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The combinator \mathbf{M} and the Mockingbird lattice

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ABSTRACT. We study combinatorial and order theoretic structures arising from the fragment of combinatory logic spanned by the basic combinator \mathbf{M} . This basic combinator, named as the Mockingbird by Smullyan, is defined by the rewrite rule $\mathbf{M}x_1 \rightarrow x_1x_1$. We prove that the reflexive and transitive closure of this rewrite relation is a partial order on terms on \mathbf{M} and that all connected components of its rewrite graph are Hasse diagram of lattices. This last result is based on the introduction of new lattices on duplicative forests, which are sorts of treelike structures. These lattices are not graded, not self-dual, and not semidistributive. We present some enumerative properties of these lattices like the enumeration of their elements, of the edges of their Hasse diagrams, and of their intervals. These results are derived from formal power series on terms and on duplicative forests endowed with particular operations.

CONTENTS

Introduction	2
1. Terms, rewrite relations, and combinatory logic systems	4
1.1. Terms, compositions, and rewrite relations	4
1.2. Combinatory logic systems	6
2. The Mockingbird lattice	9
2.1. The combinator \mathbf{M} and its poset	9
2.2. Duplicative lattices	11
2.3. Mockingbird lattices	14
3. Enumerative properties	16
3.1. Formal power series	16
3.2. Maximal, minimal, and isolated terms	17
3.3. Shortest and longest saturated chains	20
3.4. Elements, covering pairs, and intervals	21
Open questions and future work	28
References	28

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- (a') Determine if \preceq is a partial order relation;
- (b') In this case, determine if each interval of this poset is a lattice;
- (c') Enumerate the \equiv -equivalence classes of terms w.r.t. the minimal degrees of their terms.

This work fits in this general project consisting in mixing combinatory logic with combinatorics.

We choose here to start this project by studying the system made of a single combinator, the combinator \mathbf{M} , known as the Mockingbird [Smu85] or as the little omega. This combinator is a very simple and an important one (see for instance [Sta11b,Sta17]). By drawing the portions of the rewrite graph starting from terms on \mathbf{M} , the first properties that stand out are that the graph does not contain any nontrivial loops and that its connected components are finite and have exactly one minimal and one maximal element. At this stage, driven by computer exploration, we conjecture that the relation \preceq on the terms on \mathbf{M} is a partial order relation and that each \equiv -equivalence class is a lattice w.r.t. this partial order relation. This lattice property is for us a good clue for the fact that this system contains rather rich combinatorial properties.

To prove this last property, we introduce new lattices on duplicative forests, that are kinds of treelike structures, and show that each maximal interval from any term on \mathbf{M} is isomorphic as a poset to a maximal interval of a lattice of duplicative forests (Theorem 2.3.4). We define the Mockingbird lattice of order $d \geq 0$ as the lattice $M(d)$ consisting in the closed terms on \mathbf{M} equal as or greater than the right comb closed term on \mathbf{M} of degree d . We prove that each interval of the poset of duplicative forests is contained as an interval in $M(d)$ for a certain $d \geq 0$ (Theorem 2.3.5). Since any closed term on \mathbf{M} can be seen as a binary tree, this provides a new lattice structure on these objects. A lot of similar lattices have been studied on binary trees such as, among others the very famous Tamari lattice [Tam62], the Kreweras lattice [Kre72], the Stanley lattice [Sta75], the phagocyte lattice [BP06], and the pruning-grafting lattice [BP08]. However, unlike these lattices having for each order $d \geq 0$ a cardinality equal to the d -th Catalan number, the lattices $M(d)$ are enumerated by a different integer sequence. To obtain enumerative results about the Mockingbird lattices and all the posets of terms on \mathbf{M} in general, we use formal power series on terms and on duplicative forests, and several products on these. In this way, we enumerate the minimal and maximal elements of the infinite poset of the closed terms on \mathbf{M} (Propositions 3.2.1 and 3.2.2), the lengths of the shortest and longest saturated chains of $M(d)$ (Proposition 3.3.1), the cardinality of $M(d)$ (Proposition 3.4.3), the number of edges of the Hasse diagram of $M(d)$ (Proposition 3.4.6), and the number of intervals of $M(d)$ (Proposition 3.4.10). We also provide general results for systems of combinatory logic: we give in particular a necessary condition on the rewrite rules in order to have only finite connected components in the rewrite graph (Proposition 1.2.3). This relies on a combinatorial property on basic combinators, called the hierarchical property. We also discuss some consequences of this fact in order to construct models for systems having this property. These are in fact the algebras over a certain abstract clone [Tay93] defined from the system. When the relation \preceq is a partial order relation, such models have a potential nice algorithmic computational complexity.

This paper is organized as follows. Section 1 contains the preliminary notions and definitions about terms, rewrite relations, and combinatory logic systems. We show here some general properties of these systems. In Section 2, we study the combinatory logic system on \mathbf{M} and the Mockingbird lattices. Finally, Section 3 contains our enumerative results about

Mockingbird lattices. This text ends with the presentation of some open questions raised by this work.

This paper is an extended version of [Gir22] containing the proofs of the presented results and presenting new ones as discussions about some models of the combinatory logic system on \mathbf{M} and the enumeration of isolated elements of the poset of the closed terms on \mathbf{M} .

General notations and conventions. If S is a finite set $\#S$ is the cardinality of S . For any integers i and j , $[i, j]$ denotes the set $\{i, i + 1, \dots, j\}$. For any integer i , $[i]$ denotes the set $[1, i]$ and $\llbracket i \rrbracket$ denotes the set $[0, i]$. For any set A , A^* is the set of all words on A . For any $w \in A^*$, $\ell(w)$ is the length of w , for any $i \in [\ell(w)]$, $w(i)$ is the i -th letter of w , and $|w|_a$ is the number of occurrences of the letter $a \in A$ in w . The only word of length 0 is the empty word ϵ . For any statement P , the Iverson bracket $[P]$ takes 1 as value if P is true and 0 otherwise.

1. TERMS, REWRITE RELATIONS, AND COMBINATORY LOGIC SYSTEMS

In this preliminary part, we set the main definitions and notations used in the sequel. We introduce also the central notion of combinatory logic systems, which are defined from special kinds of rewrite relations. We show also some general properties of these systems.

1.1. Terms, compositions, and rewrite relations. Let us start by presenting here the notions of terms, composition, and rewrite relations.

1.1.1. Terms. An *alphabet* is a finite set \mathfrak{G} . Its elements are called *constants* or *basic combinators*. Any element of the set $\mathbb{X} := \bigcup_{n \geq 1} \mathbb{X}_n$, where $\mathbb{X}_n := \{x_1, \dots, x_n\}$, is a *variable*. The set $\mathfrak{T}(\mathfrak{G})$ of *\mathfrak{G} -terms* (or simply *terms* when the context is clear) is so that any variable of \mathbb{X} is a \mathfrak{G} -term, any constant of \mathfrak{G} is a \mathfrak{G} -term, and if t_1 and t_2 are two \mathfrak{G} -terms, then $(t_1 \star t_2)$ is a \mathfrak{G} -term. For any $n \geq 0$, we denote by $\mathfrak{T}_n(\mathfrak{G})$ the set of \mathfrak{G} -terms t having all variables belonging to \mathbb{X}_n .

From this definition, any term is a rooted planar binary tree where leaves are decorated by variables or by constants. We shall express terms concisely by removing superfluous parentheses by considering that \star associates to the left and also by removing the symbols \star (see the example given in (1.1.1), (1.1.2), and (1.1.3)).

Let t be a \mathfrak{G} -term. The *degree* $\deg(t)$ of t is the number of internal nodes (the nodes decorated by \star) of t seen as a binary tree. For any $a \in \mathbb{X} \cup \mathfrak{G}$, $\deg_a(t)$ is the number of nodes labeled by a in t . The *depth* of a node u of t is the number of internal nodes in the path connecting the root of t and u . The *height* $\text{ht}(t)$ of t is the maximal depth among all the nodes of t . The *frontier* of t is the sequence of all the variables appearing in t from the left to the right. The term t is *planar* if its frontier is of the form $x_1 x_2 \dots x_n$ where $n \geq 0$. The term t is *linear* if there are no multiple occurrences of the same variable in its frontier. A *closed term* or *combinator* is a term of $\mathfrak{T}_0(\mathfrak{G})$ having thus no occurrence of any variable.

For instance, by setting $\mathfrak{G} := \{\mathbf{A}, \mathbf{B}\}$, the \mathfrak{G} -term

$$t := (((\mathbf{A} \star x_3) \star (\mathbf{B} \star ((x_1 \star x_3) \star x_1))) \star (\mathbf{A} \star \mathbf{A})) \tag{1.1.1}$$

Observe in this example that the variable x_2 occurs in the right member of \rightarrow and not in the left member. For this reason, this variable can be replaced by any term in the context closure of \rightarrow (it is replaced by \mathbf{C} in the example).

1.1.4. Applicative term rewrite systems. An *applicative term rewrite system* (or *ATRS* for short) is a pair $(\mathcal{G}, \rightarrow)$ such that \mathcal{G} is an alphabet and \rightarrow is a rewrite relation on $\mathfrak{T}(\mathcal{G})$. Let $\mathcal{T} := (\mathcal{G}, \rightarrow)$ be an ATRS. We denote by \preceq the reflexive and transitive closure of \Rightarrow . This relation \preceq is by construction a preorder. We denote by \equiv the reflexive, symmetric, and transitive closure of \Rightarrow and by $[t]_{\equiv}$ the \equiv -equivalence class of $t \in \mathfrak{T}(\mathcal{G})$. This relation \equiv is by construction an equivalence relation. The *rewrite graph* $G_{\mathcal{T}}$ of \mathcal{T} is the digraph on the set $\mathfrak{T}(\mathcal{G})$ of vertices where there is an arc from $t \in \mathfrak{T}(\mathcal{G})$ to $t' \in \mathfrak{T}(\mathcal{G})$ if $t \Rightarrow t'$. For any $t \in \mathfrak{T}(\mathcal{G})$, we also denote by $G_{\mathcal{T}}(t)$ the subgraph of $G_{\mathcal{T}}$ restrained on the set $\{t' \in \mathfrak{T}(\mathcal{G}) : t \preceq t'\}$ of vertices. The ATRS \mathcal{T} is *locally finite* if for any term t , $[t]_{\equiv}$ is finite. This is equivalent to the fact that all the connected components of $G_{\mathcal{T}}$ are finite. A \mathcal{G} -term t is a *normal form* of \mathcal{T} if there is no arc of source t in $G_{\mathcal{T}}$. We say that t is *weakly normalizing* if there is at least one normal form in $G_{\mathcal{T}}(t)$ and that t is *strongly normalizing* if $G_{\mathcal{T}}(t)$ is finite and acyclic. When all the \mathcal{G} -terms are strongly normalizing, \mathcal{T} is *terminating*. Besides, if for any $t, s_1, s_2 \in \mathfrak{T}(\mathcal{G})$, $t \preceq s_1$ and $t \preceq s_2$ implies the existence of $t' \in \mathfrak{T}(\mathcal{G})$ such that $s_1 \preceq t'$ and $s_2 \preceq t'$, then \mathcal{T} is *confluent*.

1.1.5. Partial orders. When $\mathcal{T} := (\mathcal{G}, \rightarrow)$ is an ATRS such that \preceq is antisymmetric, \preceq is a partial order relation. In this case, \mathcal{T} has the *poset property* and we shall denote by $\mathcal{P}_{\mathcal{T}}$ the poset $(\mathfrak{T}(\mathcal{G}), \preceq)$. For any $t \in \mathfrak{T}(\mathcal{G})$, let $\mathcal{P}_{\mathcal{T}}(t)$ be the subposet of $\mathcal{P}_{\mathcal{T}}$ having t as least element. Again in this case, a \mathcal{G} -term t is *minimal* (resp. *maximal*) if t is a minimal (resp. maximal) element of $\mathcal{P}_{\mathcal{T}}$. Observe that any normal form is maximal but the converse is false because a maximal element t could satisfy $t \Rightarrow t$. A \mathcal{G} -term t is *isolated* if t is both minimal and maximal. When \mathcal{T} has the poset property, \mathcal{T} is *rooted* if for any term t , the subposet $[t]_{\equiv}$ of $\mathcal{P}_{\mathcal{T}}$ has a unique minimal element. Finally, \mathcal{T} has the *lattice property* if \mathcal{T} has the poset property and for all $t \in \mathfrak{T}(\mathcal{G})$, all posets $\mathcal{P}_{\mathcal{T}}(t)$ are lattices.

1.2. Combinatory logic systems. We begin by defining combinatory logic systems as particular ATRS and present some consequence of nonerasing and hierarchical combinatory logic systems (notions defined thereafter). We explain some of the consequences of the local finiteness, the poset property, and the lattice property on the models of combinatory logic systems.

1.2.1. Main definitions. A *combinatory logic system* (or *CLS* for short) is an ATRS $\mathcal{G} := (\mathcal{G}, \rightarrow)$ such that for each constant X of \mathcal{G} , there is exactly one rewrite rule where X appears, and this rule is of the form

$$X x_1 \dots x_n \rightarrow t_X \tag{1.2.1}$$

where $n \geq 1$ and t_X is a term having no constants and having all variables in \mathbb{X}_n . Moreover, all rewrite rules of \mathcal{G} must be of the form (1.2.1). The integer n is the *order* of X in \mathcal{G} . Some well-known terms t_X appearing among others in [Smu85] are,

- with order 1, $t_I := x_1$ (*Identity bird*), $t_M := x_1 x_1$ (*Mockingbird*);
- with order 2, $t_K := x_1$ (*Kestrel*), $t_T := x_2 x_1$ (*Thrush*), $t_{M_1} := x_1 x_1 x_2$ (*Mockingbird 1*), $t_W := x_1 x_1 x_2$ (*Warbler*), $t_L := x_1(x_2 x_2)$ (*Lark*), $t_O := x_2(x_1 x_2)$ (*Owl*), $t_U := x_2(x_1 x_1 x_2)$ (*Turing Bird*);

- with order 3, $t_C := x_1x_3x_2$ (*Cardinal*), $t_V := x_3x_1x_2$ (*Vireo*), $t_B := x_1(x_2x_3)$ (*Bluebird*), $t_S := x_1x_3(x_2x_3)$ (*Starling*);
- with order 4, $t_J := x_1x_2(x_1x_4x_3)$ (*Jay*).

The constant X is *nonerasing* if t_X contains at least one occurrence of each variable x_i for any $i \in [n]$. The constant X is *hierarchical* if for any $i \in [n]$, x_i appears in t_X at depth $n + 1 - i$. For instance, the terms t_X such that X are hierarchical and of order 3 or less are

$$x_1x_1, x_1x_1x_2, x_2(x_1x_1), x_1x_1x_2x_3, x_2(x_1x_1)x_3, x_3(x_1x_1x_2), x_3(x_2(x_1x_1)). \quad (1.2.2)$$

In particular, if X is hierarchical, then X is nonerasing. We say that \mathcal{G} is *nonerasing* (resp. *hierarchical*) if all constants of \mathcal{G} are nonerasing (resp. hierarchical). Other interesting properties of basic combinators are introduced in [Bim11] but they do not intervene directly in this work.

