



HAL
open science

Anti-windup-like Compensator Synthesis for Discrete-Time Quantized Control Systems

Samer Alsamadi, Francesco Ferrante, Sophie Tarbouriech

► **To cite this version:**

Samer Alsamadi, Francesco Ferrante, Sophie Tarbouriech. Anti-windup-like Compensator Synthesis for Discrete-Time Quantized Control Systems. ROCOND 2022 - IFAC 10th Symposium on Robust Control Design, Aug 2022, Kyoto (Online), Japan. p. 43-48, 10.1016/j.ifacol.2022.09.321 . hal-03671411

HAL Id: hal-03671411

<https://hal.science/hal-03671411v1>

Submitted on 12 Oct 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

Anti-windup-like Compensator Synthesis for Discrete-Time Quantized Control Systems

Samer Alsamadi* Francesco Ferrante**
Sophie Tarbouriech***

* *Centre de Génie Industriel, IMT Mines Albi, Albi, France.*
samer.alsamadi@mines-albi.fr

** *Department of Engineering, University of Perugia, Perugia, Italy.*
francesco.ferrante@unipg.it

*** *LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France.*
tarbour@laas.fr

Abstract: This paper addresses the problem of designing an anti-windup like compensator for discrete-time linear control systems with quantized input. The proposed compensator provides a correction signal proportional to the quantization error that is fed to the controller. The compensator is designed to ensure that solutions to the closed-loop systems converge in finite time into a compact set containing the origin that can be tuned by the designer. A numerically tractable algorithm with feasibility guarantees is provided for the design of the compensator. The proposed results are illustrated on an academic example and an open-loop unstable aircraft system.

Copyright © 2022 The Authors. This is an open access article under the CC BY-NC-ND license (<https://creativecommons.org/licenses/by-nc-nd/4.0/>)

Keywords: Quantized control, anti-windup like compensator design, LMIs, stability.

1. INTRODUCTION

1.1 Background and motivation

Recent technological enhancements have enabled the design of a new generation of control systems combining physical interactions with computational and communication abilities. The rapid spread of these technologies is related to their enormous advantages in terms of scalability, ease of maintenance, and high computational resources. This has largely impacted numerous applications such as transportation systems, autonomous robotics, and energy delivery systems, just to mention a few. This new trend has also had a strong impact in modern control systems, which are by now characterized by the interplay of digital controllers and/or digital instrumentation and physical systems (Murray et al. (2002)).

One particular problem in this context resides in the presence of quantizers in control loops, especially when the quantization affects the input of the system. Quantization is generally used to reduce data traffic load in communication channels to comply with limited bandwidth constraints. As a side effect, quantization introduces nonlinearity into the closed loop. The unwanted effects of this nonlinearity include chaotic behaviors, additional equilibrium points, and limit cycles as stated in Delchamps (1990), Ceragioli et al. (2010), Liberzon (2003b), and Tarbouriech and Gouaisbaut (2012). Over the last few years, researchers mostly focused on the analy-

sis of quantized control systems involving continuous-time plants/controllers; see, e.g., Brockett and Liberzon (2000), Liberzon (2003a), Fridman and Dambrine (2009), Ferrante et al. (2015), and Ferrante et al. (2020). On the other hand, due to the inherent relationship between digital control and quantization, the analysis of quantization in discrete-time control systems is a relevant problem in applications. This has pushed the community to address quantization in a discrete-time setting. Notable results on stability analysis and control design of discrete-time quantized control systems can be found in Picasso and Colaneri (2008), Fu and Xie (2009), Campos et al. (2018), and Ichihara et al. (2018). A novel approach for stability analysis of finite-level quantized discrete-time systems has been recently proposed in Valmorbidia and Ferrante (2020).

1.2 Contributions

The focus of this paper is on discrete-time control systems with input quantization. In this setting, we propose an approach for compensating a pre-designed closed-loop control system to reduce the effect of input quantization. The approach we pursue is inspired by the use of anti-windup compensators in saturated control systems; see Tarbouriech et al. (2011); Zaccarian and Teel (2011). In particular, we augment a standard output feedback control system with an additional static compensator loop. This loop feeds a signal that is proportional to the quantization error back into the controller. We show how this simple augmentation, if suitably tuned, can dramatically reduce the effect of the input quantization on the closed-loop system response. A similar idea has been proposed in Tarbouriech et al. (2018) in the context of control systems sub-

* This work has been supported in part by the ANR Labex CIMI (grant ANR-11-LABX- 0040) within the French State Program "Investissement d'Avenir".

ject to input backlash. In the context of continuous-time quantized control systems, the design of dynamic anti-windup loops is proposed in Sofrony and Turner (2015) to ensure specific L_2 -gain performance.

For this class of augmented control systems, the main contribution of our paper consists of establishing sufficient conditions to ensure uniform global finite-time convergence of the closed-loop system state into a compact set containing the origin with tunable shape. Later these conditions are used to devise an iterative design algorithm for the compensator that is based on semidefinite programming (SDP). Under some mild assumptions, the algorithm is guaranteed to yield a feasible solution.

