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Finite volume scheme and renormalized solutions for nonlinear elliptic Neumann problem with L^1 data

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Abstract

In this paper we study the convergence of a finite volume approximation of a convective diffusive elliptic problem with Neumann boundary conditions and L^1 data. To deal with the non-coercive character of the equation and the low regularity of the right hand-side we mix the finite volume tools and the renormalized techniques. To handle the Neumann boundary conditions we choose solutions having a null median and we prove a convergence result.

1 Introduction

In the present paper we are interested in the discretization by the cell-centered finite volume method of the following convection-diffusion equation with Neumann boundary conditions and L^1 data:

$$\begin{aligned} -\operatorname{div}(\lambda(u)\nabla u - \mathbf{v}u) &= f && \text{in } \Omega, \\ (\lambda(u)\nabla u - \mathbf{v}u) \cdot \vec{n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded polygonal connected open subset of \mathbb{R}^d , $d \geq 2$, \vec{n} is the outer unit normal to $\partial\Omega$ and λ is a continuous function such that $\lambda_\infty \geq \lambda(u) \geq \mu > 0$ with λ_∞ and μ two real numbers. The function \mathbf{v} lies in $L^p(\Omega)^d$ with $2 < p < +\infty$ if $d = 2$, $p = d$ if $d \geq 3$, and f belongs to $L^1(\Omega)$ and satisfies the compatibility condition $\int_\Omega f = 0$.

Considering elliptic equations with L^1 data requires a precise meaning of solution. Indeed we cannot expect in general to obtain a usual weak solution which belongs to $H_0^1(\Omega)$ for Dirichlet boundary conditions or to $H^1(\Omega)$ for Neumann boundary conditions. Elliptic equations with L^1 data and Dirichlet boundary conditions are widely studied in the literature. In [6] Boccardo and Gallouët have obtained the existence of a solution in the sense of distributions for a fairly class of monotone operator with measure data. However it is known that this solution is not unique in general (see the counter example of Serrin [24]). To overcome the lack of uniqueness results, it is possible to use in the linear case the duality method (see [25]) or, for general nonlinear operators, the notion of entropy solution (see [11]), the notion of solution obtained as limit of approximation (SOLA) (see [12]) or the notion of renormalized solution (see [22, 11]). The previous three notions of solution are equivalent in the L^1 case and provide existence, stability and uniqueness results for a large class of elliptic equations. As far as the approximation of elliptic equations with Dirichlet boundary conditions and L^1 data is concerned, the method of finite volume (see [18]) allows to consider such equations. In [19] the authors have studied equation (1.1) with $\mathbf{v} = 0$ and with a measure data (and Dirichlet boundary conditions). In [15] the authors

have considered a linear noncoercive equation (similar to (1.1)) with measure data (and Dirichlet boundary conditions). In both papers [19, 15] the authors have established the convergence of the finite volume approximation to a solution in the sense of distributions. More precisely for the equation $-\Delta u + \operatorname{div}(vu) = f$ in Ω with Dirichlet boundary conditions, the limit u of the finite volume scheme verifies

$$\begin{cases} u \in \bigcap_{q < d/(d-1)} W_0^{1,q}(\Omega) \\ \int_{\Omega} \nabla u \nabla \varphi \, dx - \int_{\Omega} uv \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in \bigcup_{s > d} W_0^{1,s}(\Omega). \end{cases}$$

Recently mixing the techniques of renormalized solution and the finite volume approximation has been performed in [21] for a noncoercive equation with L^1 data and Dirichlet boundary conditions: the author proves that the limit of the finite volume scheme is the renormalized solution of the equation. Concerning the finite elements approximation the model case of the equation $-\operatorname{div}(A\nabla u) = f$ with Dirichlet boundary conditions is dealt in [7].

In the present paper we have to face to a noncoercive equation, to an L^1 data and to Neumann boundary conditions. To our knowledge such a situation is less studied in the literature both in the continuous case and the discrete case. One of the difficulty in the variational and linear case is that the kernel is nontrivial and that we have to impose an additional condition on the solution to insure uniqueness result, which is in general $\int_{\Omega} u \, dx = 0$. In [16] by using the Fredholm theory the authors have been studied the operator associated to the linear version of (1.1). They prove that the linear version of (1.1) with $(H^1)'$ data verifying a compatibility condition admits a unique weak solution. Moreover they deduce existence and uniqueness results for elliptic and coercive equation of the type $-\operatorname{div}(A(x, u)\nabla u) = \mu$ with Neumann boundary condition, where μ is a bounded Radon measure. The finite volume approximation of (1.1) with f belonging to $L^2(\Omega)$ (with zero mean value) is studied in [8]. As in the continuous case the finite volume approximation requires the study of the kernel and for different approximations of the convective terms the authors prove in [8] that the finite volume approximation converges to a weak solution of (1.1). For the class of nonlinear elliptic equations $-\Delta_p u = f$ with Neumann boundary conditions, L^1 data and for small value of p it is well known that the solution is not in general a summable function so that the mean value has no meaning. To overcome this obstacle, in [1, 4] the authors have chosen the median value which is well defined instead of the mean value. In [4] an appropriate definition of renormalized solutions is given, which gives an existence result (see also [5] for the uniqueness question). The main originality of the present paper is to consider noncoercive equation (1.1) with L^1 data and to mix the techniques developed in [4] and the finite volume method. We choose here the median value instead of the mean value as in [8]. Since Poincaré-Wirtinger inequality is crucial in general we state in Proposition 2.8 an appropriate discrete Poincaré-Wirtinger inequality involving the median value (see Appendix for the proof, in the spirit of [3]). In Theorem 3.2 we prove that the finite volume approximation of (1.1) converges to the renormalized solution with a null median.

The paper is organized as follows. In Section 2 we recall some definitions, in particular the median of a measurable function. Moreover we present in Section 2 the continuous case and the notion of renormalized solution of (1.1) and, at last the finite volume tools and the scheme. The main results are stated in Section 3. Section 4 is devoted to derive the a priori estimates for the solutions of the scheme. Using Section 4 we prove the existence of a solution of the scheme in Section 5 while the convergence analysis is performed in Section 6. Finally we give in Appendix the proof of the discrete Poincaré-Wirtinger inequality involving the median (instead of the mean value).

2 Assumptions and definitions

Let Ω be a connected open bounded polygonal subset of \mathbb{R}^d , $d \geq 2$. We consider the following nonlinear elliptic problem with Neumann boundary conditions:

$$\begin{cases} -\operatorname{div}(\lambda(u)\nabla u - \mathbf{v}u) = f & \text{in } \Omega, \\ (\lambda(u)\nabla u - \mathbf{v}u) \cdot \vec{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where \vec{n} is the outer unit normal to $\partial\Omega$. We assume that

$$\mathbf{v} \in L^p(\Omega)^d \text{ with } 2 < p < +\infty \text{ if } d = 2, p = d \text{ if } d \geq 3, \quad (2.2)$$

$$\lambda \text{ is a continuous function such that } \lambda_\infty \geq \lambda(r) \geq \mu > 0, \forall r \in \mathbb{R}, \quad (2.3)$$

with λ_∞ and μ two real numbers. Moreover, we assume that

$$f \in L^1(\Omega), \quad (2.4)$$

and it satisfies the compatibility condition

$$\int_{\Omega} f \, dx = 0. \quad (2.5)$$

As explained in the Introduction we deal with solutions whose median is equal to zero. Let us recall that if u is measurable function, we define the median of u (with respect to the Lebesgue measure), denoted by $\operatorname{med}(u)$ as the set of real numbers t such that

$$\begin{aligned} \operatorname{meas}\{x \in \Omega : u(x) > t\} &\leq \frac{\operatorname{meas}(\Omega)}{2} \\ \operatorname{meas}\{x \in \Omega : u(x) < t\} &\leq \frac{\operatorname{meas}(\Omega)}{2}. \end{aligned}$$

It is known that $\operatorname{med}(u)$ is non-empty compact interval (see [26]). Let us explicitly observe that if $0 \in \operatorname{med}(u)$ then

$$\begin{aligned} \operatorname{meas}\{x \in \Omega : u(x) > 0\} &\leq \frac{\operatorname{meas}(\Omega)}{2} \\ \operatorname{meas}\{x \in \Omega : u(x) < 0\} &\leq \frac{\operatorname{meas}(\Omega)}{2}. \end{aligned}$$

We denote $\underline{\operatorname{med}}(u)$ by

$$\underline{\operatorname{med}}(u) = \inf \left\{ t \in \mathbb{R} : \operatorname{meas}\{x \in \Omega : u(x) > t\} \leq \frac{\operatorname{meas}(\Omega)}{2} \right\}, \quad (2.6)$$

and $\overline{\operatorname{med}}(u)$ by

$$\overline{\operatorname{med}}(u) = \sup \left\{ t \in \mathbb{R} : \operatorname{meas}\{x \in \Omega : u(x) > t\} \geq \frac{\operatorname{meas}(\Omega)}{2} \right\}. \quad (2.7)$$

We observe that if u is an element of $H^1(\Omega)$ (Ω being a connected domain), the median of u is uniquely determined; $\operatorname{med}(u) = \underline{\operatorname{med}}u = \overline{\operatorname{med}}(u)$. However it is not the case for the finite volume approximation of (2.18) which is a piecewise-constant function; the median is then the compact interval of \mathbb{R} $[\underline{\operatorname{med}}(u), \overline{\operatorname{med}}(u)]$.

In the whole paper, T_n , $n \geq 0$, denotes the truncation at height n that is

$$T_n(s) = \min(n, \max(s, -n)), \quad \forall s \in \mathbb{R}.$$

2.1 Continuous Case

In this subsection we precise the notion of solution of equation (2.1). Indeed as explained in the Introduction, considering elliptic equations with L^1 data requires an appropriate notion of solution which provides existence, stability and uniqueness results. There is a wide literature in the Dirichlet case. In the Neumann case, due to the lack of regularity of the solution, the mean value may not exist for nonlinear problems with L^1 data, which gives additional difficulties in deriving estimates and in defining an appropriate notion of renormalized solution. We refer mainly to [13] for linear problems using the duality method and to [23], [1] and [4] for nonlinear problems. In [1] and [4] the authors have chosen the median instead of the mean value (which may not exist if the solution is not integrable) and one of the main tool is the following Poincaré-Wirtinger inequality, see [26].

Proposition 2.1. *If $u \in W^{1,p}(\Omega)$, then*

$$\|u - \text{med}(u)\|_{L^p(\Omega)} \leq C \|\nabla u\|_{(L^p(\Omega))^d} \quad (2.8)$$

where C is a constant depending on p , d , Ω .

In [4] the authors prove the existence of a renormalized solution for a class of nonlinear problems and prove in [5] uniqueness results under additional assumptions. In the present paper we use the framework of renormalized solutions. In the particular case of equation (2.1), let us recall the following definition (see [4]).

Definition 2.2. *A real function u defined in Ω is a renormalized solution to (2.1) if*

$$u \text{ is measurable and finite almost everywhere in } \Omega, \quad (2.9)$$

$$T_n(u) \in H^1(\Omega), \text{ for any } n > 0, \quad (2.10)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{x \in \Omega, |u(x)| < n\}} \lambda(u) |\nabla u|^2 dx = 0, \quad (2.11)$$

and the following equation holds

$$\begin{aligned} & \int_{\Omega} S(u) \lambda(u) \nabla u \cdot \nabla \varphi dx + \int_{\Omega} S'(u) \lambda(u) \varphi \nabla u \cdot \nabla u dx \\ & - \int_{\Omega} u S(u) \mathbf{v} \cdot \nabla \varphi dx - \int_{\Omega} u S'(u) \varphi \mathbf{v} \cdot \nabla u dx = \int_{\Omega} f \varphi S(u) dx, \end{aligned} \quad (2.12)$$

for every $S \in W^{1,\infty}(\mathbb{R})$ having compact support and for every $\varphi \in L^\infty(\Omega) \cap H^1(\Omega)$.

By combining [4] and [5] we have the following existence and uniqueness result.

Theorem 2.3. *Let us assume that (2.2)–(2.5) hold true. Then there exists a unique renormalized solution u of (2.1) such that $\text{med}(u) = 0$.*

Remark 2.4. *As far as the uniqueness is concerned equation (2.1) is not directly in the scope of [5]. Indeed uniqueness results are mainly obtained for equations whose prototype is $-\text{div}(a(x, \nabla u) + \Phi(x, u)) = f$ with Neumann boundary conditions. The operator $a(x, \nabla u)$ does not depend on u . Due to the presence of $\lambda(u)$ in equation (2.1) the quasilinear character allows one to obtain the uniqueness by a changement of unknow. Since $\lambda(r)$ is a continuous function such that $\lambda_\infty \geq \lambda(r) \geq \mu > 0$, by defining $\tilde{\lambda}(r) = \int_0^r \lambda(s) ds$ and $w = \tilde{\lambda}(u)$, we can verify that the function w has a null median and that w is a renormalized solution of*

$$\begin{cases} -\text{div}(\nabla w - \mathbf{v} \tilde{\lambda}^{-1}(w)) = f & \text{in } \Omega, \\ (\nabla w - \mathbf{v} \tilde{\lambda}^{-1}(w)) \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.13)$$

At last since the function $\tilde{\lambda}^{-1}$ is Lipschitz continuous, Theorem 4.2 of [5] allows one to conclude that w is unique so that u is unique.

