Correlation of Rankings in Matching Markets

Rémi Castera, Patrick Loiseau, Bary Pradelski

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Rémi Castera[‡,1], Patrick Loiseau[2], and Bary S.R. Pradelski[3]

[1]Univ. Grenoble Alpes, Inria, CNRS, Grenoble INP, LIG, Grenoble, France
[2]Inria, FairPlay Team, Palaiseau, France
[3]CNRS, Maison Française d'Oxford and Department of Economics, University of Oxford, UK

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Abstract

We study the role of correlation in matching, where multiple decision-makers are simultaneously facing selection problems from the same pool of candidates. We propose a model where decision-makers have varying information on candidates from different socio-demographic groups when evaluating and ranking them, thus leading to varying correlation of a candidate's priority scores. Such *differential correlation* arises, for example, when cost of information acquisition, decision-maker preferences, or prevalence of selection criteria vary across socio-demographic groups. We show that lower correlation for one of the groups worsens the outcome for all groups, thus leading to efficiency loss. Moreover, the proportion of students from a given group remaining unmatched is increasing in its own correlation level. This implies that it is advantageous to belong to the low-correlation group. Finally, we extend results from the tie-breaking literature to multiple priority classes and intermediate levels of correlation. Overall, our results point to a previously overlooked systemic source of group inequalities in school, university, and job admissions.

1 Introduction

Outcome inequalities for different demographic or social groups are ubiquitous, for example, in college admission, job assignment, or investment allocation. Arcidiacono et al. (2022) find that Asian-American applicants have lower admission chances at Harvard than white applicants for a similar academic record, Niessen-Ruenzi and Ruenzi (2019) find significantly lower inflows in female-managed mutual funds than in male-managed mutual funds, and Bertrand and Mullainathan (2004) find race-based discrimination in callback decisions by job advertisers. Consequently, the sources of observed outcome inequalities remain the subject of frequent and continued controversy and political debate. A common concern is that the causes of outcome inequalities in matching markets are not well understood, making them difficult to address (Longhofer, 1995).

We study how different ranking correlation between different socio-demographic groups—*differential correlation* – affects outcome inequality and efficiency in matching markets. Our findings point to a before unacknowledged source of inequity between different groups that is specific to matching markets and should be included in future assessments of, for example, school, university, and job admissions. In particular, we find that differential correlation across groups leads to outcome inequalities even when the rankings by each college are fair, i.e., all groups are represented in each college's ranking as they are in the total applicant population. The resulting inequity is a form of systemic discrimination, i.e., discrimination that only arises through the interaction of decision-makers—via the matching mechanism—and is not due to either intentional or non-intentional discrimination by single decision-makers (cf. Pincus 1996; Feagin 2013; Bohren et al. 2022).

Differential correlation arises when different decision-makers—say colleges—use different information on candidates from different (socio-demographic) groups—say students—when assigning priority scores to rank and admit them. This might be the case for several reasons.

First, the cost of information acquisition may vary across groups and some information may not be available at all for some groups. Consider decentralized college admissions to competitive PhD programs and compare foreign and local candidates. Several U.S. programs virtually reserve one seat for the best foreign candidate from a given foreign school based on test scores and—due to the high cost—do not perform in-person interviews (e.g., Iran's Sharif University of Technology or India's IIT). Consequently, the priorities of foreign candidates at different programs are highly correlated. By contrast, for local candidates grades from undergraduate alone may not provide sufficient information—also due to grade inflation—and universities rely on idiosyncratic signals, such as reference letters, extracurricular activities, or interviews during campus visits (which are less costly as only local travel is required). Consequently, the priorities of local students, while correlated, are less correlated than for foreign students.

Second, differences in correlation may also arise if colleges are looking for different attributes in candidates and proxies of these attributes are more or less correlated for different groups. Consider two colleges, one admitting students for mathematics, the other admitting students for physics.

Suppose that there are two groups; one group of students come from high schools where physics is taught in a theoretical manner and the other group come from high schools where physics is taught in an experimental manner and thus less mathematical. Consequently, the students from the former group will exhibit higher correlation in their high school grades in mathematics and physics than the students from the latter group.

Third, differences in correlation may also arise when colleges use selection criteria that are more or less prevalent within different groups. Concretely, such criteria could include diversity with respect to the current student body, sibling priority, or proximity. Notably, such criteria are also used in centralized school choice mechanisms, as, for example, in Chile (Mello, 2022). Consequently, a group for which the criteria are more prevalent will exhibit higher correlation than a group for which the criteria are less prevalent.

The latter criteria—sibling priority and proximity—along others are commonly used to break ties. This allows to make a mechanism more explainable and only use random tie-breaking for a smaller number of students (for whom all criteria are the same). Such random tie-breaking has been studied in school choice problems (Abdulkadiroğlu et al., 2015; Ashlagi et al., 2019; Arnosti, 2023). When students have the same ranking at a given college a tie-breaking rule describes who should receive priority. Two natural choices are that each college breaks ties independently or colleges use a common order to break ties. Intuitively, the former leads to 0 correlation and the latter to correlation 1 (among those students for whom tie-breaking is required). As we shall see, our work also allows to extend the known theoretical results on tie-breaking to accommodate intermediate correlation levels, as often present in practice.

1.1 Our contribution

We study the college-admissions problem, where multiple decision-makers select a subset of applicants from an applicant pool with stability as the solution concept (Gale and Shapley, 1962; Azevedo and Leshno, 2016). Concretely, suppose that an infinite population of students divided into groups $G_1 \ldots G_K$ apply to colleges A and B. Groups represent, for example, protected attributes, such as gender or race. Each college assigns a priority score to each student. A given student gets priority scores W^A at college A and W^B at college B. We propose an original model for the distributions of these scores, to study the correlation between the rankings made by the different colleges.

To formalize correlation and thus capture the vague notion of "a connection between two things in which one thing changes as the other does",[1] we leverage prior work on copulas and their relation with classical notions via *coherence*. This allows us to model correlation without a specific functional form and, in particular, nest classical notions as special cases, e.g., Spearman's and Kendall's correlation indices. With this at hand, we assume that the correlation between priority scores at different colleges depends on a candidate's group—we call this feature *differential correlation*.

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[1] Oxford Advanced Learner's Dictionary, 2023

How does differential correlation impact a stable matching's efficiency, *i.e., the number of students getting their first choice and* inequality, *i.e., the difference between groups in their probability to remain unmatched?* To answer this question, we first consider comparative statics. We show that efficiency is increasing in each group's correlation level, i.e., increasing the correlation level of any group increases the amount of students getting their first choice in all groups (Theorem 1). Moreover, the proportion of students from a given group remaining unmatched is increasing in its own correlation level and decreasing in the correlation level of all other groups (Corollary 1). This implies that it is advantageous to belong to the low-correlation group. We then show that a given efficiency level can be reached by a continuum of different correlation vectors, yielding different levels of inequality, thus showing that efficiency differences cannot explain inequality (Proposition 3). Finally, our results imply extensions of known results on tie-breaking (cf. Ashlagi et al. 2019, Ashlagi and Nikzad 2020, and Arnosti 2023), in particular to multiple priority classes and intermediate levels of correlation (Proposition 4).

1.2 Related literature

Matching.   The college admission problem, i.e, how to assign prospective students to colleges given each student's preferences and colleges' priorities over students and capacities such that the outcome is stable, was introduced by Gale and Shapley (1962). A variant of this model where colleges do not have priorities over students is commonly called the school choice problem, and has been investigated in (Balinski and Sonmez 1999, Abdulkadiroğlu and Sönmez 2003, Abdulkadiroğlu 2005, Ergin and Sönmez 2006, Yenmez 2013). The idea of considering a continuum of students and a finite number of colleges has previously been exploited due to its analytical tractability (cf., Chade et al. 2014, Abdulkadiroğlu et al. 2015, Azevedo and Leshno 2016, Arnosti 2022). We follow Azevedo and Leshno (2016) who develop a supply and demand framework allowing to easily analyze the quality of a matching and deriving comparative statics.

Matching with correlated types.   We study matching in the presence of correlation between the priority scores given by each college to a given student. A special case of this problem has been studied for centralized school choice problems, where a lot of students have the same priority and therefore ties are broken at random.[2] Ashlagi et al. (2019); Ashlagi and Nikzad (2020), and Arnosti (2023) compare the welfare of students in two settings: either one common lottery is used by all colleges, or all colleges draw independent lotteries. In our model, this corresponds to correlation 1 or 0, respectively, and our results nest elements of these prior papers. Another line of work has considered correlation between other features, e.g., correlation between students' preferences and colleges' rankings. In this context, Brilliantova and Hosseini (2022) focus on one-to-one matching

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[2]The implications of this feature on students' welfare have been studied by Erdil and Ergin (2008); Abdulkadiroğlu et al. (2009) and Abdulkadiroğlu et al. (2015).

and identify a matching that does not favor one side over the other, while Che and Tercieux (2019) and Leshno and Lo (2020) study the stability-efficiency trade-off by comparing Deferred Acceptance and Top Trading Cycles (see Shapley and Scarf 1974), and how the magnitude of this trade-off depends on the correlation between agents preferences and priorities. Considering a transferable utility model, Gola (2021) studies how workers sort into two competing sectors (such that their wages are maximized) and the impact of technological change. While their model and analysis is quite distant to ours, it shares the use of copulas to model correlation and the necessary restriction to two sectors, respectively colleges.

Fairness. The computer science literature on fairness in selection problems was initiated by Kleinberg and Raghavan (2018) who study the effect of bias and the efficiency of affirmative action policies.[3] Emelianov et al. (2020, 2022) and Garg et al. (2021) study statistical discrimination. Candidates have a latent quality, and a college or company they apply to only has access to a biased and/or noisy estimator of this quality that varies dependent on their group—*differential variance*. We depart from those models by considering several decision-makers instead of one—that is, we consider the matching problem instead of the selection problem. Works on fairness in matching have considered various affirmative action policies, including upper and lower quotas, to reduce discrimination (Abdulkadiroğlu 2005; Kamada and Kojima 2015, 2023; Delacrétaz et al. 2023; Krishnaa et al. 2022; Dur et al. 2020). These works, however, focus on finding stable matchings under some constraints, accounting for different fairness notions. In contrast, we aim to explain outcome inequalities that naturally occur in stable matchings without constraints.

All of the aforementioned works consider what is generally termed *group fairness*, that is, all (relevant) groups should be treated similarly. By contrast, *individual fairness* posits that individuals with similar characteristics should be treated similarly. In this latter spirit Karni et al. (2022) broke new ground, showing that an individually fair ranking does not necessarily lead to an individually fair matching. This conclusion can also be drawn from our results in the context of group fairness. Devic et al. (2023) also consider individual fairness and adapt the classical notion to incorporate agents' preferences. In our language, they require that similar agents should be matched to a college in a similar position in their respective preference list. Our work is concerned with group fairness, but note that if the quality of applicants is similar in each group, and the notion of Devic et al. (2023)'s individual fairness is fulfilled, then group inequalities are automatically mitigated. In this sense, their notion is stricter.

Algorithmic monoculture. Finally, our work also contributes to a recent literature on algorithmic monoculture, i.e., the fact that recommendations, choices, and preferences become homogeneous with the rise of algorithmic curation and analysis. Kleinberg and Raghavan (2021) study the

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[3]Recently, reducing outcome inequalities in ranking rather than in the final selection has been actively studied, cf. Celis et al. (2020), Yang et al. (2021), andZehlike et al. (2022).

utility of multiple decision-makers who use algorithms to evaluate candidates. They show that decision-makers are sometimes better off each using a different, low-precision algorithm than all using the same high-precision one. In empirical work, Bommasani et al. (2022) find that outcomes are more homogeneous when models and training data sets are shared between decision-makers. Through our theoretical analysis we thus shed light on the impact of algorithmic monoculture from the candidates' viewpoint.

1.3 Outline

The remainder of this paper is organized as follows. Section 2 introduces the model and the concept of differential correlation. Section 3 introduces our welfare metrics and presents preliminary results. Our main results are in Section 4. Finally, Section 5 concludes with a discussion on the generality of our findings and future avenues of research.

2 Setup

We here introduce the college admission problem with a continuum of students, then formalize the notion of *correlation* in Section 2.2, and introduce the supply and demand framework to identify stable matching in Section 2.3. A table of notation is provided in Appendix A.1.1 for readers' convenience.

