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RECURRENCE PROPERTIES OF SEQUENCES OF INTEGERS

AI-HUA FAN AND DOMINIQUE SCHNEIDER

ABSTRACT. In order to study the recurrence of sequences of integers, we investigate their L^2 -exactness and Θ -Hartman property (Θ being a set of rational numbers). Two classes of sequences of integers are well studied, which are return times relative to a weakly mixing system and Bernoulli random sequences.

1. INTRODUCTION

Let $\Lambda = \{u_n\}_{n \geq 1} \subset \mathbb{N}^* = \{1, 2, \dots\}$ be a subset of integers, or a strictly increasing sequence of integers. Λ is called a *Poincaré set* or *1-recurrent set* or simply *recurrent set* if for any measure preserving system (Y, \mathcal{B}, ν, S) and any $A \in \mathcal{B}$ with $\nu(A) > 0$ there exists $n \in \Lambda$ such that $\nu(A \cap S^{-n}A) > 0$. Poincaré recurrence theorem says that \mathbb{N}^* is a recurrent set. Λ is said to be *L^2 -good* if for any measure preserving system (Y, \mathcal{B}, ν, S) and any $g \in L^2(\nu)$, the follow average

$$(1.1) \quad A_N^\Lambda g(x) := \frac{1}{N} \sum_{n=1}^N g(S^{u_n}x)$$

L^2 -converges as $N \rightarrow \infty$. If furthermore,

$$L^2\text{-}\lim_{N \rightarrow \infty} A_N^\Lambda g := \mathbb{E}^{\mathcal{J}_S}(g),$$

we say that Λ is *L^2 -exact*, where $\mathbb{E}^{\mathcal{J}_S}$ denotes the conditional expectation with respect to the invariant σ -field \mathcal{J}_S of S . These notions are different. It is well known that $2\mathbb{N}^*$ is a recurrent set, but $2\mathbb{N}^* + 1$ is not. However both are L^2 -good. On the other hand, $\mathbb{N}^{*2} := \{1, 2^2, 3^2, \dots\}$ is L^2 -good, but it is not L^2 -exact.

The L^2 -good and L^2 -exact sequences can be characterized as follows. Let

$$(1.2) \quad \widehat{A}_N^\Lambda(t) = \frac{1}{N} \sum_{n=1}^N e^{2\pi i u_n t}.$$

Theorem A. *A sequence $\Lambda = \{u_n\}_{n \geq 1} \subset \mathbb{N}^*$ is L^2 -good iff the limit $\lim_{N \rightarrow \infty} \widehat{A}_N^\Lambda(t)$ exists for every $0 \leq t < 1$. It is L^2 -exact iff the limit exists and is equal to zero for every $0 < t < 1$.*

So the study of L^2 -goodness and the L^2 -exactness is a part of Fourier analysis. The L^2 -goodness doesn't imply the recurrence, but it will be proved that the L^2 -exactness does.

Theorem B. *An L^2 -exact sequence is recurrent.*

Let us present a class of L^2 -exact sequences. Let (X, \mathcal{A}, μ, T) be a measure preserving system. Let $A \in \mathcal{A}$. For $x \in X$, we define the *set of return times* of x into A by

$$R_A(x) = \{n \geq 1 : T^n x \in A\}.$$

If the system is weakly mixing, we have the following result.

Theorem C. *Suppose (X, \mathcal{A}, μ, T) is a weakly mixing measure preserving system and $A \in \mathcal{A}$ with $\mu(A) > 0$. Then for μ -a.e. $x \in X$, $R_A(x)$ is L^2 -exact. If T is weakly mixing and uniquely ergodic, then for every $x \in X$, $R_A(x)$ is L^2 -exact.*

Without the weakly mixing condition in the theorem, the conclusion doesn't hold in general. In fact, let $0 < \alpha < 1$ be an irrational number and $I = (-\delta, \delta)$ be an interval in the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ($0 < \delta < 1/4$). The set $R_I(0)$ of n such that $n\alpha \in I$ is a set of return times (called a Sturmian sequence), which is not L^2 -exact because

$$\underline{\lim} \operatorname{Re} \frac{1}{N} \sum_{n=1}^N 1_I(n\alpha) e^{2\pi i n \alpha} \geq 2\delta \cos(2\pi\delta) > 0.$$

So, the condition in the definition of L^2 -exactness is not satisfied with S to be the α -rotation and $g(x) = e^{2\pi i x}$.

Let us present another class of recurrent sets. Let $\Theta \subset \mathbb{Q}/\mathbb{Z} \subset [0, 1)$ be a set of rational numbers containing 0. Define

$$D_\Theta = \left\{ m \geq 1 : m = [q_1, \dots, q_r] \text{ for some } \frac{p_1}{q_1}, \dots, \frac{p_r}{q_r} \in \Theta \right\}$$

where $[q_1, \dots, q_r]$ denotes the least common multiple of q_1, \dots, q_r . A set $\Lambda \subset \mathbb{N}^*$ is called a Θ -Hartman sequence if for each $m \in D_\Theta$ there is a subsequence $\{u_{n_k}\} \subset \Lambda$ such that (i) $\{u_{n_k}\} \subset m\mathbb{Z} \cap \Lambda$ and (ii)

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha u_{n_k}} = 0 \quad \forall \alpha \in (0, 1) \setminus \Theta.$$

If $\Theta = \{0\}$, the sequence $\{u_{n_k}\}$ satisfying (ii) is said to be Hartman uniformly distributed, i.e. uniformly distributed in the Bohr compactification of \mathbb{Z} . Therefore a $\{0\}$ -Hartman sequence is nothing but a sequence containing a Hartman uniformly distributed subsequence. Remark that the limit in (ii) is ensured if $\{\alpha u_{n_k}\}$ is uniformly distributed modulo 1. Also remark that $D_\Theta = \{1\}$ for $\Theta = \{0\}$, $D_\Theta = \{2^n : n \geq 0\}$ for $\Theta = \{\frac{k}{2^n} : k, n \in \mathbb{N}\}$, $D_\Theta = \mathbb{N}$ for $\Theta = \mathbb{Q}/\mathbb{Z}$. The second assertion in Theorem A means that Hartman uniformly distributed sequences are just L^2 -exact sequences. The following result shows that the Θ -Hartman property implies the recurrence.

Theorem D. *Let $\Theta \subset \mathbb{Q}/\mathbb{Z}$ contain 0. Every Θ -Hartman sequence is recurrent.*

Now let us describe the class of Bernoulli random sequences of integers. Let $\{p_n\}_{n \geq 1}$ be a sequence of real numbers such that

$$0 \leq p_n \leq 1, \quad \sum_{n=1}^{\infty} p_n = \infty.$$

Let $\{\xi_n\}_{n \geq 1}$ be a sequence of independent Bernoulli random variables such that

$$P(\xi_n = 1) = p_n = 1 - P(\xi_n = 0).$$

We are interested in the random set

$$W(\omega) = \{n \geq 1 : \xi_n(\omega) = 1\}.$$

This is almost surely (a.s. for short) an infinite subset of integers.