2.2 Finite Volume

We now introduce the discrete settings. Let us first recall the notion of admissible discretization of Ω , the definitions of the discrete norms and the space of piecewise functions associated to an admissible mesh following [18].

Definition 2.5 (Admissible mesh). *An admissible mesh \mathcal{M} of Ω is given by a finite family \mathcal{T} of disjoint open convex polygonal subsets of Ω , a finite family \mathcal{E} of disjoint subsets of Ω (the edges) consisting in non-empty open convex subsets of affine hyperplanes and a family $\mathcal{P} = (x_K)_{K \in \mathcal{T}}$ of points in Ω such that*

- $\bar{\Omega} = \cup_{K \in \mathcal{T}} \bar{K}$,
- each $\sigma \in \mathcal{E}$ is a non-empty open subset of ∂K for some $K \in \mathcal{T}$,
- by denoting $\mathcal{E}_K = \{\sigma \in \mathcal{E}, \sigma \subset \partial K\}$, $\partial K = \cup_{\sigma \in \mathcal{E}_K} \sigma$ for all $K \in \mathcal{T}$,
- for all $K \neq L$ in \mathcal{T} , either the $(d-1)$ -dimensional measure of $\bar{K} \cap \bar{L}$ is zero or $\bar{K} \cap \bar{L} = \bar{\sigma}$ for some $\sigma \in \mathcal{E}$, which is then denoted $\sigma = K|L$,
- for all $K \in \mathcal{T}$, $x_K \in K$,
- for all $\sigma = K|L \in \mathcal{E}$, the straight line (x_K, x_L) intersects and is orthogonal to σ ,
- for all $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega \cap \partial K$, the line which is orthogonal to σ and goes through x_K intersects σ .

In the whole of the present paper, we use the following notations associated with an admissible discretization. In the set of edges \mathcal{E} , we distinguish the set of interior edges \mathcal{E}_{int} and the set of boundary edges \mathcal{E}_{ext} . We denote by $m(K)$ the d -dimensional measure of a control volume K and $m(\sigma)$ the $(d-1)$ -dimensional measure of σ . For all $\sigma \in \mathcal{E}_K$, $\mathbf{n}_{K,\sigma}$ is the unit normal to σ outwards K . If $\sigma = K|L \in \mathcal{E}_{int}$, we denote by d_σ the Euclidian distance between x_K and x_L , $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ and $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$. The size of the mesh is defined by

$$h_{\mathcal{M}} = \sup_{K \in \mathcal{T}} \text{diam}(K).$$

We assume that the mesh satisfies the following assumption

$$\exists \xi > 0 \text{ such that } d(x_K, \sigma) \geq \xi d_\sigma, \quad \forall \mathcal{T} \in \mathcal{E}, \forall \sigma \in \mathcal{E}_K. \quad (2.14)$$

An example of admissible mesh in the sense of the above definition is shown in Figure 1.

The space of piecewise functions associated to an admissible mesh, denoted by $X(\mathcal{M})$, is defined as the set of functions from Ω to \mathbb{R} which are constant over each control volume of the mesh.

Definition 2.6 (Discrete $W^{1,p}$ norm). *Let Ω be an open bounded polygonal subset of \mathbb{R}^d , $d \geq 2$, and let \mathcal{M} be an admissible mesh. For $u = (u_K)_{K \in \mathcal{T}} \in X(\mathcal{T})$ and $p \in [1, +\infty[$, the discrete $W^{1,p}$ -semi-norm is defined by*

$$|u|_{1,p,\mathcal{M}} = \left(\sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{m(\sigma)}{d_\sigma^{p-1}} |u_K - u_L|^p \right)^{\frac{1}{p}}, \quad \forall u \in X(\mathcal{T})$$

and the discrete $W^{1,p}$ -norm is defined by

$$\|u\|_{1,p,\mathcal{M}} = \|u\|_{0,p} + |u|_{1,p,\mathcal{M}}, \quad \forall u \in X(\mathcal{T})$$

where $\|u\|_{0,p}$ is the L^p norm for piecewise constant functions, $\forall p \in [1, +\infty[$,

$$\|u\|_{0,p} = \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}} = \left(\sum_{K \in \mathcal{M}} m(K) |u_K|^p \right)^{\frac{1}{p}}, \quad \forall u \in X(\mathcal{T}).$$

We present now discrete functional analysis results. We refer the reader to [[8], Lemma 6.1] for a proof of the following discrete Sobolev inequality.

Proposition 2.7 (Discrete Sobolev inequality). *Let Ω be a bounded polygonal open subset of \mathbb{R}^d and let \mathcal{M} be an admissible mesh satisfying (2.14). Let $q < +\infty$ if $d = 2$ and $q = \frac{2d}{d-2}$ if $d \geq 3$. Then there exists $C = C(\Omega, \xi, q)$ such that, for all $u = (u_K)_{K \in \mathcal{T}} \in X(\mathcal{T})$,*

$$\|u\|_{0,q} \leq C (|u|_{1,2,\mathcal{M}} + \|u\|_{0,2}). \quad (2.15)$$

In the already cited references discrete Poincaré and Poincaré-Wirtinger inequalities are related to the discrete space $W_0^{1,p}(\Omega)$ and zero boundary condition or the discrete space $W^{1,p}(\Omega)$ with discrete mean value. We derive here a discrete Poincaré-Wirtinger inequality involving the median. The proof is given in the appendix.

Proposition 2.8 (Discrete Poincaré-Wirtinger median inequality). *Let Ω be an open bounded connected polyhedral domain of \mathbb{R}^d and let \mathcal{M} be an admissible mesh satisfying (2.14). Then for $1 \leq p < +\infty$ there exists a constant $C > 0$ only depending on Ω , d and p such that*

$$\|u - c\|_{0,p} \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \quad \forall u \in X(\mathcal{T}) \quad (2.16)$$

where c belongs to $\text{med}(u)$.

Theorem 2.9 (Discrete Rellich's theorem). *Let $(\mathcal{M}_m)_{m \geq 1}$ be a sequence of admissible meshes satisfying (2.14) and such that $h_{\mathcal{M}_m} \rightarrow 0$ as $m \rightarrow \infty$. If $v_m \in X(\mathcal{T}_m)$ is such that $(|v_m|_{1,2,\mathcal{M}} + \|v_m\|_{0,2})$ is bounded, then $(v_m)_{m \in \mathbb{N}}$ is relatively compact in $L^2(\Omega)$. Furthermore, any limit in $L^2(\Omega)$ of a subsequence of $(v_m)_{m \in \mathbb{N}}$ belongs to $H^1(\Omega)$.*

Let us now define a discrete finite volume gradient introduced equivalently in [[9], Lemma 4.4], [[14], Lemma 6.5] or [[17], Definition 2].

Definition 2.10 (Discrete finite volume gradient). *For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, we define the volume $D_{K,\sigma}$ as the cone of basis σ and of opposite vertex x_K . Then, we define the "diamond-cell" D_σ (see Figure 1) by*

$$\begin{aligned} D_\sigma &= D_{K,\sigma} \cup D_{L,\sigma} & \text{if } \sigma = K|L \in \mathcal{E}_{int}, \\ D_\sigma &= D_{K,\sigma} & \text{if } \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K, \end{aligned}$$

and

$$m(D_\sigma) = \frac{1}{d} d_\sigma m(\sigma).$$

The approximate gradient $\nabla_{\mathcal{M}} u$ of a function $u \in X(\mathcal{T})$ is defined as a piece-wise constant function over each diamond cell and given by

$$\begin{aligned} \forall \sigma \in \mathcal{E}_{int}, \sigma = K|L, & \quad \nabla_{\mathcal{M}} u(\mathbf{x}) = d \frac{u_L - u_K}{d_\sigma} \mathbf{n}_{K,\sigma}, & \quad \forall \mathbf{x} \in D_\sigma, \\ \forall \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K, & \quad \nabla_{\mathcal{M}} u(\mathbf{x}) = 0, & \quad \forall \mathbf{x} \in D_\sigma. \end{aligned}$$

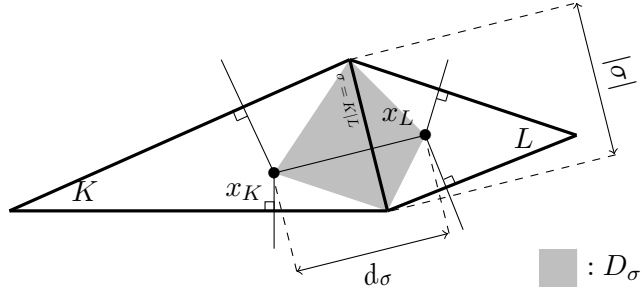


Figure 1: The diamond D_σ

Let us then give convergence property of the discrete gradient (see e.g., in the case of Dirichlet boundary condition, [9] and [17] in L^2 context, and [20] in the L^1 context).

Lemma 2.11 (Weak convergence of the finite volume gradient). *Let $(\mathcal{M}_m)_{m \geq 1}$ be a sequence of admissible meshes satisfying (2.14) and such that $h_{\mathcal{M}_m} \rightarrow 0$ as $m \rightarrow \infty$. Let $v_m \in X(\mathcal{T}_m)$ and let us assume that there exists $\alpha \in [1, +\infty[$ and $C > 0$ such that $\|v_m\|_{1,\alpha,\mathcal{M}_m} \leq C$, and that v_m converges in $L^1(\Omega)$ to $v \in W^{1,\alpha}(\Omega)$. Then $\nabla_{\mathcal{M}_m} v_m$ converges to ∇v weakly in $L^\alpha(\Omega)^d$.*

We now define the finite volume scheme. Let \mathcal{M} be an admissible mesh in the sense of definition 2.5. For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we define $\mathbf{v}_{K,\sigma}$ by

$$\mathbf{v}_{K,\sigma} = \frac{1}{m(D_\sigma)} \int_{D_\sigma} \mathbf{v} \cdot \mathbf{n}_{K,\sigma} \, dx. \quad (2.17)$$

We consider the following finite volume scheme for (1.1)

$$\forall K \in \mathcal{T}, \quad \sum_{\sigma \in \mathcal{E}_{int}(K)} \frac{m(\sigma)}{d_\sigma} \lambda(u)_\sigma (u_K - u_L) + \sum_{\sigma \in \mathcal{E}_{int}(K)} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+} = \int_K f \, dx, \quad (2.18)$$

and

$$\forall \sigma = K|L \in \mathcal{E}_{int}, \quad u_{\sigma,+} = \begin{cases} u_K & \text{if } \mathbf{v}_{K,\sigma} \geq 0, \\ u_L & \text{otherwise.} \end{cases} \quad (2.19)$$

We denote $u_{\sigma,-}$ the downstream choice of u , i.e. $u_{\sigma,-}$ is such that $\{u_{\sigma,+}, u_{\sigma,-}\} = \{u_K, u_L\}$, $\forall \sigma \in \mathcal{E}_{int}$.

Finally,

$$\forall \sigma = K|L \in \mathcal{E}_{int}, \quad \min[\lambda(u_K), \lambda(u_L)] \leq \lambda(u)_\sigma \leq \max[\lambda(u_K), \lambda(u_L)], \quad (2.20)$$

where $\lambda(u)_\sigma$ is for example the mean value of $\lambda(u_K)$ and $\lambda(u_L)$ if $\sigma \in \mathcal{E}_{int}$.

3 Main results

Our main results on the finite volume scheme are the following. The first one states that there exists at least one solution to the scheme. It is a generalization of Theorem 2.5 in [8] in the context of a quasilinear problem with a median value constraint instead of a mean value constraint. The second one gives the convergence of this solution to the unique renormalized solution of the continuous problem with null median, as the size of the mesh tends to 0.

Theorem 3.1 (Existence of the solution of the scheme). *Let us assume that (2.2)–(2.5) hold. Let \mathcal{M} be an admissible mesh in the sense of Definition 2.5 satisfying (2.14). Then there exists a solution $u_\mathcal{T} = (u_K)_{K \in \mathcal{M}}$ to (2.18)–(2.20) having $\underline{\text{med}}(u_\mathcal{T}) = 0$.*

Theorem 3.2 (Convergence of the solution of the scheme). *Let $(\mathcal{M}_m)_{m \geq 1}$ be a sequence of admissible meshes in the sense of Definition 2.5, which satisfy (2.14) and such that $h_{\mathcal{M}_m}$ goes to 0 as $m \rightarrow \infty$. Let $u_m = (u_K^m)_{K \in \mathcal{T}_m} \in X(\mathcal{T}_m)$ be a solution of (2.18) such that $\underline{\text{med}}(u_m) = 0$. Then u_m converges to the unique renormalized solution u of (1.1) having $\text{med}(u) = 0$, in the sense that*

$$\begin{aligned} &u_m \text{ converges to } u \text{ a.e. in } \Omega, \\ &\forall n \in \mathbb{N}, \nabla_{\mathcal{M}_m} T_n(u_m) \text{ converges to } \nabla T_n(u), \text{ weakly in } (L^2(\Omega))^d, \end{aligned}$$

as $m \rightarrow \infty$.