2.1 Model

Let A and B be two colleges to which a continuum unit mass of students, S, is to be matched. The mass of a subset of S is measured with a function η.[4] Colleges have maximum capacities of the mass of students they can admit, $(\alpha^A, \alpha^B) := \alpha \in (0, 1]^2$. The students are divided into K groups $G_1, \ldots, G_K$, with a fraction $\gamma_j \in [0, 1]$ of students belonging to G_j, with $\sum_{j=1}^{K} \gamma_j = 1$. Define the vector $\gamma := (\gamma_j)_{j \in [K]}$, using the notation $[K] := \{1, 2, \ldots, K\}$. We denote the group to which a student $s \in S$ belongs by $G(s)$.

Students have strict preferences over colleges, and the amount of students preferring college A might differ between groups: among group G_j, a share $\beta_j \in [0, 1]$ prefer college A to college B, the remaining $1 - \beta_j$ prefer B. When student s prefers college A to college B, we write $A \succ_s B$, and vice versa. Note that β_j is a share that is conditional on the group, and not a mass, for instance, $\eta(\{s \in G_j : A \succ_s B\}) = \gamma_j \beta_j$. We assume that all students prefer attending some college to remaining unmatched. We write $\beta := (\beta_j)_{j \in [K]}$ as we did for γ.

Each college assigns a priority score to each student, the higher the better. Each student s thus is assigned a vector of priority scores (W_s^A, W_s^B). This means that college C prefers $s \in S$ to s' if and only if $W_s^C > W_{s'}^C$. The (marginal) distribution of scores W_s^C given by college C to students in G_j

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[4]The formal definition of this measure is deferred to Appendix A.1.2.

is described by a probability density function (pdf) f_j^C defined over the support $I_j^C \subseteq \mathbb{R}$, assumed to be an interval. Let $I_j = I_j^A \times I_j^B$. We denote by $\underline{I}_j^C$ and $\bar{I}_j^C$ the lower and upper bounds of I_j^C. These bounds might be finite or not. We define $\mathbf{f} := (f_1^A, \ldots, f_K^A, f_1^B, \ldots, f_K^B)$, and denote by F_j^C the cumulative distribution function (cdf) associated to f_j^C.

Differential correlation

Consider the joint distribution of the vectors (W^A, W^B). For each group G_j, the grade vectors of students $s \in G_j$ follow some distribution with pdf $f_j : I_j \to \mathbb{R}$ and cdf F_j. A joint distribution can be characterized by its marginals, i.e., the distribution of each component of the vector, and the shape of the joint distribution, captured by a coupling function, called *copula*. A copula is a cdf over $[0,1]^n$, for some n, with uniform marginals. Sklar (1959)'s theorem states that any joint distribution can be decomposed into (independent) marginals and a unique copula:

Sklar (1959, Theorems 1, 2, and 3) *Let F be a n-dimensional cdf with marginal cdfs $F^1, \ldots, F^n$. Then there exists a unique n-dimensional copula $H : [0,1]^n \to [0,1]$ such that*

$$F(x^1, \ldots, x^n) = H(F(x^1), \ldots, F(x^n)).$$

Conversely, for any n-dimensional copula H and for any set of n 1-dimensional cdfs $F^1, \ldots, F^n$, $F(x^1, \ldots, x^n) := H(F(x^1), \ldots, F(x^n))$ is a n-dimensional cdf with marginals $F^1, \ldots, F^n$.

Each group G_j then has a joint distribution with joint pdf f_j and cdf F_j, that can be represented by its marginals F_j^A, F_j^B and a (unique) copula H_j. We assume that there exists a family of 2-dimensional copulas $(H_\theta)_{\theta \in \Theta}$ (Θ being an interval of $\mathbb{R}$) and, for all $j \in [K]$, there exists a parameter $\theta_j \in \Theta$ such that H_{θ_j} is the copula associated to G_j's distribution, i.e., $H_{\theta_j} = H_j$. This assumption is made without loss of generality, but some of our results will require additional assumptions regarding the copula family. Denote by $\theta := (\theta_j)_{j \in [K]}$ the vector containing each group's parameter. Notice that each groups has a different θ_j, and thus a different joint distribution. With some foresight to the explanations provided in Section 2.2, we call this feature of the model *differential correlation*.[5] Finally, we assume that all the copulas in $(H_\theta)_{\theta \in \Theta}$ have full support over $[0,1]^2$. We write f_{j,θ_j} and F_{j,θ_j} instead of f_j and F_j as we will consider them as functions of θ.

Given a family of copulas $(H_\theta)_{\theta \in \Theta}$, we then refer to the tuple $(\gamma, \beta, \alpha, \mathbf{f}, \theta)$ as the *college admission problem*. Note that we only assume that distributions admit a density and have full support, that the distribution family is parameterized by a scalar, and that the marginals remain the same for any θ. In Appendix A.2 we discuss implications of these assumptions and provide details of distributions that satisfy them (i.e., Gaussian copulas, Archimedean copulas and examples thereof).

This model allows for each group to have different grade distributions at each college. Notice that even though inside a given group the students preferring A have the same grade distribution as

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[5]This is in spirit of the notion of differential variance studied in Emelianov et al. (2022) and Garg et al. (2021).

students preferring B, this does not cause any loss of generality. Indeed, we can always split a given group into two groups with different distributions.

2.2 Correlation and the coherence assumption

The proxy for correlation in our model will be the parameter θ, rather than some specific functional form. We use a condition, namely coherence, on the family of distributions: whenever $(f_{j,\theta})_{\theta \in \Theta}$ is coherent, there exists a bijection between θ_j and classical measures of correlation. For details on classical correlation measures, namely Pearson's, Spearman's, and Kendall's correlation, see Appendix A.3.

Assumption 1 (Coherence). *We say that $(H_\theta)_{\theta \in \Theta}$ is* coherent *if for all $(x, y) \in \mathring{I}$, $H_\theta(x, y)$ is strictly increasing in θ on Θ, where $\mathring{I}$ is the interior of I.*

The following lemma states that under coherence θ is naturally interpreted as a measure of correlation.

Lemma 1. *If $(H_\theta)_{\theta \in \Theta}$ is coherent, and (X, Y) are random variables drawn according to H_θ, then $\forall (x, y) \in \mathring{I}$, $\mathbb{P}(X < x, Y < y)$ and $\mathbb{P}(X > x, Y > y)$ are increasing in θ, while $\mathbb{P}(X < x, Y > y)$ and $\mathbb{P}(X > x, Y < y)$ are decreasing in θ.*

Intuitively, when X and Y are highly correlated, then if, for example, X is small then Y is likely also small. The proof is provided in Appendix B.1.

Further supporting the choice of θ as our measure of correlation, we note that there is an equivalence between θ and two classical, ordinal measures of correlation, namely Spearman's correlation, denoted ρ, and Kendall's correlation, denoted τ.

Scarsini (1984, Theorems 4 and 5) *If $(H_\theta)_{\theta \in \Theta}$ is coherent, and (X_θ, Y_θ) are random variables drawn according to H_θ, then Spearman's and Kendall's correlation coefficients $\rho(X_\theta, Y_\theta)$ and $\tau(X_\theta, Y_\theta)$ are strictly increasing functions of θ.*

The above results show that, under the coherence assumption, θ is a bijection of classical measures of correlation. To illustrate the effect of θ on a coherent distribution family, Figure 1 shows draw from Gaussian copulas, which are coherent, with different values of the covariance used as θ. When $\theta = 0$ the variables are independent, when θ is positive the joint distribution gets closer to the diagonal $X = Y$, and when θ is negative the joint distribution gets closer to the diagonal $Y = -X$, which corresponds to the common idea of correlation as well as Spearman's and Kendall's correlations. Since most of our results will be qualitative, they are stated using θ but would still be true if θ was replaced by ρ or τ in the statement.

We finally introduce a technical assumption that will be required for some of our results, especially when considering comparative statics in θ.
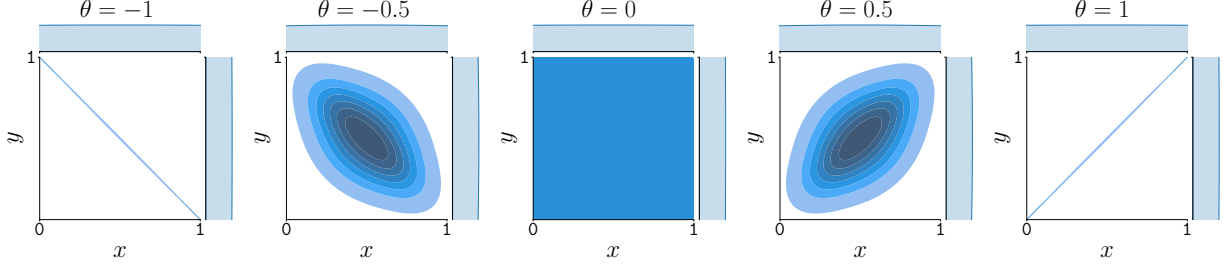
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Figure 1: Gaussian copula (i.e., bivariate Gaussian with marginals renormalized to uniform distributions, cf. Appendix A.2) for five different correlation levels, θ. The shades of blue represent the distribution density (darker means higher).

Assumption 2 (Differentiability). *We say that $(H_\theta)_{\theta \in \Theta}$ is* differentiable *if for all $(x,y) \in \mathring{I}$ and for all $\theta \in \mathring{\Theta}$, $h_\theta(x,y)$ is differentiable in θ.*

The coherence and differentiability assumptions are not particularly restrictive, for instance the Gaussian copula with covariance as the parameter verifies them, as well as Frenkel's copula and almost all example copulas mentioned in Appendix A.2.

2.3 The supply and demand framework

We now introduce the key elements of matching theory used throughout the paper. A matching μ is a mapping associating a student to a college (or themselves if they are unmatched) and a college to a subset of students. We use the common definition of stability: we say that a matching is stable if, for any student s that would prefer college C to their current match, s has a lower score at C than all the currently admitted students in that college. For the formal definitions of those notions and of the deferred acceptance algorithm, see Appendix A.4.

A matching problem can be alternatively viewed through a supply and demand lens, where a stable matching is a Walrasian equilibrium (Azevedo and Leshno 2016).

Definition 1 (Cutoffs and demand). If μ is a stable matching, define the *cutoff* at $C \in \{A, B\}$ as $P^C := \inf\{W_s^C : \mu(s) = C\}$. Given $\mathbf{P} = (P^A, P^B)$, we call the *demand* of student s, denoted $D_s(\mathbf{P}) \in \{A, B\} \cup \{s\}$, the college they prefer among those where their score is above the cutoff, or themselves if their score does not exceed the cutoff at any college. The *aggregate demand* at college C is the mass of students demanding it: $D_C(\mathbf{P}) = \eta(\{s : D_s(\mathbf{P}) = C\}$.

The cutoff of a college represents the score above which a student who applies gets admitted. Recall that $I_j = I_j^A \times I_j^B$ is the support of $f_{j,\theta}$, $\underline{I}_j^A$ and $\bar{I}_j^A$ the lower and upper bounds of I_j^A, and same for B. If $P^C = \min_{j \in [K]} \underline{I}_j^C$, then college C rejects no one, if $P^C = \max_{j \in [K]} \bar{I}_j^C$ it accepts no one. The supply associated to this demand is simply the capacity of each college.

Consider the equilibria of this problem:

Definition 2 (Market clearing). The cutoff vector $\mathbf{P}$ is *market clearing* if for $C \in \{A, B\}$, $D_C(\mathbf{P}) \leq \alpha_C$, with equality if $P^C > \min_{j \in [K]} \underline{I}_j^C$.

A cutoff vector is therefore market-clearing if it induces a demand that is equal to colleges' capacities when they reach their capacity constraint, and lower for colleges that are not full. When the constraint is reached at both colleges, i.e., when $\alpha^A + \alpha^B < 1$, the system

$$\mathbf{D}(\mathbf{P}) = \alpha \tag{1}$$

is called the *market-clearing equation*, and the market-clearing cutoffs P^A and P^B can be computed by solving the system.

The following result from Azevedo and Leshno (2016) establishes the link between market-clearing cutoffs and stable matchings:

Azevedo and Leshno (2016, Lemma 1)

1. *If μ is a stable matching, the associated cutoff vector $\mathbf{P}$ is market-clearing;*

2. *If $\mathbf{P}$ is market-clearing, we define μ such that for all $s \in S$, $\mu(s) = D_s(\mathbf{P})$. Then μ is stable.*

This allows to analyze stable matchings by studying the cutoffs of each college. Figure 2 illustrates the link between the cutoffs and the matching: students who prefer A get admitted there if and only if their score W^A is higher than the cutoff P^A. Otherwise, they get admitted to B if their score W^B is higher than P^B, and stay unmatched if it is not. The situation is symmetric for students preferring college B.