Theorem E. *The sequence $W(\omega)$ is a.s. L^2 -exact (equivalently Hartman uniformly distributed) if the following conditions are satisfied:*

$$(1) \quad \log N = o\left(\sum_{n=1}^N p_n\right), \quad (2) \quad \sum_{n=1}^N |p_n - p_{n+1}| = o\left(\sum_{n=1}^N p_n\right).$$

Without the variation condition (2), the conclusion of the theorem is not true. For example, if $p_{2n} = 0$ and $p_{2n+1} = 1$, we have the deterministic set $W(\omega) = 2\mathbb{N}^* + 1$ (the set of odd integers) which is even not recurrent. If p_n is decreasing, the condition (2) becomes redundant.

Kahane and Katznelson [9] obtained the following conditions for $W(\omega)$ to be a.s. Hartman uniformly distributed (i.e. L^2 -exact)

$$\lim np_n = \infty; \quad n^{1+\delta} p_n \uparrow, \quad n^{1-\delta} p_n \downarrow \quad \forall 0 < \delta < 1.$$

The condition stated in the above theorem seems a little bit more flexible.

Theorem E implies that if $p_n = n^{-\tau}$ with $0 < \tau < 1$, then $W(\omega)$ is a.s. Hartman uniformly distributed. This is to be compared with the following Kahane and Katznelson's result: if $p_n = n^{-1}$, then $W(\omega)$ is a.s. a Sidon set; if $p_n = n^{-\tau}$ with $\tau < 1$, then $W(\omega)$ is a.s. not a Sidon set [8]. We will discuss the L^2 -universality of $W(\omega)$.

2. PRELIMINARIES

We present in this section some preliminaries including van der Corput inequality, spectral lemma and Kronecker factor.

2.1. van der Corput inequality. We will use a simple form the van der Corput inequality (see [12]):

Lemma 2.1. *For $N \geq 1$ complex numbers a_0, a_1, \dots, a_{N-1} and any integer H with $1 \leq H \leq N$, we have*

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} a_n \right|^2 \leq \frac{C}{H} \left(\frac{1}{N} \sum_{n=0}^{N-1} |a_n|^2 + \left| \sum_{h=1}^H \frac{1}{N} \sum_{n=0}^{N-h} a_n \overline{a_{n+h}} \right| \right)$$

where C is a absolute constant.

2.2. Spectral lemma. The spectral lemma holds for contracting operators on Hilbert space (see [11]). Let (X, \mathcal{A}, μ, T) is a measure preserving system. The Koopman map $f \mapsto f \circ T$, also denoted by T , is an isometry on $L^2(\mu)$. The *spectral measure* μ_f of $f \in L^2(\mu)$ is characterized by its Fourier coefficients as follows

$$\widehat{\mu}_f(n) = \int f \circ T^n \cdot f d\mu \text{ for } n \geq 0; \quad \widehat{\mu}_f(n) = \overline{\widehat{\mu}_f(-n)} \text{ for } n < 0.$$

Lemma 2.2. *Let $p(z) = \sum a_n z^n$ be a polynomial. We have*

$$\|p(T)f\|_{L^2(T)}^2 = \int_{\mathbb{R}/\mathbb{Z}} |p(e^{2\pi i t})|^2 d\mu_f(t)$$

2.3. Kronecker factor. Let (X, \mathcal{A}, μ, T) is a measure preserving system. The *Kronecker factor* \mathcal{K} of the system is by definition the closed linear span in $L^2(\mu)$ of the eigenfunctions for T . There is a sub σ -algebra \mathcal{A}' of \mathcal{A} such that $\mathcal{K} = L^2(\mathcal{A}')$. The orthogonal complementary space \mathcal{K}^\perp consists of all functions $f \in L^2(\mu)$ whose spectral measures μ_f are continuous.

If $f = f_1 + f_2$ with $f_1 \in \mathcal{K}$ and $f_2 \in \mathcal{K}^\perp$, then $\mu_f = \mu_{f_1} + \mu_{f_2}$. Furthermore, μ_{f_2} is continuous, μ_{f_1} is discrete and $\mu_{f_1}(\{\theta\}) \neq 0$ if and only if $e^{2\pi i \theta}$ is an eigenvalue and f is not orthogonal to the corresponding eigenfunction.

For any $\theta \in \mathbb{R}/\mathbb{Z}$, denote

$$\mathcal{K}_\theta = \{f \in H : f(Tx) = e^{2\pi i \theta} f(x)\}.$$

That \mathcal{K}_θ is not trivial means that $e^{2\pi i \theta}$ is an eigenvalue for T . Thus \mathcal{K} is the direct sum of all these non-trivial \mathcal{K}_θ with $\theta \in \mathbb{R}/\mathbb{Z}$. If the system is ergodic, all eigenfunctions are in $L^\infty(\mu)$ and each eigenspace \mathcal{K}_θ is one-dimensional.

The subspace $L_0^2(\mu)$ of $L^2(\mu)$ consists of those f such that $\int f d\mu = 0$. Let $\mathcal{K}^* = \mathcal{K} \cap L_0^2(\mu)$. We have the orthogonal decomposition

$$L^2(\mu) = \mathbb{C} \oplus L_0^2(\mu) = \mathbb{C} \oplus \mathcal{K}^* \oplus \mathcal{K}^\perp.$$

For any $\Theta \subset \mathbb{R}/\mathbb{Z}$, we denote

$$\mathcal{K}_\Theta = \bigoplus_{\theta \in \Theta} \mathcal{K}_\theta.$$

In the following, we only consider the case where Θ is a set of rational numbers.

3. RECURRENCE, L^2 -GOODNESS AND L^2 -EXACTNESS

3.1. Recurrent sequences. Bertrand-Mathis [1] proved that Λ is a recurrent set if and only if it is a *set of intersectivity*, that is to say, $\Lambda \cap (S - S) \neq \emptyset$ for any set $S \subset \mathbb{N}^*$ with positive Banach density, i.e.

$$B(S) = \sup \lim_{n \rightarrow \infty} \frac{|S \cap I_n|}{|I_n|}$$

where the supremum is take over all sequence of intervals $\{I_n\}$ with length $|I_n|$ tending to infinity as n does. As was pointed out in [1], this result was also proved independently by P. Liardet, J. F. Méla and Y. Katznelson.

A necessary condition for Λ to be recurrent is that $m\mathbb{Z} \cap \Lambda \neq \emptyset$ for any $m \in \mathbb{N}^*$, which is easy to see from the definition. It follows that the set $2\mathbb{N}^* + 1$ of odd integers is not recurrent.

Kamae and Mendes France [10] proved that every van der Corput sequence is intersective and then a recurrent set. Recall the definition of a van der Corput sequence Λ : for any sequence of real numbers $\{x_n\}_{n \geq 1}$ such that $\{x_{n+h} - x_n\}_{n \geq 1}$ is uniformly distributed (mod 1) for each $h \in \Lambda$, all sequences $\{x_{an+b}\}_{n \geq 1}$ are uniformly distributed (mod 1) for all integers $a \geq 1, b > 0$.

Furstenberg proved that if $p \in \mathbb{Q}[x]$ is a polynomial taking values in \mathbb{N} such that $p(0) = 0$, then the image $p(\mathbb{N}^*)$ is a recurrent sequence. In particular, so are $2\mathbb{N}^*$ and $\mathbb{N}^{*2} = \{1, 2^2, 3^2, \dots\}$.