Remark 3.3. *As explained in Introduction we choose in the present paper a constraint on the median value instead of the mean value. It allows one to mix the techniques developed in [4] and the finite volume. Observe that the median is an appropriate choice in [1, 4] to deal with nonlinear elliptic equations with L^1 data and Neumann boundary conditions since we cannot expect to have a solution u (in the sense of distribution or in the renormalized sense) of $-\Delta_p u = f$ with p closed to 1 such that the solution belongs to $L^1(\Omega)$. However under the restriction $p > 2 - 1/N$ and using the Boccardo-Gallouët estimates it is possible to solve $-\Delta_p u = f$ with f in L^1 and Neumann boundary conditions in the sense of distributions with a mean value equal to zero, see [23]. As far as equation (1.1) is concerned a natural question is to solve its and to approximate its with $\int_{\Omega} u dx = 0$ and not $\text{med}(u) = 0$. To our knowledge the continuous case is not dealt in the literature. Starting from an approximate problem the difficulties are similar in passing to the limit in the continuous case and in the discrete case : the crucial steps are the a priori estimates stated in Section 4. Since we cannot give all the details of a possible proof we refer to [2].*

4 A priori estimates

This section is devoted to derive *a priori* estimates of the solution of the scheme (2.18), which are crucial to extract subsequences using compactness results and then to pass to the limit in the scheme. Let us observe that we adapt the strategy developed in [4] for the continuous problem (2.1) with Neumann boundary conditions to the discrete case and that we use the techniques developed in [8] for the approximation of the solution to problem (2.1) with a more regular data and Neumann boundary conditions and the ones of [15] and [21] which study the approximation of equations with L^1 (or measure data) with Dirichlet boundary conditions.

Proposition 4.1 (Estimate on $\ln(1 + |u_{\mathcal{M}}|)$ with $\underline{\text{med}}(u_{\mathcal{M}}) = 0$). *Let \mathcal{M} be an admissible mesh satisfying (2.14). If $u_{\mathcal{M}} = (u_K)_{K \in \mathcal{T}}$ is a solution to (2.18), such that $\underline{\text{med}}(u_{\mathcal{M}}) = 0$, then*

$$\|\ln(1 + |u_{\mathcal{M}}|)\|_{1,2,\mathcal{M}}^2 \leq C \left(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}} \|\mathbf{v}\|_{(L^p(\Omega))^d}^2 \right), \quad (4.1)$$

where $C = C(\Omega, \mu, \beta, p, \xi)$ is a positive constant.

Proof. A log-estimate was obtained in [Proposition 3.1, [15]] in the case of Dirichlet boundary conditions and $\mathbf{v} \in (C(\bar{\Omega}))^d$. Since we deal with Neumann boundary conditions, $\mathbf{v} \in (L^p(\Omega))^d$ and the specific choice of $\underline{\text{med}}(u_{\mathcal{M}}) = 0$, we will adapt the proof derived in [15] and explain the modifications. As in [15], let $\varphi(s) = \int_0^s \frac{dt}{(1+|t|)^2}$. Taking $\varphi(u_K)$ as a test function in the scheme (2.18) and reordering the sums yield

$$\sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_{\sigma}} \lambda(u)_{\sigma} (u_K - u_L) (\varphi(u_K) - \varphi(u_L)) \leq \|f\|_{L^1(\Omega)} + \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+} (\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})). \quad (4.2)$$

To control the second term of the right-hand side of (4.2) we introduce the set of edges \mathcal{A} (see [15]) by

$$\mathcal{A} = \{\sigma \in \mathcal{E}_{int}; u_{\sigma,+} \geq u_{\sigma,-}, u_{\sigma,+} < 0\} \cup \{\sigma \in \mathcal{E}_{int}; u_{\sigma,+} < u_{\sigma,-}, u_{\sigma,+} \geq 0\}, \quad (4.3)$$

Since φ is non-decreasing, as in [15] we obtain

$$\sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+} (\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})) \leq \sum_{\sigma \in \mathcal{A}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+} (\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})). \quad (4.4)$$

Now using Cauchy-Schwarz and Hölder inequalities, and the following inequality (see Lemma 3.1, [15]), $\forall \sigma \in \mathcal{A}$, $|u_{\sigma,+}|^2 |\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})|^2 \leq |u_{\sigma,-} - u_{\sigma,+}| |\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})|$, we obtain

$$\begin{aligned} \sum_{\sigma \in \mathcal{A}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+} |\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})| &\leq \left(\sum_{\sigma \in \mathcal{A}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{\sigma \in \mathcal{A}} \frac{m(\sigma)}{d_\sigma} |u_{\sigma,+}|^2 |\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\sigma \in \mathcal{A}} m(\sigma) d_\sigma \right)^{\frac{p-2}{p}} \left(\sum_{\sigma \in \mathcal{A}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^p \right)^{\frac{1}{p}} \\ &\quad \times \left(\sum_{\sigma \in \mathcal{A}} \frac{m(\sigma)}{d_\sigma} |u_{\sigma,+}|^2 |\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\sigma \in \mathcal{A}} m(\sigma) d_\sigma \right)^{\frac{p-2}{p}} \left(\sum_{\sigma \in \mathcal{A}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^p \right)^{\frac{1}{p}} \\ &\quad \times \left(\sum_{\sigma \in \mathcal{A}} \frac{m(\sigma)}{d_\sigma} |u_k - u_L| |\varphi(u_K) - \varphi(u_L)| \right)^{\frac{1}{2}}. \quad (4.5) \end{aligned}$$

Recalling that $\sum_{\sigma \in \mathcal{A}} m(\sigma) d_\sigma \leq \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) d_\sigma = dm(\Omega)$ and since the term $\left(\sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^p \right)^{\frac{1}{p}}$ is bounded by $d^{\frac{1}{p}} \|\mathbf{v}\|_{(L^p(\Omega))^d}$, by Young's inequality we get

$$\begin{aligned} \sum_{\sigma \in \mathcal{A}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+} |\varphi(u_{\sigma,-}) - \varphi(u_{\sigma,+})| &\leq \frac{1}{2\beta} dm(\Omega)^{\frac{p-2}{p}} \|\mathbf{v}\|_{(L^p(\Omega))^d}^2 \\ &\quad + \frac{\beta}{2} m(\Omega)^{\frac{p-2}{p}} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (u_K - u_L) (\varphi(u_K) - \varphi(u_L)), \end{aligned} \quad (4.6)$$

where $\beta > 0$. Since $0 < \mu \leq \lambda(u)$, an appropriate choice of β gives

$$\sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (u_K - u_L) (\varphi(u_K) - \varphi(u_L)) \leq C(\Omega, \mu, \beta, p) \left(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}} \|\mathbf{v}\|_{(L^p(\Omega))^d}^2 \right). \quad (4.7)$$

Moreover we have, for all $(x, y) \in \mathbb{R}^2$, $(\ln(1 + |x|) - \ln(1 + |y|))^2 \leq (x - y)(\varphi(x) - \varphi(y))$. It follows that

$$\sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (\ln(1 + |u_K|) - \ln(1 + |u_L|))^2 \leq C(\Omega, \mu, \beta, p) \left(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}} \|\mathbf{v}\|_{(L^p(\Omega))^d}^2 \right). \quad (4.8)$$

Since $\underline{\text{med}}(\ln(1 + u_{\mathcal{M}})) = 0$, the discrete Poincaré-Wirtinger inequality (2.16) implies that

$$\|\ln(1 + |u_{\mathcal{M}}|)\|_{1,2,\mathcal{M}}^2 \leq C \left(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}} \|\mathbf{v}\|_{(L^p(\Omega))^d}^2 \right).$$

□

Let us state a corollary which is a consequence of Proposition 4.1 and is necessary for the proof of the estimate of Proposition 4.3 and for Proposition 4.4. It may be found in [15] and is recalled here with its proof, for the sake of completeness.

Corollary 4.2. *Let \mathcal{M} be an admissible mesh satisfying (2.14). If $u_{\mathcal{M}} = (u_K)_{K \in \mathcal{T}} \in X(\mathcal{T})$ is a solution to (2.18) and, for $n > 0$, $E_n = \{|u_{\mathcal{M}}| > n\}$, then there exists $C > 0$ only depending on $(\Omega, \mathbf{v}, f, d, p, \xi)$ such that*

$$\text{meas}(E_n) \leq \frac{C}{(\ln(1+n))^2}. \quad (4.9)$$

Proof. On the one hand, using Proposition 4.1 we have

$$\|\ln(1 + |u_{\mathcal{M}}|)\|_{1,2,\mathcal{M}}^2 \leq C \left(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}} \|\mathbf{v}\|_{(L^p(\Omega))^d}^2 \right). \quad (4.10)$$

On the other hand, since $\underline{\text{med}}(\ln(1 + |u_{\mathcal{M}}|)) = 0$, by the discrete Poincaré-Wirtinger median inequality (2.16), we have that there exists $C > 0$ only depending on (Ω, d, ξ) such that

$$\|\ln(1 + |u_{\mathcal{M}}|)\|_{0,2} \leq C|u_{\mathcal{M}}|_{1,2,\mathcal{M}}. \quad (4.11)$$

Therefore, using (4.10) and (4.11), there exists $C > 0$ only depending on $(\Omega, \mathbf{v}, f, d, p, \xi)$ such that

$$\|\ln(1 + |u_{\mathcal{M}}|)\|_{0,2}^2 \leq C. \quad (4.12)$$

Finally, due to the fact that $\text{meas}(E_n) = \text{meas}(\{\ln(1 + |u_{\mathcal{M}}|) \geq \ln(1+n)\})$, the Chebyshev inequality and (4.12) lead to the result. \square

Proposition 4.3 (Estimate on $T_n(u_{\mathcal{M}})$). *Let \mathcal{M} be an admissible mesh satisfying (2.14). If $u_{\mathcal{M}} = (u_K)_{K \in \mathcal{T}} \in X(\mathcal{T})$ is a solution to (2.18) having $\underline{\text{med}}(u_{\mathcal{M}}) = 0$, then for any $n \geq 0$, there exists $C > 0$ only depending on $(\Omega, \mathbf{v}, f, n, d, \xi)$ such that*

$$\|T_n(u_{\mathcal{M}})\|_{1,2,\mathcal{M}} \leq C. \quad (4.13)$$

Let $(\mathcal{M}_m)_{m \geq 1}$ be a sequence of admissible meshes satisfying (2.14) and such that $h_{\mathcal{M}_m}$ goes to zero as $m \rightarrow \infty$ and let $u_m = (u_K^m)_{K \in \mathcal{T}_m} \in X(\mathcal{T}_m)$ be a solution to (2.18) having $\underline{\text{med}}(u_m) = 0$. Then there exists a measurable function u finite a.e. in Ω such that, up to a subsequence (still indexed by m),

$$T_n(u) \in H^1(\Omega), \text{ for any } n > 0, \quad (4.14)$$

$$\text{med}(u) = 0, \quad (4.15)$$

$$T_n(u_m) \rightarrow T_n(u) \text{ strongly in } L^2(\Omega) \text{ and a.e.}, \quad (4.16)$$

$$\nabla_{\mathcal{M}_m} T_n(u_m) \rightharpoonup \nabla T_n(u) \text{ in } (L^2(\Omega))^d, \forall n > 0. \quad (4.17)$$

Proof. The proof is divided into 2 steps. First, we prove that $T_n(u_{\mathcal{M}})$ satisfies the a priori estimate (4.13). In the second step, considering a sequence of admissible meshes \mathcal{M}_m , we prove that the solution u_m to the scheme (2.18) converges to a function u as m goes to infinity and that (4.14)–(4.17) hold true.

