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# Correlation of rankings in matching markets\*

Rémi Castera<sup>†</sup>, Patrick Loiseau<sup>‡</sup>, Bary S.R. Pradelski<sup>§</sup>

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We study the role of correlation in matching, where multiple decision-makers simultaneously face selection problems from the same pool of candidates. We propose a model in which decision-makers have varying information on candidates from different sociodemographic groups when evaluating and ranking them, thus leading to varying correlations among candidates' priority scores. Such *differential correlation* arises, for example, when the cost of information acquisition, decision-maker preferences, or the prevalence of selection criteria vary across sociodemographic groups. We show that a lower correlation for one of the groups worsens the outcome for all groups, thus leading to efficiency loss. Moreover, the proportion of students from a given group who remain unmatched is increasing in its own correlation level. This implies that it is advantageous to belong to the low-correlation group. Finally, we extend the extent tie-breaking literature to multiple priority classes and intermediate levels of correlation. Overall, our results point to a previously overlooked systemic source of group inequalities in school, university, and job admissions.

*Key words:* Matching, correlated priorities, inequality, tie-breaking

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## 1. Introduction

Outcome inequalities for different demographic or social groups are ubiquitous, for example, in college admission, job assignment, or investment allocation. Arcidiacono et al. (2022) find that Asian-American applicants have lower admission chances at Harvard than white applicants with similar academic records, Niessen-Ruenzi and Ruenzi (2019) find significantly lower inflows in female-managed mutual funds than in male-managed mutual funds, and Bertrand and Mullainathan (2004) find race-based discrimination in callback decisions by job advertisers. Consequently, the sources of observed outcome inequities remain the subject of frequent and continued controversy and political debate. A common concern is that the causes of outcome inequalities in matching markets are poorly understood, making them difficult to address (Longhofer 1995).

We study how different correlation of rankings between different sociodemographic groups—*differential correlation*—affects outcome inequality and efficiency in matching markets. Our findings point to a previously unacknowledged source of inequity between different groups that is specific to matching markets and should be included in future assessments of, for example, school, university, and job admissions. In particular, we find that differential correlation across groups leads to outcome inequities even when the rankings by each college are fair, i.e., all groups are represented in each college’s ranking as they are in the total applicant population. The resulting inequity is a form of systemic discrimination, i.e., discrimination that arises only through the interaction of decision-makers—via the matching mechanism—and is not due to either intentional or nonintentional discrimination by single decision-makers (cf. Pincus 1996, Feagin 2013, Bohren et al. 2022). As most policies, including those aiming to correct for biases or implement affirmative action, are designed with one decision-maker in mind, systemic discrimination is often overlooked.

Differential correlation arises when different decision-makers—such as colleges—use different information on candidates from different (sociodemographic) groups—such as students—when assigning priority scores to rank and admit them. This may be the case for several reasons.

First, the cost of information acquisition may vary across groups, or some information may not be available at all for some groups. As students latent quality is unknown to the decision maker, they rely on observable, partial information, thus only allowing for a noisy measurement of a student’s latent quality.<sup>1</sup> Concretely, consider decentralized college admissions to competitive PhD programs and compare foreign and local candidates. Several U.S. programs virtually reserve one seat for the best foreign candidate from a given foreign school based on test scores and—due to the high cost—do not perform in-person interviews (e.g., Iran’s Sharif University of Technology or India’s Institutes of Technology). Consequently, the priorities of foreign candidates at different

<sup>1</sup> The idea to study a model of latent quality plus noise can be traced to Phelps (1972). More recently, this idea has been used by Chade et al. (2014), Garg et al. (2021), Emelianov et al. (2022).

programs are highly correlated. In contrast, for local candidates, grades from undergraduate degrees alone may not provide sufficient information—also due to grade inflation—and universities rely on idiosyncratic signals, such as reference letters, extracurricular activities, or interviews during campus visits (which are less costly because only local travel is required). As a result, the priorities of local students, while correlated, are less correlated than those of foreign students.

Second, differences in correlation may also arise if colleges are looking for different attributes in candidates and proxies of these attributes are more or less correlated for different groups. Let us consider two colleges, namely, one admitting students for mathematics, and the other admitting students for physics. Suppose that there are two groups: one group of students comes from high schools where physics is taught in a theoretical manner, and the other group comes from high schools where physics is taught in an experimental manner and is thus less mathematical. As a result, students from the former group will exhibit greater correlation in their high school grades in mathematics and physics than will students from the latter group.

Third, differences in correlation may also arise when colleges use selection criteria that are more or less prevalent within different groups. Specifically, such criteria could include diversity with respect to the current student body, proximity, or sibling priority. The latter is, for example, used in the centralized school choice mechanism in Chile (Mello 2022), where in 2023 21.3% students among the economically disadvantaged received sibling priority at some school, whereas only 15.6% of the remaining students benefited from sibling priority. As a result, a group for which the criteria are more prevalent will exhibit lower correlation than a group for which the criteria are less prevalent.

Sibling priority and proximity along other criteria are commonly used to break ties, as they render a mechanism more explainable and limit the use of random tie-breaking to a small number of students (for whom all criteria are the same). Such random tie-breaking has been previously studied in school choice problems (Abdulkadiroğlu et al. 2015, Ashlagi et al. 2019, Arnosti 2023). When students have the same ranking at a given college a tie-breaking rule describes who should receive priority. Two natural choices are that each college breaks ties independently or colleges use a common order to break ties. Intuitively, the former leads to 0 correlation and the latter to correlation 1 (among those students for whom tie-breaking is required). As we shall see, our work allows us to extend the known theoretical results on tie-breaking to accommodate intermediate correlation levels, as often seen in practice.

Our results reveal differential correlation as an overlooked source of inefficiency and inequity in matching markets. Policy-makers who orchestrate matching markets need to be wary that only considering and enforcing the *fairness* of each decision-maker separately is insufficient to ensure overall equity (and efficiency). Concretely, school choice mechanisms have frequently been redesigned over the past decades, and researchers have actively contributed to these efforts (cf. e.g., Abdulkadiroğlu

and Sönmez 2003, Abdulkadiroğlu 2005, Abdulkadiroğlu et al. 2009, Correa et al. 2022, Kamada and Kojima 2023). According to our results, differential correlation should be accounted for when redesigning these markets. More generally, our results show the effect of the increasing use of similar algorithms and data sources to automate selection tasks and as such should be relevant to policy-makers and market designers in the machine learning and artificial intelligence sphere (cf. Kleinberg and Raghavan 2021, and the discussion on algorithmic monoculture in Section 1.2 below).

### 1.1. Our contribution

We study the college-admissions problem, where multiple decision-makers select a subset of applicants from an applicant pool with stability as the solution concept (Gale and Shapley 1962, Azevedo and Leshno 2016). Specifically, we suppose that an infinite population of students divided into groups  $G_1 \dots G_K$  apply to two colleges  $A$  and  $B$ . The groups represent, for example, protected attributes, such as gender or race. Each college assigns a priority score to each student. A given student receives priority scores  $W^A$  at college  $A$  and  $W^B$  at college  $B$ . We propose an original model for the distributions of these scores, to study the correlation between the rankings made by different colleges.

To formalize correlation and thus capture the vague notion of “a connection between two things in which one thing changes as the other does”,<sup>2</sup> we leverage prior work on copulas and their relation with classical notions via *coherence*. This allows us to model correlation without a specific functional form and, in particular, nest classical notions as special cases, e.g., Spearman’s and Kendall’s correlation indices. With this at hand, we assume that the correlation between the priority scores at different colleges depends on a candidate’s group; we call this feature *differential correlation*.

*How does differential correlation impact a stable matching’s efficiency, i.e., the number of students who obtain their first choice, and inequality, i.e., the difference between groups in their probability of remaining unmatched?* To answer this question, we first consider comparative statics. We show that efficiency increases with each group’s correlation level; i.e., increasing the correlation level of any group increases the number of students who obtain their first choice in all groups (Theorem 1). Moreover, the proportion of students from a given group remaining unmatched is increasing in its own correlation level and decreasing in the correlation level of all other groups (Corollary 1). This implies that it is advantageous to belong to a low-correlation group. We then show that a given efficiency level can be reached by a continuum of different correlation vectors, yielding different levels of inequality, thus proving that efficiency differences cannot explain inequality (Proposition 3). Finally, our results imply extensions of known results on tie-breaking

<sup>2</sup> Oxford Advanced Learner’s Dictionary, 2023

(cf. Ashlagi et al. 2019, Ashlagi and Nikzad 2020, and Arnosti 2023), particularly for multiple priority classes and intermediate levels of correlation (Proposition 4).

Our model allows for any number of groups, different marginals and preferences depending on the scores; however, we restrict our analysis to two colleges. In Section 5, we show by counterexample that most of our results do not extend to more than two colleges and discuss (strong) assumptions under which they extend.

## 1.2. Related literature

*Matching.* The college admission problem, i.e., how to assign prospective students to colleges given each student’s preferences and colleges’ priorities over students and capacities such that the outcome is stable, was introduced by Gale and Shapley (1962). A variant of this model where colleges do not have priorities over students is commonly called the school choice problem (cf., e.g., Balinski and Sonmez 1999, Abdulkadiroğlu and Sönmez 2003, Abdulkadiroğlu 2005, Ergin and Sönmez 2006, Yenmez 2013). The idea of considering a continuum of students and a finite number of colleges has previously been exploited due to its analytical tractability (cf., Chade et al. 2014, Abdulkadiroğlu et al. 2015, Azevedo and Leshno 2016, Arnosti 2022). We follow Azevedo and Leshno (2016), who develop a supply and demand framework that allows us to easily analyze the quality of a matching and derive comparative statics.

*Matching with correlated types.* We study matching in the presence of correlation between the priority scores given by each college to a given student. A special case of this problem has been studied for centralized school choice problems, where many students have the same priority and ties are broken at random.<sup>3</sup> Ashlagi et al. (2019), Ashlagi and Nikzad (2020), and Arnosti (2023) compare the welfare of students in two settings, i.e., that in which either one common lottery is used by all colleges or all colleges draw independent lotteries. In our model, this corresponds to a correlation of either 1 or 0, and our results nest findings of these prior papers. Another line of work has considered correlation between other features, e.g., between students’ preferences and colleges’ rankings. In this context, Brilliantova and Hosseini (2022) focus on one-to-one matching and identify a matching that does not favor one side over the other, while Che and Tercieux (2019) and Leshno and Lo (2020) study the stability-efficiency trade-off by comparing Deferred Acceptance and Top Trading Cycles (see Shapley and Scarf 1974), and how the magnitude of this trade-off depends on the correlation between agent preferences and priorities. Considering a transferable utility model, Gola (2021) studies how workers sort into two competing sectors (such that their wages are maximized) and the impact of technological change. While their model and analysis are quite distant from ours, they share the use of copulas to model correlation and the necessary restriction to two sectors, namely, colleges.

<sup>3</sup> The implications of this feature for students’ welfare have been studied by Erdil and Ergin (2008), Abdulkadiroğlu et al. (2009) and Abdulkadiroğlu et al. (2015).

*Fairness.* The computer science literature on fairness in selection problems was initiated by Kleinberg and Raghavan (2018), who study the effect of bias and the efficiency of affirmative action policies.<sup>4</sup> Emelianov et al. (2020, 2022) and Garg et al. (2021) study statistical discrimination. Candidates have a latent quality, and colleges or companies that they apply to only have access to a biased and/or noisy estimator of this quality that varies depending on their group, which is called *differential variance*. We depart from those models by considering several decision-makers instead of one; that is, we consider the matching problem instead of the selection problem. Studies on fairness in matching have considered various affirmative action policies, including upper and lower quotas, to reduce discrimination (Abdulkadiroğlu 2005, Kamada and Kojima 2015, 2023, Delacrétaz et al. 2023, Krishnaa et al. 2022, Dur et al. 2020). These works, however, focus on finding stable matchings under some constraints, accounting for different fairness notions. In contrast, we aim to explain outcome inequalities that naturally occur in stable matchings without constraints.

All of the aforementioned studies consider what is generally termed *group fairness*, that is, the idea that all (relevant) groups should be treated similarly. In contrast, *individual fairness* posits that individuals with similar characteristics should be treated similarly. In this latter spirit Karni et al. (2022) broke new ground, showing that an individually fair ranking does not necessarily lead to an individually fair matching. This conclusion can also be drawn from our results in the context of group fairness. Devic et al. (2023) also consider individual fairness and adapt the classical notion to incorporate agents' preferences; i.e., they require that similar agents be matched to a college in a similar position on their respective preference list. Our work is concerned with group fairness; however, we note that if the quality of applicants is similar in each group and the notion of individual fairness put forth by Devic et al. (2023) is fulfilled, then group inequalities are automatically mitigated. In this sense, Devic et al. (2023)'s notion is stricter.

*Algorithmic monoculture.* Finally, our work also contributes to the recent literature on algorithmic monoculture, i.e., the fact that recommendations, choices, and preferences become homogeneous with the rise of algorithmic curation and analysis. Kleinberg and Raghavan (2021) study the utility of multiple decision-makers who use algorithms to evaluate candidates. They show that decision-makers are sometimes better off using different low-precision algorithms than when using the same high-precision algorithm. Peng and Garg (2023, 2024) recently study these questions in a model where the number of decision-makers grows large. In empirical work, Bommasani et al. (2022) find that outcomes are more homogeneous when models and training datasets are shared between decision-makers. Through our theoretical analysis we thus elucidate the impact of algorithmic monoculture from the candidates' viewpoint.

<sup>4</sup> Recently, reducing outcome inequalities in ranking rather than in final selection has been actively studied, cf. Celis et al. (2020), Yang et al. (2021), and Zehlike et al. (2022).

### 1.3. Outline

The remainder of this paper is organized as follows. Section 2 introduces the model and the concept of differential correlation. Section 3 introduces our welfare metrics and presents preliminary results. Our main results are presented in Section 4. Finally, Section 5 concludes with a discussion on the generality of our findings and future avenues of research.

## 2. Setup

Next, we introduce the college admission problem with a continuum of students, formalize the notion of *correlation* in Section 2.2, and introduce the supply and demand framework to identify stable matching in Section 2.3. A table of notations is provided in Appendix A.1.1 for convenience.

### 2.1. Model

Let  $A$  and  $B$  be two colleges to which a continuum unit mass of students,  $S$ , is to be matched. The mass of a subset of  $S$  is measured with a function  $\eta$ .<sup>5</sup> Colleges have maximum capacities of the mass of students they can admit,  $(\alpha^A, \alpha^B) := \boldsymbol{\alpha} \in (0, 1]^2$ . The students are divided into  $K$  groups  $G_1, \dots, G_K$ , with a fraction  $\gamma_j \in [0, 1]$  of students belonging to  $G_j$ , with  $\sum_{j=1}^K \gamma_j = 1$ . Define the vector  $\boldsymbol{\gamma} := (\gamma_j)_{j \in [K]}$ , using the notation  $[K] := \{1, 2, \dots, K\}$ . We denote the group to which a student  $s \in S$  belongs by  $G(s)$ .

