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Shyam Sundar Ghoshal, Stéphane Junca, Akash Parmar. Fractional regularity for conservation laws with discontinuous flux. *Nonlinear Analysis: Real World Applications*, In press. hal-03669742v1

HAL Id: hal-03669742

<https://hal.science/hal-03669742v1>

Submitted on 16 May 2022 (v1), last revised 12 Jul 2023 (v2)

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Fractional regularity for conservation laws with discontinuous flux

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Abstract

This article deals with the regularity of the entropy solutions of scalar conservation laws with discontinuous flux. It is well-known [Adimurthi et al., Comm. Pure Appl. Math. 2011] that the entropy solution for such equation does not admit BV regularity in general, even when the initial data belongs to BV . Due to this phenomenon fractional BV^s spaces wider than BV are required, where the exponent $0 < s \leq 1$ and $BV = BV^1$. It is a long standing open question to find the optimal regularizing effect for the discontinuous flux with L^∞ initial data. The optimal regularizing effect in BV^s is proven on an important case using control theory. The fractional exponent s is at most $1/2$ even when the fluxes are uniformly convex.

Keywords: Conservation laws, Interface, Discontinuous flux, Cauchy problem, Regularity, BV functions, Fractional BV spaces.

2020 MSC: 35B65, 35L65, 35F25, 35L67, 26A45, 35B44.

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1. Introduction

This article deals with the regularity aspects of the solution for the following scalar conservation law with discontinuous flux:

$$\begin{cases} u_t + f(u)_x = 0, & \text{if } x > 0, t > 0, \\ u_t + g(u)_x = 0, & \text{if } x < 0, t > 0, \\ u(x, 0) = u_0(x), & \text{if } x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ is unknown, $u_0(\cdot) \in L^\infty(\mathbb{R})$ is the initial data and the fluxes f, g are $C^1(\mathbb{R})$ and strictly convex (that means f', g' are increasing functions).

The conservation law (1.1) arises in several frameworks of real-life phenomena, physical situations and applied subjects. For example, the equation (1.1) occurs naturally in the two-phase flow of a heterogeneous porous medium in the petroleum reservoir [29]. The equation (1.1) is also useful to understand the ideal clarifier thickener [14], traffic flow model with varying road surface conditions [36] and ion etching accustomed for semiconductor industry [43]. The above examples are just a little glance at the broad applicability of the equation (1.1) in the fields of applied sciences. For more details, one can see [14, 15, 20, 21].

The equation (1.1) does not have a global classical solution even for smooth initial data, so one needs to consider the following notion of a weak solution:

Definition 1.1 (Weak solution). *A function $u \in C(0, T; L^1_{loc}(\mathbb{R}))$ is said to be a weak solution of the problem (1.1) if*

$$\int_0^\infty \int_{\mathbb{R}} u \frac{\partial \phi}{\partial t} + F(x, u) \frac{\partial \phi}{\partial x} dx dt + \int_{\mathbb{R}} u_0(x) \phi(x, 0) dx = 0,$$

for all $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$, where the flux $F(x, u)$ is given as $F(x, u) = H(x)f(u) + (1 - H(x))g(u)$, and $H(x)$ is the Heaviside function.

From the above defined weak formulation it can be derived that if interface traces $u^\pm(t) = \lim_{x \rightarrow 0^\pm} u(x, t)$ exist then at $x = 0$, u satisfies Rankine-Hugoniot condition, namely, for almost all t

$$f(u^+(t)) = g(u^-(t)). \quad (1.2)$$

For equation (1.1), the left and right traces u^-, u^+ play important roles in well-posedness theory and also in determining the regularity of solutions. In [7], authors proved the existence of the interface traces via Hamilton-Jacobi type equation.

Because of the non-uniqueness of weak solutions, one needs some extra condition called the “entropy condition” to get the unique solution even for the case $f = g$ in (1.1). For $f = g$, Kruřkov [32] gave a generalized entropy condition and proved the uniqueness. But due to the discontinuity of flux at the interface, the Kruřkov entropy is not good enough to prove the uniqueness of (1.1). Hence another condition is needed near the interface, called the “interface entropy condition”.

Throughout this article, we use the following notion of the entropy solution.

Definition 1.2 (Entropy solution, [7]). *A weak solution $u \in L^\infty(\mathbb{R} \times [0, T])$ of the problem (1.1) is said to be an entropy solution if the following holds.*

1. u satisfies Kruzkov entropy conditions on each side of the interface $x = 0$, that is, in $\mathbb{R} \setminus \{0\}$.
2. The interface traces $u^\pm(t) = \lim_{x \rightarrow 0^\pm} u(x, t)$ exist for almost all $t > 0$ and they satisfy the following “interface entropy condition” for almost all $t > 0$,
$$|\{t : f'(u^+(t)) > 0 > g'(u^-(t))\}| = 0. \quad (1.3)$$

Uniqueness has been proved in [7] when interface traces exist for a weak solution and it satisfies the entropy condition (1.3). In the same article, the authors obtained the useful Lax-Oleinik type explicit formulas for (1.1). The notion of ‘A-B entropy solution’ is introduced in [5] and it coincides with (1.3) when $A = \theta_g, B = \theta_f$. The number θ_f is defined by $f(\theta_f) = \min f$ when f admits a minimum and $g(\theta_g) = \min g$. Lax-Oleinik type formula is also available [6] for the ‘A-B-entropy solutions’. It has been observed [1] that for the case $A < \theta_g$ or $B > \theta_f$, ‘A-B-entropy solutions’ belong to BV space for BV initial data and for $A = \theta_g, B = \theta_f$ total variation of entropy solution can blow up at finite time $t_0 > 0$ even for some BV initial data (see section 1.2 for more details). Therefore, we work with the choice $A = \theta_g, B = \theta_f$. In this article, we rely on the interface entropy condition (1.3), and we use the analysis of characteristics developed as in [7].

The well-posed theory from the numerical and theoretical aspects has been extensively studied. We refer to [5, 10, 12, 31, 41] and the references therein. The existence of a solution of (1.1) has been proved by several numerical schemes [4, 9, 26, 44]. Due to the absence of total variation bound of solutions even for BV data, the singular mapping technique becomes useful to show the convergence of numerical schemes (see [4, 44]). Very recently, in [26] authors generalize the Godunov type scheme in the case when discontinuities of flux may have limit point even when the set of discontinuities is dense.

Due to the lack of the BV regularity of the entropy solution of (1.1), one needs to study the regularity aspects of the solution in some bigger space than BV . More precisely, in this paper, we quantify the sharp regularity of entropy solution of (1.1) in suitable fractional spaces.

Structure of the paper

This paper is organized as follows: in sections 1.1 and 1.2, we have discussed regularity results for scalar conservation laws and for (1.1) respectively. Then, it leads to section 1.3 where we precisely state the regularity problems corresponding to the equation (1.1). In section 2 we describe our main results with some remarks. To make this article self-contained, in section 3 some definitions and preliminary results have been recalled from [7, 11]. The detailed proofs of main results are written in section 4. Proofs of these results utilize the Hopf-Lax type formula and some results from [7] and techniques from [1, 22]. Proofs here are a little more involved technically. In the last section, the construction of a counter-example shows that the main results of the present article cannot be improved. Two appendixes contain basic useful lemmas and explanations regarding our adaptation of the result from control theory [3].

1.1. Optimal regularity results in BV^s spaces for a smooth flux: $f = g$

In this subsection, we discuss the entropy solution of (1.1) for $f = g$. Even for Lipschitz flux, the theory is well-posed [30, 32, 33, 42] in the L^∞ setting and many methodologies are available to understand the regularity of entropy solutions [2, 11, 17, 18, 24, 25, 27, 33, 39, 40, 42].

Natural function space for scalar conservation law is BV since the fundamental work of A. I. Volpert [45] in 1967. This space allows to get compactness and it makes convenient to describe the structure of a shock wave with traces on each side of the singularity [8]. Information on trace helps to study finer qualitative properties of solutions. The occurrence of the BV regularity for

entropy solution appeared for the first time in [33, 42] independently by P. D. Lax and O. Oleinik. The entropy solution becomes instantaneously BV even when the data is in L^∞ and the flux is uniformly convex, i.e., $\inf f'' > 0$. This is the well known smoothing effect as a consequence of the one-sided Lipschitz-Oleinik inequality [42].

Unfortunately, the ‘ BV space is not enough’ [19] when the flux is not uniformly convex. There are many examples of entropy solutions that are not in BV [2, 16, 24]. Though non-vanishing property of second derivative of the flux is proved [25] to be necessary and sufficient condition for BV regularizing, but a smoothing effect still occurs in fractional Sobolev spaces [28, 34] for a nonlinear flux. To keep the advantages of space BV : regularity and some traces, the fractional BV spaces were introduced for conservation laws in [11]. The Lax-Oleinik smoothing effect was generalized in BV^s for a flux with power-law nonlinearity like $|u|^{p+1}$ and $p = 1/s \geq 1$, for C^1 or strictly convex flux in [11, 17, 27].

Fractional BV spaces, denoted by BV^s , $0 < s \leq 1$, were first defined for all $s \in (0, 1)$ in [35, 37, 38]. Let I be a non-empty interval of \mathbb{R} and $s \in (0, 1]$. The space of fractional bounded variation functions, denoted as $BV^s(I)$ is a generalization of space of functions with a bounded variation on I , denoted as $BV(I)$. In the sequel, we denote $S(I)$ the set of the subdivisions of I , that is the set of finite subsets $\sigma = (x_0, x_1, \dots, x_n)$ in I with $(x_0 < x_1 < x_2 < \dots < x_n)$.

Definition 1.3 (BV^s [35, 37, 38]). *Let $\sigma = (x_0, x_1, \dots, x_n)$ be in $S(I)$ and let u be real function on I . The s -total variation of u with respect to σ is*

$$TV^s u(\sigma) = \sum_{i=1}^n |u(x_i) - u(x_{i-1})|^{1/s},$$

then define,

$$TV^s u(I) = \sup_{\sigma \in S(I)} TV^s u(\sigma).$$

The set $BV^s(I)$ is the set of functions $u : I \rightarrow \mathbb{R}$ such that $TV^s u(I) < \infty$.

1.2. Previous regularity results for discontinuous flux

To study the convergence and existence of traces of the solution, BV regularity plays a very important role. Without total variation bound, the convergence of numerical scheme is difficult. It is to be noted that one can not expect the total variation diminishing of the solution to (1.1) since non-constant solution can arise from constant initial data. Despite extensive study of the equation (1.1) for several decades, optimal regularity results were missing for the solution of (1.1). There are few results known about the regularity aspects of the solution to (1.1) and we describe them in the following paragraph.

Though away from interface solution has been proved [13] to be BV in space, the regularity of solution near interface was unknown for long time. First breakthrough result [1] comes in 2009. By constructing explicit example when $\min f \neq \min g$, authors [1] have shown that total variation of entropy solution to (1.1) blows up at time $t_0 > 0$ for BV initial data. To build the example, they have utilized the failure of Lipschitz continuity of $f^{-1}g$ near the critical point of f . Here g_-^{-1} , f_+^{-1} are the inverse of g, f in appropriate domains, more precisely, they are defined as

$$g_-^{-1} :](g')^{-1}(-\infty), (g')^{-1}(0)] \rightarrow \mathbb{R} \quad \& \quad f_+^{-1} : [(f')^{-1}(0), (f')^{-1}(+\infty)[\rightarrow \mathbb{R}. \quad (1.4)$$

The key functions $f_+^{-1}g(\cdot)$ and $g_-^{-1}f(\cdot)$ transmit information via the interface from left-to-right and right-to-left respectively.

On the other hand in [22, 23] the author proved several regularity results. More surprisingly, the author showed that one can prove that solution of (1.1) belongs to BV if fluxes have the same minimum value, i.e., $\min f = f(\theta_f) = \min g = g(\theta_g)$. The author also proved that if $f(\theta_f) \neq g(\theta_g)$ and initial data is compactly supported then there exists a time T such that for all $t > T$ solution of (1.1) admits the BV regularity. The assumption of compact support can not be relaxed as it has been shown by example that there exists a sequence of time, T_n , for which the total variation of solution to (1.1) blows up.

Earlier referred publications have uniform convexity assumption on the fluxes, in [23] it has been proved that even for non-uniform convex flux (but with a special structure when the flux losses its uniform convexity) any L^∞ initial data gives the solution which is BV_{loc} near the interface when the connection (A, B) as in [6] are far from the critical point.