Bourgain [2] proved the pointwise convergence of $A_N^{p(\mathbb{N})}g$. Bourgain, Furstenberg, Katznelson and Ornstein proved the pointwise converges of $A_N^{R_A(x)}g$ for the return times $R_A(x)$ in an ergodic system (see [2]).

The L^2 -goodness doesn't imply the recurrence. But the L^2 -exactness does.

Theorem 3.1. *Suppose that $\Lambda \subset \mathbb{N}^*$ is an L^2 -exact sequence. Let (Y, \mathcal{B}, ν, S) be a measure preserving system and let $g \in L^2(\nu)$.*

- (1) *If $g \perp g \circ S^n$ for all $n \in \Lambda$, then $g \perp L^2(\mathcal{J}_S)$.*
- (2) *Λ is a recurrent sequence.*

Proof. (1) It suffices to prove $g \perp \mathbb{E}^{\mathcal{J}_S}g$. Let $\Lambda = \{u_n\}$. Recall that

$$A_N^\Lambda g(y) := \frac{1}{N} \sum_{n=1}^N g(S^{u_n}y) \rightarrow \mathbb{E}^{\mathcal{J}_S}g.$$

Write

$$(3.1) \quad \left| \int g \cdot \mathbb{E}_\nu^{\mathcal{J}_S}g d\nu \right| \leq \left| \int g[\mathbb{E}_\nu^{\mathcal{J}_S}g - A_N^\Lambda g] d\nu \right| + \left| \int g \cdot A_N^\Lambda g d\nu \right|$$

By the hypothesis, g is orthogonal to $A_N^\Lambda g$. So, it follows from (3.1) that

$$\begin{aligned} \left| \int g \cdot \mathbb{E}_\nu^{\mathcal{J}_S}g d\nu \right| &\leq \|g[\mathbb{E}_\nu^{\mathcal{J}_S}g - A_N^\Lambda g]\|_{L^1(\nu)} \\ &\leq \|g\|_{L^2(\nu)} \|\mathbb{E}_\nu^{\mathcal{J}_S} - A_N^\Lambda\|_{L^2(\nu)}. \end{aligned}$$

However the last term tends to zero when $n \rightarrow \infty$, because $A_N^\Lambda g \rightarrow \mathbb{E}^{\mathcal{J}_S}g$.

(2) Otherwise, there exist a measure preserving system (Y, \mathcal{B}, ν, S) and a $B \in \mathcal{B}$ with $\nu(B) > 0$ such that

$$\int 1_B \cdot 1_B \circ S^n d\nu = \nu(B \cap S^{-n}B) = 0, \quad \forall n \in \Lambda.$$

Applying (1) to 1_B , we get $1_B \perp L^2(\mathcal{J}_S)$. In particular, $1_B \perp 1$ which means $\nu(B) = 0$. \square

3.2. Characterizations of L^2 -goodness and L^2 -exactness.

Theorem 3.2. *A sequence $\Lambda = \{u_n\} \subset \mathbb{N}$ is L^2 -good iff the following limit exists*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i u_n t}$$

for all $t \in (0, 1)$. The sequence is L^2 -exact iff not only the above limit exists but the limit is zero.

Proof. We use the notation defined by (1.1) and (1.2).

L^2 -goodness. By considering the rotation $x \mapsto x + t$ on \mathbb{R}/\mathbb{Z} , we see that the condition is necessary. Let (Y, \mathcal{B}, S, ν) be an arbitrary measure preserving system and $g \in L^2(\nu)$. We prove the sufficiency by showing that $A_N^\Lambda g$ is a Cauchy sequence in $L^2(\nu)$. In fact, this is a consequence of the spectral lemma which implies

$$\|A_p^\Lambda g - A_q^\Lambda g\|_{L^2(\nu)}^2 \leq \int |\widehat{A}_p^\Lambda(t) - \widehat{A}_q^\Lambda(t)|^2 d\mu_g(t) \quad (\forall 1 \leq p < q),$$

and the Lebesgue dominated convergence theorem.

L^2 -exactness. To get the necessity, we consider the irrational rotation $x \mapsto x + t$ on \mathbb{R}/\mathbb{Z} if t is irrational, or the ergodic rotation $x \mapsto x + 1$ on $\mathbb{Z}/r\mathbb{Z}$ when $t = \frac{s}{r}$ is rational with $(s, r) = 1$. To prove the sufficiency, we have only to show that $A_N^\Lambda g$ tends to zero when $g \in \mathcal{K}_0^\perp$ (\mathcal{K}_0 being the space of invariant functions). It is true because

$$\|A_p^\Lambda g\|_{L^2(\nu)}^2 \leq \int |\widehat{A}_p^\Lambda(t)|^2 d\mu_g(t)$$

and the spectral measure μ_g doesn't charge the point 0. \square

The theorem shows that Λ is L^2 -exact sequences means the sequence $n \mapsto 1$ or 0 according to $n \in \Lambda$ or not is almost periodic sequences in the sense of Bohr and that L^2 -good sequences correspond to generalized almost periodic sequences in the sense of Hartman (see [6], p.72).

3.3. Some remarks. Suppose that $\Lambda = \{u_n\}_{n \geq 1}$ is L^2 -good. Let

$$c_\Lambda(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i u_n t}.$$

Suppose that $c_\Lambda(t) = 0$ for all but at most countably many points t . Let (Y, \mathcal{B}, ν, S) be a measure preserving system. Any $g \in L^2(\nu)$ can be decomposed into

$$g = \sum a_\theta g_\theta + g_{K^\perp} \quad (a_\theta \in \mathbb{C})$$

where the sum is taken over all θ 's such that $e^{2\pi i \theta}$ is an eigenvalue of S , f_θ is the orthogonal projection of g onto the eigenspace corresponding to the eigenvalue $e^{2\pi i \theta}$ and g_{K^\perp} is the projection on the orthogonal of the

Kronecker factor \mathcal{K} . In this case, we can identify the L^2 -limit of $A_N^\Lambda g$ as follows

$$\lim_{N \rightarrow \infty} A_N^\Lambda g = \sum a_\theta c_\Lambda(\theta) g_\theta.$$

We have only to show this for each component of g in the above decomposition. In fact, if $g \in K^\perp$, then the spectral measure of g is continuous, which implies $\lim_{N \rightarrow \infty} A_N^\Lambda g = 0$ through the spectral lemma and the fact that $c_\Lambda(t) \neq 0$ for at most countably many t 's. If $g = g_\theta$, it suffices to notice that

$$A_N^\Lambda g_\theta = \widehat{A}_N^\Lambda(\theta) g_\theta.$$

A set $\Lambda = \{u_n\}_{n \geq 1}$ of integers is said to be r -recurrent ($r \geq 1$) if for any measure-preserving system (Y, \mathcal{B}, ν, S) and any $A \in \mathcal{B}$ with $\nu(A) > 0$ there exists $n \in \Lambda$ such that

$$\nu(A \cap S^{-n}A \cap \cdots \cap S^{-rn}A) > 0.$$

Furstenberg proved that \mathbb{N}^* is r -recurrent for every $r \geq 1$ by showing

