Spectral equations for scattering by impedance polygons : properties and solutions

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1) Introduction

The Sommerfeld-Maliuzhinets representation of fields is not limited to the study of the diffraction by isolated impedance wedges, and we developed in [1]-[2] an analytical method concerning the determination of spectral function for the scattering by impedance polygonal object (convex or not). Our method is now quoted and used by other authors, as very recently in [3] for semi-infinite impedance polygons with three edges, but other approaches exist. They can be exact for cavities, for perfectly conducting object, or for specific geometries [4]-[8], or give asymptotic [9] or iterative [10] reduction. Our approach has the advantage to give rigorous analytical equations in complex plane, which apply for general impedance polygons with finite, but also infinite faces without being limited to single wedge. For that, we consider special features of single face expression of spectral function that we defined in [1]-[2], which leads to exact functional difference equations, and Fredholm integral equations for finite or infinite polygons that we detail here with novel properties. Existence and uniqueness of solutions are analysed in an original manner, and approximate asymptotics are discussed.

2) Generalities

Let us consider the case of diffraction in free space of a plane wave

$$u^i(\rho,\varphi) = e^{ik\rho\cos(\varphi-\varphi_\circ)}, \tag{1}$$

by a scatterer enclosed in a wedge-shaped region, defined in cylindrical coordinates (ρ,φ) as the domain complementary to the free space sector of radiation, $-\Phi_r \leq \varphi \leq \Phi_l$ (figure 1), with $-\Phi_r \leq \varphi_\circ \leq \Phi_l$. The characteristics of the scatterer are supposed to be independent of z coordinate. An implicit harmonic dependence on time $e^{i\omega t}$ is understood and henceforth suppressed. In expression (1), k denotes the wave number of the exterior medium with $|\arg(ik)| < \frac{\pi}{2}$, and $|\arg(ik)| = \frac{\pi}{2}$ is considered as a limit case. We assume that the total field in the free space region, $u = u_s + u^i$, satisfies the Helmholtz equation,

$$(\Delta + k^2)u(\rho, \varphi) = 0, \tag{2}$$

that u is analytic with respect to ρ and φ as $-\Phi_r < \varphi < \Phi_l$, except possibly at the origin O, that u is finite and $\frac{\partial u}{\rho \partial \varphi}$ is integrable with respects to ρ at any point of both faces, while there exists a constant $s_\circ$ such that $\int_0^\infty |u(\rho, \varphi) e^{-s_\circ \rho}| d\rho < \infty$.

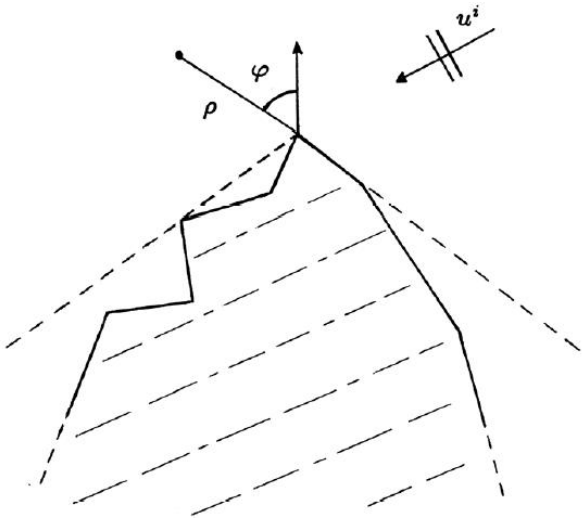


figure 1 : a polygonal surface in a wedge shaped sector

The total field u for $-\Phi_r \leq \varphi \leq \Phi_l$ can be then represented as a Sommerfeld-Maliuzhinets integral [1],[19],[20],

$$u(\rho, \varphi) = \frac{1}{2\pi i} \int_\gamma f(\alpha + \varphi) e^{ik\rho \cos\alpha} d\alpha, \tag{3}$$

which satisfies the Helmholtz equation. In this representation, f is an analytic function and the path γ consists of two branches: one, named γ_+, going from $(i\infty + \arg(ik) + (a_1 + \frac{\pi}{2}))$ to $(i\infty + \arg(ik) - (a_2 + \frac{\pi}{2}))$ with $0 < a_{1,2} < \pi$, as $\mathrm{Im}\alpha \geq d$, above all the singularities of the integrand, and the other, named γ_-, obtained by inversion of γ_+ with respect to $\alpha = 0$.
After deformation of γ to the steepest descent path SDP, we can write

$$\begin{aligned} u(\rho,\varphi) &= u_i + \sum u_r^{\pm} + \sum u_s^{\pm} \\ &+ \frac{\mathrm{e}^{-ik\rho}}{2\pi i} \int_{SDP} f(\alpha + \varphi)\, \mathrm{e}^{ik\rho(\cos\alpha + 1)}\, d\alpha, \end{aligned} \tag{4}$$

where

- the term u_i is the incident field u^i in the illuminated zone and zero in the shadow zone;

- the terms $u_r^{\pm}$ correspond to geometrical optics field, reflected by physical semi-infinite faces (when present).
- the terms $u_s^{\pm}$ are terms of guided waves, along physical semi-infinite faces (when present), which are the contribution of complex pôles $\alpha_s^{\pm}$ of $f(\alpha+\varphi)$;
- the last term, named u_d, is principally radiated cylindrically when $\rho \to \infty$. Approximating $f(\alpha+\varphi)$ on $SDP_{\pm}$ by its value at the stationnary phase points $\alpha = \pm\pi$, we obtain

$$u_d \sim \frac{-e^{-i\pi/4-ik\rho}}{\sqrt{2\pi k\rho}}[f(\pi+\varphi)-f(-\pi+\varphi)], \tag{5}$$

with $F(\varphi) = f(\pi+\varphi)-f(-\pi+\varphi)$ the diffraction (or far field) coefficient.

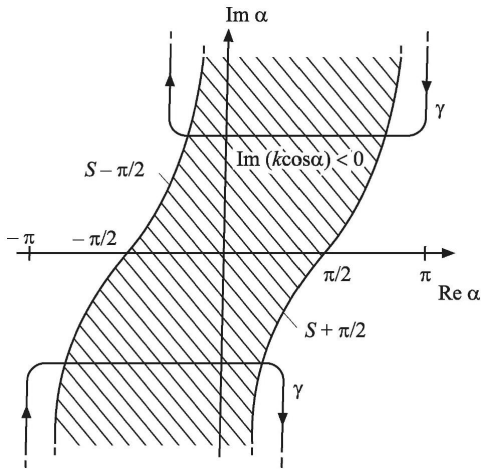


figure 2 : Complex path of integration

This representation is commonly applied for a wedge. We investigate its use for a scatterer with several discontinuities as in [1]-[2], detailing properties and ameliorating the method. This study requires us to express a shift of the origin, and for this, to use particular relations of f with radiation by one face [1].

3) Single-face expression of f for a scatterer surface with two polygonal faces

Let us consider a polygonal surface located inside the domain $|\varphi| > \Phi$ enclosing a scatterer. This surface is composed of two joined semi-infinite polygonal faces $L_{0,\infty}^{\pm}$, denoted $+$ and $-$, respectively with $m^{\pm}$ segments of lengths $d_j^{\pm}$ with tangent angles $\pm\Phi_j^{\pm}$, $j = 1, 2, ...,$ $m^{\pm}$ and a semi-infinite plane with tangent angles $\pm\Phi_e^{\pm}$, $\Phi \le \Phi_e^{\pm}$. The spectral function f satisfies a single-face expression [1], as $\frac{\pi}{2} < \Phi_e^{\pm} \mp \varphi_\circ < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \Phi_e^{\pm} \mp \varphi < \frac{3\pi}{2}$,

$$f(\pm\pi+\varphi)=\frac{1}{2}\int_{L^{\pm}_{0,\infty}}\mp\left(u(M)\nabla_M(e^{ik\hat{\tau}.OM})-\nabla_M(u(M))e^{ik\hat{\tau}.OM}\right).\hat{n}^{\pm}ds(M), \quad (6)$$

where $s(M)$ is the length abscissa on each face, at points M of $L^{\pm}_{0,\infty}$ from origin to ∞, the term $\hat{\tau}$ is the unit vector of the direction of observation with angle φ, and $\hat{n}^{\pm}$ is the outward unit normal to face at M.

Considering the geometry of $L^{\pm}_{0,\infty}$, and the notation $\hat{n}^{\pm}\nabla_M(.)=\partial(.)/\partial n=\mp\partial(.)/\rho_j\partial\varphi_j$ on straight segments $\rho_j\in[0,d_j^{\pm}]$ at $\varphi_j=\pm\Phi_j^{\pm}$, we can write the single-face expression of f [1] in the form,

$$f(\alpha)=\sum_{1\le j\le m^{\pm}}\frac{e^{-ik\sum\limits_{1\le i<j}d_i^{\pm}\cos(\alpha\mp\Phi_i^{\pm})}}{2}\int_0^{d_j^{\pm}}\Big(-iku(\rho_j',\pm\Phi_j^{\pm})\sin(\alpha\mp\Phi_j^{\pm})$$
$$\pm\frac{\partial u}{\partial n}(\rho_j',\pm\Phi_j^{\pm})\Big)e^{-ik\rho_j'\cos(\alpha\mp\Phi_j^{\pm})}d\rho_j'+e^{-ik\sum\limits_{1\le i\le m^{\pm}}d_i^{\pm}\cos(\alpha\mp\Phi_i^{\pm})}f^{\pm}_{e,m^{\pm}}(\alpha), \quad (7)$$

with $\sum\limits_{1\le i<1}\equiv 0$ and $\sum\limits_{1\le i\le 0}\equiv 0$, where $f^{\pm}_{e,m^{\pm}}(\alpha)$ is the analytic continuation of the integral expression

$$f^{\pm}_{e,m^{\pm}}(\alpha'\pm\Phi_e^{\pm})=\frac{1}{2}\int_0^{\infty}\Big(-iku(\rho_e',\pm\Phi_e^{\pm})\sin\alpha'\pm\frac{\partial u}{\partial n}(\rho_e',\pm\Phi_e^{\pm})\Big)e^{-ik\rho_e'\cos\alpha'}d\rho_e', \quad (8)$$

valid as $\mathrm{Re}(ik(\cos\alpha'-\cos(\Phi_e^{\pm}\mp\varphi_\circ)))>0$, $|\mathrm{Re}\alpha'|<\pi$, $|\arg(ik)|<\frac{\pi}{2}$.

The function $f^{\pm}_{e,m^{\pm}}$ is the spectral function corresponding to the Sommerfeld-Maliuzhinets representation of the field in cylindrical coordinates $(\rho_e^{\pm},\varphi_e^{\pm})$, with origin $Q^{\pm}_{e,m^{\pm}}$ at the edge of the semi-plane $\varphi_e^{\pm}=\pm\Phi_e^{\pm}$, radiating in the free space sector $\Phi_e^{\pm}-\delta_e^{\pm}\le\pm\varphi_e^{\pm}\le\Phi_e^{\pm}$ where $\delta_e^{\pm}$ is a strictly positive constant determined by the geometry.

The spectral functions f and $f^{\pm}_{e,m^{\pm}}$ have the following properties :

(a) $f^{\pm}_{e,m^{\pm}}(\alpha)-u^i(Q^{\pm}_{e,m^{\pm}})/(\alpha-\varphi_\circ)$ has no singularity in the band $\Phi_e^{\pm}-\delta^{\pm}\le\pm\mathrm{Re}\alpha\le\Phi_e^{\pm}$, where $u^i(Q^{\pm}_{e,m^{\pm}})=\exp(ik\sum\limits_{1\le i\le m^{\pm}}d_i^{\pm}\cos(\varphi_\circ\mp\Phi_i^{\pm}))$, while $f(\alpha)-1/(\alpha-\varphi_\circ)$ has no singularity in the band $-\Phi\le\alpha\le\Phi$, even at infinity.

(b) $f(\alpha)$ is regular in the band $-\Phi_e^-\le\mathrm{Re}(\alpha)\le\Phi_e^+$, except at $\alpha=\varphi_\circ$ and possibly as $|\mathrm{Im}\alpha|\to\infty$, and thus $e^{A^{\pm}(\alpha)}f(\alpha)+B^{\pm}(\alpha)=f^{\pm}_{e,m^{\pm}}(\alpha)$ too. As $|\mathrm{Im}\alpha|\to\infty$, the function $f(\alpha)$ is $O(1)$ with $f(\alpha)=\frac{\mp i}{2}u(\rho=0)+o(\alpha^{-1})$ as $\mathrm{Im}\alpha\to\pm\infty$ when $-\Phi\le\mathrm{Re}(\alpha)\le\Phi$, and $O(e^{ik(c_{\pm}\cos\alpha+d_{\pm}\sin\alpha)})$ when $\Phi\le\pm\mathrm{Re}(\alpha)\le\Phi_e^{\pm}$, $c_{\pm}$ and $d_{\pm}$ being constants.

The spectral functions $f^{\pm}_{a,p}$ and $f^{\pm}_{b,p}$, attached to shifts of the origin at opposite ends of the segment p on the face $\pm$, with $f^{\pm}_{b,p}(\alpha) = f^{\pm}_{a,p+1}(\alpha)$ for $1 \leq p < m^{\pm}$ and $f^{\pm}_{b,m^{\pm}}(\alpha) = f^{\pm}_{e,m^{\pm}}(\alpha)$, are the spectral functions associated with the representations of the field in coordinates $(\rho^{\pm}_{a,p}, \varphi^{\pm}_{a,p})$ and $(\rho^{\pm}_{b,p}, \varphi^{\pm}_{b,p})$ with origins at opposite ends $Q^{\pm}_{a,p}$ and $Q^{\pm}_{b,p}$ of segment p. From single-face representation of spectral function (7), we have ,

$$
\begin{aligned}
f^{\pm}_{a,p}(\alpha) &= \frac{1}{2}\int_0^{d^{\pm}_p} (- iku(\rho'_{a,p}, \varphi)\sin(\alpha - \varphi) \pm \frac{\partial u}{\partial n}(\rho'_{a,p}, \varphi)) \\
&\times e^{-ik\rho'_{a,p}\cos(\alpha-\varphi)} d\rho'_{a,p}|_{\varphi=\pm\Phi^{\pm}_p} + e^{-ik\, d^{\pm}_p \cos(\alpha \mp \Phi^{\pm}_p)} f^{\pm}_{b,p}(\alpha), \\
f^{\pm}_{b,p}(\alpha) &= \frac{1}{2}\int_0^{d^{\pm}_p} (- iku(\rho'_{b,p}, \varphi)\sin(\alpha - \varphi) \mp \frac{\partial u}{\partial n}(\rho'_{b,p}, \varphi)) \\
&\times e^{-ik\rho'_{b,p}\cos(\alpha-\varphi)} d\rho'_{b,p}|_{\varphi=\pm(\Phi^{\pm}_p-\pi)} + e^{-ik\, d^{\pm}_p \cos(\alpha \mp (\Phi^{\pm}_p - \pi))} f^{\pm}_{a,p}(\alpha).
\end{aligned} \tag{9}
$$

and we note that the function $f^{\pm}_{1,p}$ and $f^{\pm}_{2,p}$, combinations of functions $f^{\pm}_{a,p}$ and $f^{\pm}_{b,p}$ following,

$$
\begin{aligned}
f^{\pm}_{1,p}(\alpha) &= f^{\pm}_{a,p}(\alpha) - e^{-ik\, d^{\pm}_p \cos(\alpha \mp \Phi^{\pm}_p)} f^{\pm}_{b,p}(\alpha) \\
f^{\pm}_{2,p}(\alpha) &= f^{\pm}_{b,p}(\alpha) - e^{ik\, d^{\pm}_p \cos(\alpha \mp \Phi^{\pm}_p)} f^{\pm}_{a,p}(\alpha) = - e^{ik\, d^{\pm}_p \cos(\alpha \mp \Phi^{\pm}_p)} f^{\pm}_{1,p}(\alpha),
\end{aligned} \tag{10}
$$

are directly related to the radiation by finite segment p.

Remark : Let define $v^{\pm}$ so that $v^{\pm}(M) = 0$ at $M \in L^{\pm}_{0,\infty}$, with non nul finite $\nabla v^{\pm}(M)$ on regular parts of $L^{\pm}_{0,\infty}$. We have $dv^{\pm}(M) = d(OM).\nabla v^{\pm}(M) = 0$ on regular parts of each face. Considering a constant a and the unit normal $\widehat{z}$ to 2D plane, with outward unit normal to face $L^{\pm}_{0,\infty}$, $\widehat{n}^{\pm} = \frac{a\nabla v^{\pm}(M)}{|a\nabla v^{\pm}(M)|} = \frac{\mp(\widehat{z} \wedge d(\overline{OM}))}{|d(\overline{OM})|}$, we note that,

$$
ds = |d(\overline{OM})| = \mp \widehat{n}^{\pm}.(\widehat{z} \wedge d(\overline{OM})),\ \widehat{n}^{\pm} ds = \mp (\widehat{z} \wedge d(\overline{OM})). \tag{11}
$$

4) Polygonal surface with impedance boundary conditions

4.1) Functional difference equations on $f^{\pm}_{e,m^{\pm}}$ for semi-infinite planes

In the case of an infinite polygonal scatterer with impedance boundary conditions along both semi-infinite planes $\varphi^{\pm}_e = \pm \Phi^{\pm}_e$, we have,

$$
\frac{\partial u}{\partial n^{\pm}}(\rho^{\pm}_e, \pm \Phi^{\pm}_e) - ik\sin\theta^{\pm}_e u(\rho^{\pm}_e, \pm \Phi^{\pm}_e) = 0, \tag{12}
$$

where $\partial(.)/\partial n^{\pm} = \widehat{n}^{\pm}\nabla(.) = \mp\,\partial(.)/\rho_e^{\pm}\partial\varphi_e^{\pm},\ 0 < |\mathrm{Re}(\theta_e^{\pm})| \leq \pi/2,\ \widehat{n}^{\pm}$ is the outward normal to the face $\varphi_e^{\pm} = \pm\,\Phi_e^{\pm}$, and from (8), the functions $f_{e,m^{\pm}}^{\pm}$ satisfy the functional equations

$$(\sin\alpha \pm \sin\theta_e^{\pm}) f_{e,m^{\pm}}^{\pm}(\alpha \pm \Phi_e^{\pm}) - (-\sin\alpha \pm \sin\theta_e^{\pm}) f_{e,m^{\pm}}^{\pm}(-\alpha \pm \Phi_e^{\pm}) = 0. \qquad (13)$$

If the scatterer is finite, the segments of both faces form a closed surface so that we can take $Q_{e,m^+}^{+} = Q_{e,m^-}^{-}$ with $\Phi_e^{+} + \Phi_e^{-} = 2\pi$. The fields on both semi-infinite planes $\varphi_e^{\pm} = \pm\,\Phi_e^{\pm}$ are then equal and their normal derivatives are opposite,

$$u(\rho_e^{+}, \Phi_e^{+}) = u(\rho_e^{-}, -\Phi_e^{-}),\ \frac{\partial u}{\partial n^{+}}(\rho_e^{+}, \Phi_e^{+}) = -\frac{\partial u}{\partial n^{-}}(\rho_e^{-}, -\Phi_e^{-}), \qquad (14)$$

and we derive from (8) that,

$$f_{e,m^+}^{+}(\alpha + \pi) = f_{e,m^-}^{-}(\alpha - \pi). \qquad (15)$$

Considering the definition of $f(\alpha)$ and $f_{e,m^{\pm}}^{\pm}$ given in (7) with properties (a) and (b), the functional equations (13) or (15) imply [1] that these functions are meromorphic with simple poles, regular in the band $-\Phi_e^{-} \leq \mathrm{Re}\alpha \leq \Phi_e^{+}$, except for the pole at $\alpha = \varphi_{\circ}$ and possibly at infinity when $\Phi \leq \pm\,\mathrm{Re}\alpha \leq \Phi_e^{\pm}$.