In the continuous college admissions problem the same authors show that there is a unique stable matching.

Lemma 2 (Special case of Azevedo and Leshno 2016, Theorem 1). *For any college admission problem $(\gamma, \beta, \alpha, \mathbf{f}, \theta)$, there exists a unique stable matching.*

Note that the original theorem specifies conditions on the distribution of students' types, such as being continuous and having full support, which hold in our definition of a college admission problem. Unlike the finite case where typically several stable matchings exist, in the continuum model the stable matching is unique and therefore no considerations regarding selection among the set of stable matchings are necessary. From now on, we will therefore consider the cutoff vector $\mathbf{P}$ as the one uniquely determined by the parameters of the problem and the market-clearing equation. We shall say *student s goes to college C* to mean that they are matched to college C in the unique stable matching.

Azevedo and Leshno (2016) further show that the stable matching varies continuously in the parameters of the problem and that the set of stable matchings from a college admission problem with a finite number of students converges to the unique stable matching of the continuum problem
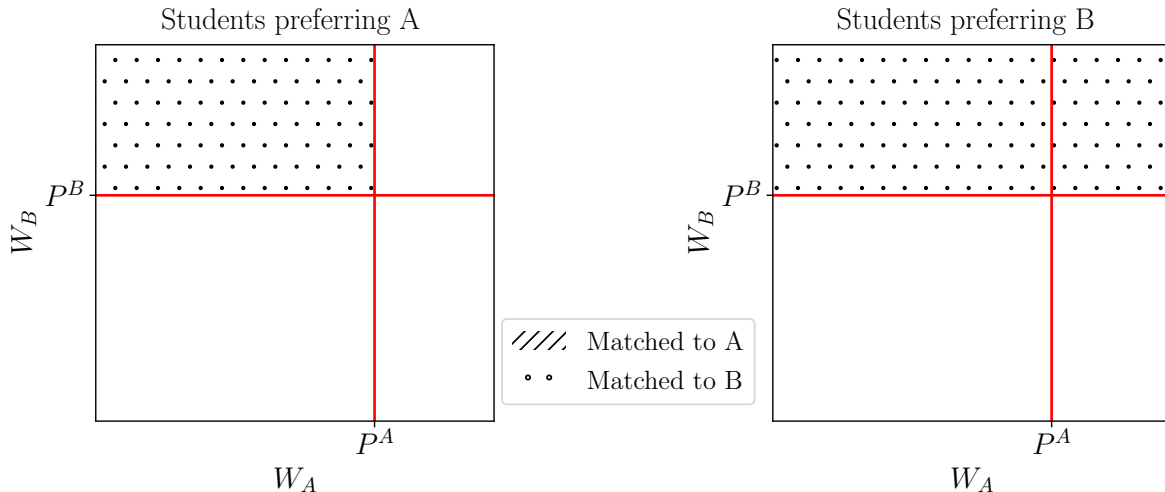
Figure 2: Illustration of the match of students depending on their preferences, priority scores, and cutoffs. Students in the hashed area are matched to college A, those in the dotted area to college B, and those in the white area remain unmatched.

with the same parameters. The latter result justifies the approximation of large finite instances by their limit.[6]

3 Welfare metrics and preliminary results

In selection problems, inequalities between groups are measured by the proportion of admitted candidates in each group. In a matching setting, the situation is more complex: on the one hand, one group might have a higher proportion of unmatched students than the other, but on the other hand, the proportion of students getting their first choice might also differ. If all students in a group get their first choice and all students in the other get their second choice, the matching may be deemed unfair. In this section, we define metrics that allow us to quantify the satisfaction of students from each group.

Consider an individual's likelihood of getting their first choice, second choice, or being rejected from both colleges in a stable matching as a function of differential correlation.

Definition 3 (Welfare metrics). Under a stable matching μ_θ induced by differential correlation parameters θ, define $V_1^{G_j,A}(\theta)$ and $V_1^{G_j,B}(\theta)$, as the proportion of students from each group-preference

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[6]For a better approximation for instances with a small number of students, Arnosti (2022) proposes a related framework.

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profile who get their first choice. Formally,

$$V_1^{G_j,A}(\theta) := \frac{1}{\gamma_j \beta_j} \eta(\{s \in G_j : A \succ_s B, \mu_\theta(s) = A\}),$$

$$V_2^{G_j,A}(\theta) := \frac{1}{\gamma_j \beta_j} \eta(\{s \in G_j : A \succ_s B, \mu_\theta(s) = B\}),$$

$$V_\emptyset^{G_j,A}(\theta) := \frac{1}{\gamma_j \beta_j} \eta(\{s \in G_j : A \succ_s B, \mu_\theta(s) = \emptyset\}).$$

The metrics for students preferring college B are defined similarly, by inverting the roles of A and B and replacing β_j by $1 - \beta_j$ in the equations.

Those metrics can be thought of in two ways: $V_1^{G_j,A}(\theta)$, for instance, is the relative mass of students getting their first choice among those in group G_j who prefer college A, or equivalently, it is the probability of a randomly drawn student to get their first choice conditionally on belonging to G_j and preferring A.[7]

We next provide expressions for these metrics for the unique (cf. Lemma 2) stable matching μ_θ via its cutoffs, $\mathbf{P}$.

Lemma 3. *Let $C \in \{A, B\}$ be a college, $\overline{C}$ be the other college, and G_j be a group. Let $\mathbf{P}$ be the cutoffs associated to μ_θ, we have:*

$$V_1^{G_j,C}(\theta) = \mathbb{P}_j(W^C \geq P^C(\theta)), \tag{2}$$

$$V_2^{G_j,C}(\theta) = \mathbb{P}_{j,\theta_j}(W^C < P^C(\theta), W^{\overline{C}} \geq P^{\overline{C}}(\theta)), \tag{3}$$

$$V_\emptyset^{G_j,C}(\theta) = \mathbb{P}_{j,\theta_j}(W^C < P^C(\theta), W^{\overline{C}} < P^{\overline{C}}(\theta)). \tag{4}$$

The notation $\mathbb{P}_{j,\theta_j}$ is used as shorthand for $\mathbb{P}_{(W^A,W^B) \sim f_{j,\theta_j}}$. Lemma 3 allows to compare probabilities of admission of different types of students, and derive comparative statics with respect to differential correlation. The proof is provided in Appendix B.2.

Regarding the probability of staying unmatched, we can derive a simple yet important result (see Appendix B.3 for its proof).

Lemma 4. *The probability that a student remains unmatched depends only on their group and is independent of their preference. Moreover, the total mass of unmatched students is constant in any group's correlation level. Formally, let $\mathbf{P}$ be associated to μ_θ. Then, for $j \in [K]$, $V_\emptyset^{G_j,A}(\theta) = V_\emptyset^{G_j,B}(\theta)$; and $\eta(\{s \in S : \mu_\theta(s) = \emptyset\}) = \max(0, 1 - \alpha^A - \alpha^B)$ (which does not depend on θ).*

With Lemma 4 at hand, we will use the notation $V_\emptyset^{G_j}$, since these quantities do not depend on

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[7]We condition over preferences because of the following observation: assume there are two groups, if students from group G_1 all prefer college A, but only half of the students from group G_2 prefer A, and A has a low capacity and B a large one. Then very few students from G_1 will get their first choice while half of the students from G_2 are likely to get their first choice since it is a less demanded college. Thus students' satisfaction might differ across groups only due to their own preferences, and not because of differential correlation.

the preference of students. For all the metrics we defined, when there is no ambiguity, we also omit the dependence on θ and write $V_i^{G_j,C}$ instead.

We now define two global metrics, i.e., metrics that are not conditioned on the groups and preferences of student, namely efficiency and inequality:

Definition 4 (Efficiency and Inequality). Define the *efficiency* $E(\theta)$ of a matching as the proportion of students getting their first choice, and the *inequality* $L^{G_i,G_j}(\theta)$ between two groups, $i,j \in [K]$ as the difference of the probability of staying unmatched between those two groups:

$$E(\theta) = \eta\left(\{s \in S : \mu_\theta(s) = C \text{ and } C \succ_s \bar{C}\}\right)$$
$$= \sum_{j \in [K]} \gamma_j \beta_j V_1^{G_j,A}(\theta) + \sum_{j \in [K]} \gamma_j (1 - \beta_j) V_1^{G_j,B}(\theta) \tag{5}$$
$$L^{G_i,G_j}(\theta) = |V_\emptyset^{G_i}(\theta) - V_\emptyset^{G_j}(\theta)|. \tag{6}$$

By Lemma 4 the mass of unmatched, and therefore matched, students is constant, and thus matched students get either their first or second choice. Therefore, ceteris paribus, it is desirable to maximize the mass of students getting their first choice E. Regarding inequality, we measure the inequality between two groups by the difference in their proportions of unassigned students.

Proposition 1. *If two groups have the same marginal distributions at some college C, then for students whose first choice is college C the probability of getting this college is the same for students of both groups. Formally, if $f_j^C = f_\ell^C$, then $V_1^{G_j,C} = V_1^{G_\ell,C}$.*

The proof is provided in Appendix B.4. Proposition 1, albeit simple, is an important property of the model. If two students prefer the same college, their probabilities of getting it only depend on their respective groups' marginals, and not on their correlation levels — so differential correlation has no effect on this metric. Proposition 1 also justifies the choice of L as a measure of inequality: the proportion of students getting their first choice is the same for two groups as long as they have the same marginals, and differences only emerge in second choice admittance versus remaining unmatched. Consequently, when the proportion of unmatched students is higher in one group than the other, then the matching is unequal.[8]

The following result shows that if there is capacity excess, differential correlation does not affect the stable matching.

Proposition 2. *If capacity is not constrained, i.e., $\alpha^A + \alpha^B \geq 1$, then correlation has no effect on the stable matching. The cutoffs P^A and P^B are constant in θ, on then so are $V_1^{G_j,C}$ and $V_2^{G_j,C}$ for all j and C. Moreover, $V_\emptyset^{G_j} = 0$, therefore $\forall i,j \in [K]$, $L^{G_i,G_j}(\theta) = 0$.*

The proof is provided in Appendix B.5, and more detail about this case can be found in Appendix A.5.

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[8]Another choice to measure inequality would be to compare the proportions of students getting their second choice in each group, however Proposition 1 implies that this quantity is equal to L.

4 Main results

This section contains our main results on the impact of differential correlation on the properties of stable matchings. We consider college admission problems where γ, β and α are assumed constant, and study the influence of differential correlation, θ, on the stable matching. Section 4.1 contains general comparative statics and Section 4.2 studies tie-breaking.

4.1 Comparative statics

We first consider how the efficiency of the matching, i.e., the probability of getting one's first choice, varies when changing the correlation for one group.