The paper is organized as follows. Section 2 states the class of systems under consideration and the problem we solve. Sufficient conditions for designing the compensator are presented in Section 3. Section 4, building upon the results in Section 3, illustrates the iterative design algorithm we present in the paper. Section 5 showcases the results on two examples borrowed from the literature. Section 6 ends the paper with some concluding remarks. Auxiliary definitions and results are reported in Appendix A.

Due to lack of space, proofs of the results are omitted and can be found in the unabridged version of the paper (Alsamadi et al. (2022)) available on-line.

1.3 Notation

The symbol \mathbb{N} denotes the set of nonnegative integers, \mathbb{R} ($\mathbb{R}_{\geq 0}$) represents the set of real (nonnegative) numbers, \mathbb{R}^n is the n -dimensional Euclidean space, $\mathbb{R}_{>0}^n$ is the positive (open) orthant in \mathbb{R}^n , and $\mathbb{R}^{n \times m}$ represents the set of $n \times m$ real matrices, and \mathbb{B} is the closed unitary ball in the Euclidean norm. The symbol \mathbb{S}_+^n (\mathbb{S}_{++}^n) stands for the set of $n \times n$ symmetric positive semidefinite (definite) matrices, \mathbb{D}_+^n denotes the set of $n \times n$ diagonal positive semidefinite matrices, and \mathbb{P}^n is the set of $n \times n$ symmetric matrices with nonnegative entries. For a vector $x \in \mathbb{R}^n$ (a matrix $A \in \mathbb{R}^{n \times m}$) x^\top (A^\top) denotes the transpose of x (A). The spectral radius of the matrix A is denoted by $\rho(A)$. Given $A \in \mathbb{R}^{n \times n}$, $\text{He}(A) = A + A^\top$. The symbol $A \preceq 0$ ($A \prec 0$) stands for seminegative (negative) definiteness of the symmetric matrix A . The symbol \star stands for symmetric blocks is symmetric matrices. Given a symmetric matrix A , $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ stand, respectively, for the largest and smallest eigenvalue of A . Given $A \in \mathbb{R}^{n \times n}$, the notation $\mathcal{E}(A) = \{x \in \mathbb{R}^n : x^\top A x \leq 1\}$ is used. The function $\text{sign}: \mathbb{R} \rightarrow \{-1, 1\}$ is defined for all $x \in \mathbb{R}$ as follows: $\text{sign}(x) = 1$ if $x \geq 0$ and -1 otherwise. The symbol $|x|_S := \inf_{y \in S} |x - y|$ denotes the distance of the point $x \in \mathbb{R}^n$ to the nonempty set $S \subset \mathbb{R}^n$. The symbol $\lfloor x \rfloor$ indicates the floor of the real number x . The symbol $L_V(c)$ denotes the c -sublevel set of the function V , i.e., $L_V(c) := \{x \in \text{dom } V : V(x) \leq c\}$.

2. PROBLEM STATEMENT

We consider the following discrete-time plant:

$$\begin{cases} x_p^+ = A_p x_p + B_p q_\Theta(u_p) \\ y_p = C_p x_p \end{cases} \quad (1)$$

where $x_p \in \mathbb{R}^{n_p}$, $u_p \in \mathbb{R}^{n_u}$, $y_p \in \mathbb{R}^{n_y}$ are, respectively, the plant state, control input, and measured output. Matrices A_p , B_p , and C_p are real and assumed to be known. The function q_Θ is the so-called *uniform quantizer*, which is defined next:

$$u \mapsto q_\Theta(u_p) = (q_{\theta_1}(u_{p1}), q_{\theta_2}(u_{p2}), \dots, q_{\theta_{n_u}}(u_{pn_u}))$$

where $\Theta = (\theta_1, \theta_2, \dots, \theta_{n_u}) \in \mathbb{R}_{>0}^{n_u}$ represents the vector of quantization levels of each channel and, for all $u \in \mathbb{R}$, $\theta \in \mathbb{R}_{>0}$

$$q_\theta(u) := \theta \text{sign}(u) \left\lfloor \frac{|u|}{\theta} \right\rfloor.$$

The plant is controlled via the following dynamic output feedback controller:

$$\begin{cases} x_c^+ = A_c x_c + B_c u_c + v \\ y_c = C_c x_c + D_c u_c \end{cases} \quad (2)$$

where $x_c \in \mathbb{R}^{n_c}$, $y_c \in \mathbb{R}^{n_u}$, and $u_c \in \mathbb{R}^{n_y}$ are, respectively, the controller state, input and output. The parameters A_c , B_c , C_c , and D_c are real matrices of adequate dimensions defining the controller dynamics. The signal $v \in \mathbb{R}^{n_c}$ is an additional input to be designed to mitigate the effect of input quantization. This signal is reminiscent of an anti-windup correction in saturated feedback control systems Zaccarian and Teel (2011). The use of anti-windup-like schemes in quantized control systems and systems subject to backlash have been investigated, respectively, in Sofrony and Turner (2015) and Tarbouriech et al. (2018). Inspired by the constructions in Tarbouriech et al. (2018), the signals v is selected as follows:

$$v = E(q_\Theta(y_c) - y_c) = E\psi_\Theta(y_c) \quad (3)$$

where $E \in \mathbb{R}^{n_c \times n_u}$ is a gain to be designed. Our goal is to design E to reduce the effect of input quantization on the closed-loop system.