Step 1. *Estimate on $T_n(u_{\mathcal{M}})$.*

After multiplying each equation of the scheme by $T_n(u_K)$, summing over each control volume and reordering the sums, we obtain $T_1 + T_2 = T_3$ with

$$T_1 = \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(u)_\sigma (u_K - u_L) (T_n(u_K) - T_n(u_L)),$$

$$T_2 = \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+} (T_n(u_K) - T_n(u_L)),$$

$$T_3 = \sum_{K \in \mathcal{T}} \int_K f T_n(u_K) dx.$$

Since T_n is bounded by n , we obtain that $|T_3| \leq n\|f\|_{L^1(\Omega)}$. Then

$$T_1 \leq n\|f\|_{L^1(\Omega)} - T_2. \quad (4.18)$$

Let $\sigma \in \mathcal{E}$. By the definition (2.19) of $u_{\sigma,+}$ and recalling that $u_{\sigma,-}$ is the downstream choice of u , if $\mathbf{v}_{K,\sigma} \geq 0$ it gives

$$\mathbf{v}_{K,\sigma}(T_n(u_K) - T_n(u_L)) = \mathbf{v}_{K,\sigma}(T_n(u_{\sigma,+}) - T_n(u_{\sigma,-})),$$

and if $\mathbf{v}_{K,\sigma} < 0$ it gives

$$\mathbf{v}_{K,\sigma}(T_n(u_K) - T_n(u_L)) = -\mathbf{v}_{K,\sigma}(T_n(u_{\sigma,+}) - T_n(u_{\sigma,-})).$$

In consequence, T_2 can be written as

$$-T_2 = \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+} (T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})). \quad (4.19)$$

As in the proof of estimate (4.1), we use the set of edges $\mathcal{A} = \{\sigma \in \mathcal{E}_{int}; u_{\sigma,+} \geq u_{\sigma,-}, u_{\sigma,+} < 0\} \cup \{\sigma \in \mathcal{E}_{int}; u_{\sigma,+} < u_{\sigma,-}, u_{\sigma,+} \geq 0\}$. Since T_n is non decreasing we have

$$-T_2 \leq \frac{1}{n} \sum_{\sigma \in \mathcal{A}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+} (T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})). \quad (4.20)$$

Due to the fact that $\forall \sigma \in \mathcal{A}$, if $|u_{\sigma,+}| \geq n$ then $|u_{\sigma,-}| \geq n$ we deduce that

$$u_{\sigma,+}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})) = T_n(u_{\sigma,+})(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})), \quad \forall \sigma \in \mathcal{A}.$$

It follows that

$$-T_2 \leq \sum_{\sigma \in \mathcal{A}} m(\sigma) |\mathbf{v}_{K,\sigma}| T_n(u_{\sigma,+}) (T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})).$$

Moreover, using Cauchy-Schwarz and Young inequalities, taking into account that

$\left(\sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^2 \right)^{\frac{1}{2}}$ is bounded by $d^{\frac{1}{2}} \|\mathbf{v}\|_{(L^2(\Omega))^d}$, we obtain

$$\begin{aligned} -T_2 &\leq \sum_{\sigma \in \mathcal{A}} m(\sigma) |\mathbf{v}_{K,\sigma}| T_n(u_{\sigma,+}) (T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})) \\ &\leq \left(\sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^2 \right)^{\frac{1}{2}} \left(\sum_{\sigma \in \mathcal{A}} \frac{m(\sigma)}{d_\sigma} T_n(u_{\sigma,+})^2 (T_n(u_{\sigma,-}) - T_n(u_{\sigma,+}))^2 \right)^{\frac{1}{2}} \\ &\leq n d^{\frac{1}{2}} \|\mathbf{v}\|_{L^2(\Omega)^d} \left(\sum_{\sigma \in \mathcal{A}} \frac{m(\sigma)}{d_\sigma} (T_n(u_{\sigma,-}) - T_n(u_{\sigma,+}))^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2\beta} n^2 d \|\mathbf{v}\|_{L^2(\Omega)^d}^2 + \frac{\beta}{2} \sum_{\sigma \in \mathcal{A}} \frac{m(\sigma)}{d_\sigma} (T_n(u_{\sigma,-}) - T_n(u_{\sigma,+}))^2 \\ &\leq \frac{1}{2\beta} n^2 d \|\mathbf{v}\|_{L^2(\Omega)^d}^2 + \frac{\beta}{2} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (T_n(u_K) - T_n(u_L))^2, \end{aligned}$$

where $\beta > 0$. Since $T_n(u_K) - T_n(u_L) \leq u_K - u_L$, we have

$$-T_2 \leq \frac{1}{2\beta} n^2 d \|\mathbf{v}\|_{L^2(\Omega)^d}^2 + \frac{\beta}{2} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (u_K - u_L) (T_n(u_K) - T_n(u_L)),$$

and we can deduce that

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}_{int}} \lambda(u)_\sigma \frac{m(\sigma)}{d_\sigma} (u_K - u_L) (T_n(u_K) - T_n(u_L)) &\leq n \|f\|_{L^1(\Omega)} + \frac{1}{2\beta} n^2 d \|\mathbf{v}\|_{(L^2(\Omega))^d}^2 \\ &+ \frac{\beta}{2} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (u_K - u_L) (T_n(u_K) - T_n(u_L)). \end{aligned}$$

Recalling that $\underline{\text{med}}(T_n(u_{\mathcal{M}})) = 0$ and $0 \leq \mu \leq \lambda(u)$, an appropriate choice of β and the Poincaré-Wirtinger inequality (2.16) lead to the result.

Step 2. *In this step we consider sequence of admissible meshes $(\mathcal{M}_m)_{m \geq 1}$ satisfying (2.14) and such that $h_{\mathcal{M}_m}$ goes to zero as $m \rightarrow \infty$. If $u_m = (u_K^m)_{K \in \mathcal{T}_m} \in X(\mathcal{T}_m)$ denotes a solution to (2.18) having $\underline{\text{med}}(u_m) = 0$, we show that there exists a measurable function u finite a.e. in Ω such that (4.14)–(4.17) hold true.*

The method is widely used for elliptic equations with L^1 (or measure data) (see e.g. [11]) and consists in proving that, up to subsequence, u_m is a Cauchy sequence in measure. For the convenience of the reader we give the complete arguments. For any n , in view of Step 1 we know that the sequence $(\|T_n(u_m)\|_{1,2,\mathcal{M}_m})_{m \geq 1}$ is bounded (uniformly with respect to m). By Theorem 2.9 and a diagonal process (n being a natural number), up to a subsequence still indexed by m , we deduce that, for any $n \in \mathbb{N}$, there exists v_n belonging to $H^1(\Omega)$ such that

$$T_n(u_m) \rightarrow v_n, \text{ a.e. in } \Omega, \text{ as } m \rightarrow \infty. \quad (4.21)$$

We now prove that u_m is a Cauchy sequence in measure. Let $\omega > 0$. For all $n > 0$, and all $m, p \geq 0$, we have

$$\{|u_m - u_p| > \omega\} \subset \{|u_m| > n\} \cup \{|u_p| > n\} \cup \{|T_n(u_m) - T_n(u_p)| > \omega\}.$$

Let $\varepsilon > 0$ fixed. By Corollary 4.2, let $n > 0$ such that, for all $m, p \geq 0$,

$$\text{meas}(\{|u_m| > n\}) + \text{meas}(\{|u_p| > n\}) < \frac{\varepsilon}{2}.$$

Once n is chosen, since $T_n(u_m)$ converges almost everywhere to v_n as m goes to infinity we obtain

$$\exists m_0 > 0; \forall m, p \geq m_0 \quad \text{meas}(\{|T_n(u_m) - T_n(u_p)| > \omega\}) \leq \frac{\varepsilon}{2}.$$

Therefore, we deduce that $\forall m, p \geq m_0$

$$\text{meas}\{|u_m - u_p| > \omega\} < \varepsilon.$$

Hence $(u_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in measure. Consequently, up to a subsequence still indexed by m , there exists a measurable function u such that

$$u_m \rightarrow u \text{ a.e. in } \Omega. \quad (4.22)$$

It follows from Corollary 4.2 that u is finite a.e. in Ω .

Moreover by the pointwise convergence (4.21) of $T_n(u_m)$ for any $n \in \mathbb{N}$ we deduce that $T_n(u) = v_n \in H^1(\Omega)$. Applying Theorem 2.11 we obtain that

$$\nabla_{\mathcal{M}_m} T_n(u_m) \rightharpoonup \nabla T_n(u) \text{ in } (L^2(\Omega))^d, \text{ as } m \rightarrow \infty.$$

It remains to prove that $\text{med}(u) = 0$. Due to the point-wise convergence of u_m to u , the sequence $\mathbb{1}_{\{u_m > 0\}} \mathbb{1}_{\{u > 0\}}$ converges to $\mathbb{1}_{\{u > 0\}}$ a.e. as $h_{\mathcal{M}}$ goes to zero. Recalling that $\underline{\text{med}}(u_m) = 0$

Fatou's lemma leads to

$$\begin{aligned} \text{meas}\{u(x) > 0\} &\leq \liminf \int_{\Omega} \mathbb{1}_{\{u_m > 0\}} \mathbb{1}_{\{u > 0\}} dx \\ &\leq \liminf \text{meas}\{u_m(x) > 0\} \\ &\leq \frac{\text{meas}(\Omega)}{2}. \end{aligned}$$

Analogously from the convergence of $\mathbb{1}_{\{u_m < 0\}} \mathbb{1}_{\{u < 0\}}$ to $\mathbb{1}_{\{u < 0\}}$ a.e. as $m \rightarrow \infty$

$$\text{meas}\{u(x) < 0\} \leq \frac{\text{meas}(\Omega)}{2}.$$

It follows that $0 \in \text{med}(u)$. Since we have for n large enough $\text{med}(T_n(u)) = \text{med}(u)$ and since $T_n(u)$ belongs to $H^1(\Omega)$, the median of u is unique and it is equal to 0. \square

Let us recall that in the renormalized framework the decay of the energy (2.11) plays an important role to derive stability or uniqueness results. In the following proposition we show a discrete version of the decay of the energy (uniformly with respect to the sequence of the admissible meshes). Having (4.23) and (4.24) is crucial to pass to the limit in the scheme.

Proposition 4.4 (Discrete estimate on the energy). *Let $(\mathcal{M}_m)_{m \geq 1}$ be a sequence of admissible meshes satisfying (2.14) and such that $h_{\mathcal{M}_m} \rightarrow 0$ as $m \rightarrow \infty$. For any $m \geq 0$, let us consider $u_m = (u_K^m)_{K \in \mathcal{T}_m} \in X(\mathcal{T}_m)$ a solution to (2.18) and let u be a measurable function finite a.e. in Ω such that, up to a subsequence still indexed by m , the second part of Proposition 4.3 holds. Then we have*

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{M}_m} \rightarrow 0} \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(u_m)_\sigma (u_K^m - u_L^m) (T_n(u_K^m) - T_n(u_L^m)) = 0, \quad (4.23)$$

where $u_L = 0$ if $\sigma \in \mathcal{E}_{ext}$, and

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{M}_m} \rightarrow 0} \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) |\mathbf{v}_{K,\sigma}| |u_{\sigma,+}^m| |T_n(u_{\sigma,+}^m) - T_n(u_{\sigma,-}^m)| = 0. \quad (4.24)$$

Proof. Let $m \geq 1$ and $u_m = (u_K^m)_{K \in \mathcal{T}_m}$ be a solution of (2.18). Multiplying each equation of the scheme by $\frac{T_n(u_K^m)}{n}$, summing over $K \in \mathcal{M}$ and gathering by edges we find

$$T_1 + T_2 = T_3$$

with

$$T_1 = \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(u_m)_\sigma (u_K^m - u_L^m) (T_n(u_K^m) - T_n(u_L^m)), \quad (4.25)$$

$$T_2 = \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m (T_n(u_K^m) - T_n(u_L^m)), \quad (4.26)$$

$$T_3 = \frac{1}{n} \sum_{K \in \mathcal{M}} \int_K f T_n(u_K^m) dx. \quad (4.27)$$

According to the definition of u_m , we have

$$T_3 = \int_{\Omega} f \frac{T_n(u_m)}{n} dx.$$

Due to (4.22), $T_n(u_m)$ converges to $T_n(u)$ as m goes to infinity in $L^\infty(\Omega)$ weak- \star and a.e. Since f belongs to $L^1(\Omega)$ it follows that

$$\lim_{h_{\mathcal{M}_m} \rightarrow 0} T_3 = \int_{\Omega} f \frac{T_n(u)}{n} dx.$$

Recalling that u is finite almost everywhere in Ω , $\frac{T_n(u)}{n}$ converges to 0 a.e. in Ω as n goes to infinity. Therefore, since f belongs to $L^1(\Omega)$ and $\left|\frac{T_n}{n}\right|$ is bounded by one, the Lebesgue dominated theorem allows one to conclude that

$$\lim_{n \rightarrow +\infty} \lim_{h, \mathcal{M}_m \rightarrow 0} T_3 = 0. \quad (4.28)$$

We now study the term T_2 . We know from (4.19) that it can be written as

$$T_2 = \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+}^m (T_n(u_{\sigma,+}^m) - T_n(u_{\sigma,-}^m)). \quad (4.29)$$

Recalling the definition of the subset of edges \mathcal{A}

$$\mathcal{A} = \{\sigma \in \mathcal{E}_{int}; u_{\sigma,+}^m \geq u_{\sigma,-}^m, u_{\sigma,+}^m < 0\} \cup \{\sigma \in \mathcal{E}_{int}; u_{\sigma,+}^m < u_{\sigma,-}^m, u_{\sigma,+}^m \geq 0\}, \quad (4.30)$$

we denote by \mathcal{B} the subset of edges such that

$$\mathcal{B} = \{\sigma \in \mathcal{E}_{int}; u_{\sigma,+}^m \geq u_{\sigma,-}^m, u_{\sigma,+}^m \geq 0\} \cup \{\sigma \in \mathcal{E}_{int}; u_{\sigma,+}^m < u_{\sigma,-}^m, u_{\sigma,+}^m < 0\}. \quad (4.31)$$

Then T_2 can be written as

$$\begin{aligned} T_2 &= \frac{1}{n} \sum_{\sigma \in \mathcal{A}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+}^m (T_n(u_{\sigma,+}^m) - T_n(u_{\sigma,-}^m)) \\ &\quad + \frac{1}{n} \sum_{\sigma \in \mathcal{B}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+}^m (T_n(u_{\sigma,+}^m) - T_n(u_{\sigma,-}^m)) \\ &= T_{2,1} + T_{2,2}. \end{aligned} \quad (4.32)$$

Using the fact that T_n is non decreasing and $T_n(0) = 0$, we notice that

$$\begin{aligned} |T_{2,1}| &= -T_{2,1} = -\frac{1}{n} \sum_{\sigma \in \mathcal{A}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+}^m (T_n(u_{\sigma,+}^m) - T_n(u_{\sigma,-}^m)), \\ |T_{2,2}| &= T_{2,2} = \frac{1}{n} \sum_{\sigma \in \mathcal{B}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+}^m (T_n(u_{\sigma,+}^m) - T_n(u_{\sigma,-}^m)), \end{aligned}$$

and $|T_2| = -T_{2,1} + T_{2,2}$.

