Spectral equations for scattering by impedance polygons : properties and solutions

J.M.L. Bernard

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J.M.L. BERNARD

1) Introduction

The Sommerfeld-Maliuzhinets representation of fields is not limited to the study of the diffraction by isolated impedance wedges, and we developed in [1]-[2] an analytical method concerning the determination of spectral function for the scattering in free space by an impedance polygonal object (convex or not). This method is now quoted and used by other authors [3], but other approaches exist, as in [4]-[13] for cavities, perfectly conducting object, and some specific geometries by asymptotic or exact reduction, or as in [14] by iterative reduction. Our approach is exact for general impedance polygons with finite or infinite surfaces. For that, we consider special features of single face expression of spectral function, that we defined in [1]-[2], to develop exact functional difference equations, then Fredholm integral equations, that can be solved exactly, but also asymptotically, considering small or large faces. We detail here novel properties. Existence and uniqueness of solutions are, in particular, analysed in a new original manner, while some refined asymptotics are presented.

2) Generalities

Let us consider the case of diffraction in free space of a plane wave

$$u^i(\rho,\varphi) = e^{ik\rho\cos(\varphi-\varphi_\circ)}, \tag{1}$$

by a scatterer enclosed in a wedge-shaped region, defined in cylindrical coordinates (ρ,φ) as the domain complementary to the free space sector of radiation, $-\Phi_r \leq \varphi \leq \Phi_l$ (figure 1), with $-\Phi_r \leq \varphi_\circ \leq \Phi_l$. The characteristics of the scatterer are supposed to be independent of z coordinate. An implicit harmonic dependence on time $e^{i\omega t}$ is understood and henceforth suppressed. In expression (1), k denotes the wave number of the exterior medium with $|\arg(ik)| < \frac{\pi}{2}$, while $|\arg(ik)| = \frac{\pi}{2}$ is considered as a limit case. We assume that the total field in the free space region, $u = u_s + u^i$, satisfies the Helmholtz equation,

$$(\Delta + k^2)u(\rho,\varphi) = 0, \tag{2}$$

with analyticity with respect to ρ and φ as $-\Phi_r < \varphi < \Phi_l$, except possibly at the origin O. Some elementary properties are assumed to hold for the field :

- (a') the only incoming plane wave, from the free space sector with origin O, $-\Phi_r \leq \varphi \leq \Phi_l$, is the incident field;
- (b') the field u tends to a finite value independent of φ as $\rho \to 0$, and the derivatives $\partial_\rho u$ and $\partial_\varphi u/\rho$ are locally summable with respect to ρ in the vicinity of the origin. This property applies for an origin taken at any point out of or upon the scatterer;
- (c') the field, except possibly its geometrical optics part when $\mathrm{Im}(k) \neq 0$, does not grow at infinity. In addition, some bounds on the far field are assumed. We consider here that the field is $O(e^{ik\rho\cos(\varphi-\varphi_\circ)})$ for large ρ, $|\arg(ik)| < \pi/2$, which is a standard assumption [1]-[2] when the scattering object has impedance boundary conditions.

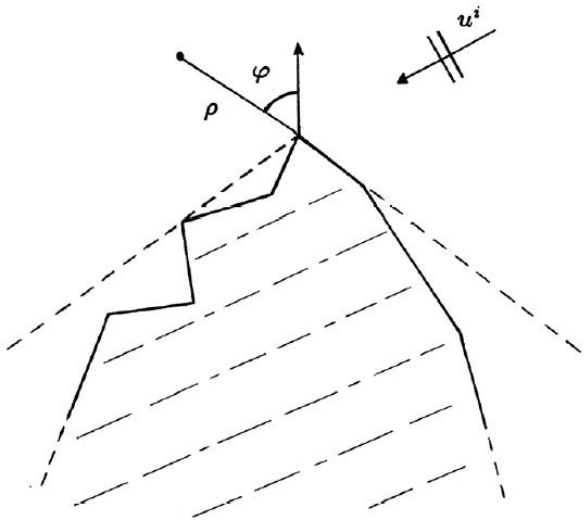

figure 1 : a polygonal surface in a wedge shaped sector

The total field u for $-\Phi_r \leq \varphi \leq \Phi_l$ can be then represented as a Sommerfeld-Maliuzhinets integral [1]-[2], [17]-[20],

$$u(\rho,\varphi) = \frac{1}{2\pi i}\int_\gamma f(\alpha+\varphi)e^{ik\rho\cos\alpha}d\alpha, \tag{3}$$

which satisfies the Helmholtz equation. In this representation, f is an analytical function and the path γ consists of two branches: one, named γ_+, going from $(i\infty + \arg(ik) + (a_1 + \frac{\pi}{2}))$ to $(i\infty + \arg(ik) - (a_2 + \frac{\pi}{2}))$ with $0 < a_{1,2} < \pi$, as $\mathrm{Im}\alpha \geq d$, above all the singularities of the integrand, and the other, named γ_-, obtained by inversion of γ_+ with respect to $\alpha = 0$.

We can develop the expression, considering a deformation of γ to the steepest descent path SDP for large $k\rho$, composed of two paths $SDP_\pm$ centered on the stationary phase points

$\alpha = \pm\pi$. Assuming that, between γ and SDP, $f(\alpha+\varphi,\varphi_\circ)e^{ik\rho\cos\alpha}$ is meromorphic and vanishes at infinity,

$$u(\rho,\varphi) = u_i + \sum u_r^{\pm} + \sum u_{tr}^{\pm} + \sum u_s^{\pm} + \frac{e^{-ik\rho}}{2\pi i}\int_{SDP} f(\alpha+\varphi)e^{ik\rho(\cos\alpha+1)}d\alpha, \quad (4)$$

where

- the terms u_i, $u_r^{\pm}$ and $u_{tr}^{\pm}$, for incident, reflected and transmitted Geometrical Optics plane waves (if any), correspond to real poles of $f(\alpha+\varphi)$ captured during deformation of γ to SDP;
- the terms $u_s^{\pm}$ correspond to complex poles of $f(\alpha+\varphi)$ captured during deformation of γ to SDP, for terms of complex plane guided waves (if any) excited by the edge;
- the last term, named u_d, is principally radiated cylindrically when $\rho \to \infty$. Approximating $f(\alpha+\varphi)$ on $SDP_{\pm}$ by its value at the stationnary phase points $\alpha = \pm\pi$, we obtain

$$u_d \sim \frac{-e^{-i\pi/4-ik\rho}}{\sqrt{2\pi k\rho}}[f(\pi+\varphi) - f(-\pi+\varphi)], \quad (5)$$

with $F(\varphi) = f(\pi+\varphi) - f(-\pi+\varphi)$ the diffraction (or far field) coefficient.

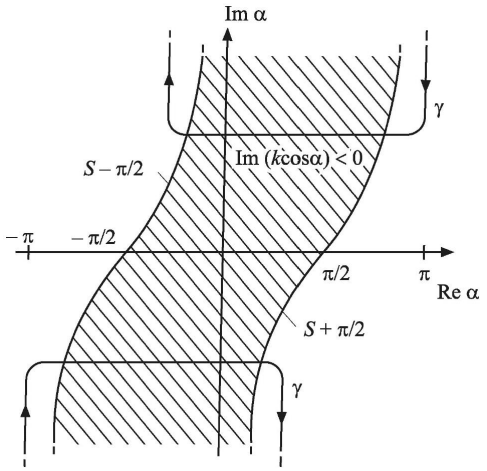

figure 2 : Complex path of integration

This representation is commonly applied for a wedge. We investigate its use for a scatterer with several discontinuities as in [1]-[2], detailing properties and ameliorating the method. This study requires us to express a shift of the origin, and for this, to use particular relations of f with radiation by one face [1]-[2]. By choice, we take in what follows $\Phi_r = \Phi_l = \Phi$.

3) Single-face expression of f for a scatterer surface with two polygonal faces

Let us consider a polygonal surface located inside the domain $|\varphi| > \Phi$ enclosing a scatterer. This surface is composed of two joined semi-infinite polygonal faces $L^{\pm}_{0,\infty}$, denoted $+$ and $-$, respectively with $m^{\pm}$ straight segments of lengths $d^{\pm}_j$ with tangent angles $\pm\,\Phi^{\pm}_j$, $j = 1$, 2, ..., ($m^{\pm} = 0$ when no finite segments), and a semi-infinite plane with tangent angles $\pm\,\Phi^{\pm}_e$, $\Phi \leq \Phi^{\pm}_e$.

The spectral function f satisfies a single-face expression [1], as $\frac{\pi}{2} < \Phi^{\pm}_e \mp \varphi_\circ < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \Phi^{\pm}_e \mp \varphi < \frac{3\pi}{2}$,

$$f(\pm\pi + \varphi) = \frac{1}{2}\int_{L^{\pm}_{0,\infty}} \mp\,(u(M)\nabla_M(e^{ik\hat{\tau}.OM}) - \nabla_M(u(M))e^{ik\hat{\tau}.OM}).\hat{n}^{\pm}ds(M), \tag{6}$$

where $s(M)$ is the length abscissa on each face, at points M of $L^{\pm}_{0,\infty}$ from origin to ∞, the term $\hat{\tau}$ is the unit vector of the direction of observation with angle φ, and $\hat{n}^{\pm}$ is the outward unit normal to face at M.

Considering the geometry of $L^{\pm}_{0,\infty}$, and the notation $\hat{n}^{\pm}\nabla_M(.) = \partial(.)/\partial n = \mp\,\partial(.)/\rho_j\partial\varphi_j$ on straight segments $\rho_j \in [0, d^{\pm}_j]$ at $\varphi_j = \pm\,\Phi^{\pm}_j$, we can write the single-face expression of f [1] in the form,

$$f(\alpha) = \sum_{1\leq j\leq m^{\pm}} \frac{e^{-ik\sum\limits_{1\leq i<j} d^{\pm}_i\cos(\alpha\mp\Phi^{\pm}_i)}}{2}\int_0^{d^{\pm}_j} (-iku(\rho'_j,\,\pm\,\Phi^{\pm}_j)\sin(\alpha\mp\Phi^{\pm}_j) \pm \frac{\partial u}{\partial n}(\rho'_j,\,\pm\,\Phi^{\pm}_j))e^{-ik\rho'_j\cos(\alpha\mp\Phi^{\pm}_j)}d\rho'_j + e^{-ik\sum\limits_{1\leq i\leq m^{\pm}} d^{\pm}_i\cos(\alpha\mp\Phi^{\pm}_i)} f^{\pm}_{e,m^{\pm}}(\alpha), \tag{7}$$

with $m^{\pm} \geq 0$ ($m^{\pm} = 0$ when no finite segments on face $\pm$), $\sum\limits_{1\leq i\leq 0} \equiv 0$, $\sum\limits_{1\leq i<1} \equiv 0$, where $f^{\pm}_{e,m^{\pm}}(\alpha)$ is the analytic continuation of the integral expression,

$$f^{\pm}_{e,m^{\pm}}(\alpha' \pm \Phi^{\pm}_e) = \frac{1}{2}\int_0^{\infty} (-iku(\rho'_e,\,\pm\,\Phi^{\pm}_e)\sin\alpha' \pm \frac{\partial u}{\partial n}(\rho'_e,\,\pm\,\Phi^{\pm}_e))e^{-ik\rho'_e\cos\alpha'}d\rho'_e, \tag{8}$$

valid as $\mathrm{Re}(ik(\cos\alpha' - \cos(\Phi^{\pm}_e \mp \varphi_\circ))) > 0$, $|\mathrm{Re}\alpha'| < \pi$, $|\arg(ik)| < \frac{\pi}{2}$.

The function $f^{\pm}_{e,m^{\pm}}$ is the spectral function corresponding to the Sommerfeld-Maliuzhinets representation of the field in cylindrical coordinates $(\rho^{\pm}_e, \varphi^{\pm}_e)$, with origin $Q^{\pm}_{e,m^{\pm}}$ at the edge

of the semi-plane $\varphi_e^{\pm} = \pm\Phi_e^{\pm}$, radiating in the free space sector $\Phi_e^{\pm} - \delta_e^{\pm} \leq \pm\varphi_e^{\pm} \leq \Phi_e^{\pm}$ where $\delta_e^{\pm}$ is a strictly positive constant determined by the geometry.
In accordance with the conditions (a'), (b'), and (c') on the field u, the spectral functions f and $f_{e,m^{\pm}}^{\pm}$ have the following properties :
(a) $f(\alpha) - 1/(\alpha - \varphi_\circ)$ has no singularity in the band $-\Phi \leq \alpha \leq \Phi$, even at infinity, while $f_{e,m^{\pm}}^{\pm}(\alpha) - u^i(Q_{e,m^{\pm}}^{\pm})/(\alpha - \varphi_\circ)$ has no singularity in the band $\Phi_e^{\pm} - \delta_e^{\pm} \leq \pm \mathrm{Re}\alpha \leq \Phi_e^{\pm}$, where $u^i(Q_{e,m^{\pm}}^{\pm}) = \exp(ik \sum_{1 \leq i \leq m^{\pm}} d_i^{\pm} \cos(\varphi_\circ \mp \Phi_i^{\pm}))$.
(b) $f(\alpha)$ is regular in the band $-\Phi_e^{-} \leq \mathrm{Re}(\alpha) \leq \Phi_e^{+}$, except at $\alpha = \varphi_\circ$ and possibly as $|\mathrm{Im}\alpha| \to \infty$, and thus $e^{A^{\pm}(\alpha)} f(\alpha) + B^{\pm}(\alpha) = f_{e,m^{\pm}}^{\pm}(\alpha)$ too (note : $A^{\pm}$ and $B^{\pm}$ are entire functions). The function $f(\alpha)$ has no singularities, except the poles of Geometrical Optics, in the region where $e^{ik\rho\cos(\alpha-\varphi)}$ diverges as $\rho \to \infty$, $|\mathrm{Re}\alpha| \leq \pi$, $|\varphi| < \Phi$, and verifies for large $\mathrm{Im}\alpha$ in this domain, $f(\alpha) = \frac{\mp i}{2} u(\rho = 0) + O(e^{\pm i c_{\pm} \alpha} \alpha^{d_{\pm}})$ as $\pm \mathrm{Im}\alpha > 0$, $c_{\pm} > 0$, $d_{\pm} \geq 0$. Thus, $f(\alpha)$ remains finite when $-\Phi - \pi/2 + \arg(ik) \leq \pm \mathrm{Re}(\alpha) \leq \pi/2 + \arg(ik) + \Phi$ as $\mathrm{Im}\alpha \to \pm\infty$. The spectral functions $f_{e,m^{\pm}}^{\pm}$, refering to geometry at origin $Q_{e,m^{\pm}}^{\pm}$, have behaviours of the same type.

The spectral functions $f_{a,p}^{\pm}$ and $f_{b,p}^{\pm}$, attached to shifts of the origin at opposite ends of the segment p on the face $\pm$, with $f_{b,p}^{\pm}(\alpha) = f_{a,p+1}^{\pm}(\alpha)$ for $1 \leq p < m^{\pm}$ and $f_{b,m^{\pm}}^{\pm}(\alpha) = f_{e,m^{\pm}}^{\pm}(\alpha)$, are the spectral functions associated with the representations of the field in coordinates $(\rho_{a,p}^{\pm}, \varphi_{a,p}^{\pm})$ and $(\rho_{b,p}^{\pm}, \varphi_{b,p}^{\pm})$ with origins at opposite ends $Q_{a,p}^{\pm}$ and $Q_{b,p}^{\pm}$ of segment p. From single-face representation of spectral function (7), we have ,

$$
\begin{aligned}
f_{a,p}^{\pm}(\alpha) &= \frac{1}{2} \int_0^{d_p^{\pm}} \Big(-iku(\rho'_{a,p}, \varphi)\sin(\alpha - \varphi) \pm \frac{\partial u}{\partial n}(\rho'_{a,p}, \varphi) \Big) \\
&\times e^{-ik\rho'_{a,p}\cos(\alpha-\varphi)} d\rho'_{a,p}|_{\varphi=\pm\Phi_p^{\pm}} + e^{-ik\, d_p^{\pm}\cos(\alpha \mp \Phi_p^{\pm})} f_{b,p}^{\pm}(\alpha), \\
f_{b,p}^{\pm}(\alpha) &= \frac{1}{2} \int_0^{d_p^{\pm}} \Big(-iku(\rho'_{b,p}, \varphi)\sin(\alpha - \varphi) \mp \frac{\partial u}{\partial n}(\rho'_{b,p}, \varphi) \Big) \\
&\times e^{-ik\rho'_{b,p}\cos(\alpha-\varphi)} d\rho'_{b,p}|_{\varphi=\pm(\Phi_p^{\pm}-\pi)} + e^{-ik\, d_p^{\pm}\cos(\alpha \mp (\Phi_p^{\pm}-\pi))} f_{a,p}^{\pm}(\alpha).
\end{aligned} \tag{9}
$$

and we note that the function $f_{1,p}^{\pm}$ and $f_{2,p}^{\pm}$, combinations of functions $f_{a,p}^{\pm}$ and $f_{b,p}^{\pm}$ following,

$$
\begin{aligned}
f_{1,p}^{\pm}(\alpha) &= f_{a,p}^{\pm}(\alpha) - e^{-ik\, d_p^{\pm}\cos(\alpha \mp \Phi_p^{\pm})} f_{b,p}^{\pm}(\alpha) \\
f_{2,p}^{\pm}(\alpha) &= f_{b,p}^{\pm}(\alpha) - e^{ik\, d_p^{\pm}\cos(\alpha \mp \Phi_p^{\pm})} f_{a,p}^{\pm}(\alpha) = -e^{ik\, d_p^{\pm}\cos(\alpha \mp \Phi_p^{\pm})} f_{1,p}^{\pm}(\alpha),
\end{aligned} \tag{10}
$$

are directly related to the radiation by the finite segment p.

Remark : Let define $v^{\pm}$ so that $v^{\pm}(M) = 0$ at $M \in L^{\pm}_{0,\infty}$, with non null finite $\nabla v^{\pm}(M)$ on regular parts of $L^{\pm}_{0,\infty}$. We have $dv^{\pm}(M) = d(OM).\nabla v^{\pm}(M) = 0$ on regular parts of each face. Considering a constant a and the unit normal $\widehat{z}$ to 2D plane, with outward unit normal to face $L^{\pm}_{0,\infty}$, $\widehat{n}^{\pm} = \frac{a\nabla v^{\pm}(M)}{|a\nabla v^{\pm}(M)|} = \frac{\mp(\widehat{z}\wedge d(\overline{OM}))}{|d(\overline{OM})|}$, we note that,

$$ds = |d(\overline{OM})| = \mp\,\widehat{n}^{\pm}.(\widehat{z}\wedge d(\overline{OM})),\ \widehat{n}^{\pm}ds = \mp\,(\widehat{z}\wedge d(\overline{OM})). \tag{11}$$

4) Polygonal surface with impedance boundary conditions

4.1) Functional difference equations on $f^{\pm}_{e,m^{\pm}}$ for semi-infinite planes

In the case of an infinite polygonal scatterer with impedance boundary conditions along both semi-infinite planes $\varphi^{\pm}_e = \pm\Phi^{\pm}_e$, we have,

$$\frac{\partial u}{\partial n^{\pm}}(\rho^{\pm}_e,\ \pm\Phi^{\pm}_e) - ik\sin\theta^{\pm}_e u(\rho^{\pm}_e,\ \pm\Phi^{\pm}_e) = 0, \tag{12}$$

where $\partial(.)/\partial n^{\pm} = \widehat{n}^{\pm}\nabla(.) = \mp\,\partial(.)/\rho^{\pm}_e\partial\varphi^{\pm}_e$, $0 < |\mathrm{Re}(\theta^{\pm}_e)| \le \pi/2$, $\widehat{n}^{\pm}$ is the outward normal to the face $\varphi^{\pm}_e = \pm\Phi^{\pm}_e$, and from (8), the functions $f^{\pm}_{e,m^{\pm}}$ satisfy the functional equations,

$$(\sin\alpha \pm \sin\theta^{\pm}_e)f^{\pm}_{e,m^{\pm}}(\alpha \pm \Phi^{\pm}_e) - (-\sin\alpha \pm \sin\theta^{\pm}_e)f^{\pm}_{e,m^{\pm}}(-\alpha \pm \Phi^{\pm}_e) = 0. \tag{13}$$

If the scatterer is finite, the segments of both faces form a closed surface so that we can take $Q^{+}_{e,m^{+}} = Q^{-}_{e,m^{-}}$ with $\Phi^{+}_e + \Phi^{-}_e = 2\pi$. The fields on both semi-infinite planes $\varphi^{\pm}_e = \pm\Phi^{\pm}_e$ are then equal and their normal derivatives are opposite,

$$u(\rho^{+}_e, \Phi^{+}_e) = u(\rho^{-}_e,\ -\Phi^{-}_e),\ \frac{\partial u}{\partial n^{+}}(\rho^{+}_e, \Phi^{+}_e) = -\frac{\partial u}{\partial n^{-}}(\rho^{-}_e,\ -\Phi^{-}_e), \tag{14}$$

and we derive from (8) that,

$$f^{+}_{e,m^{+}}(\alpha+\pi) = f^{-}_{e,m^{-}}(\alpha-\pi). \tag{15}$$

Considering the definition of $f(\alpha)$ and $f^{\pm}_{e,m^{\pm}}$ given in (7) with properties (a) and (b), the functional equations (13) or (15) imply [1] that these functions are meromorphic with simple poles, regular in the band $-\Phi^{-}_e \le \mathrm{Re}\alpha \le \Phi^{+}_e$, except for the pole at $\alpha = \varphi_\circ$ and possibly at infinity when $\Phi \le \pm\,\mathrm{Re}\alpha \le \Phi^{\pm}_e$ (note : for finite scatterers, $f(\alpha) - \frac{1}{2}\cot(\frac{1}{2}(\alpha-\varphi_\circ))$ have no poles, and this function is an entire function).

4.2) Functional equations due to boundary conditions on finite segments

Considering impedance boundary conditions on the segments in the form,

$$[\frac{\partial u}{\partial n}(\rho_{a,p},\ \pm\Phi_p^{\pm}) - ik\sin\theta_p^{\pm}u(\rho_{a,p},\ \pm\Phi_p^{\pm})]_{0\le\rho_{a,p}\le d_p^{\pm}} = 0,$$
$$[\frac{\partial u}{\partial n}(\rho_{b,p},\ \pm(\Phi_p^{\pm}-\pi)) - ik\sin\theta_p^{\pm}u(\rho_{b,p},\ \pm(\Phi_p^{\pm}-\pi))]_{0\le\rho_{b,p}\le d_p^{\pm}} = 0, \qquad (16)$$

with $0 < |\mathrm{Re}(\theta_p^{\pm})| \le \pi/2$, we can use the definition of $f_{1,p}^{\pm}$ and $f_{2,p}^{\pm}$ in (10) with the equalities given in (9), and deduce,

$$[(\sin\alpha \pm \sin\theta_p^{\pm})f_{1,p}^{\pm}(\alpha+\varphi) - (-\sin\alpha \pm \sin\theta_p^{\pm})f_{1,p}^{\pm}(-\alpha+\varphi)]|_{\varphi=\pm\Phi_p^{\pm}} = 0,$$
$$[(\sin\alpha \mp \sin\theta_p^{\pm})f_{2,p}^{\pm}(\alpha+\varphi) - (-\sin\alpha \mp \sin\theta_p^{\pm})f_{2,p}^{\pm}(-\alpha+\varphi)]|_{\varphi=\pm(\Phi_p^{\pm}-\pi)} = 0. \qquad (17)$$

4.3) A remarkable relation between $f_{a,p}^{\pm}$ and $f_{b,p}^{\pm}$ deriving from (17) when (10) applies

It is known from (10) with (9) that $f_{1,p}^{\pm}$ has a 2π period, but it is remarkable that this property also derives, in an independent manner, from the difference equation (17). Considering (17) with (10), this 2π periodicity will then implies the equivalence between the fields represented with $f_{a,p}^{\pm}$ and $f_{b,p}^{\pm}$.