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \nu(A \cap S^{-n}A \cap \cdots \cap S^{-rn}A) > 0.$$

We remark that any set $\Lambda = \{u_n\}_{n \geq 1}$ such that $\lim \frac{u_n}{n} = 1$ is also r -recurrent for every $r \geq 1$. This is a consequence of the positivity of the above liminf and the following fact: Assume $\lim \frac{u_n}{n} = 1$. For any bounded sequence of numbers $\{a_n\}$ we have

$$(3.2) \quad \lim_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N a_{u_n} - \frac{1}{N} \sum_{n=1}^N a_n \right| = 0.$$

As a consequence, Furstenberg's argument proves an improvement of Szemerédi theorem: Let $\Lambda = \{u_k\} \subset \mathbb{N}^*$ be a strictly increasing sequence such that $\frac{u_k}{k} \rightarrow 1$ as $k \rightarrow \infty$. Let $S \subset \mathbb{N}^*$ be a set with strictly positive upper Banach density. For any integer $\ell \geq 2$, there exist $h \geq 1$ and $u \in \Lambda$ such that

$$h \in S, \quad h + u \in S, \quad h + 2u \in S, \quad \cdots, \quad h + \ell u \in S.$$

Let $\Lambda = \{u_k\}$ be a given sequence with some property like L^2 -goodness or L^2 -exactness (but not the recurrence). By (3.2), a sequence $\{u_k + \delta_k\}$ which is a perturbation of $\{u_k\}$ by $\{\delta_k\}$ such that $\delta_k = o(k)$ inherits the same property from Λ . For example, a sequence $\{u_k\}$ such that $u_k = \zeta k + o(k)$ with $\zeta > 0$ is L^2 -exact; a sequence $\{u_k\}$ such that $u_k = k^2 + o(k)$ is L^2 -good and has the same c_Λ function as $\{k^2\}$.

4. WIENER-WINTNER THEOREM AND RETURN TIMES

Let $\Lambda \subset \mathbb{N}^*$. One way to study the recurrence of Λ is to investigate the limit of the average $A_N^\Lambda g$. This section is devoted to the L^2 -exactness of the set of return times in a weakly mixing system. We are actually led to recognize the limit of the Birkhoff average of a product system.

Given two measure preserving systems (X, \mathcal{A}, μ, T) and (Y, \mathcal{B}, ν, S) . Their product is the measure preserving system $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu, T \times S)$. In general, the ergodicities of both systems don't imply the ergodicity of their product. It is easy to see that

$$\mathcal{J}_T \otimes \mathcal{J}_S \subset \mathcal{J}_{T \times S}.$$

The above inclusion maybe strict. We will prove that we do get an equality when T is weakly mixing and that in this case, for any $F \in L^2(\mu \times \nu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} F(T^n x, S^n y) = \mathbb{E}_\nu^{\mathcal{J}_S} \int F(x, y) d\mu(x).$$

Moreover, for a fixed $f \in L^2(\mu)$, we can find a measurable set X_f with full μ -measure such that for any $x \in X_f$, for any measure preserving system (Y, \mathcal{B}, ν, S) and for any $g \in L^2(\nu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(S^n y) = \int f d\mu \cdot \mathbb{E}_\nu(g | \mathcal{J}_S)$$

in $L^2(\nu)$. Bourgain-Furstenberg-Katznelson-Ornstein (see [2]) proved the pointwise convergence without identifying the limit but under the condition that T is only ergodic. With the limit identified in the last equality, we can prove that if T is weakly mixing, for any set A with $\mu(A) > 0$, the set of return times $R_A(x)$ is L^2 -exact for μ -a.e. x .

4.1. Some lemmas. Let (X, \mathcal{A}, μ, T) is a measure preserving system. If $f \in L^2(\mu)$ and $1 \leq h \leq N$, write

$$J_{N,h}(x) = \frac{1}{N} \sum_{n=0}^{N-h} f(T^n x) \overline{f(T^{n+h} x)}.$$

Lemma 4.1. *Let (X, \mathcal{A}, μ, T) is a measure preserving system. If $f \in L^2(\mu)$ and $1 \leq H \leq N$, for any Borel measure ν on $[0, 1)$ we have*

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right\|_{L^2(\nu)}^2 \leq \frac{C}{H} \left(J_{N,0}(x) + \int \left| \sum_{h=1}^H e^{2\pi i h t} J_{N,h}(x) \right| d\nu(t) \right)$$

where C is the constant in the van der Corput inequality.

Proof. Apply the van der Corput inequality to $a_n = f(T^n x) e^{2\pi i n t}$, we get

$$(4.1) \quad \left| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right|^2 \leq \frac{C}{H} \left(J_{N,0}(x) + \left| \sum_{h=1}^H e^{2\pi i h t} J_{N,h}(x) \right| \right).$$

Integrate (4.1) with respect to ν to obtain the desired result. \square

Remark that the functions $J_{N,h}$ are independent of t . If the system is ergodic, by the Birkhoff ergodic theorem, for μ -a.e. x and for all $h \geq 0$ we have

$$(4.2) \quad \lim_{N \rightarrow \infty} J_{N,h}(x) = \int f \cdot \overline{f \circ T^h} d\mu.$$

The following lemma will allow us to apply the spectral lemma. The point (2) is due to Bourgain.

Lemma 4.2. *Let (X, \mathcal{A}, μ, T) is an ergodic measure preserving system.*

(1) *If $f \in \mathcal{K}^*$, for μ -a.e. x and for any Borel probability measure ν on $[0, 1)$ we have*

$$\overline{\lim}_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right\|_{L^2(\nu)}^2 \leq C \|f\|_{L^2(\mu)} \overline{\lim}_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right\|_{L^2(\mu \times \nu)}$$

where C is a constant.

(2) *If $f \in \mathcal{K}^\perp$, for μ -a.e. x we have*

$$\lim_{N \rightarrow \infty} \sup_{t \in [0,1)} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right| = 0.$$

Proof. Since the system is ergodic, for μ -a.e. x and for all $h \geq 0$ we have

$$\lim_{N \rightarrow \infty} J_{N,h}(x) = \int f \cdot \overline{f \circ T^h} d\mu.$$

In (4.1), let $N \rightarrow \infty$ then let $H \rightarrow \infty$. We obtain that for μ -a.e. x the limit

$$\overline{\lim}_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right\|_{L^2(\nu)}^2$$

is bounded up to the multiplicative constant C by

$$\begin{aligned} & \overline{\lim}_{H \rightarrow \infty} \int d\nu(t) \left| \int f(x) \left(\frac{1}{H} \sum_{h=1}^H e^{2\pi i h t} \overline{f \circ T^h(x)} \right) d\mu(x) \right| \\ & \leq \|f\|_{L^2(\mu)} \overline{\lim}_{H \rightarrow \infty} \left\| \frac{1}{H} \sum_{h=1}^H e^{2\pi i h t} \overline{f \circ T^h(x)} \right\|_{L^2(\mu \times \nu)}. \end{aligned}$$

For the last inequality, we have used the Cauchy-Schwarz inequality twice (first with respect to μ , then with respect to ν). Thus we have proved (1)

To prove (2), first we deduce from (4.1) the uniform estimate

$$\sup_{t \in [0,1)} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right| \leq \frac{C}{H} \left(J_{N,0}(x) + \sum_{h=1}^H |J_{N,h}(x)| \right).$$

Then using (4.2) and the fact $\int f \cdot \overline{f \circ T^h} d\mu = \widehat{\mu}_f(h)$, we get that μ -a.e.

$$\overline{\lim}_{N \rightarrow \infty} \sup_{t \in (0,1)} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right|^2 \leq C \lim_{H \rightarrow \infty} \sqrt{\frac{1}{H} \sum_{h=1}^H |\widehat{\mu}_f(h)|^2}.$$

The last limit is zero according to Wiener theorem because the spectral measure μ_f is continuous for $f \in \mathcal{K}^\perp$. \square

4.2. Wiener-Wintner theorem. We are now ready to prove the following theorem, a kind of Wiener-Wintner theorem which provides a well defined expression for the ergodic limit.