The interconnection of the plant (1) and the controller (2)-(3) is obtained by setting $u_c = y_p$ and $u_p = y_c$. Thus, by taking as a state $x = (x_p, x_c)$, the closed-loop system reads:

$$x^+ = \underbrace{\begin{bmatrix} A_p + B_p D_c C_p & B_p C_c \\ B_c C_p & A_c \end{bmatrix}}_{A_{CL}} x + \underbrace{\begin{bmatrix} B_p \\ E \end{bmatrix}}_{B_{CL} + RE} \psi_\Theta(Hx) \quad (4)$$

where $H := [D_c C_p \ C_c]$ and $R := \begin{bmatrix} 0_{n_p \times n_c} \\ I_{n_c} \end{bmatrix}$. The general problem we address in this paper can be formalized as follows:

Problem 1. Given plant (1) and controller (2) parameters, and a closed set \mathcal{U} containing the origin, design a gain E such that there exists a compact set $\mathcal{S} \subset \mathcal{U}$, containing the origin, that is uniform globally finite-time attractive *UGFTA* (see Definition 1 in Appendix A) for the closed-loop system (4). \diamond

Remark 1. The formulation of Problem 1 ensures that the state of the closed-loop system is bounded due to \mathcal{S} being compact. The set \mathcal{U} , which is not necessarily compact, is introduced to enable the designer to shape the response of the system for large times. For example, \mathcal{U} can be selected to ensure that the plant state converges close to zero. \diamond

3. MAIN RESULTS

We assume that the closed set \mathcal{U} introduced in Problem 1 is defined as follows:

$$\mathcal{U} = \{x \in \mathbb{R}^{n_p+n_c} : x^\top U x \leq 1\}$$

where $U \in \mathbb{S}_+^{n_p+n_c}$ is given.

The following result provides sufficient conditions for the solution to Problem 1. The result relies on the use of the sector conditions originally introduced in Ferrante et al. (2015) and later extended in Ferrante et al. (2020) for the mapping ψ_Θ .

Proposition 1. If there exist $P \in \mathbb{S}_{++}^n$, $S_1, S_2 \in \mathbb{D}_+^{n_u}$, $E \in \mathbb{R}^{n_c \times n_u}$, and $\tau \in (0, 1)$ such that the following conditions hold:

$$\begin{bmatrix} (\tau - 1)P & -H^\top S_2 & A_{CL}^\top P \\ \star & -S_1 - 2S_2 & (B_{CL} + RE)^\top P \\ \star & \star & -P \end{bmatrix} \prec 0 \quad (5a)$$

$$\Theta^\top S_1 \Theta - \tau \leq 0 \quad (5b)$$

$$U - P \preceq 0. \quad (5c)$$

Then, $\mathcal{S} = \mathcal{E}(P)$ is included in \mathcal{U} and is UGFTA for the closed-loop system (4). \square

The result given next, which plays a relevant role in the construction of the design algorithm presented in Section 4, shows that (5a) and (5b) are always feasible as long as the quantization-free uncompensated ($E = 0$) system closed-loop system (4) is asymptotically stable.

Proposition 2. If $\rho(A_{CL}) < 1$, then (5a) and (5b) are feasible with $E = 0$ and any $\tau \in (0, 1)$ such that $\sqrt{1 - \tau} > \rho(A_{CL})$. \square

4. COMPENSATOR DESIGN AND OPTIMIZATION ASPECTS

In this section, we show how the results proposed in this paper can be used to devise a computationally affordable design algorithm for the compensator gain E based on SDP tools.

4.1 Optimization

The formulation of Problem 1 is based on a preassigned set \mathcal{U} and hence on a specific selection of the matrix U . On the other hand, in practice the matrix U can be left as an extra degree of freedom and the design of the compensator gain E recast as the following optimization problem:

$$\begin{aligned} & \underset{P, E, S_1, S_2, \tau, U}{\text{maximize}} && \omega(U) \\ & \text{subject to} && (5), h(U) = 0 \\ & && U \in \mathbb{S}_+^{n_p+n_c}, P \in \mathbb{S}_{++}^{n_p+n_c}, S_1, S_2 \in \mathbb{D}_+^{n_u}, \end{aligned} \quad (6)$$

where $\omega: \mathbb{S}_+^{n_p+n_c} \rightarrow \mathbb{R}_{\geq 0}$ associates to U a suitable “size” and $h: \mathbb{S}_+^{n_p+n_c} \rightarrow \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$ prescribes structural constraints on U . As an example, if the primary objective is to keep the plant state x_p as close as possible to zero, then a possible selection of the functions ω and h is as follows:

$$h(U) = \begin{bmatrix} 0_{n_p \times n_p} & U_{12} \\ \star & U_{22} \end{bmatrix}, \omega(U) = \text{trace}(U)$$

where $U_{1,2} \in \mathbb{R}^{n_p \times n_u}$ and $U_{2,2} \in \mathbb{R}^{n_c \times n_c}$ are the corresponding blocks of the matrix U . Indeed, the selection of h implies that

$$\mathcal{U} = \{(x_p, x_c) \in \mathbb{R}^{n_p+n_c} : x_p^\top U_{1,1} x_p \leq 1\}$$

for some $U_{1,1} \in \mathbb{S}_+^{n_p}$, while the selection of ω ensures that the size of the ellipsoidal set $\mathcal{E}(U_{1,1})$ is minimized.

Remark 2. Typically, the constraint induced by the function h can be eliminated by suitably structuring the matrix U . Therefore, henceforth such a constraint will be dropped in optimization problem (6).

4.2 SDP-based compensator design

Although the function ω can be generally selected as a linear function, the fact that (5a) is bilinear in the decision variables E , τ , and P renders optimization problem (6) numerically intractable. To overcome this drawback, next we show how optimization problem (6) can be (suboptimally) solved via a sequence of semidefinite programs, i.e., optimization problems with linear objective over linear matrix inequality constraints. To this end, we rely on the convex-concave decomposition approach proposed in Dinh et al. (2011). In a nutshell, such an approach consists of expressing bilinear terms via a convex-concave decomposition (this is always possible). As second step, provided an initial feasible point is available, the concave terms are linearized around the given feasible point. Later, the resulting linearized (SDP) problem is solved and the solution obtained is used to linearize again the original concave terms. This basically leads to a sequence of SDP problems that can be solved iteratively. This approach enjoys two interesting properties that makes it appealing to devise an iterative design algorithm. If the initial point is feasible for the original problem, then the algorithm never terminates due to infeasibility; the initial point provides always a feasible solution. Another key feature of this approach is that, since any psd-concave function is upper bounded by its linearization (see Lemma 1), feasible solutions to the linearized problem are feasible for the original problem.

To deploy this approach for the solution to optimization problem (6), as first step we rewrite (5a) in the following equivalent linear-bilinear decomposed form:

$$\underbrace{\begin{bmatrix} -P & -H^\top S_2 & A_{CL}^\top P \\ \star & -S_1 - 2S_2 & B_{CL}^\top P \\ \star & \star & -P \end{bmatrix}}_{\mathcal{L}(P, S_1, S_2)} + \text{He} \left(\underbrace{\begin{bmatrix} \frac{\tau}{2} I & 0 \\ 0 & E^\top R^\top \\ 0 & 0 \end{bmatrix}}_{\mathcal{X}^\top(\tau, E)} \underbrace{\begin{bmatrix} P & 0 & 0 \\ 0 & 0 & P \end{bmatrix}}_{\mathcal{Y}(P)} \right) \prec 0.$$

The latter, by dropping the dependency on the decision variables, can be equivalently rewritten in the following psd convex-concave decomposed form:

$$\mathcal{L} + \mathcal{X}^\top \mathcal{X} + \mathcal{Y}^\top \mathcal{Y} - (\mathcal{X} - \mathcal{Y})^\top (\mathcal{X} - \mathcal{Y}) \prec 0$$

which, by Schur complement’s lemma, is equivalent to:

$$\begin{bmatrix} \mathcal{L} - \mathcal{X}^\top \mathcal{X} - \mathcal{Y}^\top \mathcal{Y} + \text{He}(\mathcal{X}^\top \mathcal{Y}) & \mathcal{X}^\top & \mathcal{Y}^\top \\ \star & -I & 0 \\ \star & \star & -I \end{bmatrix} \prec 0. \quad (7)$$

At this stage, given $q^{(0)} := (\tau_0, P_0, E_0) \in \mathbb{R} \times \mathbb{S}_{++}^{n_p+n_c} \times \mathbb{R}^{n_c \times n_u}$, the “linear inner approximation” $O^{q^{(0)}}$ of opti-

mization problem (6) around (τ_0, P_0, E_0) can be obtained by linearizing constraint (7) around $q^{(0)}$. This leads to

$$O^{q^{(0)}} : \begin{cases} \text{maximize}_{P, S_1, S_2, \tau, U, E} & \omega(U) \\ \text{s.t.} & \mathcal{M}(P, S_1, S_2, E, \tau | P_0, E_0, \tau_0) \prec 0, \\ & \Theta^\top S_1 \Theta - \tau \leq 0, \\ & U - P \preceq 0, \\ & U \in \mathbb{S}_+^{n_p+n_c}, P \in \mathbb{S}_{++}^{n_p+n_c} \\ & S_1, S_2 \in \mathbb{D}_+^{n_u}, \end{cases}$$

