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Fractional regularity for conservation laws with discontinuous flux

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Abstract

This article deals with the regularity of entropy solutions of scalar conservation laws with discontinuous flux. It is well-known [Adimurthi et al., Comm. Pure Appl. Math. 2011] that the entropy solution for such an equation does not admit BV regularity in general, even when the initial data belongs to BV . Due to this phenomenon, [fractional \$BV^s\$ spaces, where the exponent \$0 < s \leq 1\$ and \$BV = BV^1\$, are required to be wider than \$BV\$](#) . It is a long-standing open question to find the optimal regularizing effect for the discontinuous flux with L^∞ initial data. The optimal regularizing effect in BV^s is proven in an important case using control theory, [and the fractional exponent \$s\$ is at most \$1/2\$, even when the fluxes are uniformly convex](#).

Keywords: Conservation laws, Interface, Discontinuous flux, Cauchy problem, Regularity, BV functions, Fractional BV spaces.

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1. Introduction

This article deals with the regularity aspects of the [entropy](#) solution for the following scalar conservation [laws](#) with discontinuous flux:

$$\begin{cases} u_t + f(u)_x = 0, & \text{if } x > 0, t > 0, \\ u_t + g(u)_x = 0, & \text{if } x < 0, t > 0, \\ u(x, 0) = u_0(x), & \text{if } x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ is [the unknown function](#), $u_0(\cdot) \in L^\infty(\mathbb{R})$ is the initial data and the fluxes f, g are $C^1(\mathbb{R})$ and strictly convex (*i.e.*, f' and g' are increasing functions).

The conservation [laws](#) (1.1) arises in several physical situations and applied subjects. For example, [it occurs](#) naturally in the two-phase flow of a heterogeneous porous medium in petroleum reservoirs [31]. The equation (1.1) is also useful [for understanding](#) the ideal clarifier thickener [14], traffic flow [models](#) with varying road surface conditions [38], and ion etching [commonly used in](#) the semiconductor industry [45]. [These](#) examples are just a [glimpse](#) at the broad applicability of the equation (1.1) in the fields of applied sciences. For more details, see [14, 15, 20, 21].

The equation (1.1) does not have a global classical solution, even for smooth initial data. [Therefore](#), one needs to consider the following notion of a weak solution:

Definition 1.1 (Weak solution). *A function $u \in C(0, T; L^1_{loc}(\mathbb{R}))$ is said to be a weak solution of the problem (1.1) if*

$$\int_0^\infty \int_{\mathbb{R}} u \frac{\partial \phi}{\partial t} + F(x, u) \frac{\partial \phi}{\partial x} dx dt + \int_{\mathbb{R}} u_0(x) \phi(x, 0) dx = 0,$$

for all $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$, where the flux $F(x, u)$ is given as $F(x, u) = H(x)f(u) + (1 - H(x))g(u)$, and $H(x)$ is the Heaviside function.

From the above-defined weak formulation, it can be derived that if interface traces $u^\pm(t) = \lim_{x \rightarrow 0^\pm} u(x, t)$ exist, then at $x = 0$, u satisfies [the Rankine-Hugoniot condition](#), namely, for almost all t ,

$$f(u^+(t)) = g(u^-(t)). \quad (1.2)$$

For [the](#) equation (1.1), the left and right traces u^-, u^+ play important roles in the well-posedness theory and also in determining the regularity of solutions. In [7], [the](#) authors proved the existence of the interface traces via the Hamilton-Jacobi type equation.

It is well known that the conservation laws (1.1) do not have unique weak solutions. To establish uniqueness, an additional condition called the “[entropy condition](#)” is necessary even in the case of $f = g$. Kruřkov [34] provided a generalized entropy condition and proved the uniqueness [in the](#) case where $f = g$. However, (1.1) have the flux discontinuity at the interface, one also needs the “[interface entropy condition](#)” to establish uniqueness, in addition to the Kruřkov entropy. In this article, we use the following notion of the entropy solution.

Definition 1.2 (Entropy solution, [7]). *A weak solution $u \in L^\infty(\mathbb{R} \times [0, T])$ of the problem (1.1) is said to be an entropy solution if the following holds.*

1. u satisfies Kruzkov entropy conditions on each side of the interface $x = 0$, that is, in $\mathbb{R} \setminus \{0\}$.
2. The interface traces $u^\pm(t) = \lim_{x \rightarrow 0^\pm} u(x, t)$ exist for almost all $t > 0$ and they satisfy the following “interface entropy condition” for almost all $t > 0$,
$$|\{t : f'(u^+(t)) > 0 > g'(u^-(t))\}| = 0. \quad (1.3)$$

Uniqueness has been proved in [7] when interface traces exist for a weak solution and they satisfy the entropy condition (1.3). Additionally, the authors obtained useful Lax-Oleinik type explicit formulas for equation (1.1). The notion of ‘A-B entropy solution’ is introduced in [5] and it coincides with (1.3) when $A = \theta_g, B = \theta_f$. The number θ_f is defined by $f(\theta_f) = \min f$ when f admits a minimum and $g(\theta_g) = \min g$. The Lax-Oleinik type formula is also available [6] for the ‘A-B-entropy solutions’. It has been observed [1] that for the case $A < \theta_g$ or $B > \theta_f$, ‘A-B-entropy solutions’ belong to BV space for BV initial data and for $A = \theta_g, B = \theta_f$ total variation of entropy solution can blow up at finite time $t_0 > 0$ for particular BV initial data (see section 1.2 for more details). Therefore, we work with the choice $A = \theta_g, B = \theta_f$. In this article, we rely on the interface entropy condition (1.3), and we use the analysis of characteristics developed as in [7].

The well-posedness theory of the problem has been extensively studied from both numerical and theoretical aspects. We refer to [5, 10, 12, 33, 43] and the references therein. The existence of a solution of (1.1) has been proved by several numerical schemes [4, 9, 26, 47]. However, due to the absence of total variation bounds for entropy solutions, even for initial data $u_0 \in BV$, the convergence of numerical schemes is typically established using the singular mapping technique [4, 46, 47]. The singular mapping technique originates from the fundamental paper of Temple [46]. Recent works have generalized the Godunov-type scheme to handle cases where discontinuities of the flux may have a limit point [26, 27, 28].

Since the entropy solution of (1.1) lacks BV regularity, it is natural to study its regularity in a space that is bigger than BV . More precisely, in this paper, we quantify the sharp regularity of entropy solution of (1.1) in suitable fractional spaces.

Structure of the paper

This paper is organized as follows: In Sections 1.1 and 1.2, we discuss regularity results for scalar conservation laws where $f = g$ and for (1.1), respectively. This leads to Section 1.3, where we state the regularity problems corresponding to equation (1.1) in precise terms. In Section 2, we describe our main results along with some remarks. To make this article self-contained, Section 3 recalls some definitions and preliminary results from [7, 11]. The detailed proofs of the main results are presented in Section 4, which utilize the Hopf-Lax type formula and some results from [7] and techniques from [1, 22]. The construction of a counter-example is provided in the last section to show that the main results of the present article cannot be improved. Two appendices contain basic useful lemmas and explanations regarding our adaptation of the result from control theory [3].

1.1. Optimal regularity results in BV^s spaces for a smooth flux: $f = g$

In this subsection, we focus on the case where $f = g$ in (1.1). Even for Lipschitz continuous flux, the well-posedness for the entropy solution of (1.1) is established in the L^∞ setting [32, 34, 35, 44], and many methodologies are available to study the regularity of the entropy solutions [2, 11, 17, 18, 24, 25, 29, 35, 41, 42, 44].

The function space BV is considered as the natural function space for scalar conservation laws since A. I. Volpert’s fundamental work in 1967 ([48]). It allows for compactness and provides a

convenient way to describe the structure of shock waves with traces on each side of the singularity [8]. The BV regularity for entropy solutions was first observed in [35, 44] independently by P. D. Lax and O. Oleinik. In the case where the flux is uniformly convex, i.e., $\inf f'' > 0$, the entropy solution becomes BV instantaneously even when the data is in L^∞ . This well-known smoothing effect is a consequence of the one-sided Lipschitz-Oleinik inequality [44].

Unfortunately, the BV space is ‘not enough’ [19] when the flux is not uniformly convex. There are many examples of entropy solutions that are not in BV for positive time [2, 16, 24]. Although the non-vanishing property of the second derivative of the flux is necessary and sufficient for BV regularizing [25], smoothing effects can still occur in fractional Sobolev spaces [30, 36] for nonlinear flux. The fractional BV spaces preserve the advantages of the BV space, such as regularity and traces while allowing for more general flux functions [11]. The Lax-Oleinik smoothing effect was generalized in BV^s for a flux with power-law nonlinearity like $|u|^{p+1}$ and $p = 1/s \geq 1$, for C^1 or strictly convex flux in, [11, 17, 29].

Fractional BV spaces, denoted by BV^s , $0 < s \leq 1$, were first defined for all $s \in (0, 1)$ in [37, 39, 40]. Let I be a non-empty interval of \mathbb{R} and $s \in (0, 1]$. The space of fractional bounded variation functions denoted as $BV^s(I)$ is a generalization of the space of functions with a bounded variation on I , denoted as $BV(I)$. In the sequel, we denote $S(I)$ as the set of the subdivisions of I , that is the set of finite subsets $\sigma = (x_0, x_1, \dots, x_n)$ in I with $(x_0 < x_1 < x_2 < \dots < x_n)$.

Definition 1.3 (BV^s [37, 39, 40]). *Let $\sigma = (x_0, x_1, \dots, x_n)$ be in $S(I)$ and let u be real function on I . The s -total variation of u with respect to σ is*

$$TV^s u(\sigma) = \sum_{i=1}^n |u(x_i) - u(x_{i-1})|^{1/s},$$

then define,

$$TV^s u(I) = \sup_{\sigma \in S(I)} TV^s u(\sigma).$$

The set $BV^s(I)$ is the set of functions $u : I \rightarrow \mathbb{R}$ such that $TV^s u(I) < \infty$.

1.2. Previous regularity results for discontinuous flux

The regularity of the solution in the BV space is crucial for studying its convergence and the existence of its traces. Without a bound on the total variation, it is difficult to establish the convergence of numerical methods. However, it is not always possible to expect the total variation of the solution to decrease, as a non-constant solution can arise from constant initial data. Despite many decades of research on equation (1.1), optimal regularity results for its solution were not yet known. There are only a few known results regarding the regularity of the solution, which we describe below.

Although it has been proven that the solution away from the interface is BV in space [13], the regularity of the solution near the interface remained unknown for a long time. The first breakthrough result was achieved in 2009 by the authors of [1]. They constructed an explicit example where $\min f \neq \min g$, demonstrating that the total variation of the entropy solution to (1.1) blows up at time $t_0 > 0$ for BV initial data. To build this example, they exploited the lack of Lipschitz continuity of $f^{-1}g$ near the critical point of f . Here g_-^{-1} , f_+^{-1} are the inverse of g , f in appropriate domains, more precisely, they are defined as

$$g_-^{-1} :](g')^{-1}(-\infty), (g')^{-1}(0)] \rightarrow \mathbb{R} \quad \& \quad f_+^{-1} : [(f')^{-1}(0), (f')^{-1}(+\infty)[\rightarrow \mathbb{R}. \quad (1.4)$$

The key functions $f_+^{-1}g(\cdot)$ and $g_-^{-1}f(\cdot)$ transmit information via the interface from left-to-right and right-to-left respectively.

In contrast, several regularity results have been proven in [22, 23]. Surprisingly, it was shown that the solution to (1.1) belongs to BV if the fluxes have the same minimum value, i.e., $\min f = f(\theta_f) = \min g = g(\theta_g)$. The author also proved that if $f(\theta_f) \neq g(\theta_g)$ and the initial data is compactly supported, then there exists a time T such that for all $t > T$, the solution to (1.1) admits BV regularity. However, the assumption of compact support cannot be relaxed, as it has been shown by example that there exists a sequence of time, T_n , for which the total variation of the solution to (1.1) blows up.

