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# Linear random generation of Motzkin trees

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## Abstract

*Motzkin trees* are also called *unary-binary trees*. This paper proposes a linear algorithm for uniform random generation of Motzkin trees. The algorithm uses the same paradigm as this of Rémy's linear algorithm for random generation of *binary trees* and is based on a preliminary computation.

**Keywords:** combinatorics, random generation, Motzkin number, Catalan number, binary tree, unary-binary tree

## 1 Introduction

*Motzkin trees* are also called *unary-binary trees*. This paper proposes a linear algorithm for random generation of Motzkin trees. The algorithm takes the same paradigm as this of Rémy's linear algorithm for random generation of *binary trees* [19]. Assume  $n$  is the size of the trees. Rémy's algorithm is based on a bijective proof of the inductive equality:

$$(n + 1)C_n = 2(2n - 1)C_{n-1}$$

where the  $C_n$ 's are the Catalan numbers. My algorithm for random generation of Motzkin trees is based on a bijective proof due to Dulucq and Penaud [10] of the inductive equality:

$$(n + 2)M_n = (2n + 1)M_{n-1} + 3(n - 1)M_{n-2}$$

where the  $M_n$ 's are the Motzkin numbers. At some points of the algorithm, choices have to be made. To avoid an excessive cost for these steps, we propose a preprocessing. This paper is associated with a Haskell [program](#) available on GitHub which serves as an executable specification.

## 2 Motzkin numbers and Motzkin trees

The  $n^{\text{th}}$  Motzkin number  $M_n$  is the number of different ways of drawing non-intersecting chords between  $n$  points on a circle (not necessarily touching every point by a chord). Motzkin numbers count also well parenthesized expressions with a constant  $\mathbf{c}$ , called *Motzkin words*. They are words of length  $n$  in the language generated by the grammar  $M$ .

$$M = \varepsilon \mid \mathbf{c} M \mid (M) M$$

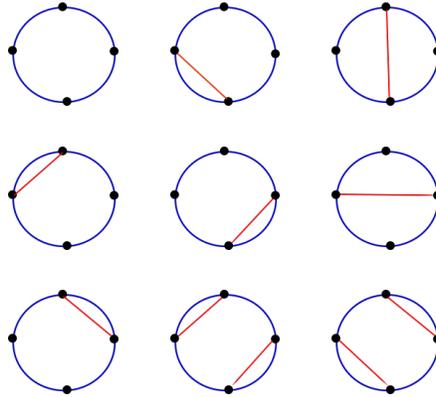
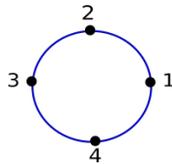


Figure 1: The 9 non-intersecting chords between 4 points on a circle

<b>c</b> <b>c</b> <b>c</b> <b>c</b>	<b>c</b> <b>(</b> <b>c</b> <b>)</b>	<b>c</b> <b>(</b> <b>c</b> <b>)</b>
<b>c</b> <b>(</b> <b>)</b> <b>c</b>	<b>(</b> <b>c</b> <b>c</b> <b>)</b>	<b>(</b> <b>c</b> <b>)</b> <b>c</b>
<b>(</b> <b>)</b> <b>c</b> <b>c</b>	<b>(</b> <b>(</b> <b>)</b> <b>)</b>	<b>(</b> <b>(</b> <b>)</b> <b>)</b>

Figure 2: The 9 parenthesized words with constant **c**.

The bijection between sets of non intersecting chords and well parenthesized words with constant **c** is as follows: first one numbers nodes on the circle counterclockwise, as follows:



A position at the beginning of a chord on the circle corresponds to an opening parenthesis. A position at the end of a chord on the circle corresponds to a closing parenthesis. A position which is neither of those corresponds to the constant **c**.

Motzkin numbers count also routes in the upper quadrant from  $(0,0)$  to  $(0,4)$  with move *up*, *down* and *straight*.

The bijection is as follows: an opening parenthesis corresponds to an *up*, a closing parenthesis corresponds to a *down* and the constant **c** corresponds to a *straight*.

Motzkin number  $M_n$  counts also the number in unary-binary planar trees with  $n$  edges, that are tree structures with nodes of arity one or two and with  $n$  edges. Let us call this number  $n$  of edges the *size* of the Motzkin tree. Notice that the number of nodes of a Motzkin tree of size  $n$  is  $n + 1$ , i.e., the size plus one. Figure 6 gives the trees for  $n = 4$ . The bijection  $f$  from well parenthesized expressions with constant **c** to Motzkin trees is as follows. Its inverse  $f^{-1}$  is also

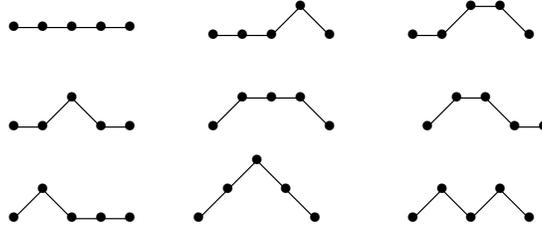


Figure 3: The 9 routes in the upper right quadrant from  $(0,0)$  to  $(0,4)$  with move *up*, *down* and *straight*

given.