This discussion leads to conclude that working in the BV space setup is not enough for scalar conservation law with discontinuous flux (1.1). Hence, working in larger space appears appropriate and we work in the space of functions of fractional bounded variation, denoted as BV^s , which is more generalised space than the BV space.

In the following subsection we discuss the questions which are answered in the present paper.

1.3. Questions on the BV^s regularity for discontinuous flux

As we discussed so far, *the entropy solution of (1.1) lacks the following properties:*

1. If $u_0 \in BV(\mathbb{R})$ then $u(\cdot, t) \in BV(\mathbb{R})$, for any $t > 0$.
2. If f, g are uniformly convex fluxes, $\min f \neq \min g$ and $u_0 \in L^\infty(\mathbb{R})$ then for any $t > 0$, $u(\cdot, t) \in BV_{loc}$.

By motivating the subject on the basis of the above facts, we settle the following issues regarding regularity of the solution of (1.1).

Question 1.1. *Can we expect that for a well chosen $0 < s \leq 1$, if the given initial data belongs to BV^s then the solution of (1.1) stays in BV^s ?*

Question 1.2. *Can we expect that for any $0 < s \leq 1$ there exists $0 < s_1$ such that if the given initial data belongs to BV^s then the solution of (1.1) belongs to BV^{s_1} ?*

In particular, when u_0 is in $BV = BV^1$, $s = 1$, in which BV^{s_1} , $0 < s_1 \leq 1$, is the solution?

Question 1.3. *What is the Lax-Oleinik type regularizing effect, for uniformly convex fluxes f and g ? In other words, does entropy solution of (1.1) belong to BV^s for some $s \in (0, 1)$ and for any given L^∞ initial data?*

Question 1.4. *Can we choose $0 < s < 1$ sharply and an initial data $u_0 \in BV^s$ space for which the generalized total variation blows up for the corresponding solution of (1.1)?*

We are able to answer all questions 1.1-1.4 affirmatively under certain assumptions on the fluxes f, g . We then show by counter-examples that the assumptions of our main results are optimal. Moreover, we provide explicit estimates of s -total variation of the solution with respect to time variable t with some sufficient conditions on initial data.

2. Main Results

Throughout the paper, f and g are C^1 strictly convex functions admitting a critical point. Let θ_f and θ_g be the unique critical points of f and g respectively, i.e., $f'(\theta_f) = 0$ and $g'(\theta_g) = 0$ and g_-^{-1} , f_+^{-1} denote the inverse of g, f for domain where $g'(u) \leq 0$ and $f'(u) \geq 0$ respectively. Notice that the existence of a minimum for f and g are always assumed in this paper as it allows the critical behaviour of admissible solution. If f and g have no minimum but both are strictly increasing or decreasing, the situation is simpler [1]. Thus, throughout the paper, it is assumed that,

$$f(\theta_f) = \min f \neq \min g = g(\theta_g). \quad (2.1)$$

In the best case when f and g are uniformly convex and (2.1) is satisfied, we obtain a smoothing in $BV^{1/2}$ instead of BV .

In the non uniformly convex case the situation is worse. The smoothing depends on the nonlinear flatness of the fluxes. Let us introduce the following *non-degeneracy flux condition* which is a power-law degeneracy condition [11], there exist two numbers $p \geq 1$, $q \geq 1$, such that, for any compact set K , there exist positive numbers C_1, C_2 such that for all $u \neq v$, $u, v \in K$,

$$\frac{|f'(u) - f'(v)|}{|u - v|^p} > C_1 > 0 \quad \text{and} \quad \frac{|g'(u) - g'(v)|}{|u - v|^q} > C_2 > 0. \quad (2.2)$$

For $p = 1$ this is the classical uniformly convex condition for f and for $p > 1$ it corresponds to a less nonlinear convex flux like $f(u) = |u|^{p+1}$.

An interesting subcase is when the loss of uniform convexity of the fluxes occurs only at the minimum. That is, if f belongs to C^2 that convex power laws, $f(u) = |u|^{p+1}$, $p > 1$ are the typical example. The same assumption can be made for the other flux g .

$$f'', g'' \text{ vanish only at } \theta_f \text{ and } \theta_g \text{ respectively.} \quad (2.3)$$

The assumption (2.3) combined with the previous one (2.2) is also called the *restricted non-degeneracy condition* and the fluxes, restricted fluxes. In the subcase (2.3) both satisfied by f and g , stronger results are obtained and first presented in following Theorem 2.1 for an L^∞ initial data and Theorem 2.2 for a BV^s initial data. Two quantities are fundamental to express the fractional regularity of the solutions, γ and ν ,

$$\gamma = \begin{cases} \frac{1}{q+1} \\ \frac{1}{p+1} \end{cases} \quad \nu = \begin{cases} \frac{1}{p} \\ \frac{1}{q} \end{cases} \quad \begin{array}{l} \text{if } \min f < \min g, \\ \text{if } \min f > \min g. \end{array} \quad (2.4)$$

The constant $\gamma \leq 1/2$ can be understood as a loss of regularity due to the interface and $\nu \leq 1$ as the smoothing effect outside the interface. More precisely, γ comes from the singular mapping technique as explained in following remark.

Remark 2.1. *Let f and g be the fluxes satisfying the non-degeneracy condition (2.2) and $f(\theta_f) \neq g(\theta_g)$. Then one of $f_+^{-1}g(\cdot)$ and $g_-^{-1}f(\cdot)$ is Lipschitz continuous and the other one is Hölder continuous with exponent γ depending on p, q from the non-degeneracy condition (2.2) and given by (2.4). The proof of the above fact is done in Lemma A.3.*

Theorem 2.1 (Smoothing effect for restricted nonlinear fluxes and L^∞ initial data). *Let f and g be two C^2 fluxes satisfying the restricted non-degeneracy condition $f(\theta_f) \neq g(\theta_g)$ (2.1), (2.2) and (2.3). Let $u(\cdot, t)$ be the entropy solution of (1.1) corresponding to an initial data $u_0 \in L^\infty(\mathbb{R})$. Then, $u(\cdot, t) \in BV^s(-M, M)$ for each $t > 0$, $M > 0$, where s is determined as*

follows

$$s = \min(\gamma, \nu) \quad (2.5)$$

and the following estimate holds with a positive constant $C_{f,g,\|u_0\|_\infty}$ depending only on the fluxes and the range of the initial data,

$$TV^s(u(\cdot, t), [-M, M]) \leq C_{f,g,\|u_0\|_\infty} + 3(2\|u_0\|_\infty)^{1/s} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \quad (2.6)$$

Remark 2.2 (Uniform convex fluxes and $BV^{1/2}$). *If the fluxes f and g are uniformly convex then the solution belongs to $BV^{1/2}$. So even for the uniformly convex case the solution goes into a fractional BV space.*

Hence in the following theorem for BV^s initial data, $0 < s \leq 1$ the previous result states as follow. Indeed, previous Theorem 2.1 can be seen as a limiting case of following Theorem 2.2 with $s = 0$ and stating $BV^0 = L^\infty$.

Theorem 2.2 (Smoothing effect for restricted nonlinear fluxes and BV^s initial data). *Let f and g be two C^2 fluxes such that $f(\theta_f) \neq g(\theta_g)$ and fluxes satisfy the restricted non-degeneracy condition (2.2) and (2.3). Let $u(\cdot, t)$ be the entropy solution of (1.1) corresponding to an initial data $u_0 \in BV^s(\mathbb{R})$ for $s \in (0, 1)$. Then, $u(\cdot, t) \in BV^{s_1}(-M, M)$ for each $t > 0$, $M > 0$ with*

$$s_1 := \min\{\gamma, \max\{\nu, s\}\} \quad (2.7)$$

the following estimate holds with a positive constant $C_{f,g,\|u_0\|_\infty}$ depending only on fluxes and the range of the initial data and a constant $D > 0$,

$$TV^{s_1}(u(\cdot, t), [-M, M]) \leq C_{f,g,\|u_0\|_\infty} + \frac{C_{f,g,\|u_0\|_\infty} M}{t} + 2\|2u_0\|_\infty^{\frac{1}{s_1}} + D \cdot TV^s(u_0). \quad (2.8)$$

We note that the assumption on vanishing points of f'' , g'' is restrictive. We can relax this assumption at the cost of smaller s_1 . More precisely, we have the following result.

Theorem 2.3 (Smoothing effect for L^∞ initial data). *Let f and g be two C^2 fluxes such that $f(\theta_f) \neq g(\theta_g)$ satisfying the non-degeneracy condition (2.2) with exponent p, q respectively. Let $u(\cdot, t)$ be the entropy solution of (1.1) corresponding to an initial data $u_0 \in L^\infty(\mathbb{R})$. Then, for each $t > 0$, $M > 0$ and there exists positive constant $C_{f,g,\|u_0\|_\infty}$ such that*

$$TV^s(u(\cdot, t), [-M, M]) \leq C_{f,g,\|u_0\|_\infty} + 3(2\|u_0\|_\infty)^{1/s} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}$$

where s is determined as follows

$$s = \gamma \nu. \quad (2.9)$$

Theorem 2.4. *With the same assumption as Theorem 2.3, if $u_0 \in BV^{s_0}$. Then, $u(\cdot, t) \in BV^{s_2}$ and there exists positive constants $C_{f,g,\|u_0\|_\infty}$ and D such that*

$$TV^{s_2}(u(\cdot, t), [-M, M]) \leq C_{f,g,\|u_0\|_\infty} + \frac{C_{f,g,\|u_0\|_\infty} M}{t} + 2\|2u_0\|_\infty^{\frac{1}{s_2}} + D \cdot TV^{s_0}(u_0), \quad (2.10)$$

where,

$$s_2 = \gamma \max(s_0, \nu). \quad (2.11)$$

In general, away from the interface, the expected fractional regularity is $\min(1/p, 1/q)$ [11] which is always bigger than s in (2.5). In particular, near the interface, for BV initial data, a BV regularity for the entropy solution cannot be expected [1]. At most, a $BV^{1/2}$ regularity is possible.

Getting BV regularity of entropy solution can be impossible near the interface. The situation is better far from the interface.

Far from the interface, the constant γ plays no role. The following theorem gives estimates which are sharp for small time.

Theorem 2.5 (Regularity outside the interface). *Let f and g be the fluxes with $f(\theta_f) \neq g(\theta_g)$. Let $u(\cdot, t)$ be the entropy solution of (1.1) corresponding to an initial data $u_0 \in BV^s(\mathbb{R})$ for $s \in (0, 1)$. Then the following holds.*

1. *If f, g satisfies (2.2) with exponent p, q respectively, for any $t > 0, \epsilon > 0$, then there exists a constant $C_{f,g,\|u_0\|_\infty} > 0$*

$$TV^{s_1}(u(\cdot, t), (-\infty, \epsilon] \cup [\epsilon, \infty)) \leq \frac{C_{f,g,\|u_0\|_\infty} t}{\epsilon} + 2TV^{s_1}(u_0) + 2(2\|u_0\|_\infty)^{1/s_1} \quad (2.12)$$

for $s_1 = \min\{p^{-1}, q^{-1}, s\}$.

2. *If we assume (2.3) that f'', g'' vanish only at θ_f and θ_g respectively then we have*

$$\begin{aligned} TV^s(u(\cdot, t), (-\infty, \epsilon] \cup [\epsilon, \infty)) &\leq \frac{C_{f,g,\|u_0\|_\infty}}{\min\{f''(v); [(f')^{-1}(\epsilon/t), S_{f,g,\|u_0\|_\infty}]\}} \frac{t}{\epsilon} + TV^s(u_0) \\ &+ \frac{C_{f,g,\|u_0\|_\infty}}{\min\{g''(v); [-S_{f,g,\|u_0\|_\infty}, (g')^{-1}(-\epsilon/t)]\}} \frac{t}{\epsilon} + 2(2\|u_0\|_\infty)^{1/s} \end{aligned} \quad (2.13)$$

for any $t > 0, \epsilon > 0$ where $S_{f,g,\|u_0\|_\infty}$ is defined as

$$S_{f,g,\|u_0\|_\infty} = \max \left\{ \|u_0\|_\infty, \sup_{|v| \leq \|u_0\|_\infty} |f_+^{-1}(g(v))|, \sup_{|v| \leq \|u_0\|_\infty} |g_-^{-1}(f(v))| \right\}.$$

Remark 2.3. *All the regularity results in Theorems 2.1, 2.2, 2.3, 2.5 are extendable to fractional Sobolev space $W^{s,p}$ with the same exponent s , up to any $\varepsilon > 0$, thanks to the embedding $BV^s \subset W^{s-\varepsilon, 1/s}$ for all $\varepsilon \in (0, s)$ [11].*

Now we discuss about the optimality result. The assumption $\min f \neq \min g$ forbids the favourable case $f = g$, that is without interface. Here, the optimality of Theorem 2.2 is proved in the best case with uniformly convex fluxes. For this purpose examples are built with the optimal regularity and not more up. The same construction is valid with a power law on one side of the interface. These examples highlight the sharpness of Theorem 2.2.