Indeed, $f_{2,p}^{\pm}(\alpha\pm\Phi_p^{\pm}) = -e^{ik\,d_p^{\pm}\cos(\alpha)}f_{1,p}^{\pm}(\alpha\pm\Phi_p^{\pm})$ from (10), and we derive from (17),

$$[(\sin\alpha \pm \sin\theta_p^{\pm})f_{1,p}^{\pm}(\alpha+\varphi) + (\sin\alpha \mp \sin\theta_p^{\pm})f_{1,p}^{\pm}(-\alpha+\varphi)]|_{\varphi=\pm\Phi_p^{\pm}} = 0,$$
$$[(\sin\alpha \mp \sin\theta_p^{\pm})f_{1,p}^{\pm}(\alpha+\varphi) + (\sin\alpha \pm \sin\theta_p^{\pm})f_{1,p}^{\pm}(-\alpha+\varphi)]|_{\varphi=\pm(\Phi_p^{\pm}-\pi)} = 0, \qquad (18)$$

then,

$$f_{1,p}^{\pm}(\alpha\pm 2\Phi_p^{\pm}) + \frac{(\sin(\alpha\pm\Phi_p^{\pm}) \mp \sin\theta_p^{\pm})}{(\sin(\alpha\pm\Phi_p^{\pm}) \pm \sin\theta_p^{\pm})}f_{1,p}^{\pm}(-\alpha) = 0,$$
$$f_{1,p}^{\pm}(\alpha\pm 2(\Phi_p^{\pm}-\pi)) + \frac{(\sin(\alpha\pm\Phi_p^{\pm}) \mp \sin\theta_p^{\pm})}{(\sin(\alpha\pm\Phi_p^{\pm}) \pm \sin\theta_p^{\pm})}f_{1,p}^{\pm}(-\alpha) = 0. \qquad (19)$$

This allows, after substraction, to deduce the 2π period of $f_{1,p}^{\pm}$,

$$f_{1,p}^{\pm}(\alpha) = f_{1,p}^{\pm}(\alpha+2\pi), \qquad (20)$$

which gives us, after writing $f_{1,p}^{\pm}(\alpha) = f_{a,p}^{\pm}(\alpha) - e^{-ik\,d_p^{\pm}\cos(\alpha\mp\Phi_p^{\pm})}f_{b,p}^{\pm}(\alpha)$ from (10),

$$f_{a,p}^{\pm}(\pi+\alpha) - f_{a,p}^{\pm}(-\pi+\alpha) = e^{ik\,d_p^{\pm}\cos(\alpha\mp\Phi_p^{\pm})}(f_{b,p}^{\pm}(\pi+\alpha) - f_{b,p}^{\pm}(-\pi+\alpha)). \qquad (21)$$

Consequently, the equations (17), when (10) applies, implies that the far field given by $f^{\pm}_{a,p}$ is the far field given by $f^{\pm}_{b,p}$ with phase center shifted by Δ, so that the field given by $f^{\pm}_{a,p}$ is the one given by $f^{\pm}_{b,p}$, relatively to two distinct origins corresponding to ends of the segment p.
In whole generality, it is remarkable that functional difference equations for boundary conditions on each segment has for additional consequence the equivalence of fields represented with any $f^{\pm}_{a,n}$ (or $f^{\pm}_{b,n}$), refering to any origins on the impedance polygons.

5) Formulation of the three-part polygonal problem : spectral functions in Sommerfeld-Maliuzhinets representation and functional difference equations in complex plane.

5.1) Definition

We consider the diffraction of an incident plane wave by a semi-infinite impedance polygon divided into three parts, each one characterized by relative surface impedances $\sin\theta_-$, $\sin\theta_1$, and $\sin\theta_+$, with $0 < \mathrm{Re}(\theta_\pm) \leq \pi/2$ and $0 < \mathrm{Re}(\theta_1) \leq \pi/2$ (strict passivity). This means that $m^+ = 1, \quad m^- = 0, \quad \Phi_1^+ = \frac{\pi}{2}, \quad \Phi_e^+ = \frac{\pi}{2} + \Phi_b, \quad \Phi_e^- = -\frac{\pi}{2} - \Phi_a, \quad f \equiv f_a = f^+_{a,1} = f^-_e$, $f_b = f^+_{b,1} = f^+_e$.

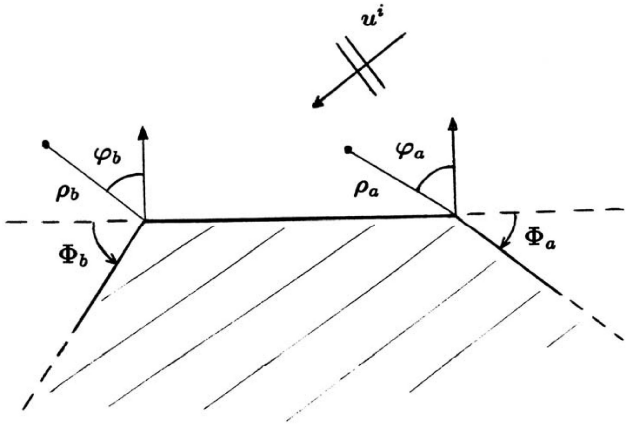

figure 3 : geometry of the three-part polygon

The functions f_a and f_b are the spectral functions associated with the Sommerfeld-Maliuzhinets representation of the field, in coordinate systems (ρ_a, φ_a) and (ρ_b, φ_b), with origins at opposite ends of the finite segment of length Δ (see geometry on figure). We have, in (ρ_a, φ_a) coordinates,

$$(\rho_a \in]0,\infty[, \varphi_a = -\frac{\pi}{2} - \Phi_a) \text{ with } \partial u/\partial n - ik\sin\theta_- u = 0,$$
$$(\rho_a \in [0,\Delta], \varphi_a = \frac{\pi}{2}) \text{ with } \partial u/\partial n - ik\sin\theta_1 u = 0, \quad (22)$$

with the incident field $u^i = e^{ik\rho_a\cos(\varphi_a - \varphi_\circ)}$, and, in coordinates (ρ_b, φ_b),

$$(\rho_b \in [0,\Delta], \varphi_b = -\frac{\pi}{2}) \text{ with } \partial u/\partial n - ik\sin\theta_1 u = 0,$$
$$(\rho_b \in]0,\infty[, \varphi_b = \frac{\pi}{2} + \Phi_b) \text{ with } \partial u/\partial n - ik\sin\theta_+ u = 0, \quad (23)$$

with the incident field $u^i = e^{ik(\rho_b\cos(\varphi_b - \varphi_\circ) + \Delta\sin\varphi_\circ)}$.
The properties (a) and (b) give us that,

$$(f_a(\alpha) - 1/(\alpha - \varphi_\circ)) \text{ and } (f_b(\alpha) - e^{ik\Delta\sin\varphi_\circ}/(\alpha - \varphi_\circ)) \quad (24)$$

are regular in the respective strips $-\frac{\pi}{2} - \Phi_a \leq \mathrm{Re}\alpha \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq \mathrm{Re}\alpha \leq \frac{\pi}{2} + \Phi_b$, with $-\frac{\pi}{2} - \Phi_a < \varphi_\circ < \frac{\pi}{2} + \Phi_b$, and verify $f_{a(b)}(\alpha) = \frac{\mp i}{2} u(\rho_{a(b)} = 0) + O(e^{\pm i c_{a(,b)\pm}\alpha}\alpha^{d_{a(,b)\pm}})$ in these regions as $\mathrm{Im}\alpha \to \pm\infty$, $c_{a(,b)\pm} > 0$.
Considering the definitions, we have $f_e^- = f_a$, $f_e^+ = f_b$, and we obtain, relatively to integration on piecewise straight faces,

$$f_a(\alpha) = e^{-ik\Delta\sin\alpha} f_b(\alpha) - \int_0^\Delta \frac{(-iku(\rho_a, \frac{\pi}{2})\cos\alpha - \frac{\partial u}{\partial n}(\rho_a, \frac{\pi}{2}))e^{-ik\rho_a\sin\alpha}}{2} d\rho_a,$$
$$f_b(\alpha) = e^{ik\Delta\sin\alpha} f_a(\alpha) - \int_0^\Delta \frac{(iku(\rho_b, -\frac{\pi}{2})\cos\alpha + \frac{\partial u}{\partial n}(\rho_b, -\frac{\pi}{2}))e^{+ik\rho_b\sin\alpha}}{2} d\rho_b, \quad (25)$$

for any complex angle α, while, relatively to integration on straight semi-infinite faces,

$$f_a(\alpha - \varphi_-) = \frac{1}{2}\int_0^\infty (-iku(\rho_a, -\varphi_-)\sin\alpha - \frac{\partial u}{\partial n}(\rho_a, -\varphi_-))e^{-ik\rho_a\cos\alpha} d\rho_a, \quad (26)$$

as $\mathrm{Re}(ik(\cos(\varphi_\circ + \varphi_-) - \cos\alpha)) < 0$, $|\mathrm{Re}\alpha| < \pi$, with $\varphi_- = \frac{\pi}{2} + \Phi_a$, and

$$f_b(\alpha + \varphi_+) = \frac{1}{2}\int_0^\infty (-iku(\rho_b, \varphi_+)\sin\alpha + \frac{\partial u}{\partial n}(\rho_b, \varphi_+))e^{-ik\rho_b\cos\alpha} d\rho_b, \quad (27)$$

as $\mathrm{Re}(ik(\cos(\varphi_\circ - \varphi_+) - \cos\alpha)) < 0$, $|\mathrm{Re}\alpha| < \pi$, with $\varphi_+ = \frac{\pi}{2} + \Phi_b$.

5.2) About functional difference equations in complex plane

5.2.1) Functional equations for the central strip and semi-infinite planes

We now derive functional equations on the spectral functions from the boundary conditions on the central strip (of finite size) and on the semi-infinite planes of the three-part impedance polygon (figure 3). Using the boundary conditions on the central strip (22), and letting

$$f_1(\alpha+\frac{\pi}{2}) = f_a(\alpha+\frac{\pi}{2}) - e^{-ik\Delta\cos\alpha} f_b(\alpha+\frac{\pi}{2}),$$
$$f_2(\alpha-\frac{\pi}{2}) = f_b(\alpha-\frac{\pi}{2}) - e^{-ik\Delta\cos\alpha} f_a(\alpha-\frac{\pi}{2}) = -e^{-ik\Delta\cos\alpha} f_1(\alpha-\frac{\pi}{2}), \qquad (28)$$

we obtain, as described in (17),

$$(\sin\alpha+\sin\theta_1) f_1(\alpha+\frac{\pi}{2}) - (-\sin\alpha+\sin\theta_1) f_1(-\alpha+\frac{\pi}{2}) = 0,$$
$$(\sin\alpha-\sin\theta_1) f_2(\alpha-\frac{\pi}{2}) - (-\sin\alpha-\sin\theta_1) f_2(-\alpha-\frac{\pi}{2}) = 0, \qquad (29)$$

while, using the boundary conditions on both semi-infinite planes (23), we obtain, as described in (13),

$$(\sin\alpha+\sin\theta_+) f_b(\alpha+\frac{\pi}{2}+\Phi_b) - (-\sin\alpha+\sin\theta_+) f_b(-\alpha+\frac{\pi}{2}+\Phi_b) = 0,$$
$$(\sin\alpha-\sin\theta_-) f_a(\alpha-\frac{\pi}{2}-\Phi_a) - (-\sin\alpha-\sin\theta_-) f_a(-\alpha-\frac{\pi}{2}-\Phi_a) = 0. \qquad (30)$$

5.2.2) A remarkable relation between f_a and f_b deriving from (29) when (28) applies

We follow a development similar to the one used to obtain (19)-(21). Considering the relation between f_1 and f_2 in (28), we have from (29),

$$(\sin\alpha+\sin\theta_1) f_1(\alpha+\frac{\pi}{2}) - (-\sin\alpha+\sin\theta_1) f_1(-\alpha+\frac{\pi}{2}) = 0,$$
$$(\sin\alpha-\sin\theta_1) f_1(\alpha-\frac{\pi}{2}) - (-\sin\alpha-\sin\theta_1) f_1(-\alpha-\frac{\pi}{2}) = 0, \qquad (31)$$

and thus,

$$(\cos\alpha+\sin\theta_1) f_1(\pi+\alpha) + (\cos\alpha-\sin\theta_1) f_1(-\alpha) = 0,$$
$$-(\cos\alpha+\sin\theta_1) f_1(-\pi+\alpha) - (\cos\alpha-\sin\theta_1) f_1(-\alpha) = 0. \qquad (32)$$

This implies, after addition,

$$f_1(\pi+\alpha) = f_1(-\pi+\alpha), \qquad (33)$$

and, from the expression of f_1 with f_a and f_b given in (28),

$$f_a(\pi+\alpha) - f_a(-\pi+\alpha) = e^{ik\Delta\sin\alpha}(f_b(\pi+\alpha) - f_b(-\pi+\alpha)). \qquad (34)$$

Consequently, the equations (29), which initially concern the impedance boundary condition

on the segment, has for additional remarkable consequence (34), which signifies that the far field given by f_a is the one given by f_b with phase center shifted by Δ, so that the field given by f_a is the one given by f_b relatively to the two origins at both ends of the segment.

Remark : Let us consider the poles of the meromorphic function f_1. They are 2π-periodic from (33), and are, by definition of f_1, also poles of f_a, or f_b, or both. The presence of these 2π-periodic poles is in contradiction with the other functional equations due to impedance conditions on the semi-infinite faces (as (a)-(b) are satisfied), and thus the integral equations derived from them (note : it is different for finite scatterers, since $f(\alpha) - \frac{1}{2}\cot(\frac{1}{2}(\alpha - \varphi_\circ))$ has no pole). It finally implies that f_1 has no pole, and is then an entire function. This property is in agreement with the integral expression of f_1 with u on the finite segment derived from (9)-(10).

5.3) Functional difference equations for $f_{br}(\alpha) = f_b(\alpha + \frac{\Phi_b}{2})$, $f_{ar}(\alpha) = f_a(\alpha - \frac{\Phi_a}{2})$

We can now derive, from (29)-(30), difference equations on $f_{br}(\alpha) = f_b(\alpha + \frac{\Phi_b}{2})$, $f_{ar}(\alpha) = f_a(\alpha - \frac{\Phi_a}{2})$, corresponding to both discontinuities when influenced each other. If we let $f_{br}(\alpha - \frac{\Phi_b}{2}) = f_b(\alpha)$ with $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$, we have

$$(\sin\alpha + \sin\theta_+) f_{br}(\alpha + \Phi_+) - (-\sin\alpha + \sin\theta_+) f_{br}(-\alpha + \Phi_+) = 0, \tag{35}$$

and

$$\begin{aligned}
&(\sin\alpha - \sin\theta_1) f_{br}(\alpha - \Phi_+) - (-\sin\alpha - \sin\theta_1) f_{br}(-\alpha - \Phi_+) \\
&= e^{-ik\Delta\cos\alpha}((\sin\alpha - \sin\theta_1) f_a(\alpha - \frac{\pi}{2}) - (-\sin\alpha - \sin\theta_1) f_a(-\alpha - \frac{\pi}{2})) \\
&= 2e^{-ik\Delta\cos\alpha} S_b^-(\alpha).
\end{aligned} \tag{36}$$

If we let $f_{ar}(\alpha + \frac{\Phi_a}{2}) = f_a(\alpha)$ with $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$, we have

$$\begin{aligned}
&(\sin\alpha + \sin\theta_1) f_{ar}(\alpha + \Phi_-) - (-\sin\alpha + \sin\theta_1) f_{ar}(-\alpha + \Phi_-) \\
&= e^{-ik\Delta\cos\alpha}((\sin\alpha + \sin\theta_1) f_b(\alpha + \frac{\pi}{2}) - (-\sin\alpha + \sin\theta_1) f_b(-\alpha + \frac{\pi}{2})) \\
&= 2e^{-ik\Delta\cos\alpha} S_a^+(\alpha),
\end{aligned} \tag{37}$$

and

$$(\sin\alpha - \sin\theta_-) f_{ar}(\alpha - \Phi_-) - (-\sin\alpha - \sin\theta_-) f_{ar}(-\alpha - \Phi_-) = 0. \tag{38}$$

From the properties of f_a and f_b, the functions

$$(f_{ar}(\alpha) - 1/(\alpha - \varphi_{\circ,a}) \text{ and } (f_{br}(\alpha) - e^{ik\Delta\sin\varphi_\circ}/(\alpha - \varphi_{\circ,b})) \tag{39}$$

are respectively regular in the bands $|\mathrm{Re}\alpha| \leq \Phi_-$ and $|\mathrm{Re}\alpha| \leq \Phi_+$, with $\varphi_{\circ,a} = \varphi_\circ + \frac{\Phi_a}{2}$ and $\varphi_{\circ,b} = \varphi_\circ - \frac{\Phi_b}{2}$, $-\frac{\pi}{2} - \Phi_a < \varphi_\circ < \frac{\pi}{2} + \Phi_b$, and, in these regions,

$$f_{ar}(\alpha) = \mp \frac{i}{2} u(\rho_a = 0) + O(e^{\pm i c_{a,\pm}\alpha}\alpha^{d_{a,\pm}}),$$
$$\text{and } f_{br}(\alpha) = \mp \frac{i}{2} u(\rho_b = 0) + O(e^{\pm i c_{b,\pm}\alpha}\alpha^{d_{b,\pm}}), \tag{40}$$

as $\mathrm{Im}\alpha \to \pm\infty$, $c_{a(,b),\pm} > 0$, $d_{a(,b),\pm} \geq 0$.

Besides, we remark that, for $\Delta = 0$, the three-part polygon is reduced to a wedge with exterior angle $2\Phi_d = (2(\Phi_+ + \Phi_-) - \pi)$ and face impedances $\sin\theta_\pm$, where $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$, $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$. In this case, we have $f_{br}(\alpha - \Phi_- + \frac{\pi}{2}) = f_{ar}(\alpha + \Phi_+ - \frac{\pi}{2}) = f_0^{ab}(\alpha)$, where f_0^{ab}, refering to the solution for an impedance wedge (see appendix), is given by

$$f_0^{ab}(\alpha, \varphi_\circ) = \frac{\pi}{2\Phi_d} \frac{\Psi_{+-}(\alpha)\cos(\frac{\pi\varphi_{\circ,d}}{2\Phi_d})}{\Psi_{+-}(\varphi_{\circ,d})(\sin(\frac{\pi\alpha}{2\Phi_d}) - \sin(\frac{\pi\varphi_{\circ,d}}{2\Phi_d}))}, \tag{41}$$

where $\varphi_{\circ,d} = \varphi_\circ + (\Phi_a - \Phi_b)/2$, $-\Phi_d < \varphi_{\circ,d} < \Phi_d$, with $\Phi_d = (\Phi_a + \Phi_b + \pi)/2$.

6) The integral expressions and integral equations for the three-part impedance polygon

6.1) Elementary integral solutions for difference equations

Considering the theory of functional equations, the analytic function $\chi(\alpha)$ satisfying,

$$\chi(\alpha \pm \Phi) - \chi(-\alpha \pm \Phi) = \vartheta^\pm(\alpha), \tag{42}$$

and regular as $|\mathrm{Re}\alpha| \leq \Phi$, is given in the strip $|\mathrm{Re}\alpha| < \Phi$ [17], [22], by,

$$\chi(\alpha) = \frac{\chi(i\infty) + \chi(-i\infty)}{2} + \frac{-i}{8\Phi}\int_{-i\infty}^{+i\infty} d\alpha' (\vartheta^+(\alpha')\tan(\frac{\pi}{4\Phi}(\alpha + \Phi - \alpha'))$$
$$- \vartheta^-(\alpha')\tan(\frac{\pi}{4\Phi}(\alpha - \Phi - \alpha'))), \tag{43}$$

when the functions $\vartheta^\pm(\alpha)$ are regular and summable on the imaginary axis, with $\chi(i\infty) - \chi(-i\infty) = 0$. The functional equations (35)-(38) can be reduced into the form (42), by using the solutions $\Psi_{+1}(\alpha, \Phi_+)$ (resp. $\Psi_{1-}(\alpha, \Phi_-)$), without pole or zero and $O(\cos(\pi\alpha/2\Phi_+))$ (resp. $O(\cos(\pi\alpha/2\Phi_-))$) in the band $|\mathrm{Re}\alpha| \leq \Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$ (resp. $|\mathrm{Re}\alpha| \leq \Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$), of the equations without second members of the type,

$$(\sin\alpha + \sin\theta_l)\Psi_{lr}(\alpha + \Phi, \Phi) - (-\sin\alpha + \sin\theta_l)\Psi_{lr}(-\alpha + \Phi, \Phi) = 0,$$
$$(\sin\alpha - \sin\theta_r)\Psi_{lr}(\alpha - \Phi, \Phi) - (-\sin\alpha - \sin\theta_r)\Psi_{lr}(-\alpha - \Phi, \Phi) = 0, \tag{44}$$

with $l \equiv +$, $r \equiv 1$, $\Phi = \Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$ for $\Psi_{+1}(\alpha, \Phi_+)$, and $l \equiv 1$, $r \equiv -$, $\Phi = \Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$ for $\Psi_{1-}(\alpha, \Phi_-)$ (see details on Ψ_{lr}, named $\Psi_{+-}(\alpha, \Phi)$ in appendix).

