mathbb{G}}(z) - 1) (\mu(dt, dz) - \nu^{\mathbb{G}}(dt, dz)) + \xi_t^0 dM_t^{\mathbb{G}}$$

and that

$$[S, L^{\mathbb{G}}]_t^{\mathbb{G}} = \int_0^t \int_{\mathbb{R}} h_s(z) (\varkappa_t^{\mathbb{G}}(z) - 1) \eta_t^{\mathbb{G}}(dz), \quad \forall t \geq 0,$$

the set $\mathcal{P}(\mathbb{G})$ of (locally) equivalent martingale measures on \mathbb{G} corresponds to the set of Radon-Nikodym density processes of the form

$$dL_t^{\mathbb{G}} = L_{t-}^{\mathbb{G}} \left(\int_{\mathbb{R}} (\varkappa_t^{\mathbb{G}}(z) - 1) (\mu(dt, dz) - \nu^{\mathbb{G}}(dt, dz)) + \xi_s^0 dM_t^{\mathbb{G}} \right) \quad (71)$$

with \varkappa such that

$$\theta_t + \int_{\mathbb{R}} h_t(z) (\varkappa_t^{\mathbb{G}}(z) - 1) \nu^{\mathbb{G}}(dt, dz) = 0, \quad \forall t \geq 0.$$

As seen in Proposition 4.4, a change of probability in has the Radon-Nikodym density process $L(\tau) = (L_t(\tau))_{t \geq 0}$ satisfying the stochastic differential equation given in (22)

$$dL_t(\tau) = L_{t-}(\tau) \int_{\mathbb{R}^*} (\Theta_t(\tau, z) - 1) (\mu(dt, dz) - \nu^{(\tau)}(dt, dz)), \quad L_0(\tau) = \ell(\tau),$$

for strictly a positive $\mathcal{P}(\mathbb{F}^{(\tau)}) \otimes \mathcal{B}(\mathbb{R}^*)$ -measurable process Θ , and a strictly positive Borel function $\ell(u)$, for $u \geq 0$. This Radon-Nikodym density process will correspond to an equivalent martingale measure if $SL(\tau)$ is an $\mathbb{F}^{(\tau)}$ -martingale. Using that

$$\begin{aligned} [S, L(\tau)]_t &= \int_0^t S_{s-} L_{s-}(\tau) \int_{\mathbb{R}^*} h_s(z) (\Theta_s(\tau, z) - 1) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)) \\ &\quad + \int_0^t S_{s-} L_{s-}(\tau) \int_{\mathbb{R}^*} h_s(z) (\Theta_s(\tau, z) - 1) f_s(\tau, z) \eta_s(dz) ds, \quad \forall t \geq 0, \end{aligned}$$

and the first term is an $\mathbb{F}^{(\tau)}$ -martingale, we obtain that the process $SL(\tau)$ is an $\mathbb{F}^{(\tau)}$ -martingale if and only if Θ satisfies the equality

$$\delta_t + \int_{\mathbb{R}^*} h_t(z) \left(f_t(\tau, z) - 1 + (\Theta_t(\tau, z) - 1) f_t(\tau, z) \right) \eta_t(dz) = 0, \quad \forall t \geq 0,$$

holds, that is,

$$\delta_t + \int_{\mathbb{R}^*} h_t(z) (\Theta_t(\tau, z) f_t(\tau, z) - 1) \eta_t(dz) = 0, \quad \forall t \geq 0. \quad (72)$$

Let \mathcal{P}^* be the set of \mathbb{G} -optional projections $L^{*,\mathbb{G}}$ of $L^*(\tau)$, which satisfies (42) being equivalent to (43), where the processes $\varkappa^{\mathbb{G}}$ and ξ^0 given by equalities (61) and (62). More precisely,

one has

$$\varkappa_t^{\mathbb{G}}(z) - 1 = \mathbb{1}_{\{\tau \geq t\}} \frac{1}{L_{t-}^{*,0} \varphi_t(z) G_{t-}} \int_t^\infty L_{t-}^*(u) (\Theta_t(u, z) f_t(u, z) - \varphi_t(z)) p_{t-}(u) \rho(du) \quad (73)$$

$$+ \mathbb{1}_{\{\tau < t\}} (\Theta_t(\tau, z) - 1), \quad \forall t \geq 0, \quad \forall z \in \mathbb{R}^*,$$

$$\xi_t^0 = \frac{p, \mathbb{F}(L_{t-}^*(t))}{L_{t-}^{*,0}} - 1, \quad \forall t \geq 0, \quad (74)$$

$$L_{t-}^{*,0} = \frac{1}{G_{t-}} \int_t^\infty L_{t-}^*(u) p_{t-}(u) \rho(du), \quad \forall t \geq 0, \quad (75)$$

where $L^{*,0}$ is the \mathbb{F} -predictable reduction of $L^{*,\mathbb{G}}$ and $L_-^{*,0}$ is its left limit. Here, each element of \mathcal{P}^* is a (locally) equivalent martingale measure on \mathbb{G} . Note that $\varkappa^{\mathbb{G}}$ does not depend on the choice of L_0^* (see (73)), whereas ξ^0 depends on it.