Theorem 1. *Suppose that $(H_\theta)_{\theta \in \Theta}$ is coherent and differentiable, and that $\alpha^A + \alpha^B < 1$. Then for all groups and all preferences the proportion of students getting their first choice is increasing in all correlation parameters θ_j, $j \in [K]$, and consequently so is the global efficiency $E(\theta)$. Formally, suppose that $\theta \in (\mathring{\Theta})^K$. Then for any $C \in \{A, B\}$, for any $j, \ell \in [K]$, $V_1^{G_j,C}(\theta)$ is differentiable and*

$$\frac{dV_1^{G_j,C}(\theta)}{d\theta_\ell} > 0.$$

The immediate consequence is that $E(\theta)$ is differentiable and for any $j \in [K]$,

$$\frac{dE(\theta)}{d\theta_j} > 0.$$

The proof relies on the following lemma:

Lemma 5. *Suppose that $\theta \in (\mathring{\Theta})^K$. Then for any $C \in \{A, B\}$, $P^C(\theta)$ is differentiable and*

$$\frac{dP^C(\theta)}{d\theta_j} < 0 \ \forall j \in [K].$$

Proof sketch. The proof follows several steps. First, we rewrite the market-clearing Equation (1) using Lemma 3. We obtain a system of two equations, where the variables are the cutoffs P^A and P^B, parameterized by θ. We then apply the implicit function theorem to a mapping whose roots are the solution of this system of equations. We next compute the partial derivatives. To characterize the sign of the derivatives with respect to θ, we use the coherence assumption. Through analytical derivations, we can conclude. The proof is provided in Appendix B.6. $\blacksquare$

Proof of Theorem 1. From Lemma 5, the cutoffs are decreasing in each θ_j. We can then conclude that for any $j \in [K]$ and for $C \in \{A, B\}$,

$$\frac{dV_1^{G_j,C}}{d\theta_j} = \frac{d\int_{P^C}^\infty f_j^C(x)\,\mathrm{d}x}{d\theta_j} = \frac{d\int_{P^C}^\infty f_j^C(x)\,\mathrm{d}x}{dP^C} \cdot \frac{dP^C}{d\theta_j} > 0. \qquad \blacksquare$$
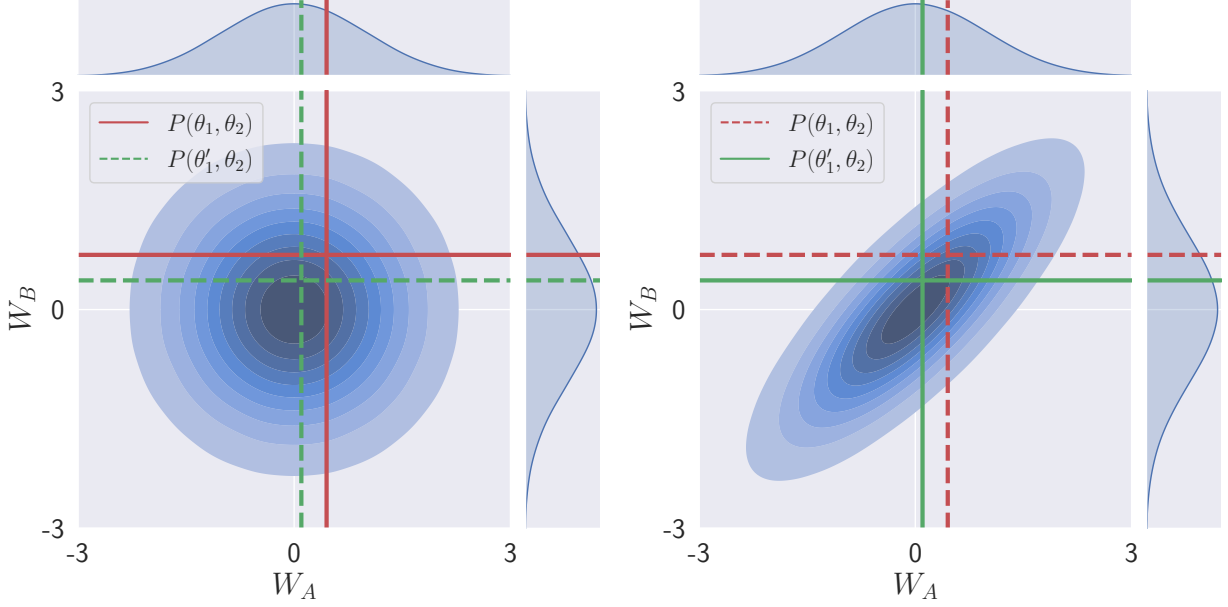
Figure 3: Illustration of change in cutoffs. The distribution of group G_1 is a bivariate Gaussian with θ equal to the covariance: $\theta_1 = 0$ in the left-hand figure and $\theta'_1 = 0.8$ in the right-hand figure. P^A is represented as a vertical line and P^B as an horizontal line. Both cutoffs decrease as θ_1 increases. In each sub-figure, the cutoffs corresponding to the current value of θ_1 are represented as full lines and the cutoffs corresponding to the other value of θ_1 as dashed lines.

Theorem 1 implies that, if the correlation decreases for one of the groups, then all groups suffer from a decrease in first-choice admittance. Conversely, increasing the correlation for one group leads to an increase in first-choice admittance for all groups.

Intuitively, when the correlation increases, students' score vectors accumulate close to the diagonal, and therefore in the lower-left and upper-right quadrants, while the other two quadrants are increasingly empty. This phenomenon is illustrated in Figure 3 with a bivariate Gaussian distribution. If the cutoffs did not change, then the amount of unmatched students would increase. As the capacities are assumed constant this would renter the resulting matching unstable. Therefore, at least one of the cutoffs decreases and, in fact, Lemma 5 implies that both decrease. As a consequence, the mass of matched students remains the same but more students get their first choice.

Remark 1. *The formal statement of Theorem 1 excludes the extremities of Θ. This assumption is made only to avoid the case where rankings are fully correlated, which would mean that $\mathbf{f}$ does not have full support. However, since V_1^A and V_1^B are continuous in θ, they are increasing on the whole interval Θ.*

Theorem 1 allows us to derive the following corollary regarding a student's probability of remaining

unmatched.

Corollary 1. *Suppose that $(H_\theta)_{\theta \in \Theta}$ is coherent and differentiable, assume $\theta \in \mathring{\Theta}^K$ and $\alpha^A + \alpha^B < 1$. Then the proportion of students from a given group remaining unmatched is increasing in its own correlation level and decreasing in the correlation level of all other groups: for $i, j \in [K]$, $i \neq j$,*

$$\frac{dV_\emptyset^{G_i}(\theta)}{d\theta_i} > 0 \quad and \quad \frac{dV_\emptyset^{G_i}(\theta)}{d\theta_{G_j}} < 0.$$

Moreover, the inequality between two groups is decreasing in the correlation level of the group with lowest rate of unmatched students and increasing in the other group's correlation level. Formally, assume $V_\emptyset^{G_i}(\theta) < V_\emptyset^{G_j}(\theta)$. Then

$$\frac{dL^{G_i,G_j}(\theta)}{d\theta_i} < 0 \quad and \quad \frac{dL^{G_i,G_j}(\theta)}{d\theta_j} > 0.$$

Proof sketch. The proof relies on Lemma 5 and leverages the fact that total capacity is constant to derive the sign of the partial derivatives of the $V_\emptyset$ terms. The proof is provided in Appendix B.7.

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Remark 2. *As a consequence, for any two groups G_i, G_j, if $\theta_i \neq \theta_j$ then the matching almost surely exhibits inequality for those groups ($L^{G_i,G_j}(\theta) > 0$). In particular, this is true even when those groups have the same marginals.*

The probability of staying unmatched is different for students from different groups, even with identical marginals. This is in contrast to Proposition 1. Different levels of correlation lead to an unequal matching. This is the case as with identical marginals the proportion of students above some cutoff is the same in every group, but for a group with high correlation, the set of students above the cutoff is almost the same at each college, while for a group with low correlation those sets are quite different at each college. Therefore, the set of matched students in group G_j, which is $\{s \in G_j | W_s^A \geq P^A\} \bigcup \{s \in G_j | W_s^B \geq P^B\}$, is larger for groups with lower correlation. This result is quite counter-intuitive: consider the point of view of some college C, which has identical marginals for all groups. From C's point of view, there is no difference between the groups, and the proportion of students with $W_s^C \geq P^C$ is the same across all groups. However, Corollary 1 implies that the groups with the lowest correlation levels are going to be overrepresented at C, and the groups with the highest correlation underrepresented. College C then ends up with a set of student that could be deemed "unfair" regarding demographic parity, while C's ranking was in fact perfectly fair.

Corollary 1 helps understand the influence of correlation on inequality. When correlation levels of two groups are equal and marginals are identical, there is no inequality. If marginals are different, there is some "baseline" inequality that can be increased or decreased by changing the correlation levels: to decrease the inequality one would need to increase the correlation of the group with

the lowest proportion of unassigned student (therefore the better-off group) and/or decrease the correlation of the worse-off group. Overall, even when colleges have fair rankings (identical marginals across all groups), the matching might still exhibit inequality.

Theorem 1 and Corollary 1 study efficiency and inequality separately. The following proposition describes their interaction. Concretely, different correlation vectors, θ, can lead to the same efficiency E, while inducing different inequality levels between groups. This shows that the effect of differential correlation cannot be solely explained via differences in efficiency.

Proposition 3. *Suppose that* $(H_\theta)_{\theta \in \Theta}$ *is coherent and differentiable, assume* Θ, *and* $\alpha^A + \alpha^B < 1$. *Let* $\theta = (\theta_1, \ldots, \theta_K)$, *and* $\hat{E} = E(\theta)$.

1. *There exist infinitely many correlation vectors achieving a given efficiency. Formally, the set of vectors* θ' *such that* $E(\theta') = \hat{E}$ *is a connected hypersurface of dimension K-1 (unless* $\theta = 0_{\mathbb{R}^K}$ *or* $(1, \ldots, 1)$, *in which case it is a singleton).*

2. *Fixing efficiency, correlation levels are substitutes. Formally, for any two groups* G_i, G_j, *there exists an interval* $U := [\underline{\theta}, \bar{\theta}] \subseteq \Theta$ *and a differentiable and decreasing function* $\phi : U \to \Theta$ *such that* $(\theta_i \in U$ *and* $\theta_j = \phi(\theta_i)) \implies E(\theta) = \hat{E}$. *The boundaries of* U *are optimal/pessimal with respect to the mass of unassigned students* $(V_\emptyset^{G_i}, V_\emptyset^{G_j})$ *for* G_i *respectively* G_j *and there is a unique* $\hat{\theta} \in U$ *such that* $\theta_1 = \hat{\theta}, \theta_j = \phi(\hat{\theta})$ *minimizes inequality* $L^{G_i, G_j}(\theta)$.

The proof is provided in Appendix B.8. Beyond the intuition that correlation favors efficiency, Proposition 3 provides a precise insight to the relation between efficiency and inequality and the trade-off between the two. The first part states that there is, in general, a continuum of correlation vectors achieving the same level of efficiency. The second part considers the comparative statics between two groups. Fixing the efficiency, correlation parameters behave as rival goods. As the correlation increases for one groups it necessarily decreases for the other group.

Figure 4 illustrates this for two groups, G_1, G_2, with standard Gaussian marginals and the parameter of the copula, θ, equal to the covariance. The left panel shows the efficiency and the right panel shows the inequality as functions of θ_1, θ_2. In the left panel, the level lines show the decreasing relation between θ_1 and θ_2 when E is kept constant. In the right panel, the inequality is minimized along the diagonal where $\theta_1 = \theta_2$ and increases as the parameters become more disparate.

4.2 Tie-Breaking

Some recent papers have studied the impact of tie-breaking rules on school choice problems, which has a strong link with correlation. In this section, we extend some of the prior results and discuss the relation to the literature.

Assume there is only one group, and each school C has n_C priority classes, i.e., there exists a partition of $S = Q_1^C \bigsqcup \cdots \bigsqcup Q_{n_C}^C$ such that for $i, j \in \{1, \ldots, n_C\}$, $s \in Q_i^C$, $s' \in Q_j^C$, s has higher
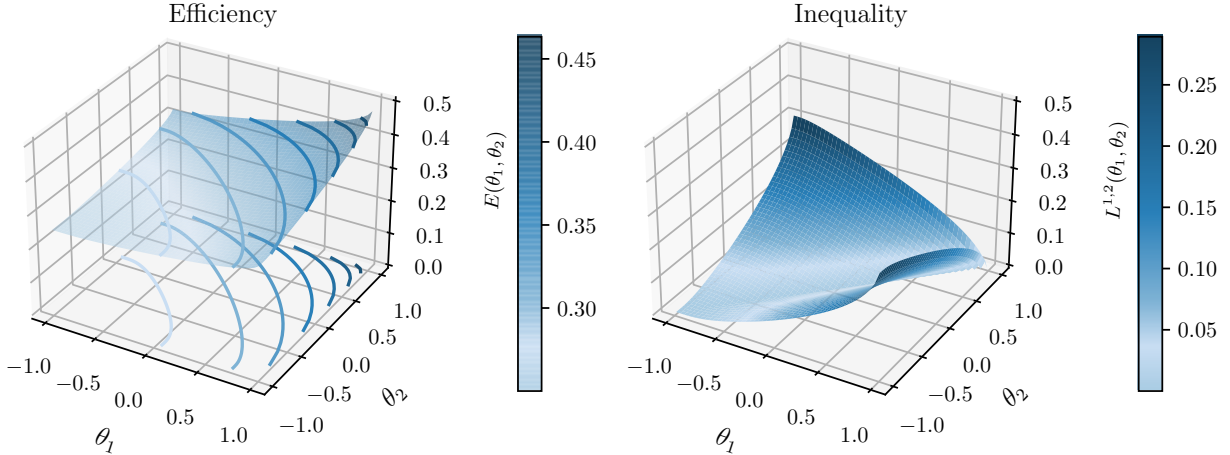
Figure 4: Variations of efficiency (E) and (L) for two groups (G_1, G_2) as a function of their respective correlation parameters (θ_1, θ_2); with standard Gaussian marginals and copula with θ equal to the covariance. Other parameters: $\alpha^A = \alpha^B = 0.25$, $\gamma_1 = \gamma_2 = 0.5$, $\beta_1 = \beta_2 = 0.5$. Left:The surface represents the efficiency and the level lines indicate constant efficiency (also projected to the bottom of the figure). Right: The surface represents the inequality.

priority than s' at C if $i < j$. Students belonging to the same priority class at a school are assumed to have the same priority at this school, but due to limited capacity the school might need to choose between them. To achieve this, schools use a random ranking of students to which they refer each time they need to choose between students from the same priority class; this random ranking is called a *tie-breaker*.