$$\begin{array}{lcl}
 f[\varepsilon] & = & \bullet \\
 f[\mathbf{c} w] & = & \begin{array}{c} \bullet \\ | \\ f[w] \end{array} \\
 f[(w_1) w_2] & = & \begin{array}{c} \bullet \\ / \quad \backslash \\ f[w_1] \quad f[w_2] \end{array}
 \end{array}
 \qquad
 \begin{array}{lcl}
 f^{-1}[\bullet] & = & \varepsilon \\
 f^{-1}\left[\begin{array}{c} \bullet \\ | \\ t \end{array}\right] & = & \mathbf{c} f^{-1}[t] \\
 f^{-1}\left[\begin{array}{c} \bullet \\ / \quad \backslash \\ t_1 \quad t_2 \end{array}\right] & = & (f^{-1}[t_1]) f^{-1}[t_2]
 \end{array}$$

Motzkin numbers fulfill the equation:

$$M_{n+1} = M_n + \sum_{i=0}^{n-1} M_i M_{n-1-i}$$

### 3 Random binary trees

*Rémy's algorithm* [19] for generation of random binary trees is linear. It is based on a constructive proof of the holonomic equation:

$$(n + 1)C_n = 2(2n - 1)C_{n-1}$$

where, roughly speaking, “holonomic” means that  $C_n$  is a linear combination of the  $C_i$ 's, for  $0 \leq i < n$  (see Flajolet and Sedgewick's book [11], Appendix B.4). Here this holonomic equation is very peculiar since  $C_n$  is equal to  $C_{n-1}$  times a factor. We will see that this is not the case for Motzkin numbers. Rémy's algorithm is described by Knuth in [16] § 7.2.1.6 (pp. 18-19) and works on *extended binary trees*, or just *binary trees* in which we distinguish *internal nodes* and *external nodes* or *leaves*. The idea of the algorithm is that a random binary tree can be built by iteratively and randomly drawing an internal node or a leaf in a random binary tree and inserting, between it and its father a new internal node and a new leaf either on the left or on the right (see Figure 4). An insertion is also possible at the root. In this case, the new inserted node becomes the root. This is not a specific case in the algorithm as we will see. The root can be seen as the child of an hypothetical node

A binary tree of size  $n$  has  $n - 1$  internal nodes and  $n$  leaves. We label binary trees with numbers between 0 and  $2n - 2$  such that internal nodes are labeled with odd numbers and leaves are labeled with even numbers. Inserting a node in a binary tree of size  $n$  requires drawing randomly a number between 0 and  $4n - 3$ . This process can be optimized by representing a

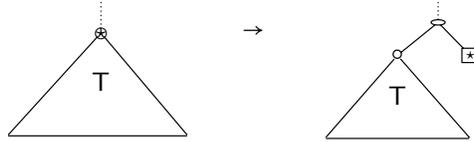
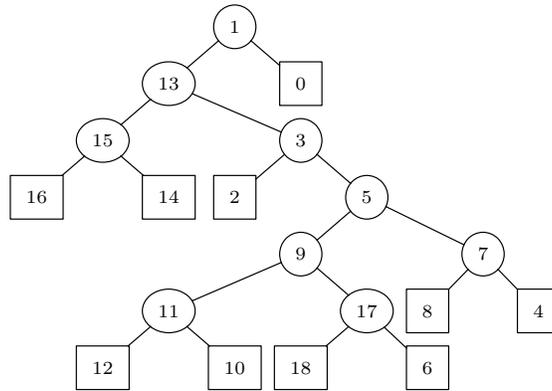


Figure 4: Rémy's right insertion of a leaf

binary tree as a list (a **vector** in **Haskell**), an idea sketched by Rémy and described by Knuth. In this vector, even values are for internal nodes and odd values are for leaves. The root is located at index 0. The left child of an internal node with label  $2k + 1$  is located at index  $2k + 1$  and its right child is located at index  $2k + 2$ . Here is a vector representing a binary tree with 10 leaves and its drawing.

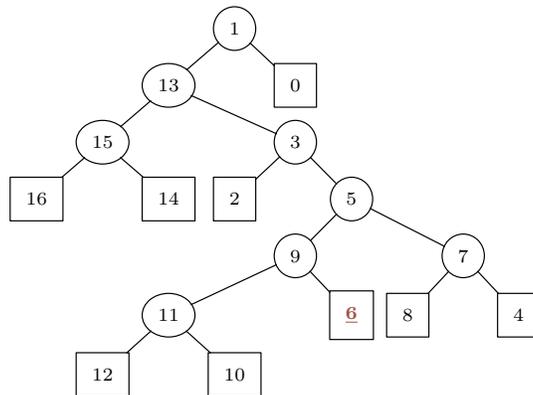
<b>indices</b>	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
<b>values</b>	1	13	0	2	5	9	7	8	4	11	17	12	10	15	3	16	14	18	6



This tree was built by inserting the node 17 together with the leaf 18 in the following vector.