6.2) Coupled integral expressions for $f_{br}(\alpha)$ and $f_{ar}(\alpha)$

Considering (35)-(38), we consider first the solution for $S_b^- \equiv 0$ (resp. $S_a^+ \equiv 0$) with a unique pole due to incident plane wave as $|\mathrm{Re}\alpha| < \Phi_+$ (resp. $|\mathrm{Re}\alpha| < \Phi_-$), which is directly related to impedance wedge case (see appendix). Substracting it from the general solution, we obtain unknowns regular as $|\mathrm{Re}\alpha| < \Phi_+$ (resp. $|\mathrm{Re}\alpha| < \Phi_-$). As previously noted, we can then use Ψ_{lr} of (44), then (42), and thus obtain integral expression of the type (43).
Therefore, we can write [1], $f_{br}(\alpha)$ for $-\Phi_+ < \mathrm{Re}\alpha < 3\Phi_+$, following,

$$\begin{aligned}
\frac{f_{br}(\alpha)}{\Psi_{+1}(\alpha)} &= \frac{i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^-(\alpha')e^{-ik\Delta\cos\alpha'}\tan(\frac{\pi}{4\Phi_+}(\alpha-\Phi_+-\alpha'))}{(\sin\alpha'-\sin\theta_1)\Psi_{+1}(\alpha'-\Phi_+)}d\alpha' + \chi_b^i(\alpha) \\
&= \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{\frac{S_b^-(\alpha')}{(\sin\alpha'-\sin\theta_1)}}{\Psi_{+1}(\alpha'-\Phi_+)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_+))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha' + \chi_b^i(\alpha) \\
&= \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{f_{ar}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{+1}(\alpha'-\Phi_+)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_+))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha' + \chi_b^i(\alpha), \qquad (45)
\end{aligned}$$

and, $f_{ar}(\alpha)$ for $-3\Phi_- < \mathrm{Re}\alpha < \Phi_-$,

$$\begin{aligned}
\frac{f_{ar}(\alpha)}{\Psi_{1-}(\alpha)} &= \frac{-i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{S_a^+(\alpha')e^{-ik\Delta\cos\alpha'}\tan(\frac{\pi}{4\Phi_-}(\alpha+\Phi_--\alpha'))}{(\sin\alpha'+\sin\theta_1)\Psi_{1-}(\alpha'+\Phi_-)}d\alpha' + \chi_a^i(\alpha) \\
&= \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{\frac{S_a^+(\alpha')}{(\sin\alpha'+\sin\theta_1)}}{\Psi_{1-}(\alpha'+\Phi_-)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_-))+\cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha' + \chi_a^i(\alpha) \\
&= \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{f_{br}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{1-}(\alpha'+\Phi_-)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_-))+\cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha' + \chi_a^i(\alpha), \qquad (46)
\end{aligned}$$

with $\varphi_{\circ,b} = \varphi_\circ - \frac{\Phi_b}{2}$, as $-\frac{\pi}{2} < \varphi_\circ < \frac{\pi}{2} + \Phi_b$, and $\varphi_{\circ,a} = \varphi_\circ + \frac{\Phi_a}{2}$, as $-\frac{\pi}{2} - \Phi_a < \varphi_\circ < \frac{\pi}{2}$, while the source terms are given by,

$$\begin{aligned}
\chi_b^i(\alpha) = e^{ik\Delta\sin\varphi_\circ}\,\chi_{+1}^i(\alpha) &= \frac{\pi}{2\Phi_+}\Big(\frac{e^{ik\Delta\sin\varphi_\circ}\cos(\frac{\pi\varphi_{\circ,b}}{2\Phi_+})}{\Psi_{+1}(\varphi_{\circ,b})(\sin(\frac{\pi\alpha}{2\Phi_+})-\sin(\frac{\pi\varphi_{\circ,b}}{2\Phi_+}))}\Big), \\
\chi_a^i(\alpha) &= \frac{\pi}{2\Phi_-}\Big(\frac{\cos(\frac{\pi\varphi_{\circ,a}}{2\Phi_-})}{\Psi_{1-}(\varphi_{\circ,a})(\sin(\frac{\pi\alpha}{2\Phi_-})-\sin(\frac{\pi\varphi_{\circ,a}}{2\Phi_-}))}\Big), \qquad (47)
\end{aligned}$$

with $\mathrm{Re}(\theta_\pm) \neq 0$, $\mathrm{Re}(\theta_1) \neq 0$, $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$, and $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$.

Remark : We use, in (45), that $(\sin\alpha' - \sin\theta_1)\Psi_{+1}(\alpha' - \Phi_+)$ is even, and thus,

$$\frac{2S_b^-(\alpha')}{(\sin\alpha' - \sin\theta_1)\Psi_{+1}(\alpha' - \Phi_+)} = \frac{f_{ar}(\alpha' - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha' - \Phi_+)} - \frac{f_{ar}(-\alpha' - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(-\alpha' - \Phi_+)}, \quad (48)$$

and in (46), that $(\sin\alpha' + \sin\theta_1)\Psi_{1-}(\alpha' + \Phi_-)$ is even, and thus,

$$\frac{2S_a^+(\alpha')}{(\sin\alpha' + \sin\theta_1)\Psi_{1-}(\alpha' + \Phi_-)} = \frac{f_{br}(\alpha' + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha' + \Phi_-)} - \frac{f_{br}(-\alpha' + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(-\alpha' + \Phi_-)}, \quad (49)$$

with $f_{br}(\alpha - \frac{\Phi_b}{2}) = f_b(\alpha),\ f_{ar}(\alpha + \frac{\Phi_a}{2}) = f_a(\alpha)$.

6.3) Integral equations when $\Phi_{a,b} > -\frac{\pi}{2}$, as $|\arg(ik)| < \pi/2$

6.3.1) Definition of integral equations

From previous integral expressions (45)-(46), we can write,

$$\frac{f_{br}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{+1}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})} = \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^-(\alpha')}{(\sin\alpha' - \sin\theta_1)} \frac{1}{\Psi_{+1}(\alpha' - \Phi_+)} \times \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})} d\alpha' + \chi_b^i(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}), \quad (50)$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) < 3\Phi_+$, and,

$$\frac{f_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{1-}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})} = \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{S_a^+(\alpha')}{(\sin\alpha' + \sin\theta_1)} \frac{1}{\Psi_{1-}(\alpha' + \Phi_-)} \times \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})} d\alpha' + \chi_a^i(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}), \quad (51)$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) < \Phi_-$, when $-\min(\frac{\pi}{2},\ \frac{\pi}{2} + \Phi_a) < \varphi_\circ < \min(\frac{\pi}{2},\ \frac{\pi}{2} + \Phi_b)$.
When $\Phi_{a,b} > -\frac{\pi}{2}$, we can take α purely imaginary, where the functions,

$$\frac{S_b^-(\alpha)}{(\sin\alpha - \sin\theta_1)} = \frac{1}{2}(f_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) + \frac{(\sin\alpha + \sin\theta_1)f_{ar}(-\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})}{(\sin\alpha - \sin\theta_1)})$$
$$\frac{S_a^+(\alpha)}{(\sin\alpha + \sin\theta_1)} = \frac{1}{2}(f_{br}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) + \frac{(\sin\alpha - \sin\theta_1)f_{br}(-\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})}{(\sin\alpha + \sin\theta_1)}), \quad (52)$$

vanish at infinity, in contrary to $f_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})$ and $f_{br}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})$ in general.
Choosing these functions as unknowns, we derive the integral equations for imaginary α,

$$\frac{S_a^+(\alpha)}{(\sin\alpha+\sin\theta_1)} = \frac{-i}{8\Phi_+}\int_{-i\infty}^{+i\infty}\frac{S_b^-(\alpha')}{(\sin\alpha'-\sin\theta_1)}\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\Psi_{+1}(\alpha'-\Phi_+)}$$
$$\times\left[\frac{\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}+\frac{\frac{(\sin\alpha-\sin\theta_1)}{(\sin\alpha+\sin\theta_1)}\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\cos(\frac{\pi}{2\Phi_+}(\alpha+\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}\right]d\alpha'$$
$$+\frac{1}{2}\left[\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})\chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})\right.$$
$$\left.+\frac{(\sin\alpha-\sin\theta_1)\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{(\sin\alpha+\sin\theta_1)}\chi_b^i(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})\right], \tag{53}$$

and

$$\frac{S_b^-(\alpha)}{(\sin\alpha-\sin\theta_1)} = \frac{i}{8\Phi_-}\int_{-i\infty}^{+i\infty}\frac{S_a^+(\alpha')}{(\sin\alpha'+\sin\theta_1)}\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\Psi_{1-}(\alpha'+\Phi_-)}$$
$$\times\left[\frac{\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-})}+\frac{\frac{(\sin\alpha+\sin\theta_1)}{(\sin\alpha-\sin\theta_1)}\Psi_{1-}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\cos(\frac{\pi}{2\Phi_-}(\alpha-\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-})}\right]d\alpha'$$
$$+\frac{1}{2}\left[\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})\right.$$
$$\left.+\frac{(\sin\alpha+\sin\theta_1)\Psi_{1-}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{(\sin\alpha-\sin\theta_1)}\chi_a^i(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})\right], \tag{54}$$

where we note that $S_b^-(-\alpha') = -S_b^-(\alpha')$, $S_a^+(-\alpha') = -S_a^+(\alpha')$.

6.3.1) Equations in $L^2(iR)\times L^2(iR)$ as $|\arg(ik)| < \pi/2$

We begin by noting that,

$$\frac{\pi}{2}-\frac{\Phi_{a(,b)}}{2} = \frac{\pi}{2}+\frac{\Phi_{a(,b)}}{2}-\Phi_{a(,b)} = \Phi_{-(,+)}-\Phi_{a(,b)},$$
$$\frac{(\sin\alpha\mp\sin\theta_1)}{(\sin\alpha\pm\sin\theta_1)} = 1-\frac{\pm 2\sin\theta_1}{(\sin\alpha\pm\sin\theta_1)}, \tag{55}$$

and,

$$\Psi_{+1,1-}(\alpha) = A_0\cos(\frac{\pi\alpha}{2\Phi_{+,-}})(1+O(\alpha^\nu e^{-|\mu\mathrm{Im}\alpha|})),$$
$$\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}) = -\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})e^{\frac{\mp i\pi\Phi_b}{\Phi_+}}(1+O(\alpha^\nu e^{-|\mu\mathrm{Im}\alpha|})),$$
$$\Psi_{1-}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}) = -\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})e^{\frac{\pm i\pi\Phi_a}{\Phi_-}}(1+O(\alpha^\nu e^{-|\mu\mathrm{Im}\alpha|})), \tag{56}$$

when Im$\alpha \to \pm\infty$, with $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$, $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$. Letting the kernels in the form,

$$K_{a,b}(\alpha', \alpha) = k_{a,b}(\alpha', \alpha) e^{-ik\Delta \cos\alpha'}, \tag{57}$$

then using (55) and (56), we note that the $k_{a,b}(\alpha', \alpha)$, independent of $k\Delta$, are bounded functions on $iR \times iR$ that verify,

$$k_{a,b}(\alpha', \alpha) = O\Big(\frac{\Big(\frac{\cos(a_1\alpha)\cos(a_1\alpha')}{\cos(a_1\alpha)+c\cos(a_1\alpha')} + \frac{d\cos(a_1\alpha)}{1+b_1\cos(\alpha)} \Big)}{\cos(a_1\alpha) + c\cos(a_1\alpha')} \Big), \tag{58}$$

as $|\text{Im}\alpha'| > A$ and $|\text{Im}\alpha| > B$, with positive real constants A, B, a_1, $b_1 > 0$, $c > 0$, and d, while the source term is $O(1/\cos(a_1\alpha))$ as $|\text{Im}\alpha| \to \infty$.

The kernels in (53)-(54) are then square integrable as $\text{Im}(k\Delta) < 0$, i.e. $|\arg(ik)| < \pi/2$ when $k\Delta \neq 0$, and we can consider the coupled equations (53)-(54) with,

$$\int_{-\infty}^{+\infty} |\frac{S_b^-(ix)}{(\sin(ix) - \sin\theta_1)}|^2 dx < \infty, \int_{-\infty}^{+\infty} |\frac{S_a^+(ix)}{(\sin(ix) + \sin\theta_1)}|^2 dx < \infty, \tag{59}$$

as a system of Fredholm equations of second kind in $L^2(iR) \times L^2(iR)$. Let us note that the solution of the functional equations (35)-(38) when $k\Delta = 0$ is the one for an impedance wedge, uniquely defined from [18] with [24].

We now explain how to extend this domain to enclose the case $|\arg(ik)| = \pi/2$, by deformation of the integration path.

6.4) Modification of integration path, and extension including k real

6.4.1) The new path of integration $\mathcal{C}_\epsilon$

We note that the exponential term $e^{-ik\Delta\cos\alpha'}$ in the kernels of (53) and (54) vanish at infinity when $\alpha' \to \pm i\infty$ as $|\arg(ik)| < \pi/2$, $k\Delta \neq 0$. However, it is possible to deform the domain of integration (possible by analyticity) so that $e^{-ik\Delta\cos\alpha'} \to 0$ at infinity on the new path even when k is real, providing to not capture any pole during the deformation, while considering that the first singularities of integrands are for $|\text{Re}\alpha| \geq |\text{Re}(\theta_1)| \neq 0$. For this extension including k real, we will modify the path, following,

$$I = \int_{-i\infty}^{+i\infty} K(\alpha', \alpha) v(\alpha') d\alpha' \to I = \int_{\mathcal{C}_\epsilon} K(\alpha', \alpha) v(\alpha') d\alpha', \tag{60}$$

changing the initial path $\alpha' = ix'$ with $x' \in R$ for the path $\mathcal{C}_\epsilon$ with $\alpha' = g_\epsilon(x')$ following,

$$\mathcal{C}_\epsilon \equiv \{\alpha' = g_\epsilon(x') \text{ with } g_\epsilon(x') = ix' + \epsilon \text{tanh} x',\ x' \in R,\ |\epsilon| < |\text{Re}(\theta_1)|\},$$
$$\text{as } |\epsilon| \leq \text{sign}(\epsilon)\text{arg}(ik\Delta) < \pi/2 + |\epsilon|/N, k\Delta \neq 0, \text{ with } N > 1, \epsilon \neq 0. \tag{61}$$

where the case $|\epsilon| = 0$ will be considered in the sense of the limit.

6.4.2) Vanishing properties on $\mathcal{C}_\epsilon$ and analysis of $\text{Re}(ik\text{cos}(g_\epsilon(x)))$

To show that $e^{-ik\Delta \text{cos}\alpha'} \to 0$ at infinity on $\mathcal{C}_\epsilon$, we analyse the behaviour of $\text{Re}(ik\text{cos}(g_\epsilon(x)))$ as $|x| \to \infty$. We can write,

$$ik\text{cos}(g_\epsilon(x)) = |ik|(e^{-x+i(\text{arg}(ik)+\epsilon\text{tanh}x)} + e^{x+i(\text{arg}(ik)-\epsilon\text{tanh}x)})/2,$$
$$\text{Re}(ik\text{cos}(g_\epsilon(x))) = |k|\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|)e^{|x|}(\frac{1 + e^{-2|x|}\frac{\text{cos}(\text{arg}(ik)+\epsilon\text{tanh}|x|)}{\text{cos}(\text{arg}(ik)-\epsilon\text{tanh}|x|)}}{2}). \tag{62}$$

We then note that we have as $0 < |\epsilon| \leq \text{sign}(\epsilon)\text{arg}(ik\Delta) \leq \pi/2$,

$$\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|) > 0,\ \partial_{|x|}\Big(\frac{\text{cos}(\text{arg}(ik) + \epsilon\text{tanh}|x|)}{\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|)}\Big) \leq 0,$$
$$\Big|\frac{\text{cos}(\text{arg}(ik) + \epsilon\text{tanh}|x|)}{\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|)}\Big| \leq 1, \tag{63}$$

while, as $\pi/2 < \text{sign}(\epsilon)\text{arg}(ik\Delta) < \pi/2 + |\epsilon|/N$, when $\text{tanh}|x| \geq \text{tanh}|x_{M,N}| = M/N$ with $1 < M < N$, we have,

$$\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|) > 0,\ \partial_{|x|}\Big(\frac{\text{cos}(\text{arg}(ik) + \epsilon\text{tanh}|x|)}{\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|)}\Big) > 0,$$
$$\frac{\text{sin}(\epsilon(1+M)/N)}{\text{sin}(\epsilon(1-M)/N)} < \frac{\text{cos}(\text{arg}(ik) + \epsilon\frac{M}{N})}{\text{cos}(\text{arg}(ik) - \epsilon\frac{M}{N})} < \frac{\text{cos}(\text{arg}(ik) + \epsilon\text{tanh}|x|)}{\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|)} < -1,$$
$$1 + e^{-2|x|}\frac{\text{cos}(\text{arg}(ik) + \epsilon\text{tanh}|x|)}{\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|)} > 1 - e^{-2|x|}\Big|\frac{\text{sin}(\epsilon(M+1)/N)}{\text{sin}(\epsilon(M-1)/N)}\Big|. \tag{64}$$

Therefore, we can write in whole generality, as $0 < |\epsilon| \leq \text{sign}(\epsilon)\text{arg}(ik\Delta) < \pi/2 + |\epsilon|/N$, when $|x| \geq \text{max}(|x_{M,N}|, \frac{1}{2}\text{ln}|\frac{\text{sin}(\epsilon(M+1)/N)}{\text{sin}(\epsilon(M-1)/N)}|) = a_{\epsilon,N}$ with $1 < M < N$,

$$\text{Re}(ik\text{cos}(g_\epsilon(x))) > |k|\text{cos}(\text{arg}(ik) - \epsilon\text{tanh}|x|)\frac{e^{|x|}}{2}(1 - e^{-2(|x|-a_{\epsilon,N})}) > 0$$
$$\text{with } \frac{e^{|x|}}{2}(1 - e^{-2(|x|-a_{\epsilon,N})}) = e^{a_{\epsilon,N}}\text{sinh}(|x'| - a_{\epsilon,N}). \tag{65}$$

Consequently, the term $e^{-ik\Delta\cos\alpha'}$ verifies as $|x'| \geq a_{\epsilon,N}$ on $\mathcal{C}_\epsilon$,

$$|e^{-ik\Delta\cos\alpha'}| \leq e^{-\gamma(k\Delta,x')e^{a_{\epsilon,N}}\sinh(|x'|-a_{\epsilon,N})} \text{ with } \gamma(k\Delta,x') = \mathrm{Re}(ik\Delta e^{-i\epsilon\tanh|x'|}) > 0, \quad (66)$$

and thus, vanishes as $|x'| \to \infty$, when ϵ satisfies,

$$\begin{aligned} &0 < |\epsilon| \leq \mathrm{sign}(\epsilon)\arg(ik\Delta) < \pi/2 + |\epsilon|/N, \\ &\text{with } N > 1,\ k\Delta \neq 0 \text{ and } 0 < |\epsilon| < |\mathrm{Re}(\theta_1)|. \end{aligned} \quad (67)$$

Reciprocally, at ϵ fixed and thus $\mathcal{C}_\epsilon$ given, we can define

$$\text{the set } \mathcal{T}_\epsilon \text{ of } k\Delta \neq 0 \text{ with } \arg(ik\Delta) \text{ verifying (67)}, \quad (68)$$

on which, by definition, the exponential term vanishes at infinity on $\mathcal{C}_\epsilon$. We can note that we have, for $k\Delta$ belonging to $\mathcal{T}_\epsilon$,

$$\begin{aligned} &\gamma(k\Delta, x') \text{ increases, as } |x'| \text{ increases and } |\arg(ik\Delta)| \text{ decreases,} \\ &\text{and as } |k\Delta| \text{ increases when } |x'| \geq a_{\epsilon,N}. \end{aligned} \quad (69)$$

Remark : we note that, as $|\arg(ik)| \leq \pi/2$,

$$\begin{aligned} &e^{-ik\Delta\cos(g_\epsilon(x))} = h(|x|, k\Delta)e^{-ik\Delta e^{-i\epsilon\tanh|x|}\sinh|x|},\ |h(|x|, k\Delta) = e^{-e^{-|x|}ik\Delta\cos(\epsilon\tanh|x|)}| \leq 1, \\ &|e^{-ik\Delta\cos\alpha'}| \leq e^{-\gamma(k\Delta,x')\sinh|x'|} \text{ with } \gamma(k\Delta,x') = \mathrm{Re}(ik\Delta e^{-i\epsilon\tanh|x'|}). \end{aligned} \quad (70)$$

6.4.3) Analyticity of integral expressions of f_{br} and f_{ar} with the new path $\mathcal{C}_\epsilon$

The integrals (45) and (46) with new path of integration $\mathcal{C}_\epsilon$ give us expressions of,

$$\frac{f_{br}(\alpha)}{\Psi_{+1}(\alpha)} \text{ and } \frac{f_{ar}(\alpha)}{\Psi_{1-}(\alpha)}, \quad (71)$$

which are regular, respectively as,

$$\begin{aligned} &\mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) - \Phi_+ < \mathrm{Re}(\alpha) < \mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) + 3\Phi_+, \\ &\text{and } \mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) - 3\Phi_- < \mathrm{Re}(\alpha) < \mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) + \Phi_-, \end{aligned} \quad (72)$$

with $\mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) = \epsilon\tanh(\mathrm{Im}(\alpha))$, $|\mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha)))| \leq |\epsilon|$, where we can notice that the functions $f_{br,ar}(\alpha)$ are $O(1)$ at infinity, with $f_{br,ar}(-i\infty) = -f_{br,ar}(+i\infty)$. These functions satisfy, by analytic continuation of integrals, the functional equations (35)-(38), and are meromorphic functions of the complex plane

Remark : the analytic continuation of complex integrals can be done by adding residues terms associated to poles crossing the integration path as a parameter varies, or by deformation of integration path.

6.4.4) Integral equations on $\mathcal{C}_\epsilon$ as $|\epsilon| \leq \text{sign}(\epsilon)\text{arg}(ik\Delta) < \pi/2 + |\epsilon|/N$

Changing the path for $\mathcal{C}_\epsilon$, the integral term I in (60) now becomes,

$$I = \int_{-\infty}^{+\infty} (K(g_\epsilon(x'), g_\epsilon(x))v(\alpha(x'))\partial_{x'} g_\epsilon(x'))dx', \tag{73}$$

for $|\epsilon| \leq \text{sign}(\epsilon)\text{arg}(ik\Delta) < \pi/2 + |\epsilon|/N,\ N > 1,$ with $k\Delta \neq 0$ and $0 < |\epsilon| < |\text{Re}(\theta_1)|$. Replacing the integration path by $\mathcal{C}_\epsilon$ in integral equations (53) and (54), we then derive,

$$\begin{aligned}
&\frac{S_a^+(g_\epsilon(x))}{(\sin(g_\epsilon(x)) + \sin\theta_1)} = \frac{-i}{8\Phi_+}\int_{-\infty}^{+\infty} \frac{S_b^-(g_\epsilon(x'))}{(\sin(g_\epsilon(x')) - \sin\theta_1)} \\
&K_{b,a}(g_\epsilon(x'), g_\epsilon(x))\partial_{x'} g_\epsilon(x')dx' + \frac{1}{2}[\Psi_{+1}(g_\epsilon(x) + \frac{\pi}{2} - \frac{\Phi_b}{2})\chi_b^i(g_\epsilon(x) + \frac{\pi}{2} - \frac{\Phi_b}{2}) \\
&+ \frac{(\sin(g_\epsilon(x)) - \sin\theta_1)\Psi_{+1}(-\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})}{(\sin(g_\epsilon(x)) + \sin\theta_1)}\chi_b^i(-g_\epsilon(x) + \frac{\pi}{2} - \frac{\Phi_b}{2})],
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
&\frac{S_b^-(g_\epsilon(x))}{(\sin(g_\epsilon(x)) - \sin\theta_1)} = \frac{i}{8\Phi_-}\int_{-\infty}^{+\infty} \frac{S_a^+(g_\epsilon(x'))}{(\sin(g_\epsilon(x')) + \sin\theta_1)} \\
&K_{a,b}(g_\epsilon(x'), g_\epsilon(x))\partial_{x'} g_\epsilon(x')dx' + \frac{1}{2}[\Psi_{1-}(g_\epsilon(x) - \frac{\pi}{2} + \frac{\Phi_a}{2})\chi_a^i(g_\epsilon(x) - \frac{\pi}{2} + \frac{\Phi_a}{2}) \\
&+ \frac{(\sin(g_\epsilon(x)) + \sin\theta_1)\Psi_{1-}(-g_\epsilon(x) - \frac{\pi}{2} + \frac{\Phi_a}{2})}{(\sin(g_\epsilon(x)) - \sin\theta_1)}\chi_a^i(-g_\epsilon(x) - \frac{\pi}{2} + \frac{\Phi_a}{2})].
\end{aligned} \tag{75}$$

for $x \in R$, with $g_\epsilon(x) = ix + \epsilon\text{tanh}x$. Following (57), the dependence on $k\Delta$ of kernels is in exponential term, and we have,

$$K_{a,b}(g_\epsilon(x'), g_\epsilon(x))\partial_{x'} g_\epsilon(x') = k_{a,b}(g_\epsilon(x'), g_\epsilon(x))\partial_{x'} g_\epsilon(x')e^{-ik\Delta\cos(g_\epsilon(x'))}, \tag{76}$$

where the $k_{a,b}(g_\epsilon(x'), g_\epsilon(x))$ are independent of $k\Delta$, and the exponential term is an entire function of $k\Delta$. Considering (55) and (56), the $k_{a,b}$ are analytic functions of $k\Delta$, regular and bounded by constants independent of $k\Delta$, which verify,

$$|k_{a,b}(g_\epsilon(x'),g_\epsilon(x))\partial_{x'}g_\epsilon(x')| = O\Big(\frac{\Big(\frac{\cosh(a_1x)\cosh(a_1x')}{\cosh(a_1x)+c\cosh(a_1x')} + \frac{d\cosh(a_1x)}{1+b_1\cosh(x)}\Big)}{\cosh(a_1x)+c\cosh(a_1x')}\Big), \tag{77}$$

as $|x| > A$ and $|x'| > A'$, for some positive real constants A, A', a_1, $b_1 > 0$, $c > 0$, and d independent of $k\Delta$, while we note that the source terms are $O(1/\cosh(ax))$ as $|x| \to \infty$.