Consider for instance the CLS $\mathcal{G} := (\mathfrak{G}, \rightarrow)$ where \mathfrak{G} contains only the constant I where t_I is defined above. It is straightforward to show that \mathcal{G} has the poset property. Nevertheless, \mathcal{G} has not the lattice property, as suggested by the Hasse diagram shown in Figure 1a. On the

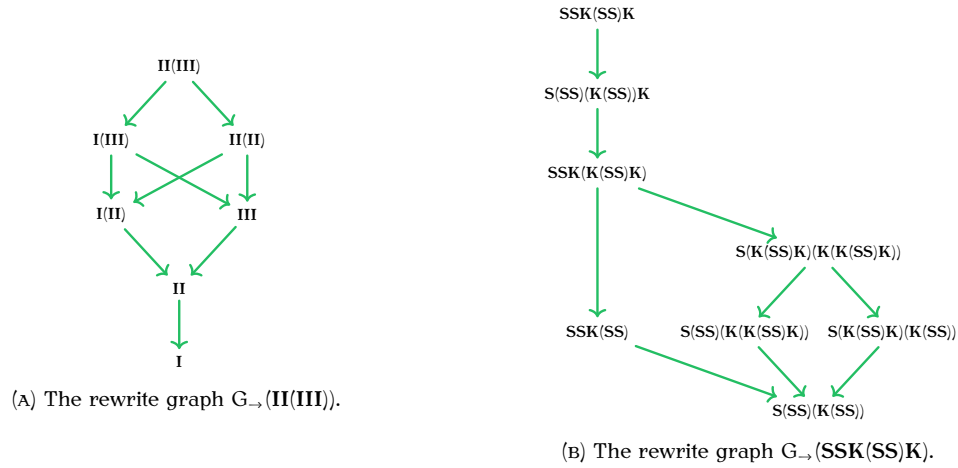


FIGURE 1. Some subgraphs of rewrite graphs of some CLS.

other side, the CLS $\mathcal{G} := (\mathfrak{G}, \rightarrow)$ where \mathfrak{G} contains only the constants K and S where t_K and t_S are defined above does not have the poset property. Indeed, \preceq is not antisymmetric because by setting $t := S(S(KS)(K(SKK)))(K(SKK))$, we have $s \preceq s'$ and $s' \preceq s$ where $s := tt(ttt)$ and $s' := ttt(ttt)$. This can be seen by noticing that $t x_1x_2 \preceq x_2x_2$. Figure 1b shows a part of the rewrite graph of \mathcal{G} .

1.2.2. *Properties of CLS.* Let us state some properties of CLS related to their properties of confluence, termination, and local finiteness.

Proposition 1.2.1. *Any CLS is confluent.*

Proof. By definition, the underlying ATRS of any CLS is orthogonal [BN98, Chapter 6]. The statement follows from the fact that all orthogonal rewrite systems are confluent [Ros73]. \square

Lemma 1.2.2. *Let $\mathcal{G} := (\mathfrak{G}, \rightarrow)$ be a CLS. If \mathcal{G} is nonerasing, then for any $n \geq 0$ and $t \in \mathfrak{T}_n(\mathfrak{G})$, $[t]_{\equiv} \subseteq \mathfrak{T}_n(\mathfrak{G})$.*

Proof. Assume that t and t' are two \mathfrak{G} -terms such that $t \Rightarrow t'$. From the definition of \Rightarrow from \rightarrow provided by (1.1.7), the fact that all constants of \mathfrak{G} are nonerasing implies that for any $x_i \in \mathbb{X}$, $\deg_{x_i}(t) \geq 1$ if and only if $\deg_{x_i}(t') \geq 1$. Therefore, both t and t' belong to $\mathfrak{T}_n(\mathfrak{G})$ where n is an integer nonsmaller than the greatest index of the variables appearing in t and t' . Since \equiv is the reflexive, symmetric, and transitive closure of \Rightarrow , the result follows. \square

Proposition 1.2.3. *Let $\mathcal{G} := (\mathfrak{G}, \rightarrow)$ be a CLS. If \mathcal{G} is hierarchical, then \mathcal{G} is locally finite and all the \mathfrak{G} -terms of a same connected component of $G_{\mathcal{G}}$ have the same height.*

Proof. Assume that t and t' are two \mathfrak{G} -terms such that $t \Rightarrow t'$. From the definition of \Rightarrow from \rightarrow provided by (1.1.7), the fact that all constants of \mathfrak{G} are hierarchical implies that $\text{ht}(t) = \text{ht}(t')$. Moreover, since \mathcal{G} is also nonerasing, by Lemma 1.2.2, for any $x_i \in \mathbb{X}$, $\deg_{x_i}(t) \geq 1$ if and only if $\deg_{x_i}(t') \geq 1$. Observe moreover that for any $n \geq 0$, the number of terms of the same height in $\mathfrak{T}_n(\mathfrak{G})$ is finite. Since \equiv is the reflexive, symmetric, and transitive closure of \Rightarrow , and since two terms t and t' are \equiv -equivalent if and only if t and t' belong to the same connected component of $G(\mathcal{G})$, the result follows. \square

Let us use these results to state some properties of a CLS \mathcal{G} . By Proposition 1.2.1 and [BKdVT03, Theorem 1.2.2], each \equiv -equivalence class of terms of \mathcal{G} admits at most one normal form. If \mathcal{G} is hierarchical and has the poset property, it follows by Proposition 1.2.3 that for each term t of \mathcal{G} , the subposet on $[t]_{\equiv}$ of $\mathcal{P}_{\mathcal{G}}$ admits exactly one maximal element. If additionally \mathcal{G} is rooted, then for each term of \mathcal{G} , the subposet on $[t]_{\equiv}$ of $\mathcal{P}_{\mathcal{G}}$ admits exactly one minimal element. In this case, there is a one-to-one correspondence between the set of the minimal terms and the set of the maximal terms of \mathcal{G} and this correspondence preserves the \equiv -equivalence classes.

1.2.3. Models. Let $\mathcal{G} := (\mathfrak{G}, \rightarrow)$ be a CLS. A *model* of \mathcal{G} is an algebra over the abstract clone defined as the quotient of $\mathfrak{T}(\mathfrak{G})$ by the clone congruence \equiv (see for instance [Tay93] for a general description of algebras over abstract clones). In more concrete terms, for this particular case of abstract clone, this is the data of a magma (\mathcal{M}, \star) such that $\mathfrak{G} \subseteq \mathcal{M}$ and, for each rule $X x_1 \dots x_n \rightarrow t_X$, the axiom $(\dots((X \star x_1) \star x_2) \dots) \star x_n = e_X$ holds, where e_X is the expression having t_X as syntax tree. For instance, a model of the CLS on the constants \mathbf{K} and \mathbf{S} is a set \mathcal{M} containing \mathbf{K} and \mathbf{S} , and endowed with an operation $\star : \mathcal{M}^2 \rightarrow \mathcal{M}$ satisfying $(\mathbf{K} \star x_1) \star x_2 = x_1$ and $((\mathbf{S} \star x_1) \star x_2) \star x_3 = (x_1 \star x_3) \star (x_2 \star x_3)$. Such structures are known as *combinatory algebras* [HS08].

The *trivial model* of \mathcal{G} is the magma (\mathcal{M}, \star) where \mathcal{M} is the set $\mathfrak{T}(\mathfrak{G})/\equiv$ and, for any $[t_1]_{\equiv}, [t_2]_{\equiv} \in \mathcal{M}$, $[t_1]_{\equiv} \star [t_2]_{\equiv}$ is the \equiv -equivalence class of $t_1 t_2$ where t_1 is any element of $[t_1]_{\equiv}$ and t_2 is any element of $[t_2]_{\equiv}$. Moreover, when \mathcal{G} is nonerasing, for any $n \geq 0$, the *n-trivial model* of \mathcal{G} is the trivial model restrained to \equiv -equivalence classes of terms of $\mathfrak{T}_n(\mathfrak{G})$. Thanks to Lemma 1.2.2, the *n-trivial model* is well-defined.

When \mathcal{G} is hierarchical and has the poset property, as already noticed, each \equiv -equivalence class contains exactly one maximal element. Therefore, the set \mathcal{M}^{\max} of the maximal terms of $\mathfrak{T}(\mathfrak{G})$ is a set of representatives of the set of the \equiv -equivalence classes and forms a model of \mathcal{G} isomorphic to the trivial model. When additionally \mathcal{G} is rooted, as already noticed, each

\equiv -equivalence class contains exactly one minimal element. For this reason, the set \mathcal{M}^{\min} of the minimal terms of $\mathcal{T}(\mathcal{G})$ is a set of representatives of the set of the \equiv -equivalence classes and forms a model of \mathcal{G} isomorphic to the previous ones. Here also, for any $n \geq 0$, we define the model \mathcal{M}_n^{\max} (resp. \mathcal{M}_n^{\min}) as the restriction of \mathcal{M}^{\max} (resp. \mathcal{M}^{\min}) on $\mathcal{T}_n(\mathcal{G})$. The interest of these two models \mathcal{M}^{\max} and \mathcal{M}^{\min} relies on considerations about algorithmic complexity for the computation of the product $t_1 \star t_2$ where t_1 and t_2 are two terms of the models.

2. THE MOCKINGBIRD LATTICE

This central section of this work concerns the study of the poset associated with the CLS containing the basic combinator \mathbf{M} . We shall prove that this CLS has the poset property, is rooted, and has also the lattice property by introducing new lattices on some kind of treelike structures, called duplicative forests.

2.1. The combinator \mathbf{M} and its poset. Let the CLS $\mathcal{G} := (\mathcal{G}, \rightarrow)$ such that $\mathcal{G} := \{\mathbf{M}\}$ and $t_{\mathbf{M}} = x_1 x_1$. Since \mathbf{M} is the Mockingbird basic combinator, we call \mathcal{G} the *Mockingbird CLS*. Observe that \mathbf{M} is hierarchical so that \mathcal{G} satisfies the properties stated by Proposition 1.2.3 and is in particular locally finite. From now, we shall simply write G instead of $G_{\mathcal{G}}$. Figure 2 provides an example of a fragment of G .

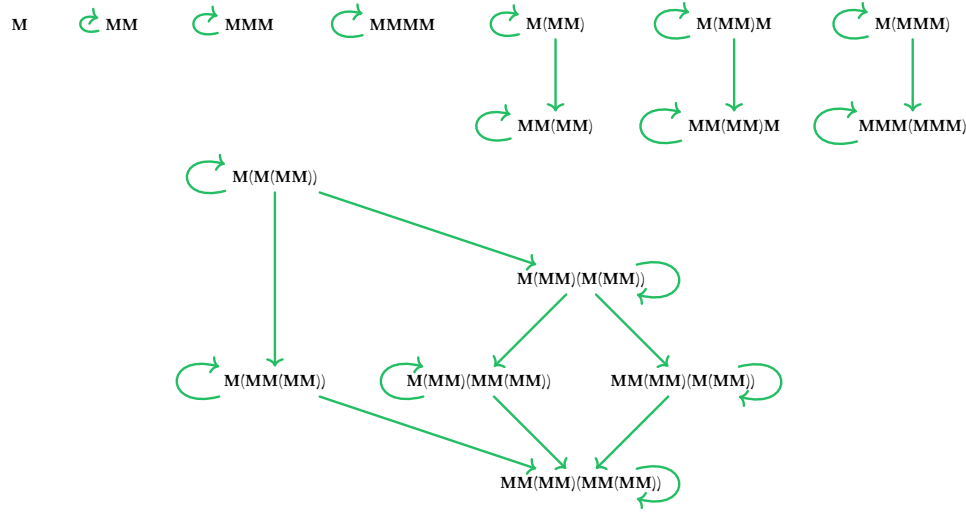


FIGURE 2. The fragment of the rewrite graph of \mathcal{G} restrained to the terms reachable from closed terms of degrees 3 or less.

2.1.1. First properties. Let us present some first properties of \mathcal{G} . We begin by providing a recursive description of the rewrite relation \Rightarrow .

Lemma 2.1.1. For any $t, t' \in \mathcal{T}(\mathcal{G})$, we have $t \Rightarrow t'$ if and only if $t = t_1 t_2$, $t' = t'_1 t'_2$ with $t_1, t_2, t'_1, t'_2 \in \mathcal{T}(\mathcal{G})$, and at least one of the following assertions holds:

- (i) $t_1 \Rightarrow t'_1$ and $t_2 = t'_2$;
- (ii) $t_2 \Rightarrow t'_2$ and $t_1 = t'_1$;
- (iii) $t_1 = \mathbf{M}$ and $t_2 = t'_1 = t'_2$.

Proof. This is a direct consequence of the definitions of the context closure \Rightarrow from the rewrite relation \rightarrow and of the term t_M . \square

Proposition 2.1.2. *The CLS \mathcal{G}*

- (i) *is locally finite;*
- (ii) *has the poset property;*
- (iii) *is rooted.*

Proof. First, since \mathbf{M} is hierarchical, by Proposition 1.2.3, \mathcal{G} is locally finite. Therefore, (i) holds.

Let us prove that \mathcal{G} has the poset property. For this, let $\theta : \mathfrak{T}(\mathcal{G}) \rightarrow \mathbb{Z}^2$ be the map defined for any $t \in \mathfrak{T}(\mathcal{G})$ by $\theta(t) := (\deg(t), -\deg_M(t))$. We denote by \leq the lexicographic order on \mathbb{Z}^2 and by $\dot{+}$ the pointwise addition on \mathbb{Z}^2 . Let us prove by structural induction on $\mathfrak{T}(\mathcal{G})$ that for any \mathcal{G} -terms t and t' , $t \neq t'$ and $t \Rightarrow t'$ implies $\theta(t) < \theta(t')$. By Lemma 2.1.1, we have $t = t_1 t_2$ and $t' = t'_1 t'_2$ for some \mathcal{G} -terms t_1, t_2, t'_1 , and t'_2 , and we have the three following cases. First, if $t_1 \Rightarrow t'_1$ and $t_2 = t'_2$, then $\theta(t) = \theta(t_1) \dot{+} \theta(t_2) \dot{+} (1, 0)$ and $\theta(t') = \theta(t'_1) \dot{+} \theta(t_2) \dot{+} (1, 0)$. Since, by induction hypothesis, $\theta(t_1) < \theta(t'_1)$, we have $\theta(t) < \theta(t')$. Second, if $t_2 \Rightarrow t'_2$ and $t_1 = t'_1$, the same arguments as the previous ones (by interchanging the indices 1 and 2 in the concerned terms) apply. Finally, if $t_1 = \mathbf{M}$ and $t_2 = t'_1 = t'_2$, then $\theta(t) = \theta(t_2) \dot{+} (1, -1)$ and $\theta(t') = \theta(t_2) \dot{+} \theta(t_2) \dot{+} (1, 0)$. Since $t \neq t'$, we have $t_2 \neq \mathbf{M}$ so that $\theta(t) < \theta(t')$. This implies that \preceq is antisymmetric, so that (ii) holds.