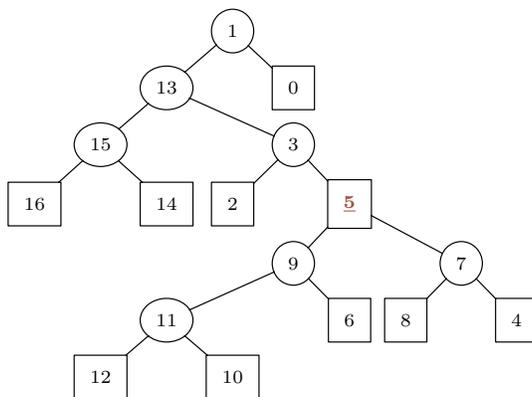
<b>indices</b>	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
<b>values</b>	1	13	0	2	5	9	7	8	4	11	6	12	10	15	3	16	14

which codes the tree

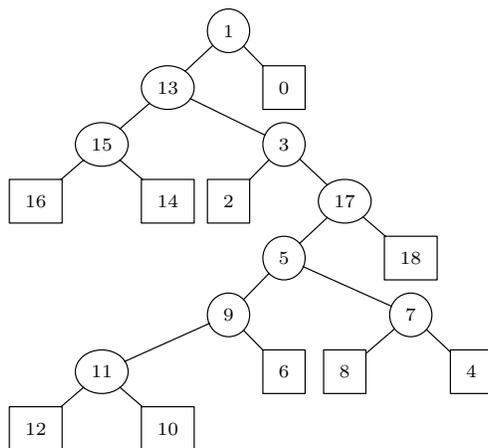


This was done by drawing a node (internal node or leaf, here the node with label 6, right child of the node with label 9) and a direction (here right) and by inserting above this node a new internal node (labeled 17) and, below the new inserted internal node, a new leaf of the left (labeled 18). This double action (inserting the internal node and attaching the leaf) is done by choosing a number in the interval  $[0..33]$  (in general, in the interval  $[0..(4n - 3)]$ ). Assume that in this case the random generator returns 21. 21 contains two pieces of information : its parity (a boolean) and floor of its half. Half of 21 is 10, which tells that the new node 17 must be inserted above the 11<sup>th</sup> node (in the vector) namely 6. Since 21 is odd, the rest of the tree (here reduced to the leaf 6) is inserted on the right (otherwise it would be inserted on the left). A new leaf 18 is inserted on the left (otherwise it would be inserted on the right).

Consider the same tree and suppose that the random value is 8. Half of 8 is 4. Hence the new internal node labeled by 17 is inserted above the node labeled by 5



and, since 8 is even, the rest of the tree is inserted on the left and a new leaf (labeled 18) is inserted on the right.



The algorithm (Figure 5) works as follows. If  $n = 0$ , Rémy’s algorithm returns the vector starting at 0 and filled with anything, since the whole algorithm works on the same vector with the same size. In general, say that, for  $n - 1$ , Rémy’s algorithm returns a vector  $v$  (**vector** is the concept used in **Haskell** for arrays that can be changed in place). One draws a random integer  $x$  between 0 and  $4n - 3$ . Let  $k$  be half of  $x$ . In the vector  $v$  one replaces the  $k^{th}$  position with  $2n - 1$  and one appends two elements, namely the  $k^{th}$  item of  $v$  followed by  $2n$  if  $x$  is even and  $2n$  followed by the  $k^{th}$  item of  $v$  if  $x$  is odd.

The algorithm builds a uniformly random *decorated binary tree*, i.e., a binary tree with its leaves numbered  $0, 2, \dots, 2n$ . We notice that the construction of a tree with such labels is unique, the labels of the internal nodes are a consequence of the construction, hence are deduced from the labels of the leaves. If we ignore the leaves, we get a uniform distribution for the undecorated binary trees (i.e., with no labels on the leaves).

In the program, `rands` is a vector of random floating numbers between 0 and 1. The reader who wants to read a better typographic presentation of the program is invited to get the version on my [web page](#).

```

1 rbt :: Vector Float -> Int -> Vector Int
2 rbt _ 0 = initialVector // [(0,0)]
3 rbt rands n =
4   let x = floor ((rands!n) * fromIntegral (4*n-3))
5       -- x is a random value between 0 and 4n-3 --
6       v = rbt rands (n-1)
7       k = x `div` 2
8   in case even x of
9     True  -> v // [(k,2*n-1),(2*n-1,v!k),(2*n,2*n)]
10    False -> v // [(k,2*n-1),(2*n-1,2*n),(2*n,v!k)]

```

Figure 5: Haskell program for Rémy’s algorithm

## 4 Dulucq-Penaud bijection proof

Motzkin numbers fulfill also the *holonomic equation* [11]:

$$(n + 2)M_n = (2n + 1)M_{n-1} + 3(n - 1)M_{n-2}.$$

Together with the equalities  $M_0 = 1$  and  $M_1 = 1$ , Motzkin numbers can be computed and form the sequence [A001006](#) in the *online encyclopedia of integer sequences* [15]. In this section, we present Dulucq and Penaud’s proof of this equality [10]. This proof relies on the exhibition of a bijection between the objects counted by the left-hand side and those counted by the right-hand side. The first idea is to consider specific binary trees called *slanting binary trees* and divide those trees into 7 subclasses.

### 4.1 Slanting binary trees

Following Dulucq and Penaud, I represent *Motzkin trees* as specific *binary trees* in which leaves  $\square$  are added. In such binary trees, only the three first configurations below are allowed and the rightmost one is not



The first configuration corresponds to a binary node, the second configuration corresponds to a unary node and the third configuration corresponds to an end node in the classical presentation (for instance in Figure 6). I call such trees *slanting trees*.