4.2) Functional equations due to boundary conditions on finite segments

Considering impedance boundary conditions on the segments in the form,

$$\begin{aligned}
&[\frac{\partial u}{\partial n}(\rho_{a,p}, \pm\,\Phi_p^{\pm}) - ik\sin\theta_p^{\pm} u(\rho_{a,p}, \pm\,\Phi_p^{\pm})]_{0 \leq \rho_{a,p} \leq d_p^{\pm}} = 0,\\
&[\frac{\partial u}{\partial n}(\rho_{b,p}, \pm\,(\Phi_p^{\pm} - \pi)) - ik\sin\theta_p^{\pm} u(\rho_{b,p}, \pm\,(\Phi_p^{\pm} - \pi))]_{0 \leq \rho_{b,p} \leq d_p^{\pm}} = 0, \qquad (16)
\end{aligned}$$

with $0 < |\mathrm{Re}(\theta_p^{\pm})| \leq \pi/2$, we can use the definition of $f_{1,p}^{\pm}$ and $f_{2,p}^{\pm}$ in (10) with the equalities given in (9), and deduce,

$$\begin{aligned}
&[(\sin\alpha \pm \sin\theta_p^{\pm}) f_{1,p}^{\pm}(\alpha + \varphi) - (-\sin\alpha \pm \sin\theta_p^{\pm}) f_{1,p}^{\pm}(-\alpha + \varphi)]|_{\varphi = \pm\Phi_p^{\pm}} = 0,\\
&[(\sin\alpha \mp \sin\theta_p^{\pm}) f_{2,p}^{\pm}(\alpha + \varphi) - (-\sin\alpha \mp \sin\theta_p^{\pm}) f_{2,p}^{\pm}(-\alpha + \varphi)]|_{\varphi = \pm(\Phi_p^{\pm} - \pi)} = 0. \qquad (17)
\end{aligned}$$

4.3) A remarkable relation between $f_{a,p}^{\pm}$ and $f_{b,p}^{\pm}$ deriving from (17) when (10) applies

It is known from (10) with (9) that $f_{1,p}^{\pm}$ has a 2π period, but it is remarkable that this property also derives, in an independent manner, from the difference equation (17). Considering (17)

with (10), this 2π periodicity will then implies the equivalence between the fields represented with $f^{\pm}_{a,p}$ and $f^{\pm}_{b,p}$.

Indeed, $f^{\pm}_{2,p}(\alpha \pm \Phi^{\pm}_p) = -e^{ik\,d^{\pm}_p \cos(\alpha)} f^{\pm}_{1,p}(\alpha \pm \Phi^{\pm}_p)$ from (10), and we derive from (17),

$$[(\sin\alpha \pm \sin\theta^{\pm}_p) f^{\pm}_{1,p}(\alpha+\varphi) - (-\sin\alpha \pm \sin\theta^{\pm}_p) f^{\pm}_{1,p}(-\alpha+\varphi)]|_{\varphi=\pm\Phi^{\pm}_p} = 0,$$
$$[(\sin\alpha \mp \sin\theta^{\pm}_p) f^{\pm}_{1,p}(\alpha+\varphi) - (-\sin\alpha \mp \sin\theta^{\pm}_p) f^{\pm}_{1,p}(-\alpha+\varphi)]|_{\varphi=\pm(\Phi^{\pm}_p-\pi)} = 0, \quad (18)$$

then,

$$(\sin(\alpha \pm \Phi^{\pm}_p) \pm \sin\theta^{\pm}_p) f^{\pm}_{1,p}(\alpha \pm 2\Phi^{\pm}_p) + (\sin(\alpha \pm \Phi^{\pm}_p) \mp \sin\theta^{\pm}_p) f^{\pm}_{1,p}(-\alpha) = 0,$$
$$(\sin(\alpha \pm \Phi^{\pm}_p) \pm \sin\theta^{\pm}_p) f^{\pm}_{1,p}(\alpha \pm 2(\Phi^{\pm}_p - \pi)) + (\sin(\alpha \pm \Phi^{\pm}_p) \mp \sin\theta^{\pm}_p) f^{\pm}_{1,p}(-\alpha) = 0 \quad (19)$$

This allows, after substraction, to deduce the 2π period of $f^{\pm}_{1,p}$,

$$f^{\pm}_{1,p}(\alpha) = f^{\pm}_{1,p}(\alpha + 2\pi), \quad (20)$$

which gives us, after writing $f^{\pm}_{1,p}(\alpha) = f^{\pm}_{a,p}(\alpha) - e^{-ik\,d^{\pm}_p \cos(\alpha \mp \Phi^{\pm}_p)} f^{\pm}_{b,p}(\alpha)$ from (10),

$$f^{\pm}_{a,p}(\pi+\alpha) - f^{\pm}_{a,p}(-\pi+\alpha) = e^{ik\,d^{\pm}_p \cos(\alpha \mp \Phi^{\pm}_p)} (f^{\pm}_{b,p}(\pi+\alpha) - f^{\pm}_{b,p}(-\pi+\alpha)). \quad (21)$$

Consequently, the equations (17), when (10) applies, implies that the far field given by $f^{\pm}_{a,p}$ is the far field given by $f^{\pm}_{b,p}$ with phase center shifted by Δ, so that the field given by $f^{\pm}_{a,p}$ is the one given by $f^{\pm}_{b,p}$ relatively to the two origins at ends of the segment p.

In whole generality, it is remarkable that functional difference equations for boundary conditions on each segment has for additional consequence the equivalence of fields represented with any $f^{\pm}_{a,m}$ (or $f^{\pm}_{b,n}$), refering to any origins on the impedance polygons.

5) Formulation of the three-part polygonal problem : spectral functions in Sommerfeld-Maliuzhinets representation and functional difference equations in complex plane.

5.1) Definition

We consider the diffraction of an incident plane wave by a semi-infinite impedance polygon divided into three parts, each one characterized by relative surface impedances $\sin\theta_-$, $\sin\theta_1$, and $\sin\theta_+$, with $0 < \mathrm{Re}(\theta_\pm) \leq \pi/2$ and $0 < \mathrm{Re}(\theta_1) \leq \pi/2$ (strict passivity). This means that $m^+ = 1, \quad m^- = 0, \quad \Phi^+_1 = \frac{\pi}{2}, \quad \Phi^+_e = \frac{\pi}{2} + \Phi_b, \quad \Phi^-_e = -\frac{\pi}{2} - \Phi_a, \quad f_a = f^+_{a,1} = f^-_e$, $f_b = f^+_{b,1} = f^+_e$.

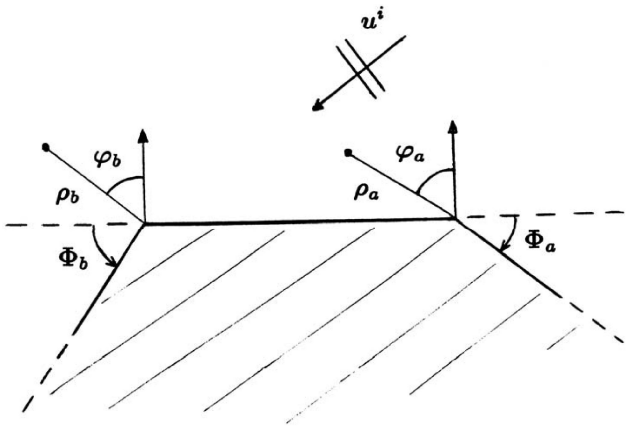


figure 3 : geometry of the three-part polygon

The functions f_a and f_b are the spectral functions associated with the Sommerfeld-Maliuzhinets representation of the field, in coordinate systems (ρ_a, φ_a) and (ρ_b, φ_b), with origins at opposite ends of the finite segment (see geometry on figure). We have, in (ρ_a, φ_a) coordinates,

$$\begin{aligned}
&(\rho_a \in]0,\infty[, \varphi_a = -\frac{\pi}{2} - \Phi_a) \text{ with } \partial u/\partial n - ik\sin\theta_- u = 0,\\
&(\rho_a \in [0,\Delta], \varphi_a = \frac{\pi}{2}) \text{ with } \partial u/\partial n - ik\sin\theta_1 u = 0,
\end{aligned} \tag{22}$$

with the incident field $u^i = e^{ik\rho_a\cos(\varphi_a - \varphi_\circ)}$, and, in coordinates (ρ_b, φ_b),

$$\begin{aligned}
&(\rho_b \in [0,\Delta], \varphi_b = -\frac{\pi}{2}) \text{ with } \partial u/\partial n - ik\sin\theta_1 u = 0,\\
&(\rho_b \in]0,\infty[, \varphi_b = \frac{\pi}{2} + \Phi_b) \text{ with } \partial u/\partial n - ik\sin\theta_+ u = 0,
\end{aligned} \tag{23}$$

with the incident field $u^i = e^{ik(\rho_b\cos(\varphi_b - \varphi_\circ) + \Delta\sin\varphi_\circ)}$.

The properties (a) and (b) give us that,

$$(f_a(\alpha) - 1/(\alpha - \varphi_\circ)) \text{ and } (f_b(\alpha) - e^{ik\Delta\sin\varphi_\circ}/(\alpha - \varphi_\circ)) \tag{24}$$

are regular in the strips $-\frac{\pi}{2} - \Phi_a \le \mathrm{Re}\alpha \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le \mathrm{Re}\alpha \le \frac{\pi}{2} + \Phi_b$ respectively, with $-\frac{\pi}{2} - \Phi_a < \varphi_\circ < \frac{\pi}{2} + \Phi_b$, with $f_{a(b)}(\alpha) = \frac{\mp i}{2} u(\rho_{a(b)} = 0) + o(\alpha^{-1})$ as $\mathrm{Im}\alpha \to \pm\infty$ in these regions from single face expressions of f_a and f_b.

Considering the definition and $f_e^- = f_a$, $f_e^+ = f_b$, we obtain, relatively to integration on piecewise straight faces,

$$f_a(\alpha) = e^{-ik\Delta\sin\alpha} f_b(\alpha) - \frac{1}{2}\int_0^{\Delta} (- iku(\rho_a, \frac{\pi}{2})\cos\alpha - \frac{\partial u}{\partial n}(\rho_a, \frac{\pi}{2}))e^{-ik\rho_a\sin\alpha} d\rho_a,$$
$$f_b(\alpha) = e^{ik\Delta\sin\alpha} f_a(\alpha) - \frac{1}{2}\int_0^{\Delta} (iku(\rho_b, -\frac{\pi}{2})\cos\alpha + \frac{\partial u}{\partial n}(\rho_b, -\frac{\pi}{2}))e^{+ik\rho_b\sin\alpha} d\rho_b, \quad (25)$$

for any complex angle α, while, relatively to integration on straight semi-infinite faces,

$$f_a(\alpha - \varphi_-) = \frac{1}{2}\int_0^{\infty} (- iku(\rho_a, -\varphi_-)\sin\alpha - \frac{\partial u}{\partial n}(\rho_a, -\varphi_-))e^{-ik\rho_a\cos\alpha} d\rho_a, \quad (26)$$

as $\mathrm{Re}(ik(\cos(\varphi_\circ + \varphi_-) - \cos\alpha)) < 0$, $|\mathrm{Re}\alpha| < \pi$, with $\varphi_- = \frac{\pi}{2} + \Phi_a$, and

$$f_b(\alpha + \varphi_+) = \frac{1}{2}\int_0^{\infty} (- iku(\rho_b, \varphi_+)\sin\alpha + \frac{\partial u}{\partial n}(\rho_b, \varphi_+))e^{-ik\rho_b\cos\alpha} d\rho_b, \quad (27)$$

as $\mathrm{Re}(ik(\cos(\varphi_\circ - \varphi_+) - \cos\alpha)) < 0$, $|\mathrm{Re}\alpha| < \pi$, with $\varphi_+ = \frac{\pi}{2} + \Phi_b$.

5.2) Functional difference equations in complex plane for f_a and f_b

5.2.1) Functional equations for the central strip and semi-infinite planes

We now derive functional equations on the spectral functions from the boundary conditions on the central strip (of finite size) and on the semi-infinite planes of the three-part impedance polygon (figure 3). Using the boundary conditions on the central strip (22), and letting

$$f_1(\alpha + \frac{\pi}{2}) = f_a(\alpha + \frac{\pi}{2}) - e^{-ik\Delta\cos\alpha} f_b(\alpha + \frac{\pi}{2}),$$
$$f_2(\alpha - \frac{\pi}{2}) = f_b(\alpha - \frac{\pi}{2}) - e^{-ik\Delta\cos\alpha} f_a(\alpha - \frac{\pi}{2}) = -e^{-ik\Delta\cos\alpha} f_1(\alpha - \frac{\pi}{2}), \quad (28)$$

we obtain, as described in (17),

$$(\sin\alpha + \sin\theta_1) f_1(\alpha + \frac{\pi}{2}) - (- \sin\alpha + \sin\theta_1) f_1(- \alpha + \frac{\pi}{2}) = 0,$$
$$(\sin\alpha - \sin\theta_1) f_2(\alpha - \frac{\pi}{2}) - (- \sin\alpha - \sin\theta_1) f_2(- \alpha - \frac{\pi}{2}) = 0, \quad (29)$$

while, using the boundary conditions on both semi-infinite planes (23), we obtain, as described in (13),

$$(\sin\alpha + \sin\theta_+) f_b(\alpha + \frac{\pi}{2} + \Phi_b) - (- \sin\alpha + \sin\theta_+) f_b(- \alpha + \frac{\pi}{2} + \Phi_b) = 0,$$
$$(\sin\alpha - \sin\theta_-) f_a(\alpha - \frac{\pi}{2} - \Phi_a) - (- \sin\alpha - \sin\theta_-) f_a(- \alpha - \frac{\pi}{2} - \Phi_a) = 0. \quad (30)$$

5.2.2) A remarkable relation between f_a and f_b deriving from (29) when (28) applies

We follow a development similar to the one used to obtain (19)-(21). Considering the relation between f_1 and f_2 in (28), we have from (29),

$$\begin{aligned}(\sin\alpha+\sin\theta_1)f_1(\alpha+\frac{\pi}{2})-(-\sin\alpha+\sin\theta_1)f_1(-\alpha+\frac{\pi}{2})=0,\\(\sin\alpha-\sin\theta_1)f_1(\alpha-\frac{\pi}{2})-(-\sin\alpha-\sin\theta_1)f_1(-\alpha-\frac{\pi}{2})=0,\end{aligned}\tag{31}$$

and thus,

$$\begin{aligned}(\cos\alpha+\sin\theta_1)f_1(\pi+\alpha)+(\cos\alpha-\sin\theta_1)f_1(-\alpha)=0,\\-(\cos\alpha+\sin\theta_1)f_1(-\pi+\alpha)-(\cos\alpha-\sin\theta_1)f_1(-\alpha)=0.\end{aligned}\tag{32}$$

This implies, after substraction,

$$f_1(\pi+\alpha)-f_1(-\pi+\alpha)=0,\tag{33}$$

and, considering the expression of f_1 with f_a and f_b given in (28),

$$f_a(\pi+\alpha)-f_a(-\pi+\alpha)=e^{ik\Delta\sin\alpha}(f_b(\pi+\alpha)-f_b(-\pi+\alpha)).\tag{34}$$

Consequently, it is worth noticing that the equations (29), when expressed with f_a and f_b from (28), which initially concern the impedance boundary condition on the segment, has for additional remarkable consequence that the far field given by f_a is the far field given by f_b with phase center shifted by Δ, so that the field given by f_a is the one given by f_b relatively to the two origins at both ends of the segment (figure 3).

5.3) Functional difference equations for $f_{br}(\alpha)=f_b(\alpha+\frac{\Phi_b}{2})$, $f_{ar}(\alpha)=f_a(\alpha-\frac{\Phi_a}{2})$

We can now derive, from (29)-(30), difference equations on $f_{br}(\alpha)=f_b(\alpha+\frac{\Phi_b}{2})$, $f_{ar}(\alpha)=f_a(\alpha-\frac{\Phi_a}{2})$, corresponding to both discontinuities when influenced each other.

If we let $f_{br}(\alpha-\frac{\Phi_b}{2})=f_b(\alpha)$ with $\Phi_+=\frac{\pi}{2}+\frac{\Phi_b}{2}$, we have

$$(\sin\alpha+\sin\theta_+)f_{br}(\alpha+\Phi_+)-(-\sin\alpha+\sin\theta_+)f_{br}(-\alpha+\Phi_+)=0,\tag{35}$$

and

$$\begin{aligned}&(\sin\alpha-\sin\theta_1)f_{br}(\alpha-\Phi_+)-(-\sin\alpha-\sin\theta_1)f_{br}(-\alpha-\Phi_+)\\&=e^{-ik\Delta\cos\alpha}((\sin\alpha-\sin\theta_1)f_a(\alpha-\frac{\pi}{2})-(-\sin\alpha-\sin\theta_1)f_a(-\alpha-\frac{\pi}{2}))\\&=2e^{-ik\Delta\cos\alpha}S_b^-(\alpha).\end{aligned}\tag{36}$$

If we let $f_{ar}(\alpha+\frac{\Phi_a}{2})=f_a(\alpha)$ with $\Phi_-=\frac{\pi}{2}+\frac{\Phi_a}{2}$, we have

$$\begin{aligned}&(\sin\alpha+\sin\theta_1)f_{ar}(\alpha+\Phi_-)-(-\sin\alpha+\sin\theta_1)f_{ar}(-\alpha+\Phi_-)\\&=e^{-ik\Delta\cos\alpha}((\sin\alpha+\sin\theta_1)f_b(\alpha+\frac{\pi}{2})-(-\sin\alpha+\sin\theta_1)f_b(-\alpha+\frac{\pi}{2}))\\&=2e^{-ik\Delta\cos\alpha}S_a^+(\alpha),\end{aligned}\tag{37}$$

and

$$(\sin\alpha-\sin\theta_-)f_{ar}(\alpha-\Phi_-)-(-\sin\alpha-\sin\theta_-)f_{ar}(-\alpha-\Phi_-)=0.\tag{38}$$

From the properties of f_a and f_b, the functions

$$(f_{ar}(\alpha)-1/(\alpha-\varphi_{\circ,a}) \text{ and } (f_{br}(\alpha)-e^{ik\Delta\sin\varphi_\circ}/(\alpha-\varphi_{\circ,b}))\tag{39}$$

are respectively regular in the bands $|\mathrm{Re}\alpha|\le\Phi_-$ and $|\mathrm{Re}\alpha|\le\Phi_+$, with $\varphi_{\circ,a}=\varphi_\circ+\frac{\Phi_a}{2}$ and $\varphi_{\circ,b}=\varphi_\circ-\frac{\Phi_b}{2},\ -\frac{\pi}{2}-\Phi_a<\varphi_\circ<\frac{\pi}{2}+\Phi_b$, and, in these regions,

$$f_{ar}(\alpha)=\mp\frac{i}{2}u(\rho_a=0)+o(\alpha^{-1}) \text{ and } f_{br}(\alpha)=\mp\frac{i}{2}u(\rho_b=0)+o(\alpha^{-1}),\tag{40}$$

as $\mathrm{Im}\alpha\to\pm\infty$. Besides, we remark that, for $\Delta=0$, the three-part polygon is reduced to a wedge with exterior angle $2\Phi_d=(2(\Phi_++\Phi_-)-\pi)$ and face impedances $\sin\theta_\pm$. In this case, we have $f_{br}(\alpha-\Phi_-+\frac{\pi}{2})=f_{ar}(\alpha+\Phi_+-\frac{\pi}{2})=f_0^{ab}(\alpha)$, where f_0^{ab}, refering to the solution for an impedance wedge (see appendix), is given by

$$f_0^{ab}(\alpha,\varphi_\circ)=\frac{\pi}{2\Phi_d}\frac{\Psi_{+-}(\alpha)\cos(\frac{\pi\varphi_{\circ,d}}{2\Phi_d})}{\Psi_{+-}(\varphi_{\circ,d})(\sin(\frac{\pi\alpha}{2\Phi_d})-\sin(\frac{\pi\varphi_{\circ,d}}{2\Phi_d}))},\tag{41}$$

where $\varphi_{\circ,d}=\varphi_\circ+(\Phi_a-\Phi_b)/2,\ -\Phi_d<\varphi_{\circ,d}<\Phi_d$.