To prove that \mathcal{G} is rooted, we consider the rewrite relation \rightarrow' obtained by inverting \rightarrow . Thus, \rightarrow' satisfies $x_1 x_1 \rightarrow' \mathbf{M} x_1$. This ATRS $(\mathfrak{T}(\mathcal{G}), \rightarrow')$ does not admit any overlapping term (see [BKdVT03, Chapter 2]). Therefore, it is confluent. A consequence of (i), is that this ATRS has only finite \equiv' -equivalence classes. By using additionally [BKdVT03, Theorem 1.2.2], each \equiv' -equivalence class admits exactly one term t such that for any $t' \in [t]_{\equiv'}$, $t' \preceq' t$. This implies that in \mathcal{G} , each \equiv -equivalence class admits exactly one minimal term. Therefore, (iii) holds. \square

By Proposition 2.1.2, $\mathcal{P}_{\mathcal{G}}$ is a well-defined poset, called *Mockingbird poset*. From now, we shall simply write \mathcal{P} instead of $\mathcal{P}_{\mathcal{G}}$. We have the following recursive description of \preceq .

Lemma 2.1.3. *For any $t, t' \in \mathfrak{T}(\mathcal{G})$, we have $t \preceq t'$ if and only if at least one of the following assertions holds:*

- (i) $t = \mathbf{M} = t'$;
- (ii) $t = x_i = t'$ for an $x_i \in \mathbb{X}$;
- (iii) $t = t_1 t_2$, $t' = t'_1 t'_2$, $t_1 \preceq t'_1$, and $t_2 \preceq t'_2$ where $t_1, t'_1, t_2, t'_2 \in \mathfrak{T}(\mathcal{G})$;
- (iv) $t = \mathbf{M} t_2$, $t' = t'_1 t'_2$, $t_2 \preceq t'_1$, and $t_2 \preceq t'_2$ where $t'_1, t_2, t'_2 \in \mathfrak{T}(\mathcal{G})$.

Proof. This is a direct consequence of Lemma 2.1.1 and of the fact that \preceq is the reflexive and transitive closure of \Rightarrow . \square

Proposition 2.1.4. *Let $t \in \mathfrak{T}(\mathcal{G})$.*

- (i) *The term t is a maximal element of \mathcal{P} if and only if t avoids $\mathbf{M}(x_1 x_2)$ and suffix-avoids $\mathbf{M} x_1$.*
- (ii) *The term t is a minimal element of \mathcal{P} if and only if t avoids $(x_1 x_2)(x_1 x_2)$ and suffix-avoids $x_1 x_1$.*

Proof. The term t is maximal in \mathcal{P} if and only if $t \Rightarrow t'$ implies that $t' = t$. By Lemma 2.1.1, this is equivalent to the fact t is maximal in \mathcal{P} if and only if when t has an internal node having \mathbf{M} as left subtree, this internal node admits necessarily \mathbf{M} as right subtree. Therefore, (i) holds.

The term t is minimal in \mathcal{P} if and only if $t' \Rightarrow t$ implies that $t' = t$. By Lemma 2.1.1, this is equivalent to the fact t is minimal in \mathcal{P} if and only if when t has an internal node such that its two children are equal, these two children are necessarily equal to \mathbf{M} . Therefore, (ii) holds. \square

2.1.2. *Some models.* We begin this discussion about models of \mathcal{G} by a very simple observation: any idempotent monoid (\mathcal{M}, \cdot, e) is a model of \mathcal{G} . Indeed, by identifying \mathbf{M} with e (the unit of \mathcal{M}) and by identifying \star with \cdot , we have for any $x \in \mathcal{M}$, $\mathbf{M} \star x = e \cdot x = x = x \cdot x = x \star x$.

Besides, since \mathcal{G} is hierarchical and has, by Proposition 2.1.2, the poset property, this CLS admits the model \mathcal{M}^{\max} described in Section 1.2.3. More specifically, this model is such that \mathcal{M}^{\max} is the set of the maximal elements of \mathcal{P} , and, by Proposition 2.1.4, for any $t_1, t_2 \in \mathcal{M}^{\max}$,

$$t_1 \star t_2 = \begin{cases} t_2 t_2 & \text{if } t_1 = \mathbf{M}, \\ t_1 t_2 & \text{otherwise.} \end{cases} \quad (2.1.1)$$

Since by Proposition 2.1.2, \mathcal{G} is also rooted, \mathcal{G} admits the model \mathcal{M}^{\min} described in Section 1.2.3. This model is such that \mathcal{M}^{\min} is the set of the minimal elements of \mathcal{P} , and, by Proposition 2.1.4, for any $t_1, t_2 \in \mathcal{M}^{\min}$,

$$t_1 \star t_2 = \begin{cases} \mathbf{M} t_2 & \text{if } t_1 = t_2, \\ t_1 t_2 & \text{otherwise.} \end{cases} \quad (2.1.2)$$

By representing \mathcal{G} -terms directly as binary trees, the computations in \mathcal{M}^{\max} have a better time complexity than in \mathcal{M}^{\min} because to compute $t_1 \star t_2$ in \mathcal{M}^{\min} , we need to decide if t_1 and t_2 are equal. Nevertheless, in return, the term $t_1 \star t_2$ produced by a computation in \mathcal{M}^{\min} has a number of internal nodes equal as or smaller than the analogous computation in \mathcal{M}^{\max} .

2.2. **Duplicative lattices.** We introduce here duplicative forests and a partial order relation on these objects. We show that this partial relation endow all intervals of this poset with lattice structures.

2.2.1. *Duplicative forests.* A *duplicative tree* is a planar rooted tree such that each internal node is either a *black node* \bullet or a *white node* \circ . A *duplicative forest* is a word $f = f(1) \dots f(\ell)$, $\ell \geq 0$, of duplicative trees. In particular, the empty word ϵ is the *empty forest*. We denote by \mathcal{D} (resp. \mathcal{D}^*) the set of such trees (resp. forests). The *height* $\text{ht}(f)$ of f is the number of internal nodes in a longest path following edges connecting a node to one of its child. In the sequel, we shall write expressions using some occurrences of \square : each such expression denotes the two expressions obtained by replacing simultaneously all \square either by \circ or by \bullet . The *grafting product* is the unary operation \square on \mathcal{D}^* such that for any $f \in \mathcal{D}^*$, $\square(f)$ is the duplicative tree obtained by grafting the roots of the duplicative trees of f on a common root node \square . The *concatenation product* is the binary operation \cdot on \mathcal{D}^* such that for any $f_1, f_2 \in \mathcal{D}^*$, $f_1 \cdot f_2$ is the duplicative forest made of the trees of f_1 and then of the trees of f_2 .

2.2.2. A poset on duplicative forests. Let \Rightarrow be the binary relation on \mathcal{D}^* defined recursively as follows. For any $f, f' \in \mathcal{D}^*$, we have $f \Rightarrow f'$ if f and f' have the same length $\ell \geq 1$ and

- either $\ell = 1$, $f = \circ(g)$ and $f' = \bullet(g \cdot g)$ where $g \in \mathcal{D}^*$;
- or $\ell = 1$, $f = \square(g)$, $f' = \square(g')$ where $g, g' \in \mathcal{D}^*$ and $g \Rightarrow g'$;
- or $\ell \geq 2$ and there is a $j \in [\ell]$ such that $f(j) \Rightarrow f'(j)$, and for all $i \in [\ell] \setminus \{j\}$, $f(i) = f'(i)$.

We observe that $f \Rightarrow f'$ if and only if f' can be obtained from f by selecting a white node of f , by turning it into black, and by duplicating its sequence of descendants. For instance, we have



Observe also that in this case, there are more black nodes in f' than in f . Hence, the reflexive and transitive closure \ll of \Rightarrow is antisymmetric so that (\mathcal{D}^*, \ll) is a poset. We call this poset the *duplicative forest poset*. For any $f \in \mathcal{D}^*$, we denote by $\mathcal{D}^*(f)$ the subposet of the duplicative forest poset on the set $\{f' \in \mathcal{D}^* : f \ll f'\}$. Figure 3 shows the Hasse diagram of the poset $\mathcal{D}^*(f)$ where f is a certain duplicative forest. According to this Hasse diagram, the duplicative forest

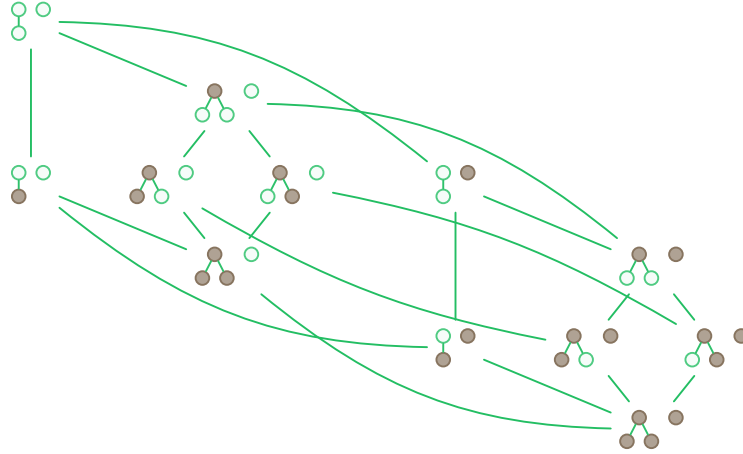


FIGURE 3. The Hasse diagram of a maximal interval of the duplicative forest poset.

poset is not graded. We have the following recursive description of \ll .

Lemma 2.2.1. For any two duplicative forests f and f' , we have $f \ll f'$ if and only if f and f' have the same length $\ell \geq 0$ and one of the following assertions holds:

- (i) $\ell = 0$;
- (ii) $\ell = 1$, $f = \circ(g)$ and $f' = \bullet(g' \cdot g'')$ where $g, g', g'' \in \mathcal{D}^*$, $g \ll g'$, and $g \ll g''$;
- (iii) $\ell = 1$, $f = \square(g)$, $f' = \square(g')$ where $g, g' \in \mathcal{D}^*$ and $g \ll g'$;
- (iv) $\ell \geq 2$ and $f(i) \ll f'(i)$ for all $i \in [\ell]$.

Proof. This is a direct consequence of the definitions of \Rightarrow and of \ll . □

A first consequence of Lemma 2.2.1 is that for any $f, f' \in \mathcal{D}^*$, $f \ll f'$ implies $\text{ht}(f) = \text{ht}(f')$.

2.2.3. *Lattices on duplicative forests.* We use in the sequel the usual notions and notations about lattices (see for instance [Sta11a]). Let \wedge and \vee be the two binary, commutative, and associative partial operations on \mathcal{D}^* defined recursively, for any $\ell \geq 0$, $f_1, \dots, f_\ell \in \mathcal{D}$, $f'_1, \dots, f'_\ell \in \mathcal{D}$, and $f, f', f'' \in \mathcal{D}^*$, by

$$f_1 \dots f_\ell \wedge f'_1 \dots f'_\ell := (f_1 \wedge f'_1) \dots (f_\ell \wedge f'_\ell), \quad (2.2.2a)$$

$$\square(f) \wedge \square(f') := \square(f \wedge f'), \quad (2.2.2b)$$

$$\circ(f) \wedge \bullet(f' \cdot f'') := \circ(f \wedge f' \wedge f'') \quad (2.2.2c)$$

and

$$f_1 \dots f_\ell \vee f'_1 \dots f'_\ell := (f_1 \vee f'_1) \dots (f_\ell \vee f'_\ell), \quad (2.2.3a)$$

$$\square(f) \vee \square(f') := \square(f \vee f'), \quad (2.2.3b)$$

$$\circ(f) \vee \bullet(f' \cdot f'') := \bullet((f \vee f') \cdot (f \vee f'')). \quad (2.2.3c)$$

Proposition 2.2.2. *Given a duplicative forest f , the poset $\mathcal{D}^*(f)$ is a lattice for the operations \wedge and \vee .*

Proof. This follows by structural induction on f by establishing the fact that for any $f', f'' \in \mathcal{D}^*(f)$, the duplicative forest $f' \wedge f''$ (resp. $f' \vee f''$) is well-defined and is the meet (resp. the join) of f' and f'' . This uses Lemma 2.2.1 in order to describe f' and f'' from f knowing that $f \ll f'$ and $f \ll f''$. \square

By Proposition 2.2.2, for any $f \in \mathcal{D}^*$, each poset $\mathcal{D}^*(f)$ is a lattice. We call it the *duplicative forest lattice* of f .

Let $\text{pr} : \mathcal{D}^* \rightarrow \mathcal{D}^*$ be the map defined recursively, for any $\ell \geq 0$, $f_1, \dots, f_\ell \in \mathcal{D}^*$, and $f \in \mathcal{D}^*$, by

$$\text{pr}(f_1 \dots f_\ell) := \text{pr}(f_1) \dots \text{pr}(f_\ell), \quad (2.2.4a)$$

$$\text{pr}(\circ(f)) := \circ(\text{pr}(f)), \quad (2.2.4b)$$

$$\text{pr}(\bullet(f)) := \text{pr}(f). \quad (2.2.4c)$$

The duplicative forest $\text{pr}(f)$ is the *pruning* of f . This forest has by construction no occurrence of any \bullet . For instance,



Lemma 2.2.3. *For any $f \in \mathcal{D}^*$, the posets $\mathcal{D}^*(f)$ and $\mathcal{D}^*(\text{pr}(f))$ are isomorphic.*

Proof. This follows by structural induction on f by establishing the fact that pr is a one-to-one correspondence between the sets $\mathcal{D}^*(f)$ and $\mathcal{D}^*(\text{pr}(f))$ and that pr is an order isomorphism. This uses Lemma 2.2.1 in order to have a recursive description of the order relation \ll . \square

For any $d \geq 0$, the *d-ladder* is the duplicative forest l_d defined recursively by $l_0 := \epsilon$ and, for any $d \geq 1$, by $l_d := \circ(l_{d-1})$. Let also

$$\mathcal{L} := \bigcup_{d \geq 0} \mathcal{D}^*(l_d). \quad (2.2.6)$$

By definition, \mathcal{L} is the set of the duplicative forests that are equal as or greater than a d -ladder for a $d \geq 0$.

Lemma 2.2.4. *For any $f \in \mathcal{D}^*$, there exists $g \in \mathcal{L}$ such that $pr(f) = pr(g)$.*

Proof. This follows by structural induction on f by showing that there is a $d \geq 0$ such that $l_d \ll g$ and $pr(f) = pr(g)$. This uses Lemma 2.2.1. \square

An important consequence of Lemmas 2.2.3 and 2.2.4 is that for any $f \in \mathcal{D}^*$, there exists $d \geq 0$ such that $\mathcal{D}^*(f)$ is isomorphic as a poset to a maximal interval of $\mathcal{D}^*(l_d)$.