6.4.4) Fredholmness on $\mathcal{C}_\epsilon \times \mathcal{C}_\epsilon$, in $L^2(R) \times L^2(R)$

Considering $\mathcal{T}_\epsilon$ defined in (68), the domain of $k\Delta$ where the integral equations can be considered on $C_\epsilon \times C_\epsilon$, we note that for any closed subset T of $\mathcal{T}_\epsilon$, it exists two complex values $\xi_{1,2}$ of $\mathcal{T}_\epsilon$ with $0 < |\xi_1| < |\xi_2|, |\arg(i\xi_1)| = |\arg(i\xi_2)|$, such that,

$$|\arg(ik\Delta)| \le |\arg(i\xi_1)|, |\xi_1| \le |k\Delta| \le |\xi_2|. \tag{78}$$

We have in T, from (66)-(69), as $|x'| \ge a_{\epsilon,N}$,

$$\begin{aligned} &|e^{-ik\Delta\cos(g_\epsilon(x'))}| \le e^{-\gamma(k\Delta,x')e^{a_{\epsilon,N}}\sinh(|x'|-a_{\epsilon,N})} \le e^{-\gamma(\xi_1,x')e^{a_{\epsilon,N}}\sinh(|x'|-a_{\epsilon,N})}, \\ &\gamma(\xi_1,x') = \mathrm{Re}(i\xi_1 e^{-i\epsilon\tanh|x'|}) > 0, \gamma(\xi_1,x') > \gamma(\xi_1,a_{\epsilon,N}) > 0, \end{aligned} \tag{79}$$

and, as $|x'| \le a_{\epsilon,N}$,

$$|e^{-ik\Delta\cos(g_\epsilon(x'))}| \le C_{\xi_{1,2}} \le C_{\xi_{1,2}} e^{-\gamma(\xi_1,a_{\epsilon,N})e^{a_{\epsilon,N}}\sinh(|x'|-a_{\epsilon,N})}, \tag{80}$$

where $C_{\xi_{1,2}}$ only depends on ϵ, ξ_1, ξ_2 and $a_{\epsilon,N}$. Thus, we have in general,

$$\begin{aligned} &|e^{-ik\Delta\cos(g_\epsilon(x'))}| \le D_{\xi_{1,2}} e^{-\gamma_1\sinh(|x'|-a_{\epsilon,N})}, \\ &\text{with } D_{\xi_{1,2}} = \max(1, C_{\xi_{1,2}}), \gamma_1 = \gamma(\xi_1,a_{\epsilon,N})e^{a_{\epsilon,N}} > 0, \end{aligned}$$

where $D_{\xi_{1,2}}$ and γ_1 depend only on the choice of the subset T.
We can then write for $k\Delta$ belonging to T,

$$\begin{aligned} &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |K_{a,b}(g_\epsilon(x'),g_\epsilon(x))\partial_{x'}g(x')|^2 dx'dx < B_0, \\ &\int_{-\infty}^{+\infty} |K_{a,b}(g_\epsilon(x'),g_\epsilon(x))\partial_{x'}g(x')|^2 dx' < F_1(x), \\ &\int_{-\infty}^{+\infty} |K_{a,b}(g_\epsilon(x'),g_\epsilon(x))\partial_{x'}g(x')|^2 dx < F_2(x'), \\ &\int_{-\infty}^{+\infty} F_1(x)dx < B_0, \int_{-\infty}^{+\infty} F_2(x')dx' < B_0, \end{aligned} \tag{81}$$

where the functions F_1 and F_2 can be chosen following,

$$F_1(x) = D^2_{\xi_{1,2}} \int_{-\infty}^{+\infty} |k_{a,b}(g_\epsilon(x'), g_\epsilon(x))\partial_{x'} g(x')|^2 e^{-2\gamma_1 \sinh(|x'|-a_{\epsilon,N})} dx',$$
$$F_2(x') = D^2_{\xi_{1,2}} e^{-2\gamma_1 \sinh(|x'|-a_{\epsilon,N})} |\partial_{x'} g(x')|^2 \int_{-\infty}^{+\infty} |k_{a,b}(g_\epsilon(x'), g_\epsilon(x))|^2 dx. \qquad (82)$$

We have then satisfied the conditions, in the theory of integral equations depending of a parameter [27] [28], to consider our equations as a system of Fredholm equations of second kind in $L^2(R) \times L^2(R)$ (on $\mathcal{C}_\epsilon \times \mathcal{C}_\epsilon$ for $\alpha = ix + \epsilon \tanh x, \quad x \in R$), as $|\epsilon| \leq \text{sign}(\epsilon)\arg(ik\Delta) < \pi/2 + |\epsilon|/N$, with $0 < |\epsilon| < |\text{Re}(\theta_1)|$, $k\Delta \neq 0$ and $N > 1$, which can be used in particular for k real.

Remark : From the known properties of analyticity of kernels in the complex plane, the solutions $S_b^-(\alpha')$ and $S_a^+(\alpha')$ are odd meromorphic functions (the only singularities are poles).

7) Existence and uniqueness for integral equations on $\mathcal{C}_\epsilon$

7.1) Uniqueness of spectral integral equations as $|\arg(ik\Delta)| \leq \pi/2$

Concerning uniqueness, we consider the solution of integral equations (74)-(75) in $L^2(R)$ along integration path $\mathcal{C}_\epsilon \equiv \{\alpha = ix + \epsilon \tanh x, \ x \in R\}$ when χ_a^i and χ_b^i are null (we have no source terms), with $|\epsilon| \leq \text{sign}(\epsilon)\arg(ik\Delta) \leq \pi/2$, as $0 < |\epsilon| < |\text{Re}(\theta_1)|$, $k\Delta \neq 0$.

7.1.1) f^a and f^b from $S_a^+(\alpha')$ and $S_b^-(\alpha')$ when $\chi_a^i = \chi_b^i = 0$

Let us take $S_a^+(\alpha')$ and $S_b^-(\alpha')$ derived from (74)-(75) when $\chi_a^i = \chi_b^i = 0$. Considering the same kernels as the ones used in (45)-(46), we now define two functions $f^{br}$and f^{ar}, following,

$$\frac{f^{br}(\alpha)}{\Psi_{+1}(\alpha)} = \frac{i}{4\Phi_+} \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \frac{S_b^-(\alpha') e^{-ik\Delta\cos\alpha'} \tan(\frac{\pi}{4\Phi_+}(\alpha - \Phi_+ - \alpha'))}{(\sin\alpha' - \sin\theta_1)\Psi_{+1}(\alpha' - \Phi_+)} d\alpha'$$
$$= \frac{-i}{4\Phi_+} \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \frac{S_b^-(\alpha')/(\sin\alpha' - \sin\theta_1)}{\Psi_{+1}(\alpha' - \Phi_+)} \frac{e^{-ik\Delta\cos\alpha'} \sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_+)) + \cos(\frac{\pi\alpha'}{2\Phi_+})} d\alpha', \qquad (83)$$

and,

$$\frac{f^{ar}(\alpha)}{\Psi_{1-}(\alpha)} = \frac{-i}{4\Phi_-}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \frac{S_a^+(\alpha')e^{-ik\Delta\cos\alpha'}\tan(\frac{\pi}{4\Phi_-}(\alpha+\Phi_- -\alpha'))}{(\sin\alpha'+\sin\theta_1)\Psi_{1-}(\alpha'+\Phi_-)}d\alpha'$$
$$= \frac{i}{4\Phi_-}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \frac{S_a^+(\alpha')/(\sin\alpha'+\sin\theta_1)}{\Psi_{1-}(\alpha'+\Phi_-)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_-))+\cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha', \qquad (84)$$

These expressions are definite in the respective bands,

$$\begin{aligned} &\mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) - \Phi_+ < \mathrm{Re}(\alpha) < \mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) + 3\Phi_+, \\ &\mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) - 3\Phi_- < \mathrm{Re}(\alpha) < \mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha))) + \Phi_-, \end{aligned} \qquad (85)$$

where, from analytical properties of kernels, the functions f^{br} and f^{ar} are analytical functions, $O(1)$ at infinity and verifying $f^{ar(br)}(i\infty) + f^{ar(br)}(-i\infty) = 0$. Considering the integral equations (74)-(75) when χ_a^i and χ_b^i are null, satisfied by S_a^+ and S_b^-, we have,

$$\begin{aligned} &f^{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) + \frac{(\sin\alpha+\sin\theta_1)f^{ar}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{(\sin\alpha-\sin\theta_1)} = \frac{2S_b^-(\alpha)}{(\sin\alpha-\sin\theta_1)}, \\ &f^{br}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) + \frac{(\sin\alpha-\sin\theta_1)f^{br}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{(\sin\alpha+\sin\theta_1)} = \frac{2S_a^+(\alpha)}{(\sin\alpha+\sin\theta_1)}. \end{aligned} \qquad (86)$$

which implies, from meromorphic character of kernels, that f^{ar} and f^{br} are also meromorphic functions of the complex plane (note : analytic continuation is done by adding residues terms associated to poles crossing the integration path as α varies). We can then consider functional equations attached to (45)-(46). We then deduce, after using elementary properties of kernels, that $f^b(\alpha) = f^{br}(\alpha - \frac{\Phi_b}{2})$ and $f^a(\alpha) = f^{ar}(\alpha + \frac{\Phi_a}{2})$, satisfy, in place of $f_b(\alpha) = f_{br}(\alpha - \frac{\Phi_b}{2})$ and $f_a(\alpha) = f_{ar}(\alpha + \frac{\Phi_a}{2})$, the functional equations (29)-(30) with (28), and thus the condition (34) which implies, in particular, that f^a and f^b refer to the same fields. Considering the zone of regularity of integral expressions, their asymptotics, and their analytical continuation, the conditions (a) and (b) are satisfied for f_a and f_b.
So defined from S_a^+ and S_b^-, given by (74)-(75) when $\chi_a^i = \chi_b^i = 0$, we then conclude that,

> f^a and f^b are two spectral functions refering to the representation of the same field u with two distinct origins of coordinates, which satisfies the boundary conditions for the impedance three-part polygon, and the elementary conditions (a'), (b') and (c') without illumination. (87)

7.1.2) Vanishing property of f^a and f^b, then uniqueness

We now consider the following properties :

(i) in absence of source terms, from uniqueness theorem on the field (see appendix), the field u and its normal derivative shall be null on the surface of impedance polygons.
(ii) because of single face expression of spectral functions (7)-(8), the functions f^a and f^b vanish when the field and its normal derivative are null on the surface of the polygon.

After use of (i) for (ii), the spectral functions f^a and f^b verifying (87), and thus S_b^- and S_a^+ satisfyng (86), have to be null, which leads us to conclude, by definition, that the solutions of our spectral integral equations (74)-(75) when $\chi_a^i = \chi_b^i = 0$ are too. This property implies by linearity that we have uniqueness of solutions for our system of spectral integral equations (74)-(75) in $L^2(R) \times L^2(R)$ as $|\arg(ik)| \leq \pi/2$, $\Delta \neq 0$.

Remark : The solution is also uniquely defined in the case $\Delta = 0$ (simple wedge) from Thuzhilin's theory [24] when conditions (a)-(b) are satisfied.

7.2) Existence of a solution

Concerning existence, we note first that the kernels are analytic functions of $k\Delta$ on $\mathcal{T}_\epsilon$ verifying (81). Considering then the theory of integral equations depending on a parameter in $L^2(R)$ (from Smirnov [27] or Tamarkin [28]), the resolvent for our system of Fredholm integral equations of second kind in $L^2(R) \times L^2(R)$ is given by $\mathcal{R}(x, x', k\Delta) = \mathcal{D}_1(x, x', k\Delta, 1)/D_2(k\Delta,1)$, where $\mathcal{D}_1(x, x', k\Delta, \mu)$ and $D_2(k\Delta,\mu)$ are given by entire series of μ, which are regular with respect to $k\Delta$ in $\mathcal{T}_\epsilon$, $k\Delta \neq 0$. It defines a unique solution in $L^2(R) \times L^2(R)$, except if $D_2(k\Delta, 1) \equiv 0$ or if $D_2(k\Delta, 1) = 0$ for some discrete values of $k\Delta$. In both cases where the resolvent is singular, the solution is not unique and we have value of $k\Delta$ where the integral equations without source terms has a non trivial (non nul) solution.
Therefore, since the uniqueness of our spectral integral equations on $C_\epsilon \times C_\epsilon$ has been shown independently in previous section, we deduce that a unique solution of (74)-(75) exists in $L^2(R) \times L^2(R)$ with $|\arg(ik\Delta)| \leq \pi/2$, $k\Delta \neq 0$. In other words, we have derived,

from our system of integral equations, the existence
of a spectral solution which is unique, when considered
in $L^2(R) \times L^2(R)$ on $C_\epsilon \times C_\epsilon$, as $|\arg(ik)| \leq \pi/2$, $k\Delta \neq 0$. (88)

Remark : Concerning the case with $k\Delta = 0$, the solution satisfies the functional difference equation for a simple impedance wedge, which is known to be uniquely defined from [18], [24].

8) Some particular features of the system of integral equations for the three-part impedance polygon and their consequences

Our spectral integral equations can be solved numerically, or analytically by approximations. In our case, the approximations principally depend on $k\Delta$, for $k\Delta$ small or $k\Delta$ large. These integral equations and the integral expressions of the spectral functions have also special features concerning the decoupling of integral equations in important cases that we first illustrate.

8.1) Decoupling in the case of the unsymmetric three-part impedance plane

In the case of the three-part impedance plane, we have $\Phi_a = \Phi_b = 0$ and $\Phi_+ = \Phi_- = \frac{\pi}{2}$. Considering that,

$$\frac{\Psi_{1-}(\alpha - \frac{\pi}{2})}{2\Psi_{+1}(\alpha - \frac{\pi}{2})} - \frac{\Psi_{1-}(-\alpha - \frac{\pi}{2})}{2\Psi_{+1}(-\alpha - \frac{\pi}{2})} =$$
$$= \frac{\Psi_{1-}(\alpha - \frac{\pi}{2})}{\Psi_{+1}(\alpha - \frac{\pi}{2})} \frac{\sin\alpha(\sin\theta_- - \sin\theta_1)}{(\sin\alpha + \sin\theta_-)(\sin\alpha - \sin\theta_1)} = \frac{C(\alpha)\sin\alpha(\sin\theta_- - \sin\theta_1)}{\sin\alpha - \sin\theta_1},$$
$$\frac{\Psi_{+1}(\alpha + \frac{\pi}{2})}{2\Psi_{1-}(\alpha + \frac{\pi}{2})} - \frac{\Psi_{+1}(\frac{\pi}{2} - \alpha)}{2\Psi_{1-}(\frac{\pi}{2} - \alpha)} =$$
$$= \frac{\Psi_{+1}(\frac{\pi}{2} - \alpha)}{\Psi_{1-}(\frac{\pi}{2} - \alpha)} \frac{-\sin\alpha(\sin\theta_+ - \sin\theta_1)}{(\sin\alpha + \sin\theta_+)(\sin\alpha - \sin\theta_1)} = \frac{-C(\alpha)\sin\alpha(\sin\theta_+ - \sin\theta_1)}{\sin\alpha - \sin\theta_1}, \quad (89)$$

and letting $N(\alpha') = \frac{iC(\alpha')\sin\alpha'/2\pi}{\sin\alpha' - \sin\theta_1}$, we then derive,

$$\frac{(\sin\theta_1 - \sin\theta_+)^{1/2} f_{br}(\alpha + \frac{\pi}{2})}{\Psi_{+1}(\alpha + \frac{\pi}{2})} = \eta_m \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} N(\alpha') \frac{(\sin\theta_1 - \sin\theta_-)^{1/2} f_{ar}(\alpha' - \frac{\pi}{2})}{\Psi_{1-}(\alpha' - \frac{\pi}{2})}$$
$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin\alpha'}{\cos\alpha + \cos\alpha'} d\alpha' + (\sin\theta_1 - \sin\theta_+)^{1/2} \chi_b^i(\alpha + \frac{\pi}{2}),$$

$$\frac{(\sin\theta_1-\sin\theta_-)^{1/2}f_{ar}(\alpha-\frac{\pi}{2})}{\Psi_{1-}(\alpha-\frac{\pi}{2})}=\eta_m\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}N(\alpha')\frac{(\sin\theta_1-\sin\theta_+)^{1/2}f_{br}(\alpha'+\frac{\pi}{2})}{\Psi_{+1}(\alpha'+\frac{\pi}{2})}$$
$$\times\frac{e^{-ik\Delta\cos\alpha'}\sin\alpha'}{\cos\alpha+\cos\alpha'}d\alpha'+(\sin\theta_1-\sin\theta_-)^{1/2}\chi_a^i(\alpha-\frac{\pi}{2}),\qquad(90)$$

where $\eta_m=(\sin\theta_1-\sin\theta_+)^{1/2}(\sin\theta_1-\sin\theta_-)^{1/2}$.
We obtain two uncoupled equations by simple addition (S) and substraction (D) :

$$(S,D)=\eta_m\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}N(\alpha')((S,-D)e^{-ik\Delta\cos\alpha'}\frac{\sin\alpha'}{\cos\alpha+\cos\alpha'}d\alpha'+(S^i,-D^i).\quad(91)$$

8.2) Decoupling for symmetric three parts polygons ($\Phi_b=\Phi_a$ and $\sin\theta_+=\sin\theta_-$)

In this case, the system (74)-(75) can also be decoupled. For this, we first express the system of integral equation (74)-(75) in a new form. We observe that $\Psi_{-1}(\alpha,\Phi_-)$ (also denoted $\Psi_{-1}(\alpha)$) satisfies $\Psi_{-1}(\alpha,\Phi_-)=\Psi_{1-}(-\alpha,\Phi_-)$. Considering (45)-(46), we can write,

$$\frac{f_{br}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}=\frac{-i}{4\Phi_+}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}\frac{f_{ar}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{+1}(\alpha'-\Phi_+)}$$
$$\times\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha+\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'+\chi_b^i(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}),$$
$$\frac{f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{-1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_a}{2})}=\frac{-i}{4\Phi_-}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}\frac{f_{br}(-\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{-1}(\alpha'-\Phi_-)}$$
$$\times\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha'+\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}),\qquad(92)$$

This form is particularly suitable for the case of a symmetric polygon. In this case $\Phi_+=\Phi_-$ (i.e. $\Phi_b=\Phi_a$) and $\sin\theta_+=\sin\theta_-$, so that $\Psi_{+1}=\Psi_{-1}$ and the equations have the same kernels. Thus, by addition and substraction, we derive a system of decoupled equations,

$$\frac{(S,D)(\alpha)}{\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}=$$
$$=\frac{-i}{4\Phi_+}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}\frac{\frac{(S,-D)(\alpha')}{\Psi_{+1}(\alpha'-\Phi_+)}e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha+\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'+(S^i,D^i)(\alpha),\qquad(93)$$

where

$$S(\alpha) = f_{br}(-\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) + f_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_b}{2}),$$
$$D(\alpha) = = f_{br}(-\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) - f_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_b}{2}),$$
$$S^i(\alpha) = \chi^i_{br}(-\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) + \chi^i_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_b}{2}),$$
$$D^i(\alpha) = \chi^i_{br}(-\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) - \chi^i_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_b}{2}), \quad (94)$$

for $|\varphi_\circ| < \min(\frac{\pi}{2},\ \frac{\pi}{2} + \Phi_b)$.