Figure 6 shows the 9 Motzkin trees with 4 edges and their corresponding slanting trees. Let us label each node of a slanting tree with a number between 1 and  $2m + 1$ , where  $m$  is the

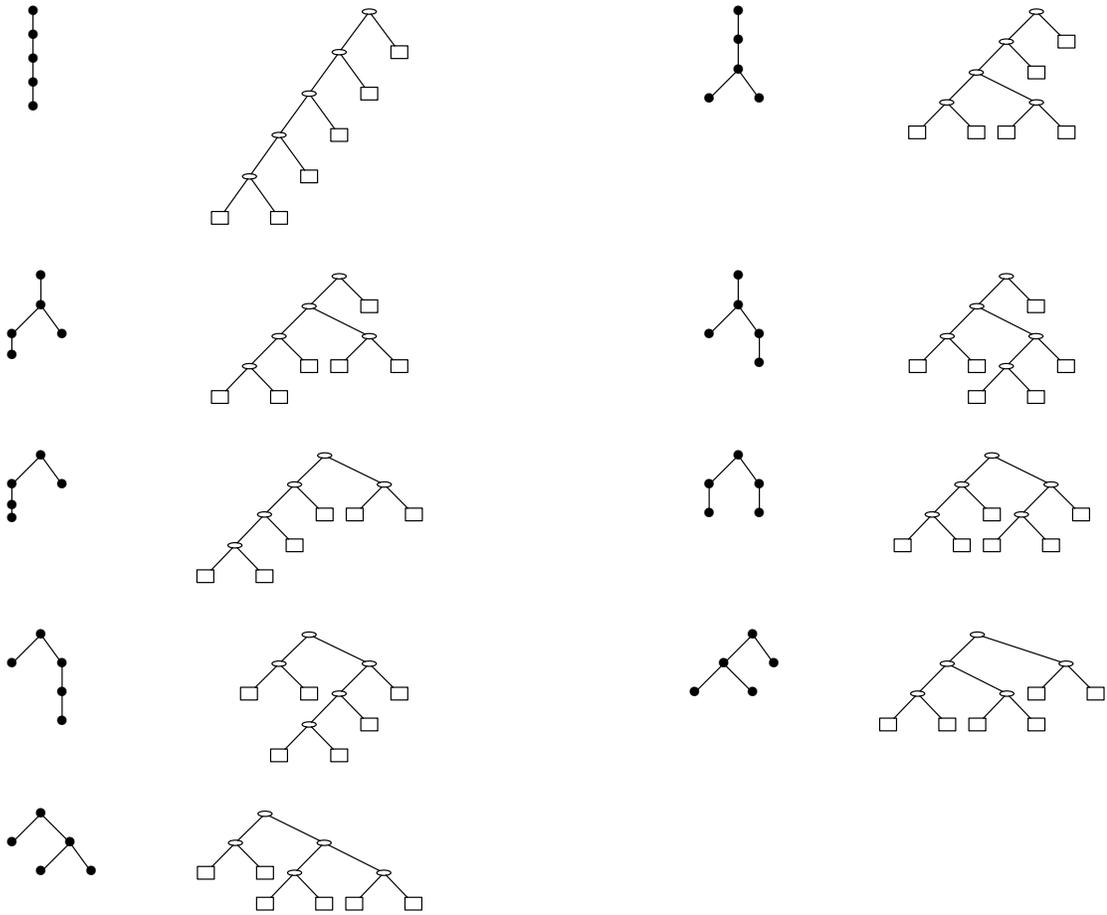
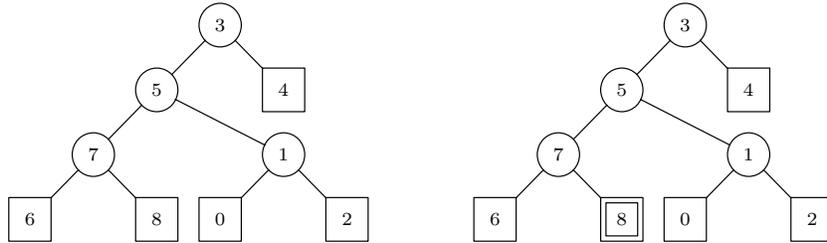


Figure 6: 9 Motzkin trees with 4 edges and their slanting trees

number of internal nodes of the slanting tree. Let us call such a labeled tree a *labeled slanting tree*. Now consider labeled slanting trees with one marked leaf. Let us call it a *leaf-marked slanting tree*. Below there is a labeled slanting tree of size 4 and a leaf-marked labeled slanting tree, where the mark is on the leaf labeled 8.



This corresponds to the vector:

$$[3, 0, 2, 5, 4, 7, 1, 6, 8]$$

How nodes and leaves are labeled by numbers will be explained below and is essentially similar to binary trees. Just notice that *internal nodes* have odd labels and *leaves* have even labels. From now on, let us forget the labels, but let us mark one of the leaves. In such trees with a marked leaf, we can distinguish 7 general patterns of subtrees containing the marked leaf (Figure 7 first column). The marked leaf is denoted by a star in a square, namely  $\square^*$ . In the first group of 4 there are the patterns where the marked leaf is a right child and in the second group of 3, there are the patterns where the marked leaf is a left child, hence, due to the constraints on slanting trees, the other child (a right child) is also a leaf.