2.3. Mockingbird lattices. We show here that each subposet $\mathcal{P}(t)$, $t \in \mathfrak{T}(\mathcal{G})$, of \mathcal{P} is a lattice. This is based on a poset isomorphism between $\mathcal{P}(t)$ and an interval of a lattice of duplicative forests. We also define for each $d \geq 0$ the Mockingbird lattice of order d as a particular maximal interval of \mathcal{P} .

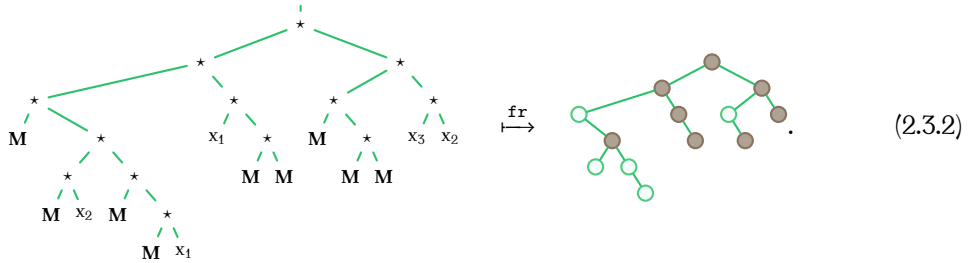
2.3.1. From terms to duplicative forests. Let $fr : \mathfrak{T}(\mathcal{G}) \rightarrow \mathcal{D}^*$ be the map defined recursively, for any $x_i \in \mathbb{X}$ and $t, t' \in \mathfrak{T}(\mathcal{G})$, by

$$fr(x_i) := \epsilon, \quad (2.3.1a)$$

$$fr(\mathbf{M}) := \epsilon, \quad (2.3.1b)$$

$$fr(tt') := \begin{cases} \circ(fr(t')) & \text{if } t = \mathbf{M} \text{ and } t' \neq \mathbf{M}, \\ \bullet(fr(t) \cdot fr(t')) & \text{otherwise.} \end{cases} \quad (2.3.1c)$$

For instance,



Intuitively, $fr(t)$ is the duplicative forest obtained from t by the following process: replace each internal node of t having \mathbf{M} as left child and having a right child different from \mathbf{M} by a \circ , replace each other internal node of t by a \bullet , and remove all the leaves. Immediately from the definition, we observe that this map is not injective. Moreover, again directly from the definition, any duplicative forest f in the image of fr is such that each white node of f has no more than one child.

Lemma 2.3.1. *For any $t \in \mathfrak{T}(\mathcal{G})$, the restriction of the map fr on the domain $\mathcal{P}(t)$ is injective.*

Proof. This follows by structural induction on t by establishing the fact that for any $s, s' \in \mathcal{P}(t)$, $fr(s) = fr(s')$ implies $s = s'$. This uses in particular Lemma 2.1.3 in order to describe s and s' from t knowing that $t \preceq s$ and $t \preceq s'$. \square

Lemma 2.3.2. *Let $t \in \mathfrak{T}(\mathcal{G})$ and $s, s' \in \mathfrak{T}(t)$ such that $s \neq s'$. We have $s \Rightarrow s'$ if and only if $fr(s) \Rightarrow fr(s')$.*

Proof. This follows by structural induction on t . This uses in particular the definition of the binary relation \Rightarrow on \mathcal{D}^* , Lemma 2.1.3 in order to describe s and s' from t knowing that $t \preceq s$ and $t \preceq s'$, and Lemma 2.1.1 in order to describe s' from s knowing that $s \neq s'$ and $s \Rightarrow s'$. \square

Proposition 2.3.3. *For any $t \in \mathfrak{T}(\mathfrak{G})$, the posets $\mathcal{P}(t)$ and $\mathcal{D}^*(fr(t))$ are isomorphic.*

Proof. Let $D := \{fr(t) : t \in \mathcal{P}(t)\}$. By Lemma 2.3.1, the sets $\mathcal{P}(t)$ and D are in one-to-one correspondence. Moreover, by Lemma 2.3.2, we have that $D = \mathcal{D}^*(fr(t))$ and that fr is an order isomorphism between $\mathcal{P}(t)$ and $\mathcal{D}^*(fr(t))$. \square

Figure 4 shows the poset $\mathcal{P}(t)$ for a \mathfrak{G} -term t and the isomorphic poset $\mathcal{D}^*(fr(t))$.

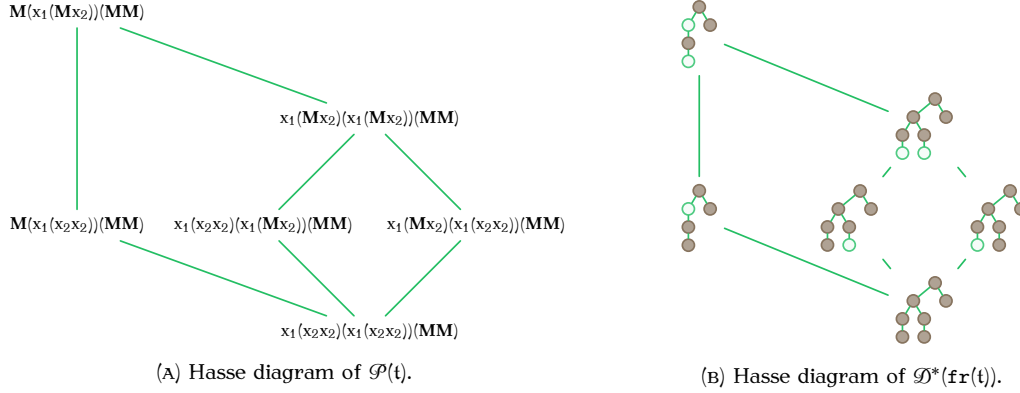


FIGURE 4. A maximal interval of the Mockingbird poset from the term $t := M(x_1(Mx_2))(MM)$ and its realization as a maximal interval in the duplicative forest poset.

Theorem 2.3.4. *For any $t \in \mathfrak{T}(\mathfrak{G})$, the poset $\mathcal{P}(t)$ is a finite lattice.*

Proof. By Proposition 2.3.3, $\mathcal{P}(t)$ is isomorphic as a poset to $\mathcal{D}^*(fr(t))$. Hence, and since by Proposition 2.2.2, $\mathcal{D}^*(fr(t))$ is a lattice, $\mathcal{P}(t)$ also is. The finiteness of $\mathcal{P}(t)$ is a consequence of the fact that by Proposition 2.1.2, \mathfrak{G} is locally finite. \square

The *Mockingbird lattice* of order $d \geq 0$ is the lattice $M(d) := \mathcal{P}(\tau_d)$ where τ_d is the term recursively defined by $\tau_0 := M$ and, for any $d \geq 1$, by $\tau_d := M\tau_{d-1}$. Figure 5 shows the Hasse diagrams of the first Mockingbird lattices.

Theorem 2.3.5. *For any $f \in \mathcal{D}^*$, the poset $\mathcal{D}^*(f)$ is isomorphic to a maximal interval of a Mockingbird lattice.*

Proof. By Lemmas 2.2.3 and 2.2.4, the poset $\mathcal{D}^*(f)$ is isomorphic to a maximal interval of $\mathcal{D}^*(l_d)$ for a $d \geq 0$. Since $pr(fr(\tau_{d+1})) = l_d$, and since by Proposition 2.3.3, the poset $\mathcal{P}(\tau_{d+1}) = M(d+1)$ is isomorphic to $\mathcal{D}^*(l_d)$, the poset $\mathcal{D}^*(f)$ is isomorphic to a maximal interval of $M(d+1)$. \square

Theorem 2.3.5 justifies the fact that the study of the Mockingbird lattices is universal enough because these lattices contain as maximal interval all duplicative forest lattices.

The Mockingbird lattices are not graded and not self-dual. Since they are not graded, they are not distributive neither. Moreover, they are not semidistributive. For instance, in $M(3)$, by setting $t_1 := M(MM(MM))$, $t_2 := MM(MM)(M(MM))$, and $t_3 := M(MM)(MM(MM))$, we have $t_1 \wedge t_2 = t_1 \wedge t_3$ but $t_1 \wedge (t_2 \vee t_3) \neq t_1 \wedge t_2$, contradicting one of the required relations to have this property.

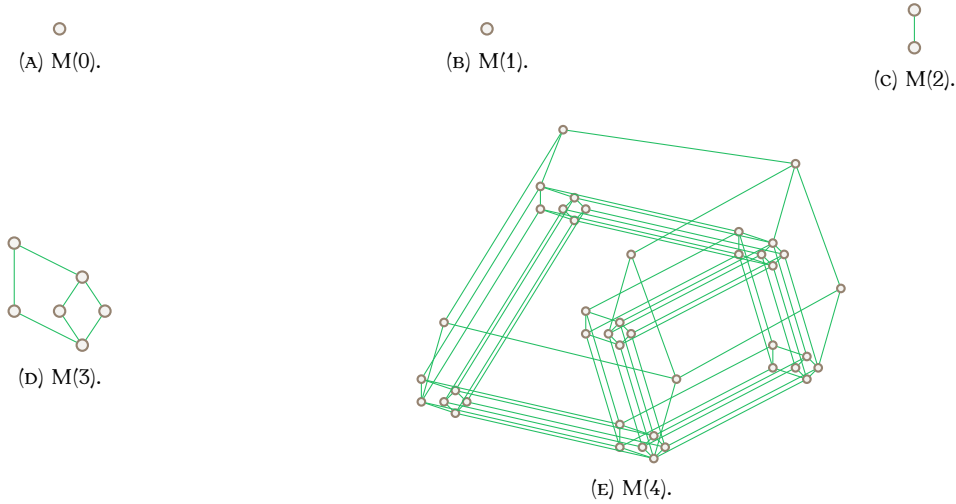


FIGURE 5. The Hasse diagrams of the Mockingbird lattices $M(d)$ for $d \in \llbracket 4 \rrbracket$.

3. ENUMERATIVE PROPERTIES

In this last central part of this work, we present some enumerative results concerning the Mockingbird poset and lattices $M(d)$, $d \geq 0$. We enumerate the maximal and minimal elements of the Mockingbird poset by degree and by height, the length of the shortest and longest saturated chains in $M(d)$, the number of elements in $M(d)$, the number of edges in the Hasse diagram of $M(d)$, and the number of intervals of $M(d)$. All this use the lattice isomorphism between the Mockingbird lattices and the duplicative forest lattices introduced in the previous sections and formal series on terms and on duplicative forests.

3.1. Formal power series. We set here some notions about formal power series and generating series. For the rest of the text, \mathbb{K} is any field of characteristic zero (as \mathbb{Q} for instance).

3.1.1. Formal power series over sets. For any set X , let $\mathbb{K}\langle X \rangle$ be the linear span of X . The dual space of $\mathbb{K}\langle X \rangle$ is denoted by $\mathbb{K}\langle\langle X \rangle\rangle$ and is by definition the space of the maps $\mathbf{F} : X \rightarrow \mathbb{K}$, called *X-series*. The coefficient $\mathbf{F}(x)$ of any $x \in X$ is denoted by $\langle x, \mathbf{F} \rangle$. The *support* of \mathbf{F} is the set $\text{Supp}(\mathbf{F}) := \{x \in X : \langle x, \mathbf{F} \rangle \neq 0\}$. The *characteristic series* of any subset X' of X is the series $\mathbf{c}(X')$ having X' as support and such that the coefficient of each $x \in X'$ is 1. For any $k \geq 0$, $\mathbb{T}^k \mathbb{K}\langle\langle X \rangle\rangle$ is the k -th tensor power of $\mathbb{K}\langle\langle X \rangle\rangle$. Elements of this space are possibly infinite linear combinations of tensors $x_1 \otimes \cdots \otimes x_k$, where for any $i \in [k]$, $x_i \in X$. The *tensor algebra* of $\mathbb{K}\langle\langle X \rangle\rangle$ is the space

$$\mathbb{T}^* \mathbb{K}\langle\langle X \rangle\rangle := \bigoplus_{k \geq 0} \mathbb{T}^k \mathbb{K}\langle\langle X \rangle\rangle. \tag{3.1.1}$$

This space is endowed with the tensor product \otimes so that $(\mathbb{T}^* \mathbb{K}\langle\langle X \rangle\rangle, \otimes)$ is a unital associative algebra, admitting $1 \in \mathbb{K}$ as unit.

A linear map $\phi : \mathbb{T}^{k_1} \mathbb{K}\langle\langle X \rangle\rangle \rightarrow \mathbb{T}^{k_2} \mathbb{K}\langle\langle X \rangle\rangle$, $k_1, k_2 \geq 0$, is a (k_1, k_2) -operation on $\mathbb{K}\langle\langle X \rangle\rangle$. To lighten the notation, when ϕ is a $(2, 1)$ -operation on $\mathbb{K}\langle\langle X \rangle\rangle$, we shall sometimes use ϕ as an infix operation by writing $\mathbf{F}_1 \phi \mathbf{F}_2$ instead of $\phi(\mathbf{F}_1 \otimes \mathbf{F}_2)$ for any $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{K}\langle\langle X \rangle\rangle$. The *diagonal coproduct* is the $(1, 2)$ -operation Δ on $\mathbb{K}\langle\langle X \rangle\rangle$ satisfying $\Delta(x) = x \otimes x$ for any $x \in X$. When X

is endowed with an n -ary operation $\bullet : X^n \rightarrow X$, $n \geq 0$, for any $k \geq 0$, the k -linearization of \bullet is the (nk, k) -operation $\bullet^{\otimes k}$ on $\mathbb{K}\langle\langle X \rangle\rangle$ satisfying

$$\bullet^{\otimes k}(x_{1,1} \otimes \cdots \otimes x_{1,k} \otimes \cdots \otimes x_{n,1} \otimes \cdots \otimes x_{n,k}) = \bullet(x_{1,1}, \dots, x_{n,1}) \otimes \cdots \otimes \bullet(x_{k,1}, \dots, x_{n,k}) \quad (3.1.2)$$

for any $x_{i,j} \in X$, $i \in [n]$, and $j \in [k]$. To lighten the notation, we shall write \bullet^{\otimes} for $\bullet^{\otimes 1}$.