As far as $-T_{2,1}$ is concerned, we observe that for any $\sigma \in \mathcal{A}$, $|u_{\sigma,+}^m| \geq n$ implies $|u_{\sigma,-}^m| \geq n$. To deal with this term, as in [21], we split the sum on $\{|u_{\sigma,+}^m| \leq r\}$ and on $\{r \leq |u_{\sigma,+}^m| \leq n\}$ where r is a positive real number which will be chosen later. We have

$$-T_{2,1} = I_1 + I_2$$

with

$$I_1 = \frac{1}{n} \sum_{\substack{\sigma \in \mathcal{A} \\ |u_{\sigma,+}^m| \leq r}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+}^m (T_n(u_{\sigma,-}^m) - T_n(u_{\sigma,+}^m)), \quad (4.33)$$

$$I_2 = \frac{1}{n} \sum_{\substack{\sigma \in \mathcal{A} \\ r \leq |u_{\sigma,+}^m| \leq n}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+}^m (T_n(u_{\sigma,-}^m) - T_n(u_{\sigma,+}^m)). \quad (4.34)$$

Recalling that $\left(\sum_{\sigma \in \mathcal{A}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^2\right)^{\frac{1}{2}}$ is bounded by $d^{\frac{1}{2}} \|\mathbf{v}\|_{(L^2(\Omega))^d}$, the Cauchy-Schwarz inequality and the Young inequality yield that

$$\begin{aligned}
|I_1| &= \frac{1}{n} \sum_{\substack{\sigma \in \mathcal{A} \\ |u_{\sigma,+}^m| \leq r}} m(\sigma) |\mathbf{v}_{K,\sigma}| u_{\sigma,+}^m (T_n(u_{\sigma,-}^m) - T_n(u_{\sigma,+}^m)) \\
&\leq \frac{1}{n} \left(\sum_{\substack{\sigma \in \mathcal{A} \\ |u_{\sigma,+}^m| \leq r}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{\sigma \in \mathcal{A} \\ |u_{\sigma,+}^m| \leq r}} \frac{m(\sigma)}{d_\sigma} (u_{\sigma,+}^m)^2 (T_n(u_{\sigma,-}^m) - T_n(u_{\sigma,+}^m))^2 \right)^{\frac{1}{2}} \\
&\leq \frac{r}{n} \left(\sum_{\substack{\sigma \in \mathcal{A} \\ |u_{\sigma,+}^m| \leq r}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{\sigma \in \mathcal{A} \\ |u_{\sigma,+}^m| \leq r}} \frac{m(\sigma)}{d_\sigma} (u_K^m - u_L^m) (T_n(u_K^m) - T_n(u_L^m)) \right)^{\frac{1}{2}} \\
&\leq \frac{1}{n} \left[\frac{r^2 d \|\mathbf{v}\|_{(L^2(\Omega))^d}^2}{2\beta} + \frac{\beta}{2} \sum_{\substack{\sigma \in \mathcal{A} \\ |u_{\sigma,+}^m| \leq r}} \frac{m(\sigma)}{d_\sigma} (u_K^m - u_L^m) (T_n(u_K^m) - T_n(u_L^m)) \right], \tag{4.35}
\end{aligned}$$

where $\beta > 0$ (to be chosen later). To control the second term I_2 , we distinguish the case $d \geq 3$ and $d = 2$. We know that for any $\sigma \in \mathcal{A}$, $|u_{\sigma,+}^m| \geq r$ implies $|u_{\sigma,-}^m| \geq r$; if $d \geq 3$ the equality

$$\frac{1}{d} + \frac{d-2}{2d} + \frac{1}{2} = 1,$$

and the Hölder inequality give

$$\begin{aligned}
|I_2| &\leq \frac{1}{n} \sum_{\substack{\sigma \in \mathcal{A} \\ r \leq |u_{\sigma,+}^m| \leq n \\ r \leq |u_{\sigma,-}^m|}} m(\sigma) |\mathbf{v}_{K,\sigma}| T_n(u_{\sigma,+}^m) (T_n(u_{\sigma,-}^m) - T_n(u_{\sigma,+}^m)) \\
&\leq \frac{1}{n} \left(\sum_{\substack{\sigma \in \mathcal{A} \\ r \leq |u_{\sigma,+}^m| \leq n \\ r \leq |u_{\sigma,-}^m|}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^d \right)^{\frac{1}{d}} \left(\sum_{\substack{\sigma \in \mathcal{A} \\ r \leq |u_{\sigma,+}^m| \leq n \\ r \leq |u_{\sigma,-}^m|}} m(\sigma) d_\sigma |T_n(u_{\sigma,+}^m)|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} \\
&\quad \times \left(\sum_{\substack{\sigma \in \mathcal{A} \\ r \leq |u_{\sigma,+}^m| \leq n \\ r \leq |u_{\sigma,-}^m|}} \frac{m(\sigma)}{d_\sigma} (T_n(u_{\sigma,-}^m) - T_n(u_{\sigma,+}^m))^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{n} \left(\sum_{\substack{\sigma \in \mathcal{A} \\ r \leq |u_{\sigma,+}^m| \leq n \\ r \leq |u_{\sigma,-}^m|}} m(\sigma) d_\sigma |\mathbf{v}_{K,\sigma}|^d \right)^{\frac{1}{d}} \left(\sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) d_\sigma |T_n(u_{\sigma,+}^m)|^{\frac{2d}{d-2}} \right)^{\frac{d-2}{2d}} \\
&\quad \times \left(\sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (T_n(u_{\sigma,-}^m) - T_n(u_{\sigma,+}^m))^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Recalling that $\underline{\text{med}}(T_n(u_m)) = 0$, the discrete Poincaré-Wirtinger inequality (2.16) and the dis-

crete Sobolev inequality (2.15) lead to

$$|I_2| \leq C_1 \frac{d^{\frac{1}{2}} \|\mathbf{v}\|_{(L^p(E_r))^d}}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (T_n(u_K^m) - T_n(u_L^m))^2, \quad (4.36)$$

where E_r is the set where $|u_{\sigma,+}^m| \geq r$ and $C_1 > 0$ is a constant independent of n and \mathcal{M} . If $d = 2$, similar arguments lead to

$$|I_2| \leq C_2 \frac{d^{\frac{1}{2}} \|\mathbf{v}\|_{(L^p(E_r))^d}}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (T_n(u_K^m) - T_n(u_L^m))^2, \quad (4.37)$$

where $C_2 > 0$ is a constant independent of n and \mathcal{M} .

In view of Corollary 4.2 and since $\mathbf{v} \in (L^p(\Omega))^d$ ($2 < p < +\infty$ if $d = 2$, $p = d$ if $d \geq 3$), the absolute continuity of the integral implies that there exists $r > 0$ (independent of m) such that for all m

$$d^{\frac{1}{2}} \|\mathbf{v}\|_{(L^p(E_r))^d} \leq \frac{1}{2}. \quad (4.38)$$

Then from (4.36), (4.37) and (4.38) we obtain

$$|I_2| \leq \frac{C_3}{2n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (u_K^m - u_L^m)(T_n(u_K^m) - T_n(u_L^m)). \quad (4.39)$$

Recalling that $-T_2 \leq -T_{2,1}$, the inequalities (4.39) and (4.35) lead to

$$\begin{aligned} -T_2 &\leq |I_1 + I_2| \\ &\leq \frac{1}{n} \frac{r^2 d \|\mathbf{v}\|_{(L^2(\Omega))^d}^2}{2\beta} + \frac{C_4}{2n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (u_K^m - u_L^m)(T_n(u_K^m) - T_n(u_L^m)), \end{aligned} \quad (4.40)$$

where C_4 is a positive constant depending on β and C_3 . Since $0 < \mu \leq \lambda(u_m)$, we choose $\beta > 0$ such that the second term of the right-hand side of (4.40) is $\leq \frac{T_1}{2}$. It follows that

$$\begin{aligned} -T_2 &\leq \frac{1}{n} \frac{r^2 d \|\mathbf{v}\|_{(L^2(\Omega))^d}^2}{2\beta} + \frac{T_1}{2} \\ &\leq R(n, h_{\mathcal{T}_m}) + \frac{T_1}{2}, \end{aligned} \quad (4.41)$$

with R verifying $\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{T}_m} \rightarrow 0} R(n, h_{\mathcal{T}_m}) = 0$.

Since $T_1 + T_2 = T_3$, (4.41) allows one to conclude that

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{M}_m} \rightarrow 0} \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(u_m)_\sigma (u_K^m - u_L^m)(T_n(u_K^m) - T_n(u_L^m)) = 0, \quad (4.42)$$

which gives (4.23). We are now in a position to prove (4.24). Recalling that $T_1 + T_2 = T_3$, we have $T_1 + T_{2,2} \leq |T_{2,1}| + T_3$. Since $T_{2,2}$ is non negative, using (4.28), (4.41) we get

$$T_1 \leq \frac{C_5}{2} T_1 + R + T_3 \text{ with}$$

$$\begin{aligned} \lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{T}_m} \rightarrow 0} R &= 0, \\ \lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{T}_m} \rightarrow 0} T_3 &= 0. \end{aligned}$$

As a consequence we obtain

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{T}_m} \rightarrow 0} |T_{2,1}| = 0. \quad (4.43)$$

Moreover, writing again $T_1 + T_{2,2} \leq |T_{2,1}| + T_3$, (4.43), (4.28) and (4.23) imply that

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{T}_m} \rightarrow 0} T_{2,2} = 0. \quad (4.44)$$

Therefore from (4.43) and (4.44) we deduce (4.24). \square

5 Existence of a solution to the scheme

In this section we prove that there exists at least one solution to the discrete scheme. Since the scheme is nonlinear we use a fixed point argument together with the study of the linear version of our problem for which we adapt the arguments developed in [8] for a linear problem with Neumann boundary conditions and mean value.

Proof of Theorem 3.1. The proof is divided into 2 steps. In Step 1 with the help of [8] we construct a map in view of the fixed point argument. In Step 2 using estimates in Proposition 4.1 we conclude with the Brouwer fixed point theorem the existence of a solution.

Step 1. Let $\tilde{u} = (\tilde{u}_K)_{K \in \mathcal{T}} \in X(\mathcal{T})$ and let us consider the linear scheme

$$\forall K \in \mathcal{M}, \quad \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(\tilde{u})_\sigma (u_K - u_L) + \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+} = \int_K f \, dx, \quad (5.1)$$

where $u = (u_K)_{K \in \mathcal{T}} \in X(\mathcal{T})$ is the unknown. Following [8], it can be rewritten as the linear system

$$\mathbb{A}U = F \quad (5.2)$$

where $U = (u_K)_{K \in \mathcal{T}}$, $F = (\int_K f \, dx)_{K \in \mathcal{T}}$ and \mathbb{A} is the square matrix of size $\text{card}(\mathcal{T}) \times \text{card}(\mathcal{T})$ with

$$\begin{cases} \mathbb{A}_{K,K} = \sum_{\sigma \in \mathcal{E}_{K,int}} m(\sigma) \left(\frac{\lambda(\tilde{u}_\sigma)}{d_\sigma} + \mathbf{v}_{K,\sigma}^+ \right), & \forall K \in \mathcal{T}, \\ \mathbb{A}_{K,L} = m(\sigma) \left(-\frac{\lambda(\tilde{u}_\sigma)}{d_\sigma} - \mathbf{v}_{K,\sigma}^- \right), & \forall K \in \mathcal{T}, \forall L \in N(K), \text{ with } \sigma = K|L, \\ \mathbb{A}_{K,L} = 0, & \forall K \in \mathcal{T}, \forall L \notin N(K). \end{cases}$$

At this step having f belonging to $L^1(\Omega)$ or $f \in L^2(\Omega)$ does not play any role. From Proposition 3.1 in [8] (see also Remark 2.4 in [8] when $-\Delta u$ is replaced by $-\text{div}(a(x)\nabla u)$, which is the isotropic case) it follows that

- $\dim(\ker(\mathbb{A})) = 1$ and any (non zero) element U belonging to $\ker(\mathbb{A})$ verifies either $u_K > 0$ for all $K \in \mathcal{T}$ or $u_K < 0$ for all $K \in \mathcal{T}$.
- $\ker(\mathbb{A}^\top) = \mathbb{R}(1, 1, \dots, 1)^\top$ and thus

$$\text{Im}(\mathbb{A}) = \left\{ (F_K)_{K \in \mathcal{T}}; \sum_{K \in \mathcal{T}} F_K = 0 \right\}.$$

Since $\int_\Omega f \, dx = 0$, F belongs to $\text{Im}(\mathbb{A})$ and then there exists at least \overline{U} solution of $\mathbb{A}\overline{U} = F$. If $V = (v_K)_{K \in \mathcal{T}}$ denotes an element of $\ker(\mathbb{A})$ such that $v_K > 0$ for any $K \in \mathcal{T}$, for any $\lambda \in \mathbb{R}$, the vector $\overline{U} + \lambda V$ is a solution of (5.2). Since $v_K > 0, \forall K \in \mathcal{T}$, the function $\lambda \mapsto \underline{\text{med}}(\overline{U} + \lambda V)$ is continuous and increasing while $\lim_{\lambda \rightarrow +\infty} \underline{\text{med}}(\overline{U} + \lambda V) = +\infty$ and $\lim_{\lambda \rightarrow -\infty} \underline{\text{med}}(\overline{U} + \lambda V) =$

$-\infty$. It follows that there exists at least one solution $u \in X(\mathcal{T})$ verifying the scheme (5.1) and $\underline{\text{med}}(u) = 0$.