6) The integral expressions and integral functional equations for the three-part impedance polygon

6.1) Elementary integral solutions for difference equations

Considering the theory of functional equations, the analytic function $\chi(\alpha)$ satisfying,

$$\chi(\alpha\pm\Phi)-\chi(-\alpha\pm\Phi)=\vartheta^\pm(\alpha),\tag{42}$$

and regular as $|\mathrm{Re}\alpha| \leq \Phi$, is given in the strip $|\mathrm{Re}\alpha| < \Phi$ [27] [31] by,

$$\chi(\alpha) = \frac{\chi(i\infty) + \chi(-i\infty)}{2} + \frac{-i}{8\Phi}\int_{-i\infty}^{+i\infty} d\alpha'(\vartheta^+(\alpha')\tan(\frac{\pi}{4\Phi}(\alpha + \Phi - \alpha')) - \vartheta^-(\alpha')\tan(\frac{\pi}{4\Phi}(\alpha - \Phi - \alpha'))), \tag{43}$$

when the functions $\vartheta^{\pm}(\alpha)$ are regular and summable on the imaginary axis, with $\chi(i\infty) - \chi(-i\infty) = 0$. The functional equations (35)-(38) can be reduced into the form (42), by using the solutions $\Psi_{+1}(\alpha, \Phi_+)$ (resp. $\Psi_{1-}(\alpha, \Phi_-)$), without pole or zero and $O(\cos(\pi\alpha/2\Phi_+))$ (resp. $O(\cos(\pi\alpha/2\Phi_-))$) in the band $|\mathrm{Re}\alpha| \leq \Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$ (resp. $|\mathrm{Re}\alpha| \leq \Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$), of the equations without second members of the type,

$$\begin{aligned}(\sin\alpha + \sin\theta_l)\Psi_{lr}(\alpha + \Phi, \Phi) - (-\sin\alpha + \sin\theta_l)\Psi_{lr}(-\alpha + \Phi, \Phi) = 0,\\ (\sin\alpha - \sin\theta_r)\Psi_{lr}(\alpha - \Phi, \Phi) - (-\sin\alpha - \sin\theta_r)\Psi_{lr}(-\alpha - \Phi, \Phi) = 0,\end{aligned} \tag{44}$$

with $l \equiv +$, $r \equiv 1$, $\Phi = \Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$ for $\Psi_{+1}(\alpha, \Phi_+)$, and $l \equiv 1$, $r \equiv -$, $\Phi = \Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$ for $\Psi_{1-}(\alpha, \Phi_-)$ (see details on Ψ_{lr}, named $\Psi_{+-}(\alpha, \Phi)$ in appendix).

6.2) Coupled integral expressions of $f_{br}(\alpha)$ and $f_{ar}(\alpha)$

Considering (35)-(38), we use the solution for $S_b^- \equiv 0$ (resp. $S_a^+ \equiv 0$) (see appendix A) with pole due to incident plane wave, and substract them from the general solution. We then obtain unknowns regular as $|\mathrm{Re}\alpha| < \Phi_+$ (resp. $|\mathrm{Re}\alpha| < \Phi_-$) for which we can use the integral expression (42)-(43).

We can then write, $f_{br}(\alpha)$ for $-\Phi_+ < \mathrm{Re}\alpha < 3\Phi_+$, and $f_{ar}(\alpha)$ for $-3\Phi_- < \mathrm{Re}\alpha < \Phi_-$, following,

$$\begin{aligned}\frac{f_{br}(\alpha)}{\Psi_{+1}(\alpha)} &= \frac{i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^-(\alpha')e^{-ik\Delta\cos\alpha'}\tan(\frac{\pi}{4\Phi_+}(\alpha - \Phi_+ - \alpha'))}{(\sin\alpha' - \sin\theta_1)\Psi_{+1}(\alpha' - \Phi_+)}d\alpha' + \chi_b^i(\alpha)\\ &= \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^-(\alpha')/(\sin\alpha' - \sin\theta_1)}{\Psi_{+1}(\alpha' - \Phi_+)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_+)) + \cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha' + \chi_b^i(\alpha)\\ &= \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{f_{ar}(\alpha' - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha' - \Phi_+)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_+)) + \cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha' + \chi_b^i(\alpha),\end{aligned} \tag{45}$$

and,

$$\frac{f_{ar}(\alpha)}{\Psi_{1-}(\alpha)} = \frac{-i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{S_a^+(\alpha')e^{-ik\Delta\cos\alpha'}\tan(\frac{\pi}{4\Phi_-}(\alpha+\Phi_- -\alpha'))}{(\sin\alpha'+\sin\theta_1)\Psi_{1-}(\alpha'+\Phi_-)}d\alpha' + \chi_a^i(\alpha)$$
$$= \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{S_a^+(\alpha')/(\sin\alpha'+\sin\theta_1)}{\Psi_{1-}(\alpha'+\Phi_-)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_-)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha' + \chi_a^i(\alpha)$$
$$= \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{f_{br}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{1-}(\alpha'+\Phi_-)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_-)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha' + \chi_a^i(\alpha), \qquad (46)$$

with $\varphi_{\circ,b} = \varphi_\circ - \frac{\Phi_b}{2}$, as $-\frac{\pi}{2} < \varphi_\circ < \frac{\pi}{2} + \Phi_b$, and $\varphi_{\circ,a} = \varphi_\circ + \frac{\Phi_a}{2}$, as $-\frac{\pi}{2} - \Phi_a < \varphi_\circ < \frac{\pi}{2}$ while the source terms are given by,

$$\chi_b^i(\alpha) = e^{ik\Delta\sin\varphi_\circ}\,\chi_{+1}^i(\alpha) = \frac{\pi}{2\Phi_+}\left(\frac{e^{ik\Delta\sin\varphi_\circ}\cos\frac{\pi\varphi_{\circ,b}}{2\Phi_+}}{\Psi_{+1}(\varphi_{\circ,b})(\sin\frac{\pi\alpha}{2\Phi_+} - \sin\frac{\pi\varphi_{\circ,b}}{2\Phi_+})}\right),$$
$$\chi_a^i(\alpha) = \frac{\pi}{2\Phi_-}\left(\frac{\cos\frac{\pi\varphi_{\circ,a}}{2\Phi_-}}{\Psi_{1-}(\varphi_{\circ,a})(\sin\frac{\pi\alpha}{2\Phi_-} - \sin\frac{\pi\varphi_{\circ,a}}{2\Phi_-})}\right), \qquad (47)$$

with $\mathrm{Re}(\theta_\pm) \neq 0$ and $\mathrm{Re}(\theta_1) \neq 0$.

6.3) Integral equations when $\Phi_{a,b} > -\frac{\pi}{2}$ (derived from coupled integral expressions)

From previous integral expressions (45)-(46), we can write,

$$\frac{f_{br}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})} = \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{S_b^-(\alpha')}{(\sin\alpha'-\sin\theta_1)} \frac{1}{\Psi_{+1}(\alpha'-\Phi_+)}$$
$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha' + \chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}), \qquad (48)$$

as $-\Phi_+ < \mathrm{Re}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}) < 3\Phi_+$, and,

$$\frac{f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})} = \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{S_a^+(\alpha')}{(\sin\alpha'+\sin\theta_1)} \frac{1}{\Psi_{1-}(\alpha'+\Phi_-)}$$
$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha' + \chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}), \qquad (49)$$

as $-3\Phi_- < \mathrm{Re}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}) < \Phi_-$, when $-\min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_a) < \varphi_\circ < \min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_b)$. When $\Phi_{a,b} > -\frac{\pi}{2}$, we can take α purely imaginary, where the functions,

$$\frac{S_b^-(\alpha)}{(\sin\alpha-\sin\theta_1)}=\frac{1}{2}(f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})\ -\frac{(-\sin\alpha-\sin\theta_1)f_{ar}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{(\sin\alpha-\sin\theta_1)})$$
$$\frac{S_a^+(\alpha)}{(\sin\alpha+\sin\theta_1)}=\frac{1}{2}(f_{br}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})-\frac{(-\sin\alpha+\sin\theta_1)f_{br}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{(\sin\alpha+\sin\theta_1)}),\quad(50)$$

vanish at infinity, contrary to $f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})$ and $f_{br}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})$ in general.

Choosing these functions as unknowns, we derive the integral equations for imaginary α,

$$\frac{S_a^+(\alpha)}{(\sin\alpha+\sin\theta_1)}=\frac{-i}{8\Phi_+}\int_{-i\infty}^{+i\infty}\frac{S_b^-(\alpha')}{(\sin\alpha'-\sin\theta_1)}\times\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\Psi_{+1}(\alpha'-\Phi_+)}$$
$$[\frac{\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}+\frac{(\sin\alpha-\sin\theta_1)}{(\sin\alpha+\sin\theta_1)}\frac{\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\cos(\frac{\pi}{2\Phi_+}(\alpha+\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}]d\alpha'$$
$$+\frac{1}{2}[\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})\chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})$$
$$+\frac{(\sin\alpha-\sin\theta_1)\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{(\sin\alpha+\sin\theta_1)}\chi_b^i(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})],\quad(51)$$

and

$$\frac{S_b^-(\alpha)}{(\sin\alpha-\sin\theta_1)}=\frac{i}{8\Phi_-}\int_{-i\infty}^{+i\infty}\frac{S_a^+(\alpha')}{(\sin\alpha'+\sin\theta_1)}\times\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\Psi_{1-}(\alpha'+\Phi_-)}$$
$$[\frac{\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-})}+\frac{(\sin\alpha+\sin\theta_1)}{(\sin\alpha-\sin\theta_1)}\frac{\Psi_{1-}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\cos(\frac{\pi}{2\Phi_-}(\alpha-\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-})}]d\alpha'$$
$$+\frac{1}{2}[\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})$$
$$+\frac{(\sin\alpha+\sin\theta_1)\Psi_{1-}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{(\sin\alpha-\sin\theta_1)}\chi_a^i(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})],\quad(52)$$

where we remark that $S_b^-(-\alpha')=-S_b^-(\alpha')$, $S_a^+(-\alpha')=-S_a^+(\alpha')$. We note that,

$$\frac{\pi}{2}-\frac{\Phi_{a(,b)}}{2}=\frac{\pi}{2}+\frac{\Phi_{a(,b)}}{2}-\Phi_{a(,b)}=\Phi_{-(,+)}-\Phi_{a(,b)},$$
$$\frac{(\sin\alpha\mp\sin\theta_1)}{(\sin\alpha\pm\sin\theta_1)}=1-\frac{\pm\,2\sin\theta_1}{(\sin\alpha\pm\sin\theta_1)},\quad(53)$$

and,

$$\Psi_{+1,1-}(\alpha) = A_0 \cos(\pi\alpha/2\Phi_{+,-})(1+O(\alpha^{\nu}e^{-|\mu \mathrm{Im}\alpha|})),$$
$$\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}) = -\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})e^{\frac{\mp i\pi\Phi_b}{\Phi_+}}(1+O(\alpha^{\nu}e^{-|\mu \mathrm{Im}\alpha|})),$$
$$\Psi_{1-}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}) = -\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})e^{\frac{\pm i\pi\Phi_a}{\Phi_-}}(1+O(\alpha^{\nu}e^{-|\mu \mathrm{Im}\alpha|})), \qquad (54)$$

when $\mathrm{Im}\alpha \to \pm\infty$, so that the kernels, bounded on the path of integration, are,

$$O\Big(\frac{|e^{-ik\Delta\cos\alpha'}|}{\cos(a\alpha)+c\cos(a\alpha')}\Big(\frac{\cos(a\alpha)\cos(a\alpha')}{\cos(a\alpha)+c\cos(a\alpha')}+\frac{d\cos(a\alpha)}{1+\cos(b\alpha)}\Big)\Big), \qquad (55)$$

as $|\mathrm{Im}\alpha'| > A$ and $|\mathrm{Im}\alpha| > B$, and the source term is $O(1/\cos(c\alpha))$ as $|\mathrm{Im}\alpha| \to \infty$, with postive real constants A, B, a, b, c, and d.

The kernels are then square integrable as $\mathrm{Im}(k\Delta) < 0$, i.e. $|\arg(ik)| < \pi/2$, when $k\Delta \neq 0$, and we can consider this set of coupled equations with,

$$\int_{-\infty}^{+\infty} |\frac{S_b^-(ix)}{(\sin(ix)-\sin\theta_1)}|^2 dx < \infty, \int_{-\infty}^{+\infty} |\frac{S_a^+(ix)}{(\sin(ix)+\sin\theta_1)}|^2 dx < \infty, \qquad (56)$$

as a system of Fredholm equations of second kind in $L^2(iR)$. Let us note that the solution of our functional equations for $k\Delta = 0$ is the one for an impedance wedge, uniquely defined from [20] with [33].

We now explain how to extend this domain to enclose the case $|\arg(ik)| = \pi/2$, by deformation of the integration path.

6.4) Modification of integration path, and extension as $|\arg(ik)-\epsilon| < \pi/2$

6.4.1) The new path of integration $\mathcal{C}_\epsilon$

We note that $e^{-ik\Delta\cos\alpha'}$ vanish to infinity when $\alpha' \to \pm i\infty$ as $|\arg(ik)| < \pi/2$, $k\Delta \neq 0$. However, it is possible to deform the domain of integration (possible by analyticity), so that $e^{-ik\Delta\cos\alpha'} \to 0$ at infinity as $\arg(ik) \leq \pi/2$ (even when k is real), without capturing any pole in particular first singularity of the integrand at $|\mathrm{Re}\alpha| = |\mathrm{Re}(\theta_1)|$ with $\mathrm{Re}(\theta_1) \neq 0$.

For that, we can modify the path, following,

$$I = \int_{-i\infty}^{+i\infty} K(\alpha',\alpha)v(\alpha')d\alpha' \to I = \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} K(\alpha',\alpha)v(\alpha')d\alpha', \qquad (57)$$

changing the initial path $\alpha' = ix'$ with $x \in R$ for the path $\mathcal{C}_\epsilon$ with $\alpha' = g_\epsilon(x')$ following,

$$\begin{aligned} &\mathcal{C}_\epsilon \equiv \{\alpha' = g_\epsilon(x') \text{ with } g_\epsilon(x') = ix' + \epsilon \text{tanh} x',\ x' \in R,\ |\epsilon| < |\text{Re}(\theta_1)|\}, \\ &ik\text{cos}(g_\epsilon(x)) = (e^{-x+i(\arg(ik)+\epsilon\text{tanh}x)} + e^{x+i(\arg(ik)-\epsilon\text{tanh}x)})/2,\ x \in R, \end{aligned} \tag{58}$$

Concerning the sign of ϵ, we note that the exponential term $e^{-ik\Delta\cos\alpha'}$ in $K(\alpha', \alpha)$ verifies,

$$|e^{-ik\Delta\cos\alpha'}| = O(e^{-\text{Re}(ike^{-i\epsilon})\cosh x'}) \text{ on } \mathcal{C}_\epsilon, \tag{59}$$

and we can then extend the domain of convergence if we take the real constant ϵ with,

$$|\arg(ik) - \epsilon| < \pi/2, \text{ i.e. } \text{sign}(\epsilon\arg(ik)) > 0,\ |\epsilon| < |\text{Re}(\theta_1)|, \tag{60}$$

that we can let tend to zero. Thereafter, $\mathcal{C}_\epsilon$ will be considered with the conditions (60) on ϵ.

6.4.2) Analyticity of integral expressions of f_{br} and f_{ar} with the new path $\mathcal{C}_\epsilon$

The integral (45) and (46) with new path of integration $\mathcal{C}_\epsilon$ give us expressions of,

$$f_{br}(\alpha) \text{ and } f_{ar}(\alpha), \tag{61}$$

which are $O(1)$ and regular, respectively as,

$$\begin{aligned} &\text{Re}(g_\epsilon(\text{Im}(\alpha))) - \Phi_+ < \text{Re}(\alpha) < \text{Re}(g_\epsilon(\text{Im}(\alpha))) + 3\Phi_+, \\ &\text{Re}(g_\epsilon(\text{Im}(\alpha))) - 3\Phi_- < \text{Re}(\alpha) < \text{Re}(g_\epsilon(\text{Im}(\alpha))) + \Phi_-, \end{aligned} \tag{62}$$

where $\text{Re}(g_\epsilon(\text{Im}(\alpha))) = \epsilon\text{tanh}(\text{Im}(\alpha)),\ \text{Im}(\alpha) \in R,\ |\text{Re}(g_\epsilon(\text{Im}(\alpha)))| \leq |\epsilon|$. We note that these functions satisfy by analytic continuation the functional equations (35)-(38), and that we have $f_{br,ar}(-i\infty) = -f_{br,ar}(+i\infty)$.