3.1.2. Generating series. The space of the usual power series on the formal parameter z is denoted by $\mathbb{K}\langle\langle z \rangle\rangle$. For any $F, F' \in \mathbb{K}\langle\langle z \rangle\rangle$, $F[z := F']$ is the series of $\mathbb{K}\langle\langle z \rangle\rangle$ obtained by substituting F' for z in F . The *Hadamard product* is the binary operation \boxtimes on $\mathbb{K}\langle\langle z \rangle\rangle$ defined linearly for any $n_1, n_2 \geq 0$ by

$$z^{n_1} \boxtimes z^{n_2} := [n_1 = n_2] z^{n_1}. \quad (3.1.3)$$

The *max product* is the binary operation \uparrow on $\mathbb{K}\langle\langle z \rangle\rangle$ defined linearly for any $n_1, n_2 \geq 0$ by $z^{n_1} \uparrow z^{n_2} := z^{\max\{n_1, n_2\}}$. Observe that for any $F \in \mathbb{K}\langle\langle z \rangle\rangle$ and $n \geq 0$,

$$\langle z^n, F \uparrow F \rangle = \langle z^n, F \rangle^2 + 2\langle z^n, F \rangle \sum_{i \in [n]} \langle z^{i-1}, F \rangle. \quad (3.1.4)$$

3.1.3. Formal power series and enumeration. If X is endowed with a map $\omega : X \rightarrow \mathbb{N}$, the *ω -enumeration map* is the partial map $\text{en}_\omega : \mathbb{T}^*\mathbb{K}\langle\langle X \rangle\rangle \rightarrow \mathbb{K}\langle\langle z \rangle\rangle$ defined linearly for any $k \geq 1$ and $x_1, \dots, x_k \in X$ by

$$\text{en}_\omega(x_1 \otimes \cdots \otimes x_k) := z^{\omega(x_1)} \uparrow \dots \uparrow z^{\omega(x_k)}. \quad (3.1.5)$$

For any $\mathbf{F} \in \mathbb{T}^*\mathbb{K}\langle\langle X \rangle\rangle$, the generating series $\text{en}_\omega(\mathbf{F})$ is the *ω -enumeration* of \mathbf{F} .

In the sequel, we shall use the following strategy to enumerate a set X w.r.t. such a map ω : we shall provide a description of $\mathbf{c}(X)$, then deduce a description of $\text{en}_\omega(\mathbf{c}(X))$, and finally deduce from this a formula to compute the coefficients $\langle z^n, \text{en}_\omega(\mathbf{c}(X)) \rangle$, $n \geq 0$.

3.2. Maximal, minimal, and isolated terms. We use here series on terms and some operations on these in order to enumerate the closed maximal, closed minimal, and closed isolated elements of the poset \mathcal{P} w.r.t. their degrees or their heights.

3.2.1. Series of terms and operations. Recall that \star is the binary operation on $\mathfrak{T}(\mathfrak{G})$ such that for any $t_1, t_2 \in \mathfrak{T}(\mathfrak{G})$, $t_1 \star t_2$ is the \mathfrak{G} -term $t_1 t_2$ having t_1 as left subtree and t_2 as right subtree. Observe that for any $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{K}\langle\langle \mathfrak{T}(\mathfrak{G}) \rangle\rangle$,

$$\text{en}_{\text{deg}}(\mathbf{F}_1 \star^{\otimes} \mathbf{F}_2) = z \text{en}_{\text{deg}}(\mathbf{F}_1) \text{en}_{\text{deg}}(\mathbf{F}_2) \quad (3.2.1)$$

and

$$\text{en}_{\text{ht}}(\mathbf{F}_1 \star^{\otimes} \mathbf{F}_2) = z (\text{en}_{\text{ht}}(\mathbf{F}_1) \uparrow \text{en}_{\text{ht}}(\mathbf{F}_2)). \quad (3.2.2)$$

Observe also that for any $\mathbf{F} \in \mathbb{K}\langle\langle \mathfrak{T}(\mathfrak{G}) \rangle\rangle$,

$$\text{en}_{\text{deg}}(\star^{\otimes}(\Delta(\mathbf{F}))) = z (\text{en}_{\text{deg}}(\mathbf{F})[z := z^2]) \quad (3.2.3)$$

and

$$\text{en}_{\text{ht}}(\star^{\otimes}(\Delta(\mathbf{F}))) = z \text{en}_{\text{ht}}(\mathbf{F}). \quad (3.2.4)$$

3.2.2. *Three series of terms.* In order to achieve the objectives described above, we begin by providing equations satisfied by the characteristic series F_{\max} of the closed maximal terms of \mathcal{P} , the characteristic series F_{\min} of the closed minimal terms of \mathcal{P} , and the characteristic series F_{iso} of the closed isolated terms of \mathcal{P} .

Proposition 3.2.1. *The characteristic series F_{\max} satisfies*

$$F_{\max} = \mathbf{M} + \mathbf{M}\mathbf{M} + F_{\max} \star^{\otimes} F_{\max} - \mathbf{M} \star^{\otimes} F_{\max}. \quad (3.2.5)$$

Proof. By using the description of Proposition 2.1.4 for the set S of closed maximal elements of \mathcal{P} , we have

$$F_{\max} = \mathbf{M} + \sum_{s_1, s_2 \in S} s_1 s_2 - \sum_{s_3 \in S \setminus \{\mathbf{M}\}} \mathbf{M} s_3 = \mathbf{M} + F_{\max} \star^{\otimes} F_{\max} - \mathbf{M} \star^{\otimes} (F_{\max} - \mathbf{M}). \quad (3.2.6)$$

This leads to (3.2.5). □

Proposition 3.2.2. *The characteristic series F_{\min} satisfies*

$$F_{\min} = \mathbf{M} + \mathbf{M}\mathbf{M} + F_{\min} \star^{\otimes} F_{\min} - \star^{\otimes} (\Delta(F_{\min})). \quad (3.2.7)$$

Proof. By using the description of Proposition 2.1.4 for the set S of closed minimal elements of \mathcal{P} , we have

$$F_{\min} = \mathbf{M} + \sum_{s_1, s_2 \in S} s_1 s_2 - \sum_{s_3 \in S \setminus \{\mathbf{M}\}} s_3 s_3 = \mathbf{M} + F_{\min} \star^{\otimes} F_{\min} - \star^{\otimes} (\Delta(F_{\min} - \mathbf{M})). \quad (3.2.8)$$

This leads to (3.2.7). □

Proposition 3.2.3. *The characteristic series F_{iso} satisfies*

$$F_{\text{iso}} = \mathbf{M} + 2\mathbf{M}\mathbf{M} + F_{\text{iso}} \star^{\otimes} F_{\text{iso}} - \mathbf{M} \star^{\otimes} F_{\text{iso}} - \star^{\otimes} (\Delta(F_{\text{iso}})). \quad (3.2.9)$$

Proof. By using the description of Proposition 2.1.4 for the set S_1 (resp. S_2) of the closed maximal (resp. minimal) elements of \mathcal{P} , the set of the closed isolated elements of \mathcal{P} is the set $S = S_1 \cap S_2$ and we have

$$\begin{aligned} F_{\text{iso}} &= \mathbf{M} + \sum_{s_1, s_2 \in S} s_1 s_2 - \sum_{s_3 \in S \setminus \{\mathbf{M}\}} \mathbf{M} s_3 - \sum_{s_4 \in S \setminus \{\mathbf{M}\}} s_4 s_4 \\ &= \mathbf{M} + F_{\text{iso}} \star^{\otimes} F_{\text{iso}} - \mathbf{M} \star^{\otimes} (F_{\text{iso}} - \mathbf{M}) - \star^{\otimes} (\Delta(F_{\text{iso}} - \mathbf{M})). \end{aligned} \quad (3.2.10)$$

This leads to (3.2.9). □

3.2.3. *deg-enumerations.* A consequence of Proposition 3.2.1 and of (3.2.1) is that the deg-enumeration D_{\max} of F_{\max} , enumerating the closed maximal elements of \mathcal{P} w.r.t. their degrees, satisfies

$$D_{\max} = 1 + z + zD_{\max}^2 - zD_{\max}. \quad (3.2.11)$$

We deduce from this that the number of these terms of degree $d \geq 0$ is $\mathbf{d}_{\max}(d)$ where \mathbf{d}_{\max} is the integer sequence satisfying $\mathbf{d}_{\max}(0) = \mathbf{d}_{\max}(1) = 1$ and, for any $d \geq 2$,

$$\mathbf{d}_{\max}(d) = \sum_{i \in [d-2]} \mathbf{d}_{\max}(i) \mathbf{d}_{\max}(d-1-i). \quad (3.2.12)$$

The first numbers are

$$1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \quad (3.2.13)$$

form Sequence **A001006** of [Slo], and are Motzkin numbers.

Besides, a consequence of Proposition 3.2.2, (3.2.1), and (3.2.3) is that the deg-enumeration D_{\min} of F_{\min} , enumerating the closed minimal elements of \mathcal{P} w.r.t. their degrees, satisfies

$$D_{\min} = 1 + z + zD_{\min}^2 - z(D_{\min}[z := z^2]). \quad (3.2.14)$$

We deduce from this that the number of these terms of degree $d \geq 0$ is $\mathbf{d}_{\min}(d)$ where \mathbf{d}_{\min} is the integer sequence satisfying $\mathbf{d}_{\min}(0) = \mathbf{d}_{\min}(1) = 1$ and, for any $d \geq 2$,

$$\mathbf{d}_{\min}(d) = \sum_{i \in \llbracket d-1 \rrbracket} \mathbf{d}_{\min}(i) \mathbf{d}_{\min}(d-1-i) - [d \text{ is odd}] \mathbf{d}_{\min}\left(\frac{d-1}{2}\right). \quad (3.2.15)$$

The first numbers are

$$1, 1, 2, 4, 12, 34, 108, 344, 1136, 3796, 12920, 44442, 154596 \quad (3.2.16)$$

and form Sequence **A343663** of [Slo].

Besides, a consequence of Proposition 3.2.3, (3.2.1), and (3.2.3) is that the deg-enumeration D_{iso} of F_{iso} , enumerating the closed isolated elements of \mathcal{P} w.r.t. their degrees, satisfies

$$D_{\text{iso}} = 1 + 2z + zD_{\text{iso}}^2 - zD_{\text{iso}} - z(D_{\text{iso}}[z := z^2]). \quad (3.2.17)$$

We deduce from this that the number of these terms of degree $d \geq 0$ is $\mathbf{d}_{\text{iso}}(d)$ where \mathbf{d}_{iso} is the integer sequence satisfying $\mathbf{d}_{\text{iso}}(0) = \mathbf{d}_{\text{iso}}(1) = 1$ and, for any $d \geq 2$,

$$\mathbf{d}_{\text{iso}}(d) = \sum_{i \in \llbracket d-2 \rrbracket} \mathbf{d}_{\text{iso}}(i) \mathbf{d}_{\text{iso}}(d-1-i) - [d \text{ is odd}] \mathbf{d}_{\text{iso}}\left(\frac{d-1}{2}\right). \quad (3.2.18)$$

The first numbers are

$$1, 1, 1, 1, 3, 5, 13, 29, 71, 171, 427, 1067, 2709. \quad (3.2.19)$$

This sequence does not appear in [Slo] for the time being.

3.2.4. ht-enumerations. Due to the fact that \mathcal{G} is hierarchical and, by Proposition 2.1.2, \mathcal{P} is rooted, Proposition 1.2.3 implies that \mathcal{P} satisfies the properties exposed at the very end of Section 1.2.2. In particular, in each \equiv -equivalence class of closed terms, there is exactly one closed maximal term, exactly one closed minimal term, and these terms have the same height. For this reason, the ht-enumerations of F_{\max} and F_{\min} are equal and enumerate also the \equiv -equivalence classes of closed terms w.r.t. the heights of their elements. By denoting by H_{mm} this generating series, by Propositions 3.2.1 and 3.2.2, H_{mm} satisfies

$$H_{\text{mm}} = 1 + z + z(H_{\text{mm}} \uparrow H_{\text{mm}}) - zH_{\text{mm}}. \quad (3.2.20)$$

We deduce from this and (3.1.4) that the number of these \equiv -equivalence classes of terms of height $h \geq 0$ is $\mathbf{h}_{\text{mm}}(h)$ where \mathbf{h}_{mm} is the integer sequence satisfying $\mathbf{h}_{\text{mm}}(0) = \mathbf{h}_{\text{mm}}(1) = 1$ and, for any $h \geq 2$,

$$\mathbf{h}_{\text{mm}}(h) = \mathbf{h}_{\text{mm}}(h-1)^2 - \mathbf{h}_{\text{mm}}(h-1) + 2\mathbf{h}_{\text{mm}}(h-1) \sum_{i \in \llbracket h-1 \rrbracket} \mathbf{h}_{\text{mm}}(i-1). \quad (3.2.21)$$

The first numbers are

$$1, 1, 2, 10, 170, 33490, 1133870930, 1285739648704587610 \quad (3.2.22)$$

and form Sequence **A063573** of [Slo].

Besides, a consequence of Proposition 3.2.3, (3.2.2), and (3.2.4) is that the ht-enumeration H_{iso} of F_{iso} , enumerating the closed isolated elements of \mathcal{P} w.r.t. their heights, satisfies

$$H_{\text{iso}} = 1 + 2z + z(H_{\text{iso}} \uparrow H_{\text{iso}}) - 2zH_{\text{iso}}. \tag{3.2.23}$$

We deduce from this and (3.1.4) that the number of these terms of height $h \geq 0$ is $\mathbf{h}_{\text{iso}}(h)$ where \mathbf{h}_{iso} is the integer sequence satisfying $\mathbf{h}_{\text{iso}}(0) = \mathbf{h}_{\text{iso}}(1) = 1$ and, for any $h \geq 2$,

$$\mathbf{h}_{\text{iso}}(h) = \mathbf{h}_{\text{iso}}(h-1)^2 - 2\mathbf{h}_{\text{iso}}(h-1) + 2\mathbf{h}_{\text{iso}}(h-1) \sum_{i \in [h-1]} \mathbf{h}_{\text{iso}}(i-1). \tag{3.2.24}$$

The first numbers are

$$1, 1, 1, 3, 21, 651, 457653, 210065930571 \tag{3.2.25}$$

and form Sequence **A001699** of [Slo], which enumerates binary trees w.r.t. the height (but with a shift). Therefore, there is a one-to-one correspondence between the set of the closed isolated elements of \mathcal{P} of height $h \geq 1$ and the set of the binary trees of height $h - 1$.

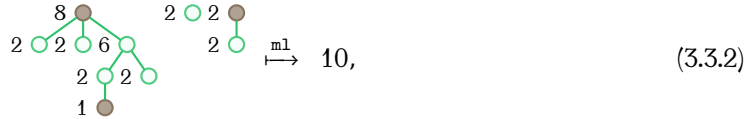
3.3. Shortest and longest saturated chains. Let $\text{ml} : \mathcal{D}^* \rightarrow \mathbb{N}$ be the statistics defined for any $\ell \geq 0$, $f_1, \dots, f_\ell \in \mathcal{D}^*$, and $f \in \mathcal{D}^*$ by

$$\text{ml}(f_1 \dots f_\ell) := 1 - \ell + \sum_{i \in [\ell]} \text{ml}(f_i), \tag{3.3.1a}$$

$$\text{ml}(\bullet(f)) := \text{ml}(f), \tag{3.3.1b}$$

$$\text{ml}(\circ(f)) := 2\text{ml}(f). \tag{3.3.1c}$$

For instance, we have



where the integer decoration of each node refers to the image by ml of the duplicative tree rooted at this node.

Proposition 3.3.1. For any $t \in \mathcal{T}_0(\mathcal{G})$,

- (i) a shortest saturated chain from t to the maximal element of $\mathcal{P}(t)$ has as length the number of \circ in $\text{fr}(t)$;
- (ii) a longest saturated chain from t to the maximal element of $\mathcal{P}(t)$ has length $\text{ml}(\text{fr}(t))$.