Students have strict preferences over colleges, and the amount of students who prefer college  $A$  might differ between groups; among group  $G_j$ , a share  $\beta_j \in [0, 1]$  prefers college  $A$  to college  $B$ , and the remaining  $1 - \beta_j$  prefer  $B$ . When student  $s$  prefers college  $A$  to college  $B$ , we write  $A \succ_s B$ , and vice versa. Note that  $\beta_j$  is a share that is conditional on the group (and thus is not a mass), for instance,  $\eta(\{s \in G_j : A \succ_s B\}) = \gamma_j \beta_j$ . We assume that all students prefer attending some college to remaining unmatched. We write  $\boldsymbol{\beta} := (\beta_j)_{j \in [K]}$  as we did for  $\boldsymbol{\gamma}$ .

Each college assigns a priority score to each student, and the higher the score is, the better a student's evaluation. Each student  $s$  is thus assigned a vector of priority scores  $(W_s^A, W_s^B)$ . This means that college  $C$  prefers  $s \in S$  to  $s'$  if and only if  $W_s^C > W_{s'}^C$ . The (marginal) distribution of scores  $W_s^C$  given by college  $C$  to students in  $G_j$  is described by a probability density function (pdf)  $f_j^C$  defined over the support  $I_j^C \subseteq \mathbb{R}$ , assumed to be an interval. Let  $I_j = I_j^A \times I_j^B$ . We denote by  $\underline{I}_j^C$  and  $\bar{I}_j^C$  the lower and upper bounds of  $I_j^C$ . These bounds may or may not be finite. We define  $\mathbf{f} := (f_1^A, \dots, f_K^A, f_1^B, \dots, f_K^B)$  and denote by  $F_j^C$  the cumulative distribution function (cdf) associated with  $f_j^C$ .

<sup>5</sup> The formal definition of this measure is found in Appendix A.1.2.



**Differential correlation.** Consider the joint distribution of the vectors  $(W^A, W^B)$ . For each group  $G_j$ , the grade vectors of students  $s \in G_j$  follow a distribution with pdf  $f_j : I_j \rightarrow \mathbb{R}$  and cdf  $F_j$ . A joint distribution can be characterized by its marginals, i.e., the distribution of each component of the vector, and the shape of the joint distribution, captured by a coupling function, called the *copula*. A copula is a cdf over  $[0, 1]^n$ , for some  $n$ , with uniform marginals. The theorem of Sklar (1959) states that any joint distribution can be decomposed into (independent) marginals and a unique copula:

**Sklar (1959, Theorems 1, 2, and 3)** *Let  $F$  be an  $n$ -dimensional cdf with marginal cdfs  $F^1, \dots, F^n$ . Then there exists a unique  $n$ -dimensional copula  $H : [0, 1]^n \rightarrow [0, 1]$  such that*

$$F(x^1, \dots, x^n) = H(F(x^1), \dots, F(x^n)).$$

*Conversely, for any  $n$ -dimensional copula  $H$  and for any set of  $n$  1-dimensional cdfs  $F^1, \dots, F^n$ ,  $F(x^1, \dots, x^n) := H(F(x^1), \dots, F(x^n))$  is an  $n$ -dimensional cdf with marginals  $F^1, \dots, F^n$ .*

Each group  $G_j$  then has a joint distribution with joint pdf  $f_j$  and cdf  $F_j$ , which can be represented by its marginals  $F_j^A, F_j^B$  and a (unique) copula  $H_j$ . We assume that there exists a family of 2-dimensional copulas  $(H_\theta)_{\theta \in \Theta}$  ( $\Theta$  being an interval of  $\mathbb{R}$ ) and, for all  $j \in [K]$ , there exists a parameter  $\theta_j \in \Theta$  such that  $H_{\theta_j}$  is the copula associated with  $G_j$ 's distribution; i.e.,  $H_{\theta_j} = H_j$ . This assumption is made without loss of generality, but some of our results will require additional assumptions regarding the copula family. Denote by  $\boldsymbol{\theta} := (\theta_j)_{j \in [K]}$  the vector containing each group's parameter. Note that each group has a different  $\theta_j$  and thus a different joint distribution. With some foresight to the explanations provided in Section 2.2, we call this feature of the model *differential correlation*.<sup>6</sup> Finally, we assume that all the copulas in  $(H_\theta)_{\theta \in \Theta}$  have full support over  $[0, 1]^2$ . We write  $f_{j, \theta_j}$  and  $F_{j, \theta_j}$  instead of  $f_j$  and  $F_j$ , as we will consider them as functions of  $\theta$ .

Given a family of copulas  $(H_\theta)_{\theta \in \Theta}$ , we refer to the tuple  $(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{f}, \boldsymbol{\theta})$  as the *college admission problem*. Note that we only assume that distributions admit a density and have full support, that the distribution family is parameterized by a scalar, and that the marginals remain the same for any  $\theta$ .

This model allows each group to have different grade distributions at each college. Notice that while inside a given group, the students preferring  $A$  have the same grade distribution as students preferring  $B$ , this does not cause any loss of generality. Indeed, the arbitrary number of groups allows to have small enough subgroups such that this assumption is satisfied.

<sup>6</sup> This is in the spirit of the notion of differential variance studied in Emelianov et al. (2022) and Garg et al. (2021).

## 2.2. Correlation and coherence

The proxy for correlation in our model will be the parameter  $\theta$ , rather than some specific functional form. We use a condition, namely coherence, on the family of distributions, such that whenever  $(f_{j,\theta})_{\theta \in \Theta}$  is coherent, there exists a bijection between  $\theta_j$  and classical measures of correlation. For details on classical correlation measures, namely, Pearson's, Spearman's, and Kendall's correlations, see Appendix A.2.

**ASSUMPTION 1 (Coherence).** *We say that  $(H_\theta)_{\theta \in \Theta}$  is coherent if for all  $(x, y) \in \mathring{I}$ ,  $H_\theta(x, y)$  is strictly increasing in  $\theta$  on  $\Theta$ , where  $\mathring{I}$  is the interior of  $I$ .*

The following lemma states that under coherence,  $\theta$  is naturally interpreted as a measure of correlation.

**LEMMA 1.** *If  $(H_\theta)_{\theta \in \Theta}$  is coherent, and  $(X, Y)$  are random variables drawn according to  $H_\theta$ , then  $\forall (x, y) \in \mathring{I}$ ,  $\mathbb{P}(X < x, Y < y)$  and  $\mathbb{P}(X > x, Y > y)$  are increasing in  $\theta$ , while  $\mathbb{P}(X < x, Y > y)$  and  $\mathbb{P}(X > x, Y < y)$  are decreasing in  $\theta$ .*

Intuitively, when  $X$  and  $Y$  are highly correlated, then, for example, if  $X$  is small,  $Y$  is likely also small. The proof is provided in Appendix B.1.

Further supporting the choice of  $\theta$  as our measure of correlation, we note that there is an equivalence between  $\theta$  and two classical, ordinal measures of correlation, namely, Spearman's correlation, which we denote  $\rho$ , and Kendall's correlation, which we denote  $\tau$ .

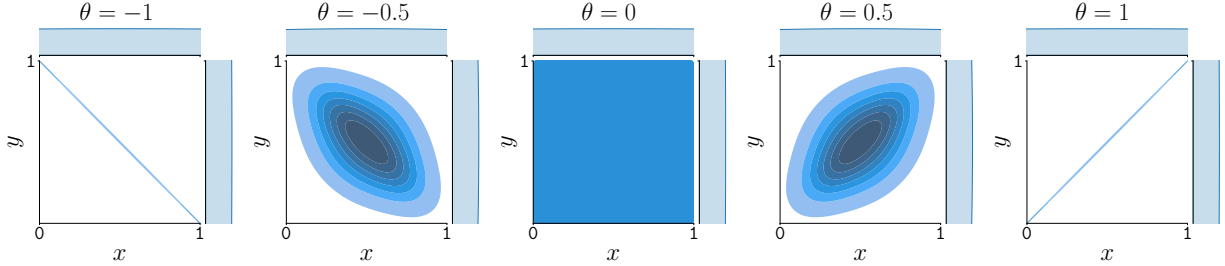
**Scarsini (1984, Theorems 4 and 5)** *If  $(H_\theta)_{\theta \in \Theta}$  is coherent, and  $(X_\theta, Y_\theta)$  are random variables drawn according to  $H_\theta$ , then Spearman's and Kendall's correlation coefficients  $\rho(X_\theta, Y_\theta)$  and  $\tau(X_\theta, Y_\theta)$  are strictly increasing functions of  $\theta$ .*

The above results show that, under the coherence assumption,  $\theta$  is a bijection of classical measures of correlation. To illustrate the effect of  $\theta$  on a coherent distribution family, Figure 1 shows the results obtained from Gaussian copulas, which are coherent, with different covariance values used as  $\theta$ . When  $\theta = 0$ , the variables are independent; when  $\theta$  is positive, the joint distribution gets closer to the diagonal  $X = Y$ ; and when  $\theta$  is negative, the joint distribution gets closer to the diagonal  $Y = -X$ , which corresponds to the common idea of correlation, as well as Spearman's and Kendall's correlations. Since most of our results will be qualitative, they are stated using  $\theta$  but would still be true if  $\theta$  was replaced by  $\rho$  or  $\tau$  in the statement.

We finally introduce a technical assumption that will be required for some of our results, especially when considering comparative statics in  $\theta$ .

**ASSUMPTION 2 (Differentiability).** *We say that  $(H_\theta)_{\theta \in \Theta}$  is differentiable if for all  $(x, y) \in \mathring{I}$  and for all  $\theta \in \mathring{\Theta}$ ,  $h_\theta(x, y)$  is differentiable in  $\theta$ .*

**Figure 1** Gaussian copula (i.e., bivariate Gaussian with marginals renormalized to uniform distributions, cf. Appendix A.3) for five different correlation levels,  $\theta$ . The shades of blue represent the distribution density (darker means higher).



The coherence and differentiability assumptions are not particularly restrictive; for instance, the Gaussian copula with covariance as the parameter verifies them, as do other commonly occurring copulas (cf. discussion on distributional assumptions in Appendix A.3). Moreover, our model nests the classic setting where students are assumed to have an—unobserved—latent quality and decision-makers only observe a noisy measurement (cf. Appendix A.4 for a discussion).

### 2.3. The supply and demand framework

We now introduce the key elements of matching theory used throughout the paper. A matching  $\mu$  is a mapping associating a student to a college (or themselves if they are unmatched) and a college to a subset of students. We use the common definition of stability; i.e., we say that a matching is stable if, for any student  $s$  who would prefer college  $C$  to their current match,  $s$  has a lower score at  $C$  than all the currently admitted students in that college. For the formal definitions of those notions and of the deferred acceptance algorithm, see Appendix A.4.

A matching problem can be alternatively viewed through a supply and demand lens, where a stable matching is a Walrasian equilibrium (Azevedo and Leshno 2016).

**DEFINITION 1 (CUTOFFS AND DEMAND).** If  $\mu$  is a stable matching, we define the *cutoff* at  $C \in \{A, B\}$  as  $P^C := \inf\{W_s^C : \mu(s) = C\}$ . Given  $\mathbf{P} = (P^A, P^B)$ , we call the *demand* of student  $s$ , which we denote  $D_s(\mathbf{P}) \in \{A, B\} \cup \{s\}$ , either the college they prefer among those at which their score is above the cutoff or themselves if their score does not exceed the cutoff at any college. The *aggregate demand* at college  $C$  is the mass of students demanding it:  $D_C(\mathbf{P}) = \eta(\{s : D_s(\mathbf{P}) = C\})$ .

The cutoff of a college represents the score above which a student who applies is admitted. Recall that  $I_j = I_j^A \times I_j^B$  is the support of  $f_{j,\theta}$ , where  $\underline{I}_j^A$  and  $\bar{I}_j^A$  are the lower and upper bounds of  $I_j^A$ , respectively, and similarly for  $B$ . If  $P^C = \min_{j \in [K]} \underline{I}_j^C$ , then college  $C$  rejects no one and if  $P^C = \max_{j \in [K]} \bar{I}_j^C$ , it accepts no one. The supply associated with this demand is simply the capacity of each college.

Consider the equilibria of this problem:

DEFINITION 2 (MARKET CLEARING). The cutoff vector  $\mathbf{P}$  is *market clearing* if for  $C \in \{A, B\}$ ,  $D_C(\mathbf{P}) \leq \alpha_C$ , with equality if  $P^C > \min_{j \in [K]} I_j^C$ .

A cutoff vector is therefore market-clearing if it induces a demand that is equal to colleges' capacities when they reach their capacity constraint and lower for colleges that are not full. When the constraint is reached at both colleges, i.e., when  $\alpha^A + \alpha^B < 1$ , the system

$$\mathbf{D}(\mathbf{P}) = \boldsymbol{\alpha} \tag{1}$$

is called the *market-clearing equation*, and the market-clearing cutoffs  $P^A$  and  $P^B$  can be computed by solving the system.

The following result from Azevedo and Leshno (2016) establishes the link between market-clearing cutoffs and stable matchings:

**Azevedo and Leshno (2016, Lemma 1)**

1. If  $\mu$  is a stable matching, then the associated cutoff vector  $\mathbf{P}$  is market-clearing;
2. If  $\mathbf{P}$  is market-clearing, then we define  $\mu$  such that for all  $s \in S$ ,  $\mu(s) = D_s(\mathbf{P})$ . Then  $\mu$  is stable.

This allows us to analyze stable matchings by studying the cutoffs of each college. Figure 2 illustrates the link between the cutoffs and the matching: students who prefer  $A$  are admitted there if and only if their score  $W^A$  is higher than the cutoff  $P^A$ . Otherwise, they are admitted to  $B$  if their score  $W^B$  is higher than  $P^B$  and stay unmatched if it is not. The situation is symmetric for students who prefer college  $B$ .

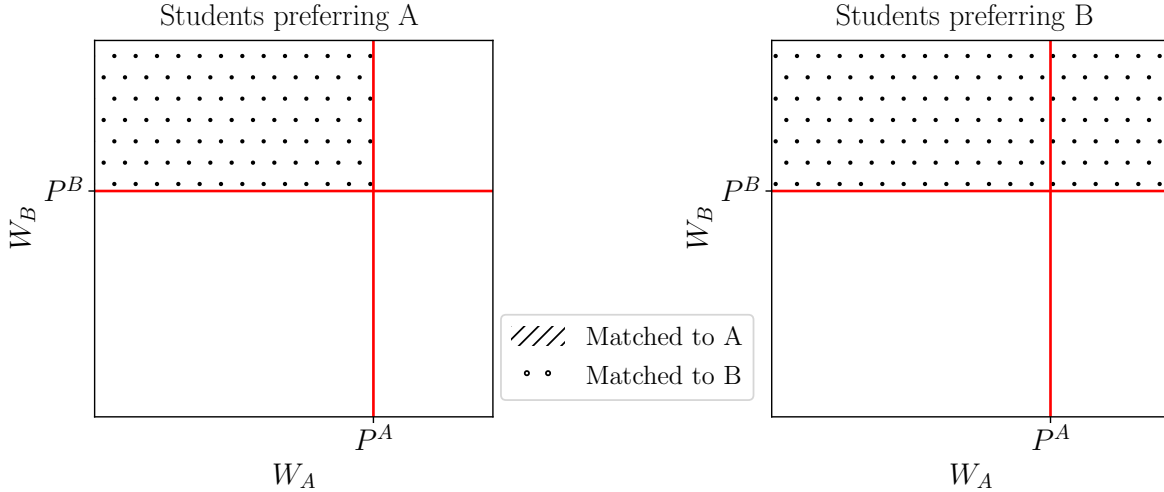
In the continuous college admissions problem, the same authors show that there is a unique stable matching.

