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ACTIONS OF $\text{Alt}(n)$ ON GROUPS OF FINITE MORLEY RANK WITHOUT INVOLUTIONS

TUNA ALTINEL AND JOSHUA WISCONS

ABSTRACT. We investigate faithful representations of $\text{Alt}(n)$ as automorphisms of a connected group G of finite Morley rank. We target a lower bound of n on the rank of such a *nonsolvable* G , and our main result achieves this in the case when G is without involutions. In the course of our analysis, we also prove a corresponding bound for solvable G by leveraging recent results on the abelian case. We conclude with an application towards establishing natural limits to the degree of generic transitivity for permutation groups of finite Morley rank.

1. INTRODUCTION

Morley rank is a fundamental, model-theoretic notion of dimension generalizing Zariski dimension from algebraic geometry. The affine algebraic groups over algebraically closed fields are among the groups of finite Morley rank as are various nonalgebraic groups. The importance of groups of finite Morley rank is highlighted by—and rooted in—their appearance as “binding groups” (analogous to Galois groups) in the analysis of certain model-theoretic classes of structures, and this connection makes the study of *permutation groups* of finite Morley rank particularly salient. The recent work of James Freitag and Rahim Moosa illustrates this well [FM].

Much of the current work on permutation groups of finite Morley rank is structured around guiding problems of Alexandre Borovik and Gregory Cherlin about the degree of generic transitivity [BC08]. This topic originated in the algebraic setting in work of Vladimir Popov [Pop07], and his findings—as well as the questions that remain—have provided much inspiration for the finite Morley rank setting. And in the other direction, results obtained in this more general (and perhaps more appropriate) context have also extended the algebraic theory.

Due in large part to work on generic transitivity, there has been much recent activity investigating *representations* in a finite Morley rank context, and one emerging theme is the relevance of understanding representations of *finite groups* in this setting. This was already present in the work of Borovik and Cherlin where the determination of the minimal Morley rank of an abelian group on which $\text{Sym}(n)$ can act faithfully and definably drives their subsequent analysis establishing a bound on the degree of generic transitivity for primitive groups of finite Morley rank. The authors of this article

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also studied representations of the symmetric group in their classification of generically 4-transitive actions on sets of rank 2 [AW], and in that work, there was a need to analyze representations where the acted upon group was not necessarily abelian. Additionally, the recent work of Ayşe Berkman and Alexandre Borovik completing the classification of representations of finite Morley rank with a sufficiently high degree of generic transitivity makes critical use of the hyperoctahedral group [BB18, BB21].

The present authors have sought to extend their analysis of actions on sets of rank 2 to sets of arbitrary rank, and additional knowledge of representations of the symmetric group appears to be essential. In particular, a lower bound on the rank of a (not necessarily abelian) group admitting a faithful and definable action of $\text{Sym}(n)$ is strongly desired, and a classification of those actions achieving the lower bound would further clarify the situation. The latter was recently addressed by Luis Jaime Corredor, Adrien Deloro, and the second author [CDW]; the following conjecture proposes a rough bound to address the former.

Conjecture (see [CDW, Section 1.2]). Suppose that $\text{Sym}(n)$ or $\text{Alt}(n)$ acts definably and faithfully by automorphisms on a connected group of finite Morley rank G , with n sufficiently large. If G is nonabelian, then $\text{rk } G \geq n - 1$, and if G is nonsolvable, then $\text{rk } G \geq n$.

The solvable case follows readily from the main result of [CDW]; we address this first (see Lemma 2.6). The nonsolvable case is much less clear to us. When G is algebraic, routine reduction to G being simple leads to an embedding of $\text{Alt}(n)$ into G , so this appears to be under control, if not already classical. However, small values of n are indeed complicated by exceptional isomorphisms, for both $\text{Sym}(n)$ and $\text{Alt}(n)$. In this paper, we establish the **Conjecture** at the other (highly nonalgebraic) extreme.

Theorem. *Suppose $\text{Alt}(n)$, with $n \geq 4$, acts definably and faithfully on a connected nonsolvable group G of finite Morley rank without involutions. Then $\text{rk } G \geq n$.*

In addition to making progress on the **Conjecture**, the **Theorem** supplements the rather small amount of existing work controlling the structure of nonsolvable groups of finite Morley rank without involutions. According to the Algebraicity Conjecture of Gregory Cherlin and Boris Zilber, no such group should exist, but this appears to be quite far from being resolved. Indeed, it was not until 2018 that Oliver Frécon established the Algebraicity Conjecture for groups of rank 3 [Fré18], and most other work on nonsolvable groups without involutions is confined to special configurations such as “bad groups.”