6.4.3) Integral equations with $\mathcal{C}_\epsilon$ become $L^2(R)$ as $|\arg(ik) - \epsilon| \leq \pi/2$

Changing the path for $\mathcal{C}_\epsilon$, the integral term I in (57) becomes,

$$\int_{-\infty}^{+\infty} (K(g_\epsilon(x'), g_\epsilon(x))v(\alpha(x'))\partial_{x'} g_\epsilon(x'))dx', \tag{63}$$

for $|\arg(ik) - \epsilon| < \pi/2$. Replacing the integration path by $\mathcal{C}_\epsilon$ in integral equations (51) and (52), we then derive,

$$\frac{S_a^+(g_\epsilon(x))}{(\sin(g_\epsilon(x))+\sin\theta_1)} = \frac{-i}{8\Phi_+}\int_{-\infty}^{+\infty}\frac{S_b^-(g_\epsilon(x'))}{(\sin(g_\epsilon(x'))-\sin\theta_1)}K_{b,a}(g_\epsilon(x'),g_\epsilon(x))\partial_{x'}g_\epsilon(x')dx'$$
$$+\frac{1}{2}[\Psi_{+1}(g_\epsilon(x)+\frac{\pi}{2}-\frac{\Phi_b}{2})\chi_b^i(g_\epsilon(x)+\frac{\pi}{2}-\frac{\Phi_b}{2})$$
$$+\frac{(\sin(g_\epsilon(x))-\sin\theta_1)\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{(\sin(g_\epsilon(x))+\sin\theta_1)}\chi_b^i(-g_\epsilon(x)+\frac{\pi}{2}-\frac{\Phi_b}{2})], \tag{64}$$

and

$$\frac{S_b^-(g_\epsilon(x))}{(\sin(g_\epsilon(x))-\sin\theta_1)} = \frac{i}{8\Phi_-}\int_{-\infty}^{+\infty}\frac{S_a^+(g_\epsilon(x'))}{(\sin(g_\epsilon(x'))+\sin\theta_1)}K_{a,b}(g_\epsilon(x'),g_\epsilon(x))\partial_{x'}g_\epsilon(x')dx'$$
$$+\frac{1}{2}[\Psi_{1-}(g_\epsilon(x)-\frac{\pi}{2}+\frac{\Phi_a}{2})\chi_a^i(g_\epsilon(x)-\frac{\pi}{2}+\frac{\Phi_a}{2})$$
$$+\frac{(\sin(g_\epsilon(x))+\sin\theta_1)\Psi_{1-}(-g_\epsilon(x)-\frac{\pi}{2}+\frac{\Phi_a}{2})}{(\sin(g_\epsilon(x))-\sin\theta_1)}\chi_a^i(-g_\epsilon(x)-\frac{\pi}{2}+\frac{\Phi_a}{2})]. \tag{65}$$

for $x\in R$, with $g_\epsilon(x)=ix+\epsilon\tanh x$. Considering (54), the kernels $K(g_\epsilon(x'),g_\epsilon(x))$ in integral terms satisfy,

$$|K(g_\epsilon(x'),g_\epsilon(x))|$$
$$=O\Big(\frac{|e^{-\gamma\cosh x'}|}{\cosh(ax)+c\cosh(ax')}\Big(\frac{\cosh(ax)\cosh(x\alpha')}{\cosh(ax)+c\cosh(ax')}+\frac{d\cosh(ax)}{1+\cosh(bx)}\Big)\Big), \tag{66}$$

as $|x|>A$ and $|x'|>A'$, with positive real constants A, A', a, b, c, d, and $\gamma=\mathrm{Re}(ike^{-i\epsilon})>0$, so that we have,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|K_{a,b}(g_\epsilon(x'),g_\epsilon(x))\partial_{x'}g(x')|^2dx'dx<\infty, \tag{67}$$

as $|\arg(ik)-\epsilon|<\pi/2,\ k\Delta\neq 0$. We then notice that we obtain a system of Fredholm equations of second kind in $L^2(R)$ on $\mathcal{C}_\epsilon$, and thus for any k with $|\arg(ik)|\leq\pi/2$.

6.5) Existence and uniqueness for integral equations on $\mathcal{C}_\epsilon$

6.5.1) Uniqueness

Concerning uniqueness, we consider the solution of integral equations (64)-(65) along integration path $\mathcal{C}_\epsilon\equiv\{\alpha=ix+\epsilon\tanh x,\ x\in R\}$ when χ_a^i and χ_b^i are nul, i.e. as we have no source terms, in $L^2(R)$ as $|\arg(ik)|\leq\pi/2$, .

From analyticity of kernels in our integrals, the solutions $S_b^-(\alpha')$ and $S_a^+(\alpha')$ are odd meromorphic functions (the only singularities are poles). Therefore, we define two functions $f^{br}$and f^{ar}, following,

$$\frac{f^{br}(\alpha)}{\Psi_{+1}(\alpha)} = \frac{i}{4\Phi_+}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \frac{S_b^-(\alpha')e^{-ik\Delta\cos\alpha'}\tan(\frac{\pi}{4\Phi_+}(\alpha-\Phi_+-\alpha'))}{(\sin\alpha'-\sin\theta_1)\Psi_{+1}(\alpha'-\Phi_+)}d\alpha'$$
$$= \frac{-i}{4\Phi_+}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \frac{S_b^-(\alpha')/(\sin\alpha'-\sin\theta_1)}{\Psi_{+1}(\alpha'-\Phi_+)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_+))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha', \qquad (68)$$

and,

$$\frac{f^{ar}(\alpha)}{\Psi_{1-}(\alpha)} = \frac{-i}{4\Phi_-}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \frac{S_a^+(\alpha')e^{-ik\Delta\cos\alpha'}\tan(\frac{\pi}{4\Phi_-}(\alpha+\Phi_--\alpha'))}{(\sin\alpha'+\sin\theta_1)\Psi_{1-}(\alpha'+\Phi_-)}d\alpha'$$
$$= \frac{i}{4\Phi_-}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \frac{S_a^+(\alpha')/(\sin\alpha'+\sin\theta_1)}{\Psi_{1-}(\alpha'+\Phi_-)} \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_-))+\cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha', \qquad (69)$$

in the respective band,

$$\begin{aligned}&\mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha)))-\Phi_+ < \mathrm{Re}(\alpha) < \mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha)))+3\Phi_+,\\ &\mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha)))-3\Phi_- < \mathrm{Re}(\alpha) < \mathrm{Re}(g_\epsilon(\mathrm{Im}(\alpha)))+\Phi_-,\end{aligned} \qquad (70)$$

where the functions f^{br} and f^{ar} are meromorphic functions, regular and $O(1)$ at infinity. Considering the integral equations (64)-(65) satisfied by S_a^+ and S_b^- when χ_a^i and χ_b^i are nul, we note that

$$f^{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}) - \frac{(-\sin\alpha-\sin\theta_1)f^{ar}(-\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{(\sin\alpha-\sin\theta_1)} = \frac{2S_b^-(\alpha)}{(\sin\alpha-\sin\theta_1)},$$
$$f^{br}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}) - \frac{(-\sin\alpha+\sin\theta_1)f^{br}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{(\sin\alpha+\sin\theta_1)} = \frac{2S_a^+(\alpha)}{(\sin\alpha+\sin\theta_1)}. \qquad (71)$$

We then deduce, after using elementary properties of kernels of (68)-(69), that $f^b(\alpha)=f^{br}(\alpha-\frac{\Phi_b}{2})$ and $f^a(\alpha)=f^{ar}(\alpha+\frac{\Phi_a}{2})$ in place of $f_b(\alpha)=f_{br}(\alpha-\frac{\Phi_b}{2})$ and $f_a(\alpha)=f_{ar}(\alpha+\frac{\Phi_a}{2})$, satisfy the functional equations (29)-(30) with (28), and thus the condition (34). This implies that f^a and f^b refer to the representation of the same field u with two distinct origins, which satisfies the boundary conditions for the impedance three-part polygon, without illumination.

Now, we can use that,
(i) in absence of source terms, from uniqueness theorem on the field u (see appendix C), the field u and its normal derivative shall be nul on the impedance polygons.

(ii) f^a and f^b vanish if the field and its normal derivative are nul on the surface of the polygon, because of single face expression of spectral function (7)-(8).

Consequently, the spectral functions f^a and f^b, and thus $S_b^-(\alpha')$ and $S_a^+(\alpha')$, have to be nul, and we have uniqueness of solutions for our system of integral equations (64)-(65) in $L^2(R)$ as $|\arg(ik) - \epsilon| < \pi/2$, and thus as $|\arg(ik)| \leq \pi/2$.

Remark : The solution is also uniquely defined in the case $\Delta = 0$ (simple wedge) from Thuzhilin's theory [33].

6.5.2) Existence

Concerning existence, we note first that the kernels are regular functions of $k\Delta$ as $\mathrm{Im}(k\Delta) \leq 0$. Considering then the theory of integral equations depending on a parameter in L^2, the resolvent for our system of Fredholm integral equations of second kind is an analytical function of the parameter $k\Delta$, which defines a unique solution in $L^2(R)$ as $\mathrm{Im}(k\Delta) \leq 0$ and $k\Delta \neq 0$, except possibly for some discrete values of $k\Delta$, where the resolvent is singular and the solution is not unique.
Since the uniqueness of our spectral integral equations has been shown independently, a unique solution exists in $L^2(R)$ for (64)-(65) as $|\arg(ik)| \leq \pi/2$, $k\Delta \neq 0$

Remark : for $\Delta = 0$, the problem corresponds to the diffraction by a simple impedance wedge where the spectral function is known to be uniquely defined [20] [33].

7) Some particular features of the system of integral equations for the three-part impedance polygon and their consequences

Our spectral integral equations can be solved numerically, or analytically by approximations. In our case, the approximations principally depend on $k\Delta$, for $k\Delta$ small or $k\Delta$ large. These integral equations and the integral expressions of the spectral functions have special features concerning the decoupling of integral equations in important cases that we now first illustrate.

7.1) Decoupling in the case of the unsymmetric three-part impedance plane

In the case of the three-part impedance plane, we have $\Phi_a = \Phi_b = 0$ and $\Phi_+ = \Phi_- = \frac{\pi}{2}$. Considering that,

$$\frac{\Psi_{1-}(\alpha-\frac{\pi}{2})}{2\Psi_{+1}(\alpha-\frac{\pi}{2})} - \frac{\Psi_{1-}(-\alpha-\frac{\pi}{2})}{2\Psi_{+1}(-\alpha-\frac{\pi}{2})} = \frac{\Psi_{1-}(\alpha-\frac{\pi}{2})}{\Psi_{+1}(\alpha-\frac{\pi}{2})}\frac{\sin\alpha(\sin\theta_- - \sin\theta_1)}{(\sin\alpha+\sin\theta_-)(\sin\alpha-\sin\theta_1)}$$
$$= \frac{C(\alpha)\sin\alpha(\sin\theta_- - \sin\theta_1)}{\sin\alpha-\sin\theta_1},$$
$$\frac{\Psi_{+1}(\alpha+\frac{\pi}{2})}{2\Psi_{1-}(\alpha+\frac{\pi}{2})} - \frac{\Psi_{+1}(\frac{\pi}{2}-\alpha)}{2\Psi_{1-}(\frac{\pi}{2}-\alpha)} = \frac{-\Psi_{+1}(\frac{\pi}{2}-\alpha)}{\Psi_{1-}(\frac{\pi}{2}-\alpha)}\frac{\sin\alpha(\sin\theta_+ - \sin\theta_1)}{(\sin\alpha+\sin\theta_+)(\sin\alpha-\sin\theta_1)}$$
$$= \frac{-C(\alpha)\sin\alpha(\sin\theta_+ - \sin\theta_1)}{\sin\alpha-\sin\theta_1}, \qquad (72)$$

and letting $N(\alpha') = \frac{iC(\alpha')\sin\alpha'/2\pi}{\sin\alpha'-\sin\theta_1}$, we then derive,

$$\frac{(\sin\theta_1-\sin\theta_+)^{1/2} f_{br}(\alpha+\frac{\pi}{2})}{\Psi_{+1}(\alpha+\frac{\pi}{2})} = \eta_m \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} N(\alpha')\frac{(\sin\theta_1-\sin\theta_-)^{1/2} f_{ar}(\alpha'-\frac{\pi}{2})}{\Psi_{1-}(\alpha'-\frac{\pi}{2})}$$
$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin\alpha'}{\cos\alpha+\cos\alpha'}d\alpha' + (\sin\theta_1-\sin\theta_+)^{1/2}\chi_b^i(\alpha+\frac{\pi}{2}),$$

$$\frac{(\sin\theta_1-\sin\theta_-)^{1/2} f_{ar}(\alpha-\frac{\pi}{2})}{\Psi_{1-}(\alpha-\frac{\pi}{2})} = \eta_m \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} N(\alpha')\frac{(\sin\theta_1-\sin\theta_+)^{1/2} f_{br}(\alpha'+\frac{\pi}{2})}{\Psi_{+1}(\alpha'+\frac{\pi}{2})}$$
$$\times \frac{e^{-ik\Delta\cos\alpha'}\sin\alpha'}{\cos\alpha+\cos\alpha'}d\alpha' + (\sin\theta_1-\sin\theta_-)^{1/2}\chi_a^i(\alpha-\frac{\pi}{2}), \qquad (73)$$

where $\eta_m = (\sin\theta_1-\sin\theta_+)^{1/2}(\sin\theta_1-\sin\theta_-)^{1/2}$.
We obtain two uncoupled equations by simple addition (S) and substraction (D) :

$$(S,D) = \eta_m \int_{-i\infty}^{+i\infty} N(\alpha')((S,-D)e^{-ik\Delta\cos\alpha'}\frac{\sin\alpha'}{\cos\alpha+\cos\alpha'}d\alpha' + (S^i,-D^i). \qquad (74)$$

7.2) Decoupling for symmetric three parts polygons ($\Phi_b = \Phi_a$ and $\sin\theta_+ = \sin\theta_-$)

In this case, the system (64)-(65) can also be decoupled. For this, we first express the system of integral equation (64)-(65) in a new form. We observe that $\Psi_{-1}(\alpha,\Phi_-)$ (also denoted $\Psi_{-1}(\alpha)$) satisfies $\Psi_{-1}(\alpha,\Phi_-) = \Psi_{1-}(-\alpha,\Phi_-)$. Then we can write

$$\frac{f_{br}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}=\frac{-i}{4\Phi_+}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}\frac{f_{ar}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{+1}(\alpha'-\Phi_+)}$$
$$\times\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha+\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'+\chi_b^i(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}),$$
$$\frac{f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{-1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_a}{2})}=\frac{-i}{4\Phi_-}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}\frac{f_{br}(-\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{-1}(\alpha'-\Phi_-)}$$
$$\times\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha'+\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}),\qquad(75)$$

This form is particularly suitable for the case of a symmetric polygon. In this case $\Phi_+=\Phi_-$ (i.e. $\Phi_b=\Phi_a$) and $\sin\theta_+=\sin\theta_-$, so that $\Psi_{+1}=\Psi_{-1}$ and the equations have the same kernels. Thus, by addition and substraction, we derive a system of decoupled equations,

$$\frac{(S,D)(\alpha)}{\Psi_{+1}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}=$$
$$=\frac{-i}{4\Phi_+}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}\frac{(S,-D)(\alpha')}{\Psi_{+1}(\alpha'-\Phi_+)}\frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha+\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'+(S^i,D^i)(\alpha),\quad(76)$$

where

$$S(\alpha)=f_{br}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})+f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_b}{2}),$$
$$D(\alpha)=\ =f_{br}(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})-f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_b}{2}),$$
$$S^i(\alpha)=\chi_{br}^i(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})+\chi_{ar}^i(\alpha-\frac{\pi}{2}+\frac{\Phi_b}{2}),$$
$$D^i(\alpha)=\chi_{br}^i(-\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})-\chi_{ar}^i(\alpha-\frac{\pi}{2}+\frac{\Phi_b}{2}),\qquad(77)$$

for $|\varphi_\circ|<\min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_b)$.

7.3) Partial inversion and new kernels for small $k\Delta$

Considering the solution known for $k\Delta=0$ [1], we can apply the identity

$$e^{-ik\Delta\cos\alpha'}=(e^{-ik\Delta\cos\alpha'}-1)+1\qquad(78)$$

and invert the part corresponding to unit term, which results in equations with kernels vanishing as $k\Delta\to 0$, suitable for approximations. To simplify, we detail the developments in appendix. We then obtain the equations with kernels vanishing as $k\Delta\to 0$:

$$f_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})=\frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty}d\alpha'(\frac{\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{\sin\alpha'-\sin\theta_1}M_+(\alpha,\alpha')$$
$$-\frac{\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{\sin\alpha'+\sin\theta_1}M_-(\alpha,\alpha')),$$
$$f_{ar0}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})=\frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty}d\alpha'(\frac{\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{\sin\alpha'-\sin\theta_1}N_+(\alpha,\alpha')$$
$$-\frac{\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{\sin\alpha'+\sin\theta_1}N_-(\alpha,\alpha')), \tag{79}$$

where $M_\pm(\alpha,\alpha')=L_\pm(\alpha+\frac{\pi}{2}+\frac{(\Phi_a-\Phi_b)}{2},\alpha')$, $N_\pm(\alpha,\alpha')=L_\pm(\alpha-\frac{\pi}{2}+\frac{(\Phi_a-\Phi_b)}{2},\alpha')$,

$$L_\pm(\alpha,\alpha')=\frac{\pi\sin\alpha'\Psi_{+-}(\alpha)}{2\Phi_d}\int_{-i\infty}^{i\infty}\frac{\cos(\frac{\pi(\varphi'+(\Phi_a-\Phi_b)/2)}{2\Phi_d})}{\Psi_{+-}(\varphi'+(\Phi_a-\Phi_b)/2)}\frac{(\cos\varphi'+\sin\theta_1)}{\cos(\varphi'\pm\frac{\pi}{2})+\cos\alpha'}$$
$$\frac{1}{(\sin(\frac{\pi\alpha}{2\Phi_d})-\sin(\frac{\pi(\varphi'+(\Phi_a-\Phi_b)/2)}{2\Phi_d}))}d\varphi', \tag{80}$$

and

$$B_{a0}(\alpha')=(\frac{f_{ar0}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{+1}(\alpha'-\Phi_+)})(e^{-ik\Delta\cos\alpha'}-1)$$
$$+(\frac{f_0^{ab}(\alpha'-\frac{\pi}{2}+(\Phi_a-\Phi_b)/2,\,\varphi_\circ)}{\Psi_{+1}(\alpha'-\Phi_+)})(e^{-ik\Delta(\cos\alpha'+\sin\varphi_\circ)}-1)e^{ik\Delta\sin\varphi_\circ},$$
$$B_{b0}(\alpha')=(\frac{f_{br0}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{1-}(\alpha'+\Phi_-)})(e^{-ik\Delta\cos\alpha'}-1)$$
$$+(\frac{f_0^{ab}(\alpha'+\frac{\pi}{2}-(\Phi_b-\Phi_a)/2,\,\varphi_\circ)}{\Psi_{1-}(\alpha'+\Phi_-)})(e^{-ik\Delta(\cos\alpha'-\sin\varphi_\circ)}-1). \tag{81}$$

7.4) Asymptotics for large $k\Delta$

The scattering diagram can been reduced to a combination of values of the spectral functions f_a and f_b in the band of validity of integral expressions, with exponential factors depending on angles. So reduced, the oscillatory nature of the diagram comes principally from exponential factors. We can write, from previous integral equations,

$$\mathcal{R}_b(\alpha) = \frac{-i}{4\Phi_+} m_1(\alpha) \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \mathcal{R}_a(\alpha') \frac{e^{-ik\Delta\cos\alpha'} \sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})} d\alpha'$$
$$+ m_1(\alpha)\chi_b^i(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}), \tag{82}$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) < 3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) = \frac{i}{4\Phi_-} m_2(\alpha) \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \mathcal{R}_b(\alpha') \frac{e^{-ik\Delta\cos\alpha'} \sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})} d\alpha'$$
$$+ m_2(\alpha)\chi_a^i(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}), \tag{83}$$

as $-3\Phi_- < \mathrm{Re}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2}) < \Phi_-$, for $-\min(\frac{\pi}{2}, \frac{\pi}{2} + \Phi_a) < \varphi_\circ < \min(\frac{\pi}{2}, \frac{\pi}{2} + \Phi_b)$, where

$$\mathcal{R}_b(\alpha) = \frac{f_{br}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha + \Phi_-)}, \mathcal{R}_a(\alpha) = \frac{f_{ar}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha - \Phi_+)}, m_1(\alpha) = \frac{\Psi_{+1}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha + \Phi_-)},$$
$$m_2(\alpha) = \frac{\Psi_{1-}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha - \Phi_+)}, \tag{84}$$

where χ_b^i and χ_a^i are smooth functions previously defined.