LEMMA 2 (**Special case of Azevedo and Leshno 2016, Theorem 1**). *For any college admission problem  $(\gamma, \beta, \alpha, \mathbf{f}, \boldsymbol{\theta})$ , there exists a unique stable matching.*

Note that the original theorem specifies conditions on the distribution of students' types, such as being continuous and having full support, which hold in our definition of a college admission problem. Unlike the finite case where several stable matchings typically exist, in the continuum model the stable matching is unique; therefore, no considerations regarding selection among the set of stable matchings are necessary. Hence, we will consider the cutoff vector  $\mathbf{P}$  to be uniquely determined by the parameters of the problem and the market-clearing equation. We say that *student  $s$  goes to college  $C$*  to mean that they are matched to college  $C$  in the unique stable matching.

Azevedo and Leshno (2016) further show that the stable matching varies continuously in the parameters of the problem and that the set of stable matchings from a college admission problem with a finite number of students converges to the unique stable matching of the continuum problem

**Figure 2** Illustration of the match of students depending on their preferences, priority scores, and cutoffs. Students in the hashed area are matched to college A, those in the dotted area to college B, and those in the white area remain unmatched.



with the same parameters. The latter result justifies the approximation of large finite instances by their limit.<sup>7</sup>

### 3. Welfare metrics and preliminary results

In selection problems, inequalities between groups are measured by the proportion of admitted candidates in each group. In a matching setting, the situation is more complex; on the one hand, one group might have a greater proportion of unmatched students than the other, while on the other hand, the proportion of students who obtain their first choice might also differ. If all students in a group obtain their first choice and all students in the other obtain their second choice, the matching may be deemed unfair. In this section, we define metrics that allow us to quantify the satisfaction of students from each group.

Consider an individual's likelihood of obtaining their first choice, second choice, or being rejected from both colleges in a stable matching as a function of differential correlation.

**DEFINITION 3 (WELFARE METRICS).** Under a stable matching  $\mu_\theta$  induced by differential correlation parameters  $\theta$ , define  $V_1^{G_j, A}(\theta)$  and  $V_1^{G_j, B}(\theta)$ , as the proportion of students from each group-preference profile who obtain their first choice. Formally,

$$V_1^{G_j, A}(\theta) := \frac{1}{\gamma_j \beta_j} \eta(\{s \in G_j : A \succ_s B, \mu_\theta(s) = A\}),$$

$$V_2^{G_j, A}(\theta) := \frac{1}{\gamma_j \beta_j} \eta(\{s \in G_j : A \succ_s B, \mu_\theta(s) = B\}),$$

<sup>7</sup> For a better approximation for instances with a small number of students, Arnosti (2022) proposes a related framework.

$$V_{\emptyset}^{G_j, A}(\boldsymbol{\theta}) := \frac{1}{\gamma_j \beta_j} \eta(\{s \in G_j : A \succ_s B, \mu_{\boldsymbol{\theta}}(s) = \emptyset\}).$$

The metrics for students preferring college  $B$  are defined similarly, by inverting the roles of  $A$  and  $B$  and replacing  $\beta_j$  with  $1 - \beta_j$  in the equations.

Those metrics can be thought of in two ways; for instance,  $V_1^{G_j, A}(\boldsymbol{\theta})$  is the relative mass of students who obtain their first choice among those in group  $G_j$  who prefer college  $A$ , or equivalently, it is the probability of a randomly drawn student obtaining their first choice conditional on belonging to  $G_j$  and preferring  $A$ .<sup>8</sup>

We next provide expressions for these metrics for the unique (cf. Lemma 2) stable matching  $\mu_{\boldsymbol{\theta}}$  via its cutoffs,  $\mathbf{P}$ .

LEMMA 3. *Let  $C \in \{A, B\}$  be a college,  $\bar{C}$  be the other college, and  $G_j$  be a group. Letting  $\mathbf{P}$  be the cutoffs associated with  $\mu_{\boldsymbol{\theta}}$ , we have:*

$$V_1^{G_j, C}(\boldsymbol{\theta}) = \mathbb{P}_j(W^C \geq P^C(\boldsymbol{\theta})), \tag{2}$$

$$V_2^{G_j, C}(\boldsymbol{\theta}) = \mathbb{P}_{j, \theta_j}(W^C < P^C(\boldsymbol{\theta}), W^{\bar{C}} \geq P^{\bar{C}}(\boldsymbol{\theta})), \tag{3}$$

$$V_{\emptyset}^{G_j, C}(\boldsymbol{\theta}) = \mathbb{P}_{j, \theta_j}(W^C < P^C(\boldsymbol{\theta}), W^{\bar{C}} < P^{\bar{C}}(\boldsymbol{\theta})). \tag{4}$$

The notation  $\mathbb{P}_{j, \theta_j}$  is used as shorthand for  $\mathbb{P}_{(W^A, W^B) \sim f_{j, \theta_j}}$ . Lemma 3 allows us to compare the probabilities of admission of different types of students and derive comparative statics with respect to differential correlation. The proof is provided in Appendix B.2.

Regarding the probability of remaining unmatched, we can derive a simple yet important result (see Appendix B.3 for its proof).

LEMMA 4. *The probability that a student remains unmatched depends only on their group and is independent of their preference. Moreover, the total mass of unmatched students is constant in any group's correlation level. Formally, let  $\mathbf{P}$  be associated with  $\mu_{\boldsymbol{\theta}}$ . Then, for  $j \in [K]$ ,  $V_{\emptyset}^{G_j, A}(\boldsymbol{\theta}) = V_{\emptyset}^{G_j, B}(\boldsymbol{\theta})$ , and  $\eta(\{s \in S : \mu_{\boldsymbol{\theta}}(s) = \emptyset\}) = \max(0, 1 - \alpha^A - \alpha^B)$  (which does not depend on  $\boldsymbol{\theta}$ ).*

With Lemma 4 at hand, we use the notation  $V_{\emptyset}^{G_j}$  since these quantities do not depend on the preferences of students. For all the metrics we defined, when there is no ambiguity, we also omit the dependence on  $\boldsymbol{\theta}$  and write  $V_i^{G_j, C}$  instead.

We now define two global metrics, i.e., metrics that are not conditioned on the groups and preferences of students, namely, efficiency and inequality:

<sup>8</sup> We condition over preferences because of the following observation. Assume there are two groups; if students from group  $G_1$  all prefer college  $A$ , but only half of the students from group  $G_2$  prefer  $A$ , and  $A$  has a low capacity and  $B$  a large capacity, then, very few students from  $G_1$  will obtain their first choice, while half of the students from  $G_2$  are likely to obtain their first choice since it is a less demanded college. Thus, students' satisfaction might differ across groups only due to their own preferences and not because of differential correlation.

DEFINITION 4 (EFFICIENCY AND INEQUALITY). Define the *efficiency*  $E(\boldsymbol{\theta})$  of a matching as the proportion of students obtaining their first choice, and the *inequality*  $L^{G_i, G_j}(\boldsymbol{\theta})$  between two groups,  $i, j \in [K]$  as the difference in the probability of remaining unmatched between those two groups:

$$\begin{aligned} E(\boldsymbol{\theta}) &= \eta(\{s \in S : \mu_{\boldsymbol{\theta}}(s) = C \text{ and } C \succ_s \bar{C}\}) \\ &= \sum_{j \in [K]} \gamma_j \beta_j V_1^{G_j, A}(\boldsymbol{\theta}) + \sum_{j \in [K]} \gamma_j (1 - \beta_j) V_1^{G_j, B}(\boldsymbol{\theta}) \end{aligned} \quad (5)$$

$$L^{G_i, G_j}(\boldsymbol{\theta}) = |V_{\emptyset}^{G_i}(\boldsymbol{\theta}) - V_{\emptyset}^{G_j}(\boldsymbol{\theta})|. \quad (6)$$

According to Lemma 4, the mass of unmatched, and therefore matched, students is constant; thus matched students obtain either their first or second choice. Therefore, ceteris paribus, it is desirable to maximize the mass of students who obtain their first choice  $E$ . Regarding inequality, we measure the inequality between two groups by the difference in their proportions of unassigned students.

PROPOSITION 1. *If two groups have the same marginal distributions at some college  $C$ , then for students whose first choice is college  $C$ , the probability of obtaining this college is the same for students in both groups. Formally, if  $f_j^C = f_{\ell}^C$ , then  $V_1^{G_j, C} = V_1^{G_{\ell}, C}$ .*

The proof is provided in Appendix B.4. Proposition 1, albeit simple, is an important property of the model. If two students prefer the same college, then their probabilities of obtaining it only depend on their respective groups' marginals and not on their correlation levels; thus, differential correlation has no effect on this metric. Proposition 1 also justifies the choice of  $L$  as a measure of inequality; the proportion of students obtaining their first choice is the same for two groups as long as they have the same marginals, and differences only emerge in second choice admittance versus remaining unmatched. Consequently, when the proportion of unmatched students is greater in one group than in the other, then the matching is unequal.<sup>9</sup>

The following result shows that if there is capacity excess, then differential correlation does not affect the stable matching.

PROPOSITION 2. *If capacity is not constrained, i.e.,  $\alpha^A + \alpha^B \geq 1$ , then correlation has no effect on the stable matching. The cutoffs  $P^A$  and  $P^B$  are constant in  $\boldsymbol{\theta}$ , as are  $V_1^{G_j, C}$  and  $V_2^{G_j, C}$  for all  $j$  and  $C$ . Moreover,  $V_{\emptyset}^{G_j} = 0$ ; therefore,  $\forall i, j \in [K]$ ,  $L^{G_i, G_j}(\boldsymbol{\theta}) = 0$ .*

The proof is provided in Appendix B.5; more detail about this case can be found in Appendix A.6.

<sup>9</sup> Another choice to measure inequality would be to compare the proportions of students obtaining their second choice in each group; however, Proposition 1 implies that this quantity is equal to  $L$ .

## 4. Main results

This section contains our main results on the impact of differential correlation on the properties of stable matchings. We consider college admission problems where  $\gamma$ ,  $\beta$  and  $\alpha$  are assumed to be constant and study the influence of differential correlation,  $\theta$ , on the stable matching. Section 4.1 contains general comparative statics, and Section 4.2 examines tie-breaking.

### 4.1. Comparative statics

We first consider how the efficiency of the matching, i.e., the probability of obtaining one's first choice, varies when changing the correlation for one group.

**THEOREM 1.** *Suppose that  $(H_\theta)_{\theta \in \Theta}$  is coherent and differentiable, and that  $\alpha^A + \alpha^B < 1$ . Then for all groups and all preferences, the proportion of students who obtain their first choice is increasing in all correlation parameters  $\theta_j$ ,  $j \in [K]$ ; consequently, so is the global efficiency  $E(\theta)$ . Formally, suppose that  $\theta \in (\overset{\circ}{\Theta})^K$ . Then, for any  $C \in \{A, B\}$ , for any  $j, \ell \in [K]$ ,  $V_1^{G_j, C}(\theta)$  is differentiable and*

$$\frac{dV_1^{G_j, C}(\theta)}{d\theta_\ell} > 0.$$

The immediate consequence is that  $E(\theta)$  is differentiable, and for any  $j \in [K]$ ,

$$\frac{dE(\theta)}{d\theta_j} > 0.$$

The proof relies on the following lemma:

**LEMMA 5.** *Suppose that  $\theta \in (\overset{\circ}{\Theta})^K$ . Then for any  $C \in \{A, B\}$ ,  $P^C(\theta)$  is differentiable and*

$$\frac{dP^C(\theta)}{d\theta_j} < 0 \quad \forall j \in [K].$$

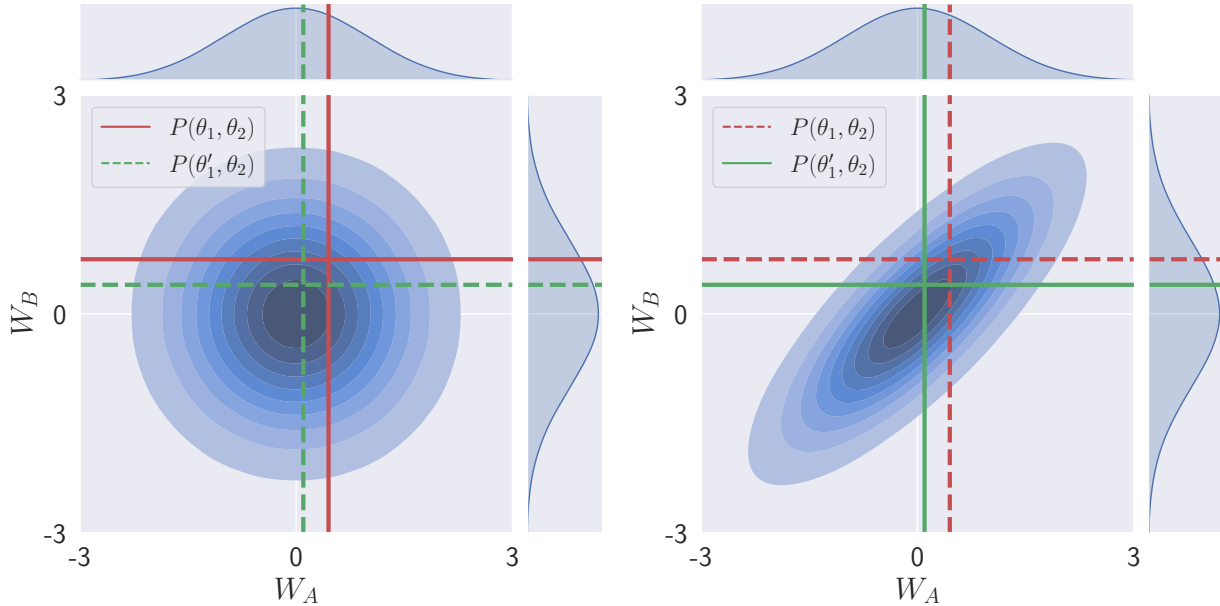
*Proof sketch.* The proof follows several steps. First, we rewrite the market-clearing Equation (1) using Lemma 3. We obtain a system of two equations, where the variables are the cutoffs  $P^A$  and  $P^B$ , parameterized by  $\theta$ . We then apply the implicit function theorem to a mapping whose roots are the solution of this system of equations. We next compute the partial derivatives. To characterize the sign of the derivatives with respect to  $\theta$ , we use the coherence assumption. Through analytical derivations, we can conclude. The proof is provided in Appendix B.6.