Theorem 2.6 (Blow-up for critical BV^s semi-norms). *Let $p \geq 1$ and $\epsilon > 0$. Then there exists fluxes f, g and an initial data $u_0 \in BV(\mathbb{R})$ such that*

1. *the flux f satisfies the non-degeneracy condition (2.2) with exponent p ,*
2. *the function g is uniformly convex,*
3. *the corresponding entropy solution $u(\cdot, T) \notin BV_{loc}^s(\mathbb{R})$ for some $T > 0$ and $s = \frac{1}{p+1} + \epsilon$.*

The proof of Theorem 2.6 is postponed in Section 5 and Appendix C.

3. Preliminaries

The fundamental paper used here is [7] where Adimurthi and Gowda settled an important foundation of the theory on scalar conservation laws with an interface and two convex fluxes. In this paper the author proposed the natural entropy condition (1.3) at the interface which means that no information comes only from the interface but crosses or go towards the interface. Such entropy condition is in the spirit of Lax-entropy conditions for shock waves. To make this paper self-contained, we recall some definitions and results.

The following theorem can be found in [7] Lemma 4.9 at page 51. It is a Lax-Oleinik or Lax-Hopf formula for the initial value problem (1.1).

Theorem 3.1 ([7]). *Let $u_0 \in L^\infty(\mathbb{R})$, then there exists the entropy solution $u(\cdot, t)$ of (1.1) corresponding to an initial data u_0 . Furthermore, there exist Lipschitz curves $R_1(t) \geq R_2(t) \geq 0$ and $L_1(t) \leq L_2(t) \leq 0$, monotone functions $z_\pm(x, t)$ non-decreasing in x and non-increasing in t and $t_\pm(x, t)$ non-increasing in x and non-decreasing in t such that the solution $u(x, t)$ can be given by the explicit formula for almost all $t > 0$,*

$$u(x, t) = \begin{cases} (f')^{-1} \left(\frac{x - z_+(x, t)}{t} \right) & \text{if } x \geq R_1(t), \\ (f')^{-1} \left(\frac{x}{t - t_+(x, t)} \right) & \text{if } 0 \leq x < R_1(t), \\ (g')^{-1} \left(\frac{x - z_-(x, t)}{t} \right) & \text{if } x \leq L_1(t), \\ (g')^{-1} \left(\frac{x}{t - t_-(x, t)} \right) & \text{if } L_1(t) < x < 0. \end{cases}$$

Furthermore, if $f(\theta_f) \geq g(\theta_g)$ then $R_1(t) = R_2(t)$ and if $f(\theta_f) \leq g(\theta_g)$ then $L_1(t) = L_2(t)$. We also have only three cases and following formula to compute the solution:

Case 1: $L_1(t) = 0$ and $R_1(t) = 0$,

$$u(x, t) = \begin{cases} u_0(z_+(x, t)) & \text{if } x > 0, \\ u_0(z_-(x, t)) & \text{if } x < 0. \end{cases}$$

Case 2: $L_1(t) = 0$ and $R_1(t) > 0$, then

$$u(x, t) = \begin{cases} f_+^{-1} g(u_0(z_+(x, t))) & \text{if } 0 < x < R_2(t), \\ f_+^{-1} g(\theta_g) & \text{if } R_2(t) \leq x \leq R_1(t), \\ u_0(z_-(x, t)) & \text{if } x < 0. \end{cases}$$

Case 3: $L_1(t) < 0$, $R_1(t) = 0$, then

$$u(x, t) = \begin{cases} g_-^{-1} f(u_0(z_-(x, t))) & \text{if } L_2(t) < x < 0, \\ u_0(z_-(x, t)) & \text{if } x \leq L_1(t), \\ g_-^{-1} f(\theta_f) & \text{if } L_1(t) < x < L_2(t). \end{cases}$$

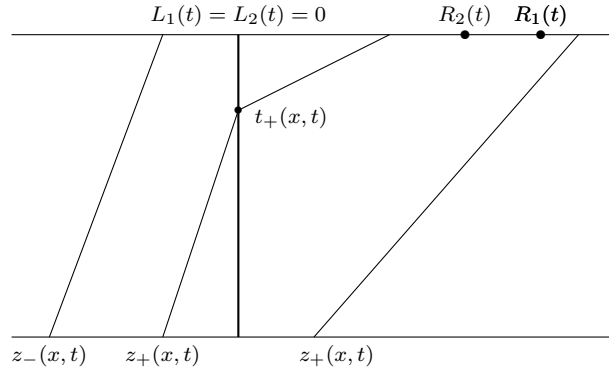


Figure 1: An illustration of solution for **Case 2** and $L_i(t)$ and $R_i(t)$ curves

There is a maximum principle for such entropy solutions, but more complicated than for $f = g$,

$$\|u\|_\infty \leq \max \left(\|u_0\|_\infty, \sup_{|v| \leq \|u_0\|_\infty} |f_+^{-1}(g(v))|, \sup_{|v| \leq \|u_0\|_\infty} |g_-^{-1}(f(v))| \right) =: S_{f,g,\|u_0\|_\infty}. \quad (3.1)$$

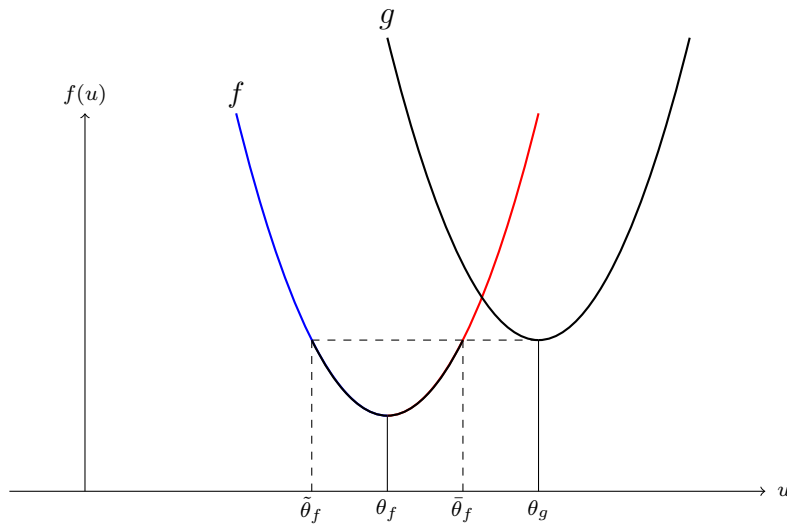


Figure 2: Illustration of the fluxes

Without loss of generality, as it is shown in the Figure 3, we can assume that $\min f < \min g$ for the proofs of all main results below. This choice enforces the values of the entropy solution at the interface being outside $(\tilde{\theta}_f, \bar{\theta}_f)$. Thus the function f' is far from 0 at the interface. Moreover, the function f'^{-1} is Lipschitz outside $(\tilde{\theta}_f, \bar{\theta}_f)$. For restricted fluxes the function f_+^{-1} is also Lipschitz outside $(\tilde{\theta}_f, \bar{\theta}_f)$. The Figure 3 illustrate that singular maps $f_+^{-1}g$ and $g_-^{-1}f$ are Lipschitz and Hölder continuous respectively which are proved in [Appendix A](#), Lemma A.3.

4. Proof of main results

This long section is devoted to prove the fractional BV regularity of the entropy solution depending on the degeneracy of the fluxes. A key point is first to estimate the regularity of the traces at the interface. In the next subsection 4.1 we start to study the fractional regularity in a favourable case when the traces at the interface are not near the critical values θ_f or θ_g . Spatial BV^s estimates for trace values issued from the interface is studied in Subsection 4.2. Moreover, only traces issuing from the initial data are considered. The crossing of the interface is studied later in Subsection 4.3.

4.1. Regularity when traces are far from critical values

We first prove fractional BV estimates when the traces at $x = 0$ are far from the critical values θ_f or θ_g .

Lemma 4.1 (Fractional BV estimate for the traces of the solution). *Let f, g be satisfying (2.2) with exponents p, q respectively. Let $0 < a < b < \infty$. Then the following holds:*

1. *If $u(0-, t) > \theta_g$ for a.e. $t \in (a, b)$, then we have*

$$TV^{\frac{1}{q}}(u(0-, \cdot), (a, b)) \leq C_g \frac{b}{a}, \quad (4.1)$$

where $C_g > 0$ is constant depending only on g .

2. *If $u(0+, t) < \theta_f$ for a.e. $t \in (a, b)$, then we have*

$$TV^{\frac{1}{p}}(u(0+, \cdot), (a, b)) \leq C_f \frac{b}{a}, \quad (4.2)$$

where $C_f > 0$ is a constant depending on f .

Proof. Since $u(0-, t) > \theta_g$ and $g' \geq 0$ on $(\theta_g, +\infty)$ the value of the left trace comes from the left. From Theorem 3.1, $u(0-, t) = (g')^{-1} \left(\frac{-z_-(0-, t)}{t} \right)$ for $t \in (a, b)$ where $t \mapsto z_-(0-, t)$ is non-increasing. Since g satisfies the non-degeneracy condition (2.2), from Lemma A.1 $(g')^{-1}$ is a $1/q$ -Hölder function and there exists a constant H_g such that

$$|u(0-, t_1) - u(0-, t_2)| \leq H_g \left| \frac{z_-(0-, t_1)}{t_1} - \frac{z_-(0-, t_2)}{t_2} \right|^{\frac{1}{q}}.$$

We observe that

$$\left| \frac{z_-(0-, t_1)}{t_1} - \frac{z_-(0-, t_2)}{t_2} \right| \leq |z_-(0-, t_1)| \left| \frac{1}{t_1} - \frac{1}{t_2} \right| + \frac{1}{t_2} |z_-(0-, t_1) - z_-(0-, t_2)|.$$

For any partition $a \leq t_1 < t_2 < \dots < t_m \leq b$,

$$\begin{aligned} & \sum_{j=1}^{m-1} |u(0-, t_j) - u(0-, t_{j+1})|^q \\ & \leq H_g^q \sum_{j=1}^{m-1} \left[|z_-(0-, t_j)| \left| \frac{1}{t_j} - \frac{1}{t_{j+1}} \right| + \frac{1}{t_{j+1}} |z_-(0-, t_j) - z_-(0-, t_{j+1})| \right]^q \\ & \leq H_g^q \left[|z_-(0-, b)| \sum_{j=1}^{m-1} \left| \frac{1}{t_j} - \frac{1}{t_{j+1}} \right| + \frac{1}{a} \sum_{i=1}^{m-1} |z_-(0-, t_j) - z_-(0-, t_{j+1})| \right]^q \\ & \leq H_g^q \left[\frac{|z_-(0-, b)| (b-a)}{ab} + \frac{|z_-(0-, a) - z_-(0-, b)|}{a} \right]^q. \end{aligned}$$

Since $|z_-(0-, a) - z_-(0-, b)| \leq |z_-(0-, b)|$ and $b - a \leq b$ we have,

$$\sum_{j=1}^{m-1} |u(0-, t_j) - u(0-, t_{j+1})|^q \leq 2H_g^q \frac{|z_-(0-, b)|}{a}.$$

From finite speed of propagation we have $|z(0-, b)| \leq M_g b$ where $K_{f,g, \|u_0\|_\infty} = \sup\{|g'(v)|; |v| \leq \|u_0\|_\infty\}$. Hence, we get a new constant C_g

$$\sum_{j=1}^{m-1} |u(0-, t_j) - u(0-, t_{j+1})|^q \leq C_g \frac{b}{a}.$$

This proves (4.1). Similarly, we can prove the (4.2). \square

Better fractional BV estimates for the traces of the solution are available for less singular fluxes.