Proof. Let $f := \text{fr}(t)$. By Proposition 2.3.3, the posets $\mathcal{P}(t)$ and $\mathcal{D}^*(f)$ are isomorphic. Therefore, to prove the statement, we consider the lengths of the shortest and longest saturated chains in $\mathcal{D}^*(f)$ from f to g where g is the maximal element of $\mathcal{D}^*(f)$.

First, immediately from the definition of \ll , a shortest saturated chain from f to g consists in selecting at each step a white node that has no white nodes as descendants and turn it into black. This establishes (i).

Let us prove (ii) by structural induction on f . We use there Lemma 2.2.1 in order to exhibit a longest saturated chain from f to g . If $f = f_1 \dots f_\ell$, $\ell \geq 0$, a longest saturated chain from f passes through the duplicative forest $f'_1 \cdot f'_2 \dots f'_\ell$ and then passes through $f'_1 \cdot f'_2 \dots f'_\ell$, ..., and ends at $f'_1 \cdot f'_2 \dots f'_\ell = g$, where for any $i \in [\ell]$, f'_i is the greatest element of $\mathcal{D}^*(f_i)$. By induction hypothesis, $\text{ml}(f_i)$ is the length of a longest saturated chain from f_i to f'_i . We deduce that the length of the former chain from f to g is $1 + \sum_{i \in [\ell]} (\text{ml}(f_i) - 1)$. Therefore,

(3.3.1a) is consistent. If $f = \bullet(f')$, $f' \in \mathcal{D}^*$, a longest saturated chain from f to g has the same length as a longest saturated chain from f' to g' where g' is the greatest element of $\mathcal{D}^*(f')$. Hence, by induction hypothesis, $\text{ml}(f)$ is consistent with (3.3.1b). If $f = \circ(f')$, $f' \in \mathcal{D}^*$, a longest saturated chain from f to g passes through the covering $\bullet(f' \cdot f')$ of f . Therefore, by induction hypothesis and by (3.3.1b) and (3.3.1a), a longest saturated chain from f to g has length $1 + \text{ml}(\bullet(f' \cdot f')) = 1 + 1 - 2 + 2\text{ml}(f') = 2\text{ml}(f')$, which is consistent with (3.3.1c). \square

By Proposition 3.3.1, for any $d \geq 1$, in $M(d)$, shortest saturated chains are of length d and longest saturated chains are of length 2^{d-1} .

3.4. Elements, covering pairs, and intervals. We use here series on duplicative forests and a palette of operations on these to enumerate the elements, covering pairs, and intervals of the Mockingbird lattices. We obtain for these three recursive formulas leading to expressions for the generating series of these numbers.

3.4.1. Series of duplicative forests and operations. We shall consider in the sequel the k -linearization $\cdot^{\otimes k}$, $k \geq 0$, of the concatenation product \cdot of duplicative forests. Observe that for any $F_1, F_2 \in \mathbb{K}\langle\langle\mathcal{D}^*\rangle\rangle$, we have in particular

$$\text{en}_{\text{ht}}(F_1 \cdot^{\otimes k} F_2) = \text{en}_{\text{ht}}(F_1) \uparrow \text{en}_{\text{ht}}(F_2). \tag{3.4.1}$$

We shall also consider the k -linearization $\square^{\otimes k}$, $k \geq 0$ of the grafting product \square . Observe that for any $F \in \mathbb{K}\langle\langle\mathcal{D}^*\rangle\rangle$, we have in particular

$$\text{en}_{\text{ht}}(\square^{\otimes k}(F)) = z \text{en}_{\text{ht}}(F). \tag{3.4.2}$$

For any $k \geq 1$ and $u \in \{\circ, \bullet\}^k$, the *merging product* is the $(k + |u|_{\bullet}, k)$ -operation mg_u on $\mathbb{K}\langle\langle\mathcal{D}^*\rangle\rangle$ satisfying, for any $f_1, \dots, f_{k+|u|_{\bullet}} \in \mathcal{D}^*$,

$$\text{mg}_{\circ}(f_1) = \circ(f_1), \tag{3.4.3a}$$

$$\text{mg}_{\circ u'}(f_1 \otimes \dots \otimes f_k) = \text{mg}_{\circ}(f_1) \otimes \text{mg}_{u'}(f_2 \otimes \dots \otimes f_k), \tag{3.4.3b}$$

$$\text{mg}_{\bullet}(f_1 \otimes f_2) = \bullet(f_1 \cdot f_2), \tag{3.4.3c}$$

$$\text{mg}_{\bullet u'}(f_1 \otimes \dots \otimes f_k) = \text{mg}_{\bullet}(f_1 \otimes f_2) \otimes \text{mg}_{u'}(f_3 \otimes \dots \otimes f_k), \tag{3.4.3d}$$

where $u' \in \{\circ, \bullet\}^*$. For instance, with $k = 3$,

$$\text{mg}_{\circ \bullet \bullet}(\bullet \otimes \circ \otimes \circ \otimes \circ \otimes \circ) = \bullet \otimes \circ \otimes \circ. \tag{3.4.4}$$

Intuitively, this product consists in grafting a \circ onto one forest or a \bullet onto a forest and its right neighbor following the letters of u . Observe that for any $u \in \{\circ, \bullet\}^*$ and $F \in \mathbb{T}^{k+|u|_{\bullet}} \mathbb{K}\langle\langle\mathcal{D}^*\rangle\rangle$, we have

$$\text{en}_{\text{ht}}(\text{mg}_u(F)) = z \text{en}_{\text{ht}}(F). \tag{3.4.5}$$

Let us finally define the *series of ladders* as the \mathcal{D}^* -series

$$\text{Id} := \sum_{h \geq 0} l_h = \epsilon + \circ + \circ \otimes \circ + \circ \otimes \circ \otimes \circ + \dots. \tag{3.4.6}$$

Observe that

$$\text{Id} = \epsilon + \circ^{\otimes}(\text{Id}). \tag{3.4.7}$$

3.4.2. *Number of elements.* Let \mathbf{gr} be the $(1, 1)$ -operation on $\mathbb{K}\langle\langle\mathcal{D}^*\rangle\rangle$ satisfying, for any $f \in \mathcal{D}^*$,

$$\mathbf{gr}(f) = \sum_{f' \in \mathcal{D}^*(f)} f'. \quad (3.4.8)$$

By definition, $\mathbf{gr}(f)$ is the characteristic series of $\mathcal{D}^*(f)$ and is therefore also the formal sum of all duplicative forests f' such that $f \ll f'$. For instance,

$$\mathbf{gr}\left(\begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array}\right) = \begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array} + \begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array} + \begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array} + \begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array} + \begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array} + \begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array} + \begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array} + \begin{array}{c} \circ \\ \circ \circ \\ \circ \circ \circ \end{array}. \quad (3.4.9)$$

Proposition 3.4.1. *For any $\ell \geq 0$, $f_1, \dots, f_\ell \in \mathcal{D}^*$, and $f \in \mathcal{D}^*$,*

$$\mathbf{gr}(f_1 \dots f_\ell) = \mathbf{gr}(f_1) \cdot^\otimes \dots \cdot^\otimes \mathbf{gr}(f_\ell), \quad (3.4.10a)$$

$$\mathbf{gr}(\bullet(f)) = \bullet^\otimes(\mathbf{gr}(f)), \quad (3.4.10b)$$

$$\mathbf{gr}(\circ(f)) = \circ^\otimes(\mathbf{gr}(f)) + \bullet^\otimes(\mathbf{gr}(f \cdot f)). \quad (3.4.10c)$$

Proof. This follows by structural induction on duplicative forests and is a consequence of Lemma 2.2.1 describing a necessary and sufficient condition for the fact that a duplicative forest f' belong to $\mathcal{D}^*(f)$ and the definitions of the operations \cdot^\otimes , \circ^\otimes , and \bullet^\otimes on \mathcal{D}^* -series. \square

Observe that $\mathbf{gr}(\mathbf{ld})$ is the characteristic series of the set \mathcal{L} (defined in Section 2.2.3). Moreover, since for any $h \geq 0$, all elements of $\mathcal{D}^*(l_h)$ have h as height, we have

$$\text{en}_{\text{ht}}(\mathbf{gr}(\mathbf{ld})) = \sum_{h \geq 0} \#\mathcal{D}^*(l_h) z^h, \quad (3.4.11)$$

so that $\text{en}_{\text{ht}}(\mathbf{gr}(\mathbf{ld}))$ is the generating series of the cardinalities of the lattices $\mathcal{D}^*(l_h)$, enumerated w.r.t. $h \geq 0$.

Theorem 3.4.2. *The series $\mathbf{gr}(\mathbf{ld})$ satisfies*

$$\mathbf{gr}(\mathbf{ld}) = \epsilon + \circ^\otimes(\mathbf{gr}(\mathbf{ld})) + \bullet^\otimes(\mathbf{gr}(\bullet^\otimes(\Delta(\mathbf{ld}))). \quad (3.4.12)$$

Proof. By (3.4.7) and by Relation (3.4.10c) of Proposition 3.4.1, we have

$$\mathbf{gr}(\mathbf{ld}) = \mathbf{gr}(\epsilon + \circ^\otimes(\mathbf{ld})) = \epsilon + \sum_{h \geq 0} (\circ^\otimes(\mathbf{gr}(l_h)) + \bullet^\otimes(\mathbf{gr}(l_h \cdot l_h))) \quad (3.4.13)$$

and the relation of the statement follows. \square

Proposition 3.4.3. *The ht-enumeration H_{gr} of $\mathbf{gr}(\mathbf{ld})$ satisfies*

$$H_{\text{gr}} = 1 + zH_{\text{gr}} + z(H_{\text{gr}} \boxtimes H_{\text{gr}}). \quad (3.4.14)$$

Proof. By (3.4.1) and by Relation (3.4.10a) of Proposition 3.4.1, we have

$$\begin{aligned} \text{en}_{\text{ht}}(\mathbf{gr}(\bullet^\otimes(\Delta(\mathbf{ld})))) &= \sum_{h \geq 0} \text{en}_{\text{ht}}(\mathbf{gr}(l_h \cdot l_h)) \\ &= \sum_{h \geq 0} \text{en}_{\text{ht}}(\mathbf{gr}(l_h) \cdot^\otimes \mathbf{gr}(l_h)) \\ &= \sum_{h \geq 0} (\text{en}_{\text{ht}}(\mathbf{gr}(l_h)) \uparrow \text{en}_{\text{ht}}(\mathbf{gr}(l_h))). \end{aligned} \quad (3.4.15)$$

Since for any $h \geq 0$, all the duplicative forests appearing in $\mathbf{gr}(l_h)$ have h as height, the last member of (3.4.15) is equal to $H_{\text{gr}} \boxtimes H_{\text{gr}}$. Now, by using this identity together with (3.4.2) and Theorem 3.4.2, we obtain the stated expression for H_{gr} . \square

By Proposition 2.3.3, for any $d \geq 1$, the cardinality of $M(d)$ is $\mathbf{h}_{\text{gr}}(h) := \langle z^h, H_{\text{gr}} \rangle$ where $h = d - 1$. By Proposition 3.4.3, \mathbf{h}_{gr} is the integer sequence satisfying $\mathbf{h}_{\text{gr}}(0) = 1$ and, for any $h \geq 1$,

$$\mathbf{h}_{\text{gr}}(h) = \mathbf{h}_{\text{gr}}(h - 1) + \mathbf{h}_{\text{gr}}(h - 1)^2. \quad (3.4.16)$$

The sequence of the cardinalities of $M(d)$, $d \geq 0$, starts by

$$1, 1, 2, 6, 42, 1806, 3263442, 10650056950806, 113423713055421844361000442 \quad (3.4.17)$$

and forms Sequence **A007018** of [Slo].

3.4.3. *Number of covering pairs.* Let \mathbf{cv} be the $(1, 1)$ -operation on $\mathbb{K}\langle\langle \mathcal{D}^* \rangle\rangle$ satisfying, for any $f \in \mathcal{D}^*$,

$$\mathbf{cv}(f) = \sum_{\substack{f' \in \mathcal{D}^* \\ f \Rightarrow f'}} f'. \quad (3.4.18)$$

Immediately from the definition of the covering relation \Rightarrow of \ll , it follows that for any $\ell \geq 0$, $f_1, \dots, f_\ell \in \mathcal{D}^*$, and $f \in \mathcal{D}^*$,

$$\mathbf{cv}(f_1 \dots f_\ell) = \sum_{i \in [\ell]} f_1 \dots f_{i-1} \otimes \mathbf{cv}(f_i) \otimes f_{i+1} \dots f_\ell, \quad (3.4.19a)$$

$$\mathbf{cv}(\bullet(f)) = \bullet \otimes (\mathbf{cv}(f)), \quad (3.4.19b)$$

$$\mathbf{cv}(\circ(f)) = \circ \otimes (\mathbf{cv}(f)) + \bullet \otimes (f \cdot f). \quad (3.4.19c)$$

Let \mathbf{ni} be the $(1, 1)$ -operation on $\mathbb{K}\langle\langle \mathcal{D}^* \rangle\rangle$ satisfying, for any $f \in \mathcal{D}^*$,

$$\mathbf{ni}(f) = \mathbf{cv}(\mathbf{gr}(f)). \quad (3.4.20)$$

By a straightforward computation, we obtain

$$\mathbf{ni}(f) = \sum_{f' \in \mathcal{D}^*(f)} \#\{f'' \in \mathcal{D}^*(f) : f'' \Rightarrow f'\} f', \quad (3.4.21)$$

so that the coefficient of each $f' \in \mathcal{D}^*(f)$ in $\mathbf{ni}(f)$ is the number of duplicative forests admitting f' as covering in $\mathcal{D}^*(f)$. For instance (see at the same time Figure 3),

$$\mathbf{ni}\left(\begin{array}{c} \circ \circ \\ \circ \circ \end{array}\right) = \begin{array}{c} \circ \bullet \\ \bullet \circ \end{array} + \begin{array}{c} \circ \circ \\ \bullet \circ \end{array} + 2 \begin{array}{c} \bullet \circ \\ \bullet \circ \end{array} + \begin{array}{c} \bullet \circ \\ \circ \circ \end{array} + 2 \begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} + \begin{array}{c} \bullet \bullet \\ \bullet \circ \end{array} + \begin{array}{c} \bullet \circ \\ \bullet \bullet \end{array} + 2 \begin{array}{c} \bullet \bullet \\ \bullet \bullet \end{array} + 2 \begin{array}{c} \bullet \bullet \\ \circ \bullet \end{array} + 3 \begin{array}{c} \bullet \bullet \\ \bullet \bullet \end{array} + 4 \begin{array}{c} \bullet \bullet \\ \bullet \bullet \end{array}. \quad (3.4.22)$$