*Proof of Theorem 1.* According to Lemma 5, the cutoffs are decreasing in each  $\theta_j$ . We can then conclude that for any  $j \in [K]$  and for  $C \in \{A, B\}$ ,

$$\frac{dV_1^{G_j, C}}{d\theta_j} = \frac{d \int_{P^C}^{\infty} f_j^C(x) dx}{d\theta_j} = \frac{d \int_{P^C}^{\infty} f_j^C(x) dx}{dP^C} \cdot \frac{dP^C}{d\theta_j} > 0. \quad \square$$



**Figure 3** Illustration of change in cutoffs. The distribution of group  $G_1$  is a bivariate Gaussian with  $\theta$  equal to the covariance:  $\theta_1 = 0$  in the left-hand figure and  $\theta'_1 = 0.8$  in the right-hand figure.  $P^A$  is represented as a vertical line and  $P^B$  as an horizontal line. Both cutoffs decrease as  $\theta_1$  increases. In each sub-figure, the cutoffs corresponding to the current value of  $\theta_1$  are represented as full lines and the cutoffs corresponding to the other value of  $\theta_1$  as dashed lines.



Theorem 1 implies that, if the correlation decreases for one of the groups, then all groups suffer from a decrease in first-choice admittance. Conversely, increasing the correlation for one group leads to an increase in first-choice admittance for all groups.

Intuitively, when the correlation increases, students' score vectors accumulate close to the diagonal, and therefore in the lower-left and upper-right quadrants, while the other two quadrants are increasingly empty. This phenomenon is illustrated in Figure 3 with a bivariate Gaussian distribution. If the cutoffs did not change, then the amount of unmatched students would increase. As the capacities are assumed to be constant, this would render the resulting matching unstable. Therefore, at least one of the cutoffs decreases, while Lemma 5 implies that both decrease. As a consequence, the mass of matched students remains the same, but more students obtain their first choice.

**REMARK 1.** The formal statement of Theorem 1 excludes the extremities of  $\Theta$ . This assumption is made only to avoid the case where rankings are fully correlated, which would mean that  $\mathbf{f}$  does not have full support. However, since  $V_1^A$  and  $V_1^B$  are continuous in  $\theta$ , they increase over the whole interval  $\Theta$ .

Theorem 1 allows us to derive the following corollary regarding a student's probability of remaining unmatched.

COROLLARY 1. *Suppose that  $(H_\theta)_{\theta \in \Theta}$  is coherent and differentiable and that  $\theta \in \overset{\circ}{\Theta}^K$  and  $\alpha^A + \alpha^B < 1$ . Then, the proportion of students from a given group remaining unmatched is increasing in its own correlation level and decreasing in the correlation level of all other groups: for  $i, j \in [K]$ ,  $i \neq j$ ,*

$$\frac{dV_\emptyset^{G_i}(\theta)}{d\theta_i} > 0 \quad \text{and} \quad \frac{dV_\emptyset^{G_i}(\theta)}{d\theta_{G_j}} < 0.$$

Moreover, the inequality between any two groups decreases in the correlation level of the group with the lowest rate of unmatched students and increases in the correlation level of the other group. Formally, assume  $V_\emptyset^{G_i}(\theta) < V_\emptyset^{G_j}(\theta)$ . Then,

$$\frac{dL^{G_i, G_j}(\theta)}{d\theta_i} < 0 \quad \text{and} \quad \frac{dL^{G_i, G_j}(\theta)}{d\theta_j} > 0.$$

*Proof sketch.* The proof relies on Lemma 5 and leverages the fact that total capacity is constant to derive the sign of the partial derivatives of the  $V_\emptyset$  terms. The proof is provided in Appendix B.7.

REMARK 2. As a consequence, for any two groups  $G_i, G_j$ , if  $\theta_i \neq \theta_j$ , then the matching almost surely exhibits inequality for those groups ( $L^{G_i, G_j}(\theta) > 0$ ). In particular, this is true even when those groups have the same marginals.

The probability of remaining unmatched is different for students from different groups, even those with identical marginals. This finding contrasts with Proposition 1. Different levels of correlation lead to an unequal matching. This is the case because with identical marginals, the proportion of students above some cutoff is the same in every group; however, for a group with high correlation, the set of students above the cutoff is almost the same at each college, while for a group with low correlation, those sets are quite different at each college. Therefore, the set of matched students in group  $G_j$ , which is  $\{s \in G_j | W_s^A \geq P^A\} \cup \{s \in G_j | W_s^B \geq P^B\}$ , is larger for groups with lower correlation. This result is counterintuitive considering the point of view of some college  $C$ , which has identical marginals for all groups. From  $C$ 's point of view, there is no difference between the groups, and the proportion of students with  $W_s^C \geq P^C$  is the same across all groups. However, Corollary 1 implies that the groups with the lowest correlation levels are overrepresented at  $C$  and that the groups with the highest correlation are underrepresented. College  $C$  then ends up with a set of students that could be deemed “unfair” regarding demographic parity, while  $C$ 's ranking was in fact perfectly fair.

Corollary 1 helps us understand the influence of correlation on inequality. When the correlation levels of two groups are equal and the marginals are identical, there is no inequality. If marginals are different, there is some “baseline” inequality that can be increased or decreased by changing the correlation levels; to decrease the inequality, one would need to increase the correlation of the group

with the lowest proportion of unassigned students (therefore the better-off group) and/or decrease the correlation of the worse-off group. Overall, even when colleges have fair rankings (identical marginals across all groups), the matching might still exhibit inequality.

Theorem 1 and Corollary 1 examine efficiency and inequality, respectively. The following proposition describes their interaction. Specifically, different correlation vectors,  $\theta$ , can lead to the same efficiency  $E$  while inducing different inequality levels between groups. This shows that the effect of differential correlation cannot be solely explained via differences in efficiency.

**PROPOSITION 3.** *Suppose that  $(H_\theta)_{\theta \in \Theta}$  is coherent and differentiable, assume  $\Theta$  is an interval, and  $\alpha^A + \alpha^B < 1$ . Let  $\theta = (\theta_1, \dots, \theta_K)$ , and  $\hat{E} = E(\theta)$ .*

1. *There exist infinitely many correlation vectors that achieve a given efficiency. Formally, the set of vectors  $\theta'$  such that  $E(\theta') = \hat{E}$  is a connected hypersurface of dimension  $K-1$  (unless  $\theta = 0_{\mathbb{R}^K}$  or  $(1, \dots, 1)$ , in which case it is a singleton).*

2. *Fixing efficiency, correlation levels are substitutes. Formally, for any two groups  $G_i, G_j$ , there exists an interval  $U := [\underline{\theta}, \bar{\theta}] \subseteq \Theta$  and a differentiable and decreasing function  $\phi: U \rightarrow \Theta$  such that  $(\theta_i \in U \text{ and } \theta_j = \phi(\theta_i)) \implies E(\theta) = \hat{E}$ . The boundaries of  $U$  are optimal/pessimal with respect to the mass of unassigned students  $(V_\emptyset^{G_i}, V_\emptyset^{G_j})$  for  $G_i$  and  $G_j$ , respectively, and there is a unique  $\hat{\theta} \in U$  such that  $\theta_1 = \hat{\theta}, \theta_j = \phi(\hat{\theta})$  minimizes inequality  $L^{G_i, G_j}(\theta)$ .*

The proof is provided in Appendix B.8. Beyond the intuition that correlation favors efficiency, Proposition 3 provides precise insight into the relation between efficiency and inequality and the trade-off between the two. The first part states that there is, in general, a continuum of correlation vectors achieving the same level of efficiency. The second part considers the comparative statics between two groups. Fixing the efficiency, the correlation parameters behave as rival goods. As the correlation increases for one group, it necessarily decreases for the other group.

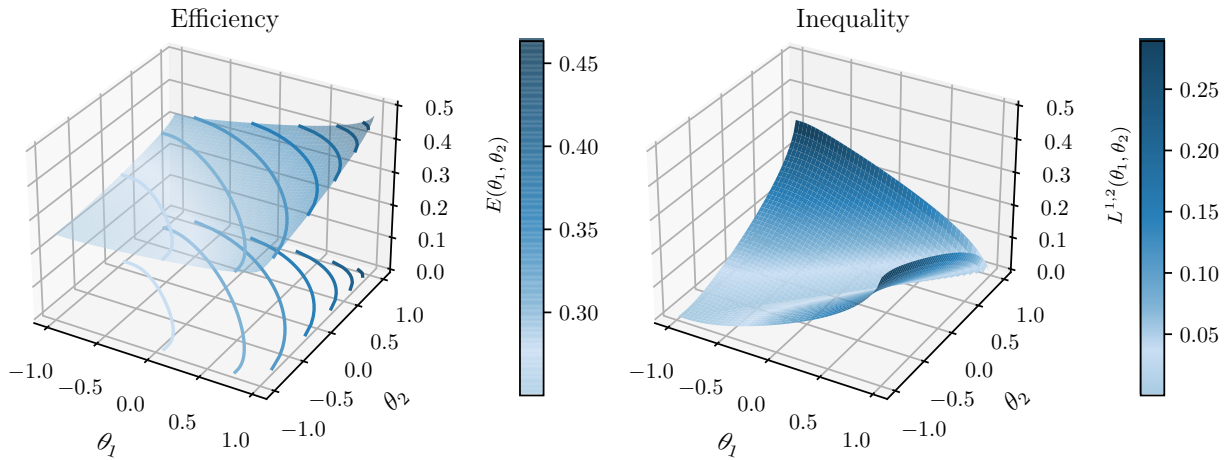
Figure 4 illustrates this for two groups,  $G_1, G_2$ , with standard Gaussian marginals and the parameter of the copula,  $\theta$ , equal to the covariance. The left panel shows the efficiency, and the right panel shows the inequality as functions of  $\theta_1, \theta_2$ . In the left panel, the level lines show the decreasing relation between  $\theta_1$  and  $\theta_2$  when  $E$  is kept constant. In the right panel, the inequality is minimized along the diagonal where  $\theta_1 = \theta_2$  and increases as the parameters become more disparate.

## 4.2. Tie-Breaking

Some recent papers have studied the impact of tie-breaking rules on school choice problems, which has a strong link with correlation. In this section, we extend some of the prior results and discuss the relation to the literature.

Assume there is only one group, and each school  $C$  has  $n_C$  priority classes; i.e., there exists a partition of  $S = Q_1^C \cup \dots \cup Q_{n_C}^C$  such that for  $i, j \in \{1, \dots, n_C\}$ ,  $s \in Q_i^C$ ,  $s' \in Q_j^C$ ,  $s$  has higher priority

**Figure 4** Variations of efficiency ( $E$ ) and ( $L$ ) for two groups ( $G_1, G_2$ ) as a function of their respective correlation parameters  $(\theta_1, \theta_2)$ , with standard Gaussian marginals and copula with  $\theta$  equal to the covariance. Other parameters:  $\alpha^A = \alpha^B = 0.25$ ,  $\gamma_1 = \gamma_2 = 0.5$ ,  $\beta_1 = \beta_2 = 0.5$ . **Left:** The surface represents the efficiency and the level lines indicate constant efficiency (also projected to the bottom of the figure). **Right:** The surface represents the inequality.



than  $s'$  at  $C$  if  $i < j$ . Students belonging to the same priority class at a school are assumed to have the same priority at that school; however, due to limited capacity, the school might need to choose between them. To achieve this, schools use a random ranking of students to which they refer each time they need to choose between students from the same priority class; this random ranking is called a *tie-breaker*.

A natural question that has been actively studied in recent years is whether there is a difference in student welfare if schools use the same tie-breaker (called a single tie-breaker, or STB) instead of each producing an independent tie-breaker (called multiple tie-breakers, or MTB). Ashlagi et al. (2019), Ashlagi and Nikzad (2020), Arnosti (2023) show—with slightly different models and assumptions (and among other results)—that when the total capacity of schools is lower than the number of students, then students are better off under STB than under MTB. To ease the comparison, we restate their results here in a simplified form.

Given  $n$  students and  $m$  schools:

*Ashlagi et al. 2019, Main Theorem:* Suppose that there is capacity shortage, students' preferences are drawn uniformly at random and there is only one priority class (the whole ranking is random); then, for any  $k < m$ , the fraction of students matched to one of their top  $k$  choices approaches 0 under MTB but approaches a positive constant under STB.

*Ashlagi and Nikzad 2020, Theorem 3.2:* Suppose there is one slot per school, only one priority class, and schools are divided into two tiers (top and bottom) with students' preferences inside

a given tier drawn uniformly at random and a capacity shortage at top schools; then, with high probability, STB stochastically Pareto-dominates MTB.

*Arnosti 2023, Theorem 2:* Suppose there is only one priority class and students only list  $l < m$  schools in a uniform random order; then, the number of students matched to their first choice is greater under STB than under MTB.

Our model, compared to prior work on tie-breaking, allows for any number of priority classes, intermediate levels of correlation or even negative correlations, and several groups of students with different tie-breaking rules. To this end, let  $(H_\theta)_{\theta \in \Theta}$  be a coherent and differentiable family of copulas such that  $\theta = 0$  gives independent random variables and  $\theta = 1$  gives fully correlated variables. Define the  $\theta$ -TB as the tie-breaker drawn according to  $H_\theta$ . Thus, MTB corresponds to  $\theta = 0$ , and STB corresponds to  $\theta = 1$ . Moreover, we can assume the existence of several groups with different  $\theta$  values.

Intermediate correlation levels can arise in tie-breaking if, for example, student characteristics are introduced into rankings to break ties, e.g., sibling priority or distance from home to school (Correa et al. 2022). This is commonly done to render algorithms more deterministic and thus explainable. Consider priority for students with shorter distance to a school and suppose that there are two villages with one school each. Then, *ceteris paribus*, a student who lives in one village exhibits a negative correlation between the grades at each of the two schools. On the other hand, a student living in neither village may exhibit any level of correlation. Note that this example also illustrates how negative correlation naturally arises.

**PROPOSITION 4.** *Let there be a continuum mass of students and assume that students prefer any school over being unmatched. Let  $A, B$  be two schools with  $n_A, n_B$  priority classes and constrained capacities  $\alpha^A + \alpha^B < 1$ . Furthermore, suppose that students are divided into  $K$  groups, such that the  $\theta_j$ -TB is used for group  $G_j$ . Then:*

1. *The mass of students who obtain their first choice*
  - *is non-decreasing in each  $\theta_j$ ,*
  - *is almost surely strictly increasing in all  $\theta_j$  if all products of priority classes  $Q_i^A \times Q_j^B$  contain a positive mass of students of each group,<sup>10</sup> and*
  - *is strictly increasing in each  $\theta_j$  if there is only one priority class.*
2. *The inequality between two groups,  $L^{G_i, G_j}(\boldsymbol{\theta})$ , is non-decreasing in the correlation  $\theta$  of the group with the lowest  $V_\theta$  and non-increasing in the other group's  $\theta$ .*

<sup>10</sup> More precisely, the set of vectors  $(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\alpha})$  such that the mass of students who obtain their first choice is constant in some  $\theta_j$  has Lebesgue measure 0.

*Proof sketch.* We build a distribution family that encompasses the priorities of students at each school, accounting for priority classes as well as tie-breakers, such that MTB and STB correspond to values of  $\theta = 0$  and  $\theta = 1$ , respectively. The obtained distribution, while complex, still satisfies most of the assumptions required by our model; with some adjustments, we are able to apply Theorem 1 and Corollary 1 and conclude. The proof is provided in Appendix B.9.