Lemma 4.2 (Fractional BV estimate for traces away from critical values). *Let $r > 0$ and f, g be satisfying (2.2) with exponent p, q respectively. Let $0 < a < b < \infty$.*

1. *If $u(0-, t) \geq \theta_g + r$ and g'' vanishes only at θ_g (2.3), then there exists a constant $C_g > 0$ independent of r such that the following inequality holds,*

$$TV(u(0-, \cdot), (a, b)) \leq \frac{C_g}{\min\{g''(v) | v \in [\theta_g + r, \|u_0\|_\infty]\}} \frac{b}{a}. \quad (4.3)$$

2. *If $u(0+, t) \leq \theta_f + r$ and f'' vanishes only at θ_f (2.3), then there exists a constant $C_f > 0$ independent of r such that the following inequality holds,*

$$TV(u(0+, \cdot), (a, b)) \leq \frac{C_f}{\min\{f''(v) | v \in [-\|u_0\|_\infty, \theta_f - r]\}} \frac{b}{a}. \quad (4.4)$$

Lemma 4.2 will be used later with constant r given by either $\theta_f - \tilde{\theta}_f$ or $\bar{\theta}_f - \theta_f$ as shown in Figure 3. The fact that r is a positive constant is crucial to get uniform estimates later.

Proof. For any $x, y \in \mathbb{R}$ consider

$$|x - y| = |g'(g'^{-1}(x)) - g'(g'^{-1}(y))| = g''(\xi) |g'^{-1}(x) - g'^{-1}(y)|,$$

where $\xi \in (x, y)$. Now for (4.3), Theorem 3.1 gives, $u(0-, t) = (g')^{-1} \left(\frac{-z_-(0-, t)}{t} \right)$ for $t \in (a, b)$

where $t \mapsto z_-(0-, t)$ is non-increasing. Thus,

$$\begin{aligned} |u(0-, t_1) - u(0-, t_2)| &= \left| g'^{-1} \left(\frac{-z_-(0-, t_1)}{t_1} \right) - g'^{-1} \left(\frac{-z_-(0-, t_2)}{t_2} \right) \right| \\ &\leq \min\{g''(v); v \in [\theta_g + r, \|u_0\|_\infty]\}^{-1} \left| \frac{z_-(0-, t_1)}{t_1} - \frac{z_-(0-, t_2)}{t_2} \right|. \end{aligned}$$

Now the similar calculation as to prove (4.1) gives (4.3). By similar arguments (4.4) can be proven for f . \square

4.2. Spatial BV^s estimates for values originating from the interface

Now, far from the interface and restricted flux, when the values of the solution are far from the critical values of f and g , a BV estimate is available. The following inequality are also valid in BV^s for free and used later with other BV^s estimates.

Lemma 4.3 (BV and BV^s estimates for the solution). *Let u be an entropy solution and $R_1(t) > 0$ for some fixed $t > 0$. Let $0 < a < b < R_1(t)$ and $S_{f,g,\|u_0\|_\infty}$ be as in (3.1). Let $r > 0$, f satisfies (2.2) and f'' vanishes only on θ_f (2.3). If $u(x, t) \geq \theta_f + r$ for $a \leq x \leq b$, then there exists a constant $C_{f,g,\|u_0\|_\infty} > 0$ such that*

$$TV^s(u(\cdot, t), [a, b]) \leq \frac{C_{f,g,\|u_0\|_\infty}}{\min\{f''(v); v \in [\theta_f + r, S_{f,g,\|u_0\|_\infty}]\}^{\frac{1}{s}}} \left(\frac{t - t_+(b, t)}{t - t_+(a, t)} \right)^{\frac{1}{s}}, \quad (4.5)$$

for all $0 < s \leq 1$.

The same result holds for the left side of the interface as follows:

Lemma 4.4 (*BV and BV^s estimate for the solution*). Let u be an entropy solution and $L_1(t) < 0$ for some $t > 0$. Let $L_1(t) < a < b < 0$ and $S_{f,g,\|u_0\|_\infty}$ be as in (3.1). Let $r > 0$, flux g satisfies (2.2) and g'' vanishes only on θ_g . If $u(x, t) \leq \theta_g - r$ for $a \leq x \leq b$, then there exists a constant $C_{f,g,\|u_0\|_\infty} > 0$ such that

$$TV^s(u(\cdot, t), [a, b]) \leq \frac{C_{f,g,\|u_0\|_\infty}}{\min\{g''(v); v \in [-S_{f,g,\|u_0\|_\infty}, \theta_g - r]\}^{\frac{1}{s}}} \left(\frac{t - t_-(b, t)}{t - t_-(a, t)} \right)^{\frac{1}{s}},$$

for all $0 < s \leq 1$.

Proof. Theorem 3.1 gives,

$$u(x, t) = (f')^{-1} \left(\frac{x}{t - t_+(x, t)} \right) \text{ for } x \in (0, R_1(t)).$$

Fix a partition $a \leq x_1 < x_2 < \dots < x_m \leq b$. Then, as in the proof of inequality (4.3), it follows,

$$\begin{aligned} \sum_{j=1}^{m-1} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{s}} &= \sum_{j=1}^{m-1} \left| (f')^{-1} \left(\frac{x_j}{t - t_+(x_j, t)} \right) - (f')^{-1} \left(\frac{x_{j+1}}{t - t_+(x_{j+1}, t)} \right) \right|^{\frac{1}{s}} \\ &\leq \frac{1}{\min\{f''(v); v \in [\theta_f + r, S_{f,g,\|u_0\|_\infty}]\}^{\frac{1}{s}}} \sum_{j=1}^{m-1} \left| \frac{x_j}{t - t_+(x_j, t)} - \frac{x_{j+1}}{t - t_+(x_{j+1}, t)} \right|^{\frac{1}{s}}. \end{aligned}$$

We calculate

$$\begin{aligned} \left| \frac{x_j}{t - t_+(x_j, t)} - \frac{x_{j+1}}{t - t_+(x_{j+1}, t)} \right| &\leq |x_j| \left| \frac{1}{t - t_+(x_j, t)} - \frac{1}{t - t_+(x_{j+1}, t)} \right| + \frac{1}{t - t_+(x_{j+1}, t)} |x_j - x_{j+1}| \\ &\leq b \left| \frac{1}{t - t_+(x_j, t)} - \frac{1}{t - t_+(x_{j+1}, t)} \right| + \frac{1}{t - t_+(a, t)} |x_j - x_{j+1}|. \end{aligned}$$

Hence, by the convexity yields, $(a + b)^{\frac{1}{s}} \leq 2^{\frac{1-s}{s}} \left(a^{\frac{1}{s}} + b^{\frac{1}{s}} \right)$ and we get

$$\begin{aligned} &\sum_{j=1}^{m-1} \left| \frac{x_j}{t - t_+(x_j, t)} - \frac{x_{j+1}}{t - t_+(x_{j+1}, t)} \right|^{\frac{1}{s}} \\ &\leq \frac{1}{2^{\frac{s-1}{s}}} \left(\sum_{j=1}^{m-1} b^{\frac{1}{s}} \left| \frac{1}{t - t_+(x_j, t)} - \frac{1}{t - t_+(x_{j+1}, t)} \right|^{\frac{1}{s}} + \sum_{j=1}^{m-1} \frac{1}{(t - t_+(a, t))^{\frac{1}{s}}} |x_j - x_{j+1}|^{\frac{1}{s}} \right) \\ &\leq \frac{1}{2^{\frac{s-1}{s}}} \left(b^{\frac{1}{s}} \left| \frac{1}{t - t_+(a, t)} - \frac{1}{t - t_+(b, t)} \right|^{\frac{1}{s}} + \left(\frac{b - a}{t - t_+(a, t)} \right)^{\frac{1}{s}} \right) \\ &\leq 2^{\frac{1}{s}} \left(\frac{b}{t - t_+(a, t)} \right)^{\frac{1}{s}}. \end{aligned}$$

In the last step we have used $b - a \leq b$ and $(t - t_+(b, t)) - (t - t_+(a, t)) \leq t - t_+(b, t)$. Note that $b \leq K_{f,g,\|u_0\|_\infty} (t - t_+(b, t))$ where $K_{f,g,\|u_0\|_\infty} = \sup\{|f'|; |v| \leq S_{f,g,\|u_0\|_\infty}\}$ where $S_{f,g,\|u_0\|_\infty}$ is defined as in (3.1). \square

The following lemma deals with spatial regularity of the entropy solution for the right side of the interface. Inequality (4.5) does not use the restricted non-degeneracy condition.

Lemma 4.5. Let u be an entropy solution and $R_1(t) > 0$ for some fixed $t > 0$. Let $0 < a < b <$

$R_1(t)$ and $S_{f,g,\|u_0\|_\infty}$ be as in (3.1). If f only satisfies (2.2) with exponent p then we have

$$TV^{\frac{1}{p}}(u(\cdot, t), [a, b]) \leq C_{f,g,\|u_0\|_\infty} \frac{t - t_+(b, t)}{t - t_+(a, t)}. \quad (4.6)$$

The same result holds for the left side of the interface as follows.

Lemma 4.6. *Let u be an entropy solution and $L_1(t) < 0$ for some $t > 0$. Let $L_1(t) < a < b < 0$. If g satisfies (2.2) with exponent q then we have*

$$TV^{\frac{1}{q}}(u(\cdot, t), [a, b]) \leq C_{f,g,\|u_0\|_\infty} \frac{t - t_-(b, t)}{t - t_-(a, t)}. \quad (4.7)$$

By a similar argument as previous Lemma 4.3 the inequality (4.6) of Lemma 4.5 can be proven, so it is not written here.

4.3. Smoothing effect for restricted nonlinear fluxes

Now we are ready to prove Theorem 2.1. To end this, an arbitrary partition is fixed and divided in several parts. Some are far from the interface and the generalized variation is estimated with a regularizing effect for a scalar conservation laws without a boundary. Some others are near the interface where the Lax-Oleinik formula for the solution [7] is used with previous lemmas.

Proof of Theorem 2.1: Since $f(\theta_f) \neq g(\theta_g)$, without loss of generality assume that $f(\theta_f) < g(\theta_g)$ as in Figure 3. It is enough to consider the following two cases, the other cases are similar.

Case(i): $L_1(t) = 0$ and $R_1(t) \geq 0$.

Consider an arbitrary partition $\{-M = x_{-n} < \dots < x_{-1} < x_0 \leq 0 < x_1 < \dots < x_l \leq R_2(t) < x_{l+1} < \dots < x_m \leq R_1(t) < x_{m+1} < \dots < x_n = M\}$. Then,

$$\begin{aligned} \sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &= \sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + \sum_{i=m+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \\ &+ \sum_{i=1}^{l-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + \sum_{i=l+1}^{m-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \\ &+ |u(x_0, t) - u(x_1, t)|^{1/s} + |u(x_l, t) - u(x_{l+1}, t)|^{1/s} + |u(x_m, t) - u(x_{m+1}, t)|^{1/s}. \end{aligned}$$

From Theorem 3.1, solution u is constant between $R_2(t)$ to $R_1(t)$ which means variation is zero for this interval. Now from the Lax-Oleinik formula in Theorem 3.1 and bounding the last three terms yield,

$$\begin{aligned} \sum_{i=-n}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &\leq \underbrace{\sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s}}_{\text{I}} + \underbrace{\sum_{i=m+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s}}_{\text{III}} \\ &+ \underbrace{\sum_{i=1}^{l-1} |f_+^{-1}g(u_0(z_+(x_i, t))) - f_+^{-1}g(u_0(z_+(x_{i+1}, t)))|^{1/s} + 3(2\|u_0\|_\infty)^{1/s}}_{\text{II}}. \end{aligned}$$

Now we wish to estimate the terms I, II, and III. The simplest terms I, III are estimated as in [11, 17]. First taking the I into the account. Since f and g are satisfying the flux non-degeneracy condition (2.2), by Lemma A.1, the maps $u \mapsto (g')^{-1}(u)$ and $u \mapsto (f')^{-1}(u)$ are Hölder continuous

with exponents q^{-1} and p^{-1} respectively. From Theorem 3.1,

$$u(x, t) = (g')^{-1} \left(\frac{x - z_-(x, t)}{t} \right), \quad \text{for } x < 0,$$

then for $-M \leq x_i < x_{i+1} \leq 0$, from Lemma A.1

$$\begin{aligned} |u(x_i, t) - u(x_{i+1}, t)|^q &= \left| (g')^{-1} \left(\frac{x_i - z_-(x_i, t)}{t} \right) - (g')^{-1} \left(\frac{x_{i+1} - z_-(x_{i+1}, t)}{t} \right) \right|^q \\ &\leq \left(C_2^{-q^{-1}} \left| \frac{x_i - z_-(x_i, t)}{t} - \frac{x_{i+1} - z_-(x_{i+1}, t)}{t} \right|^{q^{-1}} \right)^q, \end{aligned}$$

using triangle inequality we obtain,

$$|u(x_i, t) - u(x_{i+1}, t)|^q \leq C_2^{-1} \left| \frac{x_i - x_{i+1}}{t} \right| + C_2^{-1} \left| \frac{z_-(x_i, t) - z_-(x_{i+1}, t)}{t} \right|.$$

Since $|x_i|, |x_{i+1}| \leq M$ and $x = z_-(x, t) + g'(u(x, 0))t$ hence, we get

$$TV^{q^{-1}} u(\sigma \cap [-M, 0]) \leq \frac{4M}{C_2 t} + \frac{1}{C_2} \sup \left\{ |g'(v)|; |v| \leq \|u_0\|_{L^\infty(\mathbb{R})} \right\}. \quad (4.8)$$

In similar fashion, for the term III we have,

$$TV^{p^{-1}} u(\sigma \cap [R_1(t), M]) \leq \frac{4M}{C_1 t} + \frac{1}{C_1} \sup \left\{ |f'(v)|; |v| \leq \|u_0\|_{L^\infty(\mathbb{R})} \right\}. \quad (4.9)$$

Now we will estimate the II term. From the definition of s , $s \leq 1/p$ and $s \leq 1/(q+1)$. Rest of the proof for this case is divided into two sub-cases.