Earlier referred publications have uniform convexity assumption on the fluxes, in [23] it has been proved that even for non-uniform convex flux (with a special structure when the flux losses its uniform convexity) any L^∞ initial data gives the solution which is BV_{loc} near the interface when the connection (A, B) as in [6] are far from the critical point.

This discussion leads to conclude that working solely in the BV space framework is inadequate for scalar conservation laws with discontinuous flux (1.1). Hence, it is appropriate to consider a more generalized space than BV , such as the space of functions of fractional bounded variation, denoted as BV^s . In the following subsection, we outline the questions that are addressed in this paper.

1.3. Questions on the BV^s regularity for discontinuous flux

As we discussed thus far, the entropy solution of (1.1) lacks the following properties:

1. If $u_0 \in BV(\mathbb{R})$, then $u(\cdot, t) \in BV(\mathbb{R})$ for any $t > 0$.
2. If f and g are uniformly convex fluxes, $\min f \neq \min g$ and $u_0 \in L^\infty(\mathbb{R})$, then for any $t > 0$, $u(\cdot, t) \in BV_{loc}$.

Based on these issues, we aim to address the following questions regarding the regularity of the solution of (1.1):

Question 1.1. *Can we expect that if the given initial data belongs to BV^s for a well-chosen $0 < s \leq 1$, then the solution of (1.1) stays in BV^s ?*

Question 1.2. *Can we expect that for any $0 < s \leq 1$ there exists $0 < s_1$ such that if the given initial data belongs to BV^s , then the solution of (1.1) belongs to BV^{s_1} ?*

Question 1.3. *What is the Lax-Oleinik type regularizing effect for uniformly convex fluxes f and g ? In other words, does the entropy solution of (1.1) belong to BV^s for some $s \in (0, 1)$ and for any given L^∞ initial data?*

Question 1.4. *Can we choose $0 < s < 1$ sharply and an initial data $u_0 \in BV^s$ space for which the generalized total variation blows up for the corresponding solution of (1.1)?*

Under certain assumptions on the fluxes f and g , we are able to answer all of the questions from 1.1 to 1.4. We also present counterexamples that demonstrate the optimality of the assumptions in our main results. Additionally, we provide explicit estimates of s -total variation of the solution with respect to time variable t with some sufficient conditions on initial data.

2. Main Results

The paper assumes that f and g are C^1 strictly convex functions admitting a critical point. Let θ_f and θ_g be the unique critical points of f and g respectively, i.e., $f'(\theta_f) = 0$ and $g'(\theta_g) = 0$. The notation g_-^{-1} and f_+^{-1} denotes the inverse of g and f for the domain where $g'(u) \leq 0$ and $f'(u) \geq 0$, respectively. Note that the existence of a minimum for f and g are always assumed in this paper as it allows the critical behavior of the admissible solution. However, if f and g have no minimum but both are strictly increasing or decreasing, the situation is simpler [1]. Thus, throughout the paper, it is assumed that,

$$f(\theta_f) = \min f \neq \min g = g(\theta_g). \quad (2.1)$$

In the best case, when f and g are uniformly convex and satisfy (2.1), we obtain a smoothing in $BV^{1/2}$ instead of BV . In the case of non-uniformly convex fluxes, the situation is worse. The smoothing depends on the nonlinear flatness of the fluxes. To be more precise, we introduce the following *non-degeneracy flux condition* which is, there exist two numbers $p \geq 1$ and $q \geq 1$, such that for any compact set K , there exist positive numbers C_1 and C_2 such that for all $u \neq v$, where $u, v \in K$,

$$\frac{|f'(u) - f'(v)|}{|u - v|^p} > C_1 > 0 \quad \text{and} \quad \frac{|g'(u) - g'(v)|}{|u - v|^q} > C_2 > 0. \quad (2.2)$$

For $p = 1$, this is the classical uniformly convex condition for f and for $p > 1$ it corresponds to a less nonlinear convex flux such as $f(u) = |u|^{p+1}$.

An interesting subcase is when the loss of uniform convexity of the fluxes occurs only at their minimum. That is, if f belongs to C^2 and is uniformly convex except at its minimum. For example, convex power laws such as $f(u) = |u|^{p+1}$ with $p > 1$. The same assumption can be made for the other flux g .

$$f'', g'' \text{ vanish only at } \theta_f \text{ and } \theta_g \text{ respectively.} \quad (2.3)$$

The assumption (2.3) combined with the previous one (2.2) is also called the *restricted non-degeneracy condition*, and the fluxes that satisfy it are called *restricted fluxes*. In the subcase where both f and g satisfy (2.3), stronger results can be obtained, which are presented in Theorem 2.1 for initial data in L^∞ and Theorem 2.2 for initial data in BV^s . Two quantities are fundamental to express the fractional regularity of the solutions, γ and ν ,

$$\gamma = \begin{cases} \frac{1}{q+1} \\ \frac{1}{p+1} \end{cases} \quad \nu = \begin{cases} \frac{1}{p} \\ \frac{1}{q} \end{cases} \quad \begin{array}{l} \text{if } \min f < \min g, \\ \text{if } \min f > \min g. \end{array} \quad (2.4)$$

The constant $\gamma \leq 1/2$ can be understood as a loss of regularity due to the interface and $\nu \leq 1$ as the smoothing effect outside the interface. More precisely, γ comes from the singular mapping technique as explained in the following remark.

Remark 2.1. Let f and g be the fluxes satisfying the non-degeneracy condition (2.2) and $f(\theta_f) \neq g(\theta_g)$. Then either $f_+^{-1}g(\cdot)$ or $g_-^{-1}f(\cdot)$ is Lipschitz continuous while the other one is Hölder continuous with exponent γ . The value of γ depends on p and q from the non-degeneracy condition (2.2), and it is given by (2.4). The proof of this fact can be found in Lemma A.3.

Remark 2.2. In the statements of following theorems, the entropy solutions meant by the weak solutions additionally satisfy the Kružkov entropy and interface entropy conditions as mentioned in Definition 1.3.

Theorem 2.1 (Smoothing effect for restricted nonlinear fluxes and L^∞ initial data). Let f and g be two C^2 fluxes satisfying the restricted non-degeneracy condition $f(\theta_f) \neq g(\theta_g)$ (2.1), (2.2) and (2.3). Let $u(\cdot, t)$ be the entropy solution of (1.1) corresponding to an initial data $u_0 \in L^\infty(\mathbb{R})$. Then, for each $t > 0$ and $M > 0$, the entropy solution $u(\cdot, t) \in BV^s(-M, M)$, where s is determined as follows

$$s = \min(\gamma, \nu) \quad (2.5)$$

and the following estimate holds with a positive constant $C_{f,g,\|u_0\|_\infty}$ depending only on the fluxes and the range of the initial data,

$$TV^s(u(\cdot, t), [-M, M]) \leq C_{f,g,\|u_0\|_\infty} + 3(2\|u_0\|_\infty)^{1/s} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \quad (2.6)$$

Remark 2.3 (Uniform convex fluxes and $BV^{1/2}$). If the fluxes f and g are uniformly convex then the solution belongs to $BV^{1/2}$. So even for the uniformly convex case, the solution goes into a fractional BV space.

Hence, in the following theorem for BV^s initial data with $0 < s \leq 1$, the previous result can be stated as follows. The previous Theorem 2.1 can be seen as a limiting case of the following Theorem 2.2 with $s = 0$, which states that $BV^0 = L^\infty$.

Theorem 2.2 (Smoothing effect for restricted nonlinear fluxes and BV^s initial data). Let f and g be two C^2 fluxes such that $f(\theta_f) \neq g(\theta_g)$ and fluxes satisfy the restricted non-degeneracy condition (2.2) and (2.3). Let $u(\cdot, t)$ be the entropy solution of (1.1) corresponding to an initial data $u_0 \in BV^s(\mathbb{R})$ for $s \in (0, 1)$. Then, for each $t > 0$ and $M > 0$, the entropy solution $u(\cdot, t) \in BV^{s_1}(-M, M)$ where s_1 is given by

$$s_1 := \min\{\gamma, \max\{\nu, s\}\} \quad (2.7)$$

the following estimate holds with a positive constant $C_{f,g,\|u_0\|_\infty}$ depending only on fluxes and the range of the initial data and a constant $D > 0$,

$$TV^{s_1}(u(\cdot, t), [-M, M]) \leq C_{f,g,\|u_0\|_\infty} + \frac{C_{f,g,\|u_0\|_\infty} M}{t} + 2\|2u_0\|_\infty^{\frac{1}{s_1}} + D \cdot TV^s(u_0). \quad (2.8)$$

We note that the assumption on vanishing points of f'' and g'' is restrictive. We can relax this assumption at the cost of smaller s_1 . More precisely, we have the following result.

Theorem 2.3 (Smoothing effect for L^∞ initial data). Let f and g be two C^2 fluxes such that $f(\theta_f) \neq g(\theta_g)$ satisfying the non-degeneracy condition (2.2) with exponent p, q respectively. Let $u(\cdot, t)$ be the entropy solution of (1.1) corresponding to an initial data $u_0 \in L^\infty(\mathbb{R})$. Then, for each $t > 0$ and $M > 0$, there exists positive constant $C_{f,g,\|u_0\|_\infty}$ such that

$$TV^s(u(\cdot, t), [-M, M]) \leq C_{f,g,\|u_0\|_\infty} + 3(2\|u_0\|_\infty)^{1/s} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}$$

where s is determined as follows

$$s = \gamma \nu. \quad (2.9)$$

Remark 2.4. If the initial data $u_0 \in BV^{s_0}$, then the entropy solution $u(\cdot, t)$ considered in Theorem 2.3 belongs to BV^{s_2} with $s_2 = \gamma \max(s_0, \nu)$. However, we do not present a separate proof of this result as it can be obtained by following a similar argument as in the proof of Theorem 2.3.

In general, away from the interface, the expected fractional regularity is $\min(1/p, 1/q)$ [11] which is always bigger than s in (2.5). In particular, near the interface, for BV initial data, a BV

regularity for the entropy solution cannot be expected [1]. At most, a $BV^{1/2}$ regularity is possible. Getting BV regularity of entropy solution can be impossible near the interface. The situation is better far from the interface. Far from the interface, the constant γ plays no role. The following theorem gives estimates which are sharp for small time.

Theorem 2.4 (Regularity outside the interface). *Let f and g be the fluxes with $f(\theta_f) \neq g(\theta_g)$. Let $u(\cdot, t)$ be the entropy solution of (1.1) corresponding to an initial data $u_0 \in BV^s(\mathbb{R})$ for $s \in (0, 1)$. If f and g satisfy (2.2) with exponent p and q respectively, for any $t > 0$, $\epsilon > 0$, then there exists a constant $C_{f,g,\|u_0\|_\infty} > 0$*

$$TV^{s_1}(u(\cdot, t), (-\infty, -\epsilon] \cup [\epsilon, \infty)) \leq \frac{C_{f,g,\|u_0\|_\infty} t}{\epsilon} + 2TV^{s_1}(u_0) + 2(2\|u_0\|_\infty)^{1/s_1} \quad (2.10)$$

for $s_1 = \min\{p^{-1}, q^{-1}, s\}$.

Remark 2.5. *All of the regularity results in Theorems 2.1, 2.2, 2.3, and 2.4 can be extended to fractional Sobolev space $W^{s,p}$ with the same exponent s , up to any $\epsilon > 0$. This is possible due to the embedding $BV^s \subset W^{s-\epsilon, 1/s}$ for all $\epsilon \in (0, s)$ [11]. In other words, if the initial data is in a fractional BV space, then the entropy solution also belongs to a fractional Sobolev space. This is because BV^s is more suitable than Sobolev space for entropy solutions since it recovers the exact regularity and BV -like traces [16, 24].*

Now we discuss the optimality result. The assumption $\min f \neq \min g$ forbids the favorable case $f = g$, which does not have the interface. Here, the optimality of Theorem 2.2 is proved in the best case with uniformly convex fluxes. For this purpose, examples are built with optimal regularity and not more up. The same construction is valid with a power law on one side of the interface. These examples highlight the sharpness of Theorem 2.2.