Proposition 3.4.4. For any $\ell \geq 0$, $f_1, \dots, f_\ell \in \mathcal{D}^*$, and $f \in \mathcal{D}^*$,

$$\mathbf{ni}(f_1 \dots f_\ell) = \sum_{i \in [\ell]} \mathbf{gr}(f_1 \dots f_{i-1}) \otimes \mathbf{ni}(f_i) \otimes \mathbf{gr}(f_{i+1} \dots f_\ell), \quad (3.4.23a)$$

$$\mathbf{ni}(\bullet(f)) = \bullet \otimes (\mathbf{ni}(f)), \quad (3.4.23b)$$

$$\mathbf{ni}(\circ(f)) = \circ \otimes (\mathbf{ni}(f)) + \bullet \otimes (\mathbf{ni}(f \cdot f)) + \bullet \otimes (\bullet \otimes (\Delta(\mathbf{gr}(f)))). \quad (3.4.23c)$$

Proof. Relations (3.4.23a) and (3.4.23b) are direct consequences of Relations (3.4.19a) and (3.4.19b), and of Proposition 3.4.1. By Proposition 3.4.1 and Relations (3.4.19b) and (3.4.19c), we have

$$\begin{aligned}
\mathbf{ni}(\circ(f)) &= \mathbf{cv}(\circ^{\otimes}(\mathbf{gr}(f)) + \bullet^{\otimes}(\mathbf{gr}(f \cdot f))) \\
&= \sum_{f' \in \mathcal{D}^*(f)} \mathbf{cv}(\circ(f')) + \bullet^{\otimes}(\mathbf{ni}(f \cdot f)) \\
&= \sum_{f' \in \mathcal{D}^*(f)} (\circ^{\otimes}(\mathbf{cv}(f')) + \bullet^{\otimes}(f' \cdot f')) + \bullet^{\otimes}(\mathbf{ni}(f \cdot f)).
\end{aligned} \tag{3.4.24}$$

This establishes (3.4.23c). \square

Observe that

$$\text{Supp}(\mathbf{ni}(\mathbf{ld})) = \mathcal{L} \setminus \{l_h : h \geq 0\} \tag{3.4.25}$$

and that the coefficient of each duplicative forest f of this set is the number of duplicative forests of this same set covered by f . Moreover, since for any $h \geq 0$, all elements of $\mathcal{D}^*(l_h)$ have h as height,

$$\text{en}_{\text{ht}}(\mathbf{ni}(\mathbf{ld})) = \sum_{h \geq 0} \sum_{f \in \mathcal{D}^*(l_h)} \#\{f' \in \mathcal{D}^*(l_h) : f' \Rightarrow f\} z^h, \tag{3.4.26}$$

so that $\text{en}_{\text{ht}}(\mathbf{ni}(\mathbf{ld}))$ is the generating series of the number of edges of the Hasse diagrams of the lattices $\mathcal{D}^*(l_h)$, enumerated w.r.t. $h \geq 0$.

Theorem 3.4.5. *The series $\mathbf{ni}(\mathbf{ld})$ satisfies*

$$\mathbf{ni}(\mathbf{ld}) = \circ^{\otimes}(\mathbf{ni}(\mathbf{ld})) + \bullet^{\otimes}(\mathbf{ni}(\bullet^{\otimes}(\Delta(\mathbf{ld})))) + \bullet^{\otimes}(\bullet^{\otimes}(\Delta(\mathbf{gr}(\mathbf{ld}))))). \tag{3.4.27}$$

Proof. By (3.4.7) and by Relation (3.4.23c) of Proposition 3.4.4, we have

$$\begin{aligned}
\mathbf{ni}(\mathbf{ld}) &= \mathbf{ni}(\epsilon + \circ^{\otimes}(\mathbf{ld})) \\
&= 0 + \sum_{h \geq 0} \mathbf{ni}(\circ(l_h)) \\
&= \sum_{h \geq 0} (\circ^{\otimes}(\mathbf{ni}(l_h)) + \bullet^{\otimes}(\mathbf{ni}(l_h \cdot l_h)) + \bullet^{\otimes}(\bullet^{\otimes}(\Delta(\mathbf{gr}(l_h)))))
\end{aligned} \tag{3.4.28}$$

and the relation of the statement follows. \square

Proposition 3.4.6. *The ht-enumeration H_{ni} of $\mathbf{ni}(\mathbf{ld})$ satisfies*

$$H_{\text{ni}} = zH_{\text{ni}} + zH_{\text{gr}} + 2z(H_{\text{ni}} \boxtimes H_{\text{gr}}). \tag{3.4.29}$$

Proof. Observe first that we have

$$\begin{aligned}
\text{en}_{\text{ht}}(\bullet^{\otimes}(\Delta(\mathbf{gr}(\mathbf{ld})))) &= \sum_{h \geq 0} \sum_{f \in \mathcal{D}^*(l_h)} \text{en}_{\text{ht}}(f \cdot f) \\
&= \sum_{h \geq 0} \sum_{f \in \mathcal{D}^*(l_h)} \text{en}_{\text{ht}}(f) \\
&= H_{\text{gr}}.
\end{aligned} \tag{3.4.30}$$

Moreover, by (3.4.1) and by Relation (3.4.23a) of Proposition 3.4.4, we have

$$\begin{aligned} \text{en}_{\text{ht}}(\mathbf{ni}(\cdot^{\otimes}(\Delta(\mathbf{Id})))) &= \sum_{h \geq 0} \text{en}_{\text{ht}}(\mathbf{ni}(l_h \cdot l_h)) \\ &= \sum_{h \geq 0} \text{en}_{\text{ht}}(\mathbf{gr}(l_h) \cdot^{\otimes} \mathbf{ni}(l_h) + \mathbf{ni}(l_h) \cdot^{\otimes} \mathbf{gr}(l_h)) \\ &= \sum_{h \geq 0} (\text{en}_{\text{ht}}(\mathbf{gr}(l_h)) \uparrow \text{en}_{\text{ht}}(\mathbf{ni}(l_h)) + \text{en}_{\text{ht}}(\mathbf{ni}(l_h)) \uparrow \text{en}_{\text{ht}}(\mathbf{gr}(l_h))). \end{aligned} \tag{3.4.31}$$

Since for any $h \geq 0$, all duplicative forests appearing in $\mathbf{gr}(l_h)$ and in $\mathbf{ni}(l_h)$ have h as height, the last member of (3.4.31) is equal to $H_{\text{gr}} \boxtimes H_{\text{ni}} + H_{\text{ni}} \boxtimes H_{\text{gr}}$. Now by using this identity together with (3.4.30), (3.4.2), and Theorem 3.4.5, we obtain the stated expression for H_{ni} . \square

By Proposition 2.3.3, for any $d \geq 1$, the number of edges in the Hasse diagram of $M(d)$ is $\mathbf{h}_{\text{ni}}(h) := \langle z^h, H_{\text{ni}} \rangle$ where $h = d - 1$. By Proposition 3.4.6, \mathbf{h}_{ni} is the integer sequence satisfying $\mathbf{h}_{\text{ni}}(0) = 0$ and, for any $h \geq 1$,

$$\mathbf{h}_{\text{ni}}(h) = \mathbf{h}_{\text{ni}}(h - 1) + \mathbf{h}_{\text{gr}}(h - 1) + 2\mathbf{h}_{\text{ni}}(h - 1) \mathbf{h}_{\text{gr}}(h - 1), \tag{3.4.32}$$

where \mathbf{h}_{gr} is the integer sequence defined in Section 3.4.2. The sequence of the number of edges of the Hasse diagram of $M(d)$, $d \geq 0$, starts by

$$0, 0, 1, 7, 97, 8287, 29942737, 195432804247687. \tag{3.4.33}$$

This sequence does not appear in [Slo] for the time being.

3.4.4. *Number of intervals.* Let \mathbf{ns} be the $(1, 1)$ -operation on $\mathbb{K}\langle\langle \mathcal{D}^* \rangle\rangle$ satisfying, for any $f \in \mathcal{D}^*$,

$$\mathbf{ns}(f) = \mathbf{gr}(\mathbf{gr}(f)). \tag{3.4.34}$$

By a straightforward computation, we obtain

$$\mathbf{ns}(f) = \sum_{f' \in \mathcal{D}^*(f)} \# [f, f'] f', \tag{3.4.35}$$

so that the coefficient of each $f' \in \mathcal{D}^*(f)$ in $\mathbf{ns}(f)$ is the number of duplicative forests smaller than or equal as f' in $\mathcal{D}^*(f)$. For instance (see at the same time Figure 3),

$$\begin{aligned} \mathbf{ns} \left(\begin{array}{c} \circ \\ \circ \end{array} \right) &= \begin{array}{c} \circ \\ \circ \end{array} + 2 \begin{array}{c} \circ \\ \bullet \end{array} + 2 \begin{array}{c} \bullet \\ \circ \end{array} + 4 \begin{array}{c} \bullet \\ \bullet \end{array} + 2 \begin{array}{c} \circ \\ \circ \bullet \end{array} + 4 \begin{array}{c} \bullet \\ \circ \bullet \end{array} + 3 \begin{array}{c} \circ \\ \bullet \circ \end{array} + 3 \begin{array}{c} \bullet \\ \circ \circ \end{array} \\ &+ 6 \begin{array}{c} \bullet \\ \bullet \bullet \end{array} + 6 \begin{array}{c} \bullet \\ \bullet \circ \end{array} + 6 \begin{array}{c} \bullet \\ \circ \bullet \end{array} + 12 \begin{array}{c} \bullet \\ \bullet \bullet \bullet \end{array}. \end{aligned} \tag{3.4.36}$$

Contrary to what we have undertaken in Sections 3.4.2 and 3.4.3 where we have provided direct recursive expressions to compute $\mathbf{gr}(f)$ and $\mathbf{ni}(f)$, we fail to provide similar expressions for $\mathbf{ns}(f)$. The trick here consists in considering first a slightly different series depending on a parameter $k \geq 1$ which can be seen as a catalytic parameter. With this in mind, let for any $k \geq 1$, \mathbf{md}_k be the $(1, k)$ -operation on $\mathbb{K}\langle\langle \mathcal{D}^* \rangle\rangle$ satisfying, for any $f \in \mathcal{D}^*$,

$$\mathbf{md}_k(f) = \sum_{\substack{g_1, \dots, g_k \in \mathcal{D}^*(f) \\ g_1 \wedge \dots \wedge g_k = f}} g_1 \otimes \dots \otimes g_k, \tag{3.4.37}$$

where \wedge is the meet operation of the duplicative forest lattices introduced in Section 2.2.3. We call $\mathbf{md}_k(f)$ the *k-meet decomposition* of f . Observe that \mathbf{md}_1 is the identity map.

Lemma 3.4.7. For any $k \geq 1$ and $f \in \mathcal{D}^*$,

$$\mathbf{md}_k(\mathbf{ns}(f)) = \sum_{\substack{f' \in \mathcal{D}^*(f) \\ g_1, \dots, g_k \in \mathcal{D}^*(f')}} g_1 \otimes \cdots \otimes g_k. \quad (3.4.38)$$

Proof. We have

$$\begin{aligned} \mathbf{md}_k(\mathbf{ns}(f)) &= \sum_{\substack{f' \in \mathcal{D}^*(f) \\ f'' \in \mathcal{D}^*(f')}} \sum_{\substack{g_1, \dots, g_k \in \mathcal{D}^*(f) \\ g_1 \wedge \dots \wedge g_k = f''}} g_1 \otimes \cdots \otimes g_k \\ &= \sum_{f' \in \mathcal{D}^*(f)} \sum_{\substack{g_1, \dots, g_k \in \mathcal{D}^*(f) \\ g_1 \wedge \dots \wedge g_k = f'}} \#[f, f'] g_1 \otimes \cdots \otimes g_k \\ &= \sum_{g_1, \dots, g_k \in \mathcal{D}^*(f)} \#[f, g_1 \wedge \dots \wedge g_k] g_1 \otimes \cdots \otimes g_k \\ &= \sum_{g_1, \dots, g_k \in \mathcal{D}^*(f)} \sum_{\substack{f' \in \mathcal{D}^*(f) \\ f' \ll g_1, \dots, f' \ll g_k}} g_1 \otimes \cdots \otimes g_k, \end{aligned} \quad (3.4.39)$$

showing the stated identity. \square

Proposition 3.4.8. For any $k \geq 1$, $\ell \geq 0$, $f_1, \dots, f_\ell \in \mathcal{D}^*$, and $f \in \mathcal{D}^*$,

$$\mathbf{md}_k(\mathbf{ns}(f_1 \dots f_\ell)) = \mathbf{md}_k(\mathbf{ns}(f_1)) \cdot^{\otimes k} \dots \cdot^{\otimes k} \mathbf{md}_k(\mathbf{ns}(f_\ell)), \quad (3.4.40a)$$

$$\mathbf{md}_k(\mathbf{ns}(\circ(f))) = \bullet^{\otimes k}(\mathbf{md}_k(\mathbf{ns}(f))), \quad (3.4.40b)$$

$$\mathbf{md}_k(\mathbf{ns}(\circ(f))) = \sum_{u \in \{\circ, \bullet\}^k} \mathbf{mg}_u(\mathbf{md}_{k+|u|_\bullet}(\mathbf{ns}(f))) + \bullet^{\otimes k}(\mathbf{md}_k(\mathbf{ns}(f \cdot f))). \quad (3.4.40c)$$

Proof. Relations (3.4.40a) and (3.4.40b) are direct consequences of Lemmas 2.2.1 and 3.4.7. By Lemmas 2.2.1 and 3.4.7,

$$\begin{aligned} \mathbf{md}_k(\mathbf{ns}(\circ(f))) &= \sum_{\substack{f' \in \mathcal{D}^*(\circ(f)) \\ g_1, \dots, g_k \in \mathcal{D}^*(f')}} g_1 \otimes \cdots \otimes g_k \\ &= \sum_{\substack{\circ(f') \in \mathcal{D}^*(\circ(f)) \\ g_1, \dots, g_k \in \mathcal{D}^*(\circ(f'))}} g_1 \otimes \cdots \otimes g_k + \sum_{\substack{\bullet(f') \in \mathcal{D}^*(\circ(f)) \\ g_1, \dots, g_k \in \mathcal{D}^*(\bullet(f'))}} g_1 \otimes \cdots \otimes g_k \\ &= \sum_{\substack{f' \in \mathcal{D}^*(f) \\ g_1, \dots, g_k \in \mathcal{D}^*(\circ(f'))}} g_1 \otimes \cdots \otimes g_k + \sum_{\substack{f'_1, f'_2 \in \mathcal{D}^*(f) \\ g_1, \dots, g_k \in \mathcal{D}^*(\bullet(f'_1 f'_2))}} g_1 \otimes \cdots \otimes g_k \\ &= \sum_{\substack{u \in \{\circ, \bullet\}^k \\ f' \in \mathcal{D}^*(f) \\ g_1, \dots, g_k \in \mathcal{D}^*(\circ(f')) \\ \text{rts}(g_1 \dots g_k) = u}} g_1 \otimes \cdots \otimes g_k + \sum_{\substack{f'_1, f'_2 \in \mathcal{D}^*(f) \\ g'_1, \dots, g'_k \in \mathcal{D}^*(f'_1) \\ g''_1, \dots, g''_k \in \mathcal{D}^*(f'_2)}} \bullet(g'_1 \cdot g''_1) \otimes \cdots \otimes \bullet(g'_k \cdot g''_k) \\ &= \sum_{u \in \{\circ, \bullet\}^k} \mathbf{mg}_u(\mathbf{md}_{k+|u|_\bullet}(\mathbf{ns}(f))) + \bullet^{\otimes k}(\mathbf{md}_k(\mathbf{ns}(f \cdot f))), \end{aligned} \quad (3.4.41)$$

where for any duplicative forest f , $\text{rts}(f)$ is the word on $\{\circ, \bullet\}$ containing from left to right the roots of the trees forming f . This shows (3.4.40c). \square

Observe that $\text{Supp}(\mathbf{ns}(\mathbf{ld})) = \mathcal{L}$ and that the coefficient of each duplicative forest f of this set is the number of duplicative forests of this same set smaller than or equal as f . Moreover, since for any $h \geq 0$, all elements of $\mathcal{D}^*(l_h)$ have h as height,

$$\text{en}_{\text{ht}}(\mathbf{ns}(\mathbf{ld})) = \sum_{h \geq 0} \sum_{f \in \mathcal{D}^*(l_h)} \#[l_h, f] z^h, \quad (3.4.42)$$

so that $\text{en}_{\text{ht}}(\mathbf{ns}(\mathbf{ld}))$ is the generating series of the number of intervals of the lattices $\mathcal{D}^*(l_h)$, enumerated w.r.t. $h \geq 0$.