1. Consider the situation when $t_+^{min}(t) = \inf\{t_+(x, t); x \in (0, R_1(t))\} \geq t/2$. The fact $t_+^{min} > t/2 > 0$ implies that the characteristics reaching the left side of the interface at $(0-, t_+)$ has a positive speed, hence $u(0-, t_+(x, t)) > \theta_g$ for all $x \in (0, R_1(t))$ (Figure 3). Therefore, the inequality (4.1) of Lemma 4.1 gives $TV^{\frac{1}{q}}(u(0-, \cdot)(t_+^{min}, t)) \leq C_g \frac{t}{t/2} = 2C_g$. Since $s \leq \frac{1}{q+1} < \frac{1}{q}$, Lemma B.1 yields $TV^s(u(0-, \cdot)(t_+^{min}, t)) \leq \mathcal{O}(1)$ and then $\text{II} \leq \mathcal{O}(1)$.
2. Next focus on the sub-case when $t_+^{min}(t) = \inf\{t_+(x, t); x \in (0, R_1(t))\} < t/2$. As previous subcase we already have $TV^s(u(0-, \cdot)(t/2, t)) \leq 2C_g$. Let $j_0 > 0$ such that $t_+(x_j, t) \geq t/2$ for $0 < j \leq j_0$ and $t_+(x_j, t) < t/2$ for $j_0 < j \leq l-1$. Since $u(x_j, t) = u(0+, t_+(x_j, t)) = f_+^{-1}g(u(0-, t_+(x_j, t)))$ for $0 < j < l-1$, from Lemma A.3, $f_+^{-1}g$ is Lipschitz function, hence

$$\sum_{j=1}^{j_0} |u(x_j, t) - u(x_{j-1}, t)|^{\frac{1}{s}} \leq \mathcal{O}(1).$$

Let $\bar{\theta}_f > \theta_f$ be such that $f(\bar{\theta}_f) = g(\theta_g)$ as shown in Figure 3. Then by RH condition (1.2) observe that $u(x_j, t) \geq \bar{\theta}_f$. From the inequality (4.5) of Lemma 4.3 we get

$$\sum_{j=j_0+1}^{l-2} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{s}} \leq \mathcal{O}(1).$$

Subsequently, we get

$$\text{II} \leq \mathcal{O}(1). \quad (4.10)$$

Hence combining the estimates on I, II and III for constant $C_{f,g,\|u_0\|_\infty} > 0$ we have

$$\sum_{i=-n}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \leq C_{f,g,\|u_0\|_\infty} \left(1 + \frac{1}{t}\right).$$

Case (ii): $R_1(t) = 0$, $L_1(t) < 0$. Unlike previous case, this case is not as good due to the fact that $g_-^{-1}f$ is only Hölder continuous and not Lipschitz. Let us consider the partition $\sigma = \{-M = x_{-n} < \dots < x_m \leq L_2(t) = L_1(t) < x_{m+1} < \dots < x_0 \leq R_2(t) = R_1(t) = 0 < x_1 < \dots \leq x_n = M\}$ and then

$$\begin{aligned} \sum_{i=-n}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &= \sum_{i=-n}^{m-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + \sum_{i=1}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \\ &+ \sum_{i=m+1}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + |u(x_0, t) - u(x_1, t)|^{1/s} \\ &+ |u(x_m, t) - u(x_{m+1}, t)|^{1/s}. \end{aligned}$$

From Theorem 3.1 we get,

$$\begin{aligned} \sum_{i=-\infty}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &= \underbrace{\sum_{i=-n}^{m-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s}}_{\text{I}} + 2(2\|u_0\|_\infty)^{1/s} \\ &+ \underbrace{\sum_{i=m+1}^{-1} |g_-^{-1}(f(u_0(z_-(x_i, t)))) - g_-^{-1}(f(u_0(z_-(x_{i+1}, t))))|^{1/s}}_{\text{II}} \\ &+ \underbrace{\sum_{i=1}^n |u(x_i, t) - u(x_{i+1}, t)|^{1/s}}_{\text{III}}. \end{aligned}$$

Similarly to **Case (i)** we bound I, III as in (4.8), (4.9) to get

$$\text{I} + \text{III} \leq \frac{C_{f,g,\|u_0\|_\infty} M}{t}.$$

Now the term II consider as previous term II and divide into two sub-cases.

1. We first consider the situation when $t_-^{min}(t) = \inf\{t_-(x, t); x \in (L_1(t), 0)\} \geq t/2$. The RH condition (1.2) implies that $u(0+, \cdot) \leq \tilde{\theta}_f$, see Figure 3, the inequality (4.4) of Lemma 4.2 gives

$$TV(u(0+, \cdot)(t_-^{min}, t)) \leq C_{f,g,\|u_0\|_\infty}. \quad (4.11)$$

Note that $g_-^{-1} \circ f$ is Hölder continuous function with exponent $\frac{1}{q+1}$. Hence we have

$$\text{II} = \sum_{j=m+1}^{-1} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g,\|u_0\|_\infty}. \quad (4.12)$$

2. Next we focus on the sub-case when $t_-^{min}(t) = \inf\{t_-(x, t); x \in (L_1(t), 0)\} < t/2$. Let $j_0 < 0$ such that $t_+(x_j, t) \geq t/2$ for $j_0 \leq j < 0$ and $t_+(x_j, t) < t/2$ for $m+1 < j < j_0$. In previous

sub-case we have

$$\sum_{j=j_0}^{-1} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g,\|u_0\|_\infty}. \quad (4.13)$$

Note that for $m+1 < j < j_0$, $u(x_j, t) = u(0-, t_-(x_j, t)) \leq \theta_g$. From the inequality (4.7) of Lemma 4.6 we have

$$\sum_{j=m+1}^{j_0-1} |u(x_j, t) - u(x_{j+1}, t)|^{\frac{1}{q}} \leq C_{f,g,\|u_0\|_\infty}. \quad (4.14)$$

Subsequently, we get

$$\text{II} \leq C_{f,g,\|u_0\|_\infty} + \|2u_0\|_\infty^{\frac{1}{q+1}}. \quad (4.15)$$

Hence, from the estimates on I, II and III we get

$$\sum_{i=-n}^n |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g,\|u_0\|_\infty} + 3(2\|u_0\|_\infty)^{\frac{1}{q+1}} + \frac{C_{f,g}M}{t}. \quad (4.16)$$

□

4.4. Generalization for BV^s initial data

Now we are able to prove Theorem 2.2. For this, again we divide the domain in several parts. Here initial data belongs to BV^s . If s is very small then far from the interface estimates comes from the regularizing effect. If s is near to 1 then outside interface initial data regularity propagates. For the estimate on the solution near interface again we use Lax-Oleinik formula from [7].

Proof of Theorem 2.2. Since $f(\theta_f) \neq g(\theta_g)$, without loss of generality we assume that $f(\theta_f) < g(\theta_g)$, see Figure 3 because other case can be done in a similar way. Hence, from Theorem 3.1 we have $L_2(t) = L_1(t)$ then it is enough to consider the following two cases.

Case (i): If $L_1(t) = 0$ and $R_1(t) \geq 0$.

Consider the partition $\sigma = \{-M = x_{-n} \leq \dots < x_{-1} < x_0 \leq 0 < x_1 < \dots < x_l \leq R_2(t) < x_{l+1} < \dots < x_m \leq R_1(t) < x_{m+1} < \dots \leq x_n = M\}$ and

$$s_1 = \min\{\gamma, \max\{\nu, s\}\} \in (0, 1).$$

Then

$$\begin{aligned} & \sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \\ &= \sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} + \sum_{i=m+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \\ &+ \sum_{i=1}^{l-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} + \sum_{i=l+1}^{m-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \\ &+ |u(x_0, t) - u(x_1, t)|^{1/s_1} + |u(x_l, t) - u(x_{l+1}, t)|^{1/s_1} \\ &+ |u(x_m, t) - u(x_{m+1}, t)|^{1/s_1}. \end{aligned}$$

From Theorem 3.1, the entropy solution is constant between $R_2(t)$ and $R_1(t)$ which means

variation is zero for this interval. Hence,

$$\begin{aligned}
\sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} &= \underbrace{\sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1}}_{\text{I}} + 3(2\|u_0\|_\infty)^{1/s_1} \\
&\quad + \underbrace{\sum_{i=1}^{l-1} |f_+^{-1}g(u_0(z_+(x_i, t))) - f_+^{-1}g(u_0(z_+(x_{i+1}, t)))|^{1/s_1}}_{\text{II}} \\
&\quad + \underbrace{\sum_{i=m+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1}}_{\text{III}}.
\end{aligned}$$

From the choice of s_1 , we get $s_1 \leq \max\{s, 1/q\}$. If $1/q > s$, then $s_1 < 1/q$. By a similar argument as in (4.8) we have

$$\sum_{i=-n}^{-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \leq \frac{4M}{C_2 t} + \frac{1}{C_2} \sup \left\{ |g'(v)|; |v| \leq \|u\|_{L^\infty(\mathbb{R} \times [0, T])} \right\}.$$

If $s > 1/q$ then $s_1 < s$ and we use the regularity of initial data to estimate I so from Lemma B.1 I $\leq D \cdot TV^s(u_0)$. Combining both the estimate we can write

$$\text{I} \leq TV^s(u_0) + \frac{4M}{C_2 t} + \frac{1}{C_2} \sup \left\{ |g'(v)|; |v| \leq \|u\|_{L^\infty(\mathbb{R} \times [0, T])} \right\}. \quad (4.17)$$

Similarly we have

$$\text{III} \leq TV^s(u_0) + \frac{4M}{C_1 t} + \frac{1}{C_1} \sup \left\{ |f'(v)|; |v| \leq \|u\|_{L^\infty(\mathbb{R} \times [0, T])} \right\}. \quad (4.18)$$

From Lemma A.3 we know that $f_+^{-1}g(\cdot)$ is a Lipschitz continuous. Hence, the term II can be estimated as

$$\begin{aligned}
\text{II} &= \sum_{i=1}^{l-1} |f_+^{-1}g(u_0(z_+(x_i, t))) - f_+^{-1}g(u_0(z_+(x_{i+1}, t)))|^{1/s_1} \\
&\leq C \cdot \sum_{i=1}^{l-1} |u_0(z_+(x_i, t)) - u_0(z_+(x_{i+1}, t))|^{1/s_1}.
\end{aligned}$$

If $s > 1/q$ then we have $s_1 < s$, from Lemma B.1, $\text{II} \leq D \cdot TV^s(u_0)$. For the case $s < 1/q$ we do not know whether $s_1 < s$ holds or not but we surely have $s_1 < 1/q$. For this case we use the regularizing effect for solutions of conservation laws due to non-degeneracy of g [11]. For term II we note that the estimate (4.10) in proof of Theorem 2.1. Hence combining the estimates on I, II and III we get

Hence, from the estimates on I, II and III we get

$$\sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s_1} \leq D \cdot TV^s(u_0) + 3(2\|u_0\|_\infty)^{1/s_1} + \frac{C_{f,g}M}{t}.$$

Case (ii): $R_1(t) = 0$, $L_1(t) < 0$.

This case can be handled in a similar fashion as in previous case.

Only difference is the estimation of II which can be done same as in (4.15).