Recall that \mathcal{K} denotes the Kronecker factor of the system (X, \mathcal{A}, μ, T) . If the system is ergodic, the space \mathcal{K}_0 of invariant functions consists of constant functions. Also recall that $L_0^2(\mu)$ is the subspace of $L^2(\mu)$ consisting of functions f such that $\int f d\mu = 0$. We have the decomposition

$$L_0^2(\mu) = \mathcal{K}^* \oplus \mathcal{K}^\perp \quad \text{with} \quad \mathcal{K}^* = \mathcal{K} \ominus \mathcal{K}_0.$$

When T is weakly mixing, $\mathcal{K} = \mathcal{K}_0$, i.e. $L_0^2(\mu) = \mathcal{K}^\perp$.

Let (Y, \mathcal{B}, ν, S) be another measure preserving system. The theorem below provides us a kind of approximation of the projection $\mathbb{E}_\nu^{\mathcal{J}_S} g$.

Theorem 4.3. *Suppose that (X, \mathcal{A}, μ, T) is a weakly mixing measure preserving system. Let $f \in L^2(\mu)$. There exists a measure set X_f with full μ -measure such that for any $x \in X_f$, for any measure preserving system (Y, \mathcal{B}, ν, S) and for any $g \in L^2(\nu)$ we have*

$$(4.3) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(S^n y) = \int f d\mu \cdot \mathbb{E}_\nu(g | \mathcal{J}_S)$$

in $L^2(\nu)$. For any $F \in L^2(\mu \times \nu)$, we have

$$(4.4) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} F(T^n x, S^n y) = \mathbb{E}_\nu^{\mathcal{J}_S} \int F(x, y) d\mu(x)$$

in $L^2(\mu \times \nu)$. The results remain true if we assume that T is ergodic and $\mathcal{J}_{T \times S} = \mathcal{J}_T \otimes \mathcal{J}_S$ instead of assuming T is weakly mixing.

Proof. Let us first prove (4.3). By the linearity on f of both sides of the equality (4.3) and the von Neumann theorem applied to (Y, \mathcal{B}, ν, S) , in order to prove (4.3), we can assume that $f \in L_0^2(\mu)$. Since $L_0^2(\mu) = \mathcal{K}^* + \mathcal{K}^\perp$, it suffice to prove it for f in \mathcal{K}^* or in \mathcal{K}^\perp .

The spectral lemma gives

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(S^n y) \right\|_{L^2(\nu)}^2 \leq \int \left| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right|^2 d\nu_g(t).$$

Then for $f \in \mathcal{K}^\perp$, we get the desired result immediately from Lemma 4.2 (2). For $f \in \mathcal{K}^*$, there is nothing to prove when T is weakly mixing which

implies $\mathcal{K}^* = \{0\}$. Assume that T is ergodic and $\mathcal{J}_{T \times S} = \mathcal{J}_T \otimes \mathcal{J}_S$. By Lemma 4.2 (1), we get

$$\overline{\lim}_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(S^n y) \right\|_{L^2(\nu)}^2 \leq C \|f\|_{L^2} \overline{\lim}_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right\|_{L^2(\mu \times \nu_g)}.$$

Notice that, again by the spectral lemma,

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right\|_{L^2(\mu \times \nu_g)} = \left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(S^n y) \right\|_{L^2(\mu \times \nu)}.$$

Now we can conclude with von Neumann theorem applied to the product system $T \times S$ to get that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(S^n y) = \mathbb{E}^{\mathcal{J}_T} f \mathbb{E}^{\mathcal{J}_S} g = 0.$$

Now prove (4.4). Both sides of (4.4) define bounded operators on $L^2(\mu \times \nu)$. So, we have to check (4.4) for F of the form of finite sum $\sum c_i 1_{A_i}(x) 1_{B_i}(y)$ with $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$. For such simple function, the already proved (4.3) implies that the limit on the left hand side of (4.4) is equal to

$$\sum c_i \mu(A_i) \mathbb{E}^{\mathcal{J}_S} 1_{B_i} = \mathbb{E}^{\mathcal{J}_S} \sum c_i \mu(A_i) 1_{B_i} = \mathbb{E}^{\mathcal{J}_S} \int F d\mu.$$

□

4.3. Finite systems. By studying finite systems, we will show that Theorem 4.3 doesn't hold without the assumption that T is weakly mixing and that the L^2 -good sequence $\Lambda = \{n^2\}$ is not L^2 -exact (this can also be shown by computing $c_\Lambda(t)$).

For any integer $m \geq 2$, we denote by $T_m : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ the translation on the group $\mathbb{Z}/m\mathbb{Z}$ defined by

$$T_m x = x + 1 \pmod{m}.$$

T_m is a uniquely ergodic transformation with the unique invariant probability measure $\frac{1}{m} \sum_{x \in \mathbb{Z}/m\mathbb{Z}} \delta_x$. Let m_1, m_2 be two integers. Their least common multiple is denoted $[m_1, m_2]$.

Proposition 4.4. *Let $m_1, m_2 \geq 2$ be two integers. Consider the two translations T_{m_1}, T_{m_2} and their product $T_{m_1} \times T_{m_2}$.*

(1) *The orbit $O(x, y)$ of any point $(x, y) \in \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$ is a cycle of length $[m_1, m_2]$ consisting of $(x, y) + (k, k)$ with $0 \leq k < [m_1, m_2]$.*

(2) *Two orbits $O(x, y)$ and $O(x', y')$ are equal iff*

$$x' - x = y' - y \pmod{[m_1, m_2]}.$$

Therefore there are $\frac{m_1 m_2}{[m_1, m_2]}$ different cycles.

(3) *A set in $\mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$ is $T_{m_1} \times T_{m_2}$ -invariant iff it is a finite union of cycles.*

Proof. (1) Since the finite set $O(x, y)$ verifies $T_{m_1} \times T_{m_2} O(x, y) \subset O(x, y)$ and $T_{m_1} \times T_{m_2}$ is bijective, we have $T_{m_1} \times T_{m_2} O(x, y) = O(x, y)$. This implies that $O(x, y)$ is a cycle. Let ℓ be the length of the cycle. Then

$$(T_{m_1} \times T_{m_2})^\ell(x, y) = (x, y),$$

i.e. $x + \ell = x \pmod{m_1}$ and $y + \ell = y \pmod{m_2}$. It follows that $\ell = [m_1, m_2]$.