In the equations (83)-(84), we can choose to consider or to neglect the influence of complex poles of $\mathcal{R}_{a,b}(\alpha')$ (corresponding to guided waves vanishing when Δ is large) in vicinity of the integration path. To simplify the presentation, we develop here the second case, and consider the first case in appendix.

So, we assume that the principal contribution of $\mathcal{R}_{a,b}(\alpha')$ comes from the vicinity of the stationary phase point $\alpha' = 0$. Letting $\cos\alpha' = 1 - x^2/2$ with $x = 2\sin(\alpha'/2)$, and taking into account the parity of the integration path, we have

$$\mathcal{R}_b(\alpha) \sim \frac{m_1(\alpha)(\,\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0}) e^{-ik\Delta}}{2\pi i}$$
$$\times \int_{-i\infty}^{+i\infty} \frac{2e^{\frac{ik\Delta x^2}{2}} x^2}{2(\frac{\pi}{2\Phi_+})^{-2}(1 + \cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b))) - x^2} dx + m_1(\alpha)\chi_b^i(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}), \tag{85}$$

as $-\Phi_+ < \mathrm{Re}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) < 3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) \sim -\frac{m_2(\alpha)(\,\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0})e^{-ik\Delta}}{2\pi i}$$
$$\times \int_{-i\infty}^{+i\infty} \frac{2e^{\frac{ik\Delta x^2}{2}}x^2}{2(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)))-x^2}dx + m_2(\alpha)\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}), \quad (86)$$

as $-3\Phi_- < \mathrm{Re}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}) < \Phi_-$. After derivation of previous equations, we obtain

$$(\,\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0}) \sim \partial_\alpha(m_1(\alpha)\chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}))|_{\alpha=0}),$$
$$(\,\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0}) \sim \partial_\alpha(m_2(\alpha)\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}))|_{\alpha=0}), \quad (87)$$

so that we get the form

$$\mathcal{R}_b(\alpha) \sim \frac{m_1(\alpha)(\partial_\alpha(m_2(\alpha)\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}))|_{\alpha=0})e^{-ik\Delta}}{2\pi i}$$
$$\int_{-i\infty}^{+i\infty} \frac{2e^{\frac{ik\Delta x^2}{2}}x^2}{2(\frac{\pi}{2\Phi_+})^{-2}(1+\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b)))-x^2}dx + m_1(\alpha)\chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}), \quad (88)$$

as $-\Phi_+ < \mathrm{Re}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}) < 3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) \sim -\frac{m_2(\alpha)(\partial_\alpha(m_1(\alpha)\chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}))|_{\alpha=0})e^{-ik\Delta}}{2\pi i}$$
$$\int_{-i\infty}^{+i\infty} \frac{2e^{\frac{ik\Delta x^2}{2}}x^2}{2(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)))-x^2}dx + m_2(\alpha)\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}), \quad (89)$$

as $-3\Phi_- < \mathrm{Re}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}) < \Phi_-$, for $-\min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_a) < \varphi_\circ < \min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_b)$. Concerning the scattering diagram, we need, the evaluation of $\mathcal{R}_{a,b}(\alpha)$ for real $\alpha=\varphi$. Deforming the path of integration to the Steepest Descent Path (without capturing any poles), we derive

$$\mathcal{R}_b(\varphi) \sim \frac{m_1(\varphi)\partial_\alpha(m_2(\alpha)\chi_a^i(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2}))|_{\alpha=0}}{\sqrt{2ik\Delta}}2e^{-ik\Delta}(v\,e^{v^2}\mathrm{erfc}(v)-\frac{1}{\sqrt{\pi}}),$$
$$+\,m_1(\varphi)\chi_b^i(\varphi+\frac{\pi}{2}-\frac{\Phi_b}{2}), \quad (90)$$

as $-\Phi_+ < (\varphi+\frac{\pi}{2}-\frac{\Phi_b}{2}) < 3\Phi_+$, $v^2 = ik\Delta(\frac{\pi}{2\Phi_+})^{-2}(1+\cos(\frac{\pi}{2\Phi_+}(\varphi-\Phi_b)))$,and,

$$\mathcal{R}_a(\varphi) \sim -\frac{m_2(\varphi)\partial_\alpha(m_1(\alpha)\chi_b^i(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2}))|_{\alpha=0}}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}(w\,e^{w^2}\text{erfc}(w)-\frac{1}{\sqrt{\pi}}),$$
$$+\,m_2(\varphi)\chi_a^i(\varphi-\frac{\pi}{2}+\frac{\Phi_a}{2}), \qquad (91)$$

as $-3\Phi_- < (\varphi-\frac{\pi}{2}+\frac{\Phi_a}{2}) < \Phi_-$, $w^2 = ik\Delta(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\varphi+\Phi_a)))$, for $-\min(\frac{\pi}{2},\frac{\pi}{2}+\Phi_a) < \varphi_\circ < \min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_b)$, with $\Phi_+ = \frac{\pi}{2}+\frac{\Phi_b}{2}$ and $\Phi_- = \frac{\pi}{2}+\frac{\Phi_a}{2}$, where

$$ae^{a^2}\text{erfc}(a)-\frac{1}{\sqrt{\pi}} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{-t^2e^{-t^2}}{a^2+t^2}dt = \frac{-1}{2\sqrt{\pi}a^2}+O(\frac{1}{a^4}), \qquad (92)$$

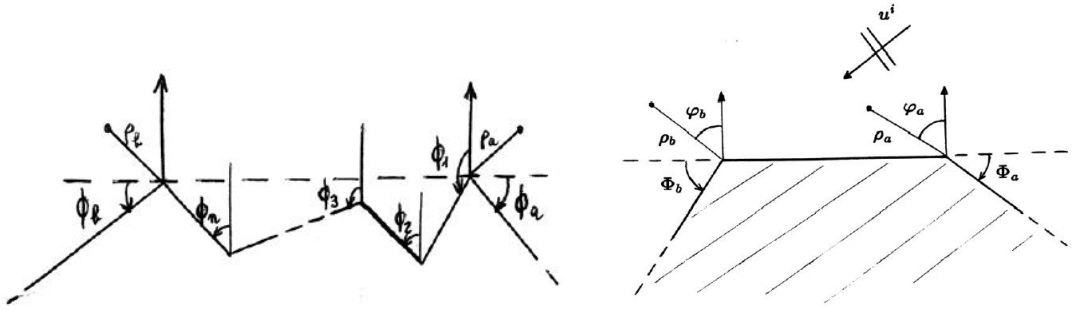
when a is real. In the right-handed sides, note that the first term corresponds to double diffraction, and the second term to single diffraction.

8) Exact first order expressions for a small cavity in a step, when $\Phi_b = -\Phi_a$ and $\sin\theta_+ = \sin\theta_-$

Let us consider a step of length Δ, when the three-part polygon is with $\Phi_b = -\Phi_a$, and thus, the right and the left-handed semi-infinite planes are paralell. We then assume a complex cavity in this step, composed of m faces of angles Φ_j between extreme edges 'a' and 'b' (see figures). We note that,

$$\sum_{1\le j\le m} d_j\cos\Phi_j = 0, \sum_{1\le j\le m} d_j\sin\Phi_j = \Delta, \qquad (93)$$

where d_j is the length of face j with angle Φ_j.



The half-planes, respectively right and left-handed, have relative impedances $\sin\theta_-$ and $\sin\theta_+$ with boundary conditions,

$$(\frac{\partial}{\partial n}-ik\sin\theta_\pm)u = 0, \qquad (94)$$

while the field on each face of angle Φ_j verifies,

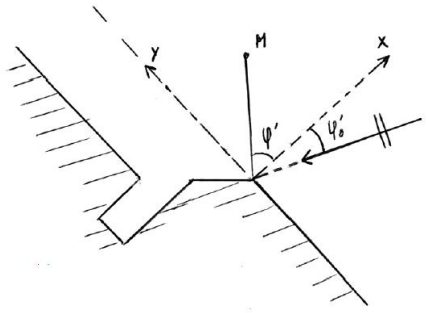
$$(\frac{\partial}{\partial n} - ik\sin\theta_j)u = 0. \tag{95}$$

We search to express the diffracted part u_d of u given by,

$$u_d(\varphi') = \frac{-e^{-i\pi/4-ik\rho_a}}{\sqrt{2\pi k\rho_a}}[f_a(\pi+\varphi_a) - f_a(-\pi+\varphi_a)] + O(1/(k\rho)^{3/2}), \tag{96}$$

when $\Phi_b = -\Phi_a$, $\Phi_d = \pi/2$. If semi-infinite planes are parallel and $\sin\theta_+ = \sin\theta_-$, we can show that first order of u_d in power of k can be determined by only studying our difference functional equations in combination of single face integral expression of spectral functions, with no need of integral equations. In what follows, we write ρ_a and φ_a without subscript, and let $\varphi = \varphi' - \Phi_a$, $\varphi_\circ = \varphi'_\circ - \Phi_a$.

8.1) Recombination of difference functional equations when $\Phi_a = -\Phi_b$



géométrie : complex step with cavity (note : $\varphi' = \varphi - \Phi_b$)

We have

$$\begin{aligned}(\sin\alpha + \sin\theta_+)f_b(\alpha+\pi/2+\Phi_b) - (-\sin\alpha + \sin\theta_+)f_b(-\alpha+\pi/2+\Phi_b) = 0,\\ (\sin\alpha - \sin\theta_-)f_a(\alpha-\pi/2-\Phi_a) - (-\sin\alpha - \sin\theta_-)f_a(-\alpha-\pi/2-\Phi_a) = 0,\end{aligned} \tag{97}$$

where,

$$f_a(\alpha) = \frac{ik}{2}\sum_{1\le j\le n} e^{-ik\sum_{1\le i<j} d_i\cos(\alpha-\Phi_i)} \int_0^{d_j} (-\sin(\alpha-\Phi_j)+\sin\theta_j)u(\rho'_j,\Phi_j)e^{-ik\rho'_j\cos(\alpha-\Phi_j)})d\rho'_j$$
$$+ e^{-ik\sum_{1\le i\le m} d_i\cos(\alpha-\Phi_i)} f_b(\alpha) = f_1(\alpha) + e^{-ik\,\Delta\cos(\alpha-\pi/2)} f_b(\alpha), \tag{98}$$

Therefore, we can write,

$$f_a(\alpha+\pi/2+\Phi_b)$$
$$-\frac{(-\sin\alpha+\sin\theta_+)}{(\sin\alpha+\sin\theta_+)} e^{ik\Delta(\cos(\alpha-\Phi_b)-\cos(\alpha+\Phi_b))} f_a(-\alpha+\pi/2+\Phi_b)$$
$$= f_1(\alpha+\pi/2+\Phi_b) - \frac{(-\sin\alpha+\sin\theta_+)}{(\sin\alpha+\sin\theta_+)} e^{ik\Delta(\cos(\alpha-\Phi_b)-\cos(\alpha+\Phi_b))} f_1(-\alpha+\pi/2+\Phi_b),$$
$$(\sin\alpha-\sin\theta_-)f_a(\alpha-\pi/2-\Phi_a) - (-\sin\alpha-\sin\theta_-)f_a(-\alpha-\pi/2-\Phi_a) = 0. \tag{99}$$

8.2) Reduction of far field function when $\sin\theta_+ = \sin\theta_-$

Considering that $f_1(\alpha) = O(k\Delta)$, we then have,

$$f_a(\pi+\varphi)$$
$$-\frac{(-\cos(\varphi-\Phi_b)+\sin\theta_+)}{(\cos(\varphi-\Phi_b)+\sin\theta_+)} e^{2ik\Delta\cos(\varphi-\Phi_b)\sin\Phi_b} f_a(-\varphi+2\Phi_b)$$
$$= f_1(\pi+\varphi) + \frac{(\cos(\varphi-\Phi_b)-\sin\theta_+)}{(\cos(\varphi-\Phi_b)+\sin\theta_+)} f_1(-\varphi+2\Phi_b) + O((k\Delta)^2),$$
$$f_a(-\pi+\varphi) - \frac{(\cos(\varphi+\Phi_a)-\sin\theta_-)}{(-\cos(\varphi+\Phi_a)-\sin\theta_-)} f_a(-\varphi-2\Phi_a) = 0, \tag{100}$$

then,

$$f_a(\pi+\varphi) - f_a(-\pi+\varphi) =$$
$$f_1(\pi+\varphi) + \frac{(\cos(\varphi-\Phi_b)-\sin\theta_+)}{\cos(\varphi-\Phi_b)+\sin\theta_+)} f_1(-\varphi+2\Phi_b)$$
$$-\frac{(\cos(\varphi-\Phi_b)-\sin\theta_+)}{(\cos(\varphi-\Phi_b)+\sin\theta_+)} e^{2ik\Delta\cos(\varphi-\Phi_b)\sin\Phi_b} f_a(-\varphi+2\Phi_b)$$
$$+\frac{(\cos(\varphi+\Phi_a)-\sin\theta_-)}{(\cos(\varphi+\Phi_a)+\sin\theta_-)} f_a(-\varphi-2\Phi_a) + O((k\Delta)^2), \tag{101}$$

For $\Phi_a = -\Phi_b$, we have,

$$\begin{aligned} & f_a(\pi+\varphi) - f_a(-\pi+\varphi) \\ & = f_1(\pi+\varphi) + \frac{(\cos(\varphi-\Phi_b)-\sin\theta_+)}{(\cos(\varphi-\Phi_b)+\sin\theta_+)} f_1(-\varphi+2\Phi_b) \\ & + f_a(-\varphi+2\Phi_b)\frac{(\cos(\varphi-\Phi_b)-\sin\theta_-)}{(\cos(\varphi-\Phi_b)+\sin\theta_-)}(1-e^{2ik\Delta\cos(\varphi-\Phi_b)\sin\Phi_b}) \\ & + O((k\Delta)^2), \end{aligned} \tag{102}$$

then, letting $\varphi' = \varphi - \Phi_b$ and considering that $\sin\theta_+ = \sin\theta_-$, we can write,

$$\begin{aligned} & f_a(\pi+\varphi'+\Phi_b) - f_a(-\pi+\varphi'+\Phi_b) \\ & = f_1(\pi+\varphi'+\Phi_b) + \frac{(\cos\varphi'-\sin\theta_+)}{(\cos\varphi'+\sin\theta_+)} f_1(-\varphi'+\Phi_b) + D_u, \end{aligned} \tag{103}$$

with,

$$\begin{aligned} & D_u = -f_a(-\varphi'+\Phi_b)\frac{(\cos\varphi'-\sin\theta_+)}{(\cos\varphi'+\sin\theta_+)} 2ik\Delta\cos\varphi'\sin\Phi_b, \\ & f_a(-\varphi'+\Phi_b) = f_0(-\varphi'+\Phi_b+(\Phi_a-\Phi_b)/2, \varphi_0) + O(k\Delta) \\ & = f_0(-\varphi', \varphi_0) + O(k\Delta), \end{aligned} \tag{104}$$

where f_0 is the spectral function for $\Delta = 0$ given by (41) with $\Phi_d = \pi/2$ and $\sin\theta_+ = \sin\theta_-$, following,

$$f_0(\alpha, \varphi_\circ) = \frac{(\cos\alpha+\sin\theta_+)\cos\varphi'_\circ}{(\cos\varphi'_\circ+\sin\theta_+)(\sin\alpha-\sin\varphi'_\circ)}. \tag{105}$$

This gives us the following expression of far field function,

$$\begin{aligned} & f_a(\pi+\varphi'+\Phi_b) - f_a(-\pi+\varphi'+\Phi_b) \\ & \sim f_1(\pi+\varphi'+\Phi_b) + \frac{(\cos\varphi'-\sin\theta_+)}{(\cos\varphi'+\sin\theta_+)} f_1(-\varphi'+\Phi_b) \\ & + \frac{(\cos\varphi'-\sin\theta_+)\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\sin\varphi'+\sin\varphi'_\circ)}\, 2ik\Delta\sin\Phi_b. \end{aligned} \tag{106}$$

8.3) Exact first order expressions for f_1 and for far field function

We can write,

$$f_1(\alpha) = \frac{ik}{2} \sum_{1\le j\le 2} e^{-ik\sum_{1\le i<j} d_i \cos(\alpha-\Phi_i)} \int_0^{d_j} (-\sin(\alpha-\Phi_j) + \sin\theta_j) u(\rho'_j, \Phi_j) e^{-ik\rho'_j \cos(\alpha-\Phi_j)}) d\rho'_j$$
$$\sim \frac{ikM_u}{2} \sum_j d_j (-\sin(\alpha-\Phi_j) + \sin\theta_j), \tag{107}$$

for small $k\Delta$, where M_u is a mean value of u for small $k\Delta$, $M_u^0 = \frac{2\cos\varphi'_\circ}{(\cos\varphi'_\circ + \sin\theta_+)}$ as $\sin\theta_+ = \sin\theta_-$. We then note that $\sum_{1\le j\le m} d_j \cos\Phi_j = 0$, $\sum_{1\le j\le m} d_j \sin\Phi_j = \Delta$, which gives us,

$$f_1(\alpha) \sim \frac{ik\cos\varphi'_\circ}{(\cos\varphi'_\circ + \sin\theta_+)} (\cos\alpha \sum_j d_j \sin\Phi_j + \sum_j d_j \sin\theta_j)$$
$$\sim \frac{ik\cos\varphi'_\circ}{(\cos\varphi'_\circ + \sin\theta_+)} (\Delta\cos\alpha + \sum_j d_j \sin\theta_j), \tag{108}$$

We then obtain for (106),

$$f_a(\pi+\varphi'+\Phi_b) - f_a(-\pi+\varphi'+\Phi_b)$$
$$\sim \frac{2ik\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)} (-\Delta\cos\Phi_b\sin\theta_+ + \sin\Phi_b\Delta\sin\varphi' + \sum_j d_j\sin\theta_j) + \frac{2(\cos\varphi'-\sin\theta_+)\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\sin\varphi'+\sin\varphi'_\circ)} ik\Delta\sin\Phi_b$$
$$\sim \frac{2ik\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)} (-\Delta\cos\Phi_b\sin\theta_+ + \sum_j d_j\sin\theta_j) + \frac{2ik\Delta\sin\Phi_b\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)(\sin\varphi'+\sin\varphi'_\circ)} \times (\sin\varphi'(\sin\varphi'+\sin\varphi'_\circ) + (\cos^2\varphi' - \sin^2\theta_+)), \tag{109}$$

and thus,

$$f_a(\pi+\varphi'+\Phi_b) - f_a(-\pi+\varphi'+\Phi_b) \sim \frac{2ik\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)} \times$$
$$\times (-\Delta\cos\Phi_b\sin\theta_+ + \sum_j d_j\sin\theta_j + \frac{\Delta\sin\Phi_b}{(\sin\varphi'+\sin\varphi'_\circ)} (\cos^2\theta_+ + \sin\varphi'\sin\varphi'_\circ)), \tag{110}$$

or more generally, letting $\sum_j d_j \sin\theta_j = \int_{O_a}^{O_b} \sin\theta(l)dl$,

$$f_a(\pi+\varphi'+\Phi_b) - f_a(-\pi+\varphi'+\Phi_b) \sim \frac{2ik\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)} \times$$
$$\times \left(-\Delta\cos\Phi_b\sin\theta_+ + \int_{O_a}^{O_b}\sin\theta(l)dl + \frac{\Delta\sin\Phi_b}{(\sin\varphi'+\sin\varphi'_\circ)}(\cos^2\theta_+ + \sin\varphi'\sin\varphi'_\circ)\right), \quad (111)$$

Using (96), we obtain for the diffracted field at large distance,

$$u_d \sim \frac{-2ik\Delta\cos\varphi'_\circ\cos\varphi' e^{-i\pi/4-ik\rho_a}}{\sqrt{2\pi k\rho}(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)}\left(-\cos\Phi_b\sin\theta_+ + \sin\theta'_1\right.$$
$$\left. + \frac{\sin\Phi_b}{(\sin\varphi'+\sin\varphi'_\circ)}(\cos^2\theta_+ + \sin\varphi'\sin\varphi'_\circ)\right), \quad (112)$$

where $l_1 = \sum_j d_j = \int_{O_a}^{O_b} dl$, $\sin\theta'_1 = \int_{O_a}^{O_b}\sin\theta(l)dl/\Delta$.