8.3) Partial inversion and new kernels for small $k\Delta$

Considering the solution known for $k\Delta = 0$ [1], we can apply the identity

$$e^{-ik\Delta\cos\alpha'} = (e^{-ik\Delta\cos\alpha'} - 1) + 1 \quad (95)$$

and invert the part corresponding to unit term, which results in equations with kernels vanishing as $k\Delta \to 0$, suitable for approximations. To simplify the reading, we will detail the method in appendix. We then obtain a system of coupled equations with kernels and source terms vanishing as $k\Delta \to 0$, following :

$$f_{br0}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) = \frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty} d\alpha'(\frac{\Psi_{1-}(\alpha' + \Phi_-)(B_{b0}(\alpha') - B_{b0}(-\alpha'))}{\sin\alpha' - \sin\theta_1}$$
$$\times M_+(\alpha, \alpha') - \frac{\Psi_{+1}(\alpha' - \Phi_+)(B_{a0}(\alpha') - B_{a0}(-\alpha'))}{\sin\alpha' + \sin\theta_1} M_-(\alpha, \alpha')),$$
$$f_{ar0}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) = \frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty} d\alpha'(\frac{\Psi_{1-}(\alpha' + \Phi_-)(B_{b0}(\alpha') - B_{b0}(-\alpha'))}{\sin\alpha' - \sin\theta_1}$$
$$\times N_+(\alpha, \alpha') - \frac{\Psi_{+1}(\alpha' - \Phi_+)(B_{a0}(\alpha') - B_{a0}(-\alpha'))}{\sin\alpha' + \sin\theta_1} N_-(\alpha, \alpha')), \quad (96)$$

where $M_\pm(\alpha, \alpha') = L_\pm(\alpha + \frac{\pi}{2} + \frac{(\Phi_a - \Phi_b)}{2}, \alpha')$, $N_\pm(\alpha, \alpha') = L_\pm(\alpha - \frac{\pi}{2} + \frac{(\Phi_a - \Phi_b)}{2}, \alpha')$,

$$L_\pm(\alpha, \alpha') = \frac{\pi\sin\alpha'\Psi_{+-}(\alpha)}{2\Phi_d}\int_{-i\infty}^{i\infty} \frac{\cos(\frac{\pi(\varphi' + (\Phi_a - \Phi_b)/2)}{2\Phi_d})}{\Psi_{+-}(\varphi' + (\Phi_a - \Phi_b)/2)}$$
$$\frac{(\cos\varphi' + \sin\theta_1)}{\cos(\varphi' \pm \frac{\pi}{2}) + \cos\alpha'} \frac{1}{(\sin(\frac{\pi\alpha}{2\Phi_d}) - \sin(\frac{\pi(\varphi' + (\Phi_a - \Phi_b)/2)}{2\Phi_d}))} d\varphi', \quad (97)$$

and

$$B_{a0}(\alpha') = (\frac{f_{ar0}(\alpha' - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha' - \Phi_+)})(e^{-ik\Delta\cos\alpha'} - 1)$$
$$+ (\frac{f_0^{ab}(\alpha' - \frac{\pi}{2} + (\Phi_a - \Phi_b)/2,\, \varphi_\circ)}{\Psi_{+1}(\alpha' - \Phi_+)})(e^{-ik\Delta(\cos\alpha' + \sin\varphi_\circ)} - 1)e^{ik\Delta\sin\varphi_\circ},$$
$$B_{b0}(\alpha') = (\frac{f_{br0}(\alpha' + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha' + \Phi_-)})(e^{-ik\Delta\cos\alpha'} - 1)$$
$$+ (\frac{f_0^{ab}(\alpha' + \frac{\pi}{2} - (\Phi_b - \Phi_a)/2,\, \varphi_\circ)}{\Psi_{1-}(\alpha' + \Phi_-)})(e^{-ik\Delta(\cos\alpha' - \sin\varphi_\circ)} - 1). \qquad (98)$$

with $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$, $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$, and $\Phi_d = \frac{\pi}{2} + \frac{\Phi_a + \Phi_b}{2}$. In the particular case $\Phi_a = \Phi_b = 0$, $\Phi_d = \frac{\pi}{2}$, the functions $L_\pm$ can be simplified so that we recover the expressions found in [2] for the three-part impedance plane.

8.4) Asymptotics for large $k\Delta$

We can write, from previous integral expressions in (45)-(46),

$$\mathcal{R}_b(\alpha) = \frac{-i}{4\Phi_+} m_1(\alpha) \int_{-i\infty}^{+i\infty} \mathcal{R}_a(\alpha') \frac{e^{-ik\Delta\cos\alpha'} \sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})} d\alpha'$$
$$+ m_1(\alpha)\chi_b^i(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}), \qquad (99)$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) < 3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) = \frac{i}{4\Phi_-} m_2(\alpha) \int_{-i\infty}^{+i\infty} \mathcal{R}_b(\alpha') \frac{e^{-ik\Delta\cos\alpha'} \sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})} d\alpha'$$
$$+ m_2(\alpha)\chi_a^i(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}), \qquad (100)$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) < \Phi_-$, for $-\min(\frac{\pi}{2}, \frac{\pi}{2} + \Phi_a) < \varphi_\circ < \min(\frac{\pi}{2}, \frac{\pi}{2} + \Phi_b)$, where

$$\mathcal{R}_b(\alpha) = \frac{f_{br}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha + \Phi_-)}, \mathcal{R}_a(\alpha) = \frac{f_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha - \Phi_+)},$$
$$m_1(\alpha) = \frac{\Psi_{+1}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha + \Phi_-)}, m_2(\alpha) = \frac{\Psi_{1-}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha - \Phi_+)}, \qquad (101)$$

To consider asymptotic evaluations of $\mathcal{R}_{a,b}(\alpha')$ from the equations (99)-(101) as $k\Delta$ is large, we can choose to take account or to neglect the influence of complex poles of $\mathcal{R}_{a,b}(\alpha')$

(corresponding to guided waves) in vicinity of the integration path. To simplify the presentation, we develop here the second case, and consider the first case in appendix.
We begin by assuming that the principal contribution of $\mathcal{R}_{a,b}(\alpha')$ comes from the vicinity of the stationary phase point $\alpha'=0$. Letting $\cos\alpha' = 1 - x^2/2$ with $x = 2\sin(\alpha'/2)$, and taking into account the parity of the integration path, we have

$$\mathcal{R}_b(\alpha) \sim m_1(\alpha)\chi_b^i(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) + \Big(\frac{m_1(\alpha)(\,\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0})e^{-ik\Delta}}{2\pi i} \times \int_{-i\infty}^{+i\infty} \frac{e^{\frac{ik\Delta x^2}{2}} x^2}{(\frac{\pi}{2\Phi_+})^{-2}(1+\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))) - x^2} dx \Big), \tag{102}$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) < 3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) \sim m_2(\alpha)\chi_a^i(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) - \Big(\frac{m_2(\alpha)(\,\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0})e^{-ik\Delta}}{2\pi i} \times \int_{-i\infty}^{+i\infty} \frac{e^{\frac{ik\Delta x^2}{2}} x^2}{(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))) - x^2} dx \Big), \tag{103}$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) < \Phi_-$. After derivation of previous equations, we obtain

$$\begin{aligned} (\,\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0}) &\sim \partial_\alpha(m_1(\alpha)\chi_b^i(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}))|_{\alpha=0}) \\ (\,\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0}) &\sim \partial_\alpha(m_2(\alpha)\chi_a^i(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}))|_{\alpha=0}) \end{aligned} \tag{104}$$

and we get,

$$\mathcal{R}_b(\alpha) \sim \frac{m_1(\alpha)(\partial_\alpha(m_2(\alpha)\chi_a^i(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}))|_{\alpha=0})e^{-ik\Delta}}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{\frac{ik\Delta x^2}{2}} x^2}{(\frac{\pi}{2\Phi_+})^{-2}(1+\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))) - x^2} dx + m_1(\alpha)\chi_b^i(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}), \tag{105}$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) < 3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) \sim -\frac{m_2(\alpha)(\partial_\alpha(m_1(\alpha)\chi_b^i(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}))|_{\alpha=0})e^{-ik\Delta}}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{\frac{ik\Delta x^2}{2}} x^2}{(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))) - x^2} dx + m_2(\alpha)\chi_a^i(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}), \tag{106}$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) < \Phi_-$, for $-\min(\frac{\pi}{2},\ \frac{\pi}{2} + \Phi_a) < \varphi_\circ < \min(\frac{\pi}{2},\ \frac{\pi}{2} + \Phi_b)$, $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$, $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$.

Concerning the diffracted field $u_d^{a,b}(\rho_{a,b},\varphi_{a,b})$, refering to origins at $\rho_{a,b}=0$, we have,

$$u_d^{a,b}(\rho_{a,b},\varphi_{a,b}) \sim \frac{-e^{-i\pi/4-ik\rho_{a,b}}}{\sqrt{2\pi k\rho_{a,b}}}[f_{a,b}(\pi+\varphi_{a,b})-f_{a,b}(-\pi+\varphi_{a,b})],$$

$$\text{with } \Psi_{1-}(\alpha+\Phi_-)\mathcal{R}_b(\alpha)=f_{br}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})=f_b(\alpha+\frac{\pi}{2}),$$

$$\text{and } \Psi_{+1}(\alpha-\Phi_+)\mathcal{R}_a(\alpha)=f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})=f_a(\alpha-\frac{\pi}{2}). \qquad (107)$$

The determination of $f_{a,b}(\pm\pi+\varphi_{a,b})$ can be reduced, after the use of functional difference equation on $f_{a,b}$, to the evaluation of $\mathcal{R}_{a,b}(\alpha)$ for real $\alpha=\varphi_r$ (see appendix on scattering diagram), provided we are in the domains of regularity of integral expressions, respectively $-\Phi_+<\text{Re}(\varphi_r+\frac{\pi}{2}-\frac{\Phi_b}{2})<3\Phi_+$ for $\mathcal{R}_b(\varphi_r)$, and $-3\Phi_-<\text{Re}(\varphi_r-\frac{\pi}{2}+\frac{\Phi_a}{2})<\Phi_-$ for $\mathcal{R}_a(\varphi_r)$.

For that, we can deform the path of integration to the Steepest Descent Path, while neglecting the contribution of complex poles, and we derive,

$$\mathcal{R}_b(\varphi_r) \sim \frac{m_1(\varphi_r)\partial_\alpha(m_2(\alpha)\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}))|_{\alpha=0}}{\sqrt{2ik\Delta}}2e^{-ik\Delta}(v\,e^{v^2}\text{erfc}(v)-\frac{1}{\sqrt{\pi}}),$$

$$+m_1(\varphi_r)\chi_b^i(\varphi_r+\frac{\pi}{2}-\frac{\Phi_b}{2}), \qquad (108)$$

as $-\Phi_+<(\varphi_r+\frac{\pi}{2}-\frac{\Phi_b}{2})<3\Phi_+$, $v^2=ik\Delta(\frac{\pi}{2\Phi_+})^{-2}(1+\cos(\frac{\pi}{2\Phi_+}(\varphi_r-\Phi_b)))$,and,

$$\mathcal{R}_a(\varphi_r) \sim -\frac{m_2(\varphi_r)\partial_\alpha(m_1(\alpha)\chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}))|_{\alpha=0}}{\sqrt{2ik\Delta}}2e^{-ik\Delta}(w\,e^{w^2}\text{erfc}(w)-\frac{1}{\sqrt{\pi}}),$$

$$+m_2(\varphi_r)\chi_a^i(\varphi_r-\frac{\pi}{2}+\frac{\Phi_a}{2}), \qquad (109)$$

as $-3\Phi_-<(\varphi_r-\frac{\pi}{2}+\frac{\Phi_a}{2})<\Phi_-$, $w^2=ik\Delta(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\varphi_r+\Phi_a)))$, for $-\min(\frac{\pi}{2},\frac{\pi}{2}+\Phi_a)<\varphi_\circ<\min(\frac{\pi}{2},\frac{\pi}{2}+\Phi_b)$, with $\Phi_+=\frac{\pi}{2}+\frac{\Phi_b}{2}$ and $\Phi_-=\frac{\pi}{2}+\frac{\Phi_a}{2}$, where

$$ae^{a^2}\text{erfc}(a)-\frac{1}{\sqrt{\pi}}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{-t^2e^{-t^2}}{a^2+t^2}dt=\frac{-1}{2\sqrt{\pi}a^2}+O(\frac{1}{a^4}), \qquad (110)$$

when a is real. In the right-handed sides, note that the first term corresponds to double diffraction, and the second term to single diffraction (note : see appendix for contributions of complex poles).

9) Exact first order expressions for a small complex cavity in a step, when $\Phi_b=-\Phi_a$ and $\sin\theta_+=\sin\theta_-$

We consider a step of length Δ, when we have $\Phi_b = -\Phi_a$ in the three-part polygon geometry, so that the right and the left-handed semi-infinite planes are paralell, then we assume a complex cavity in this step. This cavity is composed of m faces, of angles Φ_j and length d_j, between extreme edges 'a' and 'b' (see figures).

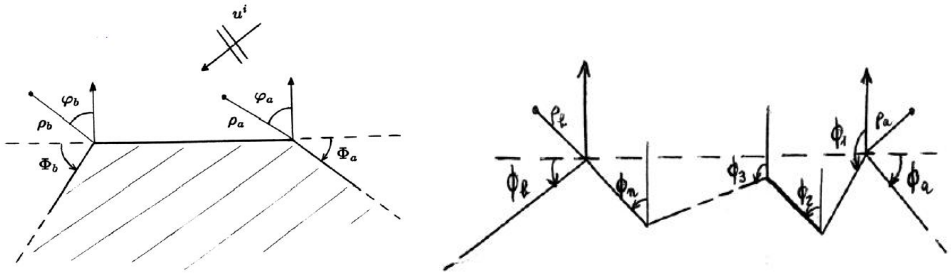

Figure 4 : geometry of complex cavity in a step

We will search to express the diffracted part u_d of u for this complex cavity, given by,

$$u_d = \frac{-e^{-i\pi/4-ik\rho_a}}{\sqrt{2\pi k\rho_a}}[f_a(\pi+\varphi_a) - f_a(-\pi+\varphi_a)] + O(1/(k\rho)^{3/2}), \qquad (111)$$

when we have impedance boundary conditions and the cavity dimension is small. We will show that, when $\sin\theta_+ = \sin\theta_-$, the first order of u_d in power of k can be determined by only studying difference functional equations, in combination of single face integral expression of spectral functions, with no need of integral equations. In what follows, we write ρ_a and φ_a without subscript as ρ and φ, and let $\varphi = \varphi' - \Phi_a$, $\varphi_\circ = \varphi'_\circ - \Phi_a$.

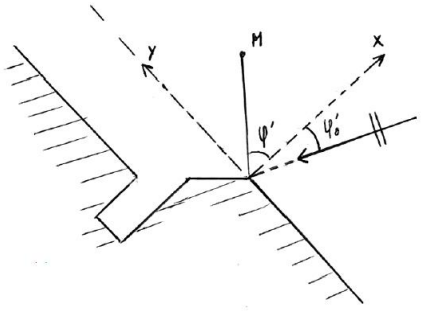

Figure 5 : complex step when $\Phi_b = -\Phi_a$ and $\varphi' = \varphi - \Phi_b$

9.1) Impedance boundary conditions and elementary equations for spectral functions

The semi-infinite planes, respectively right and left-handed, have relative impedances $\sin\theta_-$ and $\sin\theta_+$ with boundary conditions,

$$(\frac{\partial}{\partial n} - ik\sin\theta_\pm)u = 0, \tag{112}$$

while the field on each face of angle Φ_j verifies,

$$(\frac{\partial}{\partial n} - ik\sin\theta_j)u = 0, \tag{113}$$

Using single-face representation of spectral function (7) for spectral functions, and noting $f_e^- = f_a$, $f_e^+ = f_b$, we can write,

$$\begin{aligned}&(\sin\alpha + \sin\theta_+)f_b(\alpha + \pi/2 + \Phi_b) - (-\sin\alpha + \sin\theta_+)f_b(-\alpha + \pi/2 + \Phi_b) = 0,\\ &(\sin\alpha - \sin\theta_-)f_a(\alpha - \pi/2 - \Phi_a) - (-\sin\alpha - \sin\theta_-)f_a(-\alpha - \pi/2 - \Phi_a) = 0,\end{aligned} \tag{114}$$

where,

$$\begin{aligned}f_a(\alpha) = \frac{ik}{2}\sum_{1\le j\le n} e^{-ik\sum\limits_{1\le i<j} d_i\cos(\alpha-\Phi_i)}\\ \times \int_0^{d_j}(-\sin(\alpha-\Phi_j) + \sin\theta_j)u(\rho_j', \Phi_j)e^{-ik\rho_j'\cos(\alpha-\Phi_j)})d\rho_j'\\ + e^{-ik\sum\limits_{1\le i\le m} d_i\cos(\alpha-\Phi_i)} f_b(\alpha) = f_1(\alpha) + e^{-ik\Delta\cos(\alpha-\pi/2)}f_b(\alpha),\end{aligned} \tag{115}$$

with,

$$\sum_{1\le j\le m} d_j\cos\Phi_j = 0, \sum_{1\le j\le m} d_j\sin\Phi_j = \Delta, \tag{116}$$

Thus, we have,

$$\begin{aligned}&f_b(\alpha) = e^{ik\Delta\cos(\alpha-\pi/2)}(f_a(\alpha) - f_1(\alpha)),\\ &(\sin\alpha + \sin\theta_+)e^{ik\Delta\cos(\alpha+\Phi_b)} \times (f_a(\alpha + \pi/2 + \Phi_b)\\ &- f_1(\alpha + \pi/2 + \Phi_b)) + (\sin\alpha - \sin\theta_+)e^{ik\Delta\cos(\Phi_b-\alpha)}\\ &\times (f_a(-\alpha + \pi/2 + \Phi_b) - f_1(-\alpha + \pi/2 + \Phi_b)) = 0,\\ &(\sin\alpha - \sin\theta_-)f_a(\alpha - \pi/2 - \Phi_a) + (\sin\alpha + \sin\theta_-)f_a(-\alpha - \pi/2 - \Phi_a) = 0.\end{aligned} \tag{117}$$

that we can formulate following,

$$
\begin{aligned}
& f_a(\alpha+\pi/2+\Phi_b) \\
& +\frac{(\sin\alpha-\sin\theta_+)}{(\sin\alpha+\sin\theta_+)}e^{ik\Delta(\cos(\alpha-\Phi_b)-\cos(\alpha+\Phi_b))}f_a(-\alpha+\pi/2+\Phi_b) \\
& =f_1(\alpha+\pi/2+\Phi_b) \\
& +\frac{(\sin\alpha-\sin\theta_+)}{(\sin\alpha+\sin\theta_+)}e^{ik\Delta(\cos(\alpha-\Phi_b)-\cos(\alpha+\Phi_b))}f_1(-\alpha+\pi/2+\Phi_b), \\
& (\sin\alpha-\sin\theta_-)f_a(\alpha-\pi/2-\Phi_a)+(\sin\alpha+\sin\theta_-)f_a(-\alpha-\pi/2-\Phi_a)=0. \qquad (118)
\end{aligned}
$$

9.2) Reduction of far field function when $\sin\theta_+ = \sin\theta_-$ and $\Phi_a = -\Phi_b$

Considering that $f_1(\alpha) = O(k\Delta)$, we then have,

$$
\begin{aligned}
& f_a(\pi+\varphi) \\
& -\frac{(-\cos(\varphi-\Phi_b)+\sin\theta_+)}{(\cos(\varphi-\Phi_b)+\sin\theta_+)}e^{2ik\Delta\cos(\varphi-\Phi_b)\sin\Phi_b}f_a(-\varphi+2\Phi_b) \\
& =f_1(\pi+\varphi)+\frac{(\cos(\varphi-\Phi_b)-\sin\theta_+)}{(\cos(\varphi-\Phi_b)+\sin\theta_+)}f_1(-\varphi+2\Phi_b)+O((k\Delta)^2), \\
& f_a(-\pi+\varphi)-\frac{(\cos(\varphi+\Phi_a)-\sin\theta_-)}{(-\cos(\varphi+\Phi_a)-\sin\theta_-)}f_a(-\varphi-2\Phi_a)=0, \qquad (119)
\end{aligned}
$$

and thus,

$$
\begin{aligned}
& f_a(\pi+\varphi)-f_a(-\pi+\varphi)= \\
& f_1(\pi+\varphi)+\frac{\cos(\varphi-\Phi_b)-\sin\theta_+}{\cos(\varphi-\Phi_b)+\sin\theta_+}f_1(-\varphi+2\Phi_b) \\
& -\frac{\cos(\varphi-\Phi_b)-\sin\theta_+}{\cos(\varphi-\Phi_b)+\sin\theta_+}e^{2ik\Delta\cos(\varphi-\Phi_b)\sin\Phi_b}f_a(-\varphi+2\Phi_b) \\
& +\frac{\cos(\varphi+\Phi_a)-\sin\theta_-}{\cos(\varphi+\Phi_a)+\sin\theta_-}f_a(-\varphi-2\Phi_a)+O((k\Delta)^2), \qquad (120)
\end{aligned}
$$

For $\Phi_a = -\Phi_b$, $\sin\theta_+ = \sin\theta_-$, we can then write,

$$
\begin{aligned}
& f_a(\pi+\varphi)-f_a(-\pi+\varphi) \\
& =f_1(\pi+\varphi)+\frac{(\cos(\varphi-\Phi_b)-\sin\theta_+)}{(\cos(\varphi-\Phi_b)+\sin\theta_+)}f_1(-\varphi+2\Phi_b) \\
& +f_a(-\varphi+2\Phi_b)\frac{(\cos(\varphi-\Phi_b)-\sin\theta_+)}{(\cos(\varphi-\Phi_b)+\sin\theta_+)}(1-e^{2ik\Delta\cos(\varphi-\Phi_b)\sin\Phi_b}) \\
& +O((k\Delta)^2), \qquad (121)
\end{aligned}
$$

then, letting $\varphi' = \varphi - \Phi_b$,

$$f_a(\pi + \varphi' + \Phi_b) - f_a(-\pi + \varphi' + \Phi_b)$$
$$= f_1(\pi + \varphi' + \Phi_b) + \frac{(\cos\varphi' - \sin\theta_+)}{(\cos\varphi' + \sin\theta_+)} f_1(-\varphi' + \Phi_b) + D_u, \tag{122}$$

with,

$$D_u = -f_a(-\varphi' + \Phi_b)\frac{(\cos\varphi' - \sin\theta_+)}{(\cos\varphi' + \sin\theta_+)} 2ik\Delta\cos\varphi'\sin\Phi_b + o(k\Delta),$$
$$f_a(-\varphi' + \Phi_b) = f_0(-\varphi' + \Phi_b + (\Phi_a - \Phi_b)/2, \varphi_0) + O(k\Delta)$$
$$= f_0(-\varphi', \varphi_0) + O(k\Delta), \tag{123}$$

where f_0 is the spectral function for $k\Delta = 0$ given by (41) with, in our case, $\Phi_d = \pi/2$ and $\sin\theta_+ = \sin\theta_-$, which gives us,

$$f_0(\alpha, \varphi_\circ) = \frac{(\cos\alpha + \sin\theta_+)\cos\varphi'_\circ}{(\cos\varphi'_\circ + \sin\theta_+)(\sin\alpha - \sin\varphi'_\circ)}. \tag{124}$$

This leads us to the following expression of the far field function,

$$f_a(\pi + \varphi' + \Phi_b) - f_a(-\pi + \varphi' + \Phi_b)$$
$$= f_1(\pi + \varphi' + \Phi_b) + \frac{(\cos\varphi' - \sin\theta_+)}{(\cos\varphi' + \sin\theta_+)} f_1(-\varphi' + \Phi_b)$$
$$+ \frac{(\cos\varphi' - \sin\theta_+)\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ + \sin\theta_+)(\sin\varphi' + \sin\varphi'_\circ)} 2ik\Delta\sin\Phi_b + o(k\Delta). \tag{125}$$