As mentioned before, the **Theorem** has applications to permutation groups with a “high” degree of generic transitivity. This is discussed in some detail in Section 3 where we also explicitly state our **Corollary** for such actions.

2. THE THEOREM

Our proof of the **Theorem** is inductive. We first collect a handful of background results. The first two (Facts 2.1 and 2.2) allow us to quickly dispense

with the tightest, low rank configurations, and the next two (Facts 2.3 and 2.4) drive our inductive approach. Ultimately, we are forced to consider the case when the acted upon group is solvable; we treat this quickly in Lemma 2.6 by leveraging recent work on the abelian case (where the context is significantly more general than finite Morley rank and the conclusion significantly stronger than just a bound on the rank) [CDW].

General background on groups of finite Morley rank can be found in [ABC08] or [BN94], among many other sources. For the most part, the reader familiar with the algebraic setting but not the logical one may safely replace *definable* with *constructible* and *rank* with *dimension*.

2.1. Background and preliminary results. Our first two background results use the term *bad group*, which we here take to mean a nonsolvable connected group of finite Morley rank all of whose proper definable connected subgroups are nilpotent.

Fact 2.1 ([Che79, DW18, Fré18, Rei75, Wis16]). *If G is a connected nonsolvable group of finite Morley rank without involutions, then $\text{rk } G \geq 4$. Moreover, if $\text{rk } G \leq 5$, then G' is quasisimple and bad.*

In the presence of involutive automorphisms, the conclusion of Fact 2.1 is further clarified by the following classical fact.

Fact 2.2 (Delahan-Nesin, see [BN94, Proposition 13.4]). *A simple bad group of finite Morley rank has no definable involutive automorphism.*

The next two facts underlie our inductive approach to the **Theorem**.

Fact 2.3 ([BBC07, Theorem 5]). *Let G be a group of finite Morley rank without involutions and K an elementary abelian group of order 4 acting definably on G . Then $G = \langle C_G^\circ(\alpha) \mid \alpha \in K \setminus \{1\} \rangle$.*

We mention in passing a related result about actions of 8-groups; it was a part of our original proof of the **Theorem** but has since been removed (at the cost of introducing several more lines of mathematics.) The result—which is due to Jeffery Burdges, though not in any of his papers—is that if an elementary abelian group E of order 8 acts definably on a group of finite Morley rank G without involutions, then $C_G^\circ(e)$ is nonsolvable for some involution $e \in E$.

The next result is an adaptation of [BC08, Proposition 2.3]. We only need the final sentence for our proof of the **Theorem**, so we delay the definition of generically t -transitive actions until Section 3.

Fact 2.4 (see [AW18, Lemma 4.23]). *Let G be a connected group of finite Morley rank and H a maximal definable connected subgroup of corank s with finite core, i.e. $\bigcap_{g \in G} H^g$ is finite. If X is the set of right cosets of H , then for $\text{gtd}(X, G)$ the degree of generic transitivity of the action of G on X ,*

$$\text{rk } G \leq s \cdot \text{gtd}(X, G) + s(s - 1)/2.$$

In particular, if G is simple and without involutions, then $\text{rk } G \leq s(s + 1)/2$.

With notation as in Fact 2.4, we can rewrite the final statement (where G is assumed to be simple and without involutions) as

$$\mathrm{rk} H \leq \mathrm{rk} G + \frac{1}{2} - \sqrt{2 \mathrm{rk} G + \frac{1}{4}}.$$

Asymptotically, this says little, but it does create a little space for induction and will be used in the sequel.

Finally, we address the solvable analogue of the **Theorem**. As expected, it follows quickly from the recent solution of the abelian case, which we state here in a very abbreviated form.

Fact 2.5 (see [CDW, Theorem and First Geometrisation Lemma]). *If V is a faithful connected $\mathrm{Sym}(n)$ -module of finite Morley rank, then $\dim V \geq n - 2$, and equality can hold only if V is an elementary abelian p -group for some prime p dividing n .*

Moreover, the same is true for $\mathrm{Alt}(n)$ -modules provided either $n \geq 10$ or both $n \geq 7$ and V is 2-divisible.