Since in (71), there are no constraints on ξ^0 , the set $\mathcal{P}(\mathbb{G})$ is strictly greater than \mathcal{P}^* .

7 Appendix

Using the same methodology as in [11] we prove the martingale property of the two local martingales used in the proof of Proposition 5.1.

- We first prove that the $\mathbb{F}^{(\tau)}$ -local martingale $\widetilde{M}(\tau) = (\widetilde{M}_t(\tau))_{t \geq 0}$ defined by

$$\widetilde{M}_t(\tau) = \int_0^t \Upsilon_{s-} dY_s(\tau), \quad \forall t \geq 0, \quad (76)$$

is a true martingale. This will be the case when, for any $T > 0$ fixed, the property

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |\widetilde{M}_t(\tau)| \right] < \infty$$

holds (see Chapter I, Theorem 51, page 38 in [27]). By Burkholder-Davis-Gundy's inequality⁵, this condition is satisfied if

$$\mathbb{E} \left[\left(\langle \widetilde{M}(\tau) \rangle_T^{\mathbb{F}^{(\tau)}} \right)^{1/2} \right] < \infty.$$

Note that we have

$$\begin{aligned} \mathbb{E} \left[\left(\langle \widetilde{M}(\tau) \rangle_T^{\mathbb{F}^{(\tau)}} \right)^{1/2} \right] &= \mathbb{E} \left[\left(\int_0^T \Upsilon_{s-}^2 \psi_s^2(\tau, z) \nu^{(\tau)}(ds, dz) \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq s \leq T} |\Upsilon_s| \left(\int_0^T \int_{\mathbb{R}^*} \psi_s^2(\tau, z) \nu^{(\tau)}(ds, dz) \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq s \leq T} |\Upsilon_s|^2 \right] + \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^*} \psi_s^2(\tau, z) \nu^{(\tau)}(ds, dz) \right], \end{aligned}$$

⁵Burkholder-Davis-Gundy's inequality states that, if M is a local martingale, for any $p \geq 1$, then the expression

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |M_t|^p \right] \leq C_p \mathbb{E} \left[\langle M \rangle_T^{p/2} \right]$$

holds, for some $C_p > 0$ depending on p only (see, e.g., Chapter IV, Section 4, Theorem 48, page 195 in [27]).

where we have used the fact that $|ab| \leq (a^2 + b^2)$, for any $a, b \in \mathbb{R}$. Using again Burkholder-Davis-Gundy's inequality, we obtain that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |\Upsilon_s|^2 \right] \leq \tilde{C} \mathbb{E} \left[\int_0^T (\gamma_s^{\mathbb{G}})^2 \nu^{(\tau)}(ds, dz) \right] < \infty,$$

for some constant $\tilde{C} > 0$. Moreover, by the assumption of square integrability of the $\mathbb{F}^{(\tau)}$ -martingale $Y(\tau)$, we have

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^*} \psi_s^2(\tau, z) \nu^{(\tau)}(ds, dz) \right] < \infty,$$

so that the process $\widetilde{M}(\tau)$ is a martingale.

- We now prove that the $\mathbb{F}^{(\tau)}$ -local martingale $\widehat{M}(\tau) = (\widehat{M}_t(\tau))_{t \geq 0}$ defined by

$$\widehat{M}_t(\tau) = \int_0^t Y_s(\tau) d\Upsilon_s, \quad \forall t \geq 0, \quad (77)$$

is a true martingale. As above, by Burkholder-Davis-Gundy's inequality, this will be the case when, for any $T > 0$ fixed

$$\mathbb{E} \left[\left(\langle \widehat{M}(\tau) \rangle_T^{\mathbb{F}^{(\tau)}} \right)^{1/2} \right] < \infty.$$

Note that we have

$$\begin{aligned} \mathbb{E} \left[\left(\langle \widehat{M}(\tau) \rangle_T^{\mathbb{F}^{(\tau)}} \right)^{1/2} \right] &= \mathbb{E} \left[\left(\int_0^T Y_s^2(\tau) (\gamma_s^{\mathbb{G}}(z))^2 \nu^{(\tau)}(ds, dz) \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq s \leq T} |Y_s(\tau)| \left(\int_0^T \int_{\mathbb{R}^*} (\gamma_s^{\mathbb{G}}(z))^2 \nu^{(\tau)}(ds, dz) \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq s \leq T} |Y_s(\tau)|^2 \right] + \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^*} (\gamma_s^{\mathbb{G}}(z))^2 \nu^{(\tau)}(ds, dz) \right]. \end{aligned}$$

It follows, using again Burkholder-Davis-Gundy's inequality, that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |Y_s(\tau)|^2 \right] \leq \widehat{C} \mathbb{E} \left[\int_0^T \psi_s^2(\tau, z) \nu^{(\tau)}(ds, dz) \right] < \infty,$$

for some constant $\widehat{C} > 0$. Moreover, by the assumption of square integrability of the $\mathbb{F}^{(\tau)}$ -martingale Υ , we have

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^*} (\gamma_s^{\mathbb{G}})^2 \nu^{(\tau)}(ds, dz) \right] < \infty,$$

so that the process $\widehat{M}(\tau)$ is a martingale.

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