(2) According to (1), $O(x, y) = O(x', y')$ means $(x', y') = (x, y) + (k, k)$ for some $0 \leq k < [m_1, m_2]$. In other words, we have $x' - x = k \pmod{m_1}$ and $y' - y = k \pmod{m_2}$ for some $0 \leq k < [m_1, m_2]$. Equivalently $x' - x = y' - y \pmod{[m_1, m_2]}$.

(3) It is a consequence of (1). □

It follows immediately from the proposition that for any function f defined on $\mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T_{m_1}^k x, T_{m_2}^k y) = \frac{1}{[m_1, m_2]} \sum_{(a,b) \in O(x,y)} f(a, b).$$

Remark that the limit only depend on the cycle.

For $f(x, y) = 1_{\{(i,j)\}}(x, y) = 1_{\{i\}}(x)1_{\{j\}}(y)$, the characteristic function of the point $(i, j) \in \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{i\}}(T_{m_1}^k x) 1_{\{j\}}(T_{m_2}^k y) = \begin{cases} \frac{1}{[m_1, m_2]} & \text{if } (x, y) \in O(i, j) \\ 0 & \text{if } (x, y) \notin O(i, j). \end{cases}$$

This shows that Theorem 4.3 doesn't hold without the assumption that T is weakly mixing.

Let $m \geq 2$ be an integer. For $0 \leq i < m$, write

$$R_i = \{n \geq 1 : n^2 = i \pmod{m}\}.$$

Notice that R_i is a disjoint union of arithmetic sequences:

$$R_i = \bigsqcup_{0 \leq k < m, k^2 \equiv i \pmod{m}} m\mathbb{Z} + k.$$

So the density of R_i exists:

$$d_i = \lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : n \in R_i\}}{N} = \frac{\#\{0 \leq k < m : k^2 \equiv i \pmod{m}\}}{m}.$$

It follows the following proposition.

Proposition 4.5. *Consider the unique ergodic transformation $T_m : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ defined by $T_m x = x + 1 \pmod{m}$, where $m \geq 2$ is an integer. For any $j \in \mathbb{Z}/m\mathbb{Z}$ and any $x \in \mathbb{Z}/m\mathbb{Z}$, the following limit exists*

$$L_j(x) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N 1_{\{j\}}(T_m^{n^2} x) = d_{j-x}.$$

For example, when $m = 3$, we have $0^2 \equiv 0, 1^2 \equiv 1$ and $2^2 \equiv 1$, so that $d_0 = 1/3, d_1 = 2/3, d_2 = 0$. Then

$$L_0(0) = \frac{1}{3}, \quad L_0(1) = 0, \quad L_0(2) = \frac{2}{3}.$$

The limit function L_3 is not constant, however $\mathbb{E}(1_{\{0\}}|\mathcal{J}) = 1/3$ is constant because T_3 is uniquely ergodic. This shows that $\{n^2\}$ is not L^2 -exact.

The Proposition 4.5 still holds if we replace n^2 by a polynomial $P(n)$ such that $P(\mathbb{N}) \subset \mathbb{N}$. But d_i must be replaced by

$$d_i = \frac{\#\{0 \leq k < m : P(k) \equiv i \pmod{m}\}}{m}.$$

5. Θ -HARTMAN SEQUENCES ARE RECURRENT

Let $p \in \mathbb{Q}[x]$ be a polynomial such that $p(\mathbb{N}^*) \subset \mathbb{N}^*$ and $p(0) = 0$. For any irrational number t , $\{p(n)t\}$ is uniformly distributed mod 1 (Weyl theorem). It follows that $p(\mathbb{N}^*)$ is \mathbb{Q} -Hartman (see [4]).

Theorem 5.1. *Let $\Theta \subset \mathbb{Q}/\mathbb{Z}$ contain 0. Every Θ -Hartman sequence is recurrent.*

Since $D_{\mathbb{Q}/\mathbb{Z}} = \mathbb{N}$ and $D_{\{0\}} = \{1\}$, we have the following corollaries.

Corollary 5.2. *A subset $\Lambda = \{u_k\}$ of integers recurrent if the following condition is satisfied: for any integer m there is a subsequence $\{u_{n_k}\} \subset m\mathbb{Z} \cap \Lambda$ such that*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha u_{n_k}} = 0, \quad \forall \alpha \in (0, 1) \setminus \mathbb{Q}.$$

Corollary 5.3. *A subset $\Lambda = \{u_k\}$ of integers recurrent if Λ contains a subsequence which is Hartman uniformly distributed. That is to say, there is a subsequence $\{u_{n_k}\} \subset \Lambda$ such that*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha u_{n_k}} = 0, \quad \forall \alpha \in (0, 1).$$

5.1. Proof of Theorem 5.1. The proof of Theorem 5.1, inspired by [4], is divided into several lemmas, in the statements of which we will not repeat the conditions stated in Theorem 5.1.

The first lemma says that every function f in \mathcal{K}_Θ can be approximated by periodic elements under T with periods in D_Θ .

Lemma 5.4. *For any $f \in \mathcal{K}_\Theta$ and any $\epsilon > 0$, there exist an integer $m \in D_\Theta$ and a function $f' \in \mathcal{K}_\Theta$ with $T^m f' = f'$ such that $\|f - f'\| < \epsilon$.*

Proof. By the definition of \mathcal{K}_Θ , $\|f - f'\| < \epsilon$ holds for some finite linear combinations of eigenfunctions:

$$f' = \sum \alpha_j f_j, \quad \text{with } f_j \in \mathcal{K}_{\theta_j}, \theta_j \in \Theta$$

(α_j being a finite sequence of scalars). These numbers θ_j are rational. If we take m to be the least common multiple of the denominators of θ_j 's, we will have

$$T^m f' = \sum \alpha_j e^{2\pi i m \theta_j} f_j = \sum \alpha_j f_j = f'.$$

□

We denote by P_Θ the orthogonal projection onto \mathcal{K}_Θ . The next lemma show that the projection $P_\Theta f$ can be arbitrarily approximated by ergodic averages along with subsequences of Λ .

Lemma 5.5. *For any $f \in L^2(\mu)$ and any $\epsilon > 0$, there exists a subsequence $\{u_{n_k}\} \subset \Lambda$ such that*

$$\left\| \frac{1}{N} \sum_{k=1}^N T^{u_{n_k}} f - P_\Theta f \right\| < \epsilon$$

for sufficiently large N .