It is worth noticing that this expression is the exact first order of the asymptotics for small $k\Delta$, and that it satisfies the reciprocity theorem, i.e. it remains the same when we change φ' for $\varphi'_\circ$ (contrary to the one we would obtain by Physical Optics approximation).

Appendix A : about $\Psi_{+-}(\alpha,\Phi)$ and the solution for an impedance wedge

A.1) $\Psi_{+-}(\alpha)$ in passive case

The function $\Psi_{+-}(\alpha)$, also denoted $\Psi_{+-}(\alpha,\Phi)$, has been defined by Maliuzhinets, for its spectral function $f_{+-}(\alpha) = \frac{\Psi_{+-}(\alpha)}{\Psi_{+-}(\varphi')}\sigma(\alpha)$ used in its integral representation of field in presence of a passive impedance wedge illuminated by a unit plane wave with direction φ' [20]. The spectral function f_{+-} is the solution of the equations

$$(\sin\alpha+\sin\theta_+)f_{+-}(\alpha+\Phi) - (-\sin\alpha+\sin\theta_+)f_{+-}(-\alpha+\Phi) = 0,$$
$$(\sin\alpha-\sin\theta_-)f_{+-}(\alpha-\Phi) - (-\sin\alpha-\sin\theta_-)f_{+-}(-\alpha-\Phi) = 0, \quad (113)$$

regular in the strip $|\mathrm{Re}\alpha| \le \Phi$, except for the pole with unit residue at $\alpha=\varphi'$, and $O(1)$ at infinity in this band, where $\sin\theta_+$ (resp. $\sin\theta_-$) corresponds to the relative impedance attached to the face $\varphi = +\Phi$ (resp. $\varphi = -\Phi$). The function $\Psi_{+-}(\alpha)$ is the solution of (113) without poles or zeros as $|\mathrm{Re}\alpha| \le \Phi$, when $\mathrm{Re}(\sin\theta_{+,-}) > 0$ with $0 < \mathrm{Re}(\theta^{\pm}) \le \pi/2$ (passive case), $O(\cos(\pi\alpha/2\Phi))$ in this band, and $\sigma(\alpha) = \frac{\pi}{2\Phi}\cos(\frac{\pi\varphi'}{2\Phi})/(\sin(\frac{\pi\alpha}{2\Phi}) - \sin(\frac{\pi\varphi'}{2\Phi}))$ has the unit pole for incident field at $\alpha=\varphi'$ in the band $|\mathrm{Re}\alpha| \le \Phi$ [20]. The function $\Psi_{+-}(\alpha)$ has numerous

properties. We have,

$$\Psi_{+-}(\alpha) = A\Psi_\Phi(\alpha+\Phi+(\frac{\pi}{2}-\theta_+))\Psi_\Phi(\alpha-\Phi-(\frac{\pi}{2}-\theta_-))$$
$$\times\ \Psi_\Phi(\alpha+\Phi-(\frac{\pi}{2}-\theta_+))\Psi_\Phi(\alpha-\Phi+(\frac{\pi}{2}-\theta_-)),$$
$$= A\Psi_\Phi^4(\frac{\pi}{2})(\cos(\frac{\pi}{4\Phi}(\alpha+\Phi-\theta_+))\cos(\frac{\pi}{4\Phi}(\alpha-\Phi+\theta_-))$$
$$\times \frac{\Psi_\Phi(\alpha+\Phi-\frac{\pi}{2}+\theta_+)\Psi_\Phi(\alpha-\Phi+\frac{\pi}{2}-\theta_-)}{\Psi_\Phi(\alpha+\Phi-\frac{\pi}{2}-\theta_+)\Psi_\Phi(\alpha-\Phi+\frac{\pi}{2}+\theta_-)}, \qquad (114)$$

where Ψ_Φ is the Maliuzhinets function [20], $0 < \mathrm{Re}\theta_\pm \leq \frac{\pi}{2}$, A is an arbitrary constant. This even function satisfies $\Psi_\Phi(\alpha) = A_0' e^{\mp i\pi\alpha/8\Phi}(1+O(\alpha^\nu|e^{\pm i\mu'\alpha}|))$ when $\mathrm{Im}\alpha \to \pm\infty$, where A_0' and ν are constants, $\mu' = \min(\frac{\pi}{2\Phi}, 1)$ (see [32] for more details), and we can write $\Psi_{+-}(\alpha) = A_0\cos(\pi\alpha/2\Phi)(1+O(\alpha^\nu e^{-\mu'|\mathrm{Im}\alpha|}))$ for $|\mathrm{Im}\alpha|$ large, A_0 a constant. Since $\Psi_\Phi(\alpha) = \Psi_\Phi(-\alpha)$ and $\Psi_\Phi(\alpha+\frac{\pi}{2})\Psi_\Phi(\alpha-\frac{\pi}{2}) = \Psi_\Phi^2(\frac{\pi}{2})\cos(\pi\alpha/4\Phi)$, we have $\Psi_{+-}(-\alpha) = \Psi_{-+}(\alpha)$ and

$$4\Psi_{+-}(\alpha+\frac{\pi}{2})\Psi_{+-}(\alpha-\frac{\pi}{2}) = A^2\Psi_\Phi^8(\frac{\pi}{2})(\cos(\pi(\frac{\pi}{2}-\theta_+)/2\Phi)-\sin(\pi\alpha/2\Phi))$$
$$\times(\cos(\pi(\frac{\pi}{2}-\theta_-)/2\Phi)+\sin(\pi\alpha/2\Phi)), \qquad (115)$$

with $A\Psi_\Phi^4(\frac{\pi}{2}) = 2A_0$. Besides, from [27], we can write, for $|\mathrm{Re}\alpha| \leq \Phi + (N+1)\pi$,

$$\Psi_{+-}(\alpha) = B_N\prod_{\pm}(\prod_{m=0}^{N}[\Gamma(\frac{1}{2}+\frac{1}{4\Phi}(\alpha\pm\Phi+\theta_\pm+m\pi))$$
$$\Gamma(\frac{1}{2}-\frac{1}{4\Phi}(\alpha\pm\Phi-(\theta_\pm+m\pi)))\Gamma(\frac{1}{2}+\frac{1}{4\Phi}(\alpha\pm\Phi+\pi-\theta_\pm+m\pi))$$
$$\Gamma(\frac{1}{2}-\frac{1}{4\Phi}(\alpha\pm\Phi-(\pi-\theta_\pm+m\pi)))]^{(-1)^{m+1}})$$
$$\times\exp(\int_0^\infty(-e^{-\nu\pi})^{N+1}\frac{e^{-\nu\theta_\pm}+e^{-\nu(\pi-\theta_\pm)}}{(1+e^{-\nu\pi})}\frac{(1-\cosh(\nu(\alpha\pm\Phi)))}{\nu\sinh(2\nu\Phi)}\,d\nu), \qquad (116)$$

where θ_+ and θ_-, N is an arbitrary positive integer and B_N is a constant. For the applications, we choose N fixed and define Ψ_{+-} with $B_N = 1$. This expression is suitable for numerical calculus (with $N = 1$ or 2) or to derive the analytical properties of $\Psi_{+-}(\alpha)$ (with $N \to \infty$) from those of Γ.

The zeros which are the closest to the imaginary axis are $\alpha = \Phi + \theta_+$ and $\alpha = -\Phi - \theta_-$, and the closest poles are $\alpha = \Phi + \theta_+ + \pi$ and $\alpha = -\Phi - \theta_- - \pi$.

Remark : From [20], the zeros of Ψ_Φ which are the closest to $\alpha = 0$ and the corresponding poles are the points $\alpha = \pm(\frac{\pi}{2} + 2\Phi)$ and $\alpha = \pm(\frac{3\pi}{2} + 2\Phi)$. In other respects, Ψ_Φ satisfies

$$\frac{\Psi_\Phi(\alpha + 2\Phi)}{\Psi_\Phi(\alpha - 2\Phi)} = \cot(\frac{\alpha + \frac{\pi}{2}}{2}), \Psi_\Phi(\alpha + \Phi)\Psi_\Phi(\alpha - \Phi) = \Psi_\Phi^2(\Phi)\Psi_{\Phi/2}(\alpha).$$

Notice that, using (113) in (115), we deduce that,

$$\begin{aligned}
&\Psi_{+-}(\alpha \mp (\Phi - \frac{\pi}{2}))\Psi_{+-}(-\alpha \mp (\Phi - \frac{\pi}{2})) \\
&= A_0^2(\cos(\pi\alpha/2\Phi) + \cos(\pi(\frac{\pi}{2} - \theta_\pm)/2\Phi))(\cos\alpha + \sin\theta_\mp) \\
&\times \frac{(\cos(\pi\alpha/2\Phi) - \cos(\pi(\frac{\pi}{2} - \theta_\mp)/2\Phi))}{(\cos\alpha - \sin\theta_\mp)}.
\end{aligned} \tag{117}$$

A.2) Some miscellaneous properties in general case (passive or active)

The solution is more complex when we want to consider arbitrary face impedances in whole generality, passive ($0 < \mathrm{Re}(\theta^\pm) \leq \pi/2$) or active ($0 < -\mathrm{Re}(\theta^\pm) \leq \pi/2$). For that, we begin to consider,

$$\Psi_{+-}(\alpha, \sin\theta^+, \sin\theta^-) = \Psi^+(\alpha, \sin\theta^+)\Psi^-(\alpha, \sin\theta^-), \tag{118}$$

where the $\Psi^\pm$ verifies,

$$\begin{aligned}
&(\sin\alpha \pm \sin\theta_\pm)\Psi^\pm(\alpha \pm \Phi) - (-\sin\alpha \pm \sin\theta_\pm)\Psi^\pm(-\alpha \pm \Phi) = 0, \\
&\Psi^\pm(\alpha \mp \Phi) - \Psi^\pm(-\alpha \mp \Phi) = 0,
\end{aligned} \tag{119}$$

with Ψ_{+-} remaining without any zero or pole in the band $|\mathrm{Re}\alpha| \leq \Phi$, and thus,

$$\Psi^\pm(\alpha, \sin\theta^\pm) = 1/\Psi^\pm(\alpha, -\sin\theta^\pm) \text{ as } \mathrm{Re}(\theta^\pm) \neq 0. \tag{120}$$

The function $\Psi^\pm$ with an active $\sin\theta^\pm$ then directly derives from $\Psi^\pm$ with passive one, and, noting $s^\pm = \mathrm{sign}(\sin\theta^\pm)$, we remark that, when $s^\pm = 1$ (resp. $s^\pm = -1$), $\Psi^\pm(\alpha, \sin\theta^\pm)$ has zeros (resp. poles) at $\alpha = \pm(s^\pm\theta^\pm + \Phi)$ and $\alpha = \pm(\pi - s^\pm\theta^\pm + \Phi)$, and poles (resp. zeros) at $\alpha = \pm(\pi + s^\pm\theta^\pm + \Phi)$,

$$\Psi^\pm(\alpha) = A_\pm(\cos(\frac{\pi\alpha}{4\Phi}))^{s^\pm}(1 + O(\alpha^\nu e^{-\mu'|\mathrm{Im}\alpha|})), \tag{121}$$

for large $|\mathrm{Im}\alpha|$, with constants $A_\pm$ and $\mu' > 0$.

A.3) The solution for the diffraction by a wedge with passive or active impedance

We can then express the spectral solution $f_0(\alpha)$ for the diffraction by a wedge with passive or active impedance faces with the following general definition,

$$f_0(\alpha) = \Psi_{+-}(\alpha, \sin\theta^+, \sin\theta^-)\sigma(\alpha)/\Psi_{+-}(\varphi_\circ, \sin\theta^+, \sin\theta^-),$$
$$\sigma(\alpha) = \mu\cos(\mu\varphi_\circ)P_n(\sin(\mu\alpha))/(\sin(\mu\alpha) - \sin(\mu\varphi_\circ)), \tag{122}$$

where P_n is a polynomial of order n, $\mu = \pi/2\Phi$. Since we have $f_0(\alpha) \sim 1/(\alpha - \varphi_\circ)$ in vicinity of incidence pole $\alpha = \varphi_\circ$, and, from unique definition of u at $\rho = 0$, $f_0(\alpha) = O(1)$ as $|\mathrm{Im}\alpha| \to \infty$, we can write,

$$P_n(\sin(\mu\varphi_\circ)) = 1,\ n = \frac{1}{2}\sum_{\pm}(1 - s^\pm), \tag{123}$$

with $s^\pm = \mathrm{sign}(\mathrm{Re}(\sin\theta^\pm))$, and thus $n = 0$ when both impedances are passive. In addition, the n zeros of $P_n(\sin(\mu\alpha))$ have to suppress, when $s^\pm = -1$, the pole $\alpha_s^\pm$ of $\Psi^\pm(\alpha, \sin\theta^\pm)$ with $\pm\,\mathrm{sign}(\arg(ik))\mathrm{Im}\alpha_s^\pm > 0$, among $\alpha_1^\pm = \pm(s^\pm\theta^\pm + \Phi)$ and $\alpha_2^\pm = \pm(\pi - s^\pm\theta^\pm + \Phi)$, which give poles of $f_0(\alpha + \varphi)$ within the region $|\mathrm{Re}\alpha| < \pi$ with $\mathrm{Re}(ik\cos\alpha) > 0$, as $|\mathrm{Re}\varphi| < \Phi$, and thus non physical contribution $e^{ik\rho\cos(\alpha_s^\pm - \varphi)}$ which can diverge at infinity as $|\mathrm{Re}\varphi| < \Phi$. We then have,

$$P_n(\sin(\mu\alpha)) = \prod_{\pm}\left(\frac{(\sin(\mu\alpha) - \sin(\mu\alpha_s^\pm)}{(\sin(\mu\varphi_\circ) - \sin(\mu\alpha_s^\pm)}\right)^{\frac{1-s^\pm}{2}}, \tag{124}$$

which completes the definition of f_0 for arbitrary impedance (passive or active).

Appendix B : Principle of semi-inversion for our system of integral equations

We can modify equations and derive integral equations with kernels vanishing as $k\Delta \to 0$ for the three-part semi-infinite impedance polygon, for approximations when $k\Delta$ is small.
For this, we begin with changing the unknowns in the equations, vanishing when $\Delta = 0$, and take,

$$f_{ar0}(\alpha) = [f_{ar}(\alpha) - f_0^{ab}(\alpha - (\Phi_+ - \frac{\pi}{2}))],$$
$$f_{br0}(\alpha) = [f_{br}(\alpha) - e^{ik\Delta\sin\varphi_\circ}f_0^{ab}(\alpha + (\Phi_- - \frac{\pi}{2}))], \tag{125}$$

with $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$, $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$. The function $f_0^{ab}(\alpha, \varphi_\circ)$, corresponds to the solution for $\Delta = 0$, i.e. for a simple wedge of exterior angle $2\Phi_d = \Phi_a + \Phi_b$. It is given by

$$f_0^{ab}(\alpha,\,\varphi_\circ)=\frac{\pi}{2\Phi_d}\frac{\Psi_{+-}(\alpha)\cos(\frac{\pi\varphi_{\circ,d}}{2\Phi_d})}{\Psi_{+-}(\varphi_{\circ,d})(\sin(\frac{\pi\alpha}{2\Phi_d})-\sin(\frac{\pi\varphi_{\circ,d}}{2\Phi_d}))},\tag{126}$$

with $\varphi_{\circ,d}=\varphi_\circ+(\Phi_a-\Phi_b)/2,\ -\Phi_d<\varphi_{\circ,d}<\Phi_d$. The functions f_{ar0} and f_{br0} vanish as $\Delta=0$, and satisfy, from (45)-(46),

$$\begin{aligned}\frac{f_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}&=\frac{-i}{4\Phi_+}(\int_{-i\infty}^{+i\infty}(\frac{f_{ar0}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{+1}(\alpha'-\Phi_+)})\\&\times\frac{\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'\\&+\int_{-i\infty}^{+i\infty}\frac{B_{a0}(\alpha')\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha'),\end{aligned}\tag{127}$$

where,

$$\begin{aligned}B_{a0}(\alpha')&=(\frac{f_{ar0}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{+1}(\alpha'-\Phi_+)})(e^{-ik\Delta\cos\alpha'}-1)\\&+(\frac{f_0^{ab}(\alpha'-\frac{\pi}{2}+(\Phi_a-\Phi_b)/2,\,\varphi_\circ)}{\Psi_{+1}(\alpha'-\Phi_+)})(e^{-ik\Delta(\cos\alpha'+\sin\varphi_\circ)}-1)e^{ik\Delta\sin\varphi_\circ},\end{aligned}\tag{128}$$

as $-\Phi_+<\mathrm{Re}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})<3\Phi_+$, and

$$\begin{aligned}\frac{f_{ar0}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}&=\frac{i}{4\Phi_-}(\int_{-i\infty}^{+i\infty}(\frac{f_{br0}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{1-}(\alpha'+\Phi_-)})\\&\times\frac{\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha'\\&+\int_{-i\infty}^{+i\infty}\frac{B_{b0}(\alpha')\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha'),\end{aligned}\tag{129}$$

where

$$\begin{aligned}B_{b0}(\alpha')&=(\frac{f_{br0}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{1-}(\alpha'+\Phi_-)})(e^{-ik\Delta\cos\alpha'}-1)\\&+(\frac{f_0^{ab}(\alpha'+\frac{\pi}{2}-(\Phi_b-\Phi_a)/2,\,\varphi_\circ)}{\Psi_{1-}(\alpha'+\Phi_-)})(e^{-ik\Delta(\cos\alpha'-\sin\varphi_\circ)}-1),\end{aligned}\tag{130}$$

as $-3\Phi_-<\mathrm{Re}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})<\Phi_-$, for $-\min(\frac{\pi}{2},\frac{\pi}{2}+\Phi_a)<\varphi_\circ<\min(\frac{\pi}{2},\frac{\pi}{2}+\Phi_b)$, with $\Phi_+=\frac{\pi}{2}+\frac{\Phi_b}{2}$ and $\Phi_-=\frac{\pi}{2}+\frac{\Phi_a}{2}$.