where:

$$\underbrace{\begin{bmatrix} \mathcal{R}(P, S_1, S_2, E, \tau | P_0, E_0, \tau_0) & \mathcal{W}(\tau, P, E) \\ \star & -I \end{bmatrix}}_{\mathcal{M}(P, S_1, S_2, E, \tau | P_0, E_0, \tau_0)}$$

$$\begin{aligned} \mathcal{R}(P, S_1, S_2, E, \tau | P_0, E_0, \tau_0) &:= \mathcal{L}(P, S_1, S_2, P) \\ &+ \mathcal{Q}(\tau_0, P_0, E_0) + (D\mathcal{Q}(\tau_0, P_0, E_0))(\tau - \tau_0, P - P_0, E - E_0), \\ \mathcal{W}(\tau, P, E) &:= [\mathcal{X}^\top(\tau, E) \quad \mathcal{Y}^\top(P)], \end{aligned}$$

in which

$$(\tau, P, E) \mapsto \mathcal{Q}(\tau, P, E) := -\mathcal{X}^\top \mathcal{X} - \mathcal{Y}^\top \mathcal{Y} + \text{He}(\mathcal{X}^\top \mathcal{Y})$$

As mentioned earlier, the applicability of the convex-concave decomposition approach in Dinh et al. (2011) requires the knowledge of an initial feasible solution to optimization problem (6). To this end, we make use of Proposition 2 and select the feasible initial point as the solution to the following optimization problem:

$$\begin{aligned} & \text{maximize}_{P, S_1, S_2, \tau, U} && \omega(U) \\ & \text{subject to} && \begin{bmatrix} (\tau - 1)P & -H^\top S_2 & A_{CL}^\top P \\ \star & -S_1 - 2S_2 & B_{CL}^\top P \\ \star & \star & -P \end{bmatrix} \prec 0 \\ & && \Theta^\top S_1 \Theta - \tau \leq 0 \\ & && U - P \preceq 0 \\ & && U \in \mathbb{S}_+^{n_p+n_c}, P \in \mathbb{S}_{++}^{n_p+n_c}, S_1, S_2 \in \mathbb{D}_+^{n_u}. \end{aligned} \quad (8)$$

Proposition 2 ensures that (8) is always feasible provided that $\rho(A_{CL}) < 1$. Moreover, since whenever τ is fixed (and ω is linear) (8) is an SDP program, a feasible solution to (8) can be easily computed by performing a line search on the variable τ in the interval $(0, 1)$.

Based on the steps presented so far, our approach to solve optimization problem (6) is summarized in Algorithm 1.

Remark 3. An alternative approach to handle the bilinear constraint (5a) consists of the use of a coordinate descent-type algorithm, i.e., alternatively and iteratively fixing some of the decision variables and optimizing with respect to the others; see, e.g., Peaucelle and Arzelier (2001) and the references therein. However, numerical experiments show that this latter approach lead to more conservative results.

5. NUMERICAL EXAMPLES

In this section, we showcase the application of the methodology proposed in the paper in two examples. The first example, which is more academic, pertains to an unstable nonminimum phase plant. The second example is of practical interest and concerns a linearized model of an

Algorithm 1: Optimal compensator synthesis

Input: Matrices A_{CL}, B_{CL} , quantization levels vector Θ , a linear function $\omega: \mathbb{S}^{n_p+n_c} \rightarrow \mathbb{R}$, $k_{\max} \in \mathbb{N}_{>0}$, and $\varepsilon > 0$.

- 1 **Initial solution:** Solve (8) via a line search on $\tau \in (0, 1)$. Let P_0, τ_0 , and U_0 the values associated to the corresponding solution. Set $k = 0$ and $E_0 = 0$;
 - 2 **while** $k < k_{\max}$ **do**
 - 3 Solve SDP problem $O(\tau_k, P_k, E_k)$;
 - 4 $\tau_{k+1} \leftarrow \tau, P_{k+1} \leftarrow P, U_{k+1} \leftarrow U,$
 $E_{k+1} \leftarrow E$;
 - 5 **if** $|\omega(U_{k+1}) - \omega(U_k)| \leq \varepsilon$ **then**
 - 6 **break**;
 - 7 **end**
 - 8 $k \leftarrow k + 1$;
 - 9 **return** E
-

aircraft. Numerical solutions to LMIs are obtained through the solver *MOSEK ApS* (2019) and coded in Matlab via *YALMIP* Löfberg (2004).