Proof. Write $f = f_\Theta + f_\Theta^\perp$ with $f_\Theta = P_\Theta f \in \mathcal{K}_\Theta$ and $f_\Theta^\perp \in \mathcal{K}_\Theta^\perp$. Apply Lemma 5.4 to f_Θ and $\epsilon/4$. Then there exist $m \in D_\Theta$ and $f'_\Theta \in \Theta$ such that

$$T^m f'_\Theta = f'_\Theta, \quad \|f_\Theta - f'_\Theta\| < \frac{\epsilon}{4}.$$

By the hypothesis on Θ and Lemma 5.4, we can find a subsequence $\{u_{n_k}\} \subset \Lambda$ which has the properties

- (i) $T^{u_{n_k}} f'_\Theta = f'_\Theta$;
- (ii) for all $\alpha \in \Theta^c$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha u_{n_k}} = 0.$$

By (i) and the contractivity of T , we have

$$\|T^{u_{n_k}} f_\Theta - f_\Theta\| \leq \|T^{u_{n_k}}(f_\Theta - f'_\Theta) - (f_\Theta - f'_\Theta)\| \leq \frac{\epsilon}{2}.$$

It follows immediately

$$\left\| \frac{1}{N} \sum_{k=1}^N T^{u_{n_k}} f_\Theta - f_\Theta \right\| \leq \frac{\epsilon}{2}.$$

On the other hand, by (ii) and the spectral lemma, we have

$$\left\| \frac{1}{N} \sum_{k=1}^N T^{u_{n_k}} f_\Theta^\perp \right\|^2 \leq \int_{\Theta^c} \left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha u_{n_k}} \right|^2 d\mu_{f_\Theta^\perp} < \frac{\epsilon^2}{4}$$

if N is sufficiently large. Thus we conclude by combining the last two estimations. □

For $f \in L^2(\mu)$, let H_f^Λ be the subspace spanned by $T^u f$ with $u \in \Lambda$.

Lemma 5.6. *If $f \perp H_f^\Lambda$, then $f \perp \mathcal{K}_\Theta$.*

Proof. What we have to prove is that $\langle f, f_\Theta \rangle = 0$ where $f_\Theta = P_\Theta f$. For any $\epsilon > 0$, take the sequence $\{u_{n_k}\}$ in Lemma 5.5 such that $\|A_N f - f_\Theta\| < \epsilon$ for large N where $A_N f = N^{-1} \sum_{k=1}^N T^{u_{n_k}} f$. Write

$$\langle f, f_\Theta \rangle = \langle f, A_N f \rangle + \langle f, f_\Theta - A_N f \rangle.$$

Since the hypothesis of the lemma implies $\langle f, A_N f \rangle = 0$, we have

$$|\langle f, f_\Theta \rangle| \leq \|f\| \cdot \|f_\Theta - A_N f\| < \|f\| \epsilon.$$

□

Proof of Theorem 5.1. We prove the theorem by contradiction. Suppose that Λ is not a Poincaré set. Then there exists a measure preserving system (X, \mathcal{A}, T, μ) and a measurable set $A \in \mathcal{A}$ such that

$$\mu(A) > 0, \quad \mu(A \cap T^{-u} A) = 0 \quad (\forall u \in \Lambda).$$

Thus we have $1_A \perp H_f^\Lambda$ because

$$\mu(A \cap T^{-u} A) = \int 1_A \cdot 1_A \circ T^u d\mu.$$

By Lemma 5.6, we have $1_A \perp \mathcal{K}_\Theta$. Since $1 \in \mathcal{K}_\Theta$ (Θ contains 0), we have $\mu(A) = \langle 1_A, 1 \rangle = 0$, which is a contradiction. □

6. RANDOM SEQUENCES

Let $\{p_n\}_{n \geq 1}$ be a sequence of real numbers such that

$$0 \leq p_n \leq 1, \quad \sum_{n=1}^{\infty} p_n = \infty.$$

Let $\{\xi_n\}_{n \geq 1}$ be a sequence of independent Bernoulli random variables such that

$$P(\xi_n = 1) = p_n = 1 - P(\xi_n = 0).$$

The study object in this section is the random set of integers

$$W(\omega) = \{n \geq 1 : \xi_n(\omega) = 1\}.$$

This is almost surely (a.s. for short) an infinite subset of integers. The following is the main theorem in this section.

Theorem 6.1. *The sequence $W(\omega)$ is a.s. Hartman uniformly distributed if the following conditions are satisfied:*

$$(1) \quad \log N = o\left(\sum_{n=1}^N p_n\right), \quad (2) \quad \sum_{n=1}^N |p_n - p_{n+1}| = o\left(\sum_{n=1}^N p_n\right).$$

Let us first present some elementary properties of $W(\omega)$.

6.1. Basic properties of $W(\omega)$. The classical law of large numbers of Kolmogorov concerns independent sequences of identically distributed variables. The following proposition is a kind of law of large numbers. It can be proved by Doob's martingale convergence theorem.

Proposition 6.2. *Almost surely*

$$\lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N \xi_n}{\sum_{n=1}^N p_n} = 1.$$

Proof. Let $s_n = \sum_{k=1}^n p_k$ and $s_0 = p_1$. Consider the martingale

$$M_n = \sum_{k=1}^n \frac{\xi_k - p_k}{s_k}.$$

It is easy to compute that

$$\mathbb{E}M_n^2 = \sum_{k=1}^n \frac{p_k(1-p_k)}{s_k^2} \leq \sum_{k=1}^n \frac{p_k}{s_k s_{k-1}} = \frac{1}{p_1} + \sum_{k=2}^n \left(\frac{1}{s_{k-1}} - \frac{1}{s_k} \right) \leq \frac{2}{p_1}.$$

By the martingale convergence theorem of Doob, the L^2 -bounded martingale M_n converges almost surely. In other words, the series $\sum_{k=1}^{\infty} \frac{\xi_k - p_k}{s_k}$ converges. We conclude by the Kronecker lemma. \square

Consequently, $W(\omega)$ admits a.s. its upper density

$$\overline{D}(W(\omega)) = \overline{\lim}_{N \rightarrow \infty} \frac{\#W(\omega) \cap [1, N]}{N} = \overline{\lim}_{N \rightarrow \infty} \frac{\sum_{n=1}^N p_n}{N}.$$

The density of $W(\omega)$ exists iff $\lim_{N \rightarrow \infty} N^{-1} \sum_{n=1}^N p_n$ exists.

An increasing sequence $\{u_n\} \subset \mathbb{N}^*$ is said to be *syndetic* if

$$\sup_n (u_{n+1} - u_n) < \infty.$$

Proposition 6.3. *Almost surely $W(\omega)$ is syndetic iff there exists an integer $\ell \geq 1$ such that*

$$\sum_{n=1}^{\infty} \prod_{j=0}^{\ell-1} (1 - p_{n+j}) < \infty.$$

Proof. Notice that $P(n \notin W(\omega)) = P(\xi_n = 0) = 1 - p_n$. Let

$$A_{n,\ell} = \{\omega : n \notin W(\omega), n+1 \notin W(\omega), \dots, n+\ell-1 \notin W(\omega)\}.$$

Then $P(A_{n,\ell}) = \prod_{j=0}^{\ell-1} (1 - p_{n+j})$. So, the convergence of the series, together with the Borel-Cantelli lemma, shows that $P(\overline{\lim}_n A_{n,\ell}) = 0$. That means, almost sure when n is sufficiently large, each interval $[n, n+\ell-1]$ contains a term of $W(\omega)$. So, $W(\omega)$ is syndetic.