Hence, we have proven that $u(\cdot, t) \in BV^{s_1}(-M, M)$. To show that $u(\cdot, t) \in BV^{s_1}(\mathbb{R})$, we consider a partition $-\infty < x_{-n} < \dots < x_n < \infty$ which is not necessarily contained in $[-M, M]$. We can choose $M = t \sup\{|f'(v)|, |g'(v)|; |v| \leq \|u_0\|_\infty\}$. Suppose $|x_j| \leq M$ for $-m_1 \leq j \leq m_2$ for some $0 < m_1, m_2 \leq n$. From (4.16) we get

$$\sum_{i=-m_1}^{m_2} |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g} + 2(2\|u_0\|_\infty)^{\frac{1}{q+1}}.$$

From the choice of M , we can see that $R_1(t) \leq M, L_1(t) \geq -M$. Hence for $i \leq -m_1$, $u(x_i, t) = u_0(z_-(x_i, t))$ and for $i \geq m_2$, $u(x_i, t) = u_0(z_+(x_i, t))$. Subsequently,

$$\sum_{i=-n}^{-m_1-2} |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{s}} + \sum_{i=m_2+1}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{s}} \leq TV^s(u_0).$$

Therefore, we obtain

$$\sum_{i=-n}^{n-1} |u(x_i, t) - u(x_{i+1}, t)|^{\frac{1}{q+1}} \leq C_{f,g,\|u_0\|_\infty} + 4(2\|u_0\|_\infty)^{\frac{1}{q+1}} + TV^s(u_0).$$

This completes the proof of Theorem 2.2. \square

4.5. Non restricted fluxes

Now we assume the weaker non-degeneracy on flux the estimates on solution near interface so Lemma 4.2, 4.2 and 4.3 can not be used here. So the regularity is weaker here.

Proof of Theorem 2.3: Fix a time $t > 0$. We only show for the case when $R_1(t) > 0$. Note that in this case $L_1(t) = L_2(t) = 0$. Suppose $t_0 = \lim_{x \rightarrow R_1(t)-} t_+(x, t)$. First consider $t_0 > t/2$. From Lemma 4.1, we have

$$TV^{\frac{1}{q}}(u(0-, \cdot), (t_0, t)) \leq \frac{C_g t}{t_0} \leq 2C_g.$$

Since $u \mapsto f_+^{-1}(g(u))$ is Hölder continuous with exponent $\frac{1}{p+1}$, we get

$$|u(0+, t_1) - u(0+, t_2)| \leq C_{f,g,\|u_0\|_\infty} |u(0-, t_1) - u(0-, t_2)|^{\frac{1}{p+1}}.$$

Subsequently, we have

$$TV^s(u(0+, \cdot), (t_0, t)) \leq C_{f,g,\|u_0\|_\infty} \text{ where } s = \frac{1}{q(p+1)}.$$

Note that for $x \in (0, R_1(t))$ we have $u(x, t) = u(0+, t_+(x, t))$. Therefore,

$$TV^s(u(\cdot, t), (0, R_1(t))) \leq C_{f,g,\|u_0\|_\infty}. \quad (4.19)$$

For $x > R_1(t)$ we have $u(x, t) = (f')^{-1}\left(\frac{x - z_+(x, t)}{t}\right)$ for a non-decreasing $x \mapsto z_+(x, t)$. By using flux condition (2.2) of f , we obtain

$$TV^{\frac{1}{p}}(u(\cdot, t), (R_1(t), M)) \leq \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \quad (4.20)$$

Hence,

$$TV^s(u(\cdot, t), (0, M)) \leq TV^s(u(\cdot, t), (0, R_1(t))) + \|2u\|_{L^\infty(\mathbb{R})}^{\frac{1}{s}} + TV^s(u(\cdot, t), (R_1(t), M))$$

$$\begin{aligned}
&\leq C_{f,g,\|u_0\|_\infty} + \|2u_0\|_{L^\infty(\mathbb{R})}^{\frac{1}{s}} + TV^{\frac{1}{p}}(u(\cdot, t), (R_1(t), M)) \\
&\leq C_{f,g,\|u_0\|_\infty} + \|2u_0\|_{L^\infty(\mathbb{R})}^{\frac{1}{s}} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}.
\end{aligned}$$

Next we consider the case when $t_0 < t/2$. Let $x_0 = \sup\{x; t_+(x, t) \geq t/2\}$. By Lemma 4.3 we have

$$TV^{\frac{1}{p}}(u(\cdot, t); (x_0, M)) \leq C_{f,g,\|u_0\|_\infty} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \quad (4.21)$$

Similar to (4.19) we get

$$TV^s(u(\cdot, t), (0, x_0)) \leq C_{f,g,\|u_0\|_\infty} \text{ with } s = \frac{1}{q(p+1)}.$$

Subsequently, we obtain

$$TV^s(u(\cdot, t), (0, M)) \leq C_{f,g,\|u_0\|_\infty} + \|2u_0\|_{L^\infty(\mathbb{R})}^{\frac{1}{s}} + \frac{C_{f,g,\|u_0\|_\infty} M}{t}.$$

Note that for $x < 0$ we have $u(x, t) = (g')^{-1}\left(\frac{x - z_-(x, t)}{t}\right)$. Then by using flux condition (2.2) we can show that

$$TV^{\frac{1}{q}}(u(\cdot, t); (-M, 0)) \leq \frac{C_{f,g,\|u_0\|_\infty} M}{t}. \quad (4.22)$$

The other case when $L_1(t) < 0$ follows from a similar argument. This completes the proof of Theorem 2.3. \square

4.6. Propagation of the initial regularity outside the interface

The regularity of entropy solutions outside the interface is better than at the interface. It is proven in this section.

Proof of Theorem 2.5. We consider the partition $\epsilon \leq x_0 < x_1 < \dots < x_l \leq R_1(t) \leq x_{l+1} < \dots$. Then

$$\sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} = \sum_{i=0}^{l-1} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} + \sum_{i=l}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s}.$$

Now from Theorem 3.1 we get,

$$\begin{aligned}
\sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} &\leq \sum_{i=0}^{l-1} \left| (f')^{-1}\left(\frac{x_i}{t - t_+(x_i, t)}\right) - (f')^{-1}\left(\frac{x_{i+1}}{t - t_+(x_{i+1}, t)}\right) \right|^{1/s} \\
&\quad + |u(x_l, t) - u(x_{l+1}, t)|^{1/s} + \sum_{i=l+1}^{\infty} |u_0(y(x_i, t)) - u_0(y(x_{i+1}, t))|^{1/s}.
\end{aligned}$$

Since $t_+(x, t)$ is monotone function and has bound and infimum of $t - t_+(x, t)$ positive. Hence, we get

$$\frac{\epsilon}{T} \leq \frac{x}{t - t_+(x, t)} \leq \frac{M}{h(\epsilon, T)},$$

where $h(\epsilon, T) = \inf\{t - t_+(x, t) : \epsilon \leq x \leq R_1(t), 0 < t \leq T\}$, which also implies that $(f')^{-1}$ is Lipschitz continuous function on interval $\left[\frac{\epsilon}{T}, \frac{M}{h(\epsilon, T)}\right]$. Then,

$$\sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \leq C(\epsilon, t) \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s}$$

$$\begin{aligned}
& + |u(x_l, t) - u(x_{l+1}, t)|^{1/s} \\
& + \sum_{i=l+1}^{\infty} |u_0(y(x_i, t)) - u_0(y(x_{i+1}, t))|^{1/s}.
\end{aligned}$$

The estimate on first sum follow from,

$$\begin{aligned}
& \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \\
& = \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_i}{t - t_+(x_{i+1}, t)} + \frac{x_i}{t - t_+(x_{i+1}, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s},
\end{aligned}$$

from triangle inequality we get,

$$\begin{aligned}
& \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \\
& \leq \sum_{i=0}^{l-1} \left(\left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_i}{t - t_+(x_{i+1}, t)} \right| + \left| \frac{x_i}{t - t_+(x_{i+1}, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right| \right)^{1/s},
\end{aligned}$$

now from the inequality $a^{1/s} + b^{1/s} \leq (a + b)^{1/s}$ we get,

$$\begin{aligned}
& \sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \\
& \leq \left(\sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_i}{t - t_+(x_{i+1}, t)} \right| + \left| \frac{x_i}{t - t_+(x_{i+1}, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right| \right)^{1/s}.
\end{aligned}$$

Therefore, we get the following estimate,

$$\sum_{i=0}^{l-1} \left| \frac{x_i}{t - t_+(x_i, t)} - \frac{x_{i+1}}{t - t_+(x_{i+1}, t)} \right|^{1/s} \leq \left(\frac{R_1(t) - \epsilon}{|t - t_+(\epsilon, t)|} + \frac{R_1(t)|t_+(\epsilon, t) - t_+(R_1(t), t)|}{|t - t_+(\epsilon, t)|^2} \right)^{1/s}.$$

Thus we have,

$$\begin{aligned}
\sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} & \leq C \sup_{0 \leq t \leq T} \left(\frac{R_1(t) - \epsilon}{|t - t_+(\epsilon, t)|} + \frac{R_1(t)|t_+(\epsilon, t) - t_+(R_1(t), t)|}{|t - t_+(\epsilon, t)|^2} \right)^{1/s} \\
& + TV^s(u_0) + (2\|u_0\|)^{1/s}, \\
& \leq C(\epsilon, t) + TV^s(u_0) + (2\|u_0\|)^{1/s}.
\end{aligned}$$

In a similar way the other case $x \leq -\epsilon$ can be handle,

$$\sum_{i=0}^{\infty} |u(x_i, t) - u(x_{i+1}, t)|^{1/s} \leq C(\epsilon, t) + TV^s(u_0) + 2(2\|u_0\|)^{1/s}.$$

□

5. Construction of counter-example

Next we proceed to construct a counter-example so that initial data is in BV but the corresponding solution is not in BV^s at a fixed positive time $T > 0$ and for some specific choice of flux. We refer to the backward construction for conservation laws with discontinuous flux introduced in [3]. In order to apply the method of backward construction we need to recall some notations and results from [3]. Next result is borrowed from [3] which says that given h_+, z functions we can construct an entropy solution satisfying Hopf-Lax type formula for (1.1) with h_+, z .

Proposition 5.1 (Backward construction, [3]). *Let f, g are C^1 strictly convex functions. Let $R > 0$ and $z : [0, R] \rightarrow (-\infty, 0]$ be a non-decreasing function with $z_0 = z(0+)$ and $z_1 = z(R-)$. Suppose*

$$\begin{aligned} h_+ \left(\frac{R}{T - t_1} \right) &= -\frac{z_1}{t_1}, \\ g'(u_-) &= \frac{z_0}{T}, \quad g'(v_-) = -\frac{z_1}{t_1}, \quad \bar{v}_- = f_+^{-1}(g(v_-)), \end{aligned} \quad (5.1)$$

where h_+ is defined as

$$h_+ := g' \circ g_+^{-1} \circ f \circ (f')^{-1}. \quad (5.2)$$

We additionally assume that h_+ is a locally Lipschitz function. Then there exists an initial data $u_0 \in L^\infty(\mathbb{R})$ and the corresponding entropy solution u to (1.1) such that

$$u(x, T) = (f')^{-1} \left(\frac{x}{T - t_+(x)} \right) \quad \text{where} \quad -\frac{z(x)}{t_+(x)} = h_+ \left(\frac{x}{T - t_+(x)} \right) \quad \text{for } x \in [0, R] \quad (5.3)$$

and additionally, it holds $u(x, T) = u_-$ for $x < 0$ and $u(x, T) = \bar{v}_-$ for $x > R$.

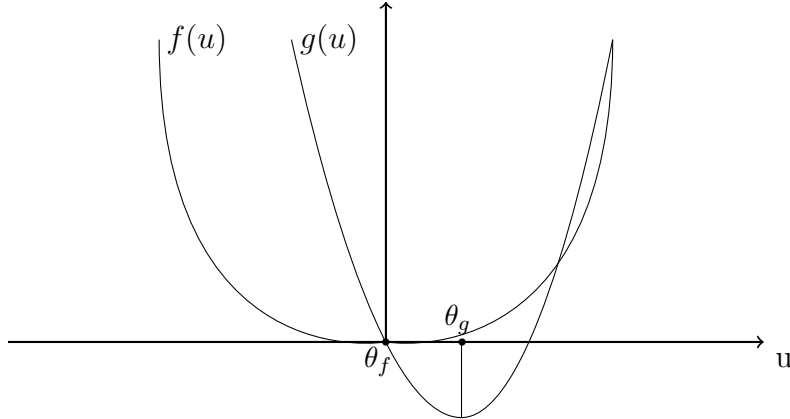


Figure 3: An illustration of fluxes

To be self-contained the main ingredients of the proof is given in [Appendix C](#). Now the proof of [Theorem 2.6](#) can be done.