Example 1. We consider the following nonminimum phase unstable discrete-time plant borrowed from Fu and Xie (2005):

$$\begin{aligned} x_p^+ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ -3 & 1 & 0 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} q_\Theta(u_p) \\ y_p &= [-3 \ 1 \ 0] x \end{aligned} \quad (9)$$

We assume that $\Theta = 0.5$ and that the plant is controlled by the following LQG feedback stabilizing controller:

$$\begin{aligned} x_c^+ &= \begin{bmatrix} -4.6 & 2.53 & 0 \\ -9.2 & 3.39 & 0 \\ -0.0609 & 0.0203 & 0 \end{bmatrix} x_c + \begin{bmatrix} -1.53 \\ -3.07 \\ 0.98 \end{bmatrix} u_c \\ y_c &= [0 \ -1.67 \ 0] x_c. \end{aligned}$$

In this example, we select the following structure of the matrix U :

$$U = c \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where $c \geq 0$ is decision variable and $\omega(U) = \text{trace}(U)$. Setting $\varepsilon = 10^{-4}$, Algorithm 1 terminates in 57 iterations and returns the following gain for the compensator:

$$E = \begin{bmatrix} 0.0379 \\ 1.0645 \\ 0.01 \end{bmatrix}.$$

In Fig. 1 we report the evolution of the plant state, from the initial condition $x_p(0) = (1, 2, -1), x_c(0) = 0$, obtained with and without the use of the compensator. The picture clearly shows that the proposed compensation strategy leads to a dramatic improvement in the plant state response.

Example 2. We consider the discretized¹ linearized model of the short period longitudinal dynamics of TAFE (Tailless Advanced Fighter Aircraft) in Cristofaro et al. (2019)

¹ System (10) is obtained by performing a ZOH-discretization of the model in Cristofaro et al. (2019) with a sampling period $T_s = 0.1$.

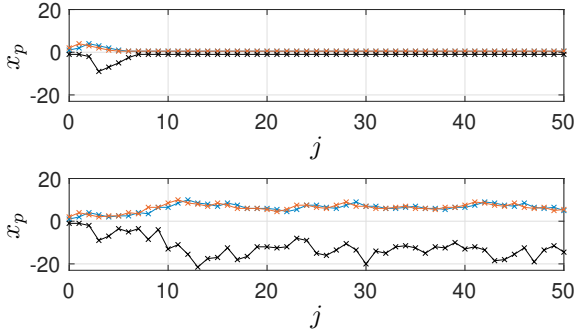


Fig. 1. Example 1. Evolution of the plant state from $x_p(0) = (1, 2, -1)$, $x_c(0) = 0$. With compensation (top) and without compensation (bottom).

$$\begin{bmatrix} \alpha^+ \\ \psi^+ \end{bmatrix} = \begin{bmatrix} 0.94 & 0.087 \\ 0.516 & 0.836 \end{bmatrix} \begin{bmatrix} \alpha \\ \psi \end{bmatrix} + \begin{bmatrix} 0.0364 \\ 0.729 \end{bmatrix} u \quad (10)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \psi \end{bmatrix}.$$

The variables ψ (rad s^{-1}) and α (rad) represent, respectively, the body axis pitch rate and the deviation of the angle of attack. The control input u (rad) corresponds to the deviation of the elevator deflection. We focus on a scenario in which the control input is quantized via a uniform quantizer with quantization level $\Theta = 0.0035$, which corresponds to a quantization of 0.2 [deg] of the elevator deflection. The system is controlled via the following output feedback stabilizing controller:

$$x_c^+ = \begin{bmatrix} 0.706 & -1.58 \\ -4.17 & -1.88 \end{bmatrix} x_c + \begin{bmatrix} 1.62 \\ 1.68 \end{bmatrix} u_c$$

$$y_c = \begin{bmatrix} -6.43 & -1.43 \end{bmatrix} x_c.$$

In this example, we select $U = P$ and $\omega(U) = \text{trace}(U)$. Setting $\varepsilon = 10^{-3}$, Algorithm 1 terminates in 736 iterations and returns the following gain for the compensator:

$$E = \begin{bmatrix} -0.0775 \\ 0.7222 \end{bmatrix}.$$

In Fig. 2 we report a simulation of the evolution of the closed-loop plant state and of the control input from the initial condition $x_p(0) = (\frac{\pi}{180}, 0)$, $x_c(0) = 0$. To further emphasize the benefits of the proposed compensation strategy, in this simulation the compensation is activated at $j = 50$ and deactivated again at $j = 100$. It is interesting to notice that the use of the compensator not only leads to an improved plant state response but also to a reduced control effort.

6. CONCLUSION

This paper addressed the design of a static anti-windup-like loop for linear closed-loop control systems subject to quantized input. The compensator was designed so as to reduce the size of the closed-loop attractor. The approach relied on matrix inequality stability conditions and resulted in an iterative algorithm based on a sequence of semidefinite programs for the design of the compensator gain. The effectiveness of the compensation mechanism was validated via two numerical examples.