A natural question that has been actively studied in recent years is whether there is a difference in students' welfare if schools use the same tie-breaker (called single tie-breaker, or STB), instead of each producing an independent one (multiple tie-breakers, or MTB). Ashlagi et al. (2019); Ashlagi and Nikzad (2020); Arnosti (2023) show — with slightly different models and assumptions (and among other results) — that when the total capacity of schools is lower than the number of students, then students are better off under STB than MTB. To ease the comparison, we restate their results here in a simplified form. Given n students and m schools:

- *Ashlagi et al. 2019, Main Theorem*: Suppose that there is capacity shortage, students' preferences are drawn uniformly at random and there is only one priority class (the whole ranking is random), then for any $k < m$ the fraction of students matched to one of their top k choices approaches 0 under MTB but approaches a positive constant under STB.

- *Ashlagi and Nikzad 2020, Theorem 3.2*: Suppose there is one slot per school, only one priority class, and schools are divided into two tiers (top and bottom) with students' preferences inside a given tier drawn uniformly at random and a capacity shortage at top schools, then, with high probability, STB stochastically Pareto-dominates MTB.

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- *Arnosti 2023, Theorem 2*: Suppose there is only one priority class and students only list $l < m$ schools in a uniform random order, then the number of students matched to their first choice is greater under STB than under MTB.

Our model, compared to prior work on tie-breaking, allows to for any number of priority classes, intermediate levels of correlation or even negative correlations, and several groups of students with different tie-breaking rules. To this end, let $(H_\theta)_{\theta \in \Theta}$ be a coherent and differentiable family of copulas such that $\theta = 0$ gives independent random variables, and $\theta = 1$ gives fully correlated variables. Define the θ-TB as the tie-breaker drawn according to H_θ. Thus, MTB corresponds to $\theta = 0$ and STB to $\theta = 1$. Moreover, we can assume the existence of several groups with different θs.

Intermediate correlation levels can arise in tie-breaking if, for example, student characteristics are introduced into rankings to break ties, e.g., sibling priority or distance to from home to school (Correa et al., 2022). This is commonly done to render algorithms more deterministic and thus understandable. Consider priority for students with lower distance to a school and suppose that there are two villages with one school each. Then, ceteris paribus, a student who lives in one village exhibits negative correlation between the grades at each of the two schools. On the other hand, a student living in neither village may exhibit any level of correlation. Note that this example also illustrates how negative correlation naturally arises.

Proposition 4. *Let there be a continuum mass of students and assume that students prefer any school over being unmatched. Let A, B be two schools with n_A, n_B priority classes, and constrained capacities $\alpha^A + \alpha^B < 1$. Further suppose that students are divided into K groups, such that the θ_j-TB is used for group G_j. Then:*

1. The mass of students getting their first choice

   - *is non-decreasing in each θ_j,*
   - *is almost surely strictly increasing in all θ_j, if all products of priority classes $Q_i^A \times Q_j^B$ contain a positive mass of students of each group,[9] and*
   - *is strictly increasing in each θ_j, if there is only one priority class.*

2. The inequality between two groups, $L^{G_i, G_j}(\theta)$, is non-decreasing in the correlation θ of the group with the lowest $V_\emptyset$, and non-increasing in the other group's θ.

Proof. Proof sketch. We build a distribution family that encompasses the priorities of students at each school taking into account priority classes as well as tie-breakers, such that MTB and STB correspond to values of $\theta = 0$ and $\theta = 1$. The obtained distribution, while being complex, still satisfies most of the assumptions required by our model, and with some adjustments we are able to apply Theorem 1 and Corollary 1 and conclude. The proof is provided in Appendix B.9. ∎

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[9]More precisely, the set of vectors (γ, β, α) such that the mass of students getting their first choice is constant in some θ_j has Lebesgue measure 0.

This result shows that increasing the correlation of tie-breakers, for one or several groups, increases the amount of students getting their first choice. Moreover, it also shows that a policy maker able to change the correlation of tie-breakers for some groups can use it to mitigate the inequalities between groups.

Proposition 4 is in some regards more restrictive than the results from the literature presented above, because it only applies to two schools and assumes students prefer either school over being unmatched. On the other hand, it is more general in that it applies to cases with several priority classes and does not require students preferences to be uniform (in our model, we can have any fraction β of students preferring school A). It also allows to have several groups with different tie-breaking rules. Finally, Proposition 4 allows for intermediate tie-breaking rules interpolating between MTB and STB, and also for negatively correlated tie-breaking rules.

5 Discussion

We have introduced a tractable model to study the impact of differential ranking correlation between different groups and studied its effect on on outcome inequality and efficiency in matching markets. Our framework is general in that it accommodates almost any grade distribution, any number of groups with different distributions and different student preferences, and colleges of any capacity.

However, a limitation is our focus on two colleges, which we shall discuss here. First, our model can be extended to any number of colleges; in particular copulas can be defined for any number of variables and the coherence definition extends. Our welfare metrics also carry over and it would be natural to also consider further metrics, e.g., the probability of getting one of the top k choices. Turning to our results, Proposition 1 remains true. That is, whenever two groups have the same marginal at some college, they have the same probability of getting this college as a first choice. On the other hand, Proposition 2 does not hold anymore: for any pair of colleges whose joint capacity is less than 1, correlation between their rankings will have an effect on the matching, even if the total capacity is more than 1. The other results all rely on Lemma 5. We found, through numerical experiments, that Lemma 5 generally does not extend beyond two colleges. See Figure 5 for a counterexample with four colleges, one group, standard Gaussian marginals at every college, a Gaussian copula, and $\alpha = (0.05, 0.05, 0.2, 0.5)$. The cutoff P^C of the third college is increasing for high values of θ.

It follows that Theorem 1 also does not extend beyond two colleges as it entirely relies on Lemma 5. The other results (Corollary 1, and Propositions 3 and 4) may remain true, but different proof techniques would be required. Our experiments suggest that the statement of Lemma 5 may still be true for large classes of correlation, thus suggesting the possibility to extend our results under some additional assumptions.

To conclude, we believe that there is ample scope to study themes that have already been considered in the single decision-maker settings in the matching context. Our analysis suggests that
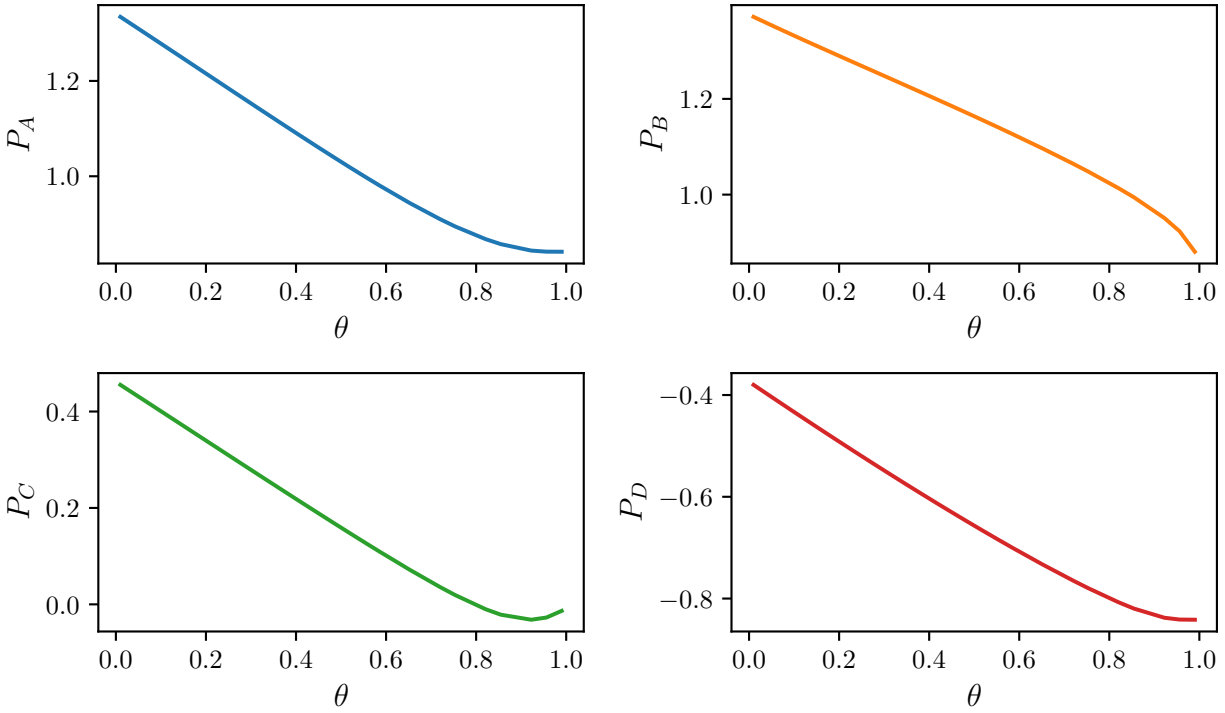
Figure 5: Counterexample to the extension of Lemma 5 to more than two colleges: Four colleges, one group, bivariate Gaussian distribution with correlation θ, and $\alpha = (0.05, 0.05, 0.2, 0.5)$. Note that P_C is not monotone decreasing.

in matching new phenomena arise and it is important to further understand them. In models where decision-makers use noisy estimates of applicants' latent quality, existing results about algorithmic monoculture could be extended, providing a theoretical foundation to experimental results such as those of Bommasani et al. (2022). Using the noise structure, interesting variations could be allowing applicants to invest in accurate assessment, e.g., via acquiring certifications or doing in-person interviews, or considering the effects of risk aversion. Other possible directions could include making applications costly, or allowing applicants to not list all colleges in their preferences, and a more thorough study of colleges' utility.

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A Definitions and technical details

A.1 Definitions

A.1.1 Table of notation.

Table 1 provides a summary of the notation used throughout the paper.

A.1.2 Definition of the mass η.

Here we formally define the notion of mass for a subset of students. This section is self-contained and is not necessary to understand the results of the paper; the notations introduced here are not used elsewhere. We identify S to $\Sigma := \mathbb{R}^2 \times \{G1, \ldots, G_K\} \times \{A, B\}$. We partition Σ into several subsets: $\Sigma_{G_j,C} := \{s \in \Sigma : s = ((x, y), G_j, C), x, y \in \mathbb{R}\}$ is the subset of students belonging to group G and preferring college C. Given a vector of parameters θ and priorities W^A, W^B distributed according to f_{j,θ_j} for G_j students, we say that a subset $J \subseteq \Sigma$ is measurable if and only if $\{(W_s^A, W_s^B) : s \in J\}$ is Borel-measurable in $\mathbb{R}^2$. We can partition J into subsets $J_{G_j,C} := J \cap \Sigma_{G_j,C}$. On each $\Sigma_{G_j,C}$ we define a measure $\eta_{G_j,C}$ as follows: for $J \subseteq \Sigma$ measurable,

$$
\begin{aligned}
\eta_{G_j,A}(J_{G_j,A}) &= \gamma_j \beta_j \mathbb{P}_{\theta_j}((W^A, W^B) \in \{(W_s^A, W_s^B) : s \in J_{G_j,A}\}), \\
\eta_{G_j,B}(J_{G_j,B}) &= \gamma_j (1 - \beta_j) \mathbb{P}_{\theta_j}((W^A, W^B) \in \{(W_s^A, W_s^B) : s \in J_{G_j,B}\}),
\end{aligned}
\tag{7}
$$

Agents:

A, B	Colleges (generic: C)
s	An arbitrary student
S	Students set
$G_1, \ldots G_K$	Groups of students, partition of S
η	Measure for student masses

Agents' features:

α^A, α^B	Colleges' capacities ($\in (0,1)$)
γ_j	Mass of students in group G_j ($\in [0,1]$)
β_j	Share of students in group G_j preferring college A ($\in [0,1]$)

Priority scores:

W_s^C	Score at C of student s (generic: W)
f_j^C, F_j^C	Marginal pdf and cdf of college C for group G_j
$(H_\theta)_{\theta \in \Theta}, (h_\theta)_{\theta \in \Theta}$	Copula family and associated pdfs, indexed by θ
θ	Parameter for a copula family
$f_{j,\theta_j}, F_{j,\theta_j}$	Group G_j's score vectors' joint pdf and cdf, $F_{j,\theta_j} = H_{\theta_j}(F_j^A, F_j^B)$
Θ	Set of possible values for θ
I_j^C, I_j	Support of f_j^C and f_{j,θ_j} respectively. $I_j = I_j^A \times I_j^B$
$\underline{I}_j^C, \bar{I}_j^C$	Lower and upper bounds of I_j^C

Correlation:

r	Pearson's correlation
ρ	Spearman's correlation
τ	Kendall's correlation

Matching:

μ	Matching
$V_1^{G_j,C}$	Share of students of group G_j and preferring C who get their first choice
$V_2^{G_j,C}$	Share of students of group G_j and preferring C who get their second choice
$V_\emptyset^{G_j}$	Share of students of group G who are unassigned
V_1	Total mass of students getting their first choice
$L^{G_i,G_j}(\theta)$	Inequality between G_i and G_j, equal to $\lvert V_\emptyset^{G_i}(\theta) - V_\emptyset^{G_j}(\theta) \rvert$

Let $\mathcal{B}(S)$ be the set of measurable subsets of S. We define over $\mathcal{B}(S)$ the probability measure $\eta : \mathcal{B}(S) \to [0,1]$ such that for any measurable subset J of S,

$$\eta(J) = \sum_{j \in [K]} \eta_{G_j,A}(J_{G_j,A}) + \eta_{G_j,B}(J_{G_j,B}). \tag{8}$$

This definition is consistent with the definition of the parameters, as it verifies $\eta(G_j) = \gamma_j$, $\eta(\{s \in G_j : A \succ_s B\}) = \gamma_j \beta_j$ and so on.

A.2 Discussion on distributional assumptions

We assume that distributions admit a density and have full support, and that they can be represented using a copula family and marginals that remain the same for any θ, and that this copula is coherent and differentiable. We here explain why these assumptions are not very restrictive by presenting canonical examples of classical copulas satisfying our assumptions.

1. *Gaussian copula*: The Gaussian copula is obtained by composing the cdf Φ_θ of a bivariate Gaussian with covariance matrix $\begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}$ and the univariate cdf ϕ of the standard Gaussian: $H_\theta(x, y) = \Phi_\theta(\phi(x), \phi(y))$. Here, the parameter θ controls the covariance.

2. *Archimedean copulas*: Archimedean copulas are a broad family of copulas, each element of this family being itself a parametric family of copulas with parameter θ. The general formula is

$$H_\theta(x, y) = \psi_\theta^{-1}\left(\psi_\theta(x) + \psi_\theta(y)\right)$$

where $\psi_\theta : [0, 1] \to \mathbb{R}_+$ is a continuous strictly decreasing and convex function such that $\psi_\theta(1) = 0$. Examples include:

- Clayton: $H_\theta(x, y) = \left(\max\{x^{-\theta} + y^{-\theta} - 1; 0\}\right)^{-1/\theta}$
- Frank: $H_\theta(x, y) = -\frac{1}{\theta} \log\left(1 + \frac{(\exp(-\theta x) - 1)(\exp(-\theta y) - 1)}{\exp(-\theta) - 1}\right)$
- Gumbel: $H_\theta(x, y) = \exp\left(-((-\log(x))^\theta + (-\log(y))^\theta)^{1/\theta}\right)$

The Gaussian copula, as well as Clayton's, Frank's, Gumbel's and other Archimedean copulas, all satisfy our coherence and differentiability assumptions.

The only assumption our model makes on the marginals is that they are continuous. This is not particularly restrictive as long as there are no ties (see Section 4.2 for a treatment of ties).

A.3 Elements of correlation theory

In this section, we present common measures of correlation used in the literature, and some of their properties.

Definition 5 (Common measures of correlation). Let (X, Y) be two random variables with respective cdfs F_X, F_Y. Define:

1. Pearson's correlation: assume X, Y have finite standard deviations σ_X and σ_Y. Then $r_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$.

2. Spearman's correlation: let $rk_X = F_X(X)$ and $rk_Y = F_Y(Y)$. We can think of rk_X as describing the ranking of X inside a sample. Then Spearman's correlation is $\rho_{X,Y} = r_{rk_X,rk_Y}$.

3. Kendall's correlation: let (X_1, Y_1) and (X_2, Y_2) be two independent pairs of random variables with the same joint distribution as (X, Y). Then Kendall's correlation is

$$
\begin{aligned}
\tau_{X,Y} = \ & \mathbb{P}\left[(X_1 > X_2 \cap Y_1 > Y_2) \cup (X_1 < X_2 \cap Y_1 < Y_2)\right] - \\
& \mathbb{P}\left[(X_1 > X_2 \cap Y_1 < Y_2) \cup (X_1 < X_2 \cap Y_1 > Y_2)\right].
\end{aligned}
$$

We use the same letter r for the covariance of the standard bivariate Gaussian and for Pearson's correlation as they are equal. Moreover, for this distribution simple expressions exist for the two other correlation coefficients:

$$
\rho = \frac{6}{\pi}\arcsin(r/2), \tau = \frac{2}{\pi}\arcsin(r).
$$

A correlation measure should be zero when variables are independent, and reach its maximaum when the variables are totally dependent on each other. The following lemma provides these properties for the measures we just introduced.

Lemma 6 (Scarsini 1984, Theorems 1, 4, and 5). *Let X, Y be two real random variables.*

1. *$r_{X,Y}, \rho_{X,Y}, \tau_{X,Y} \in [-1, 1]$.*

2. *$\rho_{X,Y} = 1$ if and only if $Y = g(X)$ with $g : \mathbb{R} \to \mathbb{R}$ increasing. The same holds for $\tau_{X,Y}$. $r_{X,Y} = 1$ if and only if the relation is affine.*

3. *If X and Y are independent, then $r_{X,Y} = \rho_{X,Y} = \tau_{X,Y} = 0$.*

A.4   Stable matching

We now introduce some elements of matching theory used throughout the paper.

To define matching in a continuum context, we follow Azevedo and Leshno (2016).

Definition 6. A *matching* is an assignment of students to colleges, described by a mapping $\mu : S \cup \{A, B\} \to 2^S \cup C \cup S$, with the following properties:

1. for all $s \in S$, $\mu(s) \in \{A, B\} \cup \{s\}$;

2. for $C \in \{A, B\}$, $\mu(C) \subseteq S$ is measurable and $\eta(\mu(C)) \leq \alpha_C$;

3. $C = \mu(s)$ if and only if $s \in \mu(C)$;

4. for $C \in \{A, B\}$, the set $\{s \in S : \mu(s) \preceq_s C\}$ is open.

The first three conditions are common to almost all definitions of matching in discrete or continuous models. Condition (1) ensures that a student is either matched to a college or to themselves, which means that they remain unmatched. Condition (2) ensures that colleges are assigned to a subset of students that respects the capacity constraints. Condition (3) ensures that the matching is consistent, i.e., if a student is matched to a college, then this college is also matched to the student. Condition (4) was introduced by Azevedo and Leshno (2016) and is necessary to ensure that there do not exist several stable matchings that only differ by a set of students of measure 0.

We next define the notions of blocking and stability.

Definition 7 (Stability). The pair (s, C) *blocks* a matching μ if s would prefer C to her current match, and either C has remaining capacity or it admitted a student with a lower score than s; formally, if $\mu(s) \prec_s C$ and either $\eta(\mu(C)) < \alpha_C$ or $\exists s' \in \mu(C)$ such that $W_{s'}^C < W_s^C$. A matching is *stable* if it is not blocked by any student-college pair.

To produce a stable matching, one can extend the classic deferred acceptance algorithm by Gale and Shapley (1962) to the continuum model. This algorithm is described in Algorithm 1.

Algorithm 1 Deferred acceptance algorithm (DA)

First step: All students apply to their favorite college, they are temporarily accepted. If the mass of students applying to college C is greater than its capacity α_C, then C only keeps the α_C best
while A positive mass of students are unmatched and have not yet been rejected from every college **do**

Each student who has been rejected at the previous step proposes to her preferred college among those which have not rejected them yet

Each college C keeps the best α_C mass of students among those it had temporarily accepted and those who just applied, and rejects the others
end while
End: If the mass of students that are either matched or rejected from every college is 1, the algorithm stops. However it could happen that it takes an infinite number of steps to converge.

If the algorithm stops, the matching it outputs is stable; Abdulkadiroğlu et al. (2015) show that even when the number of steps is infinite, the algorithm converges to a stable matching.

Remark 3. *Note that stable matchings do not only result from centralized algorithms but are often the result of a decentralized process (see, e.g., Roth and Vande Vate 1990).*

A.5 Excess capacity

In the case where $\alpha^A + \alpha^B \geq 1$, we can compute the steps of the deferred acceptance algorithm (see Algorithm 1 in the Appendix). We consider three (partitioning) cases:

(1) *There is not enough room in college A for all students preferring it to college B, i.e., $\sum_j \gamma_j \beta_j \geq \alpha^A$.* In this case, there is necessarily enough room in college B for all students preferring it, since

$\alpha^A + \alpha^B \geq 1$. Therefore, following the steps of DA, we find:

(i) At step one, $\sum\limits_{j} \gamma_j \beta_j$ students preferring A apply there and the best α^A are temporarily admitted, and $\sum\limits_{j} \gamma_j (1 - \beta_j)$ students preferring B apply there and are all temporarily admitted.

(ii) At step two, the $\sum\limits_{j} \gamma_j \beta_j - \alpha^A$ students rejected from A apply to B, and are admitted since there is enough room for them (considering the students previously admitted).

This results in the following probabilities of a student to get their first or second choice:

$$V_1^{G_j,A} = 1 - F_j^A(P^A), \qquad\qquad V_1^{G_j,B} = 1,$$
$$V_2^{G_j,A} = F_j^A(P^A), \qquad\qquad V_2^{G_j,B} = 0,$$
$$P^A = \left(\sum_j \gamma_j \beta_j \left(1 - F_j^A\right) \right)^{-1} (\alpha^A).$$

Finally, as every student is admitted somewhere, $V_\emptyset^{G_j} = 0$.

(2) *There is not enough room in college B for all students preferring it to A, i.e., $\sum\limits_{j} \gamma_j (1 - \beta_j) \geq \alpha^B$.*
Symmetrically we get

$$V_1^{G_j,A} = 1, \qquad\qquad V_1^{G_j,B} = 1 - F_j^B(P^B),$$
$$V_2^{G_j,A} = 0, \qquad\qquad V_2^{G_j,B} = F_j^B(P^B),$$
$$P^B = \left(\sum_j \gamma_j (1 - \beta_j) \left(1 - F_j^B\right) \right)^{-1} (\alpha^B), \qquad V_\emptyset^{G_j} = 0.$$

(3) *There is enough room in each college to admit all students who prefer attending it, i.e., $\sum\limits_{j} \gamma_j \beta_j \leq \alpha^A$ and $\sum\limits_{j} \gamma_j (1 - \beta_j) \leq \alpha^B$.* It follows that everyone gets their first choice: for $j \in [K]$ and $C \in \{A, B\}$,

$$V_1^{G_j,C} = 1,$$
$$V_2^{G_j,C} = V_\emptyset^{G_j} = 0.$$

B Omitted proofs

B.1 Proof of Lemma 1.

Since $H_\theta(x,y) = \mathbb{P}(X < x, Y < y)$ by definition, then the first part of the lemma is just a rewriting of the definition of coherence. For the second part, we have

$$\mathbb{P}(X > x, Y > y) = \mathbb{P}(Y > y) - \mathbb{P}(X < x, Y > y)$$
$$= \mathbb{P}(Y > y) - \mathbb{P}(X < x) + \mathbb{P}(X < x, Y < y)$$

and $\mathbb{P}(Y > y), \mathbb{P}(X < x)$ are constant in θ (H_θ are copulas therefore they all have uniform marginals) while $\mathbb{P}(X < x, Y < y)$ is increasing, so $\mathbb{P}(X > x, Y > y)$ is also increasing. Finally, we also get that

$$\mathbb{P}(X > x, Y < y) = \mathbb{P}(Y < y) - \mathbb{P}(X < x, Y < y)$$

and

$$\mathbb{P}(X < x, Y > y) = \mathbb{P}(X < x) - \mathbb{P}(X < x, Y < y)$$

and both are thus decreasing.