This result shows that increasing the correlation of tie-breakers, for one or several groups increases the amount of students who obtain their first choice. Moreover, it also shows that a policy-maker who is able to change the correlation of tie-breakers for some groups can use it to mitigate the inequalities between groups.

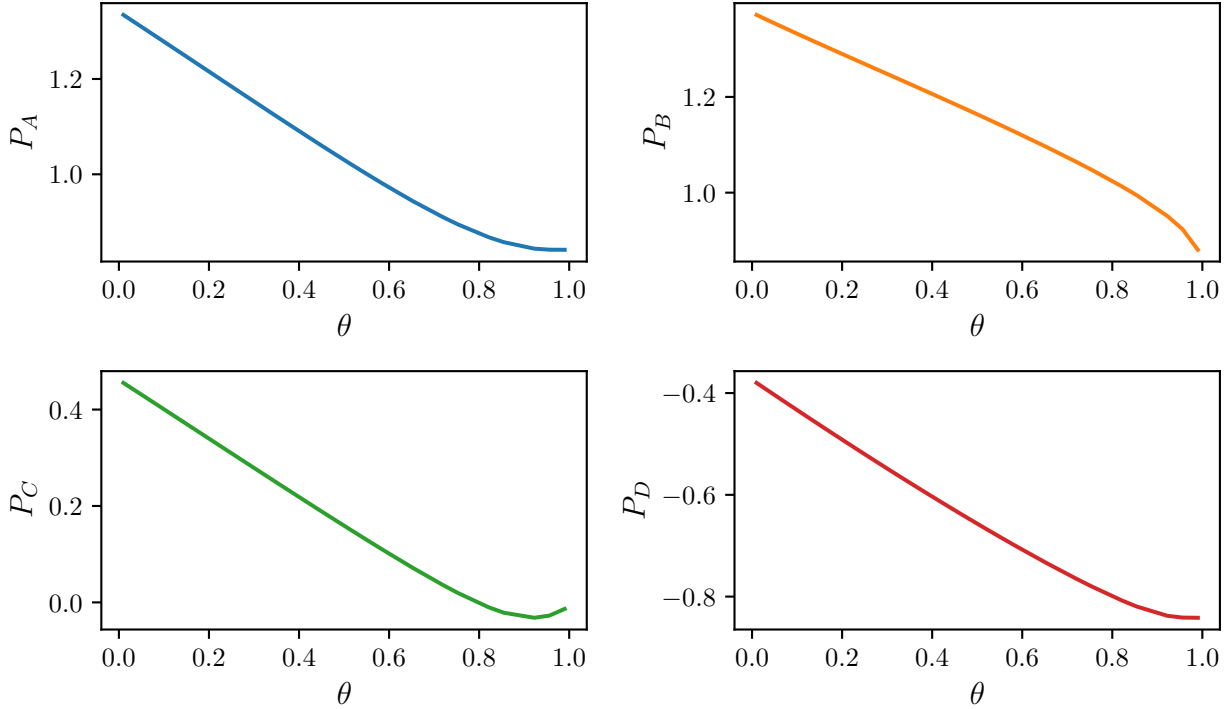
Proposition 4 is in some regards more restrictive than the results from the literature presented above, because it only applies to two schools and assumes that students prefer either school over being unmatched. On the other hand, it is more general in that it applies to cases with several priority classes and does not require students' preferences to be uniform (in our model, we can have any fraction  $\beta$  of students preferring school  $A$ ). It also allows several groups with different tie-breaking rules to exist. Finally, Proposition 4 allows for intermediate tie-breaking rules that interpolate between MTB and STB, and for negatively correlated tie-breaking rules.

## 5. Discussion

We have introduced a tractable model to study the impact of differential ranking correlation between different groups and studied its effect on outcome inequality and efficiency in matching markets. Our framework is general in that it accommodates almost any grade distribution, any number of groups with different distributions and different student preferences, and colleges of any capacity.

However, a limitation is our focus on two colleges, which we will discuss here. First, our model can be extended to any number of colleges; in particular, copulas can be defined for any number of variables, and the coherence definition can be extended. Our welfare metrics also carry over, and it would be natural to also consider further metrics, e.g., the probability of obtaining one of the top  $k$  choices. Turning to our results, Proposition 1 remains true. That is, whenever two groups have the same marginal distribution at some college, they have the same probability of obtaining this college as a first choice. On the other hand, Proposition 2 no longer holds; for any pair of colleges whose joint capacity is less than 1, the correlation between their rankings will have an effect on the matching, even if the total capacity is more than 1. The other results all rely on Lemma 5. Through numerical experiments, we find that Lemma 5 generally does not extend beyond two colleges. Figure 5 shows a counterexample with four colleges, one group, standard Gaussian marginals at every college, a Gaussian copula, and  $\alpha = (0.05, 0.05, 0.2, 0.5)$ . The cutoff  $P^C$  of the third college is increasing for high values of  $\theta$ .

**Figure 5** Counterexample to the extension of Lemma 5 to more than two colleges: Four colleges, one group, bivariate Gaussian distribution with correlation  $\theta$ , and  $\alpha = (0.05, 0.05, 0.2, 0.5)$ . Note that  $P_C$  is not monotone decreasing.



It follows that Theorem 1 also does not extend beyond two colleges without further assumptions, as it relies on Lemma 5; the same holds for Corollary 1 and Propositions 3 and 4. These results can be easily recovered under strong assumptions, for instance if all colleges have the same capacity and students' preferences are uniform; (Peng and Garg (2023) use similar assumptions in a subsequent study focusing on algorithmic monoculture). If this is the case, then each college is equally demanded and therefore the cutoff at each college is the same. Moreover, our experiments suggest that Lemma 5 may be true even in the general setting for correlation parameters that are in some interior interval.

To conclude, we believe that there is ample scope to study themes that have already been considered in single decision-maker settings in the matching context. Our analysis suggests that in matching new phenomena arise, and it is important to further understand them. In models where decision-makers use noisy estimates of applicants' latent quality, the existing results about algorithmic monoculture could be extended, providing a theoretical foundation for experimental findings, such as those of Bommasani et al. (2022). Using this noise structure, interesting variations could include allowing applicants to invest in accurate assessment, e.g., by acquiring certifications or participating in in-person interviews, or considering the effects of risk aversion. Other possible

research directions include making applications costly, allowing applicants to not list all colleges in their preferences, and an analysis of colleges' utility.

## Appendix A: Definitions and technical details

### A.1. Definitions

**A.1.1. Table of notation.** Table 1 provides a summary of the notation used throughout the paper.

**Table 1** Notation

<u>Agents:</u>	
$A, B$	Colleges (generic: $C$ )
$s$	An arbitrary student
$S$	Students set
$G_1, \dots, G_K$	Groups of students, partition of $S$
$\eta$	Measure for student masses
<u>Agents' features:</u>	
$\alpha^A, \alpha^B$	Colleges' capacities ( $\in (0, 1)$ )
$\gamma_j$	Mass of students in group $G_j$ ( $\in [0, 1]$ )
$\beta_j$	Share of students in group $G_j$ preferring college $A$ ( $\in [0, 1]$ )
<u>Priority scores:</u>	
$W_s^C$	Score at $C$ of student $s$ (generic: $W$ )
$f_j^C, F_j^C$	Marginal pdf and cdf of college $C$ for group $G_j$
$(H_\theta)_{\theta \in \Theta}, (h_\theta)_{\theta \in \Theta}$	Copula family and associated pdfs, indexed by $\theta$
$\theta$	Parameter for a copula family
$f_{j,\theta_j}, F_{j,\theta_j}$	Group $G_j$ 's score vectors' joint pdf and cdf, $F_{j,\theta_j} = H_{\theta_j}(F_j^A, F_j^B)$
$\Theta$	Set of possible values for $\theta$
$I_j^C, \bar{I}_j^C$	Support of $f_j^C$ and $f_{j,\theta_j}$ respectively. $I_j = I_j^A \times I_j^B$
$\underline{I}_j^C, \bar{I}_j^C$	Lower and upper bounds of $I_j^C$
<u>Correlation:</u>	
$r$	Pearson's correlation
$\rho$	Spearman's correlation
$\tau$	Kendall's correlation
<u>Matching:</u>	
$\mu$	Matching
$V_1^{G_j, C}$	Share of students of group $G_j$ and preferring $C$ who get their first choice
$V_2^{G_j, C}$	Share of students of group $G_j$ and preferring $C$ who get their second choice
$V_\emptyset^{G_j}$	Share of students of group $G_j$ who are unassigned
$V_1$	Total mass of students getting their first choice
$L^{G_i, G_j}(\boldsymbol{\theta})$	Inequality between $G_i$ and $G_j$ , equal to $ V_\emptyset^{G_i}(\boldsymbol{\theta}) - V_\emptyset^{G_j}(\boldsymbol{\theta}) $



**A.1.2. Definition of the mass  $\eta$ .** Here we formally define the notion of mass for a subset of students. This section is self-contained and is not necessary to understand the results of the paper; the notations introduced here are not used elsewhere. We identify  $S$  to  $\Sigma := \mathbb{R}^2 \times \{G_1, \dots, G_K\} \times \{A, B\}$ . We partition  $\Sigma$  into several subsets:  $\Sigma_{G_j, C} := \{s \in \Sigma : s = ((x, y), G_j, C), x, y \in \mathbb{R}\}$  is the subset of students belonging to group  $G$  and preferring college  $C$ . Given a vector of parameters  $\theta$  and priorities  $W^A, W^B$  distributed according to  $f_{j, \theta_j}$  for  $G_j$  students, we say that a subset  $J \subseteq \Sigma$  is measurable if and only if  $\{(W_s^A, W_s^B) : s \in J\}$  is Borel-measurable in  $\mathbb{R}^2$ . We can partition  $J$  into subsets  $J_{G_j, C} := J \cap \Sigma_{G_j, C}$ . On each  $\Sigma_{G_j, C}$  we define a measure  $\eta_{G_j, C}$  as follows: for  $J \subseteq \Sigma$  measurable,

$$\begin{aligned} \eta_{G_j, A}(J_{G_j, A}) &= \gamma_j \beta_j \mathbb{P}_{\theta_j}((W^A, W^B) \in \{(W_s^A, W_s^B) : s \in J_{G_j, A}\}), \\ \eta_{G_j, B}(J_{G_j, B}) &= \gamma_j (1 - \beta_j) \mathbb{P}_{\theta_j}((W^A, W^B) \in \{(W_s^A, W_s^B) : s \in J_{G_j, B}\}), \end{aligned} \quad (7)$$

Let  $\mathcal{B}(S)$  be the set of measurable subsets of  $S$ . We define over  $\mathcal{B}(S)$  the probability measure  $\eta : \mathcal{B}(S) \rightarrow [0, 1]$  such that for any measurable subset  $J$  of  $S$ ,

$$\eta(J) = \sum_{j \in [K]} \eta_{G_j, A}(J_{G_j, A}) + \eta_{G_j, B}(J_{G_j, B}). \quad (8)$$

This definition is consistent with the definition of the parameters, as it verifies  $\eta(G_j) = \gamma_j$ ,  $\eta(\{s \in G_j : A \succ_s B\}) = \gamma_j \beta_j$  and so on.

## A.2. Elements of correlation theory

In this section, we present common measures of correlation used in the literature, and some of their properties.

**DEFINITION 5 (COMMON MEASURES OF CORRELATION).** Let  $(X, Y)$  be two random variables with respective cdfs  $F_X, F_Y$ . Define:

1. Pearson's correlation: assume  $X, Y$  have finite standard deviations  $\sigma_X$  and  $\sigma_Y$ . Then  $r_{X, Y} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$ .
2. Spearman's correlation: let  $rk_X = F_X(X)$  and  $rk_Y = F_Y(Y)$ . We can think of  $rk_X$  as describing the ranking of  $X$  inside a sample. Then Spearman's correlation is  $\rho_{X, Y} = r_{rk_X, rk_Y}$ .
3. Kendall's correlation: let  $(X_1, Y_1)$  and  $(X_2, Y_2)$  be two independent pairs of random variables with the same joint distribution as  $(X, Y)$ . Then Kendall's correlation is

$$\begin{aligned} \tau_{X, Y} &= \mathbb{P}[(X_1 > X_2 \cap Y_1 > Y_2) \cup (X_1 < X_2 \cap Y_1 < Y_2)] - \\ &\quad \mathbb{P}[(X_1 > X_2 \cap Y_1 < Y_2) \cup (X_1 < X_2 \cap Y_1 > Y_2)]. \end{aligned}$$

We use the same letter  $r$  for the covariance of the standard bivariate Gaussian and for Pearson's correlation as they are equal. Moreover, for this distribution simple expressions exist for the two other correlation coefficients:

$$\rho = \frac{6}{\pi} \arcsin(r/2), \tau = \frac{2}{\pi} \arcsin(r).$$

A correlation measure should be zero when variables are independent, and reach its maximum when the variables are totally dependent on each other. The following lemma provides these properties for the measures we just introduced.

**LEMMA 6 (Scarsini 1984, Theorems 1, 4, and 5).** *Let  $X, Y$  be two real random variables.*

1.  $r_{X, Y}, \rho_{X, Y}, \tau_{X, Y} \in [-1, 1]$ .
2.  $\rho_{X, Y} = 1$  if and only if  $Y = g(X)$  with  $g : \mathbb{R} \rightarrow \mathbb{R}$  increasing. The same holds for  $\tau_{X, Y}$ .  $r_{X, Y} = 1$  if and only if the relation is affine.
3. If  $X$  and  $Y$  are independent, then  $r_{X, Y} = \rho_{X, Y} = \tau_{X, Y} = 0$ .

### A.3. Discussion on distributional assumptions

We assume that distributions admit a density and have full support, and that they can be represented using a copula family and marginals that remain the same for any  $\theta$ , and that this copula is coherent and differentiable. We here explain why these assumptions are not very restrictive by presenting canonical examples of classical copulas satisfying our assumptions.

1. *Gaussian copula*: The Gaussian copula is obtained by composing the cdf  $\Phi_\theta$  of a bivariate Gaussian with covariance matrix  $\begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}$  and the univariate cdf  $\phi$  of the standard Gaussian:  $H_\theta(x, y) = \Phi_\theta(\phi(x), \phi(y))$ . Here, the parameter  $\theta$  controls the covariance.

2. *Archimedean copulas*: Archimedean copulas are a broad family of copulas, each element of this family being itself a parametric family of copulas with parameter  $\theta$ . The general formula is

$$H_\theta(x, y) = \psi_\theta^{-1}(\psi_\theta(x) + \psi_\theta(y))$$

where  $\psi_\theta : [0, 1] \rightarrow \mathbb{R}_+$  is a continuous strictly decreasing and convex function such that  $\psi_\theta(1) = 0$ . Examples include:

- Clayton:  $H_\theta(x, y) = (\max\{x^{-\theta} + y^{-\theta} - 1; 0\})^{-1/\theta}$
- Frank:  $H_\theta(x, y) = -\frac{1}{\theta} \log \left( 1 + \frac{(\exp(-\theta x) - 1)(\exp(-\theta y) - 1)}{\exp(-\theta) - 1} \right)$
- Gumbel:  $H_\theta(x, y) = \exp(-((-\log(x))^\theta + (-\log(y))^\theta)^{1/\theta})$

The Gaussian copula, as well as Clayton's, Frank's, Gumbel's and other Archimedean copulas, all satisfy our coherence and differentiability assumptions.