This work opens the door to several research lines. In particular, the addition of a supplementary correction

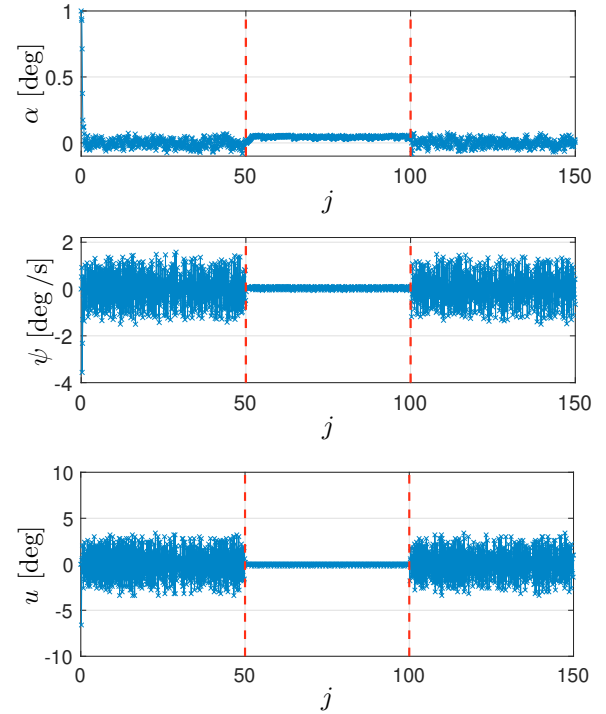


Fig. 2. Example 2. Evolution of the plant states from $x_p(0) = (\frac{\pi}{180}, 0)$, $x_c(0) = 0$. The compensation is activated at $j = 50$ and deactivated at $j = 100$.

signal affecting the output of the controller is under study. This modification is particularly challenging since it leads to discontinuous algebraic loops, which cannot be addressed via existing results for saturated systems. In addition, the extension of the proposed approach to sampled-data systems is currently part of our research.

REFERENCES

- Alsamadi, S., Ferrante, F., and Tarbouriech, S. (2022). Anti-windup-like compensator synthesis for discrete-time quantized control systems. Technical report, arXiv. URL <https://arxiv.org/pdf/2202.04610.pdf>.
- ApS, M. (2019). *The MOSEK optimization toolbox for MATLAB manual. Version 9.2*. URL <http://docs.mosek.com/9.0/toolbox/index.html>.
- Brockett, R. and Liberzon, D. (2000). Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control*, 45(7), 1279–1289.
- Campos, G.C., da Silva, J.M.G., Tarbouriech, S., and Pereira, C.E. (2018). Stabilisation of discrete-time systems with finite-level uniform and logarithmic quantisers. *IET Control Theory & Applications*, 12(8), 1125–1132.
- Ceragioli, F., De Persis, C., and Frasca, P. (2010). Discontinuities and hysteresis in quantized average consensus. In *8th IFAC Symposium on Nonlinear Control Systems, NOLCOS 2010*. Bologna, Italy.
- Cristofaro, A., Galeani, S., Onori, S., and Zaccarian, L. (2019). A switched and scheduled design for model recovery anti-windup of linear plants. *European Journal of Control*, 46, 23–35.
- Delchamps, D.F. (1990). Stabilizing a linear system with quantized state feedback. *IEEE Transactions on Automatic Control*, 35(8), 916–924.

- Dinh, Q.T., Gumussoy, S., Michiels, W., and Diehl, M. (2011). Combining convex–concave decompositions and linearization approaches for solving BMIs, with application to static output feedback. *IEEE Transactions on Automatic Control*, 57(6), 1377–1390.
- Ferrante, F., Gouaisbaut, F., and Tarbouriech, S. (2015). Stabilization of continuous-time linear systems subject to input quantization. *Automatica*, 58, 167–172.
- Ferrante, F., Gouaisbaut, F., and Tarbouriech, S. (2020). On sensor quantization in linear control systems: Krasovskii solutions meet semidefinite programming. *IMA Journal of Mathematical Control and Information*, 37(1), 395–417.
- Fridman, E. and Dambrine, M. (2009). Control under quantization, saturation and delay: An LMI approach. *Automatica*, 45, 2258–2264.
- Fu, M. and Xie, L. (2005). The sector bound approach to quantized feedback control. *IEEE Transactions on Automatic control*, 50(11), 1698–1711.
- Fu, M. and Xie, L. (2009). Finite-level quantized feedback control for linear systems. *IEEE Transactions on Automatic Control*, 54(5), 1165–1170.
- Ichihara, H., Sawada, K., and Tarbouriech, S. (2018). Invariant set analysis for SISO discrete-time polynomial systems with dynamic quantizers. *International Journal of Robust and Nonlinear Control*, 28(17), 5495–5508.
- Liberzon, D. (2003a). Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39, 1543–1554.
- Liberzon, D. (2003b). *Switching in Systems and Control*. Birkhauser, Boston, USA.
- Löfberg, J. (2004). Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*. Taipei, Taiwan.
- Murray, R.M., Astrom, K., Boyd, S., Brockett, R., and Stein, G. (2002). Control in an information rich world. *Report of the Panel on Future Directions in Control Theory*.
- Peaucelle, D. and Arzelier, D. (2001). An efficient numerical solution for H2 static output feedback synthesis. In *Proceedings of the European control conference*, 3800–3805. Porto, Portugal.
- Picasso, B. and Colaneri, P. (2008). Stabilization of discrete-time quantized linear systems: an H_∞/l_1 approach. In *Proceedings of the 47th IEEE Conference on Decision and Control*, 2868–2873. Cancun, Mexico.
- Shapiro, A. (1997). First and second order analysis of nonlinear semidefinite programs. *Mathematical Programming*, 77(1), 301–320.
- Sofrony, J. and Turner, M. (2015). Anti-windup design for systems with input quantization. In *Proceedings of the 54th IEEE Conference on Decision and Control*, 7586–7591. IEEE, Osaka, Japan.
- Tarbouriech, S., Queinnec, I., and Prieur, C. (2018). Non-standard use of anti-windup loop for systems with input backlash. *IFAC Journal of Systems and Control*, 6, 33–42.
- Tarbouriech, S., Garcia, G., da Silva Jr, J.M.G., and Queinnec, I. (2011). *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer Science & Business Media.
- Tarbouriech, S. and Gouaisbaut, F. (2012). Control design for quantized linear systems with saturations. *IEEE Transactions on Automatic Control*, 57(7), 1883–1889.
- Valmorbidia, G. and Ferrante, F. (2020). On quantization in discrete-time control systems: Stability analysis of ternary controllers. In *Proceedings of the 59th IEEE Conference on Decision and Control*, 2543–2548. Jeju Island, Republic of Korea.
- Zaccarian, L. and Teel, A.R. (2011). *Modern Anti-windup Synthesis*. Princeton University Press.