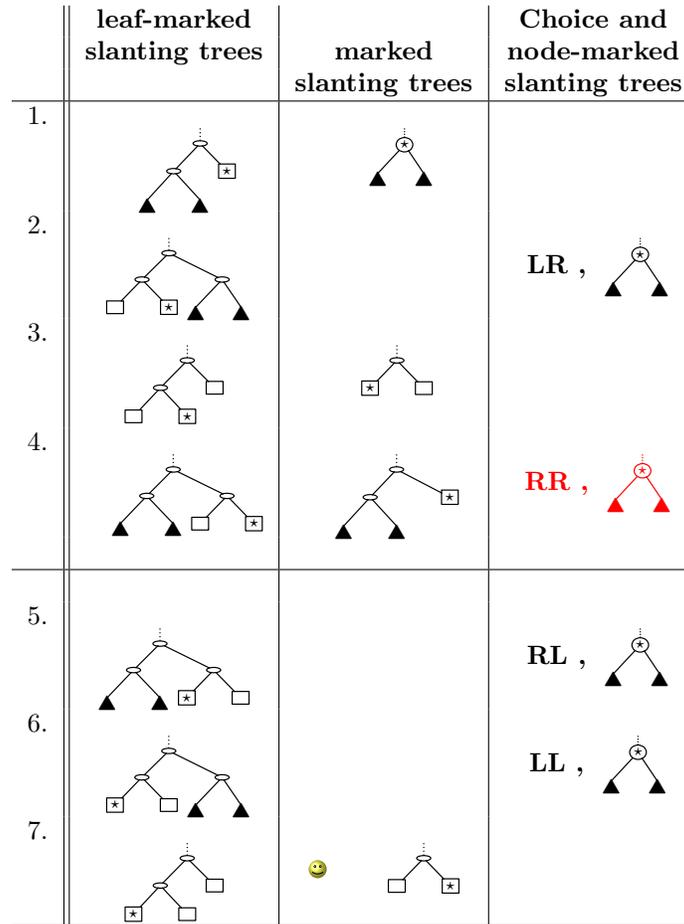


Figure 7: The 7 patterns of leaf-marked slanting trees

Let us call *node-marked* a slanting tree in which one internal node is marked. Let us call *marked tree*, a slanting tree in which either a leaf or an internal node is marked. Recall Rémy’s algorithm which consists in inserting, at the marked place (internal node or leaf), a leaf in a marked tree (see Figure 4). After the insertion the formerly marked node becomes unmarked and the inserted leaf becomes marked. Here, since we are interested in Motzkin trees, we insert a leaf on the right, above the marked node in the marked slanting tree and like in *Rémy’s algorithm*, a leaf insertion on a marked tree is performed, but unlike *Rémy’s algorithm*

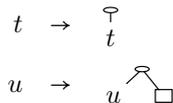
the insertion is performed only on the right and a leaf-marked slanting tree is produced. This corresponds to what is done to *pattern1*, *pattern3* and *pattern4* in the middle column of Figure 7. But as we will see for the other patterns, there are other ways to increase a slanting tree when it does not correspond to one of these three patterns.

### 4.2 How many leaves in a Motzkin tree?

The slanting tree associated with a Motzkin tree of size  $n$  (number of its edges) has  $n + 2$  leaves. This can be shown by induction.

**Basic case:** If the Motzkin tree is  $\bullet$ , its size is 0 and its associated slanting tree  has 2 leaves.

**Adding a unary node:** Assume we add a unary node above a Motzkin tree  $t$  of size  $n$ , this yields a Motzkin tree  $t'$  of size  $n + 1$ . The slanting tree associated with  $t'$  has  $n + 2$  leaves (the number of the leaf of the slanting tree  $u$  associated with  $t$ ) plus a new one added, then all together  $n + 3$ .



**Adding a binary node:** Assume we add a binary node above two Motzkin trees  $t_1$  and  $t_2$  of size  $n_1$  and  $n_2$ , this yields a Motzkin tree  $t'$  of size  $n_1 + n_2 + 2$ . The slanting trees associated with  have  $n_1 + n_2 + 4$  leaves.

## 5 The bijection

What makes the random generation of Motzkin trees trickier than Rémy’s algorithm is the structure of the holonomic equation:

$$(n + 2)M_n = (2n + 1)M_{n-1} + 3(n - 1)M_{n-2}$$

when compared to the equation:

$$(n + 1)C_n = 2(2n - 1)C_{n-1}$$

First, if we use a construction based on that equation, a Motzkin tree of size  $n$  can be built from a Motzkin tree of size  $n - 1$  or from a Motzkin tree of size  $n - 2$ . Thus there are at least two cases to consider. Actually 6 cases as we will see. Notice that  $M_n$  counts both the number of Motzkin trees of size  $n$  and the slanting trees with  $n + 2$  leaves.