Theorem 2.5 (Blow-up for critical BV^s semi-norms). *Let $p \geq 1$ and $\epsilon > 0$. Then there exist fluxes f, g and an initial data $u_0 \in BV(\mathbb{R})$ such that*

1. the flux f satisfies the non-degeneracy condition (2.2) with exponent p ,
2. the function g is uniformly convex,
3. the corresponding entropy solution $u(\cdot, T) \notin BV_{loc}^s(\mathbb{R})$ for some $T > 0$ and $s = \frac{1}{p+1} + \epsilon$.

The proof of Theorem 2.5 is postponed in Section 5 and Appendix C.

3. Preliminaries

The present work builds upon the fundamental work of Adimurthi and Gowda in [7], where they established an important foundation of the theory on scalar conservation laws with an interface and two convex fluxes. The author proposed the natural entropy condition (1.3) at the interface which means that no information comes only from the interface but crosses or goes towards the interface. Such entropy condition is in the spirit of Lax-entropy conditions for shock waves. To make this paper self-contained, we recall some definitions and results from their paper.

The following theorem, which provides a Lax-Oleinik or Lax-Hopf formula for the initial value problem (1.1), can be found in [7, Lemma 4.9, pp. 51]. The notations introduced in Theorem 3.1 will be used in the statements and proofs throughout the paper.

Theorem 3.1 ([7]). Let $u_0 \in L^\infty(\mathbb{R})$, then there exists the entropy solution $u(\cdot, t)$ of (1.1) corresponding to an initial data u_0 . Furthermore, there exist Lipschitz curves $R_1(t) \geq R_2(t) \geq 0$ and $L_1(t) \leq L_2(t) \leq 0$, monotone functions $z_\pm(x, t)$ non-decreasing in x and non-increasing in t and $t_\pm(x, t)$ non-increasing in x and non-decreasing in t such that the solution $u(x, t)$ can be given by the explicit formula for almost all $t > 0$,

$$u(x, t) = \begin{cases} (f')^{-1} \left(\frac{x - z_+(x, t)}{t} \right) & \text{if } x \geq R_1(t), \\ (f')^{-1} \left(\frac{x}{t - t_+(x, t)} \right) & \text{if } 0 \leq x < R_1(t), \\ (g')^{-1} \left(\frac{x - z_-(x, t)}{t} \right) & \text{if } x \leq L_1(t), \\ (g')^{-1} \left(\frac{x}{t - t_-(x, t)} \right) & \text{if } L_1(t) < x < 0. \end{cases}$$

Furthermore, if $f(\theta_f) \geq g(\theta_g)$ then $R_1(t) = R_2(t)$ and if $f(\theta_f) \leq g(\theta_g)$ then $L_1(t) = L_2(t)$. We also have only three cases and following formula to compute the solution:

Case 1: $L_1(t) = 0$ and $R_1(t) = 0$,

$$u(x, t) = \begin{cases} u_0(z_+(x, t)) & \text{if } x > 0, \\ u_0(z_-(x, t)) & \text{if } x < 0. \end{cases}$$

Case 2: $L_1(t) = 0$ and $R_1(t) > 0$, then

$$u(x, t) = \begin{cases} f_+^{-1} g(u_0(z_+(x, t))) & \text{if } 0 < x < R_2(t), \\ f_+^{-1} g(\theta_g) & \text{if } R_2(t) \leq x \leq R_1(t), \\ u_0(z_-(x, t)) & \text{if } x < 0. \end{cases}$$

Case 3: $L_1(t) < 0$, $R_1(t) = 0$, then

$$u(x, t) = \begin{cases} g_-^{-1} f(u_0(z_-(x, t))) & \text{if } L_2(t) < x < 0, \\ u_0(z_-(x, t)) & \text{if } x \leq L_1(t), \\ g_-^{-1} f(\theta_f) & \text{if } L_1(t) < x < L_2(t). \end{cases}$$

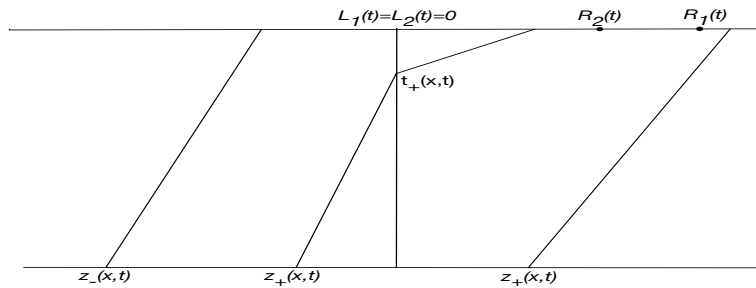


Figure 1: An illustration of solution for **Case 2** and $L_i(t)$ and $R_i(t)$ curves

There is a maximum principle for such entropy solutions, but more complicate than for $f = g$,

$$\|u\|_\infty \leq \max \left(\|u_0\|_\infty, \sup_{|v| \leq \|u_0\|_\infty} |f_+^{-1}(g(v))|, \sup_{|v| \leq \|u_0\|_\infty} |g_-^{-1}(f(v))| \right) =: S_{f,g, \|u_0\|_\infty}. \quad (3.1)$$

Without loss of generality, we may assume that $\min f < \min g$ for the proofs of the main results

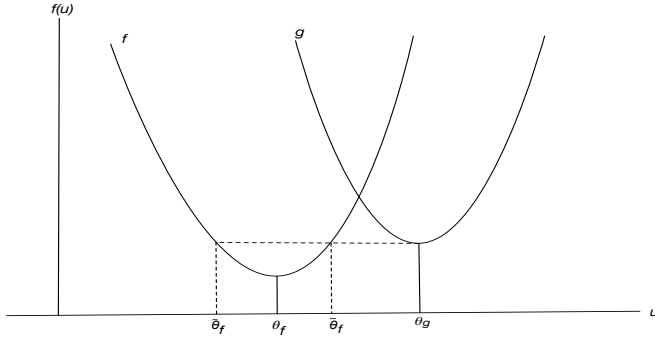


Figure 2: Illustration of the fluxes

below, as illustrated in Figure 2. This choice enforces the values of the entropy solution at the interface lie outside $(\tilde{\theta}_f, \bar{\theta}_f)$. Thus the function f' is far from 0 at the interface. Moreover, the function f'^{-1} is Lipschitz outside $(\tilde{\theta}_f, \bar{\theta}_f)$, and for restricted fluxes, the function f_+^{-1} is also Lipschitz outside $(\tilde{\theta}_f, \bar{\theta}_f)$. The singular maps $f_+^{-1}g$ and $g_-^{-1}f$ are shown to be Lipschitz and Hölder continuous, respectively, in Lemma A.3 in Appendix A.

4. Proof of main results

This long section aims to establish the fractional BV regularity of the entropy solution, which depends on the degeneracy of the fluxes. A crucial aspect of this is estimating the regularity of the traces at the interface. In subsection 4.1, we study the fractional regularity in a favorable case when the traces at the interface are not near the critical values θ_f or θ_g . Here, spatial BV^s estimates for trace values issued from the interface are studied in Subsection 4.2. Moreover, only traces issued from the initial data are considered. The crossing of the interface is studied later in Subsection 4.3.

4.1. Regularity when traces are far from critical values

Our first objective is to establish fractional BV estimates for the solution in the time variable, assuming that the traces at $x = 0$ are far from the critical values θ_f or θ_g .

Lemma 4.1 (Fractional BV estimate for the traces of the solution). *Let f, g be satisfying (2.2) with exponents p, q respectively. Let $0 < a < b < \infty$. Then the following holds:*

1. If $u(0-, t) > \theta_g$ for a.e. $t \in (a, b)$, then we have

$$TV^{\frac{1}{q}}(u(0-, \cdot), (a, b)) \leq C_g \frac{b}{a}, \quad (4.1)$$

where $C_g > 0$ is constant depending only on g .

2. If $u(0+, t) < \theta_f$ for a.e. $t \in (a, b)$, then we have

$$TV^{\frac{1}{p}}(u(0+, \cdot), (a, b)) \leq C_f \frac{b}{a}, \quad (4.2)$$

where $C_f > 0$ is a constant depending on f .

Proof. Since $u(0-, t) > \theta_g$ and $g' \geq 0$ on $(\theta_g, +\infty)$, the value of the left trace comes from the left. Using Theorem 3.1, we have $(g')^{-1} \left(\frac{x - z_-(x, t)}{t} \right)$ that gives $u(0-, t) = (g')^{-1} \left(\frac{-z_-(0-, t)}{t} \right)$ for $t \in (a, b)$, where $t \mapsto z_-(0-, t)$ is non-increasing. Since g satisfies the non-degeneracy condition (2.2), from Lemma A.1 $(g')^{-1}$ is a $1/q$ -Hölder function with a constant H_g such that

$$|u(0-, t_1) - u(0-, t_2)| \leq H_g \left| \frac{z_-(0-, t_1)}{t_1} - \frac{z_-(0-, t_2)}{t_2} \right|^{\frac{1}{q}}.$$

We observe that

$$\left| \frac{z_-(0-, t_1)}{t_1} - \frac{z_-(0-, t_2)}{t_2} \right| \leq |z_-(0-, t_1)| \left| \frac{1}{t_1} - \frac{1}{t_2} \right| + \frac{1}{t_2} |z_-(0-, t_1) - z_-(0-, t_2)|.$$

For any partition $a \leq t_1 < t_2 < \dots < t_m \leq b$,

$$\begin{aligned} & \sum_{j=1}^{m-1} |u(0-, t_j) - u(0-, t_{j+1})|^q \\ & \leq H_g^q \sum_{j=1}^{m-1} \left[|z_-(0-, t_j)| \left| \frac{1}{t_j} - \frac{1}{t_{j+1}} \right| + \frac{1}{t_{j+1}} |z_-(0-, t_j) - z_-(0-, t_{j+1})| \right] \\ & \leq H_g^q \left[|z_-(0-, b)| \sum_{j=1}^{m-1} \left| \frac{1}{t_j} - \frac{1}{t_{j+1}} \right| + \frac{1}{a} \sum_{i=1}^{m-1} |z_-(0-, t_i) - z_-(0-, t_{i+1})| \right] \\ & \leq H_g^q \left[\frac{|z_-(0-, b)| (b-a)}{ab} + \frac{|z_-(0-, a) - z_-(0-, b)|}{a} \right]. \end{aligned}$$

Since $|z_-(0-, a) - z_-(0-, b)| \leq |z_-(0-, b)|$ and $b-a \leq b$ we have,

$$\sum_{j=1}^{m-1} |u(0-, t_j) - u(0-, t_{j+1})|^q \leq 2H_g^q \frac{|z_-(0-, b)|}{a}.$$

We obtain an upper bound on $|z(0-, b)|$ from the finite speed of propagation, which gives $|z(0-, b)| \leq M_g b$ where $K_{f,g, \|u_0\|_\infty} = \sup\{|g'(v)|; |v| \leq \|u_0\|_\infty\}$. Hence, we get a new constant C_g

$$\sum_{j=1}^{m-1} |u(0-, t_j) - u(0-, t_{j+1})|^q \leq C_g \frac{b}{a}.$$

This proves (4.1). Similarly, we can prove the (4.2). \square

Better fractional BV estimates for the traces of the solution can be obtained when the fluxes are less singular.

Lemma 4.2 (Fractional BV estimate for traces away from critical values). *Let $r > 0$ and f, g be satisfying (2.2) with exponent p, q respectively. Let $0 < a < b < \infty$.*

1. *If $u(0-, t) \geq \theta_g + r$ and g'' vanishes only at θ_g (2.3), then there exists a constant $C_g > 0$ independent of r such that the following inequality holds,*

$$TV(u(0-, \cdot), (a, b)) \leq \frac{C_g}{\min\{g''(v) | v \in [\theta_g + r, \|u_0\|_\infty]\}} \frac{b}{a}. \quad (4.3)$$

2. *If $u(0+, t) \leq \theta_f + r$ and f'' vanishes only at θ_f (2.3), then there exists a constant $C_f > 0$*

independent of r such that the following inequality holds,

$$TV(u(0+, \cdot), (a, b)) \leq \frac{C_f}{\min\{f''(v); v \in [-\|u_0\|_\infty, \theta_f - r]\}} \frac{b}{a}. \quad (4.4)$$

Lemma 4.2 will be used later with constant r given by either $\theta_f - \tilde{\theta}_f$ or $\bar{\theta}_f - \theta_f$ as shown in (see Fig 2). The fact that r is a positive constant is crucial to get uniform estimates later.