Now suppose que the series diverges for each $\ell \geq 1$. Then, for each fixed ℓ , there exists $0 \leq r < \ell$ such that

$$\sum_{k=1}^{\infty} \prod_{j=0}^{\ell-1} (1 - p_{r+k\ell+j}) = \infty.$$

Observe that the events $A_{r+k\ell}$ ($k = 1, 2, \dots$) are independent and $P(A_{r+k\ell}) = \prod_{j=0}^{\ell-1} (1 - p_{r+k\ell+j})$. Then the divergence of the last series, together with the Borel-Cantelli lemma, implies $P(\overline{\lim}_k A_{r+k\ell}) = 1$. That means, almost sure there exists k (actually infinitely many), such that the interval $[r + k\ell, r + k\ell - 1]$ contains no term of $W(\omega)$. Since ℓ is arbitrary, $W(\omega)$ is not syndetic. \square

6.2. Uniform norm of some random trigonometric polynomials.

The following theorem on the uniform norm of a random trigonometric polynomials will be useful in the proof of Theorem 6.1.

Theorem 6.4. *Let $\{\xi_n\}$ be a sequence of independent Bernoulli variables such that $P(\xi_n = 1) = p_n = 1 - P(\xi_n = 0)$. Suppose that*

$$\log N = o\left(\sum_{n=1}^N p_n\right).$$

There exists a constant $C > 0$ such that

$$P\left(\max_{t \in [0,1)} \left| \sum_{n=1}^N (\xi_n - p_n) e^{2\pi i n t} \right| \geq C \sqrt{\sum_{n=1}^N p_n \log N}\right) \leq \frac{1}{N^2}.$$

Proof. Similar estimates were first obtained by Salem and Zygmund when Rademacher sequence or Steinhaus sequence take the place of $\{\xi_n - p_n\}$. Kahane considered subnormal or subgaussian sequences. Fan and Schneider have replaced the subgaussian condition by another integrability condition. But for the centralized but biased Bernoulli sequence $\{\xi_n - p_n\}$, we can not directly apply these known results in [6, 3]. However, we can imitate [6].

First estimate the Laplace transform of $\xi_n - p_n$:

$$\mathbb{E}e^{\lambda(\xi_n - p_n)} = e^{-\lambda p_n} (1 + p_n(e^{-\lambda} - 1)) \leq e^{p_n(e^{-\lambda} - 1 - \lambda)}.$$

Using the fact that $e^{-\lambda} - 1 - \lambda \leq A\lambda^2/2$ for $|\lambda| \leq 1$ (A being a constant). Thus we have

$$\mathbb{E}e^{\lambda(\xi_n - p_n)} \leq e^{Ap_n\lambda^2/2} \quad \text{if } |\lambda| \leq 1,$$

which could be called a local subnormal property. As in [6], we will take (with the notation in [6])

$$\lambda^2 = \frac{\log(2\rho\kappa)}{Ar} = \frac{\log(4\pi N^4)}{A \sum_{n=1}^N p_n},$$

We should notice that $\lambda \rightarrow 0$ when $N \rightarrow \infty$. \square

6.3. Proof of Theorem 6.1.

What we have to prove is

$$a.s. \quad \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N \xi_n e^{2\pi i n t}}{\sum_{n=1}^N \xi_n} = 0 \quad \forall t \in (0, 1).$$

By Lemma 6.3, we have only to show

$$(6.1) \quad a.s. \quad \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N (\xi_n - p_n) e^{2\pi i n t}}{\sum_{n=1}^N p_n} = 0 \quad \forall t \in (0, 1);$$

$$(6.2) \quad \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N p_n e^{2\pi i n t}}{\sum_{n=1}^N p_n} = 0 \quad \forall t \in (0, 1).$$

By Theorem 6.4, we have

$$(6.3) \quad a.s. \quad \max_{t \in [0, 1]} \left| \sum_{n=1}^N (\xi_n - p_n) e^{2\pi i n t} \right| = O \left(\sqrt{\sum_{n=1}^N p_n \log N} \right).$$

So, (6.1) follows immediately from (6.3) and the hypothesis $\log N = o(\sum_{n=1}^N p_n)$.

An Abel summation leads to

$$\left| \sum_{n=1}^N p_n e^{2\pi i n t} \right| \leq \frac{1}{|\sin \pi t|} \left(\sum_{n=1}^{N-1} |p_n - p_{n+1}| + p_N \right).$$

Then (6.2) follows from this estimate and the hypothesis $\sum_{n=1}^N |p_n - p_{n+1}| = o(\sum_{n=1}^N p_n)$. \square

Assume that $\sum_{n=1}^N |p_n - p_{n+1}| = o(\sum_{n=1}^N p_n)$. It is interesting to find a condition for $W(\omega)$ to be a.s. non L^2 -exact. In other words, we would like to find conditions such that a.s. there exists t such that

$$(6.4) \quad \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N (\xi_n - p_n) e^{2\pi i u_n t}}{\sum_{n=1}^N p_n} \neq 0.$$

Notice that for t fixed, the above limit is a.s. equal to zero (the same proof for Proposition 6.3 works). So, the point t in (6.4) must be a random point depending on ω . Such point does exist when $p_n = 1/n$ [8]. See [13, 14] for the study of $W(\omega)$ as thin sets in harmonic analysis. Another interesting problem is to study the multiple recurrence of $W(\omega)$.

6.4. L^2 -universality of $W(\omega)$. A sequence $\Lambda = \{u_k\} \in \mathbb{N}^*$ is said to be L^2 -universal if for any measure-preserving system (X, \mathcal{A}, μ, T) and any $f \in L^2(\mu)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{u_k} \quad \text{exists } \mu\text{-a.e.}$$

Bourgain and Boshernitzan proved the L^2 -universality of $W(\omega)$ under the conditions that the sequence $\{p_k\}$ decreasing and for any geometric sequence $\mathcal{N}_\rho = \{[\rho^k], k \geq 1\}$ ($\rho > 1$), we have

$$(6.5) \quad \sum_{N \in \mathcal{N}_\rho} \frac{\log N}{\sum_{k=1}^N p_k} < +\infty.$$

We remark that the monotonicity of $\{p_k\}$ can be replaced by the weaker condition

$$(6.6) \quad \overline{\lim}_{N \rightarrow +\infty} \frac{1}{\sum_{k=1}^N p_k} \left(N p_N + \sum_{k=1}^{N-1} k |p_k - p_{k+1}| \right) < \infty.$$

Recall that $\sum_1^n \xi_k \sim \sum_1^n p_k$ (Lemma 6.3). Write

$$\frac{1}{\sum_{k=1}^N p_k} \sum_{k=1}^N \xi_k f \circ T^k = \frac{1}{\sum_{k=1}^N p_k} \sum_{k=1}^N (\xi_k - p_k) f \circ T^k + \frac{1}{\sum_{k=1}^N p_k} \sum_{k=1}^N p_k f \circ T^k.$$

The condition (6.5) ensures the convergence of the first term on the right hand side. The monotonicity was used to prove the convergence of the second term as a consequence of the Birkhoff theorem. The monotonicity can be replaced by (6.6) because under the condition (6.6) we have

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n = \ell \Rightarrow \lim_{n \rightarrow \infty} \frac{\sum_{n=1}^N p_n a_n}{\sum_{n=1}^N p_n} = \ell$$

for any sequence (a_n) .

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