B.2 Proof of Lemma 3.

Consider student $s \in G_j$ who prefers college A. By Lemma 1 from Azevedo and Leshno (2016) (cf. Section 2.3), s is admitted to A if and only if $s \in D_A(\mathbf{P})$, i.e., if and only if their score at A is greater than P^A. Then by definition of η,

$$V_1^{G_j,A}(\theta) = \frac{\eta(\{s \in G_j : A \succ_s B, \mu(s) = A\})}{\gamma_j \beta_j} = \mathbb{P}_{j,\theta_j}((W^A, W^B) \in [P^A, +\infty) \times \mathbb{R}) = \mathbb{P}_j(W^A > P^A).$$

The same reasoning applies to $V_1^{G_j,B}$ if s prefers B, which proves (2).

The same student s is admitted to B if and only if $s \in D_B(P^A, P^B)$, i.e., if and only if $W_s^B \geq P^B$ and $W_s^A < P^A$. Then we have

$$V_2^{G_j,A}(\theta) = \frac{\eta(\{s \in G_j : A \succ_s B, \mu(s) = B\})}{\gamma_j \beta_j} = \mathbb{P}_{j,\theta_j}((W^A, W^B) \in (-\infty, P^A) \times [P^B, +\infty)).$$

The same reasoning applies to $V_2^{G_j,B}$, which proves (3).

Student s remains unmatched if and only if $W_s^A < P^A$ and $W_s^B < P^B$. Then we have

$$V_\emptyset^{G_j,A}(\theta) = \frac{\eta(\{s \in G_j : A \succ_s B, \mu(s) = s\})}{\gamma_j \beta_j} = \mathbb{P}_{j,\theta_j}((W^A, W^B) \in (-\infty, P^A) \times (-\infty, P^B)).$$

which proves (4).

B.3 Proof of Lemma 4

It is sufficient to notice that Equation (4) is symmetric in C and $\bar{C}$ to obtain the first part of the lemma. The second one follows from the fact that either there is excess capacity and everyone is matched, or both colleges are full and the mass of matched students is the sum of the capacities.

Remark 4. *The first part of Lemma 4 could also be derived from the strategy-proofness for students of the student-proposing deferred acceptance algorithm (Roth, 1985). Indeed, the fact that students cannot improve their outcome by modifying the order of their preferences implies that them being unmatched or not does not depend on which college they reported to be their first choice.*

B.4 Proof of Proposition 1

The result follows directly by applying (2) to both groups, and by observing that the integral in (2) depends on θ only through the cutoff vector $\mathbf{P}$. Therefore, if groups G_i and G_j have the same marginal F_j^C at college C, then $V_1^{G_i,C} = V_1^{G_j,C}$.

B.5 Proof of Proposition 2

The value of V_1 comes from Lemma 3. If $\alpha^A + \alpha^B \geq 1$, then all students are admitted to some college, therefore $V_\emptyset^{G_j} = 0$ for all j. Moreover, either $P^A = \min_j \underline{I}_j^A$ or $P^B = \min_j \underline{I}_j^B$, or both. Let us suppose the former holds. Then

$$V_1^{G_j,A} = 1 - F_j^A(\min_j \underline{I}_j^A) = 1$$
$$V_2^{G_j,A} = 0$$

and

$$V_1^{G_j,B} = 1 - F_j^B(P^B)$$
$$V_2^{G_j,B} = F_j^B(P^B).$$

It remains to show that P^B is constant in θ. Define

$$H : x \in \mathbb{R} \mapsto \sum_j \gamma_j (1 - \beta_j) \left(1 - F_j^B(x)\right).$$

Note that since all F_j^B are invertible H is invertible too. Since all students preferring A get it, the market clearing equation for B becomes $H(P^B) = \alpha^B$, i.e., $P^B = H^{-1}(\alpha^B)$. From this relation

it appears that P^B is indeed constant in θ. The same reasoning applies if $P^A \neq \min\limits_{j} \underline{I}^A_j$ and $P^B = \min\limits_{j} \underline{I}^B_j$.

B.6 Proof of Lemma 5.

Assume $\alpha^A + \alpha^B < 1$, and $\theta \in \mathring{\Theta}^K$. Let $P^A, P^B \in \mathbb{R}$ be the cutoffs of colleges A and B.

By definition of the quantities V_1 and V_2, the market-clearing equation (1) can be written as

$$
\begin{cases}
\sum\limits_{j \in [K]} \left(\gamma_j \beta_j V_1^{G_j,A} + \gamma_j(1-\beta_j) V_2^{G_j,B} \right) &= \alpha^A, \\
\sum\limits_{j \in [K]} \left(\gamma_j \beta_j V_2^{G_j,A} + \gamma_j(1-\beta_j) V_1^{G_j,B} \right) &= \alpha^B.
\end{cases}
$$

Then, using Lemma 3, we can rewrite it as

$$
\begin{cases}
\sum\limits_{j \in [K]} \left(\gamma_j \beta_j \mathbb{P}_j(W^A \geq P^A) + \gamma_j(1-\beta_j)\mathbb{P}_{j,\theta_j}(W^A \geq P^A, W^B < P^B) \right) &= \alpha^A, \\
\sum\limits_{j \in [K]} \left(\gamma_j \beta_j \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B \geq P^B) + \gamma_j(1-\beta_j)\mathbb{P}_j(W^B \geq P^B) \right) &= \alpha^B,
\end{cases}
$$

which is finally equivalent to

$$
\begin{cases}
\sum\limits_{j \in [K]} \left(\gamma_j \beta_j \int_{P^A}^{\infty} f_j^A(x)\,\mathrm{d}x + \gamma_j(1-\beta_j) \int_{P^A}^{\infty} \int_{-\infty}^{P^B} f_{j,\theta_j}(x,y)\,\mathrm{d}x\mathrm{d}y \right) &= \alpha^A, \\
\sum\limits_{j \in [K]} \left(\gamma_j \beta_j \int_{-\infty}^{P^A} \int_{P^B}^{\infty} f_{j,\theta_j}(x,y)\,\mathrm{d}x\mathrm{d}y + \gamma_j(1-\beta_j) \int_{P^B}^{\infty} f_j^B(x)\,\mathrm{d}x \right) &= \alpha^B.
\end{cases}
\tag{9}
$$

We fix θ, and we want to study how the solution (P^A, P^B) of the above equation varies as a function of θ_j for some $j \in [K]$. Let us define $Z : \mathbb{R}^2 \times \Theta \to \mathbb{R}^2$, $(P^A, P^B, \theta_j) \mapsto (D_A(P^A, P^B) - \alpha^A, D_B(P^A, P^B) - \alpha^B)$. (We will denote by Z_1 and Z_2 its two components.)

$$
Z(P^A, P^B, \theta_j) = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \sum\limits_{j \in [K]} \left(\gamma_j \beta_j \int_{P^A}^{\infty} f_j^A(x)\,\mathrm{d}x + \gamma_j(1-\beta_j) \int_{P^A}^{\infty} \int_{-\infty}^{P^B} f_{j,\theta_j}(x,y)\,\mathrm{d}x\mathrm{d}y \right) - \alpha^A \\ \sum\limits_{j \in [K]} \left(\gamma_j \beta_j \int_{-\infty}^{P^A} \int_{P^B}^{\infty} f_{j,\theta_j}(x,y)\,\mathrm{d}x\mathrm{d}y + \gamma_j(1-\beta_j) \int_{P^B}^{\infty} f_j^B(x)\,\mathrm{d}x \right) - \alpha^B \end{pmatrix}
$$

$$
\tag{10}
$$

Then for each $\theta_j \in \Theta$, (P^A, P^B) is the solution of the equation $Z(P^A, P^B, \theta_j) = (0,0)$. In order to show that P^A and P^B are decreasing in θ_j, we wish to apply the implicit function theorem. Let $P^A, P^B \in \mathbb{R}$ and $\theta_j \in \Theta$ such that $Z(P^A, P^B, \theta_j) = 0$. Function Z is of class C^1. We first verify

that the partial Jacobian $J_{Z,(P^A,P^B)}(P^A, P^B, \theta_j)$ is invertible, where

$$J_{Z,(P^A,P^B)}(P^A, P^B, \theta_j) = \begin{pmatrix} \dfrac{\partial Z_1}{\partial P^A} & \dfrac{\partial Z_1}{\partial P^B} \\ \dfrac{\partial Z_2}{\partial P^A} & \dfrac{\partial Z_2}{\partial P^B} \end{pmatrix}. \tag{11}$$

To show that the determinant $\frac{\partial Z_1}{\partial P^A}\frac{\partial Z_2}{\partial P^B} - \frac{\partial Z_1}{\partial P^B}\frac{\partial Z_2}{\partial P^A} \neq 0$, we will show that it is in fact strictly positive. From (9), it is clear that Z_1 is decreasing in P^A and increasing in P^B, and that Z_2 is increasing in P^A and decreasing in P^B. Therefore, to prove that $\frac{\partial Z_1}{\partial P^A}\frac{\partial Z_2}{\partial P^B} - \frac{\partial Z_1}{\partial P^B}\frac{\partial Z_2}{\partial P^A} > 0$, we only need to prove that $\left|\frac{\partial Z_1}{\partial P^A}\right| > \frac{\partial Z_2}{\partial P^A}$ and $\left|\frac{\partial Z_2}{\partial P^B}\right| > \frac{\partial Z_1}{\partial P^B}$.

By symmetry, we will only prove the first one. We can compute each term separately:

$$\frac{\partial Z_1}{\partial P^A} = \sum_{j \in [K]} \left(\gamma_j \beta_j \frac{\partial \, \mathbb{P}_j(W^A \geq P^A)}{\partial P^A} + \gamma_j(1 - \beta_j)\frac{\partial \, \mathbb{P}_{j,\theta_j}(W^A \geq P^A, W^B < P^B)}{\partial P^A} \right),$$

$$\frac{\partial Z_2}{\partial P^A} = \sum_{j \in [K]} \gamma_j \beta_j \frac{\partial \, \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B \geq P^B)}{\partial P^A}.$$

All terms of Z_1 are decreasing in P^A and all terms of Z_2 are increasing in P^A, therefore we can proceed term by term:

$$\left| \gamma_j \beta_j \frac{\partial \, \mathbb{P}_j(W^A \geq P^A)}{\partial P^A} \right| = \gamma_j \beta_j \frac{\partial \, \mathbb{P}_j(W^A < P^A)}{\partial P^A},$$

$$= \gamma_j \beta_j \left(\frac{\partial \, \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B < P^B)}{\partial P^A} + \frac{\partial \, \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B \geq P^B)}{\partial P^A} \right), \tag{12}$$

$$> \gamma_j \beta_j \frac{\partial \, \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B \geq P^B)}{\partial P^A}.$$

We conclude that $\left|\frac{\partial Z_1}{\partial P^A}\right| > \frac{\partial Z_2}{\partial P^A}$, and similarly $\left|\frac{\partial Z_2}{\partial P^B}\right| > \frac{\partial Z_1}{\partial P^B}$. Therefore the Jacobian in (11) has positive determinant and is invertible.