Proof. For any $x, y \in \mathbb{R}$ consider

$$|x - y| = |g'(g'^{-1}(x)) - g'(g'^{-1}(y))| = g''(\xi) |g'^{-1}(x) - g'^{-1}(y)|,$$

where $\xi \in (x, y)$. Now for (4.3), Theorem 3.1 gives, $u(0-, t) = (g')^{-1} \left(\frac{-z_-(0-, t)}{t} \right)$ for $t \in (a, b)$ where $t \mapsto z_-(0-, t)$ is non-increasing. Thus,

$$\begin{aligned} |u(0-, t_1) - u(0-, t_2)| &= \left| g'^{-1} \left(\frac{-z_-(0-, t_1)}{t_1} \right) - g'^{-1} \left(\frac{-z_-(0-, t_2)}{t_2} \right) \right| \\ &\leq \min\{g''(v); v \in [\theta_g + r, \|u_0\|_\infty]\}^{-1} \left| \frac{z_-(0-, t_1)}{t_1} - \frac{z_-(0-, t_2)}{t_2} \right|. \end{aligned}$$

Now the similar calculation as to prove (4.1) gives (4.3). By similar arguments (4.4) can be proven for f . \square

4.2. Spatial BV^s estimates for values originating from the interface

Now, far from the interface and restricted flux, when the values of the solution are far from the critical values of f and g , a BV estimate is available. The following [inequalities](#) are also valid in BV^s and used later [along](#) with other BV^s estimates.

Lemma 4.3 (*BV and BV^s estimates for the solution*). *Let u be an entropy solution and $R_1(t) > 0$ for some fixed $t > 0$. Let $0 < a < b < R_1(t)$ and $S_{f,g,\|u_0\|_\infty}$ be as in (3.1). Let $r > 0$, f satisfies (2.2) and f'' vanishes only on θ_f (2.3). If $u(x, t) \geq \theta_f + r$ for $a \leq x \leq b$, then there exists a constant $C_{f,g,\|u_0\|_\infty} > 0$ such that*

$$TV^s(u(\cdot, t), [a, b]) \leq \frac{C_{f,g,\|u_0\|_\infty}}{\min\{f''(v); v \in [\theta_f + r, S_{f,g,\|u_0\|_\infty}]\}^{\frac{1}{s}}} \left(\frac{t - t_+(b, t)}{t - t_+(a, t)} \right)^{\frac{1}{s}}, \quad (4.5)$$

for all $0 < s \leq 1$.

The same result holds for the left side of the interface as follows:

Lemma 4.4 (*BV and BV^s estimate for the solution*). *Let u be an entropy solution and $L_1(t) < 0$ for some $t > 0$. Let $L_1(t) < a < b < 0$ and $S_{f,g,\|u_0\|_\infty}$ be as in (3.1). Let $r > 0$, flux g satisfies (2.2) and g'' vanishes only on θ_g . If $u(x, t) \leq \theta_g - r$ for $a \leq x \leq b$, then there exists a constant $C_{f,g,\|u_0\|_\infty} > 0$ such that*

$$TV^s(u(\cdot, t), [a, b]) \leq \frac{C_{f,g,\|u_0\|_\infty}}{\min\{g''(v); v \in [-S_{f,g,\|u_0\|_\infty}, \theta_g - r]\}^{\frac{1}{s}}} \left(\frac{t - t_-(b, t)}{t - t_-(a, t)} \right)^{\frac{1}{s}},$$

for all $0 < s \leq 1$.

Proof. Theorem 3.1 gives,

$$u(x, t) = (f')^{-1} \left(\frac{x}{t - t_+(x, t)} \right) \text{ for } x \in (0, R_1(t)).$$

Fix a partition $a \leq x_1 < x_2 < \dots < x_m \leq b$. Then, as in the proof of inequality (4.3), it follows,

$$\begin{aligned} \sum_{j=1}^{m-1} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{s}} &= \sum_{j=1}^{m-1} \left| (f')^{-1} \left(\frac{x_j}{t - t_+(x_j, t)} \right) - (f')^{-1} \left(\frac{x_{j+1}}{t - t_+(x_{j+1}, t)} \right) \right|^{\frac{1}{s}} \\ &\leq \frac{1}{\min\{f''(v); v \in [\theta_f + r, S_{f,g,\|u_0\|_\infty}]\}^{\frac{1}{s}}} \sum_{j=1}^{m-1} \left| \frac{x_j}{t - t_+(x_j, t)} - \frac{x_{j+1}}{t - t_+(x_{j+1}, t)} \right|^{\frac{1}{s}}. \end{aligned}$$

We calculate

$$\begin{aligned} \left| \frac{x_j}{t - t_+(x_j, t)} - \frac{x_{j+1}}{t - t_+(x_{j+1}, t)} \right| &\leq |x_j| \left| \frac{1}{t - t_+(x_j, t)} - \frac{1}{t - t_+(x_{j+1}, t)} \right| + \frac{1}{t - t_+(x_{j+1}, t)} |x_j - x_{j+1}| \\ &\leq b \left| \frac{1}{t - t_+(x_j, t)} - \frac{1}{t - t_+(x_{j+1}, t)} \right| + \frac{1}{t - t_+(a, t)} |x_j - x_{j+1}|. \end{aligned}$$

Hence, by the convexity yields, $(a + b)^{\frac{1}{s}} \leq 2^{\frac{1-s}{s}} \left(a^{\frac{1}{s}} + b^{\frac{1}{s}} \right)$ and we get

$$\begin{aligned} &\sum_{j=1}^{m-1} \left| \frac{x_j}{t - t_+(x_j, t)} - \frac{x_{j+1}}{t - t_+(x_{j+1}, t)} \right|^{\frac{1}{s}} \\ &\leq \frac{1}{2^{\frac{s-1}{s}}} \left(\sum_{j=1}^{m-1} b^{\frac{1}{s}} \left| \frac{1}{t - t_+(x_j, t)} - \frac{1}{t - t_+(x_{j+1}, t)} \right|^{\frac{1}{s}} + \sum_{j=1}^{m-1} \frac{1}{(t - t_+(a, t))^{\frac{1}{s}}} |x_j - x_{j+1}|^{\frac{1}{s}} \right) \\ &\leq \frac{1}{2^{\frac{s-1}{s}}} \left(b^{\frac{1}{s}} \left| \frac{1}{t - t_+(a, t)} - \frac{1}{t - t_+(b, t)} \right|^{\frac{1}{s}} + \left(\frac{b - a}{t - t_+(a, t)} \right)^{\frac{1}{s}} \right) \\ &\leq 2^{\frac{1}{s}} \left(\frac{b}{t - t_+(a, t)} \right)^{\frac{1}{s}}. \end{aligned}$$

In the last step we have used $b - a \leq b$ and $(t - t_+(b, t)) - (t - t_+(a, t)) \leq t - t_+(b, t)$. Note that $b \leq K_{f,g,\|u_0\|_\infty} (t - t_+(b, t))$ where $K_{f,g,\|u_0\|_\infty} = \sup\{|f'|; |v| \leq S_{f,g,\|u_0\|_\infty}\}$ where $S_{f,g,\|u_0\|_\infty}$ is defined in (3.1). \square

The following lemma deals with the spatial regularity of the entropy solution on the right side of the interface. Inequality (4.5) does not use the restricted non-degeneracy condition.

Lemma 4.5. *Let u be an entropy solution and $R_1(t) > 0$ for some fixed $t > 0$. Let $0 < a < b < R_1(t)$ and $S_{f,g,\|u_0\|_\infty}$ be as in (3.1). If f only satisfies (2.2) with exponent p then we have*

$$TV^{\frac{1}{p}}(u(\cdot, t), [a, b]) \leq C_{f,g,\|u_0\|_\infty} \frac{t - t_+(b, t)}{t - t_+(a, t)}. \quad (4.6)$$

The same result holds for the left side of the interface as follows.

Lemma 4.6. *Let u be an entropy solution and $L_1(t) < 0$ for some $t > 0$. Let $L_1(t) < a < b < 0$. If g satisfies (2.2) with exponent q then we have*

$$TV^{\frac{1}{q}}(u(\cdot, t), [a, b]) \leq C_{f,g,\|u_0\|_\infty} \frac{t - t_-(b, t)}{t - t_-(a, t)}. \quad (4.7)$$

Using a similar argument to that of the previous Lemma 4.3, we can prove the inequality (4.6) of Lemma 4.5, so we omit the proof here.

4.3. Smoothing effect for restricted nonlinear fluxes

We are now ready to prove Theorem 2.1. Accordingly, we first fix an arbitrary partition and divide it into several parts. Some of these parts are far from the interface, where we estimate the generalized variation using the regularizing effect for scalar conservation laws without a boundary. For the parts near the interface, we apply the Lax-Oleinik formula (explicit formula in Theorem 3.1) for the solution, along with previous lemmas.

Proof of Theorem 2.1: Since $f(\theta_f) \neq g(\theta_g)$, without loss of generality assume that $f(\theta_f) < g(\theta_g)$ as in (see Fig 2). It is enough to consider the following two cases, the other cases are similar.

Case(i): $L_1(t) = 0$ and $R_1(t) \geq 0$.

Consider an arbitrary partition $\{-M = x_{-n} < \dots < x_{-1} < x_0 \leq 0 < x_1 < \dots < x_l \leq R_2(t) < x_{l+1} < \dots < x_m \leq R_1(t) < x_{m+1} < \dots < x_n = M\}$. Then,

$$\begin{aligned} \sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &= \sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + \sum_{i=m+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \\ &+ \sum_{i=1}^{l-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + \sum_{i=l+1}^{m-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \\ &+ |u(x_0, t) - u(x_1, t)|^{1/s} + |u(x_l, t) - u(x_{l+1}, t)|^{1/s} + |u(x_m, t) - u(x_{m+1}, t)|^{1/s}. \end{aligned}$$

From Theorem 3.1, it follows that the solution u is constant between $R_2(t)$ to $R_1(t)$, which means variation is zero for this interval. Using the Lax-Oleinik formula from Theorem 3.1 and bounding the last three terms yield

$$\begin{aligned} \sum_{i=-n}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &\leq \underbrace{\sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s}}_{\text{I}} + \underbrace{\sum_{i=m+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s}}_{\text{III}} \\ &+ \underbrace{\sum_{i=1}^{l-1} |f_+^{-1}g(u_0(z_+(x_i, t))) - f_+^{-1}g(u_0(z_+(x_{i+1}, t)))|^{1/s} + 3(2\|u_0\|_\infty)^{1/s}}_{\text{II}}. \end{aligned}$$

Now we wish to estimate the terms I, II, and III. The simplest terms I, III are estimated as in [11, 17]. First, let us take into account the term I. Since f and g are satisfying the flux non-degeneracy condition (2.2), by Lemma A.1, the maps $u \mapsto (g')^{-1}(u)$ and $u \mapsto (f')^{-1}(u)$ are Hölder continuous with exponents q^{-1} and p^{-1} , respectively. From Theorem 3.1, we have that

$$u(x, t) = (g')^{-1} \left(\frac{x - z_-(x, t)}{t} \right), \quad \text{for } x < 0,$$

then for $-M \leq x_i < x_{i+1} \leq 0$, from Lemma A.1

$$\begin{aligned} |u(x_i, t) - u(x_{i+1}, t)|^q &= \left| (g')^{-1} \left(\frac{x_i - z_-(x_i, t)}{t} \right) - (g')^{-1} \left(\frac{x_{i+1} - z_-(x_{i+1}, t)}{t} \right) \right|^q \\ &\leq \left(C_2^{-q^{-1}} \left| \frac{x_i - z_-(x_i, t)}{t} - \frac{x_{i+1} - z_-(x_{i+1}, t)}{t} \right|^{q^{-1}} \right)^q, \end{aligned}$$

using triangle inequality we obtain,

$$|u(x_i, t) - u(x_{i+1}, t)|^q \leq C_2^{-1} \left| \frac{x_i - x_{i+1}}{t} \right| + C_2^{-1} \left| \frac{z_-(x_i, t) - z_-(x_{i+1}, t)}{t} \right|.$$

Since $|x_i|, |x_{i+1}| \leq M$ and $x = z_-(x, t) + g'(u(x, 0))t$ hence, we get

$$TV^{q-1}u(\sigma \cap [-M, 0]) \leq \frac{4M}{C_2 t} + \frac{1}{C_2} \sup \left\{ |g'(v)|; |v| \leq \|u_0\|_{L^\infty(\mathbb{R})} \right\}. \quad (4.8)$$

In similar fashion, for the term III we have,

$$TV^{p-1}u(\sigma \cap [R_1(t), M]) \leq \frac{4M}{C_1 t} + \frac{1}{C_1} \sup \left\{ |f'(v)|; |v| \leq \|u_0\|_{L^\infty(\mathbb{R})} \right\}. \quad (4.9)$$

Now we will estimate the term II. From the definition of s , $s \leq 1/p$ and $s \leq 1/(q+1)$. The rest of the proof for this case is divided into two sub-cases.