9.3) Exact first order expressions for f_1 and for the far field function

We can write,

$$f_1(\alpha) = \frac{ik}{2} \sum_{1\le j\le 2} e^{-ik\sum_{1\le i<j} d_i\cos(\alpha - \Phi_i)} \int_0^{d_j} (-\sin(\alpha - \Phi_j)$$
$$+ \sin\theta_j) u(\rho'_j, \Phi_j) e^{-ik\rho'_j\cos(\alpha - \Phi_j)}) d\rho'_j$$
$$\sim \frac{ikM_u}{2} \sum_j d_j(-\sin(\alpha - \Phi_j) + \sin\theta_j), \tag{126}$$

for small $k\Delta$, where M_u is a mean value of u for small $k\Delta$. Considering the solution for $k\Delta = 0$, we have $M_u \sim M_u^0 = \frac{2\cos\varphi'_\circ}{(\cos\varphi'_\circ + \sin\theta_+)}$ as $\sin\theta_+ = \sin\theta_-$, and using $\sum_{1\le j\le m} d_j\cos\Phi_j = 0$ and $\sum_{1\le j\le m} d_j\sin\Phi_j = \Delta$, we have,

$$f_1(\alpha) \sim \frac{ik\cos\varphi'_\circ}{(\cos\varphi'_\circ + \sin\theta_+)} (\cos\alpha \sum_j d_j \sin\Phi_j + \sum_j d_j \sin\theta_j)$$
$$\sim \frac{ik\cos\varphi'_\circ}{(\cos\varphi'_\circ + \sin\theta_+)} (\Delta\cos\alpha + \sum_j d_j \sin\theta_j). \quad (127)$$

We then obtain for (125),

$$f_a(\pi + \varphi' + \Phi_b) - f_a(-\pi + \varphi' + \Phi_b)$$
$$\sim \frac{2ik\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ + \sin\theta_+)(\cos\varphi' + \sin\theta_+)}(-\Delta\cos\Phi_b\sin\theta_+ + \sin\Phi_b\Delta\sin\varphi'$$
$$+ \sum_j d_j \sin\theta_j) + \frac{2(\cos\varphi' - \sin\theta_+)\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ + \sin\theta_+)(\sin\varphi' + \sin\varphi'_\circ)}\, ik\Delta\sin\Phi_b$$
$$\sim \frac{2ik\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ + \sin\theta_+)(\cos\varphi' + \sin\theta_+)}(-\Delta\cos\Phi_b\sin\theta_+ + \sum_j d_j \sin\theta_j)$$
$$+ \frac{2ik\Delta\sin\Phi_b\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ + \sin\theta_+)(\cos\varphi' + \sin\theta_+)(\sin\varphi' + \sin\varphi'_\circ)}$$
$$\times (\sin\varphi'(\sin\varphi' + \sin\varphi'_\circ) + (\cos^2\varphi' - \sin^2\theta_+)), \quad (128)$$

and thus,

$$f_a(\pi + \varphi' + \Phi_b) - f_a(-\pi + \varphi' + \Phi_b) \sim \frac{2ik\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ + \sin\theta_+)(\cos\varphi' + \sin\theta_+)} \times$$
$$\times (-\Delta\cos\Phi_b\sin\theta_+ + \sum_j d_j \sin\theta_j + \frac{\Delta\sin\Phi_b}{(\sin\varphi' + \sin\varphi'_\circ)}(\cos^2\theta_+ + \sin\varphi'\sin\varphi'_\circ)), \quad (129)$$

or more generally, letting $\sum_j d_j \sin\theta_j = \int_{O_a}^{O_b} \sin\theta(l)dl$,

$$f_a(\pi + \varphi' + \Phi_b) - f_a(-\pi + \varphi' + \Phi_b) \sim \frac{2ik\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ + \sin\theta_+)(\cos\varphi' + \sin\theta_+)} \times$$
$$\times (-\Delta\cos\Phi_b\sin\theta_+ + \int_{O_a}^{O_b} \sin\theta(l)dl + \frac{\Delta\sin\Phi_b}{(\sin\varphi' + \sin\varphi'_\circ)}(\cos^2\theta_+ + \sin\varphi'\sin\varphi'_\circ)). \quad (130)$$

9.4) Exact first order expressions for the diffracted field

Using (130) in (111), we then obtain for the diffracted field at large distance,

$$u_d \sim \frac{-2ik\Delta\cos\varphi'_\circ\cos\varphi' e^{-i\pi/4-ik\rho_a}}{\sqrt{2\pi k\rho}(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)}(-\cos\Phi_b\sin\theta_+ + \sin\theta'_1 + \frac{\sin\Phi_b}{(\sin\varphi'+\sin\varphi'_\circ)}(\cos^2\theta_+ + \sin\varphi'\sin\varphi'_\circ)), \quad (131)$$

where

$$l_1 = \sum_j d_j = \int_{O_a}^{O_b} dl,\ \sin\theta'_1 = \int_{O_a}^{O_b} \sin\theta(l)dl/\Delta,$$
$$\varphi_a = \varphi' + \Phi_b,\ \varphi_\circ = \varphi'_\circ + \Phi_b,\ \Phi_b = -\Phi_a,\ \sin\theta_+ = \sin\theta_-. \quad (132)$$

It is worth noticing that this expression is the exact first order of the asymptotics for small $k\Delta$, and that it satisfies the reciprocity theorem, i.e. it remains the same when we change φ' for $\varphi'_\circ$ (in contrary to the one that we would obtain by Physical Optics approximation).

Appendix A : about $\Psi_{+-}(\alpha,\Phi)$ and the solution for an impedance wedge

A.1) $\Psi_{+-}(\alpha)$ in passive case

The function $\Psi_{+-}(\alpha)$, also denoted $\Psi_{+-}(\alpha,\Phi)$, has been defined by Maliuzhinets [18], for the spectral function $f_{+-}(\alpha) = \frac{\Psi_{+-}(\alpha)}{\Psi_{+-}(\varphi')}\sigma(\alpha)$ attached to the representation of the field, in presence of a passive impedance wedge illuminated by a unit plane wave of direction φ'. The spectral function f_{+-} is the solution of the equations

$$(\sin\alpha+\sin\theta_+)f_{+-}(\alpha+\Phi) - (-\sin\alpha+\sin\theta_+)f_{+-}(-\alpha+\Phi) = 0,$$
$$(\sin\alpha-\sin\theta_-)f_{+-}(\alpha-\Phi) - (-\sin\alpha-\sin\theta_-)f_{+-}(-\alpha-\Phi) = 0, \quad (133)$$

regular in the strip $|\mathrm{Re}\alpha| \le \Phi$, except for the pole with unit residue at $\alpha = \varphi'$, and $O(1)$ at infinity in this band, where $\sin\theta_+$ (resp. $\sin\theta_-$) corresponds to the relative impedance attached to the face $\varphi = +\Phi$ (resp. $\varphi = -\Phi$). The function $\Psi_{+-}(\alpha)$ is the solution of (133) without poles or zeros as $|\mathrm{Re}\alpha| \le \Phi$, when $\mathrm{Re}(\sin\theta_{+,-}) > 0$ with $0 < \mathrm{Re}(\theta^{\pm}) \le \pi/2$ (passive case), $O(\cos(\pi\alpha/2\Phi))$ in this band, and $\sigma(\alpha) = \frac{\pi}{2\Phi}\cos(\frac{\pi\varphi'}{2\Phi})/(\sin(\frac{\pi\alpha}{2\Phi}) - \sin(\frac{\pi\varphi'}{2\Phi}))$ has the unit pole for incident field at $\alpha = \varphi'$ in the band $|\mathrm{Re}\alpha| \le \Phi$ [18].

The function $\Psi_{+-}(\alpha)$ has numerous properties. We have,

$$\Psi_{+-}(\alpha) = A\Psi_\Phi(\alpha+\Phi+(\frac{\pi}{2}-\theta_+))\Psi_\Phi(\alpha-\Phi-(\frac{\pi}{2}-\theta_-))$$
$$\times\ \Psi_\Phi(\alpha+\Phi-(\frac{\pi}{2}-\theta_+))\Psi_\Phi(\alpha-\Phi+(\frac{\pi}{2}-\theta_-)),$$
$$= A\Psi_\Phi^4(\frac{\pi}{2})(\cos(\frac{\pi}{4\Phi}(\alpha+\Phi-\theta_+))\cos(\frac{\pi}{4\Phi}(\alpha-\Phi+\theta_-))$$
$$\times \frac{\Psi_\Phi(\alpha+\Phi-\frac{\pi}{2}+\theta_+)\Psi_\Phi(\alpha-\Phi+\frac{\pi}{2}-\theta_-)}{\Psi_\Phi(\alpha+\Phi-\frac{\pi}{2}-\theta_+)\Psi_\Phi(\alpha-\Phi+\frac{\pi}{2}+\theta_-)}, \qquad (134)$$

where Ψ_Φ is the Maliuzhinets function [18], $0 < \mathrm{Re}\theta_\pm \leq \frac{\pi}{2}$, A is an arbitrary constant. This even function satisfies $\Psi_\Phi(\alpha) = A_0' e^{\mp i\pi\alpha/8\Phi}(1+O(\alpha^\nu|e^{\pm i\mu'\alpha}|))$ when $\mathrm{Im}\alpha \to \pm\infty$, where A_0' and ν are constants, $\mu' = \min(\frac{\pi}{2\Phi}, 1)$ (see [23] for more details), and we can write $\Psi_{+-}(\alpha) = A_0\cos(\pi\alpha/2\Phi)(1+O(\alpha^\nu e^{-\mu'|\mathrm{Im}\alpha|}))$ for $|\mathrm{Im}\alpha|$ large, A_0 a constant. Since $\Psi_\Phi(\alpha) = \Psi_\Phi(-\alpha)$ and $\Psi_\Phi(\alpha+\frac{\pi}{2})\Psi_\Phi(\alpha-\frac{\pi}{2}) = \Psi_\Phi^2(\frac{\pi}{2})\cos(\pi\alpha/4\Phi)$, we have $\Psi_{+-}(-\alpha) = \Psi_{-+}(\alpha)$ and

$$4\Psi_{+-}(\alpha+\frac{\pi}{2})\Psi_{+-}(\alpha-\frac{\pi}{2}) = A^2\Psi_\Phi^8(\frac{\pi}{2})(\cos(\pi(\frac{\pi}{2}-\theta_+)/2\Phi)-\sin(\pi\alpha/2\Phi))$$
$$\times(\cos(\pi(\frac{\pi}{2}-\theta_-)/2\Phi)+\sin(\pi\alpha/2\Phi)), \qquad (135)$$

with $A\Psi_\Phi^4(\frac{\pi}{2}) = 2A_0$. Besides, from [27], we can write, for $|\mathrm{Re}\alpha| \leq \Phi+(N+1)\pi$,

$$\Psi_{+-}(\alpha) = B_N\prod_\pm(\prod_{m=0}^{N}[\Gamma(\frac{1}{2}+\frac{1}{4\Phi}(\alpha\pm\Phi+\theta_\pm+m\pi))$$
$$\Gamma(\frac{1}{2}-\frac{1}{4\Phi}(\alpha\pm\Phi-(\theta_\pm+m\pi)))\Gamma(\frac{1}{2}+\frac{1}{4\Phi}(\alpha\pm\Phi+\pi-\theta_\pm+m\pi))$$
$$\Gamma(\frac{1}{2}-\frac{1}{4\Phi}(\alpha\pm\Phi-(\pi-\theta_\pm+m\pi)))]^{(-1)^{m+1}})$$
$$\times\exp(\int_0^\infty(-e^{-\nu\pi})^{N+1}\frac{e^{-\nu\theta_\pm}+e^{-\nu(\pi-\theta_\pm)}}{(1+e^{-\nu\pi})}\frac{(1-\cosh(\nu(\alpha\pm\Phi)))}{\nu\sinh(2\nu\Phi)}\,d\nu), \qquad (136)$$

where θ_+ and θ_-, N is an arbitrary positive integer and B_N is a constant. For the applications, we choose N fixed and define Ψ_{+-} with $B_N = 1$. This expression is suitable for numerical calculus (with $N = 1$ or 2) or to derive the analytical properties of $\Psi_{+-}(\alpha)$ (with $N \to \infty$) from those of Γ.

The zeros which are the closest to the imaginary axis are $\alpha = \Phi+\theta_+$ and $\alpha = -\Phi-\theta_-$, and the closest poles are $\alpha = \Phi+\theta_++\pi$ and $\alpha = -\Phi-\theta_--\pi$.

A.2) Some miscellaneous properties in general case (passive or active)

The solution is more complex when we want to consider arbitrary face impedances in whole generality, passive ($0 < \mathrm{Re}(\theta^{\pm}) \leq \pi/2$) or active ($0 < -\mathrm{Re}(\theta^{\pm}) \leq \pi/2$). For that, we begin to consider,

$$\Psi_{+-}(\alpha, \sin\theta^{+}, \sin\theta^{-}) = \Psi^{+}(\alpha, \sin\theta^{+})\Psi^{-}(\alpha, \sin\theta^{-}), \tag{137}$$

where the $\Psi^{\pm}$ verifies,

$$\begin{aligned}&(\sin\alpha \pm \sin\theta_{\pm})\Psi^{\pm}(\alpha \pm \Phi) - (-\sin\alpha \pm \sin\theta_{\pm})\Psi^{\pm}(-\alpha \pm \Phi) = 0,\\ &\Psi^{\pm}(\alpha \mp \Phi) - \Psi^{\pm}(-\alpha \mp \Phi) = 0,\end{aligned} \tag{138}$$

with Ψ_{+-} remaining without any zero or pole in the band $|\mathrm{Re}\alpha| \leq \Phi$, and thus,

$$\Psi^{\pm}(\alpha, \sin\theta^{\pm}) = 1/\Psi^{\pm}(\alpha, -\sin\theta^{\pm}) \text{ as } \mathrm{Re}(\theta^{\pm}) \neq 0. \tag{139}$$

The function $\Psi^{\pm}$ with an active $\sin\theta^{\pm}$ then directly derives from $\Psi^{\pm}$ with passive one, and, noting $s^{\pm} = \mathrm{sign}(\sin\theta^{\pm})$, we remark that, when $s^{\pm} = 1$ (resp. $s^{\pm} = -1$), $\Psi^{\pm}(\alpha, \sin\theta^{\pm})$ has zeros (resp. poles),

$$\begin{aligned}&\alpha = \mp\Phi + \epsilon(s^{\pm}\theta^{\pm} + (4p+2)\Phi + 2m\pi),\\ &\text{and } \alpha = \mp\Phi + \epsilon(\pi - s^{\pm}\theta^{\pm} + (4p+2)\Phi + 2m\pi),\end{aligned}$$

and poles (resp. zeros) at,

$$\begin{aligned}&\alpha = \mp\Phi + \epsilon(\pi + s^{\pm}\theta^{\pm} + (4p+2)\Phi + 2m\pi),\\ &\text{and } \alpha = \mp\Phi + \epsilon(2\pi - s^{\pm}\theta^{\pm} + (4p+2)\Phi + 2m\pi),\end{aligned}$$

for $m \geq 0$, $p \geq 0$, $\epsilon = 1$ and -1, while we have,

$$\Psi^{\pm}(\alpha) = A_{\pm}(\cos(\frac{\pi\alpha}{4\Phi}))^{s^{\pm}}(1 + O(\alpha^{\nu}e^{-\mu'|\mathrm{Im}\alpha|})), \tag{140}$$

for large $|\mathrm{Im}\alpha|$, with constants $A_{\pm}$ and $\mu' > 0$.

A.3) The solution for the diffraction by a wedge with passive or active impedance

We can then express the spectral solution $f_0(\alpha)$ for the diffraction by a wedge with passive or active impedance faces with the following general definition,

$$\begin{aligned}&f_0(\alpha) = \chi_0(\alpha) - \frac{1}{2}(\chi_0(i\infty) + \chi_0(-i\infty)),\\ &\chi_0(\alpha) = \Psi_{+-}(\alpha, \sin\theta^{+}, \sin\theta^{-})P_n(\sin(\mu\alpha))\sigma(\alpha)/\Psi_{+-}(\varphi_{\circ}, \sin\theta^{+}, \sin\theta^{-}),\\ &\sigma(\alpha) = \mu\cos(\mu\varphi')/(\sin(\mu\alpha) - \sin(\mu\varphi')),\end{aligned} \tag{141}$$

where P_n is a polynomial of order n, $\mu = \pi/2\Phi$, and φ' is the incidence angle. Since we have $f_0(\alpha) \sim 1/(\alpha - \varphi')$ in vicinity of incidence pole $\alpha = \varphi'$, and, from unique definition of u at $\rho = 0$, $f_0(\alpha) = O(1)$ as $|\mathrm{Im}\alpha| \to \infty$, we can write,

$$P_n(\sin(\mu\varphi')) = 1,\ n = \frac{1}{2}\sum_{\pm}(1 - s^{\pm}), \tag{142}$$

with $s^{\pm} = \mathrm{sign}(\mathrm{Re}(\sin\theta^{\pm}))$, and thus $n = 0$ when both impedances are passive. In addition, the n zeros of $P_n(\sin(\mu\alpha))$ have to suppress, for each active face with $s^{\pm} = -1$, the pole $\alpha_s^{\pm}$ of $\Psi^{\pm}(\alpha, \sin\theta^{\pm})$ with $\pm\,\mathrm{sign}(\arg(ik))\mathrm{Im}\alpha_s^{\pm} > 0$, among $\alpha_1^{\pm} = \pm(s^{\pm}\theta^{\pm} + \Phi)$ and $\alpha_2^{\pm} = \pm(\pi - s^{\pm}\theta^{\pm} + \Phi)$, that give poles of $f_0(\alpha + \varphi)$ within the region $|\mathrm{Re}\alpha| < \pi$ with $\mathrm{Re}(ik\cos\alpha) > 0$, as $|\mathrm{Re}\varphi| < \Phi$, which leads to cancel non physical contribution $e^{ik\rho\cos(\alpha_s^{\pm} - \varphi)}$ that can diverge at infinity as $|\mathrm{Re}\varphi| < \Phi$. We then have,

$$P_n(\sin(\mu\alpha)) = \prod_{\pm}\left(\frac{(\sin(\mu\alpha) - \sin(\mu\alpha_s^{\pm}))}{(\sin(\mu\varphi') - \sin(\mu\alpha_s^{\pm}))}\right)^{\frac{1-s^{\pm}}{2}}, \tag{143}$$

which completes the definition of f_0 for arbitrary impedance (passive or active).

Remark : From [18], the zeros of Ψ_Φ which are the closest to $\alpha = 0$ and the corresponding poles are the points $\alpha = \pm(\frac{\pi}{2} + 2\Phi)$ and $\alpha = \pm(\frac{3\pi}{2} + 2\Phi)$. In other respects, Ψ_Φ satisfies

$$\frac{\Psi_\Phi(\alpha + 2\Phi)}{\Psi_\Phi(\alpha - 2\Phi)} = \cot(\frac{\alpha + \frac{\pi}{2}}{2}),\ \Psi_\Phi(\alpha + \Phi)\Psi_\Phi(\alpha - \Phi) = \Psi_\Phi^2(\Phi)\Psi_{\Phi/2}(\alpha). \tag{144}$$

Using (133) in (135), we deduce that,

$$\begin{aligned}
&\Psi_{+-}(\alpha \mp (\Phi - \frac{\pi}{2}))\Psi_{+-}(-\alpha \mp (\Phi - \frac{\pi}{2})) \\
&= A_0^2(\cos(\pi\alpha/2\Phi) + \cos(\pi(\frac{\pi}{2} - \theta_{\pm})/2\Phi))(\cos\alpha + \sin\theta_{\mp}) \\
&\times \frac{(\cos(\pi\alpha/2\Phi) - \cos(\pi(\frac{\pi}{2} - \theta_{\mp})/2\Phi))}{(\cos\alpha - \sin\theta_{\mp})}.
\end{aligned} \tag{145}$$

and,

$$\begin{aligned}
&\Psi_{+-}(-\alpha \mp (\Phi - \pi)) = \\
&= A_0^2(\cos(\pi(\alpha \mp \pi/2)/2\Phi) + \cos(\pi(\frac{\pi}{2} - \theta_{\pm})/2\Phi))(\pm\sin\alpha + \sin\theta_{\mp}) \\
&\times \frac{(\cos(\pi(\alpha \mp \pi/2)/2\Phi) - \cos(\pi(\frac{\pi}{2} - \theta_{\mp})/2\Phi))}{\pm\,\Psi_{+-}(\alpha \mp \Phi)(\sin\alpha \mp \sin\theta_{\mp})},
\end{aligned} \tag{146}$$

then,

$$\frac{\Psi_{+-}(-\alpha\mp(\Phi-\pi))}{\Psi_{+-}(\alpha\mp(\Phi-\pi))}=\frac{(\cos(\pi(\alpha\mp\pi/2)/2\Phi)+\cos(\pi(\frac{\pi}{2}-\theta_{\pm})/2\Phi))}{(\cos(\pi(\alpha\pm\pi/2)/2\Phi)+\cos(\pi(\frac{\pi}{2}-\theta_{\pm})/2\Phi))}$$
$$\times\frac{(\cos(\pi(\alpha\mp\pi/2)/2\Phi)-\cos(\pi(\frac{\pi}{2}-\theta_{\mp})/2\Phi))}{(\cos(\pi(\alpha\pm\pi/2)/2\Phi)-\cos(\pi(\frac{\pi}{2}-\theta_{\mp})/2\Phi))}\frac{(\sin\alpha\pm\sin\theta_{\mp})}{(-\sin\alpha\pm\sin\theta_{\mp})}. \qquad (147)$$

Appendix B : Principle of semi-inversion for our system of integral equations

We can modify equations and derive integral equations with kernels vanishing as $k\Delta\to 0$ for the three-part semi-infinite impedance polygon, which allows approximations when $k\Delta$ is small. For this, we begin with changing the unknowns in the equations, vanishing when $\Delta=0$. For that, we consider,

$$f_{ar0}(\alpha)=[f_{ar}(\alpha)-f_0^{ab}(\alpha-(\Phi_+-\frac{\pi}{2}))],$$
$$f_{br0}(\alpha)=[f_{br}(\alpha)-e^{ik\Delta\sin\varphi_\circ}f_0^{ab}(\alpha+(\Phi_--\frac{\pi}{2}))], \qquad (148)$$

with $\Phi_+=\frac{\pi}{2}+\frac{\Phi_b}{2}$, $\Phi_-=\frac{\pi}{2}+\frac{\Phi_a}{2}$. The function $f_0^{ab}(\alpha,\,\varphi_\circ)$ corresponds to the solution for $\Delta=0$, i.e. for a simple wedge of exterior angle $2\Phi_d=\Phi_a+\Phi_b+\pi,$ given by

$$f_0^{ab}(\alpha,\,\varphi_\circ)=\frac{\pi}{2\Phi_d}\frac{\Psi_{+-}(\alpha)\cos(\frac{\pi\varphi_{\circ,d}}{2\Phi_d})}{\Psi_{+-}(\varphi_{\circ,d})(\sin(\frac{\pi\alpha}{2\Phi_d})-\sin(\frac{\pi\varphi_{\circ,d}}{2\Phi_d}))}, \qquad (149)$$

with $\varphi_{\circ,d}=\varphi_\circ+(\Phi_a-\Phi_b)/2,\ -\Phi_d<\varphi_{\circ,d}<\Phi_d$, which, we note, satisfies the system of coupled equations as $\Delta=0$. Multiplying first one by $e^{ik\Delta\sin\varphi_\circ}$ and second one by 1, we can substract them from initial equations, and then deduce a new system. So, the functions f_{ar0} and f_{br0} which vanish as $\Delta=0$, then satisfy,

$$\frac{f_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}=\frac{-i}{4\Phi_+}(\int_{-i\infty}^{+i\infty}(\frac{f_{ar0}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{+1}(\alpha'-\Phi_+)})$$
$$\times\frac{\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'$$
$$+\int_{-i\infty}^{+i\infty}\frac{B_{a0}(\alpha')\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'), \qquad (150)$$

where,

$$B_{a0}(\alpha') = (\frac{f_{ar0}(\alpha' - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha' - \Phi_+)})(e^{-ik\Delta\cos\alpha'} - 1)$$
$$+ (\frac{f_0^{ab}(\alpha' - \frac{\pi}{2} + (\Phi_a - \Phi_b)/2,\, \varphi_\circ)}{\Psi_{+1}(\alpha' - \Phi_+)})(e^{-ik\Delta(\cos\alpha' + \sin\varphi_\circ)} - 1)e^{ik\Delta\sin\varphi_\circ}, \qquad (151)$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) < 3\Phi_+$, and

$$\frac{f_{ar0}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{1-}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})} = \frac{i}{4\Phi_-}(\int_{-i\infty}^{+i\infty}(\frac{f_{br0}(\alpha' + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha' + \Phi_-)})$$
$$\times \frac{\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha'$$
$$+ \int_{-i\infty}^{+i\infty}\frac{B_{b0}(\alpha')\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha'), \qquad (152)$$

where

$$B_{b0}(\alpha') = (\frac{f_{br0}(\alpha' + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha' + \Phi_-)})(e^{-ik\Delta\cos\alpha'} - 1)$$
$$+ (\frac{f_0^{ab}(\alpha' + \frac{\pi}{2} - (\Phi_b - \Phi_a)/2,\, \varphi_\circ)}{\Psi_{1-}(\alpha' + \Phi_-)})(e^{-ik\Delta(\cos\alpha' - \sin\varphi_\circ)} - 1), \qquad (153)$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) < \Phi_-$, for $-\min(\frac{\pi}{2}, \frac{\pi}{2} + \Phi_a) < \varphi_\circ < \min(\frac{\pi}{2}, \frac{\pi}{2} + \Phi_b)$, with $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$ and $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$.