Proof of Theorem 2.6. Let $f(u) = |u|^{p+1}$ and $g(u) = u^2 - 1$. Note that by [Lemma A.4](#) f satisfies the non-degeneracy condition (2.2) with exponent p and g is uniformly convex.

Let $\{a_k\}_{k \geq 1}$ be a sequence defined as $a_{2i} = i^{-\beta}$ and $a_{2i+1} = i^{-\alpha}$ with $\beta > \alpha > 0$ which will be chosen later. Consider an increasing sequence $\{t_k\}$ such that $t_k \rightarrow 1$ and

$$1 - t_{2k+1} = \frac{1}{k^{\beta-\alpha}}(1 - t_{2k}) \quad \text{and} \quad t_{2k+2} - t_{2k+1} = k^{-\lambda} \quad (5.4)$$

where $\lambda > 1$ will be chosen later. Then we have

$$\frac{t_{2k+2} - t_{2k+1}}{t_{2k+1}} = \frac{1}{k^\lambda} \frac{1}{t_{2k+1}} \geq \frac{1}{k^\lambda}. \quad (5.5)$$

We define $\{x_i\}$ as follows

$$x_i = (1 - t_{2i})a_{2i} = (1 - t_{2i+1})a_{2i+1}. \quad (5.6)$$

Since $\{t_{2i}\}_{i \geq 1}$ is increasing and $\{a_{2i}\}_{i \geq 1}$ is decreasing sequence, $\{x_i\}_{i \geq 1}$ is a decreasing sequence. Let $h : [0, \infty) \rightarrow \mathbb{R}$ be defined as

$$h(u) = 2\sqrt{1 + (p+1)^{-1-\frac{1}{p}}u^{1+\frac{1}{p}}}$$

for $u \geq 0$. Observe that

$$\frac{h(a_{2i+1})}{h(a_{2i+2})} - 1 = \frac{\sqrt{1 + \left(\frac{i-\alpha}{p+1}\right)^{1+\frac{1}{p}}} - \sqrt{1 + \left(\frac{(i+1)-\beta}{p+1}\right)^{1+\frac{1}{p}}}}{\sqrt{1 + \left(\frac{(i+1)-\beta}{p+1}\right)^{1+\frac{1}{p}}}} \leq \frac{1}{i^{\frac{p+1}{p}\alpha}} - \frac{1}{i^{\frac{p+1}{p}\beta}}. \quad (5.7)$$

Then if $\lambda < \frac{p+1}{p}\alpha$ we get

$$\frac{h(a_{2i+1})}{h(a_{2i+2})} - 1 < \frac{t_{2i+2}}{t_{2i+1}} - 1. \quad (5.8)$$

Therefore, we have

$$t_{2i+1}h(a_{2i+1}) < t_{2i+2}h(a_{2i+2}). \quad (5.9)$$

Note that

$$\frac{1 - t_{2i+1}}{1 - t_{2i}} = \frac{1}{i^{\beta-\alpha}} < 1. \quad (5.10)$$

Hence, $t_{2i+1} > t_{2i}$. Since $h(a_{2i+1}) > h(a_{2i})$ we have $t_{2i+1}h(a_{2i+1}) > h(a_{2i})t_{2i}$. Let $\xi(x)$ be solving the following problem

$$\left(\frac{x}{1 - \xi(x)}\right)^{1+\frac{1}{p}} = \left(\frac{C}{\xi(x) + d}\right)^2 - 1 \quad (5.11)$$

$$\xi(x_i) = t_{2i+1}, \quad (5.12)$$

$$\xi(x_{i+1}) = t_{2i+2}. \quad (5.13)$$

Note that $C, d > 0$ is determined by (5.12) and (5.13). Next we show that $\xi' < 0$. To this end we differentiate both side of (5.11) and get the following

$$\begin{aligned} 0 < \left(1 + \frac{1}{p}\right)x^{\frac{1}{p}} &= -\xi'(x) \left(1 + \frac{1}{p}\right) (1 - \xi(x))^{\frac{1}{p}} \left[\left(\frac{C}{\xi(x) + d}\right)^2 - 1 \right] \\ &\quad - \xi'(x) (1 - \xi(x))^{1+\frac{1}{p}} \frac{2C}{(\xi(x) + d)^2} \left(\frac{C}{\xi(x) + d}\right). \end{aligned} \quad (5.14)$$

Therefore, we get $\xi'(x) < 0$. Let $\Phi(x)$ be defined as

$$\Phi(x) := \xi(x) \sqrt{1 + \left(\frac{x}{1 - \xi(x)}\right)^{1+\frac{1}{p}}} = \frac{C\xi(x)}{\xi(x) + d}. \quad (5.15)$$

Observe that

$$\Phi'(x) = \xi'(x) \left[\frac{C}{\xi(x) + d} - \frac{C\xi(x)}{(\xi(x) + d)^2} \right] = \xi'(x) \frac{Cd}{(\xi(x) + d)^2} < 0. \quad (5.16)$$

Finally we define the function $t(x)$ such that $t(x_{i+}) = t_{2i}$ and $t(x_{i-}) = t_{2i+1}$ for $i \geq i_0$ and t satisfies (5.11)–(5.13) for $x \in (x_{i+1}, x_i)$. Let $\rho : (0, \infty) \rightarrow \mathbb{R}$ be defined as

$$\rho(x) = -t(x)h\left(\frac{x}{1-t(x)}\right). \quad (5.17)$$

By (5.9) and (5.16), $x \mapsto \rho(x)$ is increasing. By Proposition 5.1 with $R = x_1$, there exists an entropy solution u such that

$$u(x_{i+}, 1) = \left(\frac{x_i}{(p+1)(1-t_{2i})}\right)^{\frac{1}{p}} \text{ and } u(x_{i-}, 1) = \left(\frac{x_i}{(p+1)(1-t_{2i+1})}\right)^{\frac{1}{p}}. \quad (5.18)$$

By (5.6) we get

$$u(x_{i+}, 1) = \left(\frac{a_{2i}}{p+1}\right)^{\frac{1}{p}} \text{ and } u(x_{i-}, 1) = \left(\frac{a_{2i+1}}{p+1}\right)^{\frac{1}{p}}. \quad (5.19)$$

Therefore,

$$\begin{aligned} |u(x_{i-}, 1) - u(x_{i+}, 1)| &= \left| \left(\frac{a_{2i}}{p+1}\right)^{\frac{1}{p}} - \left(\frac{a_{2i+1}}{p+1}\right)^{\frac{1}{p}} \right| \\ &= (1+p)^{-\frac{1}{p}} \left[i^{-\frac{\alpha}{p}} - i^{-\frac{\beta}{p}} \right]. \end{aligned} \quad (5.20)$$

Let $\epsilon > 0$. Then, we have

$$|u(x_{i-}, 1) - u(x_{i+}, 1)|^{\frac{p+1}{1+\epsilon}} \geq C(p) \left[i^{-\frac{\alpha(p+1)}{p(1+\epsilon)}} - i^{-\frac{\beta(p+1)}{p(1+\epsilon)}} \right]. \quad (5.21)$$

Now, we set

$$\lambda = 1 + \frac{2p}{3(2p+1)}\epsilon, \quad \frac{p+1}{p}\alpha = 1 + \frac{4p+2}{3(2p+1)}\epsilon \text{ and } \frac{p+1}{p}\beta = 1 + \frac{2(3p+2)}{3(2p+1)}\epsilon. \quad (5.22)$$

We check that $\beta - \alpha = \lambda - 1$ and $\frac{p+1}{p}\beta > 1 + \epsilon$. Hence, $u(\cdot, 1) \notin BV_{loc}^s(\mathbb{R})$ for $s = \frac{1}{p+1} + \frac{\epsilon}{p+1}$.

Note that by Proposition 5.1 initial data $u_0 \in L^\infty(\mathbb{R})$. Now we find a data which is in $BV(\mathbb{R})$. From the construction we have $x_1 < R_2(1)$ where $R_2(t)$ is as in Theorem 3.1. Choose a point $x_0 \in (x_1, R_2(1))$. Note that $0 < t_+(x_0, 1) < 1$ and $u(x, t_+(x_0, 1)) = \bar{v}_-$ for $x \geq 0$. We also observe that $L_1(t) = 0$ and $R_2(t) > 0$ for $t = t_+(x_0, 1)$. Therefore, for $t = t_+(x_0)$ we have

$$u(x, t) = (g')^{-1}\left(\frac{x - z_-(x, t)}{t}\right) \text{ for } x < 0. \quad (5.23)$$

Since g is uniformly convex we have $u(\cdot, t_+(x_0, 1)) \in BV((-\infty, 0))$. To conclude the Theorem 2.6 we set $v_0(x) := u(x, t_0(x_0, 1))$. Let $v(x, t)$ be the entropy solution to (1.1) with initial data v_0 . Note that $v(x, 1 - t_0(x_0, 1)) = u(x, 1)$ for all $x \in \mathbb{R}$. Hence, the proof of Theorem 2.6 is completed. \square

Appendix A. Hölder continuity of singular maps

In this section useful lemma on Hölder exponent and non degeneracy of fluxes are collected and used throughout the paper. Some commentaries are added for all lemmas.

The following lemma recall that the non uniform convexity of a flux function corresponds to a loss of the Lipschitz regularity for the reciprocal function of the derivative. This key point inforces a BV^s (or generalized BV regularity [17, 27]) instead of BV regularity [33, 42] for the entropy solutions.

Lemma A.1. *Let $g \in C^1(\mathbb{R})$ be satisfying the non-degeneracy (2.2) with exponent q . Then $(g')^{-1}$ is Hölder continuous with exponent $1/q$.*

Proof. Fix a compact set K . Let x and y is in $g'(K)$. There exist \tilde{x}, \tilde{y} such that $\tilde{x} = (g')^{-1}(x)$ and $\tilde{y} = (g')^{-1}(y)$. Then,

$$\frac{|(g')^{-1}(x) - (g')^{-1}(y)|}{|x - y|^{1/q}} = \frac{|\tilde{x} - \tilde{y}|}{|g'(\tilde{x}) - g'(\tilde{y})|^{1/q}} = \frac{|\tilde{x} - \tilde{y}|}{|g'(\tilde{x}) - g'(\tilde{y})|^{1/q}} \leq \frac{1}{C_2^{1/q}}.$$

This proves the Lemma A.1. \square

The interface condition (1.2) needs the use of some reciprocal functions of the flux g or f . The fact that the reciprocal function of g is never Lipschitz near $\min g$ forbids the classical Lax-Oleinik BV smoothing effect for an uniform convex flux.

Lemma A.2. *Let g be a C^2 function satisfying (2.2) with exponent q then g_+ satisfies (2.2) with exponent $q + 1$ on domain (θ_g, ∞) .*

Proof. Since θ_g is the critical point of g hence, $g'(\theta_g) = 0$, then we consider

$$\begin{aligned} g(x) - g(y) &= (x - y) \int_0^1 g'(\lambda x + (1 - \lambda)y) d\lambda, \\ &= (x - y) \int_0^1 (g'(\lambda x + (1 - \lambda)y) - g'(\theta_g)) d\lambda. \end{aligned}$$

We know that $g'(\cdot)$ is a increasing function and g satisfies the non-degeneracy condition (2.2). Let $x > y \geq \theta_g$, then

$$\begin{aligned} |g(x) - g(y)| &= |x - y| \int_0^1 (g'(\lambda x + (1 - \lambda)y) - g'(\theta_g)) d\lambda \\ &\geq C_2 |x - y| \int_0^1 (\lambda x + (1 - \lambda)y - \theta_g)^q d\lambda \\ &\geq \frac{1}{q + 1} C_2 ((x + (1 - \lambda)y) - \theta_g)^{q+1} \Big|_0^1 \\ &\geq \frac{1}{q + 1} C_2 ((x - \theta_g)^{q+1} - (y - \theta_g)^{q+1}) \\ &\geq \frac{C_2}{q + 1} |x - y|^{q+1}. \end{aligned} \tag{A.1}$$

\square

The previous commentary of Lemma A.2 is even more important for the non Lipschitz regularity of the singular map.