The only assumption our model makes on the marginals is that they are continuous. This is not particularly restrictive as long as there are no ties (see Section 4.2 for a treatment of ties).

### A.4. The latent quality plus noise setting

We here discuss the classical model where an (unknown) latent quality is observed with an added noise term (cf. Phelps 1972, Emelianov et al. 2020, 2022, Garg et al. 2021). This provides an example of how differential correlation can arise.

For concreteness, assume that all students have a latent quality  $W$ , and their score  $\widehat{W}^A, \widehat{W}^B$  are the sum of the latent quality and a noise term  $\varepsilon$  drawn independently at each college, that is,

$$\forall s \in S, \text{ for } C \in \{A, B\}, \widehat{W}_s^C = W_s + \varepsilon_s^C.$$

Further, assume that the latent qualities of all students are drawn from a (group-independent) normal distribution, and that the noises are also normally distributed and depend on the group:

$$\forall s \in S, \text{ for } C \in \{A, B\}, W_s \sim \mathcal{N}(0, \chi^2), \varepsilon_s^C \sim \mathcal{N}(0, \sigma_{G(s)}^2).$$

The fact that the noise's variance is different for each group can be interpreted as colleges having different accuracies when evaluating students from different groups. Consider two examples: First,  $G_1$  could consist of students from well-known high schools, which colleges can evaluate well since they have a lot of applicants from there each year; and  $G_2$  could consist of students from unknown high schools, for which colleges do not

have a lot of prior information. Second, each group  $G_j$  could consist of students from a different demographic groups, defined by sensitive attributes such as gender, ethnicity, or social class.

Suppose that colleges know each student's group, and are aware of the difference that exists in noise variance across groups. Further, assume that colleges implement equal opportunity policies, that is, *ceteris paribus* the rank distribution of students must be the same for all groups.<sup>11</sup> To do so while maximizing the expected quality of admitted candidates, it is optimal for a college to not change the order of grades within groups, but to only fit each group's grades  $\widehat{W}^C$  to the same, standardized distribution. Thus, suppose that colleges divide each student's score by the standard deviation of their group's scores:

$$\forall s \in S, \text{ for } C \in \{A, B\}, \widetilde{W}_s^C = \frac{\widehat{W}_s^C}{\sqrt{\chi^2 + \sigma_{G(s)}^2}}.$$

With these new standardized scores, the marginal priority distribution of each group is  $\mathcal{N}(0, 1)$  at each college. The priority vectors then follow a centered bivariate normal distribution with variance 1 and a correlation that is different between the two groups: formally,

$$\left( \widetilde{W}_s^A, \widetilde{W}_s^B \right) \sim \mathcal{N} \left( (0, 0), \begin{pmatrix} 1 & r_{G(s)} \\ r_{G(s)} & 1 \end{pmatrix} \right),$$

with

$$r_{G(s)} = \frac{\chi^2}{\chi^2 + \sigma_{G(s)}^2}.$$

This setting satisfied the assumptions of our general model, and thus our analysis applies. The parameter  $\theta_j$  for each group  $G_j$  can then be chosen as  $-\sigma_{G_j}^2$ , or  $r_{G_j}$ , or any increasing function of one of those quantities.

Beyond this example, note that any grade vector can be decomposed into an unknown latent quality and the remainder, that is, the noise. Therefore, as long as the observed grade vectors are drawn from a coherent distribution family our results apply.

### A.5. Stable matching

We now introduce some elements of matching theory used throughout the paper.

To define matching in a continuum context, we follow Azevedo and Leshno (2016).

**DEFINITION 6.** A *matching* is an assignment of students to colleges, described by a mapping  $\mu : S \cup \{A, B\} \rightarrow 2^S \cup C \cup S$ , with the following properties:

1. for all  $s \in S$ ,  $\mu(s) \in \{A, B\} \cup \{s\}$ ;
2. for  $C \in \{A, B\}$ ,  $\mu(C) \subseteq S$  is measurable and  $\eta(\mu(C)) \leq \alpha_C$ ;
3.  $C = \mu(s)$  if and only if  $s \in \mu(C)$ ;
4. for  $C \in \{A, B\}$ , the set  $\{s \in S : \mu(s) \preceq_s C\}$  is open.

The first three conditions are common to almost all definitions of matching in discrete or continuous models. Condition (1) ensures that a student is either matched to a college or to themselves, which means that they remain unmatched. Condition (2) ensures that colleges are assigned to a subset of students that

<sup>11</sup> Without the equal opportunity assumption, colleges would compute the expected true qualities based on the different variances for each group, see Emelianov et al. (2022), Garg et al. (2021).

respects the capacity constraints. Condition (3) ensures that the matching is consistent, i.e., if a student is matched to a college, then this college is also matched to the student. Condition (4) was introduced by Azevedo and Leshno (2016) and is necessary to ensure that there do not exist several stable matchings that only differ by a set of students of measure 0.

We next define the notions of blocking and stability.

**DEFINITION 7 (STABILITY).** The pair  $(s, C)$  *blocks* a matching  $\mu$  if  $s$  would prefer  $C$  to her current match, and either  $C$  has remaining capacity or it admitted a student with a lower score than  $s$ ; formally, if  $\mu(s) \prec_s C$  and either  $\eta(\mu(C)) < \alpha_C$  or  $\exists s' \in \mu(C)$  such that  $W_{s'}^C < W_s^C$ . A matching is *stable* if it is not blocked by any student-college pair.

To produce a stable matching, one can extend the classic deferred acceptance algorithm by Gale and Shapley (1962) to the continuum model. This algorithm is described in Algorithm 1.

---

**Algorithm 1** Deferred acceptance algorithm (DA)

---

**First step:** All students apply to their favorite college, they are temporarily accepted. If the mass of students applying to college  $C$  is greater than its capacity  $\alpha_C$ , then  $C$  only keeps the  $\alpha_C$  best

**while** A positive mass of students are unmatched and have not yet been rejected from every college **do**

Each student who has been rejected at the previous step proposes to her preferred college among those which have not rejected them yet

Each college  $C$  keeps the best  $\alpha_C$  mass of students among those it had temporarily accepted and those who just applied, and rejects the others

**end while**

**End:** If the mass of students that are either matched or rejected from every college is 1, the algorithm stops. However it could happen that it takes an infinite number of steps to converge.

---

If the algorithm stops, the matching it outputs is stable; Abdulkadiroğlu et al. (2015) show that even when the number of steps is infinite, the algorithm converges to a stable matching.

**REMARK 3.** Note that stable matchings do not only result from centralized algorithms but are often the result of a decentralized process (see, e.g., Roth and Vande Vate 1990).

### A.6. Excess capacity

In the case where  $\alpha^A + \alpha^B \geq 1$ , we can compute the steps of the deferred acceptance algorithm (see Algorithm 1 in the Appendix). We consider three (partitioning) cases:

- (1) *There is not enough room in college A for all students preferring it to college B, i.e.,  $\sum_j \gamma_j \beta_j \geq \alpha^A$ .* In this case, there is necessarily enough room in college B for all students preferring it, since  $\alpha^A + \alpha^B \geq 1$ . Therefore, following the steps of DA, we find:

(i) At step one,  $\sum_j \gamma_j \beta_j$  students preferring  $A$  apply there and the best  $\alpha^A$  are temporarily admitted, and  $\sum_j \gamma_j (1 - \beta_j)$  students preferring  $B$  apply there and are all temporarily admitted.

(ii) At step two, the  $\sum_j \gamma_j \beta_j - \alpha^A$  students rejected from  $A$  apply to  $B$ , and are admitted since there is enough room for them (considering the students previously admitted).

This results in the following probabilities of a student to get their first or second choice:

$$\begin{aligned} V_1^{G_j, A} &= 1 - F_j^A(P^A), & V_1^{G_j, B} &= 1, \\ V_2^{G_j, A} &= F_j^A(P^A), & V_2^{G_j, B} &= 0, \\ P^A &= \left( \sum_j \gamma_j \beta_j (1 - F_j^A) \right)^{-1} (\alpha^A). \end{aligned}$$

Finally, as every student is admitted somewhere,  $V_\emptyset^{G_j} = 0$ .

(2) *There is not enough room in college  $B$  for all students preferring it to  $A$ , i.e.,  $\sum_j \gamma_j (1 - \beta_j) \geq \alpha^B$ .* Symmetrically we get

$$\begin{aligned} V_1^{G_j, A} &= 1, & V_1^{G_j, B} &= 1 - F_j^B(P^B), \\ V_2^{G_j, A} &= 0, & V_2^{G_j, B} &= F_j^B(P^B), \\ P^B &= \left( \sum_j \gamma_j (1 - \beta_j) (1 - F_j^B) \right)^{-1} (\alpha^B), & V_\emptyset^{G_j} &= 0. \end{aligned}$$

(3) *There is enough room in each college to admit all students who prefer attending it, i.e.,  $\sum_j \gamma_j \beta_j \leq \alpha^A$  and  $\sum_j \gamma_j (1 - \beta_j) \leq \alpha^B$ .* It follows that everyone gets their first choice: for  $j \in [K]$  and  $C \in \{A, B\}$ ,

$$\begin{aligned} V_1^{G_j, C} &= 1, \\ V_2^{G_j, C} &= V_\emptyset^{G_j} = 0. \end{aligned}$$

## Appendix B: Omitted proofs

### B.1. Proof of Lemma 1.

Since  $H_\theta(x, y) = \mathbb{P}(X < x, Y < y)$  by definition, then the first part of the lemma is just a rewriting of the definition of coherence. For the second part, we have

$$\begin{aligned} \mathbb{P}(X > x, Y > y) &= \mathbb{P}(Y > y) - \mathbb{P}(X < x, Y > y) \\ &= \mathbb{P}(Y > y) - \mathbb{P}(X < x) + \mathbb{P}(X < x, Y < y) \end{aligned}$$

and  $\mathbb{P}(Y > y), \mathbb{P}(X < x)$  are constant in  $\theta$  ( $H_\theta$  are copulas therefore they all have uniform marginals) while  $\mathbb{P}(X < x, Y < y)$  is increasing, so  $\mathbb{P}(X > x, Y > y)$  is also increasing. Finally, we also get that

$$\mathbb{P}(X > x, Y < y) = \mathbb{P}(Y < y) - \mathbb{P}(X < x, Y < y)$$

and

$$\mathbb{P}(X < x, Y > y) = \mathbb{P}(X < x) - \mathbb{P}(X < x, Y < y)$$

and both are thus decreasing. □

### B.2. Proof of Lemma 3.

Consider student  $s \in G_j$  who prefers college  $A$ . By Lemma 1 from Azevedo and Leshno (2016) (cf. Section 2.3),  $s$  is admitted to  $A$  if and only if  $s \in D_A(\mathbf{P})$ , i.e., if and only if their score at  $A$  is greater than  $P^A$ . Then by definition of  $\eta$ ,

$$V_1^{G_j,A}(\boldsymbol{\theta}) = \frac{\eta(\{s \in G_j : A \succ_s B, \mu(s) = A\})}{\gamma_j \beta_j} = \mathbb{P}_{j,\theta_j}((W^A, W^B) \in [P^A, +\infty) \times \mathbb{R}) = \mathbb{P}_j(W^A > P^A).$$

The same reasoning applies to  $V_1^{G_j,B}$  if  $s$  prefers  $B$ , which proves (2).

The same student  $s$  is admitted to  $B$  if and only if  $s \in D_B(P^A, P^B)$ , i.e., if and only if  $W_s^B \geq P^B$  and  $W_s^A < P^A$ . Then we have

$$V_2^{G_j,A}(\boldsymbol{\theta}) = \frac{\eta(\{s \in G_j : A \succ_s B, \mu(s) = B\})}{\gamma_j \beta_j} = \mathbb{P}_{j,\theta_j}((W^A, W^B) \in (-\infty, P^A) \times [P^B, +\infty)).$$

The same reasoning applies to  $V_2^{G_j,B}$ , which proves (3).

Student  $s$  remains unmatched if and only if  $W_s^A < P^A$  and  $W_s^B < P^B$ . Then we have

$$V_0^{G_j,A}(\boldsymbol{\theta}) = \frac{\eta(\{s \in G_j : A \succ_s B, \mu(s) = s\})}{\gamma_j \beta_j} = \mathbb{P}_{j,\theta_j}((W^A, W^B) \in (-\infty, P^A) \times (-\infty, P^B)).$$

which proves (4).  $\square$

### B.3. Proof of Lemma 4

It is sufficient to notice that Equation (4) is symmetric in  $C$  and  $\bar{C}$  to obtain the first part of the lemma. The second one follows from the fact that either there is excess capacity and everyone is matched, or both colleges are full and the mass of matched students is the sum of the capacities.  $\square$

REMARK 4. The first part of Lemma 4 could also be derived from the strategy-proofness for students of the student-proposing deferred acceptance algorithm (Roth 1985). Indeed, the fact that students cannot improve their outcome by modifying the order of their preferences implies that them being unmatched or not does not depend on which college they reported to be their first choice.

### B.4. Proof of Proposition 1

The result follows directly by applying (2) to both groups, and by observing that the integral in (2) depends on  $\theta$  only through the cutoff vector  $\mathbf{P}$ . Therefore, if groups  $G_i$  and  $G_j$  have the same marginal  $F_j^C$  at college  $C$ , then  $V_1^{G_i,C} = V_1^{G_j,C}$ .  $\square$

### B.5. Proof of Proposition 2

The value of  $V_1$  comes from Lemma 3. If  $\alpha^A + \alpha^B \geq 1$ , then all students are admitted to some college, therefore  $V_0^{G_j} = 0$  for all  $j$ . Moreover, either  $P^A = \min_j \underline{I}_j^A$  or  $P^B = \min_j \underline{I}_j^B$ , or both. Let us suppose the former holds. Then

$$\begin{aligned} V_1^{G_j,A} &= 1 - F_j^A(\min_j \underline{I}_j^A) = 1 \\ V_2^{G_j,A} &= 0 \end{aligned}$$

and

$$\begin{aligned} V_1^{G_j,B} &= 1 - F_j^B(P^B) \\ V_2^{G_j,B} &= F_j^B(P^B). \end{aligned}$$

It remains to show that  $P^B$  is constant in  $\theta$ . Define

$$H : x \in \mathbb{R} \mapsto \sum_j \gamma_j (1 - \beta_j) (1 - F_j^B(x)).$$

Note that since all  $F_j^B$  are invertible  $H$  is invertible too. Since all students preferring  $A$  get it, the market clearing equation for  $B$  becomes  $H(P^B) = \alpha^B$ , i.e.,  $P^B = H^{-1}(\alpha^B)$ . From this relation it appears that  $P^B$  is indeed constant in  $\theta$ . The same reasoning applies if  $P^A \neq \min_j \underline{I}_j^A$  and  $P^B = \min_j \underline{I}_j^B$ .  $\square$

### B.6. Proof of Lemma 5.

Assume  $\alpha^A + \alpha^B < 1$ , and  $\theta \in \overset{\circ}{\Theta}^K$ . Let  $P^A, P^B \in \mathbb{R}$  be the cutoffs of colleges  $A$  and  $B$ .