Theorem 3.4.9. *The series $\mathbf{ns}(\mathbf{ld})$ satisfies $\mathbf{ns}(\mathbf{ld}) = \mathbf{md}_1(\mathbf{ns}(\mathbf{ld}))$ where, for any $k \geq 1$, the series $\mathbf{md}_k(\mathbf{ns}(\mathbf{ld}))$ satisfies*

$$\mathbf{md}_k(\mathbf{ns}(\mathbf{ld})) = \underbrace{\epsilon \otimes \cdots \otimes \epsilon}_{k \text{ terms}} + \sum_{u \in \{\circ, \bullet\}^k} \text{mg}_u(\mathbf{md}_{k+|u|_\bullet}(\mathbf{ns}(\mathbf{ld}))) + \bullet^{\otimes k}(\mathbf{md}_k(\mathbf{ns}(\bullet^{\otimes}(\Delta(\mathbf{ld}))))). \quad (3.4.43)$$

Proof. First, the relation $\mathbf{ns}(\mathbf{ld}) = \mathbf{md}_1(\mathbf{ns}(\mathbf{ld}))$ holds since \mathbf{md}_1 is the identity map. By (3.4.7) and by Proposition 3.4.8, we have

$$\begin{aligned} \mathbf{md}_k(\mathbf{ns}(\mathbf{ld})) &= \mathbf{md}_k(\mathbf{ns}(\epsilon + \bullet^{\otimes}(\mathbf{ld}))) \\ &= \underbrace{\epsilon \otimes \cdots \otimes \epsilon}_{k \text{ terms}} + \sum_{h \geq 0} \left(\sum_{u \in \{\circ, \bullet\}^k} \text{mg}_u(\mathbf{md}_{k+|u|_\bullet}(\mathbf{ns}(l_h))) + \bullet^{\otimes k}(\mathbf{md}_k(\mathbf{ns}(l_h \cdot l_h))) \right) \end{aligned} \quad (3.4.44)$$

and the relation of the statement follows. \square

Proposition 3.4.10. *The ht-enumeration H_{ns} of $\mathbf{ns}(\mathbf{ld})$ satisfies $H_{\text{ns}} = H_{\text{ns}}^{(1)}$ where, for any $k \geq 1$, $H_{\text{ns}}^{(k)}$ is the ht-enumeration of $\mathbf{md}_k(\mathbf{ns}(\mathbf{ld}))$ which satisfies*

$$H_{\text{ns}}^{(k)} = 1 + z \left(H_{\text{ns}}^{(k)} \boxtimes H_{\text{ns}}^{(k)} \right) + z \sum_{i \in \llbracket k \rrbracket} \binom{k}{i} H_{\text{ns}}^{(k+i)}. \quad (3.4.45)$$

Proof. By Relation (3.4.40a) of Proposition 3.4.8, we have

$$\begin{aligned} \text{en}_{\text{ht}}(\mathbf{md}_k(\mathbf{ns}(\bullet^{\otimes}(\Delta(\mathbf{ld})))))) &= \sum_{h \geq 0} \text{en}_{\text{ht}}(\mathbf{md}_k(\mathbf{ns}(l_h \cdot l_h))) \\ &= \sum_{h \geq 0} \text{en}_{\text{ht}}(\mathbf{md}_k(\mathbf{ns}(l_h)) \bullet^{\otimes k} \mathbf{md}_k(\mathbf{ns}(l_h))) \\ &= \sum_{h \geq 0} (\text{en}_{\text{ht}}(\mathbf{md}_k(\mathbf{ns}(l_h))) \uparrow \text{en}_{\text{ht}}(\mathbf{md}_k(\mathbf{ns}(l_h)))). \end{aligned} \quad (3.4.46)$$

Since for any $h \geq 0$, all the duplicative forests of the tensors $f_1 \otimes \cdots \otimes f_k$ appearing in $\mathbf{md}_k(\mathbf{ns}(l_h))$ have h as height, the last member of (3.4.46) is equal to $H_{\text{ns}}^{(k)} \boxtimes H_{\text{ns}}^{(k)}$. Moreover, by (3.4.5), for any $u \in \{\circ, \bullet\}^k$,

$$\text{en}_{\text{ht}}(\text{mg}_u(\mathbf{md}_{k+|u|_\bullet}(\mathbf{ns}(\mathbf{ld})))) = z \text{en}_{\text{ht}}(\mathbf{md}_{k+|u|_\bullet}(\mathbf{ns}(\mathbf{ld}))) = z H_{\text{ns}}^{(k+|u|_\bullet)}. \quad (3.4.47)$$

Now, by using these two identities together with Theorem 3.4.9, and by the fact that the number of words of length k on $\{\circ, \bullet\}$ having exactly $i \in \llbracket k \rrbracket$ occurrences of \bullet is $\binom{k}{i}$, we obtain the stated expression for $H_{\text{ns}}^{(k)}$. Finally, we have $H_{\text{ns}} = H_{\text{ns}}^{(1)}$ since $\mathbf{ns}(\mathbf{ld}) = \mathbf{md}_1(\mathbf{ns}(\mathbf{ld}))$. \square

By Proposition 2.3.3, for any $d \geq 1$, the number of intervals in $M(d)$ is $\mathbf{h}_{\text{ns}}^{(1)}(h) := \langle z^h, H_{\text{ns}} \rangle$ where $h = d - 1$. By Proposition 3.4.10, for any $k \geq 1$, $\mathbf{h}_{\text{ns}}^{(k)}$ is the integer sequence satisfying $\mathbf{h}_{\text{ns}}^{(k)}(0) = 1$ and, for any $h \geq 1$,

$$\mathbf{h}_{\text{ns}}^{(k)}(h) = \mathbf{h}_{\text{ns}}^{(k)}(h-1)^2 + \sum_{i \in \llbracket k \rrbracket} \binom{k}{i} \mathbf{h}_{\text{ns}}^{(k+i)}(h-1). \quad (3.4.48)$$

The sequence of the cardinalities of $M(d)$, $d \geq 0$, starts by

$$1, 1, 3, 17, 371, 144513, 20932611523, 438176621806663544657. \quad (3.4.49)$$

This sequence does not appear in [Slo] for the time being.

OPEN QUESTIONS AND FUTURE WORK

We have studied in this work a CLS having a lot of rich combinatorial properties despite its simplicity. This can be considered as the prototypical example for this kind of investigation. We could expect to have also similar combinatorial properties (related to poset and lattice properties, and quantitative data) for more complex CLS. In addition to the obvious possible future investigations consisting in studying in a similar fashion some other CLS, we describe here three open questions raised by this work.

The description of minimal and maximal elements of \mathcal{P} uses a notion of pattern avoidance in terms (see Proposition 2.1.4). This is a general fact: when a CLS $(\mathcal{G}, \rightarrow)$ has the poset property, its minimal (resp. maximal) elements are the terms avoiding some terms deduced from the ones appearing as right-hand (resp. left-hand) members of \rightarrow . Such an enumerative problem has been considered in [KP15, Gir20] for the particular case of linear terms. The question here concerns the general enumeration of terms avoiding a set of other terms wherein multiple occurrences of a same variable are allowed.

We have shown that the Mockingbird CLS has the poset property and is rooted (Proposition 2.1.2), and has the lattice property (Theorem 2.3.4) by employing some specific reasoning from the definition of the Mockingbird basic combinator \mathbf{M} . A question here concerns the existence of a general criterion to decide if a CLS has the poset (resp. lattice) property and if it is rooted. All this seems independent from the property of a CLS to be hierarchical because, among others, the CLS containing a basic combinator \mathbf{X} such that \mathbf{X} has order 3 and $t_{\mathbf{X}} := x_3(x_1x_1x_2)$ has not the poset property because we have $t \Rightarrow t'$ and $t' \Rightarrow t$ with $t := \mathbf{X}\mathbf{X}\mathbf{X}(\mathbf{X}\mathbf{X}(\mathbf{X}\mathbf{X}))$ and $t' := \mathbf{X}\mathbf{X}(\mathbf{X}\mathbf{X})(\mathbf{X}\mathbf{X}\mathbf{X})$ while \mathbf{X} is hierarchical.

Finally, in Proposition 1.2.3, we have shown that being hierarchical is a sufficient condition for a CLS \mathcal{G} to be locally finite. The question in this context consists in obtaining a necessary and sufficient condition for this last property. Observe that being nonerasing is necessary because, by assuming that \mathcal{G} contains a basic combinator \mathbf{X} such that \mathbf{X} has order $n \geq 2$ and that there is $i \in [n]$ such that x_i does not appear in $t_{\mathbf{X}}$, we have $\mathbf{X}x_1 \dots x_{i-1}s x_{i+1} \dots x_n \Rightarrow t_{\mathbf{X}}$ for any term s , showing that the connected component of $t_{\mathbf{X}}$ in $\mathcal{G}_{\mathcal{G}}$ is infinite.

REFERENCES

- [BEKW17] H. Barendregt, J. Endrullis, J. W. Klop, and J. Waldmann. Dance of the starlings. In *Raymond Smullyan on self reference*, volume 14 of *Outst. Contrib. Log.*, pages 67–111. Springer, Cham, 2017. 2
- [BGZ17] M. Bendkowski, K. Grygiel, and M. Zaionc. On the likelihood of normalization in combinatory logic. *J. Logic Comput.*, 27(7):2251–2269, 2017. 2

- [Bim11] K. Bimbó. *Combinatory Logic: Pure, Applied and Typed*. Discrete Mathematics and Its Applications. Taylor & Francis, 2011. 2, 7
- [BKdVT03] M. Bezem, J. W. Klop, R. de Vrijer, and Terese. *Term Rewriting Systems*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2003. 2, 8, 10
- [BN98] F. Baader and T. Nipkow. *Term rewriting and all that*. Cambridge University Press, Cambridge, 1998. 2, 7
- [BP06] J. L. Baril and J. M. Pallo. The phagocyte lattice of Dyck words. *Order*, 23(2-3):97–107, 2006. 3
- [BP08] J. L. Baril and J. M. Pallo. The pruning-grafting lattice of binary trees. *Theor. Comput. Sci.*, 409(3):382–393, 2008. 3
- [CF58] H. B. Curry and R. Feys. *Combinatory Logic*, volume 1. North-Holland Publishing Company, 1958. 2
- [Cur30] H. B. Curry. Grundlagen der Kombinatorischen Logik. *Am. J. Math.*, 52(4):789–834, 1930. 2
- [DGK⁺13] R. David, K. Grygiel, J. Kozik, C. Raffalli, G. Theyssier, and M. Zaionc. Asymptotically almost all λ -terms are strongly normalizing. *Log. Meth. Comput. Sci.*, 9(1):1:02, 30, 2013. 2
- [Gir20] S. Giraud. Tree series and pattern avoidance in syntax trees. *J. Comb. Theory A*, 176, 2020. 28
- [Gir22] S. Giraud. Mockingbird lattices. *To appear in Formal Power Series and Algebraic combinatorics*, 2022. 4
- [HS08] J. R. Hindley and J. P. Seldin. *Lambda-calculus and combinators, an introduction*. Cambridge University Press, Cambridge, 2008. 2, 8
- [KP15] A. Khoroshkin and D. Piontkovski. On generating series of finitely presented operads. *J. Algebra*, 426:377–429, 2015. 28
- [Kre72] G. Kreweras. Sur les partitions non croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1972. 3
- [PS01] D. Probst and T. Studer. How to normalize the Jay. *Theoret. Comput. Sci.*, 254(1-2):677–681, 2001. 2
- [Ros73] B. K. Rosen. Tree-manipulating systems and Church-Rosser theorems. *J. Assoc. Comput. Mach.*, 20:160–187, 1973. 7
- [Sch24] M. Schönfinkel. Über die Bausteine der mathematischen Logik. *Mathematische Annalen*, 92:305–316, 1924. 2
- [Slo] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. <https://oeis.org/>. 19, 20, 23, 25, 28
- [Smu85] R. Smullyan. *To Mock a Mockingbird*. Alfred A. Knopf, Inc., 1985. 2, 3, 6
- [Sta75] R. P. Stanley. The Fibonacci lattice. *Fibonacci Quart.*, 13(3):215–232, 1975. 3
- [Sta89] R. Statman. The word problem for Smullyan's lark combinator is decidable. *J. Symbolic Comput.*, 7(2):103–112, 1989. 2
- [Sta00] R. Statman. On the Word Problem for Combinators. *Lect. Notes Comp. Sci., Rewriting Techniques and Applications, 11th International Conference*, 1833:203–213, 2000. 2
- [Sta11a] R. P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, second edition, 2011. 13
- [Sta11b] R. Statman. To type a Mockingbird. *Unpublished*, 2011. Draft available at <http://tlca.di.unito.it/PAPER/TypeMock.pdf>. 3
- [Sta17] R. Statman. Some tweets about Mockingbirds. In *Raymond Smullyan on self reference*, volume 14 of *Outst. Contrib. Log.*, pages 113–122. Springer, Cham, 2017. 3
- [SWB93] M. Sprenger and M. Wymann-Böni. How to decide the lark. *Theoret. Comput. Sci.*, 110(2):419–432, 1993. 2
- [Tam62] D. Tamari. The algebra of bracketings and their enumeration. *Nieuw Arch. Wisk.*, 10(3):131–146, 1962. 3
- [Tay93] W. Taylor. Abstract clone theory. In *Algebras and orders*, volume 389 of *NATO Adv. Sci. I. C-Mat.*, pages 507–530. Kluwer Acad. Publ., Dordrecht, 1993. 3, 5, 8
- [Wal00] J. Waldmann. The combinator S. *Inform. Comput.*, 159(1-2):2–21, 2000. 2