Lemma A.3. *Suppose fluxes f and g are $C^1(\mathbb{R})$ and convex functions with $f(\theta_f) < g(\theta_g)$ which additionally satisfies the non-degeneracy condition (2.2) and let K is any compact set of \mathbb{R} . Then for $x \in K$, $f_+^{-1}g(\cdot)$ is a Lipschitz continuous function and $g_-^{-1}f(\cdot)$ is a Hölder continuous function.*

Proof. Since $f(\theta_f) < g(\theta_g)$, there exist $a_1 < \theta_f < a_2$ such that $f(a_1) = g(\theta_g) = f(a_2)$. Hence, we have

$$\bar{c} := \min \{|f'(a)|; a \in (-\infty, a_1] \cup [a_2, \infty)\} > 0. \tag{A.2}$$

Without loss generality we can assume that $g(x) \neq g(y)$ because if $g(x) = g(y)$ then result holds anyway. There exist $\tilde{x}, \tilde{y} > \theta_f$ such that $f(\tilde{x}) = g(x)$ and $f(\tilde{y}) = g(y)$. As f_+^{-1} is increasing, we get $\tilde{x}, \tilde{y} > a_2$. Consider the following

$$\begin{aligned} \frac{|f_+^{-1}g(x) - f_+^{-1}g(y)|}{|x - y|} &= \frac{|f_+^{-1}g(x) - f_+^{-1}g(y)|}{|g(x) - g(y)|} \cdot \frac{|g(x) - g(y)|}{|x - y|}, \\ &= \frac{|f_+^{-1}f(\tilde{x}) - f_+^{-1}f(\tilde{y})|}{|f(\tilde{x}) - f(\tilde{y})|} \cdot \frac{|g(x) - g(y)|}{|x - y|}, \\ &= \frac{|\tilde{x} - \tilde{y}|}{|f(\tilde{x}) - f(\tilde{y})|} \cdot \frac{|g(x) - g(y)|}{|x - y|}, \\ &= \frac{1}{f'(c_0)} \cdot \frac{|g(x) - g(y)|}{|x - y|}, \end{aligned}$$

for some c_0 in between \tilde{x}, \tilde{y} . Note that $c_0 \geq a_2$ and $f' \geq \bar{c}$. As g is Lipschitz continuous function, we have $|g(x) - g(y)| \leq c_1|x - y|$ where c_1 depends on g and K . Therefore we get,

$$\frac{|f_+^{-1}g(x) - f_+^{-1}g(y)|}{|x - y|} \leq C. \quad (\text{A.3})$$

We know that for $f(x) \geq g(\theta_g)$ there exists \tilde{x} such that $f(x) = g(\tilde{x})$ and $g'(\tilde{x}) > 0$, without loss of generality we can assume that $g(x) \neq g(y)$ because if $g(x) = g(y)$ then result holds.

$$\begin{aligned} \frac{|g_-^{-1}f(x) - g_-^{-1}f(y)|^{q+1}}{|x - y|} &= \frac{|g_-^{-1}f(x) - g_-^{-1}f(y)|^{q+1}}{|f(x) - f(y)|} \cdot \frac{|f(x) - f(y)|}{|x - y|}, \\ &= \frac{|g_-^{-1}g(\tilde{x}) - g_-^{-1}g(\tilde{y})|^{q+1}}{|g(\tilde{x}) - g(\tilde{y})|} \cdot \frac{|f(x) - f(y)|}{|x - y|}, \\ &= \frac{|\tilde{x} - \tilde{y}|^{q+1}}{|g(\tilde{x}) - g(\tilde{y})|} \cdot \frac{|f(x) - f(y)|}{|x - y|}, \end{aligned}$$

Now from the Lipschitz continuity f and (A.1),

$$\frac{|g_-^{-1}f(x) - g_-^{-1}f(y)|^{q+1}}{|x - y|} \leq C. \quad (\text{A.4})$$

Hence, it implies that

$$|g_-^{-1}f(x) - g_-^{-1}f(y)| \leq C|x - y|^{1/q+1}.$$

□

Next lemma shows that power law fluxes satisfies the non-degeneracy condition (2.2).

Lemma A.4. *Let $M > 0$ and $g : [-M, M] \rightarrow \mathbb{R}$ be defined as $g(u) = |u|^p$ for $p \geq 2$. Then g satisfies the non-degeneracy condition (2.2) with exponent $p - 1$.*

This is the simplest example with power-law degeneracy $p - 1$ [11, 16].

Proof. We calculate $g'(u) = \text{sign}(u)p|u|^{p-1}$. Then we show that the non-degeneracy condition (2.2) is satisfied for g case by case. Hence, we consider

Case(I): If $u, v \geq 0$, since $(|u| + |v|)^p > (|u|^p + |v|^p)$, we get

$$\frac{|g'(u) - g'(v)|}{|u - v|^{p-1}} = p \frac{|u^{p-1} - v^{p-1}|}{|u - v|^{p-1}} \geq p. \quad (\text{A.5})$$

Now for $u, v \leq 0$ can be handled similarly as Case I.

Case(II): If $u \leq 0$ and $v \geq 0$, then we also get

$$\frac{|g'(u) - g'(v)|}{|u - v|^{p-1}} = p \frac{|u|^{p-1} + |v|^{p-1}}{|u - v|^{p-1}} \geq p. \quad (\text{A.6})$$

Again for $u \leq 0$ and $v \geq 0$ can be handled in similar way as Case II. \square

Appendix B. BV^s embedding

The continuous embedding between fractional BV spaces is explicated using the L^∞ norm or, more precisely, the oscillation in the next lemma. Recall that the oscillation of the function u on I is,

$$\text{osc}(u) := \sup_{x < y} \{|u(x) - u(y)|\} \leq 2\|u\|_\infty.$$

Lemma B.1. *Let $u : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be bounded function on a given interval I and $0 < s < t$ such that $u \in BV^t \subset BV^s$. Let $p = \frac{1}{s} \geq q = \frac{1}{t}$, then,*

$$TV^s u(I) \leq \text{osc}(u)^{p-q} TV^t u(I). \quad (\text{B.1})$$

Proof. When $\text{osc}(u) \leq 1$, the inequality $y^p \leq y^q$ for all $y \in [0, 1]$ gives a direct estimate. More precisely, let $\sigma = (x_1, \dots, x_n)$ be any partition of I ,

$$\sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^p \leq \sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^q \leq TV^t u(I).$$

This inequality can be improved as follows if u is non constant, that is $\text{osc}(u) > 0$. For this purpose, consider $v = u/\text{osc}(u)$ so $\text{osc}(v) \leq 1$. Now, on a subdivision, we have,

$$\begin{aligned} \text{osc}(u)^{-p} \sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^p &= \sum_{i=1}^{n-1} |v(x_i) - v(x_{i+1})|^p \\ &\leq \sum_{i=1}^{n-1} |v(x_i) - v(x_{i+1})|^q = \text{osc}(u)^{-q} \sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^q. \end{aligned}$$

That is to say, the following inequality which is also valid when $\text{osc}(u) = 0$,

$$\sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^p \leq \text{osc}(u)^{p-q} \sum_{i=1}^{n-1} |u(x_i) - u(x_{i+1})|^q.$$

This is enough to conclude the lemma. \square

Appendix C. Backward construction

The proof of the optimality presented in section 5 needs a construction of an initial data and solution by borrowing ideas and techniques from control. We only give a sketch of the existence of such solution along with initial data that stated in Proposition 5.1. The complete construction can be found in [3].

Proof of Theorem 5.1. We first approximate $z(x)$ by piece-wise constant increasing function as

follows

$$\left\{ \begin{array}{l} z_0 = w_0 < w_1 < \cdots < w_k = z_1, \\ |w_{i+1} - w_i| < \frac{1}{N}, \\ 0 = x_0 < x_1 < \cdots < x_k = R, \\ z(x_i) = w_i \text{ for } 1 \leq i \leq k-1, \\ \text{with } z_0 = z(0) \text{ and } z_1 = z(R-). \end{array} \right. \quad (\text{C.1})$$

We set $t_0 = T$ and t_i , $1 \leq i \leq 2k$, c_i, d_i $1 \leq i \leq k$ as follows

$$\begin{aligned} h_+ \left(\frac{x_i}{T - t_{2i-1}} \right) &= -\frac{w_{i-1}}{t_{2i-1}}, \quad h_+ \left(\frac{x_i}{T - t_{2i}} \right) = -\frac{w_i}{t_{2i}}, \\ f'(c_{2i-1}) &= \frac{x_i}{T - t_{2i-1}}, \quad f'(c_{2i}) = \frac{x_i}{T - t_{2i}} \text{ and } d_i = g_+^{-1}(f(a_i)). \end{aligned} \quad (\text{C.2})$$

Then we observe that $c_{2i-1} > c_{2i}$, $d_{2i-1} > d_{2i}$, $T = t_0 > t_1 > \cdots > t_{2k} = T_1$. Consider Lipschitz curves $r_i, \tilde{r}_i, a_i, b_i$ defined as follows

$$\begin{aligned} s_i &= \frac{f(c_{2i-1}) - f(c_{2i})}{c_{2i-1} - c_{2i}}, \quad S_i = \frac{g(d_{2i-1}) - g(d_{2i})}{d_{2i-1} - d_{2i}}, \quad 1 \leq i \leq k, \\ r_i(t) &= g'(d_i)(t - t_i), \quad \tilde{r}_i(t) = f'(c_i)(t - t_i), \quad 1 \leq i \leq 2k, \\ a_i(t) &= x_i + s_i(t - T), \quad b_i(t) = S_i(t - q_i), \quad a_i(q_i) = 0, \quad 1 \leq i \leq 2k, \end{aligned} \quad (\text{C.3})$$

$$r_0(t) = g'(b_0)(t - T) = g'(u_-)(t - t_0). \quad (\text{C.4})$$

Now, we define u_0^N as below

$$u_0^N := \begin{cases} u_- & \text{if } x < w_0, \\ d_{2i-1} & \text{if } w_{i-1} < x < b_i(0), 1 \leq i \leq k, \\ d_{2i} & \text{if } b_i(0) < x < w_i, 1 \leq i \leq k, \\ v_- & \text{if } w_{2k} < x < 0, \\ \bar{v}_- & \text{if } x > 0. \end{cases} \quad (\text{C.5})$$

Let $\tilde{t}_i(x)$ be the unique solution to

$$h_+ \left(\frac{x}{T - \tilde{t}_i(x, t)} \right) = -\frac{z_i}{\tilde{t}_i(x, t)} \text{ for } x \in (x_i, x_{i+1}), 1 \leq i \leq k-1. \quad (\text{C.6})$$

Corresponding entropy solution u^N is the following

$$u^N(x, t) = \begin{cases} u_- & \text{if } x < r_0(t), \\ (g')^{-1} \left(\frac{x - z_i}{t} \right) & \text{if } r_{2i}(t) < x < \min\{r_{2i+1}(t), 0\}, \\ (f')^{-1} \left(\frac{x}{t - \tilde{t}_i(x, t)} \right) & \text{if } \max\{\tilde{r}_{2i+1}(t), 0\} < x < \tilde{r}_{2i-1}(t), \\ d_{2i-1} & \text{if } r_{2i-1}(t) < x < \min\{S_i(t), 0\}, 1 \leq i \leq k, \\ d_{2i} & \text{if } S_{2i}(t) < x < \min\{r_{2i}(t), 0\}, 1 \leq i \leq k, \\ c_{2i-1} & \text{if } \max\{\tilde{r}_{2i-1}(t), 0\} < x < s_i(t), 1 \leq i \leq k, \\ c_{2i} & \text{if } \max\{s_i(t), 0\} < x < \tilde{r}_{2i}, 1 \leq i \leq k, \\ v_- & \text{if } r_{2k}(t) < x < 0, \\ \bar{v}_- & \text{if } x > \max\{\tilde{r}_{2k}, 0\}. \end{cases} \quad (\text{C.7})$$

By assumption we have h_+ is a locally Lipschitz continuous function and we can prove TV bound of $g'(u_0^N)$ (see [3] for more details). Then, by applying Helly's Theorem we can find a $u_0 \in L^\infty(\mathbb{R})$ and corresponding entropy solution u satisfying (5.3). This completes the proof of Proposition 5.1. \square

Acknowledgement. Authors thank IFCAM project “Conservation laws: BV^s , interface and control”. SSG and AP thank the Department of Atomic Energy, Government of India, under project no. 12-R&D-TFR-5.01-0520 for support. SSG acknowledges Inspire faculty-research grant DST/INSPIRE/04/2016/00-0237.

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