1. Consider the situation when $t_+^{min}(t) = \inf\{t_+(x, t); x \in (0, R_1(t))\} \geq t/2$. The fact that $t_+^{min} > t/2 > 0$ implies that the characteristics reaching the left side of the interface at $(0-, t_+)$ have a positive speed. Hence, $u(0-, t_+(x, t)) > \theta_g$ for all $x \in (0, R_1(t))$ (see Fig 2). Therefore, the inequality (4.1) of Lemma 4.1 gives $TV^{\frac{1}{q}}(u(0-, \cdot)(t_+^{min}, t)) \leq C_g \frac{t}{t/2} = 2C_g$.

Since $s \leq \frac{1}{q+1} < \frac{1}{q}$ Lemma B.1 yields

$$\begin{aligned} TV^s(u(0-, \cdot)(t_+^{min}, t)) &\leq \text{osc}(u)^{1/s-q} \cdot TV^{\frac{1}{q}}(u(0-, \cdot)(t_+^{min}, t)) \\ &\leq \text{osc}(u)^{1/s-q} C_g, \end{aligned}$$

that gives

$$\text{II} \leq \text{osc}(u)^{1/s-q} C_g.$$

2. Next, we focus on the sub-case when $t_+^{min}(t) = \inf\{t_+(x, t); x \in (0, R_1(t))\} < t/2$. As previous subcase, we already have $TV^s(u(0-, \cdot)(t/2, t)) \leq 2C_g$. Let $j_0 > 0$ such that $t_+(x_j, t) \geq t/2$ for $0 < j \leq j_0$ and $t_+(x_j, t) < t/2$ for $j_0 < j \leq l-1$. Since $u(x_j, t) = u(0+, t_+(x_j, t)) = f_+^{-1}g(u(0-, t_+(x_j, t)))$ for $0 < j < l-1$, from Lemma A.3, $f_+^{-1}g$ is Lipschitz function, hence

$$\sum_{j=1}^{j_0} |u(x_j, t) - u(x_{j-1}, t)|^{\frac{1}{s}} \leq 2C_{f,g}.$$

Let $\bar{\theta}_f > \theta_f$ be such that $f(\bar{\theta}_f) = g(\theta_g)$ as shown in (see Fig 2). Then by Rankine-Hugoniot condition (1.2) observe that $u(x_j, t) \geq \bar{\theta}_f$. From the inequality (4.5) of Lemma 4.3 we get

$$\sum_{j=j_0+1}^{l-2} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{s}} \leq C_{f,g,\|u_0\|_\infty}.$$

Subsequently, we get

$$\text{II} \leq C_{f,g,\|u_0\|_\infty}. \quad (4.10)$$

Hence combining the estimates on I, II and III for constant $C_{f,g,\|u_0\|_\infty} > 0$ we have

$$\sum_{i=-n}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \leq C_{f,g,\|u_0\|_\infty} \left(1 + \frac{1}{t}\right).$$

Case (ii): $R_1(t) = 0, L_1(t) < 0$. Unlike the previous case, this case is not as good due to the fact that $g_-^{-1}f$ is only Hölder continuous and not Lipschitz. Let us consider the partition $\sigma = \{-M = x_{-n} < \dots < x_m \leq L_2(t) = L_1(t) < x_{m+1} < \dots < x_0 \leq R_2(t) = R_1(t) = 0 < x_1 < \dots \leq x_n = M\}$

and then

$$\begin{aligned}
\sum_{i=-n}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &= \sum_{i=-n}^{m-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + \sum_{i=1}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \\
&+ \sum_{i=m+1}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + |u(x_0, t) - u(x_1, t)|^{1/s} \\
&+ |u(x_m, t) - u(x_{m+1}, t)|^{1/s}.
\end{aligned}$$

From Theorem 3.1 we get,

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &= \underbrace{\sum_{i=-n}^{m-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s}}_{\text{I}} + 2(2\|u_0\|_{\infty})^{1/s} \\
&+ \underbrace{\sum_{i=m+1}^{-1} |g_-^{-1}(f(u_0(z_-(x_i, t)))) - g_-^{-1}(f(u_0(z_-(x_{i+1}, t))))|^{1/s}}_{\text{II}} \\
&+ \underbrace{\sum_{i=1}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s}}_{\text{III}}.
\end{aligned}$$

Similarly to **Case (i)** we bound I, III as in (4.8), (4.9) to get

$$\text{I} + \text{III} \leq \frac{C_{f,g,\|u_0\|_{\infty}} M}{t}.$$

We now estimate the term II, and similar to **Case (i)**, we divide this analysis into two sub-cases.

1. We first consider the situation when $t_-^{\min}(t) = \inf\{t_-(x, t); x \in (L_1(t), 0)\} \geq t/2$. The Rankine-Hugoniot condition (1.2) implies that $u(0+, \cdot) \leq \theta_f$, (see Fig 2). The inequality (4.4) of Lemma 4.2 gives

$$TV(u(0+, \cdot)(t_-^{\min}, t)) \leq C_{f,g,\|u_0\|_{\infty}}. \quad (4.11)$$

Note that $g_-^{-1} \circ f$ is Hölder continuous function with exponent $\frac{1}{q+1}$. Hence we have

$$\text{II} = \sum_{j=m+1}^{-1} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g,\|u_0\|_{\infty}}. \quad (4.12)$$

2. Next we focus on the sub-case when $t_-^{\min}(t) = \inf\{t_-(x, t); x \in (L_1(t), 0)\} < t/2$. Let $j_0 < 0$ such that $t_+(x_j, t) \geq t/2$ for $j_0 \leq j < 0$ and $t_+(x_j, t) < t/2$ for $m+1 < j < j_0$. In the previous sub-case we have

$$\sum_{j=j_0}^{-1} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g,\|u_0\|_{\infty}}. \quad (4.13)$$

Note that for $m+1 < j < j_0$, $u(x_j, t) = u(0-, t_-(x_j, t)) \leq \theta_g$. From the inequality (4.7) of Lemma 4.6 we have

$$\sum_{j=m+1}^{j_0-1} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{q}} \leq C_{f,g,\|u_0\|_{\infty}}. \quad (4.14)$$

Subsequently, we get

$$\text{II} \leq C_{f,g,\|u_0\|_\infty} + \|2u_0\|_\infty^{\frac{1}{q+1}}. \quad (4.15)$$

Hence, from the estimates on I, II and III we get

$$\sum_{i=-n}^n |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g,\|u_0\|_\infty} + 3(2\|u_0\|_\infty)^{\frac{1}{q+1}} + \frac{C_{f,g}M}{t}. \quad (4.16)$$

□

4.4. Generalization for BV^s initial data

Now we can prove Theorem 2.2. To do this, we divide the domain into several parts. Here, the initial data belongs to BV^s . If s is very small, then far from the interface estimates come from the regularizing effect. If s is near 1, then outside interface initial data regularity propagates. For the estimate on the solution near the interface, again we use the Theorem 3.1 (Lax-Oleinik formula from [7]).

Proof of Theorem 2.2. Since $f(\theta_f) \neq g(\theta_g)$, without loss of generality we assume that $f(\theta_f) < g(\theta_g)$, (see Fig 2) because other cases can be done in a similar way. Hence, from Theorem 3.1 we have $L_2(t) = L_1(t)$ then it is enough to consider the following two cases.

Case (i): If $L_1(t) = 0$ and $R_1(t) \geq 0$.

Consider the partition $\sigma = \{-M = x_{-n} \leq \dots < x_{-1} < x_0 \leq 0 < x_1 < \dots < x_l \leq R_2(t) < x_{l+1} < \dots < x_m \leq R_1(t) < x_{m+1} < \dots \leq x_n = M\}$ and

$$s_1 = \min\{\gamma, \max\{\nu, s\}\} \in (0, 1).$$

Then

$$\begin{aligned} & \sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \\ &= \sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} + \sum_{i=m+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \\ &+ \sum_{i=1}^{l-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} + \sum_{i=l+1}^{m-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \\ &+ |u(x_0, t) - u(x_1, t)|^{1/s_1} + |u(x_l, t) - u(x_{l+1}, t)|^{1/s_1} \\ &+ |u(x_m, t) - u(x_{m+1}, t)|^{1/s_1}. \end{aligned}$$

From Theorem 3.1, the entropy solution is constant between $R_2(t)$ and $R_1(t)$ which means variation is zero for this interval. Hence,

$$\begin{aligned} \sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} &= \underbrace{\sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1}}_{\text{I}} + 3(2\|u_0\|_\infty)^{1/s_1} \\ &+ \underbrace{\sum_{i=1}^{l-1} |f_+^{-1}g(u_0(z_+(x_i, t))) - f_+^{-1}g(u_0(z_+(x_{i+1}, t)))|^{1/s_1}}_{\text{II}} \end{aligned}$$

$$+ \underbrace{\sum_{i=m+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1}}_{\text{III}}.$$

From the choice of s_1 , we get $s_1 \leq \max\{s, 1/q\}$. If $1/q > s$, then $s_1 < 1/q$. By a similar argument as in (4.8) we have

$$\sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \leq \frac{4M}{C_2 t} + \frac{1}{C_2} \sup \left\{ |g'(v)|; |v| \leq \|u\|_{L^\infty(\mathbb{R} \times [0, T])} \right\}.$$

If $s > 1/q$ then $s_1 < s$ and we use the regularity of initial data to estimate I so from Lemma B.1 $I \leq D \cdot TV^s(u_0)$. Combining both the estimates we can write

$$I \leq TV^s(u_0) + \frac{4M}{C_2 t} + \frac{1}{C_2} \sup \left\{ |g'(v)|; |v| \leq \|u\|_{L^\infty(\mathbb{R} \times [0, T])} \right\}. \quad (4.17)$$

Similarly we have

$$\text{III} \leq TV^s(u_0) + \frac{4M}{C_1 t} + \frac{1}{C_1} \sup \left\{ |f'(v)|; |v| \leq \|u\|_{L^\infty(\mathbb{R} \times [0, T])} \right\}. \quad (4.18)$$

From Lemma A.3 we know that $f_+^{-1}g(\cdot)$ is a Lipschitz continuous. Hence, the term II can be estimated as

$$\begin{aligned} \text{II} &= \sum_{i=1}^{l-1} |f_+^{-1}g(u_0(z_+(x_i, t))) - f_+^{-1}g(u_0(z_+(x_{i+1}, t)))|^{1/s_1} \\ &\leq C \cdot \sum_{i=1}^{l-1} |u_0(z_+(x_i, t)) - u_0(z_+(x_{i+1}, t))|^{1/s_1}. \end{aligned}$$

If $s > 1/q$, then we have $s_1 < s$. Using Lemma B.1, we obtain $\text{II} \leq D \cdot TV^s(u_0)$. For the case $s < 1/q$, it is uncertain whether $s_1 < s$, but we certainly have $s_1 < 1/q$. In this case, we use the regularizing effect for solutions of conservation laws due to the non-degeneracy of g [11]. Thus, for term II, we obtain the estimate (4.10) similarly as in the proof of Theorem 2.1. Hence, combining the estimates on I, II and III we get

$$\sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \leq D \cdot TV^s(u_0) + 3(2\|u_0\|_\infty)^{1/s_1} + \frac{C_{f,g}M}{t}.$$

Case (ii): $R_1(t) = 0, L_1(t) < 0$.