By definition of the quantities  $V_1$  and  $V_2$ , the market-clearing equation (1) can be written as

$$\begin{cases} \sum_{j \in [K]} (\gamma_j \beta_j V_1^{G_j, A} + \gamma_j (1 - \beta_j) V_2^{G_j, B}) = \alpha^A, \\ \sum_{j \in [K]} (\gamma_j \beta_j V_2^{G_j, A} + \gamma_j (1 - \beta_j) V_1^{G_j, B}) = \alpha^B. \end{cases}$$

Then, using Lemma 3, we can rewrite it as

$$\begin{cases} \sum_{j \in [K]} (\gamma_j \beta_j \mathbb{P}_j(W^A \geq P^A) + \gamma_j (1 - \beta_j) \mathbb{P}_{j, \theta_j}(W^A \geq P^A, W^B < P^B)) = \alpha^A, \\ \sum_{j \in [K]} (\gamma_j \beta_j \mathbb{P}_{j, \theta_j}(W^A < P^A, W^B \geq P^B) + \gamma_j (1 - \beta_j) \mathbb{P}_j(W^B \geq P^B)) = \alpha^B, \end{cases}$$

which is finally equivalent to

$$\begin{cases} \sum_{j \in [K]} \left( \gamma_j \beta_j \int_{P^A}^{\infty} f_j^A(x) dx + \gamma_j (1 - \beta_j) \int_{P^A}^{\infty} \int_{-\infty}^{P^B} f_{j, \theta_j}(x, y) dx dy \right) = \alpha^A, \\ \sum_{j \in [K]} \left( \gamma_j \beta_j \int_{-\infty}^{P^A} \int_{P^B}^{\infty} f_{j, \theta_j}(x, y) dx dy + \gamma_j (1 - \beta_j) \int_{P^B}^{\infty} f_j^B(x) dx \right) = \alpha^B. \end{cases} \quad (9)$$

We fix  $\theta$ , and we want to study how the solution  $(P^A, P^B)$  of the above equation varies as a function of  $\theta_j$  for some  $j \in [K]$ . Let us define  $Z : \mathbb{R}^2 \times \Theta \rightarrow \mathbb{R}^2$ ,  $(P^A, P^B, \theta_j) \mapsto (D_A(P^A, P^B) - \alpha^A, D_B(P^A, P^B) - \alpha^B)$ . (We will denote by  $Z_1$  and  $Z_2$  its two components.)

$$Z(P^A, P^B, \theta_j) = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \sum_{j \in [K]} \left( \gamma_j \beta_j \int_{P^A}^{\infty} f_j^A(x) dx + \gamma_j (1 - \beta_j) \int_{P^A}^{\infty} \int_{-\infty}^{P^B} f_{j, \theta_j}(x, y) dx dy \right) - \alpha^A \\ \sum_{j \in [K]} \left( \gamma_j \beta_j \int_{-\infty}^{P^A} \int_{P^B}^{\infty} f_{j, \theta_j}(x, y) dx dy + \gamma_j (1 - \beta_j) \int_{P^B}^{\infty} f_j^B(x) dx \right) - \alpha^B \end{pmatrix} \quad (10)$$

Then for each  $\theta_j \in \Theta$ ,  $(P^A, P^B)$  is the solution of the equation  $Z(P^A, P^B, \theta_j) = (0, 0)$ . In order to show that  $P^A$  and  $P^B$  are decreasing in  $\theta_j$ , we wish to apply the implicit function theorem. Let  $P^A, P^B \in \mathbb{R}$  and  $\theta_j \in \Theta$  such that  $Z(P^A, P^B, \theta_j) = 0$ . Function  $Z$  is of class  $C^1$ . We first verify that the partial Jacobian  $J_{Z, (P^A, P^B)}(P^A, P^B, \theta_j)$  is invertible, where

$$J_{Z, (P^A, P^B)}(P^A, P^B, \theta_j) = \begin{pmatrix} \frac{\partial Z_1}{\partial P^A} & \frac{\partial Z_1}{\partial P^B} \\ \frac{\partial Z_2}{\partial P^A} & \frac{\partial Z_2}{\partial P^B} \end{pmatrix}. \quad (11)$$

To show that the determinant  $\frac{\partial Z_1}{\partial P^A} \frac{\partial Z_2}{\partial P^B} - \frac{\partial Z_1}{\partial P^B} \frac{\partial Z_2}{\partial P^A} \neq 0$ , we will show that it is in fact strictly positive. From (9), it is clear that  $Z_1$  is decreasing in  $P^A$  and increasing in  $P^B$ , and that  $Z_2$  is increasing in  $P^A$  and decreasing in  $P^B$ . Therefore, to prove that  $\frac{\partial Z_1}{\partial P^A} \frac{\partial Z_2}{\partial P^B} - \frac{\partial Z_1}{\partial P^B} \frac{\partial Z_2}{\partial P^A} > 0$ , we only need to prove that  $\left| \frac{\partial Z_1}{\partial P^A} \right| > \frac{\partial Z_2}{\partial P^A}$  and  $\left| \frac{\partial Z_2}{\partial P^B} \right| > \frac{\partial Z_1}{\partial P^B}$ .

By symmetry, we will only prove the first one. We can compute each term separately:

$$\begin{aligned} \frac{\partial Z_1}{\partial P^A} &= \sum_{j \in [K]} \left( \gamma_j \beta_j \frac{\partial \mathbb{P}_j(W^A \geq P^A)}{\partial P^A} + \gamma_j (1 - \beta_j) \frac{\partial \mathbb{P}_{j, \theta_j}(W^A \geq P^A, W^B < P^B)}{\partial P^A} \right), \\ \frac{\partial Z_2}{\partial P^A} &= \sum_{j \in [K]} \gamma_j \beta_j \frac{\partial \mathbb{P}_{j, \theta_j}(W^A < P^A, W^B \geq P^B)}{\partial P^A}. \end{aligned}$$

All terms of  $Z_1$  are decreasing in  $P^A$  and all terms of  $Z_2$  are increasing in  $P^A$ , therefore we can proceed term by term:

$$\begin{aligned} \left| \gamma_j \beta_j \frac{\partial \mathbb{P}_j(W^A \geq P^A)}{\partial P^A} \right| &= \gamma_j \beta_j \frac{\partial \mathbb{P}_j(W^A < P^A)}{\partial P^A}, \\ &= \gamma_j \beta_j \left( \frac{\partial \mathbb{P}_{j, \theta_j}(W^A < P^A, W^B < P^B)}{\partial P^A} + \frac{\partial \mathbb{P}_{j, \theta_j}(W^A < P^A, W^B \geq P^B)}{\partial P^A} \right), \\ &> \gamma_j \beta_j \frac{\partial \mathbb{P}_{j, \theta_j}(W^A < P^A, W^B \geq P^B)}{\partial P^A}. \end{aligned} \quad (12)$$

We conclude that  $\left| \frac{\partial Z_1}{\partial P^A} \right| > \frac{\partial Z_2}{\partial P^A}$ , and similarly  $\left| \frac{\partial Z_2}{\partial P^B} \right| > \frac{\partial Z_1}{\partial P^B}$ . Therefore the Jacobian in (11) has positive determinant and is invertible.

By the implicit function theorem, there exists a neighborhood  $U$  of  $(P^A, P^B)$ , a neighborhood  $V$  of  $\theta_j$ , and a function  $\psi : V \rightarrow U$  such that for all  $(x, y) \in \mathbb{R}^2$ ,  $\theta \in \Theta$ ,

$$((x, y, \theta) \in U \times V \text{ and } Z(x, y, \theta) = 0) \Leftrightarrow (\theta \in V \text{ and } (x, y) = \psi(\theta)).$$

In particular,  $(P^A, P^B) = \psi(\theta_j)$ , and we can compute the derivative of  $\psi$ :

$$\begin{aligned} J_\psi(\theta_j) &= -J_{Z, (P^A, P^B)}(P^A, P^B, \theta_j)^{-1} J_{Z, \theta_j}(P^A, P^B, \theta_j), \\ &= \frac{-1}{\frac{\partial Z_1}{\partial P^A} \frac{\partial Z_2}{\partial P^B} - \frac{\partial Z_1}{\partial P^B} \frac{\partial Z_2}{\partial P^A}} \begin{pmatrix} \frac{\partial Z_2}{\partial P^B} & -\frac{\partial Z_1}{\partial P^B} \\ -\frac{\partial Z_2}{\partial P^A} & \frac{\partial Z_1}{\partial P^A} \end{pmatrix} \begin{pmatrix} \frac{\partial Z_1}{\partial \theta_j} \\ \frac{\partial Z_2}{\partial \theta_j} \end{pmatrix}, \\ &= \frac{-1}{\frac{\partial Z_1}{\partial P^A} \frac{\partial Z_2}{\partial P^B} - \frac{\partial Z_1}{\partial P^B} \frac{\partial Z_2}{\partial P^A}} \begin{pmatrix} \frac{\partial Z_2}{\partial P^B} \frac{\partial Z_1}{\partial \theta_j} - \frac{\partial Z_1}{\partial P^B} \frac{\partial Z_2}{\partial \theta_j} \\ -\frac{\partial Z_2}{\partial P^A} \frac{\partial Z_1}{\partial \theta_j} + \frac{\partial Z_1}{\partial P^A} \frac{\partial Z_2}{\partial \theta_j} \end{pmatrix}. \end{aligned} \quad (13)$$

We only need to know the sign of each term to conclude about the variations of  $\psi$ . We already know the sign of the derivatives in  $P^A$  and  $P^B$ , so we only need those in  $\theta_j$ . The terms of  $Z_1$  that depend on  $\theta_j$  are

$\sum_{j \in [K]} \gamma_j (1 - \beta_j) \mathbb{P}_{j, \theta_j}(W^A \geq P^A, W^B < P^B)$ . By Lemma 1,  $\mathbb{P}_{j, \theta_j}(W^A \geq P^A, W^B < P^B)$  is decreasing in  $\theta_j$ , and thus  $\frac{\partial Z_1}{\partial \theta_j} < 0$ . By the same argument,  $\frac{\partial Z_2}{\partial \theta_j}$  is also negative. (Note that here the implicit functions theorem requires that we compute the partial derivatives of  $Z$  as if  $P^A$  and  $P^B$  were not functions of  $\theta_j$ .)



If we replace each term of Equation (13) by its signs, we get

$$-\frac{1}{+} \begin{pmatrix} (- \times -) - (+ \times -) \\ - (+ \times -) + (- \times -) \end{pmatrix} = \begin{pmatrix} - \\ - \end{pmatrix}.$$

We conclude that  $\psi$  and therefore  $P^A$  and  $P^B$  are decreasing in  $\theta_j$ .

Note that we require  $\theta_j \in \overset{\circ}{\Theta}$  because if one of the  $\theta_j$  is maximal, i.e., the distribution is fully correlated ( $W^B$  is a deterministic function of  $W^A$ ), then the  $V_1$  metrics are not differentiable at this point. However, they are continuous, therefore they are increasing on the whole interval  $\Theta$ . Moreover, if the distribution is not fully correlated when  $\theta$  is maximal, then we can replace  $\overset{\circ}{\Theta}$  by  $\Theta$  in the statement of the lemma.  $\square$

### B.7. Proof of Corollary 1.

By Lemma 5,  $P^A$  and  $P^B$  are decreasing in both  $\theta_1$  and  $\theta_2$ , thus for  $i \neq j \in [K]$ :

$$\begin{aligned} \frac{dV_\theta^{G_j}}{d\theta_i} &= \frac{d\mathbb{P}_{j,\theta_j}(W^A < P^A, W^B < P^B)}{d\theta_i} \\ &= \left( \frac{\partial \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B < P^B)}{dP^A}, \frac{\partial \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B < P^B)}{dP^B} \right) \cdot \begin{pmatrix} dP^A \\ d\theta_i \end{pmatrix}, \begin{pmatrix} dP^B \\ d\theta_i \end{pmatrix}^T \\ &< 0. \end{aligned}$$

Since the total capacity (of the two colleges) is constant, the mass of unmatched student must also be constant. Therefore, we have

$$\gamma_i V_\theta^{G_i} + \sum_{j \neq i} \gamma_j V_\theta^{G_j} = 1 - \alpha^A - \alpha^B.$$

By differentiating this equation we get

$$\begin{aligned} \gamma_i \frac{dV_\theta^{G_i}}{d\theta_i} + \sum_{j \neq i} \gamma_j \frac{dV_\theta^{G_j}}{d\theta_i} &= 0 \\ \Leftrightarrow \frac{dV_\theta^{G_i}}{d\theta_i} &= -\frac{1}{\gamma_i} \sum_{j \neq i} \gamma_j \frac{dV_\theta^{G_j}}{d\theta_i} \\ \Rightarrow \frac{dV_\theta^{G_i}}{d\theta_i} &> 0 \end{aligned}$$

which proves the first part of Corollary 1. Moreover, since  $L(G_i, G_j) = |V_\theta^{G_i} - V_\theta^{G_j}|$ , assume that  $V_\theta^{G_i} < V_\theta^{G_j}$ , then  $L(G_i, G_j) = V_\theta^{G_j} - V_\theta^{G_i}$ , and using the first part of the result we get that  $\frac{dL(G_i, G_j)}{d\theta_i} < 0$  and  $\frac{dL(G_i, G_j)}{d\theta_j} > 0$ , which proves the second part.  $\square$

### B.8. Proof of Proposition 3.

$V_1$  is a convex combination of the first choice functions that are increasing in all  $\theta_j$ . Moreover it is continuous, and we assumed  $\Theta$  to be an interval, so the set of possible values for  $V_1$  is an interval, say  $[V_1^{\min}, V_1^{\max}]$ . Fix  $V \in (V_1^{\min}, V_1^{\max})$ , and consider the solutions of the equation  $V(\boldsymbol{\theta}) = V$ . By continuity, this equation has a solution. The implicit function theorem applied to express some  $\theta_j$  (the choice of  $j$  does not matter) as a function  $\phi$  of all the other elements of  $\boldsymbol{\theta}$  shows that the solutions of  $V(\boldsymbol{\theta}) = V$  is a connected subset of  $\Theta^K$ ,

and also an hypersurface because the function  $\phi$  is monotonous in all  $\theta_i$  (this comes from the fact that  $V_1$  is itself monotonous). This proves the first part of the proposition.