The uniqueness is a consequence of the characterization of $\ker(\mathbb{A})$. As a conclusion we can define the map Γ from $X(\mathcal{T})$ into $X(\mathcal{T})$ by

$$\forall \tilde{u} \in X(\mathcal{T}), \Gamma(\tilde{u}) = u$$

where $u \in X(\mathcal{T})$ is the unique solution of (5.1) such that $\underline{\text{med}}(u) = 0$.

Step 2. In this step we prove that

$$\exists C > 0, \forall \tilde{u} \in X(\mathcal{T}), \quad \|\Gamma(\tilde{u})\|_{L^\infty} \leq C, \quad (5.3)$$

$$\Gamma \text{ is a continuous map} \quad (5.4)$$

in order to apply the Brouwer fixed point theorem.

The boundedness of Γ relies on Proposition 4.1. Indeed by replacing $\lambda(u)_\sigma$ by $\lambda(\tilde{u})_\sigma$ in the proof of Proposition 4.1 it can be shown that there exists $C > 0$ (not depending on \tilde{u}) such that

$$\|\ln(1 + |\Gamma(\tilde{u})|)\|_{1,2,\mathcal{M}}^2 \leq C \left(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}} \|\mathbf{v}\|_{(L^p(\Omega))^d}^2 \right), \quad (5.5)$$

Since $\Gamma(\tilde{u})$ lies in a finite dimension vector space we obtain that (5.3) holds true.

We now prove that Γ is a continuous map. Let $(\tilde{u}_n)_{n \in \mathbb{N}}$ and \tilde{u} belonging to $X(\mathcal{T})$ such that \tilde{u}_n goes to \tilde{u} as n goes to infinity. In view of (5.3) up to a subsequence, still indexed by n , there exists $w \in X(\mathcal{T})$ such that $\Gamma(\tilde{u}_n)$ tends to w as n goes to infinity. Recalling that the coefficient of the matrix \mathbb{A} are continuous with respect to \tilde{u} we obtain that w is a solution to the scheme

$$\forall K \in \mathcal{M}, \quad \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(\tilde{u})_\sigma (w_K - w_L) + \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) \mathbf{v}_{K,\sigma} w_{\sigma,+} = \int_K f \, dx. \quad (5.6)$$

The mesh \mathcal{T} being fixed since we consider constant piecewise functions the fact that $\underline{\text{med}}(\Gamma(\tilde{u}_n)) = 0$ for any $n \in \mathbb{N}$ implies that $\underline{\text{med}}(w) = 0$. Recalling that $u = \Gamma(\tilde{u})$ is the unique solution of (5.1) with $\underline{\text{med}}(u) = 0$ we conclude that $u = w$. Since the limit does not depend on the subsequence we obtain that Γ is a continuous map.

In view of (5.3) and (5.4) the Brouwer fixed point theorem allows one to conclude the proof of Theorem 3.1. \square

6 Convergence of the scheme

In this section we prove using Section 4 that the function u obtained in Proposition 4.3 is a renormalized solution to equation (2.1). The main difficulty is the equation (2.12): in short the test functions in (2.12) are of the kind $\varphi S(u)$ which is nonlinear in u . The strategy is to mix renormalized techniques and finite volume with the use of a discrete version of $\varphi S_n(u_{\mathcal{M}_m})$ (see below the definition of S_n) where φ belongs to $\mathcal{D}(\bar{\Omega})$ (see [21] in the case of Dirichlet boundary condition). Before proving Theorem 3.2 we give in Lemma 6.1 a convergence result concerning $S_n(u_{\mathcal{M}_m})$ (see e.g. [20, 21]) and in Corollary 6.2 the asymptotic behavior of an extra term (with respect to the continuous case) which appears when we pass to the limit in the discrete equation.

First, let us define the function S_n , $n \geq 1$, by

$$S_n(s) = \begin{cases} 0 & \text{if } s \leq -2n, \\ \frac{s}{n} + 2 & \text{if } -2n \leq s \leq -n, \\ 1 & \text{if } -n \leq s \leq n, \\ \frac{-s}{n} + 2 & \text{if } n \leq s \leq 2n, \\ 0 & \text{if } s \geq 2n. \end{cases} \quad (6.1)$$

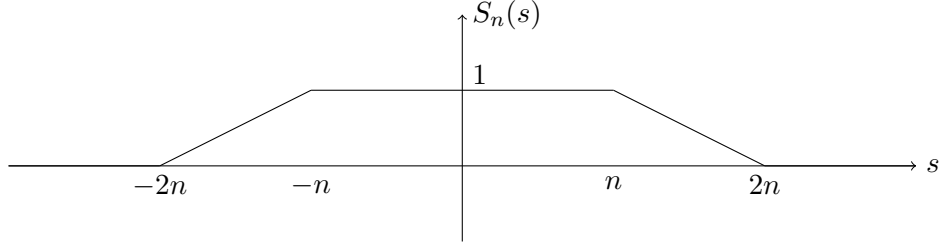


Figure 2: The function S_n

Lemma 6.1. *Let $(\mathcal{M}_m)_{m \geq 1}$ be a sequence of admissible meshes satisfying (2.14) such that $h_{\mathcal{M}_m} \rightarrow 0$ as $m \rightarrow \infty$ and let $u_m = (u_K^m)_{K \in \mathcal{T}_m} \in X(\mathcal{T}_m)$ be a sequence of solution of (2.18) such that the conclusions of Proposition 4.3 hold true. We define the function \bar{S}_n^m over the diamonds by*

$$\begin{aligned} \forall \sigma = K|L \in \mathcal{E}_{int}, \forall x \in D_\sigma, \quad \bar{S}_n^m(x) &= \frac{S_n(u_K^m) + S_n(u_L^m)}{2}, \\ \forall \sigma \in \mathcal{E}_{ext}, \forall x \in D_\sigma, \quad \bar{S}_n^m(x) &= S_n(u_K^m), \end{aligned}$$

and the function $\bar{\lambda}^m$ by

$$\forall \sigma \in \mathcal{E}, \forall x \in D_\sigma \quad \bar{\lambda}^m(x) = \lambda(u_m)_\sigma.$$

Then the functions \bar{S}_n^m and $\bar{\lambda}$ converge respectively to $S_n(u)$ and $\lambda(u)$ in $L^q(\Omega) \forall q \in [1, +\infty[$ and in $L^\infty(\Omega)$ weak-*, as $h_{\mathcal{M}_m} \rightarrow 0$, where u is the limit of $u_{\mathcal{M}_m}$.

Proof. Since the functions \bar{S}_n^m is bounded with respect to m it is sufficient to prove that the convergence holds true in $L^2(\Omega)$. By Proposition 4.3 u_m converges to u a.e. in Ω . Since S_n is a continuous and bounded function, the Lebesgue dominated convergence theorem gives that $S_n(u_m) \rightarrow S_n(u)$ in $L^2(\Omega)$, so that it is sufficient to study the behavior of $\bar{S}_n^m - S_n(u_m)$. Recalling that S_n is Lipschitz continuous and has a compact support, we have

$$\begin{aligned} \|\bar{S}_n^m - S_n(u_m)\|_{L^2(\Omega)}^2 &= \int_{\Omega} |\bar{S}_n^m - S_n(u_m(x))|^2 dx \\ &= \sum_{\sigma \in \mathcal{E}} \int_{D_\sigma} |\bar{S}_n^m(x) - S_n(u_m(x))|^2 dx \\ &= \sum_{\sigma \in \mathcal{E}_{int}} \int_{D_\sigma} \left| \frac{S_n(u_K^m) + S_n(u_L^m)}{2} - S_n(u_m(x)) \right|^2 dx \\ &= \frac{1}{4} \sum_{\sigma \in \mathcal{E}_{int}} m(D_\sigma) |S_n(u_K^m) - S_n(u_L^m)|^2 \\ &\leq \frac{1}{4n^2} \sum_{\sigma \in \mathcal{E}_{int}} m(D_\sigma) |T_{2n}(u_K^m) - T_{2n}(u_L^m)|^2 \\ &= \frac{1}{4dn^2} \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) d_\sigma \left| \frac{T_{2n}(u_K^m) - T_{2n}(u_L^m)}{d_\sigma} \right|^2 (d_\sigma)^2 \\ &\leq \frac{1}{4dn^2} |T_{2n}(u_m)|_{1,2,\mathcal{M}}^2 (h_{\mathcal{M}_m})^2, \end{aligned}$$

so that $\|\bar{S}_n^m - S_n(u_m)\|_{L^2(\Omega)}^2$ goes to zero as $h_{\mathcal{M}_m} \rightarrow 0$.

As far as $\bar{\lambda}^m$ is concerned let us first define the function \bar{u}^m by

$$\begin{aligned} \forall \sigma = K|L \in \mathcal{E}_{int}, \forall x \in D_\sigma, \quad \bar{u}^m(x) &= \frac{u_K^m + u_L^m}{2}, \\ \forall \sigma \in \mathcal{E}_{ext}, \forall x \in D_\sigma, \quad \bar{u}^m(x) &= u_K^m. \end{aligned}$$

In view of the definition of $\lambda(u)_\sigma$ we have in Ω

$$\min(\lambda(u_m), \lambda(2\bar{u}^m - u_m)) \leq \bar{\lambda}^m \leq \max(\lambda(u_m), \lambda(2\bar{u}^m - u_m)). \quad (6.2)$$

In view of already used arguments for \bar{S}^m , since the function T_n is Lipschitz continuous, $T_n(\bar{u}^m)$ converges to $T_n(u)$ in $L^q(\Omega)$, $\forall q \in [1, +\infty[$ and in L^∞ weak-*. By the diagonal process, up to a subsequence still index by m , \bar{u}^m goes to u a.e. in Ω as $h_{\mathcal{M}_m}$ goes to zero. Since λ is a bounded continuous function we obtain, up to a subsequence, that $\bar{\lambda}^m$ converges to $\lambda(u)$ in $L^q(\Omega)$, $\forall q \in [1, +\infty[$ and in L^∞ weak-* as $h_{\mathcal{M}_m}$ goes to zero. To conclude it is sufficient to observe that the limit of $\bar{\lambda}^m$ is independent of the subsequence. \square

Corollary 6.2. *Let $(\mathcal{M}_m)_{m \geq 1}$ be a sequence of admissible meshes satisfying (2.14) such that $h_{\mathcal{M}_m} \rightarrow 0$ as $m \rightarrow \infty$. Let $u_m = (u_K^m)_{K \in \mathcal{T}_m} \in X(\mathcal{T}_m)$ be a sequence of solution of (2.18) such that the conclusions of Proposition 4.3 hold true. Then we have*

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{M}_m} \rightarrow 0} \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ |u_K^m| \leq 2n \\ |u_L^m| > 4n}} \lambda(u)_\sigma \frac{m(\sigma)}{d_\sigma} |u_L^m| = 0. \quad (6.3)$$

Proof. For $m \in \mathbb{N}$, let us consider $K \in \mathcal{T}_m$ and $n \in \mathbb{N}$. On one hand if $|u_K^m| \leq 2n$ and $u_L^m > 4n$ then

$$(u_K^m - u_L^m)(T_{4n}(u_K^m) - T_{4n}(u_L^m)) \geq \frac{u_L^m}{2} 2n \geq 0.$$

On the other hand, if $|u_K^m| \leq 2n$ and $u_L^m < -4n$ then

$$(u_K^m - u_L^m)(T_{4n}(u_K^m) - T_{4n}(u_L^m)) \geq \frac{-u_L^m}{2} 2n \geq 0.$$

It follows that

$$\sum_{\substack{\sigma \in \mathcal{E}_{int} \\ |u_K^m| \leq 2n \\ |u_L^m| > 4n}} \frac{m(\sigma)}{d_\sigma} \lambda(u_m)_\sigma |u_L^m| \leq \frac{1}{4n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(u_m)_\sigma (u_L^m - u_K^m)(T_{4n}(u_K^m) - T_{4n}(u_L^m)). \quad (6.4)$$

Using the discrete estimate on energy (4.23), we have

$$\lim_{n \rightarrow +\infty} \lim_{h_{\mathcal{M}_m} \rightarrow 0} \frac{1}{4n} \sum_{\sigma \in \mathcal{E}_{int}} \lambda(u_m)_\sigma \frac{m(\sigma)}{d_\sigma} (u_L^m - u_K^m)(T_{4n}(u_K^m) - T_{4n}(u_L^m)) = 0, \quad (6.5)$$

so that (6.4) and (6.5) give (6.3). \square

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2 Let $m \geq 1$ and let us consider $u_m = (u_K^m)_{K \in \mathcal{T}_m}$ be a solution of the scheme (2.18).