This case can be handled in a similar fashion as in the previous case.

Only difference is the estimation of II which can be done same as in (4.15).

Hence, we have proven that $u(\cdot, t) \in BV^{s_1}(-M, M)$. To show that $u(\cdot, t) \in BV^{s_1}(\mathbb{R})$, we consider a partition $-\infty < x_{-n} < \dots < x_n < \infty$ which is not necessarily contained in $[-M, M]$. We can choose $M = t \sup\{|f'(v)|, |g'(v)|; |v| \leq \|u_0\|_\infty\}$. Suppose $|x_j| \leq M$ for $-m_1 \leq j \leq m_2$ for some $0 < m_1, m_2 \leq n$. From (4.16) we get

$$\sum_{i=-m_1}^{m_2} |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g} + 2(2\|u_0\|_\infty)^{\frac{1}{q+1}}.$$

From the choice of M , we can see that $R_1(t) \leq M, L_1(t) \geq -M$. Hence for $i \leq -m_1, u(x_i, t) =$

$u_0(z_-(x_i, t))$ and for $i \geq m_2$, $u(x_i, t) = u_0(z_+(x_i, t))$. Subsequently,

$$\sum_{i=-n}^{-m_1-2} |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{s}} + \sum_{i=m_2+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{s}} \leq TV^s(u_0).$$

Therefore, we obtain

$$\sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g,\|u_0\|_\infty} + 4(2\|u_0\|_\infty)^{\frac{1}{q+1}} + TV^s(u_0).$$

This completes the proof of Theorem 2.2. \square

4.5. Non restricted fluxes

We now consider the case of weaker non-degeneracy conditions on the fluxes. In this case, we cannot utilize Lemma 4.2, 4.3, and 4.4 to obtain estimates on the solution near the interface. As a result, the regularity of the solution is weaker here.

Proof of Theorem 2.3: Fix a time $t > 0$. We only show for the case when $R_1(t) > 0$. Note that in this case $L_1(t) = L_2(t) = 0$. Suppose $t_0 = \lim_{x \rightarrow R_1(t)-} t_+(x, t)$. First consider $t_0 > t/2$. From Lemma 4.1, we have

$$TV^{\frac{1}{q}}(u(0-, \cdot), (t_0, t)) \leq \frac{C_g t}{t_0} \leq 2C_g.$$

Since $u \mapsto f_+^{-1}(g(u))$ is Hölder continuous with exponent $\frac{1}{p+1}$, we get

$$|u(0+, t_1) - u(0+, t_2)| \leq C_{f,g,\|u_0\|_\infty} |u(0-, t_1) - u(0-, t_2)|^{\frac{1}{p+1}}.$$

Subsequently, we have

$$TV^s(u(0+, \cdot), (t_0, t)) \leq C_{f,g,\|u_0\|_\infty} \text{ where } s = \frac{1}{q(p+1)}.$$

Note that for $x \in (0, R_1(t))$ we have $u(x, t) = u(0+, t_+(x, t))$. Therefore,

$$TV^s(u(\cdot, t), (0, R_1(t))) \leq C_{f,g,\|u_0\|_\infty}. \quad (4.19)$$

For $x > R_1(t)$ we have $u(x, t) = (f')^{-1}\left(\frac{x - z_+(x, t)}{t}\right)$ for a non-decreasing $x \mapsto z_+(x, t)$. By using flux condition (2.2) of f , we obtain

$$TV^{\frac{1}{p}}(u(\cdot, t), (R_1(t), M)) \leq \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \quad (4.20)$$

Hence,

$$\begin{aligned} TV^s(u(\cdot, t), (0, M)) &\leq TV^s(u(\cdot, t), (0, R_1(t))) + \|2u\|_{L^\infty(\mathbb{R})}^{\frac{1}{s}} + TV^s(u(\cdot, t), (R_1(t), M)) \\ &\leq C_{f,g,\|u_0\|_\infty} + \|2u_0\|_{L^\infty(\mathbb{R})}^{\frac{1}{s}} + TV^{\frac{1}{p}}(u(\cdot, t), (R_1(t), M)) \\ &\leq C_{f,g,\|u_0\|_\infty} + \|2u_0\|_{L^\infty(\mathbb{R})}^{\frac{1}{s}} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \end{aligned}$$

Next we consider the case when $t_0 < t/2$. Let $x_0 = \sup\{x; t_+(x, t) \geq t/2\}$. By Lemma 4.3 we have

$$TV^{\frac{1}{p}}(u(\cdot, t); (x_0, M)) \leq C_{f,g,\|u_0\|_\infty} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \quad (4.21)$$

Similar to (4.19) we get

$$TV^s(u(\cdot, t), (0, x_0)) \leq C_{f,g,\|u_0\|_\infty} \text{ with } s = \frac{1}{q(p+1)}.$$

Subsequently, we obtain

$$TV^s(u(\cdot, t), (0, M)) \leq C_{f,g,\|u_0\|_\infty} + \|2u_0\|_{L^\infty(\mathbb{R})}^{\frac{1}{s}} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}.$$

Note that for $x < 0$ we have $u(x, t) = (g')^{-1} \left(\frac{x - z_-(x, t)}{t} \right)$. Then by using flux condition (2.2) we can show that

$$TV^{\frac{1}{q}}(u(\cdot, t); (-M, 0)) \leq \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \quad (4.22)$$

The other case when $L_1(t) < 0$ follows from a similar argument. This completes the proof of Theorem 2.3. \square

4.6. Propagation of the initial regularity outside the interface

In this section, we show the regularity of entropy solutions outside the interface is better than at the interface.

Proof of Theorem 2.4. We consider the partition $\epsilon \leq x_0 < x_1 < \dots < x_l \leq R_1(t) \leq x_{l+1} < \dots$. Then

$$\sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} = \sum_{i=0}^{l-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + \sum_{i=l}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s}.$$

Now from Theorem 3.1 we get,

$$\begin{aligned} \sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &\leq \sum_{i=0}^{l-1} \left| (f')^{-1} \left(\frac{x_i}{t - t_+(x_i, t)} \right) - (f')^{-1} \left(\frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right) \right|^{1/s} \\ &\quad + |u(x_l, t) - u(x_{l+1}, t)|^{1/s} + \sum_{i=l+1}^{\infty} |u_0(y(x_i, t)) - u_0(y(x_{i+1}, t))|^{1/s}. \end{aligned}$$

Since $t_+(x, t)$ is a monotone and bounded function such that $t_+(x, t) < t$ for all $x \geq \epsilon$, it follows that the infimum of $t - t_+(x, t)$ positive. Hence, we obtain

$$\frac{\epsilon}{T} \leq \frac{x}{t - t_+(x, t)} \leq \frac{M}{h(\epsilon, T)},$$

where $h(\epsilon, T) = \inf\{t - t_+(x, t) : \epsilon \leq x \leq R_1(t), 0 < t \leq T\}$, which also implies that $(f')^{-1}$ is Lipschitz continuous function on interval $\left[\frac{\epsilon}{T}, \frac{M}{h(\epsilon, T)} \right]$. Then,

$$\begin{aligned} \sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &\leq C(\epsilon, t) \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \\ &\quad + |u(x_l, t) - u(x_{l+1}, t)|^{1/s} \\ &\quad + \sum_{i=l+1}^{\infty} |u_0(y(x_i, t)) - u_0(y(x_{i+1}, t))|^{1/s}. \end{aligned}$$

The estimate on first sum follow from,

$$\begin{aligned} & \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \\ &= \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_i}{t - t_+(x_{i+1}, t)} + \frac{x_i}{t - t_+(x_{i+1}, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s}, \end{aligned}$$

from triangle inequality we get,

$$\begin{aligned} & \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \\ & \leq \sum_{i=0}^{l-1} \left(\left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_i}{t - t_+(x_{i+1}, t)} \right| + \left| \frac{x_i}{t - t_+(x_{i+1}, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right| \right)^{1/s}, \end{aligned}$$

now from the inequality $a^{1/s} + b^{1/s} \leq (a + b)^{1/s}$ we get,

$$\begin{aligned} & \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \\ & \leq \left(\sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_i}{t - t_+(x_{i+1}, t)} \right| + \left| \frac{x_i}{t - t_+(x_{i+1}, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right| \right)^{1/s}. \end{aligned}$$

Therefore, we get the following estimate,

$$\sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \leq \left(\frac{R_1(t) - \epsilon}{|t - t_+(\epsilon, t)|} + \frac{R_1(t)|t_+(\epsilon, t) - t_+(R_1(t), t)|}{|t - t_+(\epsilon, t)|^2} \right)^{1/s}.$$

Thus we have,

$$\begin{aligned} \sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} & \leq C \sup_{0 \leq t \leq T} \left(\frac{R_1(t) - \epsilon}{|t - t_+(\epsilon, t)|} + \frac{R_1(t)|t_+(\epsilon, t) - t_+(R_1(t), t)|}{|t - t_+(\epsilon, t)|^2} \right)^{1/s} \\ & + TV^s(u_0) + (2\|u_0\|)^{1/s}, \\ & \leq C(\epsilon, t) + TV^s(u_0) + (2\|u_0\|)^{1/s}. \end{aligned}$$

In a similar way the other case $x \leq -\epsilon$ can be handled, to yield

$$\sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \leq C(\epsilon, t) + TV^s(u_0) + 2(2\|u_0\|)^{1/s}.$$

□

5. Construction of counter-example

We now proceed to construct a counterexample to show that when the initial data is in BV , the corresponding solution may not be in BV^s at a fixed positive time $T > 0$, for some specific choice of flux. To do this, we refer to the backward construction for conservation laws with discontinuous flux introduced in [3]. However, before we apply this method, we need to recall some notations and results from [3]. In particular, we use the following result from [3]: given functions h_+ and z , we can construct an entropy solution that satisfies the Hopf-Lax type formula for (1.1) with h_+ and z .

Proposition 5.1 (Backward construction, [3]). *Let f, g are C^1 strictly convex functions. Let $R > 0$ and $z : [0, R] \rightarrow (-\infty, 0]$ be a non-decreasing function with $z_0 = z(0+)$ and $z_1 = z(R-)$. Suppose*

$$\begin{aligned} h_+ \left(\frac{R}{T - t_1} \right) &= -\frac{z_1}{t_1}, \\ g'(u_-) &= \frac{z_0}{T}, \quad g'(v_-) = -\frac{z_1}{t_1}, \quad \bar{v}_- = f_+^{-1}(g(v_-)), \end{aligned} \quad (5.1)$$

where h_+ is defined as

$$h_+ := g' \circ g_+^{-1} \circ f \circ (f')^{-1}. \quad (5.2)$$

We additionally assume that h_+ is a locally Lipschitz function. Then there exists an initial data $u_0 \in L^\infty(\mathbb{R})$ and the corresponding entropy solution u to (1.1) such that

$$u(x, T) = (f')^{-1} \left(\frac{x}{T - t_+(x)} \right) \quad \text{where} \quad -\frac{z(x)}{t_+(x)} = h_+ \left(\frac{x}{T - t_+(x)} \right) \quad \text{for } x \in [0, R] \quad (5.3)$$

and additionally, it holds $u(x, T) = u_-$ for $x < 0$ and $u(x, T) = \bar{v}_-$ for $x > R$.