We then notice some similarity with the equations satisfied when $\Delta = 0$, related to spectral function f_0^{ab} for an impedance wedge. Thus, we let,

$$f'_{br0}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) = \int_{-i\infty}^{+i\infty} G(\varphi')f_0^{ab}(\alpha + \frac{\pi}{2} + (\Phi_a - \Phi_b)/2, \varphi')d\varphi',$$
$$f'_{ar0}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) = \int_{-i\infty}^{+i\infty} G(\varphi')f_0^{ab}(\alpha - \frac{\pi}{2} + (\Phi_a - \Phi_b)/2, \varphi')d\varphi', \qquad (154)$$

as $|\mathrm{Re}(\alpha)| < \frac{\pi}{2}$. The functions $f_0^{ab}(\alpha \pm \frac{\pi}{2} + (\Phi_a - \Phi_b)/2, \varphi')$ are regular and $O(1/\cos(\pi\varphi'/2\Phi_d))$ on the imaginary axis, and a pole at $\varphi' = \alpha \pm \frac{\pi}{2}$ ensures that we have, even if $\Phi_a = \Phi_b = 0$, generally $f'_{br0} \neq f'_{ar0}$.

We search to define $G(\varphi')$, regular in the band $|\mathrm{Re}(\varphi')| \leq \frac{\pi}{2}$, so that $f'_{br0} \equiv f_{br0}$ and $f'_{ar0} \equiv f_{ar0}$ verifies (150)-(153) for arbitrary Δ. For that, we consider the integral expressions (45)-(46) for f_{ar} and f_{br} as $\Delta = 0$, then multiply by $G(\varphi')$, integrate following (154), and we obtain,

$$\frac{f'_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}=\int_{-i\infty}^{+i\infty}\frac{\frac{-i}{4\Phi_+}f'_{ar0}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})\sin(\frac{\pi\alpha'}{2\Phi_+})}{\Psi_{+1}(\alpha'-\Phi_+)(\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+}))}d\alpha'$$
$$+\frac{\pi}{2\Phi_+}\int_{-i\infty}^{+i\infty}\frac{G(\varphi')}{\Psi_{+1}(\varphi'-\frac{\Phi_b}{2})}\frac{\sin\frac{\pi(\varphi'+\frac{\pi}{2})}{2\Phi_+}}{(\cos\frac{\pi(\alpha-\Phi_b)}{2\Phi_+}+\cos\frac{\pi(\varphi'+\frac{\pi}{2})}{2\Phi_+})}d\varphi', \tag{155}$$

and,

$$\frac{f'_{ar0}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}=\int_{-i\infty}^{+i\infty}\frac{\frac{i}{4\Phi_-}f'_{br0}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})\sin(\frac{\pi\alpha'}{2\Phi_-})}{\Psi_{1-}(\alpha'+\Phi_-)(\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-}))}d\alpha'$$
$$+\frac{\pi}{2\Phi_-}\int_{-i\infty}^{i\infty}\frac{G(\varphi')}{\Psi_{1-}(\varphi'+\frac{\Phi_a}{2})}\frac{\sin\frac{\pi(\varphi'-\frac{\pi}{2})}{2\Phi_-}}{(\cos\frac{\pi(\alpha+\Phi_a)}{2\Phi_-}+\cos\frac{\pi(\varphi'-\frac{\pi}{2})}{2\Phi_-})}d\varphi'. \tag{156}$$

In the case where $G(\varphi')$ is regular in the band $|\mathrm{Re}(\varphi')|\le\frac{\pi}{2}$, we can shift the integral paths in the integrals containing $G(\varphi')$. Comparing (150)-(153) with (155)-(156), we notice that (f'_{br0}, f'_{ar0}) is a solution of the system of equations (150)-(153) if G satisfies the conditions

$$\frac{G(\alpha'+\frac{\pi}{2})}{\Psi_{1-}(\alpha'+\Phi_-)}-\frac{G(-\alpha'+\frac{\pi}{2})}{\Psi_{1-}(-\alpha'+\Phi_-)}=\frac{i}{2\pi}(B_{b0}(\alpha')-B_{b0}(-\alpha')),$$
$$\frac{G(\alpha'-\frac{\pi}{2})}{\Psi_{+1}(\alpha'-\Phi_+)}-\frac{G(-\alpha'-\frac{\pi}{2})}{\Psi_{+1}(-\alpha'-\Phi_+)}=\frac{-i}{2\pi}(B_{a0}(\alpha')-B_{a0}(-\alpha')). \tag{157}$$

Taking account of the properites of Ψ_{+1} and Ψ_{1-} (see appendix), and letting $G(\alpha')=(\cos\alpha'+\sin\theta_1)g(\alpha')$, (157) can be written

$$g(\alpha'+\frac{\pi}{2})-g(-\alpha'+\frac{\pi}{2})=\frac{i\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{2\pi(-\sin\alpha'+\sin\theta_1)},$$
$$g(\alpha'-\frac{\pi}{2})-g(-\alpha'-\frac{\pi}{2})=\frac{i\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{2\pi(-\sin\alpha'-\sin\theta_1)}. \tag{158}$$

Since $G(\alpha')$ is regular in the band $|\mathrm{Re}\alpha'|\le\frac{\pi}{2}$ and $\mathrm{Re}(\sin\theta_1)>0$, $g(\alpha')$ is regular in this band. We can then use (42)-(43), and write, as $|\mathrm{Re}\alpha|<\frac{\pi}{2}$,

$$g(\alpha)=\frac{i}{4\pi}\int_{-i\infty}^{+i\infty}d\alpha'(\frac{i\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{2\pi(\sin\alpha'-\sin\theta_1)}\tan(\frac{1}{2}(\alpha+\frac{\pi}{2}-\alpha'))$$
$$-\frac{i\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{2\pi(\sin\alpha'+\sin\theta_1)}\tan(\frac{1}{2}(\alpha-\frac{\pi}{2}-\alpha'))), \tag{159}$$

Using (154) and (159), we obtain the equations with kernels vanishing as $k\Delta\to 0$:

$$f_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})=\int_{-i\infty}^{+i\infty}d\alpha'(\frac{\frac{\Psi_{1-}(\alpha'+\Phi_-)}{8\pi^2}(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{\sin\alpha'-\sin\theta_1}M_+(\alpha,\alpha')$$
$$-\frac{\frac{\Psi_{+1}(\alpha'-\Phi_+)}{8\pi^2}(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{\sin\alpha'+\sin\theta_1}M_-(\alpha,\alpha')), \tag{160}$$

and,

$$f_{ar0}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})=\int_{-i\infty}^{+i\infty}d\alpha'(\frac{\frac{\Psi_{1-}(\alpha'+\Phi_-)}{8\pi^2}(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{\sin\alpha'-\sin\theta_1}N_+(\alpha,\alpha')$$
$$-\frac{\frac{\Psi_{+1}(\alpha'-\Phi_+)}{8\pi^2}(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{\sin\alpha'+\sin\theta_1}N_-(\alpha,\alpha')), \tag{161}$$

where $M_\pm(\alpha,\alpha')=L_\pm(\alpha+\frac{\pi}{2}+\frac{(\Phi_a-\Phi_b)}{2},\alpha')$, $N_\pm(\alpha,\alpha')=L_\pm(\alpha-\frac{\pi}{2}+\frac{(\Phi_a-\Phi_b)}{2},\alpha')$,

$$L_\pm(\alpha,\alpha')=\frac{\pi\sin\alpha'\Psi_{+-}(\alpha)}{2\Phi_d}\int_{-i\infty}^{+i\infty}\frac{\cos(\frac{\pi(\varphi'+(\Phi_a-\Phi_b)/2)}{2\Phi_d})}{\Psi_{+-}(\varphi'+(\Phi_a-\Phi_b)/2)}$$
$$\frac{(\cos\varphi'+\sin\theta_1)}{\cos(\varphi'\pm\frac{\pi}{2})+\cos\alpha'}\frac{1}{(\sin(\frac{\pi\alpha}{2\Phi_d})-\sin(\frac{\pi(\varphi'+(\Phi_a-\Phi_b)/2)}{2\Phi_d}))}d\varphi', \tag{162}$$

In the particular case $\Phi_a=\Phi_b=0$, $\Phi_d=\frac{\pi}{2}$, the functions $L_\pm$ can be simplified so that we recover the expressions found in [2] for the three-part impedance plane (see remark below).

Remark : It exists analytical expressions of $L_\pm$ when $\Phi_a=\Phi_b=0$. For this, we consider Ψ_{+-} for $\Phi=\pi/2$, where we have,

$$\frac{(-\sin\alpha+\sin\theta_-)}{\Psi_{+-}(\alpha+\frac{\pi}{2})}-\frac{(\sin\alpha+\sin\theta_-)}{\Psi_{+-}(-\alpha+\frac{\pi}{2})}$$
$$=(\sin\theta_--\sin\theta_+)(\frac{1}{\Psi_{+-}(\alpha+\frac{\pi}{2})}-\frac{1}{\Psi_{+-}(-\alpha+\frac{\pi}{2})}),$$
$$\frac{(\sin\alpha+\sin\theta_-)}{\Psi_{+-}(\alpha-\frac{\pi}{2})}-\frac{(-\sin\alpha+\sin\theta_-)}{\Psi_{+-}(-\alpha-\frac{\pi}{2})}=0. \tag{163}$$

Using that $\Psi_{+-}(-\alpha)\Psi_{+-}(\alpha)=A_0^2(\cos\alpha+\sin\theta_+)(\cos\alpha+\sin\theta_-)$, and $\Psi_{+-}(\alpha)/\cos\alpha\to A_0$ as $\alpha\to i\infty$, we can let $\chi(\alpha)=\frac{A_0(\cos\alpha+\sin\theta_-)}{\Psi_{+-}(\alpha)}$ in (42)-(43), and derive,

$$\frac{(\cos\alpha+\sin\theta_-)}{\Psi_{+-}(\alpha)}-\frac{1}{A_0}=\frac{(\sin\theta_--\sin\theta_+)}{4\pi i}\int_{-i\infty}^{+i\infty}H(\alpha')\tan(\frac{1}{2}(\alpha+\frac{\pi}{2}-\alpha'))d\alpha',$$
$$\frac{\Psi_{+-}(-\alpha)/A_0^2}{(\cos\alpha+\sin\theta_+)}-\frac{1}{A_0}=\frac{(\sin\theta_--\sin\theta_+)}{4\pi i}\int_{-i\infty}^{+i\infty}H(\alpha')\tan(\frac{1}{2}(\alpha+\frac{\pi}{2}-\alpha'))d\alpha', \tag{164}$$

for $\frac{-3\pi}{2} < \mathrm{Re}(\alpha) < \frac{\pi}{2}$, where $H(\alpha') = 2\sin\alpha'/(\Psi_{+-}(\alpha'+\frac{\pi}{2})(\sin\alpha'+\sin\theta_+))$ and

$$H(\alpha) = \frac{1}{\Psi_{+-}(\alpha+\frac{\pi}{2})} - \frac{1}{\Psi_{+-}(-\alpha+\frac{\pi}{2})}$$
$$= \frac{\sin\alpha/\sin\theta_+}{\Psi_{+-}(\alpha+\frac{\pi}{2})} + \frac{\sin\alpha/\sin\theta_+}{\Psi_{+-}(-\alpha+\frac{\pi}{2})}. \qquad (165)$$

From oddness of $H(\alpha)$, $\tan(\frac{1}{2}(\alpha+\frac{\pi}{2}-\alpha'))$ can be replaced by $\frac{-\sin\alpha'}{(\cos(\alpha+\frac{\pi}{2})+\cos\alpha')}$, and we note that, after deformation of integration path, we have,

$$\int_{-i\infty}^{+i\infty} \frac{1}{\Psi_{+-}(\varphi')} \frac{\cos\varphi'}{\cos\alpha+\sin\varphi'} d\varphi' = -2\pi i \frac{\frac{\sin\alpha+\sin\theta_-}{\Psi(\alpha-\frac{\pi}{2})} - \frac{1}{A_0}}{(\sin\theta_+ - \sin\theta_-)},$$
$$\int_{-i\infty}^{+i\infty} \frac{\cos\varphi'}{\Psi_{+-}(\varphi')} \frac{\cos\varphi'}{\cos\alpha+\sin\varphi'} d\varphi' = 2\pi i \sin\theta_+ \frac{\frac{\sin\alpha+\sin\theta_-}{\Psi(\alpha-\frac{\pi}{2})} - \frac{1}{A_0}}{(\sin\theta_+ - \sin\theta_-)} + \frac{i\pi}{A_0}, \qquad (166)$$

as $|\mathrm{Re}\alpha| < \frac{\pi}{2}$. Similar expressions with $-\sin\varphi'$ in place of $\sin\varphi'$ can be obtained from $\Psi_{+-}(-\alpha) \to \Psi_{-+}(\alpha)$ when $\theta_\pm \to \theta_\mp$ (or by continuation with capture of poles). After noticing that we have,

$$\frac{1}{(z+\sin\varphi')(z'+\sin\varphi')} = (\frac{1}{z+\sin\varphi'} - \frac{1}{z'+\sin\varphi'}) \frac{1}{z'-z}, \qquad (167)$$

the equalities (166) permits, by elementary combinations, to obtain analytical expressions of $L_\pm$ terms when $\Phi_a = \Phi_b = 0$.

Appendix C : Uniqueness of fields when impedance boundary conditions on piecewise regular geometry

C.1) A use of first Green theorem for uniqueness : nullity of field outside S' when $|\arg(ik)| < \pi/2$, and nullity of surface fields on any regular parts of S' when $|\arg(ik)| = \pi/2$

We analyse here the uniqueness of solutions for boundary value problem, that we previously use in [21], in the domain Ω outside a picewise regular boundary S' in 2D and 3D (finite or infinite), when the field is a real-analytic function of coordinates satisfying the Helmholtz equation in Ω, except possibly at discontinuity of S'. In [30], Levine develops an uniqueness theorem for the scalar acoustic field p, i.e. a proof that $p_{inc} \equiv 0$ implies $p \equiv 0$, in the case of a scatterer with impedance boundary conditions on S', when no primary source is present. He considers piecewise $C^{(2+\lambda)}$ boundary S' (with no zero exterior angle), $\lambda > 0$, without auxiliary 'edge conditions' at edges or corner points, except that p is continuous. He studies at

first bounded scatterers, but he also gives, in section 7 of his paper, the elements to generalize his results to scatterers with infinite boundaries and general conical points (detailed in [31]), in particular by the use of Jones' uniqueness theorem [32].

We begin to notice first that the conditions given by Levine to apply the Green's first theorem are : the scatterer surface is piecewise analytic (with no zero exterior angle), the field is countinuous at discontinuities of S' and satisfies impedance boundary conditions,

$$(\partial p/\partial n - ikg(s)p)|_{s\in S'} = 0, \tag{168}$$

with $\mathrm{Re}(g) \geq 0$, and $\mathrm{Re}(g) \neq 0$ on a regular part S_0 of S', the conditions at infinity, no sources outside S', and harmonic dependence on time $e^{i\omega t}$. From Green's first theorem, we can write,

$$\int_\Omega (p^*(r)\Delta p(r) + \mathrm{grad}p^*(r)\mathrm{grad}p(r))d\Omega = -\int_{S'} p^*(r)(\widehat{n}\,\mathrm{grad}(p(r)))dS + \\ + \lim_{a\to\infty}\int_{|r|=a,\, r\in\Omega} p^*(r)(\frac{\partial p(r)}{\partial |r|})dS, \tag{169}$$

where Ω is the infinite domain outside S', $\widehat{n}$ is the inward normal to Ω. Besides, from the impedance boundary conditions, we have,

$$\mathrm{Re}(\int_\Omega -ik|p(r)|^2 + \frac{|\mathrm{grad}p(r)|^2}{-ik}d\Omega) = \int_{S'} \mathrm{Re}(g)|p(r)|^2 dS + I_\infty, \tag{170}$$

where,

$$I_\infty = \mathrm{Re}(\lim_{a\to\infty} I_a) \text{ with } I_a = \int_{|r|=a,\, r\in\Omega} p^*(r)(\frac{\partial p(r)}{-ik\partial|r|})dS$$

As $|\arg(ik)| < \pi/2$ (space with losses), we can assume,

$$I_\infty = \mathrm{Re}(\lim_{a\to\infty} O(e^{-\delta a})) = 0 \text{ as } |\arg(ik)| < \pi/2. \tag{171}$$

As $|\arg(ik)| = \pi/2$ (without losses), the field at large distance (when no primary source) is principally the sum of a diffracted term radiating longitudinally from origin and a term p_s which includes possible guided wave along S', following,

$$p(r) = F(\frac{r}{|r|})\frac{e^{-ik|r|}}{|r|^{(m-1)/2}} + p_s(\frac{r}{|r|}, |r|) \text{ with } F(\frac{r}{|r|}) = O(1) \text{ in } \Omega,$$
$$|r^{(m-1)/2}p_s(r)| \text{ and } |r^{(m-1)/2}\partial_{|r|}p_s(r)| \to 0 \text{ as } |r| \to \infty \text{ in } \Omega\backslash S',$$
$$|p_s(r)| \text{ and } |\partial_{|r|}p_s(r)| = O(1) \text{ on } S' \text{ as } |r| > a_0,$$

with $m = 2$ in 2D and $m = 3$ in 3D, and thus,

$$I_\infty = \lim_{a\to\infty} \int_{|r|=a,\, r\in\Omega} |p(r)|^2 dS > 0.$$

For $\mathrm{Re}(g) \geq 0$ and $|\arg(ik)| \leq \pi/2$, the left-hand term of (170) is negative since $\mathrm{Re}(ik) \geq 0$, while the right-handed term is positive, and thus both terms vanish. Consequently, we have, when $|\arg(ik)| < \pi/2$,

$$p(r) = 0 \text{ in } \Omega, \text{ for } \mathrm{Re}(g) \geq 0, \tag{172}$$

while, when $|\arg(ik)| = \pi/2$, the far field radiation function F like I_∞ is nul, $\mathrm{Re}(g)|p(r)|^2$ vanish on S', and thus,

$$\partial_n p(r) = ikg\, p(r) = 0 \text{ on any regular part } S_0 \text{ of } S' \text{ with } \mathrm{Re}(g) > 0. \tag{173}$$

In the latter case, we can use the second Green theorem to conclude that p vanishes in Ω (see next section) to complete the proof of uniqueness. We could also use when $S_0 \equiv S'$, as suggested by Levine, the Jones' uniqueness theorem theorem [32] for surfaces conical at infinity (conical outside a sphere of large radius), when Neumann boundary condition ($\partial_n p(r)|_{S'} = 0$) is satisfied, which implies $p \equiv 0$ in the entire domain Ω. This uniqueness proof is also possible for electromagnetic field, from the use of expressions with surface tangential components (note : the use of Cauchy-Kovalevskaia theorem, the Colton results for elliptic operator [33], or the demonstration given in [3] lead to the same result).

C.2) Uniqueness for $|\arg(ik)| \leq \pi/2$ comes from nullity on a regular part of boundaries

Let a field p' satisfying $\Delta p' + k^2 p' = 0$ in a subdomain $\Omega_1 \subset \Omega_2$ without sources, bounded by a surface $\partial\Omega_1$, and a Green function $\mathcal{G}$ verifying,

$$(\Delta + k^2)\mathcal{G}(k|r - r'|) = \delta(r - r'), \tag{174}$$

like $\mathcal{G}(k|r|) = -\frac{e^{-ik|r|}}{4\pi|r|}$ as $|r| = \sqrt{x^2+y^2+z^2}$ and $\delta(r) \equiv \delta(x)\delta(y)\delta(z)$ in 3D for $|\arg(ik)| \leq \pi/2$, or like $\mathcal{G}(k|r|) = \frac{i}{4}H^{(2)}(k|r|)$ as $|r| = \sqrt{x^2+y^2}$ and $\delta(r) \equiv \delta(x)\delta(y)$ in 2D. We can then use the second Green theorem (or identity), to express the field p' as a combination of simple and double layer expressions on $\partial\Omega_1$ following,

$$\int_{\Omega_1} (p'\Delta\mathcal{G} - \mathcal{G}\Delta p')d\Omega = -\int_{\partial\Omega_1} (p'\frac{\partial\mathcal{G}}{\partial n} - \mathcal{G}\frac{\partial p'}{\partial n})dS = \begin{vmatrix} p'(r'), r'\in\Omega_1 \\ 0, r'\notin\Omega_1 \end{vmatrix}, \tag{175}$$

with the normal vector here directed inside $\partial\Omega_1$, where it is worth noticing the nullity outside $\partial\Omega_1$. If we have $\partial_n p(r) = p(r) = 0$ on a analytic part $\mathcal{H}$ of $\partial\Omega_1$, we have a hole in the surface field expression, and we can write,

$$-\int_{\partial\Omega_1\setminus\mathcal{H}}(p'\frac{\partial\mathcal{G}}{\partial n}-\mathcal{G}\frac{\partial p'}{\partial n})dS=\Big|_{0,\,r'\notin\Omega_1}^{p'(r'),\,r'\in\Omega_1}, \tag{176}$$

which implies by analyticity that nullity outside $\partial\Omega_1$ continues inside $\partial\Omega_1$, i.e. $p'=0$ in Ω_1 and thus everywhere in Ω_2.