By the implicit function theorem, there exists a neighborhood U of (P^A, P^B), a neighborhood V of θ_j, and a function $\psi : V \to U$ such that for all $(x, y) \in \mathbb{R}^2$, $\theta \in \Theta$,

$$(\, (x, y, \theta) \in U \times V \text{ and } Z(x, y, \theta) = 0 \,) \Leftrightarrow (\, \theta \in V \text{ and } (x, y) = \psi(\theta) \,).$$

In particular, $(P^A, P^B) = \psi(\theta_j)$, and we can compute the derivative of ψ:

$$J_\psi(\theta_j) = -J_{Z,(P^A,P^B)}(P^A, P^B, \theta_j)^{-1} J_{Z,\theta_j}(P^A, P^B, \theta_j),$$

$$= \frac{-1}{\frac{\partial Z_1}{\partial P^A}\frac{\partial Z_2}{\partial P^B} - \frac{\partial Z_1}{\partial P^B}\frac{\partial Z_2}{\partial P^A}} \begin{pmatrix} \frac{\partial Z_2}{\partial P^B} & -\frac{\partial Z_1}{\partial P^B} \\ -\frac{\partial Z_2}{\partial P^A} & \frac{\partial Z_1}{\partial P^A} \end{pmatrix} \begin{pmatrix} \frac{\partial Z_1}{\partial \theta_j} \\ \frac{\partial Z_2}{\partial \theta_j} \end{pmatrix},$$

$$= \frac{-1}{\frac{\partial Z_1}{\partial P^A}\frac{\partial Z_2}{\partial P^B} - \frac{\partial Z_1}{\partial P^B}\frac{\partial Z_2}{\partial P^A}} \begin{pmatrix} \frac{\partial Z_2}{\partial P^B}\frac{\partial Z_1}{\partial \theta_j} - \frac{\partial Z_1}{\partial P^B}\frac{\partial Z_2}{\partial \theta_j} \\ -\frac{\partial Z_2}{\partial P^A}\frac{\partial Z_1}{\partial \theta_j} + \frac{\partial Z_1}{\partial P^A}\frac{\partial Z_2}{\partial \theta_j} \end{pmatrix}. \tag{13}$$

We only need to know the sign of each term to conclude about the variations of ψ. We already know the sign of the derivatives in P^A and P^B, so we only need those in θ_j. The terms of Z_1 that depend on θ_j are $\sum_{j\in[K]} \gamma_j(1 - \beta_j)\mathbb{P}_{j,\theta_j}(W^A \geq P^A, W^B < P^B)$. By Lemma 1, $\mathbb{P}_{\theta_j}(W^A \geq P^A, W^B < P^B)$ is decreasing in θ_j, and thus $\frac{\partial Z_1}{\partial \theta_j} < 0$. By the same argument, $\frac{\partial Z_2}{\partial \theta_j}$ is also negative. (Note that here the implicit functions theorem requires that we compute the partial derivatives of Z as if P^A and P^B were not functions of θ_j.)

If we replace each term of the last line of Equation (13) by its signs, we get

$$-\frac{1}{+} \begin{pmatrix} c(- \times -) - (+ \times -) \\ -(+ \times -) + (- \times -) \end{pmatrix} = \begin{pmatrix} c- \\ - \end{pmatrix}.$$

We conclude that ψ and therefore P^A and P^B are decreasing in θ_j.

Note that we require $\theta_j \in \mathring{\Theta}$ because if one of the θ_j is maximal, i.e., the distribution is fully correlated (W^B is a deterministic function of W^A), then the V_1 metrics are not differentiable at this point. However, they are continuous, therefore they are increasing on the whole interval Θ. Moreover, if the distribution is not fully correlated when θ is maximal, then we can replace $\mathring{\Theta}$ by Θ in the statement of the lemma.

B.7   Proof of Corollary 1.

By Lemma 5, P^A and P^B are decreasing in both θ_1 and θ_2, thus for $i \neq j \in [K]$:

$$\frac{dV_\emptyset^{G_j}}{\partial \theta_i} = \frac{d\mathbb{P}_{j,\theta_j}(W^A < P^A, W^B < P^B)}{d\theta_i}$$

$$= \left(\frac{\partial \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B < P^B)}{dP^A}, \frac{\partial \mathbb{P}_{j,\theta_j}(W^A < P^A, W^B < P^B)}{dP^B} \right) \cdot \left(\frac{dP^A}{d\theta_i}, \frac{dP^B}{d\theta_i} \right)^T$$

$$< 0.$$

Since the total capacity (of the two colleges) is constant, the mass of unmatched student must also be constant. Therefore, we have

$$\gamma_i V_\emptyset^{G_i} + \sum_{j \neq i} \gamma_j V_\emptyset^{G_j} = 1 - \alpha^A - \alpha^B.$$

By differentiating this equation we get

$$\gamma_i \frac{dV_\emptyset^{G_i}}{d\theta_i} + \sum_{j \neq i} \gamma_j \frac{dV_\emptyset^{G_j}}{d\theta_i} = 0$$

$$\Leftrightarrow \frac{dV_\emptyset^{G_i}}{d\theta_i} = -\frac{1}{\gamma_i} \sum_{j \neq i} \gamma_j \frac{dV_\emptyset^{G_j}}{d\theta_i}$$

$$\Rightarrow \frac{dV_\emptyset^{G_i}}{d\theta_i} > 0$$

which proves the first part of Corollary 1. Moreover, since $L(G_i, G_j) = |V_\emptyset^{G_i} - V_\emptyset^{G_j}|$, assume that $V_\emptyset^{G_i} < V_\emptyset^{G_j}$, then $L(G_i, G_j) = V_\emptyset^{G_j} - V_\emptyset^{G_i}$, and using the first part of the result we get that $\frac{dL(G_i, G_j)}{d\theta_i} < 0$ and $\frac{dL(G_i, G_j)}{d\theta_j} > 0$, which proves the second part.

B.8 Proof of Proposition 3.

V_1 is a convex combination of the first choice functions that are increasing in all θ_j. Moreover it is continuous, and we assumed Θ to be an interval, so the set of possible values for V_1 is an interval, say $[V_1^{\min}, V_1^{\max}]$. Fix $V \in (V_1^{\min}, V_1^{\max})$, and consider the solutions of the equation $V(\theta) = V$. By continuity, this equation has a solution. The implicit function theorem applied to express some θ_j (the choice of j does not matter) as a function ϕ of all the other elements of θ shows that the solutions of $V(\theta) = V$ is a connected subset of Θ^K, and also an hypersurface because the function ϕ is monotonous in all θ_i (this comes from the fact that V_1 is itself monotonous). This proves the first part of the proposition.

Let us choose two groups G_i, G_j, and fix all θ_ℓ for $\ell \neq i, j$. We apply the implicit function theorem to express θ_j as a function of θ_i, which shows that there exists an interval $U := [\underline{\theta}, \bar{\theta}] \subseteq \Theta$ and a differentiable function $\phi : U \to \Theta$ such that $(\theta_i \in U$ and $\theta_j = \phi(\theta_i)) \implies V_1(\theta) = V$. Since $V(\theta)$ is increasing in all arguments, ϕ is necessarily decreasing. If we keep $\theta_j = \phi(\theta_i)$, then $\frac{dV_\emptyset^{G_i}}{d\theta_i} = \frac{\partial V_\emptyset^{G_i}}{\partial \theta_i} + \frac{\partial V_\emptyset^{G_i}}{\partial \theta_j} \phi'(\theta_i)$, which is positive by Corollary 1, so $(\theta_i, \theta_j) = (\underline{\theta}, \phi(\underline{\theta}))$ minimizes $V_\emptyset^{G_i}$, and $(\theta_i, \theta_j) = (\bar{\theta}, \phi(\bar{\theta}))$ maximizes it. The same reasoning shows that those two points respectively maximize and minimize $V_\emptyset^{G_j}$. Finally, since $V_\emptyset^{G_i}$ is increasing and $V_\emptyset^{G_j}$ decreasing, $L(G_i, G_j) = |V_\emptyset^{G_i} - V_\emptyset^{G_j}|$ has a unique local (and therefore global) minimum.
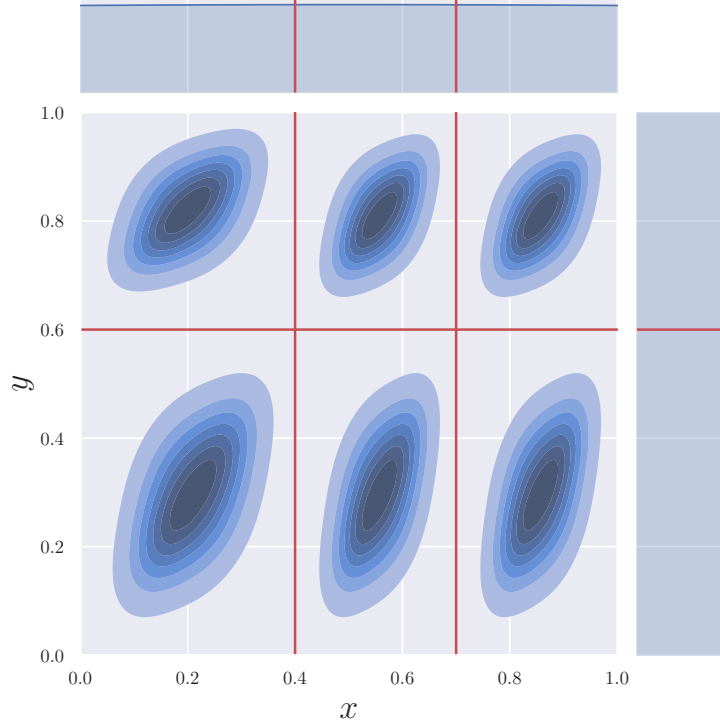
Figure 6: Illustration of the distribution f_θ with three priority classes at A (30% of applicants in the first class, 30% in the second, 40% in the third), two priority classes at B (40% in the first class, 60% in the second), and correlation $\theta = 0.5$.

B.9   Proof of Proposition 4.

We start by building a distribution family that can represent both STB and MTB for two values of the parameter. The priority classes are $Q_1^A, \ldots Q_{n_A}^A$ and $Q_1^B, \ldots Q_{n_B}^B$, and we denote by $\kappa_C^j = \eta(Q_j^C)$ the mass of students inside class j of college C. Let $a_0 = 0, a_1 = \kappa_A^1, a_2 = \kappa_A^1 + \kappa_A^2, \ldots, a_{n_A} = 1$, such that they form a partition of $[0, 1]$ with the j-th segment having length κ_A^j. Define $b_0, \ldots, b_{n_B}$ similarly. Finally, for any $i \leq n_A, j \leq n_B$, let $\kappa^{i,j} = \eta(Q_i^A \times Q_j^B)$ be the mass of students belonging to class i for A and class j for B.

Let ϕ_θ be the pdf of the Gaussian copula with uniform marginals on $[0, 1]^2$ and covariance θ. For any $\theta \in [-1, 1]$, let $f_\theta : [-1, 1]^2 \to \mathbb{R}$ be defined as:

$$f_\theta(x, y) = \kappa^{i,j} \phi_\theta\left(\frac{x - a_{i-1}}{\kappa_A^i}, \frac{y - b_{j-1}}{\kappa_B^j}\right) \text{ with } a_{i-1} \leq x \leq a_i, \, b_{j-1} \leq y \leq b_j$$

Defined this way, f_r is a pdf since it is non-negative and has integral 1. The marginals are uniform and do not depend on θ. Moreover, the integral of f_r over each rectangle $Q_i^A \times Q_j^B$ is $\kappa^{i,j}$, and each rectangle contains a "copy" of the Gaussian copula adjusted to its dimensions. There is no "spill" between classes: if student s is in a higher priority class at college C than student s', then s

will have a higher score with probability 1. If for all i, j, $\kappa^{i,j} > 0$, and $\theta \notin \{-1, 1\}$, then f_θ has full support. This distribution is depicted in Figure 6.

We can verify that this definition recovers MTB and STB: if $\theta = 0$, if two students are in the same priority class for a college, they have the same ex-ante probability of getting a seat there, and if they also are in the same priority class for the other college (i.e., they are in the same rectangle $Q_i^A \times Q_j^B$), the result of this second tie-breaking is independent from the first one. When $\theta = 1$, if two students are in the same priority class for a college, have the same ex-ante probability of getting a seat there, but if they also are in the same priority class for the other college (i.e., they are in the same rectangle $Q_i^A \times Q_j^B$), the winner of the tie-breaking is the same as for the first college since scores inside the rectangle are perfectly correlated. In that case, the distribution does not have full support but this is not an issue as explained in Remark 1. Therefore MTB is the case $\theta = 0$ and STB $\theta = 1$.

Let us now prove the proposition:

1.
   - The family $(f_\theta)_{\theta \in [-1,1]}$ is differentiable by differentiability of the Gaussian copula. It is also coherent (except for the (x, y) such that $x = a_i$ or $y = b_j$, i.e., on the sides of rectangles, in which case the cdf is constant and not increasing). Therefore by applying Theorem 1, E is either increasing or constant.

   - Moreover, the case where it could be constant can only happen if there are several priority classes, so if there is only one it is strictly increasing.

   - Let us look into the multiple priority classes case. Suppose that $\exists \theta \in \Theta$ such that $P^A(\theta) \neq a_i$ and $P^B(\theta) \neq b_j$ for all i, j. We can then apply Theorem 1, and deduce that V_1^A and V_1^B are increasing in all θ_j. If there exists no such θ, it implies that P^A, P^B are constant in θ and so are V_1^A and V_1^B. However, as any perturbation of either $\gamma, \beta \, or \, \alpha$ would change the cutoffs, and resolve the issue, the set of problematic values of (γ, β, α) has Lebesgue measure 0.

2. Finally, Corollary 1 can be applied with the same adjustments, which gives the fourth point.