We then notice some similarity with the integral expressions (45)-(46) when $\Delta=0$, directly related to spectral function f_0^{ab} for an impedance wedge. Thus, we let,

$$f'_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})=\int_{-i\infty}^{+i\infty}G(\varphi')f_0^{ab}(\alpha+\frac{\pi}{2}+(\Phi_a-\Phi_b)/2,\varphi')d\varphi',$$
$$f'_{ar0}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})=\int_{-i\infty}^{+i\infty}G(\varphi')f_0^{ab}(\alpha-\frac{\pi}{2}+(\Phi_a-\Phi_b)/2,\varphi')d\varphi', \qquad (131)$$

as $|\mathrm{Re}(\alpha)|<\frac{\pi}{2}$. The functions $f_0^{ab}(\alpha\pm\frac{\pi}{2}+(\Phi_a-\Phi_b)/2,\varphi')$ are regular and $O(1/\cos(\pi\varphi'/2\Phi_d))$ on the imaginary axis, and a pole at $\varphi'=\alpha\pm\frac{\pi}{2}$ ensures that we have, even if $\Phi_a=\Phi_b=0$, generally $f'_{br0}\neq f'_{ar0}$.

We search to define $G(\varphi')$, regular in the band $|\mathrm{Re}(\varphi')|\leq\frac{\pi}{2}$, so that $f'_{br0}\equiv f_{br0}$ and $f'_{ar0}\equiv f_{ar0}$ verifies (127)-(130) for arbitrary Δ. We consider the integral expressions (45)-(46) of f_{ar} and f_{br} as $\Delta=0$, when $f_{br}(\alpha-\Phi_-+\frac{\pi}{2})=f_{ar}(\alpha+\Phi_+-\frac{\pi}{2})=f_0(\alpha)$ and,

$$f_{ar}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})=f_0^{ab}(\alpha-\frac{\pi}{2}+(\Phi_a-\Phi_b)/2,\varphi'),$$
$$f_{br}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})=f_0^{ab}(\alpha+\frac{\pi}{2}+(\Phi_a-\Phi_b)/2,\varphi'), \qquad (130)$$

We then multiply by $G(\varphi')$, integrate following (131), and we obtain,

$$\frac{f'_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}=\frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty}\frac{f'_{ar0}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})\sin(\frac{\pi\alpha'}{2\Phi_+})}{\Psi_{+1}(\alpha'-\Phi_+)(\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+}))}d\alpha'$$
$$+\frac{\pi}{2\Phi_+}\int_{-i\infty}^{+i\infty}\frac{G(\varphi')}{\Psi_{+1}(\varphi'-\frac{\Phi_b}{2})}\frac{\sin\frac{\pi(\varphi'+\frac{\pi}{2})}{2\Phi_+}}{(\cos\frac{\pi(\alpha-\Phi_b)}{2\Phi_+}+\cos\frac{\pi(\varphi'+\frac{\pi}{2})}{2\Phi_+})}d\varphi', \qquad (132)$$

and,

$$\frac{f'_{ar0}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}=\frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty}\frac{f'_{br0}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})\sin(\frac{\pi\alpha'}{2\Phi_-})}{\Psi_{1-}(\alpha'+\Phi_-)(\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-}))}d\alpha'$$
$$+\frac{\pi}{2\Phi_-}\int_{-i\infty}^{i\infty}\frac{G(\varphi')}{\Psi_{1-}(\varphi'+\frac{\Phi_a}{2})}\frac{\sin\frac{\pi(\varphi'-\frac{\pi}{2})}{2\Phi_-}}{(\cos\frac{\pi(\alpha+\Phi_a)}{2\Phi_-}+\cos\frac{\pi(\varphi'-\frac{\pi}{2})}{2\Phi_-})}d\varphi'. \qquad (133)$$

In the case where $G(\varphi')$ is regular in the band $|\mathrm{Re}(\varphi')|\leq\frac{\pi}{2}$, we can shift the integral paths in the integrals containing $G(\varphi')$. Comparing (127)-(130) with (132)-(106), we notice that (f'_{br0},f'_{ar0}) is a solution of the system of equations (127)-(130) if G satisfies the conditions

$$\frac{G(\alpha'+\frac{\pi}{2})}{\Psi_{1-}(\alpha'+\Phi_-)}-\frac{G(-\alpha'+\frac{\pi}{2})}{\Psi_{1-}(-\alpha'+\Phi_-)}=\frac{i}{2\pi}(B_{b0}(\alpha')-B_{b0}(-\alpha')),$$
$$\frac{G(\alpha'-\frac{\pi}{2})}{\Psi_{+1}(\alpha'-\Phi_+)}-\frac{G(-\alpha'-\frac{\pi}{2})}{\Psi_{+1}(-\alpha'-\Phi_+)}=\frac{-i}{2\pi}(B_{a0}(\alpha')-B_{a0}(-\alpha')), \qquad (134)$$

Taking account of the properites of Ψ_{+1} and Ψ_{1-} (see appendix A), and letting

$G(\alpha')=(\cos\alpha'+\sin\theta_1)g(\alpha')$, (134) can be written

$$g(\alpha'+\frac{\pi}{2})-g(-\alpha'+\frac{\pi}{2})=\frac{i\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{2\pi(-\sin\alpha'+\sin\theta_1)},$$
$$g(\alpha'-\frac{\pi}{2})-g(-\alpha'-\frac{\pi}{2})=\frac{i\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{2\pi(-\sin\alpha'-\sin\theta_1)}. \qquad (135)$$

Since $G(\alpha')$ is regular in the band $|\mathrm{Re}\alpha'|\leq\frac{\pi}{2}$ and $\mathrm{Re}(\sin\theta_1)>0$, $g(\alpha')$ is regular in this band. We can then use (42)-(43), and write, as $|\mathrm{Re}\alpha|<\frac{\pi}{2}$,

$$g(\alpha)=\frac{i}{4\pi}\int_{-i\infty}^{+i\infty}d\alpha'\Big(\frac{i\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{2\pi(\sin\alpha'-\sin\theta_1)}\tan\big(\frac{1}{2}(\alpha+\frac{\pi}{2}-\alpha')\big)$$
$$-\frac{i\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{2\pi(\sin\alpha'+\sin\theta_1)}\tan\big(\frac{1}{2}(\alpha-\frac{\pi}{2}-\alpha')\big)\Big), \qquad (136)$$

Using (131) and (136), we obtain the equations with kernels vanishing as $k\Delta\to 0$:

$$f_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})=\frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty}d\alpha'\Big(\frac{\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{\sin\alpha'-\sin\theta_1}M_+(\alpha,\alpha')$$
$$-\frac{\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{\sin\alpha'+\sin\theta_1}M_-(\alpha,\alpha')\Big), \qquad (137)$$

$$f_{ar0}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})=\frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty}d\alpha'\Big(\frac{\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{\sin\alpha'-\sin\theta_1}N_+(\alpha,\alpha')$$
$$-\frac{\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{\sin\alpha'+\sin\theta_1}N_-(\alpha,\alpha')\Big), \qquad (138)$$

where $M_\pm(\alpha,\alpha')=L_\pm(\alpha+\frac{\pi}{2}+\frac{(\Phi_a-\Phi_b)}{2},\alpha')$, $N_\pm(\alpha,\alpha')=L_\pm(\alpha-\frac{\pi}{2}+\frac{(\Phi_a-\Phi_b)}{2},\alpha')$,

$$L_\pm(\alpha,\alpha')=\frac{\pi\sin\alpha'\Psi_{+-}(\alpha)}{2\Phi_d}\int_{-i\infty}^{+i\infty}\frac{\cos(\frac{\pi(\varphi'+(\Phi_a-\Phi_b)/2)}{2\Phi_d})}{\Psi_{+-}(\varphi'+(\Phi_a-\Phi_b)/2)}$$
$$\frac{(\cos\varphi'+\sin\theta_1)}{\cos(\varphi'\pm\frac{\pi}{2})+\cos\alpha'}\frac{1}{(\sin(\frac{\pi\alpha}{2\Phi_d})-\sin(\frac{\pi(\varphi'+(\Phi_a-\Phi_b)/2)}{2\Phi_d}))}d\varphi', \qquad (139)$$

In the particular case $\Phi_a=\Phi_b=0$, $\Phi_d=\frac{\pi}{2}$, the functions $L_\pm$ can be simplified so that we recover the expressions found in [2] for the three-part impedance plane (see remark below).

Remark : It exists analytical expressions of $L_\pm$ when $\Phi_a=\Phi_b=0$. For this, we consider Ψ_{+-} for $\Phi=\pi/2$, where we have,

$$\frac{(-\sin\alpha+\sin\theta_-)}{\Psi_{+-}(\alpha+\frac{\pi}{2})}-\frac{(\sin\alpha+\sin\theta_-)}{\Psi_{+-}(-\alpha+\frac{\pi}{2})}$$
$$=(\sin\theta_--\sin\theta_+)(\frac{1}{\Psi_{+-}(\alpha+\frac{\pi}{2})}-\frac{1}{\Psi_{+-}(-\alpha+\frac{\pi}{2})}),$$
$$\frac{(\sin\alpha+\sin\theta_-)}{\Psi_{+-}(\alpha-\frac{\pi}{2})}-\frac{(-\sin\alpha+\sin\theta_-)}{\Psi_{+-}(-\alpha-\frac{\pi}{2})}=0. \qquad (140)$$

Using that $\Psi_{+-}(-\alpha)\Psi_{+-}(\alpha)=A_0^2(\cos\alpha+\sin\theta_+)(\cos\alpha+\sin\theta_-)$, and $\Psi_{+-}(\alpha)/\cos\alpha\to A_0$ as $\alpha\to i\infty$, we can let $\chi(\alpha)=\frac{A_0(\cos\alpha+\sin\theta_-)}{\Psi_{+-}(\alpha)}$ in (42)-(43), and derive,

$$\frac{(\cos\alpha+\sin\theta_-)}{\Psi_{+-}(\alpha)}-\frac{1}{A_0}=\frac{(\sin\theta_--\sin\theta_+)}{4\pi i}\int_{-i\infty}^{+i\infty}H(\alpha')\tan(\frac{1}{2}(\alpha+\frac{\pi}{2}-\alpha'))d\alpha',$$
$$\frac{\Psi_{+-}(-\alpha)/A_0^2}{(\cos\alpha+\sin\theta_+)}-\frac{1}{A_0}=\frac{(\sin\theta_--\sin\theta_+)}{4\pi i}\int_{-i\infty}^{+i\infty}H(\alpha')\tan(\frac{1}{2}(\alpha+\frac{\pi}{2}-\alpha'))d\alpha', \quad (141)$$

for $\frac{-3\pi}{2}<\mathrm{Re}(\alpha)<\frac{\pi}{2}$, where $H(\alpha')=2\sin\alpha'/(\Psi_{+-}(\alpha'+\frac{\pi}{2})(\sin\alpha'+\sin\theta_+))$ and

$$H(\alpha)=\frac{1}{\Psi_{+-}(\alpha+\frac{\pi}{2})}-\frac{1}{\Psi_{+-}(-\alpha+\frac{\pi}{2})}=\frac{\sin\alpha/\sin\theta_+}{\Psi_{+-}(\alpha+\frac{\pi}{2})}+\frac{\sin\alpha/\sin\theta_+}{\Psi_{+-}(-\alpha+\frac{\pi}{2})}. \quad (142)$$

From oddness of $H(\alpha)$, $\tan(\frac{1}{2}(\alpha+\frac{\pi}{2}-\alpha'))$ can be replaced by $\frac{-\sin\alpha'}{(\cos(\alpha+\frac{\pi}{2})+\cos\alpha')}$, and we note that, after deformation of integration path, we have,

$$\int_{-i\infty}^{+i\infty}\frac{1}{\Psi_{+-}(\varphi')}\frac{\cos\varphi'}{\cos\alpha+\sin\varphi'}d\varphi'=-2\pi i\frac{\frac{\sin\alpha+\sin\theta_-}{\Psi(\alpha-\frac{\pi}{2})}-\frac{1}{A_0}}{(\sin\theta_l-\sin\theta_r)},$$
$$\int_{-i\infty}^{+i\infty}\frac{\cos\varphi'}{\Psi_{+-}(\varphi')}\frac{\cos\varphi'}{\cos\alpha+\sin\varphi'}d\varphi'=2\pi i\sin\theta_l\frac{\frac{\sin\alpha+\sin\theta_-}{\Psi(\alpha-\frac{\pi}{2})}-\frac{1}{A_0}}{(\sin\theta_+-\sin\theta_-)}+\frac{i\pi}{A_0}, \qquad (143)$$

as $|\mathrm{Re}\alpha|<\frac{\pi}{2}$. Similar expressions with $-\sin\varphi'$ in place of $\sin\varphi'$ can be obtained from $\Psi_{+-}(-\alpha)\to\Psi_{-+}(\alpha)$ when $\theta_\pm\to\theta_\mp$ (or by continuation with capture of pole). This permits, by elementary combinations, to have analytical expressions of $L_\pm$ terms when $\Phi_a=\Phi_b=0$.

Appendix C : Uniqueness of fields in the boundary value problem for piecewise regular geometry

We analyse here the uniqueness of solutions for boundary value problem, that we give in [28] for picewise regular geometry in 2D and 3D. In [41], Levine develops an uniqueness theorem for the field p, i.e. a proof that $p_{inc}\equiv 0$ implies $p\equiv 0$, in the case of a scatterer with

impedance boundary conditions. He considers piecewise $C^{(2+\lambda)}$ surface (with no zero exterior angle), $\lambda > 0$, without auxiliary 'edge conditions' at edges or corner points, except that p is continuous. He studies at first bounded scatterers, but he also gives, in section 7 of his paper, the elements to generalize his results to scatterers with infinite boundaries, in particular by the use of Jones' uniqueness theorem [42], that we follow.

We begin to notice first that the conditions given by Levine to apply the Green's first theorem are satisfied: the scatterer surface is piecewise analytic (with no zero exterior angle), the field is countinuous and satisfies impedance boundary conditions

$$(\partial u/\partial n - ikg(s)u)|_{s\in S} = 0, \tag{144}$$

on it, and the conditions at infinity. So, we can write,

$$\int_\Omega (p^*(r)\Delta p(r) + \text{grad}p^*(r)\text{grad}p(r))d\Omega = -\int_S p^*(r)(\widehat{n}\,\text{grad}(p(r)))dS + \\ + \lim_{a\to\infty}\int_{r=a,\, z\geq 0} p^*(r)(\frac{\partial p(r)}{\partial r})dS, \tag{145}$$

where Ω is the infinite domain bounded by S, $\widehat{n}$ is the inward normal to Ω, and, from impedance boundary conditions,

$$\text{Re}(\int_\Omega -ik|p(r)|^2 + \frac{|\text{grad}p(r)|^2}{-ik}d\Omega) = \int_S \text{Re}(g)|p(r)|^2 dS + I_\infty, \tag{146}$$

where

$$I_\infty = \text{Re}\lim_{a\to\infty} O(e^{-\delta a}) = 0 \text{ for } |\arg(ik)| < \pi/2, \\ I_\infty = \lim_{a\to\infty}\int_{r=a,\, z\geq 0} |p(r)|^2 dS > 0 \text{ for } |\arg(ik)| = \pi/2. \tag{147}$$

For $\text{Re}(g) \geq 0$ and $|\arg(ik)| \leq \pi/2$, the left-hand term is negative since $\text{Re}(ik) \geq 0$, while the right-handed term is positive, and thus both terms vanish. Consequently, we have, when $|\arg(ik)| < \pi/2$,

$$p(r) = 0 \text{ in } \Omega, \text{ for } \text{Re}(g) \geq 0, \tag{148}$$

and, when $|\arg(ik)| = \pi/2$,

$$p(r) = 0 \text{ on } S, \text{ for } \text{Re}(g) > 0, \\ \partial_n p(r) = 0 \text{ on } S, \text{ for } \text{Re}(g) > 0, \text{ or for } g = g_c = 0. \tag{149}$$

In the latter case, we can use, as suggested by Levine, the Jones' uniqueness theorem [42] for surfaces conical at infinity, when Neumann boundary condition ($\partial_n p(r)|_S = 0$) is satisfied, which implies $u \equiv 0$ in the entire domain Ω, and thus completes the proof of uniqueness.

Appendix D : Evaluation of $\mathcal{R}_a(\alpha)$ and $\mathcal{R}_b(\alpha)$ taking account of complex poles

For large $k\Delta$, we consider (45)-(46) and use the method of Steepest Descent Path taking account of the influence of complex poles of $\mathcal{R}_{a,b}(\alpha')$. During the deformation of the initial path of integration along imaginary axis to SDP, where $\mathrm{Im}(ik(\sin(\alpha'/2))^2) = 0$ and $\mathrm{Re}(ik(\sin(\alpha'/2))^2) \le 0$, poles of $\mathcal{R}_a(\alpha')$ and $\mathcal{R}_b(\alpha')$ can be captured, and we have

$$\begin{aligned}\mathcal{R}_b(\alpha) &= \frac{-i}{4\Phi_+} m_1(\alpha) \int_{SDP} \mathcal{R}_a(\alpha') \frac{e^{-ik\Delta\cos\alpha'} \sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})} d\alpha' \\ &+ \frac{2\pi}{4\Phi_+} \sum_s \frac{e^{-ik\Delta\cos\alpha'_s} \sin(\frac{\pi\alpha'_s}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'_s}{2\Phi_+})} \epsilon_{a,s} \mathrm{Residue}(\mathcal{R}_a(\alpha'))|_{\alpha'=\alpha_s, \mathrm{Re}\alpha_s<0} \\ &+ m_1(\alpha) \chi_b^i(\alpha + \pi/2 - \Phi_b/2) \end{aligned} \tag{150}$$

as $\alpha + \pi/2 - \Phi_b/2$ is in the domain limited by $SDP - \Phi_+$ and $SDP + 3\Phi_+$, and,

$$\begin{aligned}\mathcal{R}_a(\alpha) &= \frac{i}{4\Phi_-} m_2(\alpha) \int_{SDP} \mathcal{R}_b(\alpha') \frac{e^{-ik\Delta\cos\alpha'} \sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})} d\alpha' \\ &+ \frac{2\pi}{4\Phi_-} \sum_s \frac{e^{-ik\Delta\cos\alpha'_s} \sin(\frac{\pi\alpha'_s}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'_s}{2\Phi_-})} \epsilon_{b,s} \mathrm{Residue}(\mathcal{R}_b(\alpha'))|_{\alpha'=\alpha_s, \mathrm{Re}\alpha_s>0} \\ &+ m_2(\alpha) \chi_a^i(\alpha - \pi/2 + \Phi_a/2), \end{aligned} \tag{151}$$

as $\alpha - \pi/2 + \Phi_a/2$ is in the domain limited by $SDP - 3\Phi_-$ and $SDP + \Phi_-$, where

$$\begin{aligned}&\mathcal{R}_b(\alpha) = (\frac{f_{br}(\alpha + \pi/2 - \Phi_b/2)}{\Psi_{1-}(\alpha + \Phi_-)}),\ \mathcal{R}_a(\alpha) = (\frac{f_{ar}(\alpha - \pi/2 + \Phi_a/2)}{\Psi_{+1}(\alpha - \Phi_+)}), \\ &m_1(\alpha) = \frac{\Psi_{+1}(\alpha + \pi/2 - \Phi_b/2)}{\Psi_{1-}(\alpha + \Phi_-)},\ m_2(\alpha) = \frac{\Psi_{1-}(\alpha - \pi/2 + \Phi_a/2)}{\Psi_{+1}(\alpha - \Phi_+)}, \\ &\chi_b^i(\alpha) = \frac{\pi}{2\Phi_+}(\frac{e^{ik\Delta\sin\varphi_\circ} \cos\frac{\pi\varphi_{\circ,b}}{2\Phi_+}}{\Psi_{+1}(\varphi_{\circ,b})(\sin\frac{\pi\alpha}{2\Phi_+} - \sin\frac{\pi\varphi_{\circ,b}}{2\Phi_+})}), \\ &\chi_a^i(\alpha) = \frac{\pi}{2\Phi_-}(\frac{\cos\frac{\pi\varphi_{\circ,a}}{2\Phi_-}}{\Psi_{1-}(\varphi_{\circ,a})(\sin\frac{\pi\alpha}{2\Phi_-} - \sin\frac{\pi\varphi_{\circ,a}}{2\Phi_-})}), \end{aligned} \tag{152}$$

with $\varphi_{\circ,b} = \varphi_\circ - \Phi_b/2$, $\varphi_{\circ,a} = \varphi_\circ + \Phi_a/2$, and $\epsilon_{a(b),s} = 1$ if the pole s is captured, $\epsilon_{a(b),s} = 0$ else.