Let us choose two groups  $G_i, G_j$ , and fix all  $\theta_\ell$  for  $\ell \neq i, j$ . We apply the implicit function theorem to express  $\theta_j$  as a function of  $\theta_i$ , which shows that there exists an interval  $U := [\underline{\theta}, \bar{\theta}] \subseteq \Theta$  and a differentiable function  $\phi : U \rightarrow \Theta$  such that  $(\theta_i \in U \text{ and } \theta_j = \phi(\theta_i)) \implies V_1(\boldsymbol{\theta}) = V$ . Since  $V(\boldsymbol{\theta})$  is increasing in all arguments,  $\phi$  is necessarily decreasing. If we keep  $\theta_j = \phi(\theta_i)$ , then  $\frac{dV_\theta^{G_i}}{d\theta_i} = \frac{\partial V_\theta^{G_i}}{\partial \theta_i} + \frac{\partial V_\theta^{G_i}}{\partial \theta_j} \phi'(\theta_i)$ , which is positive by Corollary 1, so  $(\theta_i, \theta_j) = (\underline{\theta}, \phi(\underline{\theta}))$  minimizes  $V_\theta^{G_i}$ , and  $(\theta_i, \theta_j) = (\bar{\theta}, \phi(\bar{\theta}))$  maximizes it. The same reasoning shows that those two points respectively maximize and minimize  $V_\theta^{G_j}$ . Finally, since  $V_\theta^{G_i}$  is increasing and  $V_\theta^{G_j}$  decreasing,  $L(G_i, G_j) = |V_\theta^{G_i} - V_\theta^{G_j}|$  has a unique local (and therefore global) minimum.  $\square$

### B.9. Proof of Proposition 4.

We start by building a distribution family that can represent both STB and MTB for two values of the parameter. The priority classes are  $Q_1^A, \dots, Q_{n_A}^A$  and  $Q_1^B, \dots, Q_{n_B}^B$ , and we denote by  $\kappa_C^j = \eta(Q_C^j)$  the mass of students inside class  $j$  of college  $C$ . Let  $a_0 = 0, a_1 = \kappa_A^1, a_2 = \kappa_A^1 + \kappa_A^2, \dots, a_{n_A} = 1$ , such that they form a partition of  $[0, 1]$  with the  $j$ -th segment having length  $\kappa_A^j$ . Define  $b_0, \dots, b_{n_B}$  similarly. Finally, for any  $i \leq n_A, j \leq n_B$ , let  $\kappa^{i,j} = \eta(Q_i^A \times Q_j^B)$  be the mass of students belonging to class  $i$  for  $A$  and class  $j$  for  $B$ .

Let  $\phi_\theta$  be the pdf of the Gaussian copula with uniform marginals on  $[0, 1]^2$  and covariance  $\theta$ . For any  $\theta \in [-1, 1]$ , let  $f_\theta : [-1, 1]^2 \rightarrow \mathbb{R}$  be defined as:

$$f_\theta(x, y) = \kappa^{i,j} \phi_\theta\left(\frac{x - a_{i-1}}{\kappa_A^i}, \frac{y - b_{j-1}}{\kappa_B^j}\right) \text{ with } a_{i-1} \leq x \leq a_i, b_{j-1} \leq y \leq b_j$$

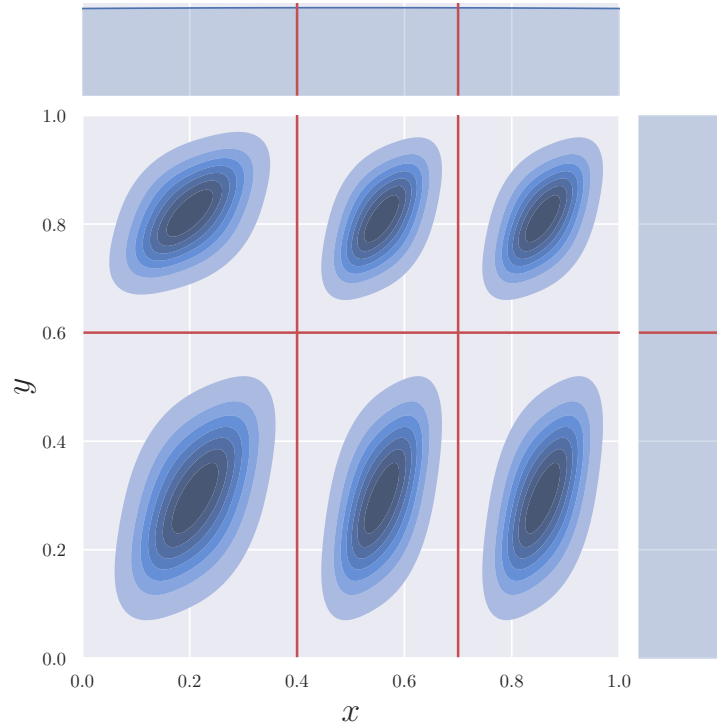
Defined this way,  $f_\theta$  is a pdf since it is non-negative and has integral 1. The marginals are uniform and do not depend on  $\theta$ . Moreover, the integral of  $f_\theta$  over each rectangle  $Q_i^A \times Q_j^B$  is  $\kappa^{i,j}$ , and each rectangle contains a ‘‘copy’’ of the Gaussian copula adjusted to its dimensions. There is no ‘‘spill’’ between classes: if student  $s$  is in a higher priority class at college  $C$  than student  $s'$ , then  $s$  will have a higher score with probability 1. If for all  $i, j$ ,  $\kappa^{i,j} > 0$ , and  $\theta \notin \{-1, 1\}$ , then  $f_\theta$  has full support. This distribution is depicted in Figure 6.

We can verify that this definition recovers MTB and STB: if  $\theta = 0$ , if two students are in the same priority class for a college, they have the same ex-ante probability of getting a seat there, and if they also are in the same priority class for the other college (i.e., they are in the same rectangle  $Q_i^A \times Q_j^B$ ), the result of this second tie-breaking is independent from the first one. When  $\theta = 1$ , if two students are in the same priority class for a college, have the same ex-ante probability of getting a seat there, but if they also are in the same priority class for the other college (i.e., they are in the same rectangle  $Q_i^A \times Q_j^B$ ), the winner of the tie-breaking is the same as for the first college since scores inside the rectangle are perfectly correlated. In that case, the distribution does not have full support but this is not an issue as explained in Remark 1. Therefore MTB is the case  $\theta = 0$  and STB  $\theta = 1$ .

Let us now prove the proposition:

1.  $\bullet$  The family  $(f_\theta)_{\theta \in [-1, 1]}$  is differentiable by differentiability of the Gaussian copula. It is also coherent (except for the  $(x, y)$  such that  $x = a_i$  or  $y = b_j$ , i.e., on the sides of rectangles, in which case the cdf is constant and not increasing). Therefore by applying Theorem 1,  $E$  is either increasing or constant.

**Figure 6** Illustration of the distribution  $f_\theta$  with three priority classes at  $A$  (30% of applicants in the first class, 30% in the second, 40% in the third), two priority classes at  $B$  (40% in the first class, 60% in the second), and correlation  $\theta = 0.5$ .



- Moreover, the case where it could be constant can only happen if there are several priority classes, so if there is only one it is strictly increasing.

- Let us look into the multiple priority classes case. Suppose that  $\exists \theta \in \Theta$  such that  $P^A(\theta) \neq a_i$  and  $P^B(\theta) \neq b_j$  for all  $i, j$ . We can then apply Theorem 1, and deduce that  $V_1^A$  and  $V_1^B$  are increasing in all  $\theta_j$ . If there exists no such  $\theta$ , it implies that  $P^A, P^B$  are constant in  $\theta$  and so are  $V_1^A$  and  $V_1^B$ . However, as any perturbation of either  $\gamma, \beta$  or  $\alpha$  would change the cutoffs, and resolve the issue, the set of problematic values of  $(\gamma, \beta, \alpha)$  has Lebesgue measure 0.

2. Finally, Corollary 1 can be applied with the same adjustments, which gives the fourth point.

## References

- Abdulkadiroğlu A (2005) College admissions with affirmative action. *International Journal of Game Theory* 33(4):535–549.
- Abdulkadiroğlu A, Che YK, Yasuda Y (2015) Expanding “Choice” in School Choice. *American Economic Journal: Microeconomics* 7(1):1–42.
- Abdulkadiroğlu A, Pathak PA, Roth AE (2009) Strategy-Proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match. *American Economic Review* 99(5):1954–1978.
- Abdulkadiroğlu A, Sönmez T (2003) School Choice: A Mechanism Design Approach. *American Economic Review* 93(3):729–747.

- Arcidiacono P, Kinsler J, Ransom T (2022) Asian american discrimination in harvard admissions. *European Economic Review* 144:104079.
- Arnosti N (2022) A continuum model of stable matching with finite capacities. *Proceedings of the 23rd ACM Conference on Economics and Computation*, 960.
- Arnosti N (2023) Lottery design for school choice. *Management Science* 69(1):244–259.
- Ashlagi I, Nikzad A (2020) What matters in school choice tie-breaking? how competition guides design. *Journal of Economic Theory* 190:105120.
- Ashlagi I, Nikzad A, Romm A (2019) Assigning more students to their top choices: A comparison of tie-breaking rules. *Games and Economic Behavior* 115:167–187.
- Azevedo EM, Leshno JD (2016) A Supply and Demand Framework for Two-Sided Matching Markets. *Journal of Political Economy* 124(5):1235–1268.
- Balinski M, Sonmez T (1999) A Tale of Two Mechanisms: Student Placement. *Journal of Economic Theory* 84(1):73–94.
- Bertrand M, Mullainathan S (2004) Are Emily and Greg more employable than Lakisha and Jamal? A field experiment on labor market discrimination. *American Economic Review* 94(4):991–1013.
- Bohren JA, Hull P, Imas A (2022) Systemic discrimination: Theory and measurement. National Bureau of Economic Research working paper.
- Bommasani R, Creel KA, Kumar A, Jurafsky D, Liang PS (2022) Picking on the same person: Does algorithmic monoculture lead to outcome homogenization? Koyejo S, Mohamed S, Agarwal A, Belgrave D, Cho K, Oh A, eds., *Advances in Neural Information Processing Systems*, volume 35, 3663–3678 (Curran Associates, Inc.).
- Brilliantova A, Hosseini H (2022) Fair stable matching meets correlated preferences. *CoRR* abs/2201.12484.
- Celis LE, Mehrotra A, Vishnoi NK (2020) Interventions for ranking in the presence of implicit bias. *Proceedings of the 2020 Conference on Fairness, Accountability, and Transparency (FAT\* '20)*.
- Chade H, Lewis G, Smith L (2014) Student Portfolios and the College Admissions Problem. *The Review of Economic Studies* 81(3):971–1002.
- Che YK, Tercieux O (2019) Efficiency and stability in large matching markets. *Journal of Political Economy* 127(5):2301–2342.
- Correa J, Epstein N, Epstein R, Escobar J, Rios I, Aramayo N, Bahamondes B, Bonet C, Castillo M, Cristi A, Epstein B, Subiabre F (2022) School choice in Chile. *Operations Research* 70(2):1066–1087.
- Delacrétaz D, Kominers SD, Teytelboym A (2023) Matching mechanisms for refugee resettlement. *American Economic Review* 113(10):2689–2717.
- Devic S, Kempe D, Sharan V, Korolova A (2023) Fairness in matching under uncertainty. *Proceedings of the 40th International Conference on Machine Learning, ICML'23*.

- Dur U, Pathak PA, Sönmez T (2020) Explicit vs. statistical targeting in affirmative action: Theory and evidence from Chicago's exam schools. *Journal of Economic Theory* 187:104996.
- Emelianov V, Gast N, Gummadi KP, Loiseau P (2020) On fair selection in the presence of implicit variance. *Proceedings of the 21st ACM Conference on Economics and Computation (EC)*.
- Emelianov V, Gast N, Gummadi KP, Loiseau P (2022) On fair selection in the presence of implicit and differential variance. *Artificial Intelligence* 302:103609.
- Erdil A, Ergin H (2008) What's the matter with tie-breaking? improving efficiency in school choice. *American Economic Review* 98(3):669–89.
- Ergin H, Sönmez T (2006) Games of school choice under the Boston mechanism. *Journal of Public Economics* 90(1-2):215–237.
- Feagin J (2013) *Systemic racism: A theory of oppression* (Routledge).
- Gale D, Shapley LS (1962) College Admissions and the Stability of Marriage. *The American Mathematical Monthly* 69(1):9–15.
- Garg N, Li H, Monachou F (2021) Standardized tests and affirmative action: The role of bias and variance. *Proceedings of the 2021 ACM Conference on Fairness, Accountability, and Transparency (FAccT '21)*.
- Gola P (2021) Supply and demand in a two-sector matching model. *Journal of Political Economy* 129(3):940–978.
- Kamada Y, Kojima F (2015) Efficient Matching under Distributional Constraints: Theory and Applications. *American Economic Review* 105(1):67–99.
- Kamada Y, Kojima F (2023) Fair Matching under Constraints: Theory and Applications. *The Review of Economic Studies* rdad046.
- Karni G, Rothblum GN, Yona G (2022) On Fairness and Stability in Two-Sided Matchings. *13th Innovations in Theoretical Computer Science Conference (ITCS 2022)*, 92:1–92:17, Leibniz International Proceedings in Informatics (LIPIcs).
- Kleinberg J, Raghavan M (2018) Selection Problems in the Presence of Implicit Bias. *Proceedings of the 9th Innovations in Theoretical Computer Science Conference (ITCS '18)*.
- Kleinberg J, Raghavan M (2021) Algorithmic monoculture and social welfare. *Proceedings of the National Academy of Sciences* 118(22).
- Krishnaa P, Limaye G, Nasre M, Nimbhorkar P (2022) Envy-freeness and relaxed stability: hardness and approximation algorithms. *Journal of Combinatorial Optimization* 45.
- Leshno JD, Lo I (2020) The Cutoff Structure of Top Trading Cycles in School Choice. *The Review of Economic Studies* 88(4):1582–1623.
- Longhofer SD (1995) Rooting out discrimination in home mortgage lending. *Economic Commentary, Federal Reserve Bank of Cleveland* (Nov).

- 
- Mello U (2022) Centralized admissions, affirmative action, and access of low-income students to higher education. *American Economic Journal: Economic Policy* 14(3):166–97.
- Niessen-Ruenzi A, Ruenzi S (2019) Sex matters: Gender bias in the mutual fund industry. *Management Science* 65(7):3001–3025.
- Peng K, Garg N (2023) Monoculture in matching markets.
- Peng K, Garg N (2024) Wisdom and foolishness of noisy matching markets.
- Phelps E (1972) The statistical theory of racism and sexism. *American Economic Review* 62(4):659–661.
- Pincus FL (1996) Discrimination comes in many forms: Individual, institutional, and structural. *American Behavioral Scientist* 40(2):186–194.
- Roth AE (1985) The college admissions problem is not equivalent to the marriage problem. *Journal of Economic Theory* 36(2):277–288.
- Roth AE, Vande Vate H (1990) Random paths to stability in two-sided matching. *Econometrica* 58:1475–1480.
- Scarsini M (1984) On measures of concordance. *Stochastica* 8(3):201–218.
- Shapley L, Scarf H (1974) On cores and indivisibility. *Journal of Mathematical Economics* 1(1):23–37.
- Sklar M (1959) Fonctions de répartition à N dimensions et leurs marges. *Annales de l'ISUP* VIII(3):229–231.
- Yang K, Loftus JR, Stoyanovich J (2021) Causal intersectionality for fair ranking. *Proceedings of the Symposium on Foundations of Responsible Computing (FORC 2021)*.
- Yenmez MB (2013) Incentive-Compatible Matching Mechanisms: Consistency with Various Stability Notions. *American Economic Journal: Microeconomics* 5(4):120–41.
- Zehlike M, Yang K, Stoyanovich J (2022) Fairness in ranking, part i: Score-based ranking. *ACM Comput. Surv.* 55(6).