Our statement of the solvable case requires a definition.

Definition. Let G be a connected group of finite Morley rank, and suppose a group H acts definably on G . We say that $G = G_\ell > G_{\ell-1} > \cdots > G_0 = 1$ is an H -minimal series for G if for all $i > 0$,

- G_i is definable, nontrivial, connected, normal, and H -invariant; and
- each G_i/G_{i-1} possesses no proper definable, nontrivial, connected, normal, H -invariant subgroups.

Lemma 2.6 (Solvable Lemma). *Suppose $\mathrm{Sym}(n)$ acts faithfully and definably on a connected solvable group B of finite Morley rank. If $B = B_\ell > B_{\ell-1} > \cdots > B_0 = 1$ is any $\mathrm{Sym}(n)$ -minimal abelian series, then $\mathrm{rk} B \geq n + \ell - 3$; in particular, $\mathrm{rk} B \geq n - 1$ whenever B is nonabelian.*

Moreover, the same is true for an action of $\mathrm{Alt}(n)$ provided either $n \geq 10$ or both $n \geq 7$ and B is 2-divisible.

Proof. Let $S := \mathrm{Sym}(n)$, $A := S'$, and $r = \mathrm{rk} B$. There are ℓ quotients in the series, each of rank at least 1, so $s_m := \mathrm{rk}(B_m/B_{m-1}) \leq r - \ell + 1$.

Towards a contradiction, assume $r < n + \ell - 3$ (which also implies $n \geq 4$). Then for all $1 \leq m \leq \ell$, we have $s_m \leq r - \ell + 1 < n - 2$, and Fact 2.5 implies that A does not act faithfully on any B_m/B_{m-1} .

If $n \geq 5$, then A is simple and $[A, \dots, A, B] = 1$, and (as A is simple) this easily implies $[A, B] = 1$, contradicting faithfulness. It remains to consider when the acting group is $S = \mathrm{Sym}(4)$ in which case $s_m = 1$ for all m . It is easily seen that each involution in S much either centralize or invert any connected rank 1 section on which it acts (see [ABC08, I, Lemma 10.3]), so in this case we again find that $[A, \dots, A, B] = 1$. This condition, for a faithful action, in fact implies that A is nilpotent (see for example [Kal53]), again a contradiction. \square

2.2. The proof. Our main analysis starts here. We first deal with the cases of lowest rank.

Lemma 2.7. *Suppose a Klein four-group K acts definably and faithfully on a connected nonsolvable group G without involutions. Then $\mathrm{rk} G \geq 6$.*

Proof. In light of Fact 2.1, we need only consider when $4 \leq \text{rk } G \leq 5$ and G' is quasisimple and bad, which further implies $\text{rk}(G') \geq 4$. Crucially, Fact 2.2 tells us that $[K, G'] \leq Z(G')$, and as G' is connected, we in fact have that $[K, G'] \leq Z^\circ(G')$ (see [BN94, Corollary 5.29]).

First assume $G' = G$. We study the action of K on the series $1 \leq Z^\circ(G) < G$, and we already know $[K, G] \leq Z^\circ(G)$. By Fact 2.1, $\text{rk}(G/Z(G)) \geq 4$, so $\text{rk } Z^\circ(G) \leq 1$. Consequently, each involution in K either centralizes or inverts $Z^\circ(G)$, so some involution $\alpha \in K$ must centralize $Z^\circ(G)$. Thus, α acts quadratically on G (i.e. $[\alpha, \alpha, G] = 1$), which implies $[\alpha, G]$ has exponent 2. As G is without involutions, $[\alpha, G] = 1$, contradicting faithfulness.

Next suppose that $G' \neq G$. We now consider $1 < G' < G$. Then G' has rank 4, and Fact 2.1 forces $Z(G')$ to be finite. So here we have $[K, G'] = 1$, and consideration of the rank 1 quotient G/G' shows (as before) that some $\alpha \in K$ acts quadratically on G , a contradiction (as before). \square

Proof of the Theorem. Assume the theorem is false. Choose n minimal such that the theorem fails, and then choose G witnessing this failure that minimizes $\text{rk } G$ among all such witnesses.

Set $A := \text{Alt}(n)$ and $r := \text{rk } G$. As $n \geq 4$, A contains a Klein 4-group, so $r \geq 6$ by Lemma 2.7. As we are working with a counterexample, $n \geq 7$.