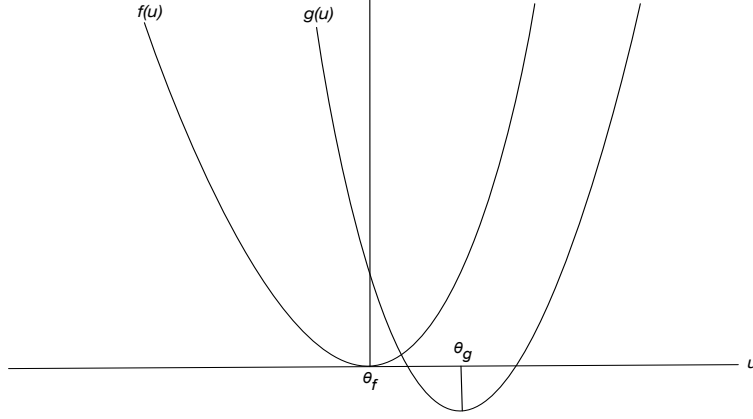


Figure 3: An illustration of fluxes

To be self-contained the main ingredients of the proof are given in [Appendix C](#). We now proceed with the proof of [Theorem 2.5](#).

Proof of Theorem 2.5. Let $f(u) = |u|^{p+1}$ and $g(u) = u^2 - 1$. Note that by [Lemma A.4](#) f satisfies the non-degeneracy condition (2.2) with exponent p and g is uniformly convex.

Let $\{a_k\}_{k \geq 1}$ be a sequence defined as $a_{2i} = i^{-\beta}$ and $a_{2i+1} = i^{-\alpha}$ with $\beta > \alpha > 0$ which will be chosen later. Consider an increasing sequence $\{t_k\}$ such that $t_k \rightarrow 1$ and

$$1 - t_{2k+1} = \frac{1}{k^{\beta-\alpha}}(1 - t_{2k}) \quad \text{and} \quad t_{2k+2} - t_{2k+1} = k^{-\lambda} \quad (5.4)$$

where $\lambda > 1$ will be chosen later. Then we have

$$\frac{t_{2k+2} - t_{2k+1}}{t_{2k+1}} = \frac{1}{k^\lambda} \frac{1}{t_{2k+1}} \geq \frac{1}{k^\lambda}. \quad (5.5)$$

We define $\{x_i\}$ as follows

$$x_i = (1 - t_{2i})a_{2i} = (1 - t_{2i+1})a_{2i+1}. \quad (5.6)$$

Since $\{t_{2i}\}_{i \geq 1}$ is increasing and $\{a_{2i}\}_{i \geq 1}$ is decreasing sequence, $\{x_i\}_{i \geq 1}$ is a decreasing sequence.

Let $h : [0, \infty) \rightarrow \mathbb{R}$ be defined as

$$h(u) = 2\sqrt{1 + (p+1)^{-1-\frac{1}{p}}u^{1+\frac{1}{p}}}$$

for $u \geq 0$. Observe that

$$\frac{h(a_{2i+1})}{h(a_{2i+2})} - 1 = \frac{\sqrt{1 + \left(\frac{i-\alpha}{p+1}\right)^{1+\frac{1}{p}}} - \sqrt{1 + \left(\frac{(i+1)-\beta}{p+1}\right)^{1+\frac{1}{p}}}}{\sqrt{1 + \left(\frac{(i+1)-\beta}{p+1}\right)^{1+\frac{1}{p}}}} \leq \frac{1}{i^{\frac{p+1}{p}\alpha}} - \frac{1}{i^{\frac{p+1}{p}\beta}}. \quad (5.7)$$

Then if $\lambda < \frac{p+1}{p}\alpha$ we get

$$\frac{h(a_{2i+1})}{h(a_{2i+2})} - 1 < \frac{t_{2i+2}}{t_{2i+1}} - 1. \quad (5.8)$$

Therefore, we have

$$t_{2i+1}h(a_{2i+1}) < t_{2i+2}h(a_{2i+2}). \quad (5.9)$$

Note that

$$\frac{1 - t_{2i+1}}{1 - t_{2i}} = \frac{1}{i^{\beta-\alpha}} < 1. \quad (5.10)$$

Hence, $t_{2i+1} > t_{2i}$. Since $h(a_{2i+1}) > h(a_{2i})$ we have $t_{2i+1}h(a_{2i+1}) > h(a_{2i})t_{2i}$. Let $\xi(x)$ be solving the following problem

$$\left(\frac{x}{1 - \xi(x)}\right)^{1+\frac{1}{p}} = \left(\frac{C}{\xi(x) + d}\right)^2 - 1 \quad (5.11)$$

$$\xi(x_i) = t_{2i+1}, \quad (5.12)$$

$$\xi(x_{i+1}) = t_{2i+2}. \quad (5.13)$$

Note that $C > 0$ and $d > 0$ are determined by (5.12) and (5.13). In particular, the existence of such ξ can be shown from Lemma 3.4 and 3.5 of [3]. Next, we show that $\xi' < 0$. To do this, we differentiate both sides of (5.11) and get the following

$$\begin{aligned} 0 < \left(1 + \frac{1}{p}\right)x^{\frac{1}{p}} &= -\xi'(x) \left(1 + \frac{1}{p}\right) (1 - \xi(x))^{\frac{1}{p}} \left[\left(\frac{C}{\xi(x) + d}\right)^2 - 1 \right] \\ &\quad - \xi'(x) (1 - \xi(x))^{1+\frac{1}{p}} \frac{2C}{(\xi(x) + d)^2} \left(\frac{C}{\xi(x) + d}\right). \end{aligned} \quad (5.14)$$

Therefore, we get $\xi'(x) < 0$. Let $\Phi(x)$ be defined as

$$\Phi(x) := \xi(x) \sqrt{1 + \left(\frac{x}{1 - \xi(x)}\right)^{1+\frac{1}{p}}} = \frac{C\xi(x)}{\xi(x) + d}. \quad (5.15)$$

Observe that

$$\Phi'(x) = \xi'(x) \left[\frac{C}{\xi(x) + d} - \frac{C\xi(x)}{(\xi(x) + d)^2} \right] = \xi'(x) \frac{Cd}{(\xi(x) + d)^2} < 0. \quad (5.16)$$

Finally we define the function $t(x)$ such that $t(x_i+) = t_{2i}$ and $t(x_i-) = t_{2i+1}$ for $i \geq i_0$ and t satisfies (5.11)–(5.13) for $x \in (x_{i+1}, x_i)$. Let $\rho : (0, \infty) \rightarrow \mathbb{R}$ be defined as

$$\rho(x) = -t(x)h\left(\frac{x}{1 - t(x)}\right). \quad (5.17)$$

By (5.9) and (5.16), $x \mapsto \rho(x)$ is increasing. By Proposition 5.1 with $R = x_1$, there exists an

entropy solution u such that

$$u(x_{i+}, 1) = \left(\frac{x_i}{(p+1)(1-t_{2i})} \right)^{\frac{1}{p}} \text{ and } u(x_{i-}, 1) = \left(\frac{x_i}{(p+1)(1-t_{2i+1})} \right)^{\frac{1}{p}}. \quad (5.18)$$

By (5.6) we get

$$u(x_{i+}, 1) = \left(\frac{a_{2i}}{p+1} \right)^{\frac{1}{p}} \text{ and } u(x_{i-}, 1) = \left(\frac{a_{2i+1}}{p+1} \right)^{\frac{1}{p}}. \quad (5.19)$$

Therefore,

$$\begin{aligned} |u(x_{i-}, 1) - u(x_{i+}, 1)| &= \left| \left(\frac{a_{2i}}{p+1} \right)^{\frac{1}{p}} - \left(\frac{a_{2i+1}}{p+1} \right)^{\frac{1}{p}} \right| \\ &= (1+p)^{-\frac{1}{p}} \left[i^{-\frac{\alpha}{p}} - i^{-\frac{\beta}{p}} \right]. \end{aligned} \quad (5.20)$$

Let $\epsilon > 0$. Then, we have

$$|u(x_{i-}, 1) - u(x_{i+}, 1)|^{\frac{p+1}{1+\epsilon}} \geq C(p) \left[i^{-\frac{\alpha(p+1)}{p(1+\epsilon)}} - i^{-\frac{\beta(p+1)}{p(1+\epsilon)}} \right]. \quad (5.21)$$

Now, we set

$$\lambda = 1 + \frac{2p}{3(2p+1)}\epsilon, \quad \frac{p+1}{p}\alpha = 1 + \frac{4p+2}{3(2p+1)}\epsilon \text{ and } \frac{p+1}{p}\beta = 1 + \frac{2(3p+2)}{3(2p+1)}\epsilon. \quad (5.22)$$

We check that $\beta - \alpha = \lambda - 1$ and $\frac{p+1}{p}\beta > 1 + \epsilon$. Hence, $u(\cdot, 1) \notin BV_{loc}^s(\mathbb{R})$ for $s = \frac{1}{p+1} + \frac{\epsilon}{p+1}$.

Note that by Proposition 5.1 initial data $u_0 \in L^\infty(\mathbb{R})$. Now we find a data which is in $BV(\mathbb{R})$. From the construction we have $x_1 < R_2(1)$ where $R_2(t)$ is as in Theorem 3.1. Choose a point $x_0 \in (x_1, R_2(1))$. Note that $0 < t_+(x_0, 1) < 1$ and $u(x, t_+(x_0, 1)) = \bar{v}_-$ for $x \geq 0$. We also observe that $L_1(t) = 0$ and $R_2(t) > 0$ for $t = t_+(x_0, 1)$. Therefore, for $t = t_+(x_0)$ we have

$$u(x, t) = (g')^{-1} \left(\frac{x - z_-(x, t)}{t} \right) \text{ for } x < 0. \quad (5.23)$$

Since g is uniformly convex we have $u(\cdot, t_+(x_0, 1)) \in BV((-\infty, 0))$. To conclude the Theorem 2.5 we set $v_0(x) := u(x, t_0(x_0, 1))$. Let $v(x, t)$ be the entropy solution to (1.1) with initial data v_0 . Note that $v(x, 1 - t_0(x_0, 1)) = u(x, 1)$ for all $x \in \mathbb{R}$. Hence, the proof of Theorem 2.5 is completed. \square

Appendix A. Hölder continuity of singular maps

In this section, we collect useful lemmas on the Hölder exponent and non-degeneracy of fluxes, which are used throughout the paper. Some commentaries are added for all lemmas. The following lemma recall that the non-uniform convexity of a flux function corresponds to a loss of the Lipschitz regularity for the reciprocal function of the derivative. This key point enforces a BV^s (or generalized BV regularity [17, 29]) instead of BV regularity [35, 44] for the entropy solutions.

Lemma A.1. *Let $g \in C^1(\mathbb{R})$ be satisfying the non-degeneracy (2.2) with exponent q . Then $(g')^{-1}$ is Hölder continuous with exponent $1/q$.*

Proof. Fix a compact set K . Let x and y is in $g'(K)$. There exist \tilde{x}, \tilde{y} such that $\tilde{x} = (g')^{-1}(x)$ and $\tilde{y} = (g')^{-1}(y)$. Then,

$$\frac{|(g')^{-1}(x) - (g')^{-1}(y)|}{|x - y|^{1/q}} = \frac{|\tilde{x} - \tilde{y}|}{|g'(\tilde{x}) - g'(\tilde{y})|^{1/q}} = \frac{|\tilde{x} - \tilde{y}|}{|g'(\tilde{x}) - g'(\tilde{y})|^{1/q}} \leq \frac{1}{C_2^{1/q}}.$$

This proves the Lemma [A.1](#). □

The interface condition [\(1.2\)](#) requires the use of some reciprocal functions of the fluxes g or f . The fact that the reciprocal function of g is never Lipschitz near $\min g$ forbids the classical Lax-Oleinik BV smoothing effect for a uniform convex flux.

Lemma A.2. *Let g be a C^2 function satisfying [\(2.2\)](#) with exponent q then g_+ satisfies [\(2.2\)](#) with exponent $q + 1$ on domain (θ_g, ∞) .*

Proof. Since θ_g is the critical point of g hence, $g'(\theta_g) = 0$, then we consider

$$\begin{aligned} g(x) - g(y) &= (x - y) \int_0^1 g'(\lambda x + (1 - \lambda)y) d\lambda, \\ &= (x - y) \int_0^1 (g'(\lambda x + (1 - \lambda)y) - g'(\theta_g)) d\lambda. \end{aligned}$$

We know that $g'(\cdot)$ is an increasing function and g satisfies the non-degeneracy condition [\(2.2\)](#). Let $x > y \geq \theta_g$, then

$$\begin{aligned} |g(x) - g(y)| &= |x - y| \int_0^1 (g'(\lambda x + (1 - \lambda)y) - g'(\theta_g)) d\lambda \\ &\geq C_2 |x - y| \int_0^1 (\lambda x + (1 - \lambda)y - \theta_g)^q d\lambda \\ &\geq \frac{1}{q + 1} C_2 \left((x + (1 - \lambda)y - \theta_g)^{q+1} \right) \Big|_0^1 \\ &\geq \frac{1}{q + 1} C_2 ((x - \theta_g)^{q+1} - (y - \theta_g)^{q+1}) \\ &\geq \frac{C_2}{q + 1} |x - y|^{q+1}. \end{aligned} \tag{A.1}$$

□

The previous [comment](#) of Lemma [A.2](#) is even more important for the non-Lipschitz regularity of the singular map.