Appendix A. AUXILIARY DEFINITIONS AND RESULTS

A.1 Discrete-time systems

In this paper, we consider autonomous nonlinear discrete-time systems of the form:

$$x^+ = g(x) \quad (\text{A.1})$$

where $x \in \mathbb{R}^n$ is the system state, $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$, and x^+ stands for the value of x after a jump. A function $\phi: \text{dom } \phi \rightarrow \mathbb{R}^n$ is a solution to (A.1) if $\text{dom } \phi = \mathbb{N} \cap \{0, 1, \dots, J\}$ for some $J \in \mathbb{N} \cup \{\infty\}$ and for all $j \in \text{dom } \phi$ such that $j+1 \in \text{dom } \phi$, $\phi(j+1) = g(\phi(j))$. A solution ϕ to (A.1) is said to be maximal if its domain cannot be extended and complete if $\text{dom } \phi$ is unbounded. In particular, notice that maximal solutions to (A.1) are complete.

Definition 1. Let $S \subset \mathbb{R}^n$ be closed. We say that S is uniformly globally finite-time attractive (UGFTA) for (A.1) if there exists a locally bounded function $T: \mathbb{R}_{\geq 0} \rightarrow \mathbb{N}_{\geq 0}$ such that for any maximal solution ϕ to (A.1), $j \geq T(\|\phi(0)\|_S)$ implies that $\phi(j) \in S$. \diamond

A.2 Preliminaries on matrix-valued functions

In this paper, we consider matrix valued functions of the form:

$$X: \mathcal{S} \rightarrow \mathcal{Y} \quad (\text{A.2})$$

where \mathcal{S} is a finite dimensional real linear vector space and $\mathcal{Y} \subset \mathbb{R}^{n \times m}$.

Definition 2. (Differential) Let X be defined as in (A.2). We say that X is differentiable at $x \in \mathcal{S}$ if there exists a linear map $DX(x): \mathcal{S} \rightarrow \mathcal{Y}$ such that:

$$\lim_{\|h\|_{\mathcal{S}} \rightarrow 0} \frac{\|X(x+h) - X(x) - DX(x)h\|_{\mathcal{Y}}}{\|h\|_{\mathcal{S}}} = 0$$

where $\|\cdot\|_{\mathcal{S}}$ and $\|\cdot\|_{\mathcal{Y}}$ are any norms, respectively, on \mathcal{S} and \mathcal{Y} . \diamond

Definition 3. (Shapiro (1997)). Let $\mathcal{C} \subset \mathcal{S}$ be convex and \mathbb{S}^n be the set of $n \times n$ symmetric matrices. A function $X: \mathcal{C} \rightarrow \mathbb{S}^n$ is said to be positive semidefinite convex (psd-convex) on \mathcal{C} if for all $x, y \in \mathcal{C}$ and $t \in [0, 1]$ the following holds:

$$X(tx + (1-t)y) \preceq tX(x) + (1-t)X(y).$$

Furthermore, we say that X is positive semidefinite concave (psd-concave) if $-X$ is psd-convex. \diamond

Lemma 1. (Dinh et al. (2011)) Let $\mathcal{C} \subset \mathcal{S}$ be convex, \mathbb{S}^n be the set of $n \times n$ symmetric matrices, and $X: \mathcal{C} \rightarrow \mathbb{S}^n$ be differentiable on an open neighborhood of \mathcal{C} . Then, X is psd-convex on \mathcal{C} if and only if for all $x, y \in \mathcal{C}$

$$X(y) - X(x) \succeq DX(x)(y - x).$$

\diamond