C.3) Some extension in electromagnetism

In electromagnetism, we have similar results. Let us consider fields E and H verifying,

$$\begin{aligned}&\text{rot}(H)=i\omega\epsilon E,\ \text{rot}(E)=\ -i\omega\mu H,\\&(\Delta+k^2)H=(\Delta+k^2)E=0,\text{div}(E)=\text{div}(H)=0,\end{aligned} \tag{177}$$

in a domain Ω_2 without sources, with homogenous isotropic media of permittivity ϵ and permeability μ, $k=\omega\sqrt{\epsilon\mu}$, and a subdomain $\Omega_1\subset\Omega_2$. Considering the Helmholtz equations alone, we could write,

$$\int_{\Omega_1}(\Big|_H^E\Delta\mathcal{G}-\mathcal{G}\Delta\Big|_H^E)d\Omega=\ -\int_{\partial\Omega_1}(\Big|_H^E\frac{\partial\mathcal{G}}{\partial n}-\mathcal{G}\frac{\partial\Big|_H^E}{\partial n})dS_r=\Big|_{0,\,r'\notin\Omega_1}^{\Big|_H^E(r'),\,r'\in\Omega_1}, \tag{178}$$

but we can also obtain expressions only related to tangential surface components. For that, we use the electromagnetic Green functions E_G and H_G radiated by dirac sources $j_0\delta$ and $m_0\delta$ in Ω_2 which satisfy,

$$\text{rot}(H_G)=i\omega\epsilon E_G+j_0\delta(r-r'),\ \text{rot}(E_G)=\ -i\omega\mu H_G-m_0\delta(r-r'), \tag{179}$$

then the identities $\text{div}(e_l\wedge h_q)=h_q.\text{rot}e_l-e_l.\text{rot}h_q$ and $\int_\Omega\text{div}(a)dV=\int_{\partial\Omega}n_s.a\ dS$, and we obtain,

$$\begin{aligned}&\int_{\partial\Omega_1}(E_G.(n_s\wedge H)+H_G.(n_s\wedge E))dS=\\&\int_{\Omega_1}(m_0\delta.H-j_0\delta.E)\,d\Omega=\Big|_{0,\,r'\notin\Omega_1}^{m_0.H(r')-j_0.E(r'),\,r'\in\Omega_1},\end{aligned} \tag{180}$$

where n_s is the unit outward normal to Ω_1 ($\widehat{n}=\ -n_s$ is the unit inward normal to Ω_1). In a similar manner than in scalar case, we then obtain that if the tangential components of E and H vanishes on a regular part of $\partial\Omega_1$, then we have a hole in expression (180) on $\partial\Omega_1$, and by analyticity, the fields continue to be zero from outside to inside $\partial\Omega_1$, and thus are null in all parts of Ω_2.

Remark : When E_G and H_G are the free space Green functions in electromagnetism, we note that we have,

$$(\Delta + k^2)E_G = (\text{grad}(\text{div}(\frac{ij_0(.)}{\epsilon\omega})) + i\omega\mu j_0(.))\delta + \text{rot}(m_0(.))\delta,$$
$$(\Delta + k^2)H_G = (\text{grad}(\text{div}(\frac{im_0(.)}{\mu\omega})) + i\omega\epsilon m_0(.))\delta - \text{rot}(j_0(.))\delta, \quad (181)$$

This implies, after using $\mathcal{G}$ verifying $(\Delta + k^2)\mathcal{G}(k|r - r'|) = \delta(r - r')$, that,

$$E_G = [(\text{grad}_r(\text{div}_r(\frac{ij_0(.)}{\epsilon\omega})) + i\omega\mu j_0(.)) + \text{rot}_r(m_0(.))]\mathcal{G}(k|r - r'|),$$
$$j(r).E_G = [\frac{ij_0}{\epsilon\omega}.(\text{grad}_{r'}(\text{div}_{r'}(j(r)(.))) + k^2 j(r))$$
$$+ m_0.\text{rot}_{r'}(j(r)(.))]\mathcal{G}(k|r - r'|),$$
$$H_G = [(\text{grad}_r(\text{div}_r(\frac{im_0(.)}{\mu\omega})) + i\omega\epsilon m_0(.)) - \text{rot}_r(j_0(.))]\mathcal{G}(k|r - r'|),$$
$$m(r).H_G = [\frac{im_0}{\mu\omega}.(\text{grad}_{r'}(\text{div}_{r'}(m(r)(.)) + k^2 m(r)).$$
$$- j_0[\text{rot}_{r'}(m(r)(.))]\mathcal{G}(k|r - r'|) \quad (182)$$

Appendix D : Asymptotics for $\mathcal{R}_a(\alpha)$ and $\mathcal{R}_b(\alpha)$ taking account of complex poles

When $k\Delta$ is large, the highly oscillatory nature of the exponential terms in kernels leads to simple approximations. We can then consider (99)-(101) and use the method of Steepest Descent Path, taking, this time, account of the influence of complex poles of $\mathcal{R}_{a,b}(\alpha')$. During the deformation of the initial path of integration along imaginary axis to SDP, where $\text{Im}(ik(\sin(\alpha'/2))^2) = 0$ and $\text{Re}(ik(\sin(\alpha'/2))^2) \leq 0$, poles of $\mathcal{R}_a(\alpha')$ and $\mathcal{R}_b(\alpha')$ can be captured, and we have,

$$\mathcal{R}_b(\alpha) = \frac{-i}{4\Phi_+} m_1(\alpha)\Big(\int_{SDP} \mathcal{R}_a(\alpha')\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'$$
$$+ 2\pi i\sum_s \frac{e^{-ik\Delta\cos\alpha'_s}\sin(\frac{\pi\alpha'_s}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'_s}{2\Phi_+})}\epsilon_{a,s}\text{Residue}(\mathcal{R}_a(\alpha'))|_{\alpha'=\alpha_s,\text{Re}\alpha_s<0}\Big)$$
$$+ m_1(\alpha)\chi_b^i(\alpha + \pi/2 - \Phi_b/2) \quad (183)$$

as $\alpha + \pi/2 - \Phi_b/2$ is in the domain limited by $SDP - \Phi_+$ and $SDP + 3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) = \frac{i}{4\Phi_-} m_2(\alpha)\Big(\int_{SDP} \mathcal{R}_b(\alpha') \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})} d\alpha'$$
$$+ 2\pi i \sum_s \frac{e^{-ik\Delta\cos\alpha'_s}\sin(\frac{\pi\alpha'_s}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)) + \cos(\frac{\pi\alpha'_s}{2\Phi_-})} \epsilon_{b,s} \text{Residue}(\mathcal{R}_b(\alpha'))|_{\alpha'=\alpha_s, \text{Re}\alpha_s>0}\Big)$$
$$+ m_2(\alpha)\chi^i_a(\alpha - \pi/2 + \Phi_a/2), \tag{184}$$

as $\alpha - \pi/2 + \Phi_a/2$ is in the domain limited by $SDP - 3\Phi_-$ and $SDP + \Phi_-$, where

$$\mathcal{R}_b(\alpha) = (\frac{f_{br}(\alpha + \pi/2 - \Phi_b/2)}{\Psi_{1-}(\alpha + \Phi_-)}),\ \mathcal{R}_a(\alpha) = (\frac{f_{ar}(\alpha - \pi/2 + \Phi_a/2)}{\Psi_{+1}(\alpha - \Phi_+)}),$$
$$m_1(\alpha) = \frac{\Psi_{+1}(\alpha + \pi/2 - \Phi_b/2)}{\Psi_{1-}(\alpha + \Phi_-)},\ m_2(\alpha) = \frac{\Psi_{1-}(\alpha - \pi/2 + \Phi_a/2)}{\Psi_{+1}(\alpha - \Phi_+)},$$
$$\chi^i_b(\alpha) = \frac{\pi}{2\Phi_+}(\frac{e^{ik\Delta\sin\varphi_\circ}\cos\frac{\pi\varphi_{\circ,b}}{2\Phi_+}}{\Psi_{+1}(\varphi_{\circ,b})(\sin\frac{\pi\alpha}{2\Phi_+} - \sin\frac{\pi\varphi_{\circ,b}}{2\Phi_+})}),$$
$$\chi^i_a(\alpha) = \frac{\pi}{2\Phi_-}(\frac{\cos\frac{\pi\varphi_{\circ,a}}{2\Phi_-}}{\Psi_{1-}(\varphi_{\circ,a})(\sin\frac{\pi\alpha}{2\Phi_-} - \sin\frac{\pi\varphi_{\circ,a}}{2\Phi_-})}), \tag{185}$$

with $\varphi_{\circ,b} = \varphi_\circ - \Phi_b/2$, $\varphi_{\circ,a} = \varphi_\circ + \Phi_a/2$, $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$, $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$, and $\epsilon_{a(b),s} = 1$ or -1 if the pole s is captured, $\epsilon_{a(b),s} = 0$ else.

For large $k\Delta$, the residue terms are principally given by

$$\text{Residue}(\mathcal{R}_a(\alpha'))|_{\alpha'=\alpha_s} \sim \text{Residue}(m_2(\alpha')\chi^i_a(\alpha' - \pi/2 + \Phi_a/2))|_{\alpha'=-\theta_1}$$
$$\text{Residue}(\mathcal{R}_b(\alpha'))|_{\alpha'=\alpha_s} \sim \text{Residue}(m_1(\alpha')\chi^i_b(\alpha' + \pi/2 - \Phi_b/2))|_{\alpha'=\theta_1} \tag{186}$$

where $\alpha = -\theta_1$ and $\alpha = \theta_1$ are the zeros of $\Psi_{+1}(\alpha - \Phi_+)$ and $\Psi_{1-}(\alpha + \Phi_-)$ the closest to the imaginary axis, that can be captured. Letting $t = -i\sqrt{2ik\Delta}\sin(\alpha'/2)$, we then derive,

$$\mathcal{R}_b(\alpha) \sim \frac{m_1(\alpha)(\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0})}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}(v\, e^{v^2}\text{erfc}(v) - \frac{1}{\sqrt{\pi}})$$
$$+ \frac{2\pi}{4\Phi_+} \frac{e^{-ik\Delta\cos\theta_1}\sin(\frac{-\pi\theta_1}{2\Phi_+})\chi^i_a(-\theta_1 - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\theta_1}{2\Phi_+})} \epsilon_a \text{Residue}(m_2(\alpha'))|_{\alpha'=-\theta_1}$$
$$+ m_1(\alpha)\chi^i_b(\alpha + \pi/2 - \Phi_b/2), \tag{187}$$

with $v^2 = ik\Delta(\frac{\pi}{2\Phi_+})^{-2}(1 + \cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)))$, $\alpha + \pi/2 - \Phi_b/2$ in the domain limited by $SDP - \Phi_+$ and $SDP + 3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) \sim -\frac{m_2(\alpha)(\,\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0})}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}(w\,e^{w^2}\text{erfc}(w) - \frac{1}{\sqrt{\pi}})$$
$$- \frac{2\pi}{4\Phi_-}\frac{e^{-ik\Delta\cos\alpha'_s}\sin(\frac{\pi\theta_1}{2\Phi_-})\chi_b^i(\theta_1+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\theta_1}{2\Phi_-})}\epsilon_b \text{Residue}(m_1(\alpha'))|_{\alpha'=\theta_1}$$
$$+ m_2(\alpha)\chi_a^i(\alpha-\pi/2+\Phi_a/2), \tag{188}$$

with $w^2 = ik\Delta(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)))$, $\alpha-\pi/2+\Phi_a/2$ in the domain limited by $SDP-3\Phi_-$ and $SDP+\Phi_-$, for $-\min(\pi/2,\ \pi/2+\Phi_a) < \varphi_\circ < \min(\pi/2,\ \pi/2+\Phi_b)$. To obtain $\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0}$ and $\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0}$, we derivate the previous expressions and solve the linear system of equations. We obtain, for large $k\Delta$,

$$\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0} \sim \partial_\alpha(m_1(\alpha)\chi_b^i(\alpha+\pi/2-\Phi_b/2))|_{\alpha=0},$$
$$\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0} \sim \partial_\alpha(m_2(\alpha)\chi_a^i(\alpha-\pi/2+\Phi_a/2))|_{\alpha=0}. \tag{189}$$

In consequence, we can write,

$$\mathcal{R}_b(\alpha) \sim \frac{m_1(\alpha)\partial_\alpha(m_2(\alpha)\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}))|_{\alpha=0}}{\sqrt{ik\Delta/2}} e^{-ik\Delta}(v\,e^{v^2}\text{erfc}(v) - \frac{1}{\sqrt{\pi}})$$
$$+ \frac{2\pi}{4\Phi_+}\frac{e^{-ik\Delta\cos\theta_1}\sin(\frac{-\pi\theta_1}{2\Phi_+})\chi_a^i(-\theta_1-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\theta_1}{2\Phi_+})}\epsilon_a \text{Residue}(m_2(\alpha'))|_{\alpha'=-\theta_1}$$
$$+ m_1(\alpha)\chi_b^i(\alpha+\pi/2-\Phi_b/2), \tag{190}$$

with $v^2 = ik\Delta(\frac{\pi}{2\Phi_+})^{-2}(1+\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b)))$, $\alpha+\pi/2-\Phi_b/2$ in the domain limited by $SDP-\Phi_+$ and $SDP+3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) \sim -\frac{m_2(\alpha)\partial_\alpha(m_1(\alpha)\chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}))|_{\alpha=0}}{\sqrt{ik\Delta/2}} e^{-ik\Delta}(w\,e^{w^2}\text{erfc}(w) - \frac{1}{\sqrt{\pi}})$$
$$- \frac{2\pi}{4\Phi_-}\frac{e^{-ik\Delta\cos\alpha'_s}\sin(\frac{\pi\theta_1}{2\Phi_-})\chi_b^i(\theta_1+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\theta_1}{2\Phi_-})}\epsilon_b \text{Residue}(m_1(\alpha'))|_{\alpha'=\theta_1}$$
$$+ m_2(\alpha)\chi_a^i(\alpha-\pi/2+\Phi_a/2), \tag{191}$$

with $w^2 = ik\Delta(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)))$, $\alpha-\pi/2+\Phi_a/2$ in the domain limited by $SDP-3\Phi_-$ and $SDP+\Phi_-$, while $-\min(\pi/2,\ \pi/2+\Phi_a) < \varphi_\circ < \min(\pi/2,\ \pi/2+\Phi_b)$. In right-handed sides of previous expressions, the first term corresponds to double diffraction mechanism, the second term to the influence of guided waves, and the third term to single diffraction mechanism.

Remark :

The integral term is calculated, using that,

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{t^2e^{-t^2}}{z^2-t^2}dt=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{z^2e^{-t^2}}{z^2-t^2}dt-\frac{i}{\sqrt{\pi}}=ze^{-z^2}\text{erfc}(-iz)-\frac{i}{\sqrt{\pi}}, \qquad (192)$$

as Im$z>0$, with erfc$(a)=1-\text{erf}(a)\sim e^{-a^2}(1/a-1/(2a^3))/\sqrt{\pi}$ for large a, erf(a) being the error function [25],[26].

Appendix E : The scattering diagram from the solutions of the integral equations

In a previous section, we have reduced the problem to a system of non-singular integral equations. When the solutions of the integral equations are known, the different elements of the decomposition of the field can be evaluated from the integral expressions (45)-(46), where the integral terms can be considered as smooth coupling terms between both edges. In this case, the difference functional equations (28)-(30) have to be used to reduce the calculus in the band of validity of integral expressions.

Here, we study the reduction of the field term with radial dependence $\exp(-ik\rho_{a,b})/\sqrt{2\pi k\rho_{a,b}}$ for large $\rho_{a,b}$. Its angular dependence $\mathcal{F}_{a,b}$ in the direction $\varphi_a=\varphi_b=\varphi$ is commonly called the scattering diagram (or directivity) [15][16][29], given from (5) by,

$$\mathcal{F}_{a,b}(\varphi)=-e^{-i\pi/4}(f_{a,b}(\pi+\varphi)-f_{a,b}(-\pi+\varphi)), \qquad (193)$$

where, from (25), $\mathcal{F}_a(\varphi)=e^{ik\Delta\sin\varphi}\mathcal{F}_b(\varphi)$.

We illustrate the development by the one of $\mathcal{F}_b(\varphi_b)$ in the case where $\Phi_b>-\frac{\pi}{2}$, $\Phi_a>0$, and $-\min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_a)<\varphi_\circ<\min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_b)$, when $-\frac{\pi}{2}<\varphi_b<\frac{\pi}{2}+\Phi_b$ (similar to $\mathcal{F}_a(\varphi_a)$ when $-\frac{\pi}{2}<-\varphi_a<\frac{\pi}{2}+\Phi_a$, $\Phi_a>-\frac{\pi}{2}$, $\Phi_b>0$).

We consider at first the reduction of $f_b(\pi+\varphi_b)$ in (193). If $\Phi_b>0$, then $-\frac{\pi}{2}<\pi+\varphi_b<\frac{3\pi}{2}+2\Phi_b$, and we are in the domain of validity of (45)-(46). If $\Phi_b<0$, then we have a concave part and there are two cases. If $-\frac{\pi}{2}<\pi+\varphi_b<\frac{3\pi}{2}+2\Phi_b$, (45)-(46) directly applies. In contrary, if $\frac{3\pi}{2}+2\Phi_b<\pi+\varphi_b<\frac{3\pi}{2}+\Phi_b$, we use (30) and write

$$f_b(\pi+\varphi_b)=\frac{(-\sin(\frac{\pi}{2}-\Phi_b+\varphi_b)+\sin\theta_+)}{(\sin(\frac{\pi}{2}-\Phi_b+\varphi_b)+\sin\theta_+)}f_b(-\varphi_b+2\Phi_b), \qquad (194)$$

Then we use (28)-(29),

$$f_b(-\varphi_b+2\Phi_b)=\frac{(\sin(\frac{\pi}{2}-\varphi_b+2\Phi_b)+\sin\theta_1)}{(-\sin(\frac{\pi}{2}-\varphi_b+2\Phi_b)+\sin\theta_1)}(f_b(-\pi+\varphi_b-2\Phi_b)$$
$$-e^{ik\Delta\sin(2\Phi_b-\varphi_b)}f_a(-\pi+\varphi_b-2\Phi_b))+e^{ik\Delta\sin(2\Phi_b-\varphi_b)}f_a(-\varphi_b+2\Phi_b), \quad (195)$$

where the terms in right handed-side of (195) can be expressed with (45)-(46).
Then, we reduce $f_b(-\pi+\varphi_b)$ in (193). If $-\frac{\pi}{2}<-\pi+\varphi_b<\frac{3\pi}{2}+2\Phi_b$, i.e. $\varphi_b>\frac{\pi}{2}$, which implies $\Phi_b>0$, we are in the domain of validity of (45)-(46). In contrary, when $-\frac{\pi}{2}<\varphi_b<\frac{\pi}{2}$, which is obliged when $\Phi_b<0$, we first have to use (28)-(29),

$$f_b(-\pi+\varphi_b)=\frac{(\sin(\frac{\pi}{2}-\varphi_b)-\sin\theta_1)}{(-\sin(\frac{\pi}{2}-\varphi_b)-\sin\theta_1)}$$
$$\times(f_b(-\varphi_b)-e^{-ik\Delta\sin\varphi_b}f_a(-\varphi_b))+e^{-ik\Delta\sin\varphi_b}f_a(-\pi+\varphi_b). \quad (196)$$

In this case, since $-\frac{3\pi}{2}-2\Phi_a<-\varphi_b<\frac{\pi}{2}$ and $-\frac{3\pi}{2}-2\Phi_a<-\pi+\varphi_b<\frac{\pi}{2}$ when $\Phi_a>0$, and $-\frac{\pi}{2}<-\varphi_b<\frac{3\pi}{2}+2\Phi_b$ when $\Phi_b>-\frac{\pi}{2}$, $f_a(-\pi+\varphi_b)$ and $f_a(-\varphi_b)$ can be expressed with (46), and $f_b(-\varphi_b)$ with (45), which ends the reduction.

Appendix F : Approximated second order ameliorations for small cavity

F.1) Amelioration of equivalent cavity impedance $\sin\theta_1'$

Using our works in [21], we can modifiy and add a second order to the expressions of $\sin\theta_1'$. We can write,

$$\sin\theta_1'=\sum_j d_j\sin\theta_j/\Delta+ik(S_+-S_-)/\Delta, \quad (197)$$

where S_+ is the total surface of cavities below the straight line between the exterior discontinuities 'a' and 'b', while S_- is the total surface of cavities above the straight line. In the case where some elements would be filled with materials of impedance $Z_j=\sqrt{\epsilon_2\mu_2}/\epsilon_2$ (mode H when $u\equiv H_z$) or admittance $Y_j=\sqrt{\epsilon_2\mu_2}/\mu_2$ (mode E when $u\equiv E_z$), and with wave number k_m, we can write,

$$\sin\theta_1'|_{\text{mode }H}=\sum_j d_j\sin\theta_j/\Delta+\sum_m i\frac{Z_m}{Z_0}k_m(S_{m+}-S_{m-})/\Delta,$$
$$\sin\theta_1'|_{\text{mode }E}=\sum_j d_j\sin\theta_j/\Delta+\sum_m i\frac{Y_m}{Y_0}k_m(S_{m+}-S_{m-})/\Delta, \quad (198)$$

where the $\sin\theta_j$ remain the relative surface in free space.

F.2) A change of M_u

We can refine in (126), from [1],[2],[21], the expression of the mean value M_u of u on the cavity,

$$M_u \sim \frac{(1-\frac{i}{\pi}B_0\sin\varphi'_\circ\sin\Phi_b)}{1+\frac{i}{\pi}B_0(\sin\theta'_1-\sin\theta_+\cos\Phi_b)}\frac{2\cos\varphi'_\circ}{(\cos\varphi'_\circ+\sin\theta_+)}=D_0M_u^0$$
$$B_0=\int_{-i\infty}^{+i\infty}d\alpha'\frac{(e^{-ik\Delta\cos\alpha'}-1)}{2\cos\alpha'}\sim -k\Delta(\ln(k\Delta)+\gamma_0-1+i\pi/2)$$
$$D_0=\frac{(1-\frac{i}{\pi}B_0\sin\varphi'_\circ\sin\Phi_b)}{1+\frac{i}{\pi}B_0(\sin\theta'_1-\sin\theta_+\cos\Phi_b)},\gamma_0\approx .577, \qquad (199)$$

and then write, in place of (130),

$$f_a(\pi+\varphi'+\Phi_b)-f_a(-\pi+\varphi'+\Phi_b)$$
$$\sim\frac{2ik\Delta\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)}(-\cos\Phi_b\sin\theta_++\sin\Phi_b\sin\varphi'$$
$$+\sin\theta'_1)\times D_0+\frac{2ik\Delta\sin\Phi_b(\cos\varphi'-\sin\theta_+)\cos\varphi'\cos\varphi'_\circ}{(\cos\varphi'_\circ+\sin\theta_+)(\sin\varphi'+\sin\varphi'_\circ)}. \qquad (200)$$

A more refined expression could be found by semi-inversion (see previous appendix).

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