Lemma A.3. *Suppose fluxes f and g are $C^1(\mathbb{R})$ and convex functions with $f(\theta_f) < g(\theta_g)$ which additionally satisfies the non-degeneracy condition [\(2.2\)](#) and let K is any compact set of \mathbb{R} . Then for $x \in K$, $f_+^{-1}g(\cdot)$ is a Lipschitz continuous function and $g_-^{-1}f(\cdot)$ is a Hölder continuous function.*

Proof. Since $f(\theta_f) < g(\theta_g)$, there exist $a_1 < \theta_f < a_2$ such that $f(a_1) = g(\theta_g) = f(a_2)$. Hence, we have

$$\bar{c} := \min \{ |f'(a)| ; a \in (-\infty, a_1] \cup [a_2, \infty) \} > 0. \tag{A.2}$$

Without loss of generality, we can assume that $g(x) \neq g(y)$ because if $g(x) = g(y)$, then [the](#) result holds anyway. There exist $\tilde{x}, \tilde{y} > \theta_f$ such that $f(\tilde{x}) = g(x)$ and $f(\tilde{y}) = g(y)$. As f_+^{-1} is increasing, we get $\tilde{x}, \tilde{y} > a_2$. Consider the following

$$\begin{aligned} \frac{|f_+^{-1}g(x) - f_+^{-1}g(y)|}{|x - y|} &= \frac{|f_+^{-1}g(x) - f_+^{-1}g(y)|}{|g(x) - g(y)|} \cdot \frac{|g(x) - g(y)|}{|x - y|}, \\ &= \frac{|f_+^{-1}f(\tilde{x}) - f_+^{-1}f(\tilde{y})|}{|f(\tilde{x}) - f(\tilde{y})|} \cdot \frac{|g(x) - g(y)|}{|x - y|}, \end{aligned}$$

$$\begin{aligned}
&= \frac{|\tilde{x} - \tilde{y}|}{|f(\tilde{x}) - f(\tilde{y})|} \cdot \frac{|g(x) - g(y)|}{|x - y|}, \\
&= \frac{1}{f'(c_0)} \cdot \frac{|g(x) - g(y)|}{|x - y|},
\end{aligned}$$

for some c_0 in between \tilde{x}, \tilde{y} . Note that $c_0 \geq a_2$ and $f' \geq \bar{c}$. Since g is a Lipschitz continuous function, we have $|g(x) - g(y)| \leq c_1 |x - y|$, where c_1 depends on g and K . Therefore, we get

$$\frac{|f_+^{-1}g(x) - f_+^{-1}g(y)|}{|x - y|} \leq C. \quad (\text{A.3})$$

We know that for $f(x) \geq g(\theta_g)$ there exists \tilde{x} such that $f(x) = g(\tilde{x})$ and $g'(\tilde{x}) > 0$, without loss of generality we can assume that $g(x) \neq g(y)$ because if $g(x) = g(y)$ then result holds.

$$\begin{aligned}
\frac{|g_-^{-1}f(x) - g_-^{-1}f(y)|^{q+1}}{|x - y|} &= \frac{|g_-^{-1}f(x) - g_-^{-1}f(y)|^{q+1}}{|f(x) - f(y)|} \cdot \frac{|f(x) - f(y)|}{|x - y|}, \\
&= \frac{|g_-^{-1}g(\tilde{x}) - g_-^{-1}g(\tilde{y})|^{q+1}}{|g(\tilde{x}) - g(\tilde{y})|} \cdot \frac{|f(x) - f(y)|}{|x - y|}, \\
&= \frac{|\tilde{x} - \tilde{y}|^{q+1}}{|g(\tilde{x}) - g(\tilde{y})|} \cdot \frac{|f(x) - f(y)|}{|x - y|},
\end{aligned}$$

Now from the Lipschitz continuity f and (A.1),

$$\frac{|g_-^{-1}f(x) - g_-^{-1}f(y)|^{q+1}}{|x - y|} \leq C. \quad (\text{A.4})$$

Hence, it implies that

$$|g_-^{-1}f(x) - g_-^{-1}f(y)| \leq C|x - y|^{1/q+1}.$$

□

The following lemma shows that power law fluxes satisfy the non-degeneracy condition (2.2).

Lemma A.4. *Let $M > 0$ and $g : [-M, M] \rightarrow \mathbb{R}$ be defined as $g(u) = |u|^p$ for $p \geq 2$. Then g satisfies the non-degeneracy condition (2.2) with exponent $p - 1$.*

This is the simplest example with power-law degeneracy $p - 1$ [11, 16].

Appendix B. BV^s embedding

The continuous embedding between fractional BV spaces is explicitly expressed using the L^∞ norm or, more precisely, the oscillation in the following lemma. It is important to recall that the oscillation of a function u on an interval I is defined as follows

$$\text{osc}(u) := \sup_{x < y} \{|u(x) - u(y)|\} \leq 2\|u\|_\infty.$$

Lemma B.1. *Let $u : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be bounded function on a given interval I and $0 < s < t$ such that $u \in BV^t \subset BV^s$. Let $p = \frac{1}{s} \geq q = \frac{1}{t}$, then,*

$$TV^s u(I) \leq \text{osc}(u)^{p-q} TV^t u(I). \quad (\text{B.1})$$

Proof. When $\text{osc}(u) \leq 1$, the inequality $y^p \leq y^q$ for all $y \in [0, 1]$ gives a direct estimate. More precisely, let $\sigma = (x_1, \dots, x_n)$ be any partition of I ,

$$\sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^p \leq \sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^q \leq TV^t u(I).$$

This inequality can be improved as follows if u is non constant, that is $\text{osc}(u) > 0$. For this purpose, consider $v = u/\text{osc}(u)$ so $\text{osc}(v) \leq 1$. Now, on a subdivision, we have,

$$\begin{aligned} \text{osc}(u)^{-p} \sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^p &= \sum_{i=1}^{n-1} |v(x_i) - v(x_{i+1})|^p \\ &\leq \sum_{i=1}^{n-1} |v(x_i) - v(x_{i+1})|^q = \text{osc}(u)^{-q} \sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^q. \end{aligned}$$

That is to say, the following inequality which is also valid when $\text{osc}(u) = 0$,

$$\sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^p \leq \text{osc}(u)^{p-q} \sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^q.$$

This is enough to conclude the lemma. \square

Appendix C. Backward construction

The proof of optimality presented in section 5 needs a construction of initial data and solution by borrowing ideas and techniques from control. We only give a sketch of the existence of such solution along with initial data that is stated in Proposition 5.1. The complete construction can be found in [3].

Proof of Proposition 5.1. We first approximate $z(x)$ by piece-wise constant increasing function as follows

$$\left\{ \begin{array}{l} z_0 = w_0 < w_1 < \dots < w_k = z_1, \\ |w_{i+1} - w_i| < \frac{1}{N}, \\ 0 = x_0 < x_1 < \dots < x_k = R, \\ z(x_i) = w_i \text{ for } 1 \leq i \leq k-1, \\ \text{with } z_0 = z(0) \text{ and } z_1 = z(R-). \end{array} \right. \quad (\text{C.1})$$

We set $t_0 = T$ and t_i , $1 \leq i \leq 2k$, c_i, d_i $1 \leq i \leq k$ as follows

$$\begin{aligned} h_+ \left(\frac{x_i}{T - t_{2i-1}} \right) &= -\frac{w_{i-1}}{t_{2i-1}}, \quad h_+ \left(\frac{x_i}{T - t_{2i}} \right) = -\frac{w_i}{t_{2i}}, \\ f'(c_{2i-1}) &= \frac{x_i}{T - t_{2i-1}}, \quad f'(c_{2i}) = \frac{x_i}{T - t_{2i}} \text{ and } d_i = g_+^{-1}(f(a_i)). \end{aligned} \quad (\text{C.2})$$

Then we observe that $c_{2i-1} > c_{2i}, d_{2i-1} > d_{2i}, T = t_0 > t_1 > \dots > t_{2k} = T_1$. Consider Lipschitz curves $r_i, \tilde{r}_i, a_i, b_i$ defined as follows

$$s_i = \frac{f(c_{2i-1}) - f(c_{2i})}{c_{2i-1} - c_{2i}}, \quad S_i = \frac{g(d_{2i-1}) - g(d_{2i})}{d_{2i-1} - d_{2i}}, \quad 1 \leq i \leq k, \quad (\text{C.3})$$

$$\begin{aligned} r_i(t) &= g'(d_i)(t - t_i), \quad \tilde{r}_i(t) = f'(c_i)(t - t_i), \quad 1 \leq i \leq 2k, \\ a_i(t) &= x_i + s_i(t - T), \quad b_i(t) = S_i(t - q_i), \quad a_i(q_i) = 0, \quad 1 \leq i \leq 2k, \end{aligned}$$

$$r_0(t) = g'(b_0)(t - T) = g'(u_-)(t - t_0). \quad (\text{C.4})$$

Now, we define u_0^N as below

$$u_0^N := \begin{cases} u_- & \text{if } x < w_0, \\ d_{2i-1} & \text{if } w_{i-1} < x < b_i(0), 1 \leq i \leq k, \\ d_{2i} & \text{if } b_i(0) < x < w_i, 1 \leq i \leq k, \\ v_- & \text{if } w_{2k} < x < 0, \\ \bar{v}_- & \text{if } x > 0. \end{cases} \quad (\text{C.5})$$

Let $\tilde{t}_i(x)$ be the unique solution to

$$h_+ \left(\frac{x}{T - \tilde{t}_i(x, t)} \right) = -\frac{z_i}{\tilde{t}_i(x, t)} \text{ for } x \in (x_i, x_{i+1}), 1 \leq i \leq k-1. \quad (\text{C.6})$$

Corresponding entropy solution u^N is the following

$$u^N(x, t) = \begin{cases} u_- & \text{if } x < r_0(t), \\ (g')^{-1} \left(\frac{x - z_i}{t} \right) & \text{if } r_{2i}(t) < x < \min\{r_{2i+1}(t), 0\}, \\ (f')^{-1} \left(\frac{x}{t - \tilde{t}_i(x, t)} \right) & \text{if } \max\{\tilde{r}_{2i+1}(t), 0\} < x < \tilde{r}_{2i-1}(t), \\ d_{2i-1} & \text{if } r_{2i-1}(t) < x < \min\{S_i(t), 0\}, 1 \leq i \leq k, \\ d_{2i} & \text{if } S_{2i}(t) < x < \min\{r_{2i}(t), 0\}, 1 \leq i \leq k, \\ c_{2i-1} & \text{if } \max\{\tilde{r}_{2i-1}(t), 0\} < x < s_i(t), 1 \leq i \leq k, \\ c_{2i} & \text{if } \max\{s_i(t), 0\} < x < \tilde{r}_{2i}, 1 \leq i \leq k, \\ v_- & \text{if } r_{2k}(t) < x < 0, \\ \bar{v}_- & \text{if } x > \max\{\tilde{r}_{2k}, 0\}. \end{cases} \quad (\text{C.7})$$

By assumption we have h_+ is a locally Lipschitz continuous function and we can prove TV bound of $g'(u_0^N)$ (see [3] for more details). Then, by applying Helly's Theorem we can find a $u_0 \in L^\infty(\mathbb{R})$ and corresponding entropy solution u satisfying (5.3). This completes the proof of Proposition 5.1. \square

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