### Interpreting the holonomic equation

We conclude that  $(n + 2)M_n$  counts the number of *leaf-marked* slanting trees of size  $n$ , that  $(2n + 1)M_{n-1}$  counts the number of *marked* slanting trees of size  $n - 1$  and that  $(n - 1)M_{n-2}$  counts the number of *node-marked* slanting trees of size  $n - 2$ . Therefore looking at the equation, we see that we should be able to build a leaf-marked slanting tree of size  $n$  from either a marked slanting tree of size  $n - 1$  or from a pair made of an item which can take one of three values and of a *node-marked* slanting tree of size  $n - 2$ . Let us see how Dulucq and Penaud propose to proceed.

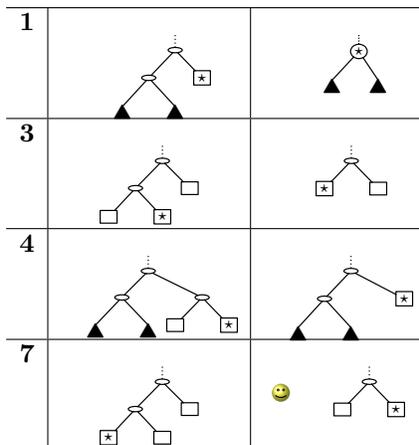


Figure 8: Contribution of marked slanting tree of size  $n - 1$ . 😊 for *pattern7* marks a specific case explained in Section *The lower part*

### A taxonomy of leaf-marked slanting trees

Leaf-marked slanting trees can be sorted according to the position of their mark. This is done in the first column of Figure 7. This column has two parts.

#### The upper part

In the upper part, we have four patterns in which the marked leaf is a right child. Let us call them *pattern1*, *pattern2*, *pattern3* and *pattern4*. Three of them *pattern1*, *pattern3* and *pattern4* are obtained by Rémy’s right insertion of a leaf (Figure 4) in a marked slanting tree. The other *pattern2* (Figure 9) is not. Indeed if the marked leaf is removed, the tree that is obtained has a leaf on the left and a node on the right, which is forbidden. This pattern will be obtained another way.

#### The lower part

In the lower part, there are three patterns which correspond to the case where the marked leaf is a left child, hence the sister of a leaf (Figure 9); *pattern7* cannot be obtained by a right insertion of a leaf (Figure 4), this is why we mark it by 😊. But Dulucq and Penaud noticed that pattern among marked slanting trees is not taken into consideration. Thus they propose to associate this pattern with *pattern7*, as shown in Figure 7.

### The bijection by cases

#### Case $(2n + 1)M_{n-1}$

The previously explained contribution to leaf-marked slanting trees of size  $n$  from marked slanting tree of size  $n - 1$  is summarized in Figure 8. All the patterns of marked slanting trees are taken into account.

**Case  $3(n-1)M_{n-2}$**

Let us now look at the three remaining patterns: *pattern2*, *pattern5* and *pattern6*; forming the lines of Figure 9. Those three patterns have the same model, namely an internal node with two children, one is an internal node and the other one is an internal node whose children are two leaves, one of which is marked, the other is not. Depending on the position of the marked leaf, we distinguish three cases.

- **LL** corresponds to the case where the marked node is on the left of the top node and on the left of its father node.
- **LR** corresponds to the case where the marked node is on the left of the top node and on the right of its father node.
- **RL** corresponds to the case where the marked node is on the right of the top node and on the left of its father node.

One notices that there is no case **RR**, because this would correspond to *pattern4* considered in the previous section. As a matter of fact, the three cases **LL**, **LR** and **RL** correspond to the multiplicative factor 3 in the holonomic equation.

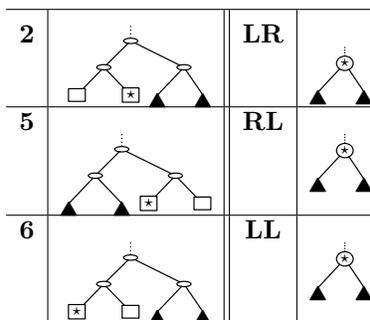


Figure 9: Contribution of node-marked slanting tree of size  $n - 2$

**Forgetting the marks on leaves**

As Rémy noticed for binary trees, since we generate the leaf-marked slanting trees of size  $n$  uniformly, we also get a uniform distribution of slanting trees of size  $n$ . Thus we can forget the marks, which we do in the concrete algorithm.

**6 A concrete algorithm for random generation of Motzkin trees**

The Haskell program of Figure 10 can be considered as abstract enough to present the algorithm for random generation of Motzkin trees. Thus, in what follows, I make basically no distinction between the algorithm and the program and I consider the Haskell program as a executable specification.

Assume that there is a function `motzkin` that returns the  $n^{\text{th}}$  Motzkin number. Like for Rémy's algorithm, one represents a labeled slanting tree by a vector. In this vector, odd labels are for internal nodes and even labels are for leaves. Notice that the algorithm preserves two properties:

1. the vector codes a slanting tree,
2. the vector of a Motzkin tree of size  $n$  has a length  $2n + 3$ .