For φ a function belonging to $\mathcal{D}(\bar{\Omega})$ we denote by φ_m the function defined by $\varphi_K = \varphi(x_K)$ for all $K \in \mathcal{T}_m$. For $n \in \mathbb{N}$, multiplying each equation of the scheme by $\varphi(x_K) S_n(u_K^m)$ (which is a discrete version of the test function used in the renormalized formulation), summing over the control volumes and gathering by edges, we get

$$T_1 + T_2 = T_3$$

with

$$\begin{aligned}
T_1 &= \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma (u_K^m - u_L^m) (\varphi(x_K) S_n(u_K^m) - \varphi(x_L) S_n(u_L^m)), \\
T_2 &= \sum_{\sigma \in \mathcal{E}_{int}} \mathfrak{m}(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m (\varphi(x_K) S_n(u_K^m) - \varphi(x_L) S_n(u_L^m)), \\
T_3 &= \sum_{K \in \mathcal{T}} \int_K f \varphi(x_K) S_n(u_K^m).
\end{aligned}$$

Since $S_n(u_m) \rightarrow S_n(u)$ a.e. and L^∞ weak \star , by the regularity of φ , $\varphi_m \rightarrow \varphi$ uniformly and $|f \varphi_m S_n(u_m)| \leq C_\varphi |f| \in L^1(\Omega)$, the Lebesgue theorem ensures that

$$T_3 = \int_\Omega f \varphi_m S_n(u_m) \, dx \xrightarrow{h_{\mathcal{M}_m} \rightarrow 0} \int_\Omega f \varphi S_n(u) \, dx. \quad (6.6)$$

We now study the convergence of the diffusion term. We write

$$\begin{aligned}
T_1 &= \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma (u_K^m - u_L^m) (\varphi(x_K) S_n(u_K^m) - \varphi(x_L) S_n(u_L^m)) \\
&= T_{1,1} + T_{1,2}
\end{aligned}$$

with

$$\begin{aligned}
T_{1,1} &= \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma S_n(u_K^m) (u_K^m - u_L^m) (\varphi(x_K) - \varphi(x_L)), \\
T_{1,2} &= \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma \varphi(x_L) (u_K^m - u_L^m) (S_n(u_K^m) - S_n(u_L^m)).
\end{aligned}$$

According to the definition of S_n we have

$$|T_{1,2}| \leq \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma \varphi(x_L) (u_K^m - u_L^m) (T_{2n}(u_K^m) - T_{2n}(u_L^m)),$$

so that (4.23) give

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{M}_m} \rightarrow 0} T_{1,2} = 0. \quad (6.7)$$

The main difference with respect to the continuous case is that u_K^m is truncated while u_L is not in $T_{1,1}$. To control this term we have to write

$$\begin{aligned}
T_{1,1} &= \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma S_n(u_K^m) (T_{2n}(u_K^m) - u_L^m) (\varphi(x_K) - \varphi(x_L)), \\
&= I + II + III
\end{aligned}$$

with

$$\begin{aligned}
I &= \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma \frac{S_n(u_K^m) + S_n(u_L^m)}{2} (T_{4n}(u_K^m) - T_{4n}(u_L^m)) (\varphi(x_K) - \varphi(x_L)), \\
II &= \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma S_n(u_K^m) (T_{4n}(u_L^m) - u_L^m) (\varphi(x_K) - \varphi(x_L)), \\
III &= \sum_{\sigma \in \mathcal{E}_{int}} \frac{\mathfrak{m}(\sigma)}{d_\sigma} \lambda(u_m)_\sigma \frac{S_n(u_K^m) - S_n(u_L^m)}{2} (T_{4n}(u_K^m) - T_{4n}(u_L^m)) (\varphi(x_K) - \varphi(x_L))
\end{aligned}$$

We first study the asymptotic behavior of II and III as the parameter $h_{\mathcal{M}_n}$ goes to zero. Since

$$\begin{aligned} |II| &\leq \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(u)_\sigma |S_n(u_K^m)| |T_{4n}(u_L^m) - u_L^m| |\varphi(x_K) - \varphi(x_L)| \\ &\leq 2\|\varphi\|_{L^\infty(\Omega)} \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ |u_K^m| \leq 2n \\ |u_L^m| > 4n}} \frac{m(\sigma)}{d_\sigma} \lambda(u_m)_\sigma |u_L^m|, \end{aligned}$$

and due to Corollary 6.2 we obtain that

$$\overline{\lim}_{h_{\mathcal{M}} \rightarrow 0} |II| \leq \omega(n) \|\varphi\|_{L^\infty(\Omega)}, \quad (6.8)$$

where $\omega(n)$ tends to zero as n goes to infinity. In the sequel of the present proof $\omega(n)$ is a generic positive quantity such that $\lim_{n \rightarrow +\infty} \omega(n) = 0$. According to the definition of S_n we have

$$|III| \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} \lambda(u_m)_\sigma |u_K^m - u_L^m| |T_n(u_K^m) - T_n(u_L^m)|,$$

recalling (4.23) of Proposition 4.4 we obtain that

$$\overline{\lim}_{h_{\mathcal{T}} \rightarrow 0} |III| \leq \omega(n) \|\varphi\|_{L^\infty(\Omega)}. \quad (6.9)$$

We now turn to I . By rewriting I as integral over the diamonds D_σ (see e.g. [9, 17, 14]) we have

$$\begin{aligned} I &= \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) d_\sigma \lambda(u_m)_\sigma \frac{S_n(u_K^m) + S_n(u_L^m)}{2} \frac{T_{4n}(u_K^m) - T_{4n}(u_L^m)}{d_\sigma} \frac{\varphi(x_K) - \varphi(x_L)}{d_\sigma} \\ &= \sum_{\sigma \in \mathcal{E}_{int}} \int_{D_\sigma} \bar{\lambda}^m(x) \bar{S}_n^m(x) \nabla_{\mathcal{M}_m} T_{4n}(u_m) \cdot \nabla \varphi \, dx \\ &\quad + \sum_{\sigma \in \mathcal{E}_{int}} dm(D_\sigma) \lambda(u_m)_\sigma \frac{S_n(u_K^m) + S_n(u_L^m)}{2} \frac{T_{4n}(u_K^m) - T_{4n}(u_L^m)}{d_\sigma} \\ &\quad \quad \times \left[\frac{\varphi(x_K) - \varphi(x_L)}{d_\sigma} + \frac{1}{m(D_\sigma)} \int_{D_\sigma} \nabla \varphi \cdot \eta_{K,\sigma} \, dx \right] \\ &= I_1 + I_2. \end{aligned}$$

By Lemma 6.1 $\bar{S}_n^m \rightarrow S_n(u)$ in $L^q(\Omega)$ for all $q \in [1, +\infty[$ and $\bar{\lambda}^m \rightarrow \lambda(u)$ in L^∞ weak \star , while $\nabla_{\mathcal{M}_m} T_{4n}(u_m)$ tends to $\nabla T_{4n}(u)$ weakly in $(L^2(\Omega))^d$. Since φ belongs to $C_c^\infty(\Omega)$ we conclude that

$$\lim_{h_{\mathcal{M}_m} \rightarrow 0} I_1 = \int_{\Omega} \lambda(u) S_n(u) \nabla T_{4n}(u) \cdot \nabla \varphi \, dx.$$

By the regularity of φ we see that

$$\begin{aligned} |I_2| &\leq \sum_{\sigma \in \mathcal{E}_{int}} dm(D_\sigma) \lambda(u_m)_\sigma \left| \frac{S_n(u_K^m) + S_n(u_L^m)}{2} \right| \frac{|T_{4n}(u_K^m) - T_{4n}(u_L^m)|}{d_\sigma} \\ &\quad \times \left| \frac{\varphi(x_K) - \varphi(x_L)}{d_\sigma} + \frac{1}{m(D_\sigma)} \int_{D_\sigma} \nabla \varphi \cdot \eta_{K,\sigma} \, dx \right| \\ &\leq \lambda_\infty \|\varphi\|_{W^{2,\infty}(\Omega)} h_{\mathcal{M}} \|T_{4n}(u_m)\|_{1,1,\mathcal{M}}, \end{aligned}$$

thus

$$\lim_{h_{\mathcal{M}_m} \rightarrow 0} I_2 = 0. \quad (6.10)$$

We now study the convergence of the convection term T_2 . We have

$$\begin{aligned}
T_2 &= \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m (\varphi(x_K) S_n(u_K^m) - \varphi(x_L) S_n(u_L^m)) \\
&= \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \mathbf{v}_{K,\sigma} \geq 0}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m (\varphi(x_K) S_n(u_{\sigma,+}^m) - \varphi(x_L) S_n(u_{\sigma,-}^m)) \\
&\quad + \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \mathbf{v}_{K,\sigma} < 0}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m (\varphi(x_K) S_n(u_{\sigma,-}^m) - \varphi(x_L) S_n(u_{\sigma,+}^m)) \\
&= \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \mathbf{v}_{K,\sigma} \geq 0}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m S_n(u_{\sigma,+}^m) (\varphi(x_K) - \varphi(x_L)) \\
&\quad + \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \mathbf{v}_{K,\sigma} \geq 0}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m \varphi(x_L) (S_n(u_{\sigma,+}^m) - S_n(u_{\sigma,-}^m)) \\
&\quad - \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \mathbf{v}_{K,\sigma} < 0}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m S_n(u_{\sigma,+}^m) (\varphi(x_L) - \varphi(x_K)) \\
&\quad - \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \mathbf{v}_{K,\sigma} < 0}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m \varphi(x_K) (S_n(u_{\sigma,+}^m) - S_n(u_{\sigma,-}^m)) \\
&= T_{2,1} + T_{2,2} + T_{2,3}
\end{aligned}$$

with

$$\begin{aligned}
T_{2,1} &= \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m S_n(u_{\sigma,+}^m) (\varphi(x_K) - \varphi(x_L)) \\
T_{2,2} &= \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \mathbf{v}_{K,\sigma} \geq 0}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m \varphi(x_L) (S_n(u_{\sigma,+}^m) - S_n(u_{\sigma,-}^m)) \\
T_{2,3} &= - \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \mathbf{v}_{K,\sigma} < 0}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m \varphi(x_K) (S_n(u_{\sigma,+}^m) - S_n(u_{\sigma,-}^m))
\end{aligned}$$

Since

$$|T_{2,2} + T_{2,3}| \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{n} \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) |\mathbf{v}_{K,\sigma}| |u_{\sigma,+}^m| |T_{2n}(u_{\sigma,+}^m) - T_{2n}(u_{\sigma,-}^m)|,$$

we deduce from (4.24) that

$$\lim_{n \rightarrow +\infty} \overline{\lim}_{h_{\mathcal{M}_m} \rightarrow 0} T_{2,2} + T_{2,3} = 0. \quad (6.11)$$

For the term $T_{2,1}$, we have

$$\begin{aligned}
T_{2,1} &= \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) \mathbf{v}_{K,\sigma} u_{\sigma,+}^m S_n(u_{\sigma,+}^m) (\varphi(x_K) - \varphi(x_L)) \\
&= \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma) d_\sigma}{d m(D_\sigma)} u_{\sigma,+}^m S_n(u_{\sigma,+}^m) d \frac{\varphi(x_K) - \varphi(x_L)}{d_\sigma} \int_{D_\sigma} \mathbf{v} \cdot \eta_{K,\sigma} dx \\
&= - \sum_{\sigma \in \mathcal{E}_{int}} \int_{D_\sigma} T_{2n}(u_{\sigma,+}^m) S_n(u_{\sigma,+}^m) \mathbf{v} \cdot \nabla_{\mathcal{M}} \varphi_{\mathcal{M}} dx.
\end{aligned}$$

We define the function \overline{G}_n^m defined over the diamonds by

$$\forall \sigma = K|L \in \mathcal{E}_{int}, \forall x \in D_\sigma, \quad \overline{G}_n^m(x) = T_{2n}(u_{\sigma,+}^m) S_n(u_{\sigma,+}^m)$$