For large $k\Delta$, the residue terms are principally given by

$$\begin{aligned}&\text{Residue}(\mathcal{R}_a(\alpha'))|_{\alpha'=\alpha_s} \sim \text{Residue}(m_2(\alpha')\chi_a^i(\alpha'-\pi/2+\Phi_a/2))|_{\alpha'=-\theta_1}\\&\text{Residue}(\mathcal{R}_b(\alpha'))|_{\alpha'=\alpha_s} \sim \text{Residue}(m_1(\alpha')\chi_b^i(\alpha'+\pi/2-\Phi_b/2))|_{\alpha'=\theta_1}\end{aligned} \tag{153}$$

where $\alpha = -\theta_1$ and $\alpha=\theta_1$ are the zeros of $\Psi_{+1}(\alpha-\Phi_+)$ and $\Psi_{1-}(\alpha+\Phi_-)$ the closest to the imaginary axis, that can be captured. Letting $t = -i\sqrt{2ik\Delta}\sin(\alpha'/2)$, we then derive,

$$\begin{aligned}\mathcal{R}_b(\alpha) \sim\ & \frac{m_1(\alpha)(\,\partial_\alpha \mathcal{R}_a(\alpha)|_{\alpha=0})}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}(v\,e^{v^2}\text{erfc}(v)-\frac{1}{\sqrt{\pi}})\\&+\frac{2\pi}{4\Phi_+}\frac{e^{-ik\Delta\cos\theta_1}\sin(\frac{\pi\theta_1}{2\Phi_+})\chi_a^i(\theta_1-\pi/2+\Phi_a/2)}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\theta_1}{2\Phi_+})}\epsilon_a\text{Residue}(m_2(\alpha'))|_{\alpha'=-\theta_1}\\&+m_1(\alpha)\chi_b^i(\alpha+\pi/2-\Phi_b/2),\end{aligned} \tag{154}$$

with $v^2 = ik\Delta(\frac{\pi}{2\Phi_+})^{-2}(1+\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b)))$, $\alpha+\pi/2-\Phi_b/2$ in the domain limited by $SDP-\Phi_+$ and $SDP+3\Phi_+$, and,

$$\begin{aligned}\mathcal{R}_a(\alpha) \sim\ & -\frac{m_2(\alpha)(\,\partial_\alpha \mathcal{R}_b(\alpha)|_{\alpha=0})}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}(w\,e^{w^2}\text{erfc}(w)-\frac{1}{\sqrt{\pi}})\\&-\frac{2\pi}{4\Phi_-}\frac{e^{-ik\Delta\cos\alpha'_s}\sin(\frac{\pi\theta_1}{2\Phi_-})\chi_b^i(-\theta_1+\pi/2-\Phi_b/2)}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\theta_1}{2\Phi_-})}\epsilon_b\text{Residue}(m_1(\alpha'))|_{\alpha'=-\theta_1}\\&+m_2(\alpha)\chi_a^i(\alpha-\pi/2+\Phi_a/2),\end{aligned} \tag{155}$$

with $w^2 = ik\Delta(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)))$, $\alpha-\pi/2+\Phi_a/2$ in the domain limited by $SDP-3\Phi_-$ and $SDP+\Phi_-$, for $-\min(\pi/2,\ \pi/2+\Phi_a) < \varphi_\circ < \min(\pi/2,\ \pi/2+\Phi_b)$. To obtain $\partial_\alpha\mathcal{R}_b(\alpha)|_{\alpha=0}$ and $\partial_\alpha\mathcal{R}_a(\alpha)|_{\alpha=0}$, we derivate the previous expressions and solve the linear system of equations. In this case, $v|_{\alpha=0}$ and $w|_{\alpha=0}$ are large, and using

$$z\,e^{z^2}\text{erfc}(z)-\frac{1}{\sqrt{\pi}} = \frac{-1}{2\sqrt{\pi}z^2}+O(\frac{1}{z^4}), \tag{156}$$

we obtain, for large $k\Delta$,

$$\begin{aligned}&\partial_\alpha\mathcal{R}_b(\alpha)|_{\alpha=0} \sim\ \partial_\alpha(m_1(\alpha)\chi_b^i(\alpha+\pi/2-\Phi_b/2))|_{\alpha=0}\\&\partial_\alpha\mathcal{R}_a(\alpha)|_{\alpha=0} \sim\ \partial_\alpha(m_2(\alpha)\chi_a^i(\alpha-\pi/2+\Phi_a/2))|_{\alpha=0}\end{aligned} \tag{157}$$

In consequence, we can write

$$\mathcal{R}_b(\alpha) \sim \frac{m_1(\alpha)\partial_\alpha(m_2(\alpha)\chi_a^i(\alpha-\pi/2+\Phi_a/2))|_{\alpha=0}}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}(v\, e^{v^2}\text{erfc}(v) - \frac{1}{\sqrt{\pi}})$$
$$+ \frac{2\pi}{4\Phi_+} \frac{e^{-ik\Delta\cos\theta_1}\sin(\frac{\pi\theta_1}{2\Phi_+})\chi_a^i(\theta_1-\pi/2+\Phi_a/2)}{\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b)) + \cos(\frac{\pi\theta_1}{2\Phi_+})} \epsilon_a \text{Residue}(m_2(\alpha'))|_{\alpha'=\theta_1}$$
$$+ m_1(\alpha)\chi_b^i(\alpha+\pi/2-\Phi_b/2), \qquad (158)$$

with $v^2 = ik\Delta(\frac{\pi}{2\Phi_+})^{-2}(1+\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b)))$, $\alpha+\pi/2-\Phi_b/2$ in the domain limited by $SDP-\Phi_+$ and $SDP+3\Phi_+$, and,

$$\mathcal{R}_a(\alpha) \sim -\frac{m_2(\alpha)\partial_\alpha(m_1(\alpha)\chi_b^i(\alpha+\pi/2-\Phi_b/2))|_{\alpha=0}}{\sqrt{2ik\Delta}} 2e^{-ik\Delta}(w\, e^{w^2}\text{erfc}(w) - \frac{1}{\sqrt{\pi}})$$
$$- \frac{2\pi}{4\Phi_-} \frac{e^{-ik\Delta\cos\alpha'_s}\sin(\frac{\pi\theta_1}{2\Phi_-})\chi_b^i(-\theta_1+\pi/2-\Phi_b/2)}{\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)) + \cos(\frac{\pi\theta_1}{2\Phi_-})} \epsilon_b \text{Residue}(m_1(\alpha'))|_{\alpha'=-\theta_1}$$
$$+ m_2(\alpha)\chi_a^i(\alpha-\pi/2+\Phi_a/2), \qquad (159)$$

with $w^2 = ik\Delta(\frac{\pi}{2\Phi_-})^{-2}(1+\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a)))$, $\alpha-\pi/2+\Phi_a/2$ in the domain limited by $SDP-3\Phi_-$ and $SDP+\Phi_-$, while $-\min(\pi/2,\ \pi/2+\Phi_a) < \varphi_\circ < \min(\pi/2,\ \pi/2+\Phi_b)$. In right-handed sides of previous expressions, the first term corresponds to double diffraction mechanism, the second term to the influence of guided waves, and the third term to single diffraction mechanism.

Remark :

When $k\Delta$ is large, the highly oscillatory nature of the kernels exponential term leads to simple approximations. The spectral functions as the source terms are smooth functions, except when we approach the limits of validity of integrals, and the Steepest Descent Path method [1],[2] can be used to approximate analytically the integral terms. Letting $\cos\alpha' = 1-x^2/2$ with $x = 2\sin(\alpha'/2)$, we have elementary integrals of the type

$$e^{-ik\Delta}\int_{-i\infty}^{+i\infty} \frac{l(x)}{(x^2+x_0^2)} x e^{ik\Delta x^2/2} dx = \frac{1}{2}\int_{-i\infty}^{+i\infty} \frac{l(x)-l(-x)}{(x^2+x_0^2)} x e^{ik\Delta x^2/2} dx, \qquad (160)$$

The integration path being deformed to SDP, certain poles $\alpha=\alpha_i$ are captured during the deformation and the integral term is developed in vicinity of the saddle point, which gives us

$$e^{-ik\Delta}\partial_x(l(x))|_{x=0}\int_{SDP} \frac{x^2}{(x^2+x_0^2)} e^{\frac{ik\Delta x^2}{2}} dx + \sum_{\alpha_i} c_i e^{-ik\Delta\cos\alpha_i}, \qquad (161)$$

The integral term can be calculated, using that,

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{t^2e^{-t^2}}{z^2-t^2}dt=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{z^2e^{-t^2}}{z^2-t^2}dt-\frac{i}{\sqrt{\pi}}=ze^{-z^2}\text{erfc}(-iz)-\frac{i}{\sqrt{\pi}},\qquad(162)$$

where $\text{erfc}(z)=1-\text{erf}(z)\sim e^{-z^2}(1/z-1/(2z^3))/\sqrt{\pi}$ for large z, $\text{erf}(z)$ being the error function [34],[39] $\text{Im}z>0$. Notice that, in the approximation (161), the contribution of complex poles α_i with non nul imaginary part correspond to exponentially decreasing waves guiding along the central plate. They are negligible in first approximation when $k\Delta$ is large.

Appendix E : The scattering diagram from the solutions of the integral equations

In a previous section, we have reduced the problem to a system of non-singular integral equations. When the solutions of the integral equations are known, the different elements of the decomposition of the field can be evaluated from the integral expressions (45)-(46), where the integral terms can be considered as smooth coupling terms between both edges. In this case, the difference functional equations (0)-(28) have to be used to reduce the calculus in the band of validity of integral expressions.

Here, we study the reduction of the field term with radial dependence $\exp(-ik\rho_{a,b})/\sqrt{2\pi k\rho_{a,b}}$ for large $\rho_{a,b}$. Its angular dependence $\mathcal{F}_{a,b}$ in the direction $\varphi_a=\varphi_b=\varphi$ is commonly called the scattering diagram (or directivity), given, from (5), by

$$\mathcal{F}_{a,b}(\varphi)=-e^{-i\pi/4}(f_{a,b}(\pi+\varphi)-f_{a,b}(-\pi+\varphi)),\qquad(163)$$

where, from (25), $\mathcal{F}_a(\varphi)=e^{ik\Delta\sin\varphi}\mathcal{F}_b(\varphi)$.

We illustrate the development by the one of $\mathcal{F}_b(\varphi_b)$ in the case where $\Phi_b>-\frac{\pi}{2}$, $\Phi_a>0$, and $-\min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_a)<\varphi_\circ<\min(\frac{\pi}{2},\ \frac{\pi}{2}+\Phi_b)$, when $-\frac{\pi}{2}<\varphi_b<\frac{\pi}{2}+\Phi_b$ (similar to $\mathcal{F}_a(\varphi_a)$ when $-\frac{\pi}{2}<-\varphi_a<\frac{\pi}{2}+\Phi_a$, $\Phi_a>-\frac{\pi}{2}$, $\Phi_b>0$).

We consider at first the reduction of $f_b(\pi+\varphi_b)$ in (163). If $\Phi_b>0$, then $-\frac{\pi}{2}<\pi+\varphi_b<\frac{3\pi}{2}+2\Phi_b$, and we are in the domain of validity of (45)-(46). If $\Phi_b<0$, then we have a concave part and there are two cases. If $-\frac{\pi}{2}<\pi+\varphi_b<\frac{3\pi}{2}+2\Phi_b$, (45)-(46) directly applies. In contrary, if $\frac{3\pi}{2}+2\Phi_b<\pi+\varphi_b<\frac{3\pi}{2}+\Phi_b$, we use (30) and write

$$f_b(\pi+\varphi_b)=\frac{(-\sin(\frac{\pi}{2}-\Phi_b+\varphi_b)+\sin\theta_+)}{(\sin(\frac{\pi}{2}-\Phi_b+\varphi_b)+\sin\theta_+)}f_b(-\varphi_b+2\Phi_b),\qquad(164)$$

Then we use (0),

$$f_b(-\varphi_b+2\Phi_b)=\frac{(\sin(\frac{\pi}{2}-\varphi_b+2\Phi_b)+\sin\theta_1)}{(-\sin(\frac{\pi}{2}-\varphi_b+2\Phi_b)+\sin\theta_1)}(f_b(-\pi+\varphi_b-2\Phi_b)$$
$$-e^{ik\Delta\sin(2\Phi_b-\varphi_b)}f_a(-\pi+\varphi_b-2\Phi_b))+e^{ik\Delta\sin(2\Phi_b-\varphi_b)}f_a(-\varphi_b+2\Phi_b), \quad (165)$$

where the terms in right handed-side of (165) can be expressed with (45)-(46).
Then, we reduce $f_b(-\pi+\varphi_b)$ in (163). If $-\frac{\pi}{2}<-\pi+\varphi_b<\frac{3\pi}{2}+2\Phi_b$, i.e. $\varphi_b>\frac{\pi}{2}$, which implies $\Phi_b>0$, we are in the domain of validity of (45)-(46). In contrary, when $-\frac{\pi}{2}<\varphi_b<\frac{\pi}{2}$, which is obliged when $\Phi_b<0$, we first have to use (30),

$$f_b(-\pi+\varphi_b)=\frac{(\sin(\frac{\pi}{2}-\varphi_b)-\sin\theta_1)}{(-\sin(\frac{\pi}{2}-\varphi_b)-\sin\theta_1)}$$
$$\times(f_b(-\varphi_b)-e^{-ik\Delta\sin\varphi_b}f_a(-\varphi_b))+e^{-ik\Delta\sin\varphi_b}f_a(-\pi+\varphi_b). \quad (166)$$

In this case, since $-\frac{3\pi}{2}-2\Phi_a<-\varphi_b<\frac{\pi}{2}$ and $-\frac{3\pi}{2}-2\Phi_a<-\pi+\varphi_b<\frac{\pi}{2}$ when $\Phi_a>0$, and $-\frac{\pi}{2}<-\varphi_b<\frac{3\pi}{2}+2\Phi_b$ when $\Phi_b>-\frac{\pi}{2}$, $f_a(-\pi+\varphi_b)$ and $f_a(-\varphi_b)$ can be expressed with (46), and $f_b(-\varphi_b)$ with (45), which ends the reduction.

Appendix F : Approximated second order ameliorations

F.1) Amelioration of equivalent cavity impedance $\sin\theta_1'$

Using our works in [28], we can modifiy and add a second order to the expressions of $\sin\theta_1'$. We can write,

$$\sin\theta_1'=\sum_j d_j\sin\theta_j/\Delta+ik(S_+-S_-)/\Delta, \quad (167)$$

where S_+ is the total surface of cavities below the straight line between the exterior discontinuities 'a' and 'b', while S_- is the total surface of cavities above the straight line. In the case where some elements would be filled with materials of impedance $Z_j=\sqrt{\epsilon_2\mu_2}/\epsilon_2$ (mode H when $u\equiv H_z$) or admittance $Y_j=\sqrt{\epsilon_2\mu_2}/\mu_2$ (mode E when $u\equiv E_z$), and with wave number k_m, we can write,

$$\sin\theta_1'|_{\text{mode }H}=\sum_j d_j\sin\theta_j/\Delta+\sum_m i\frac{Z_m}{Z_0}k_m(S_{m+}-S_{m-})/\Delta,$$
$$\sin\theta_1'|_{\text{mode }E}=\sum_j d_j\sin\theta_j/\Delta+\sum_m i\frac{Y_m}{Y_0}k_m(S_{m+}-S_{m-})/\Delta, \quad (168)$$

where the $\sin\theta_j$ remain the relative surface in free space.

F.2) A change of M_u

We can refine, from [1],[2],[28], the expression of the mean value M_u of u on the cavity,

$$M_u \sim \frac{(1-\frac{i}{\pi}B_0\sin\varphi'_\circ\sin\Phi_b)}{1+\frac{i}{\pi}B_0(\sin\theta'_1-\sin\theta_+\cos\Phi_b)}\frac{2\cos(\varphi'_\circ)}{(\cos(\varphi'_\circ)+\sin\theta_+)} = D_0 M_u^0$$
$$B_0 = \int_{-i\infty}^{+i\infty} d\alpha' \frac{(e^{-ik\Delta\cos\alpha'}-1)}{2\cos\alpha'} \sim -k\Delta(\ln(k\Delta)+\gamma_0-1+i\pi/2)$$
$$D_0 = \frac{(1-\frac{i}{\pi}B_0\sin\varphi'_\circ\sin\Phi_b)}{1+\frac{i}{\pi}B_0(\sin\theta'_1-\sin\theta_+\cos\Phi_b)}, \gamma_0 \approx .577, \quad (169)$$

and write,

$$f_a(\pi+\varphi'+\Phi_b) - f_a(-\pi+\varphi'+\Phi_b)$$
$$\sim \frac{2ik\Delta\cos\varphi'_\circ\cos\varphi'}{(\cos\varphi'_\circ+\sin\theta_+)(\cos\varphi'+\sin\theta_+)}(-\cos\Phi_b\sin\theta_+ + \sin\Phi_b\sin\varphi'$$
$$+\sin\theta'_1)\times D_0 + \frac{2ik\Delta\sin\Phi_b(\cos\varphi'-\sin\theta_+)\cos\varphi'\cos\varphi'_\circ}{(\cos\varphi'_\circ+\sin\theta_+)(\sin\varphi'+\sin\varphi'_\circ)}. \quad (170)$$

A more refined expression could be found by semi-inversion (see appendix B).

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