In order to choose which case to consider, namely  $(2n + 1)M_{n-1}$  (**case1**) or  $3(n - 1)M_{n-2}$  (**case2**), the algorithm `rMt` requires a random value between 0 and  $(n + 2)M_n$  which we call  $r$ . If  $r$  is less than or equal to  $(2n + 1)M_{n-1}$ , we are in **case1**, else we are in **case2**. Said otherwise, given a random value  $c$  between 0 and 1 if  $c \leq \frac{(2n+1)M_{n-1}}{(n+2)M_n}$  we choose **case1**, if not we choose **case2**.

- **case1**: one draws at random a leaf or an internal node in a slanting tree of size  $n - 1$ . This means choosing at random an index  $k$  in the vector  $v$ . We get *pattern7* if three conditions are fulfilled.
  1. **The marked item, should be a right child.** This means that  $k$  is even, since the left child of a node of index  $2p + 1$  is  $2p + 1$  and the right child of this node is  $2p + 2$ .
  2. **The marked item is a leaf.** This means that  $v[k]$  is even, since leaves have even labels. Notice that Haskell uses the notation  $v!k$  for our mathematical notation  $v[k]$ .
  3. **The sister item of the marked item is a leaf** (a left child by the way). This means that  $v[k - 1]$  is even.

In this case ( $k$  is even,  $v[k]$  is even and  $v[k - 1]$  is even) one inserts a node and a leaf as shown by Figure 8, which corresponds in the code to:

```
v // [(k-1, 2*n+1), (2*n+1, v!(k-1)), (2*n+2, 2*n+2)]
```

In Haskell, the operator `//` updates vectors at once, it is called a **bulk update**.  $(2*n+1, v!(k-1))$  means that the left child is a new node, at index  $2 * n + 1$ , which points to the former value of  $v!(k-1)$ . The right child is a new leaf.

The other cases (*pattern1*, *pattern3*, *pattern4*) correspond to cases when one of  $k$ ,  $v[k]$  or  $v[k - 1]$  is odd. The update is then

```
v // [(k, 2*n+1), (2*n+1, v!k), (2*n+2, 2*n+2)]
```

which corresponds to the first lines of Figure 8.

- **case2**: In this case we consider a random node-marked slanting trees of size  $n - 2$  and a random values among **LR**, **RL**, **LL**. For that we draw a number  $r$  between 0 and  $3n - 6$ , from which  $r \div 3$  gives a random number between 0 and  $n - 2$  (a random node) and  $r \bmod 3$  yields a random number among 0, 1 and 2. We notice that **LR** and **LL** correspond to the same transformation, while **RL** corresponds to another transformation. In each case one adds four nodes, with labels  $2n - 1$ ,  $2n$ ,  $2n + 1$  and  $2n + 2$ . Thus,

```
v // [(2*k+1, 2*n-1), (2*k+2, 2*n+1), (2*n-1, 2*n), (2*n, 2*n+2),
      (2*n+1, v!(2*k+1)), (2*n+2, v!(2*k+2))]
```

is the transformation for **LR** and **LL** and

$$v // [(2*k+1, 2*n-1), (2*k+2, 2*n+1), (2*n-1, v!(2*k+1)), (2*n, v!(2*k+2)), (2*n+1, 2*n), (2*n+2, 2*n+2)]$$

corresponds to **RL**. We let the reader check that the code corresponds to the pictures of Figure 9.

```

1 rMt :: Vector Float -> Int -> Vector Int
2 rMt _ 0 = initialVector // [(0,1),(1,0),(2,2)]
3 rMt _ 1 = initialVector // [(0,1),(1,3),(2,0),(3,2),(4,4)]
4 rMt rands n = -- (rands!n) a Float between 0 and 1
5   let floatToInteger x n = floor (x * (fromIntegral n))
6       r = floatToInteger (rands!(3*n)) ((fromIntegral (n+2))*(motzkin n))
7   in case r <= (fromIntegral (2*n+1)) * (motzkin (n-1)) of
8       True -> case1 rands n
9       False -> case2 rands n
10      False -> case2 rands n
11
12 -- (even k) means that the marked item is a right child
13 -- (even (v!k)) means that its label is even, hence it is a leaf
14 -- (even (v!(k-1))) means that the label of its left sister is even,
15 --             hence it is a leaf
16 -- Therefore {(even k) &&& (even (v!k) &&& (even (v!(k-1)))} is configuration 7
17 case1 :: Vector Float -> Int -> Vector Int
18 case1 _ 0 = initialVector
19 case1 _ 1 = initialVector -- 0, 1 are never invoqued and are irrelevant
20 case1 rands n =
21   let k = floor ((rands!(3*n+1)) * (fromIntegral (2*n)))
22       v = rMt rands (n-1)
23   in case odd k || odd (v!k) || odd (v!(k-1)) of
24       True -> v // [(k,2*n+1),(2*n+1,v!k),(2*n+2,2*n+2)]
25       False -> v // [(k-1,2*n+1),(2*n+1,v!(k-1)),(2*n+2,2*n+2)]
26
27 case2 :: Vector Float -> Int -> Vector Int
28 case2 _ 0 = initialVector
29 case2 _ 1 = initialVector -- 0, 1 are never invoqued and are irrelevant
30 case2 rands n =
31   let r = floor ((rands!(3*n+2)) * (fromIntegral (3*n-6)))
32       k = r 'div' 3
33       c = r 'rem' 3
34       v = rMt rands (n-2)
35   in case c < 2 of
36   True -> v // [(2*k+1,2*n-1),(2*k+2,2*n+1),(2*n-1,2*n),
37               (2*n,2*n+2),(2*n+1,v!(2*k+1)),(2*n+2,v!(2*k+2))]
38   False -> v // [(2*k+1,2*n-1),(2*k+2,2*n+1),(2*n-1,v!(2*k+1)),
39                 (2*n,v!(2*k+2)),(2*n+1,2*n),(2*n+2,2*n+2)]