Then $T_{2,1}$ reads as

$$T_{2,1} = - \int_{\Omega} \overline{G}_n^m \mathbf{v} \cdot \nabla_{\mathcal{M}_m} \varphi_m \, dx.$$

Since the function $r \mapsto T_{2n}(r)S_n(r)$ is Lipschitz continuous and bounded, with the help of arguments already used in the proof of Lemma 6.1, we can show that \overline{G}_n^m converges to $T_{2n}(u)S_n(u)$ in L^∞ weak \star , as $h_{\mathcal{M}_m} \rightarrow 0$. Recalling that $\nabla_{\mathcal{M}_m} \varphi_m$ converges weakly in $(L^2(\Omega))^d$ we obtain

$$\lim_{h_{\mathcal{M}_m} \rightarrow 0} T_{2,1} = - \int_{\Omega} T_{2n}(u)S_n(u) \mathbf{v} \cdot \nabla \varphi \, dx. \quad (6.12)$$

We now pass to the limit in the scheme first as $h_{\mathcal{M}_m}$ goes to zero and then as n goes to infinity. Gathering equations (6.6) to (6.12), allows one to conclude that

$$\int_{\Omega} \lambda(u)S_n(u) \nabla u \cdot \nabla \varphi \, dx - \int_{\Omega} u S_n(u) \mathbf{v} \cdot \nabla \varphi \, dx - \int_{\Omega} f \varphi S_n(u) \, dx = \lim_{h_{\mathcal{M}_m} \rightarrow 0} T(n, \varphi) \quad (6.13)$$

where $\lim_{h_{\mathcal{M}_m} \rightarrow 0} |T(n, \varphi)| \leq \|\varphi\|_{L^\infty(\Omega)} \omega(n)$ with $\omega(n) \rightarrow 0$ as $n \rightarrow +\infty$. Since $S_n(u)\lambda(u)\nabla u$, $uS_n(u)\mathbf{v}$ and $fS_n(u)$ belongs respectively to $(L^2(\Omega))^d$, $L^2(\Omega)$ and $L^1(\Omega)$ a density argument gives that (6.13) holds true for any φ lying in $H^1(\Omega) \cap L^\infty(\Omega)$.

Let S be a function in $W^{1,\infty}(\mathbb{R})$ with compact support, contained in the interval $[-k, k]$, $k > 0$ and let $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$. Using the function $S(u)\psi$ in (6.13), we deduce that

$$\begin{aligned} & \left| \int_{\Omega} \lambda(u)\nabla u S_n(u)S(u)\nabla \psi \, dx + \int_{\Omega} \lambda(u)\nabla u S_n(u)\psi \nabla u S'(u) \, dx \right. \\ & \quad - \int_{\Omega} u S_n(u) S(u) \mathbf{v} \cdot \nabla \psi \, dx - \int_{\Omega} u S_n(u) S'(u) \psi \mathbf{v} \cdot \nabla u \, dx \\ & \quad \left. - \int_{\Omega} \psi S(u)S_n(u)f \, dx \right| \leq \|\varphi\|_{L^\infty(\Omega)} \omega(n). \end{aligned}$$

By observing that $S_n(u)S(u) = S(u)$ and $S_n(u)S'(u) = S'(u)$ a.e. in Ω for n sufficiently large, by passing to the limit as n goes to infinity we obtain the condition (2.12) of Definition 2.2, that is

$$\begin{aligned} & \int_{\Omega} \lambda(u)\nabla u S(u)\nabla \psi \, dx + \int_{\Omega} \lambda(u)\nabla u \psi \nabla u S'(u) \, dx \\ & \quad - \int_{\Omega} u S(u) \mathbf{v} \cdot \nabla \psi \, dx - \int_{\Omega} u S'(u) \psi \mathbf{v} \cdot \nabla u \, dx = \int_{\Omega} \psi S(u) f \, dx. \end{aligned}$$

We now turn to the decay of the energy. As a consequence of (4.23) we get

$$\lim_{n \rightarrow \infty} \overline{\lim}_{h_{\mathcal{M}_m} \rightarrow 0} \frac{1}{n} \sum_{\sigma \in \mathcal{E}_{int}} \lambda(u) \frac{m(\sigma)}{d_\sigma} (T_{2n}(u_K^m) - T_{2n}(u_L^m))^2 = 0,$$

and

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}_{int}} \frac{m(\sigma)}{d_\sigma} (T_{2n}(u_K^m) - T_{2n}(u_L^m))^2 &= \sum_{\sigma \in \mathcal{E}_{int}} m(\sigma) d_\sigma \left(\frac{T_{2n}(u_K^m) - T_{2n}(u_L^m)}{d_\sigma} \right)^2 \\ &= \sum_{\sigma \in \mathcal{E}_{int}} dm(D_\sigma) \left(\frac{T_{2n}(u_K^m) - T_{2n}(u_L^m)}{d_\sigma} \right)^2 \\ &= \frac{1}{d} \int_{\Omega} |\nabla_{\mathcal{M}_m} T_{2n}(u_m)|^2 \, dx. \end{aligned}$$

so that $\lim_{n \rightarrow \infty} \overline{\lim}_{h, \mathcal{M}_m \rightarrow 0} \frac{1}{n} \int_{\Omega} |\nabla_{\mathcal{M}_m} T_{2n}(u_m)|^2 dx = 0$. Since $\nabla_{\mathcal{M}_m} T_{2n}(u_m)$ converges weakly in $(L^2(\Omega))^d$, we have also

$$\frac{1}{n} \int_{\Omega} |\nabla T_{2n}(u)|^2 dx \leq \liminf_{h, \mathcal{M} \rightarrow 0} \frac{1}{n} \int_{\Omega} |\nabla_{\mathcal{M}} T_{2n}(u_{\mathcal{M}})|^2 dx,$$

which leads to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \lambda(u) |\nabla T_{2n}(u)|^2 dx = 0.$$

Since u is finite almost everywhere in Ω and since $T_n(u) \in H^1(\Omega)$ for any $n > 0$ we can conclude $u_{\mathcal{M}_m}$ converges to u which is the unique renormalized solution with null median. \square

A Appendix

Discrete functional inequalities are useful for the study of finite volume schemes. Discrete Sobolev inequalities are proved in [10] for Dirichlet boundary conditions and in [8] for non Dirichlet boundary conditions. In [3] discrete Gagliardo-Nirenberg-Sobolev and Poincaré-Sobolev inequalities for some finite volume schemes are proved. There does not seem to be any proof of Discrete Poincaré-Wirtinger median inequality. The authors in [3] use the continuous embedding of the space $BV(\Omega)$ into $L^{\frac{d}{d-1}}(\Omega)$ for a Lipschitz domain $\Omega \subset \mathbb{R}^d$, with $d \geq 2$ to establish discrete inequalities. We will use this method to prove the Discrete Poincaré-Wirtinger median inequality (2.16).

Let us first recall some results concerning functions of bounded variation (more details about these functions can be found in [26]). Let Ω be an open set of \mathbb{R}^d and $u \in L^1(\Omega)$. The total variation of u in Ω , denoted by $TV_{\Omega}(u)$, is defined by

$$TV_{\Omega}(u) = \sup \left\{ \int_{\Omega} u(x) \operatorname{div}(\phi(x)) dx, \phi \in C_c^1(\Omega), |\phi(x)| \leq 1, \forall x \in \Omega \right\}.$$

The function $u \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if $TV_{\Omega}(u) < +\infty$. The space $BV(\Omega)$ is endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + TV_{\Omega}(u).$$

The space $BV(\Omega)$ is a natural space to study finite volume approximations. Indeed, for $u = (u_K)_{K \in \mathcal{T}} \in X(\mathcal{T})$, we have

$$TV_{\Omega}(u) = \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} m(\sigma) |u_L - u_K| = |u|_{1,1,\mathcal{M}} < +\infty.$$

The discrete space $X(\mathcal{T})$ is included in $L^1(\Omega) \cap BV(\Omega)$ and we have

$$\|u\|_{BV(\Omega)} = \|u\|_{1,1,\mathcal{M}}.$$

Our starting point for the discrete Poincaré-Wirtinger median inequality is the continuous embedding of $BV(\Omega)$ into $L^{\frac{d}{d-1}}(\Omega)$ for Lipschitz bounded connected domain Ω of \mathbb{R}^d , $d \geq 2$, written in the following theorem (see [26] for more details).

Theorem A.1 ([26]). *There exists a constant $C(\Omega) > 0$ only depending on Ω such that, for all $u \in BV(\Omega)$,*

$$\left(\int_{\Omega} |u - m|^{\frac{d}{d-1}} dx \right)^{\frac{d-1}{d}} \leq C(\Omega) TV_{\Omega}(u), \quad (\text{A.1})$$

where $m \in \operatorname{med}(u)$.

In the spirit of [3] (which studied Discrete Poincaré-Wirtinger mean inequality), let us prove now the following proposition

Proposition A.2 (Discrete Poincaré-Wirtinger median inequality). *Let Ω be an open bounded connected polyhedral domain of \mathbb{R}^d and let \mathcal{M} be an admissible mesh satisfying (2.14). Then for $1 \leq p < +\infty$ there exists a constant $C > 0$ only depending on Ω , d and p such that*

$$\|u - c\|_{0,p} \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \quad \forall u \in X(\mathcal{T}) \quad (\text{A.2})$$

where c is in $\text{med}(u)$.

Proof of Proposition A.2. Let $u = (u_K)_{K \in \mathcal{T}}$ be a function of $X(\mathcal{T})$ and let m be an element of $\text{med}(u)$. We define $v \in X(\mathcal{T})$ by $v_K = (u_k - m)|u_k - m|^{p-1}$ for all $K \in \mathcal{T}$. Since $m \in \text{med}(u)$, we have $0 \in \text{med}(v)$, using inequality (A.1), we obtain

$$\|v\|_{0,\frac{d}{d-1}} \leq C(\Omega) |v|_{1,1,\mathcal{M}}, \quad (\text{A.3})$$

and using the inclusion of $L^{\frac{d}{d-1}}(\Omega)$ into $L^1(\Omega)$, we get

$$\|v\|_{0,1} \leq C(\Omega, d) |v|_{1,1,\mathcal{M}}, \quad (\text{A.4})$$

where the constant C depends on Ω and d .

Moreover, for all $K, L \in \mathcal{T}$, we have

$$\begin{aligned} |v_K - v_L| &= |u_K - u_L| |v'(w_{LK})|, \quad \forall w_{LK} \in [u_K, u_L], \\ &\leq p |u_K - u_L| |w_{LK} - m|^{p-1} \\ &\leq p |u_K - u_L| \left(|u_K - m|^{p-1} + |u_L - m|^{p-1} \right). \end{aligned} \quad (\text{A.5})$$

Therefore, gathering (A.4) and (A.5), we obtain

$$\begin{aligned} \|v\|_{0,1} &= \| |u - m|^p \|_{0,1} = \|u - m\|_{0,p}^p \\ &\leq C \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \mathfrak{m}(\sigma) p |u_K - u_L| \left(|u_K - m|^{p-1} + |u_L - m|^{p-1} \right). \end{aligned} \quad (\text{A.6})$$

Using Hölder's inequality we get,

$$\begin{aligned} \|u - m\|_{0,p}^p &\leq pC \left(\sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{\mathfrak{m}(\sigma)}{d_\sigma^{p-1}} |u_K - u_L|^p \right)^{\frac{1}{p}} \\ &\quad \times \left(\sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \mathfrak{m}(\sigma) \left(d_\sigma^{\frac{p-1}{p}} \right)^{\frac{p}{p-1}} \left(|u_K - m|^{p-1} + |u_L - m|^{p-1} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq pC |u - m|_{1,p,\mathcal{M}} \left(\sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \mathfrak{m}(\sigma) d_\sigma \frac{p}{p-1} \left(|u_K - m|^p + |u_L - m|^p \right) \right)^{\frac{p-1}{p}}. \end{aligned} \quad (\text{A.7})$$

The regularity constraint (2.14) on the mesh ensures that

$$\sum_{\sigma \in \mathcal{E}_{int}} \mathfrak{m}(\sigma) d_\sigma \leq \frac{1}{\xi} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \mathfrak{m}(\sigma) d(x_K, \sigma) = \frac{N}{\xi} \sum_{K \in \mathcal{T}} \mathfrak{m}(K). \quad (\text{A.8})$$

Then applying the previous inequality (A.8) and a discrete integration by parts, we get

$$\begin{aligned} \|u - m\|_{0,p}^p &\leq pC(\Omega)|u - m|_{1,p,\mathcal{M}} \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_\sigma |u_K - m|^p\right)^{\frac{p-1}{p}} \\ &\leq \frac{C(\Omega, p, d)}{\xi^{\frac{p-1}{p}}} |u - m|_{1,p,\mathcal{M}} \|u - m\|_{0,p}^{p-1}. \end{aligned} \quad (\text{A.9})$$

Then we obtain the general result

$$\|u - m\|_{0,p} \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \quad \forall u \in X(\mathcal{T}).$$

□

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