```

Figure 10: Haskell program for random generation of Motzkin trees

## 7 Linear algorithm

In the above algorithm, choosing between `case1` and `case2` requires computing  $M_n$  and  $M_{n-1}$ , numbers of order  $3^n$ . Since for each instance of the algorithm, one uses the same values  $(2n + 1)M_{n-1}/(n + 2)M_n$ , these values can be precomputed once for all and stored in a vector, hence making the algorithm linear.

### Arithmetic complexity

In arithmetic complexity, in which operation  $+$ ,  $-$ ,  $*$  and  $/$  take time  $O(1)$ , the table can be precomputed in time  $O(n)$ . Therefore the full algorithm is linear.

### Bit complexity

In bit complexity, the precomputation takes time  $O(n^2)$  and then the algorithm itself is linear.

### Results

I precomputed the values, using SAGE [7], until  $n = 100\,000$  and the HASKELL program (Figure 12) produces random Motzkin trees in times given by the arrays of Figure 11.

size	time
1 000	1.4s
2 000	3.1s
3 000	5.3s
4 000	6.6s
5 000	8.4s

size	time
6 000	8.5s
7 000	10.8s
8 000	13.0s
9 000	12.7s

size	time
10 000	15s
20 000	32s
30 000	54s
40 000	74s
50 000	85s

size	time
60 000	101s
70 000	131s
80 000	160s
90 000	172s
100 000	179s

Figure 11: Benchmarks

```

1 rMtFast :: Vector Double -> Int -> Vector Int
2 rMtFast _ 0 = initialVector // [(0,1),(1,0),(2,2)]
3 rMtFast _ 1 = initialVector // [(0,1),(1,3),(2,0),(3,2),(4,4)]
4 rMtFast rands n = -- (rands!n) a Float between 0 and 1
5   case ratioM!n <= rands!n of
6     True -> case1Fast rands n
7     False -> case2Fast rands n

```

Figure 12: A program with processing (vector ratioM)

## 8 Related Works

Our work was inspired by Jean-Luc Rémy’s algorithm [19]. Laurent Alonso [1] proposed an algorithm for generating uniformly Motzkin trees. His method consists in generating the number  $k$  of binary nodes with the correct probability law; he uses then standard techniques to

generate a unary-binary tree with  $k$  binary nodes among  $n$  nodes. The number of trees with  $k$  binary nodes is over-approximated by values that follow a binomial distribution: choosing  $k$  is therefore done using random generations for a binomial law and rejections. Therefore his algorithm is *linear on the average*, with possible but rare long sequences of rejection. Dominique Gouyou-Beauchamps and Cyril Nicaud [14] propose a random generation for color Motzkin trees which is linear on the average and Srečko Brlek et al. [5] propose an extension of Alonso’s algorithm to generalized versions of Motzkin trees.

Axel Bacher, Olivier Bodini and Alice Jacquot [2] propose an algorithm with similar ideas. Especially their Figure 2 shares similarity with our Figure 9. Their “operations”  $G_3$ ,  $G_4$  and  $G_5$  are connected with our cases **RL**, **LR** and **LL** respectively.  $\circlearrowright$  corresponds to  and  $\circlearrowleft$  corresponds to . However the algorithm they propose has a linear expected complexity, like Alonso’s. Let us also mention generic Boltzmann’s samplers with exact-size which apply among others to Motzkin trees [4, 17] and generic algorithms [8, 18] with linear expected complexity.

Denise and Zimmermann [6] discuss what can be done on floating-point arithmetic when generating random structures. The authors focus on decomposable labeled structures [12] and address the problem of choice, with a specific section on Motzkin trees and a non linear algorithm.

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## 9 Conclusion

Generating Motzkin trees has many potential applications [9, 3]. Moreover my algorithm has a simple code and is linear. Among the possible extensions of my method which could be explored, there is the generation of extended versions of Motzkin structures like Motzkin trees with colored leaves [14] or Motzkin paths with  $k$  long straights [5].  $k = 1$  corresponds to Motzkin paths and  $k = 2$  to Schröder paths. I am currently working on a linear algorithm for random generation of Schröder paths or trees, since Dominique Foata and Doron Zeilberger [13] proposed a constructive proof of the holonomic equation for Schröder numbers, which inspired Dulucq and Penaud.

On another hand, Bracucci et al. [3] study a family of sets of permutations:  $\mathcal{M}_1$ ,  $\mathcal{M}_2$ , ...,  $\mathcal{M}_\infty$ , in which  $\mathcal{M}_1$  is for Motzkin permutations (that are Motzkin trees up to a bijection) and  $\mathcal{M}_\infty$  is for Catalan permutations (that are binary trees up to a bijection) an interpolation of our method seems doable.

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