Claim 1. We may assume G is simple.

Proof of Claim. Let $G = G_\ell > G_{\ell-1} > \dots > G_0 = 1$ be an A -minimal series for G ; thus, each factor G_i/G_{i-1} is either abelian or nonsolvable.

We show $\ell = 1$. Assume not; then $\text{rk}(G_i/G_{i-1}) < r$ for each i . Moreover, if a factor G_i/G_{i-1} is solvable, then $\text{rk}(G_i/G_{i-1}) < r - 3$ as otherwise G would be solvable (since connected groups of rank at most 2 are solvable). Consequently, Fact 2.5 ensures that each abelian factor of the series is centralized by A , and minimality of r ensures that each nonsolvable factor is also centralized by A . Thus, in this case, $[A, \dots, A, G] = 1$, implying that $[A, G] = 1$, a contradiction.

We now show G is quasisimple. As $\ell = 1$, G is equal to its generalized Fitting subgroup (which also equals the socle of G), so there exist definable, normal, connected, quasisimple subgroups Q_1, \dots, Q_k of G such that $G = Q_1 \oplus \dots \oplus Q_k$. All Q_i are connected, so $k \leq r < n$. (In fact, $4k \leq r < n$ by Fact 2.1.) Since A acts trivially on sets of size less than n , A normalizes each Q_i , so as $\ell = 1$, $G = Q_1$ is quasisimple.

Again using that $\ell = 1$, G is quasisimple with finite center. Since $[A, G]$ is connected, A must act faithfully on $G/Z(G)$ as otherwise A would centralize all of G . Hence, we may replace G with this quotient. \diamond

For $\sigma, \psi \in A$, we use the notation $C_\sigma := C_G^\circ(\sigma)$ and $C_{\sigma, \psi} := C_G^\circ(\sigma, \psi)$. We write A_σ to denote the subgroup of permutations whose support is contained in the support of σ ; A_{σ^\perp} denotes the subgroup of permutations with support disjoint from the support of σ . So, if σ has support of size m , then $A_\sigma \cong \text{Alt}(m)$ and $A_{\sigma^\perp} \cong \text{Alt}(n - m)$.

Claim 2. If $\alpha \in A$ is a bitransposition, then A_{α^\perp} acts faithfully on C_α .

Proof of Claim. Suppose A_{α^\perp} is not faithful on C_α . Let $\{1, \alpha_1, \alpha_2, \alpha_3\}$ be the Klein group on the support of α . Since A_α permutes the C_{α_i} transitively

while centralizing A_{α^\perp} , A_{α^\perp} is not faithful on *any* C_{α_i} . Thus, by Fact 2.3 and the normal subgroup structure of A_{α^\perp} , A_{α^\perp} does not act faithfully on G , a contradiction (since $n \geq 7$ implies A_{α^\perp} is nontrivial). \diamond

Claim 3. If $\alpha \in A$ is a bitransposition, then C_α is solvable but nonabelian.

Proof of Claim. Suppose C_α is nonsolvable. We know $A_{\alpha^\perp} \cong \text{Alt}(n-4)$ acts faithfully on C_α . By minimality of n , we must have $\text{rk } C_\alpha \geq n-4 > r-4$, so the corank of C_α in G is at most 3. Thus, by Fact 2.4, $r \leq 6$, so by the same fact we find that $\text{rk } C_\alpha \leq 3$, contradicting Fact 2.1. Thus C_α is solvable.

We now show C_α is nonabelian. We may assume $\alpha = (12)(34)$. Let $\beta = (34)(56)$. We claim that $C_{\alpha,\beta}$ is nontrivial. If $C_{\alpha,\beta} = 1$, then β inverts C_α . The same is then true (by conjugacy under the action of A_{α^\perp}) for all bitranspositions $(34)(ij)$ with $5 \leq i < j$, so A_{α^\perp} centralizes C_α . This contradicts Claim 2, so $\text{rk } C_{\alpha,\beta} \geq 1$. Now, if C_α is abelian, then $C_{\alpha,\beta} = C_{\alpha,\beta,\alpha\beta}$ is a nontrivial central subgroup of $\langle C_\alpha, C_\beta, C_{\alpha\beta} \rangle = G$, against simplicity. \diamond

The next claim handles the generic case, after which we will have only a handful of low rank cases to address.

Claim 4. Let $\alpha = (ab)(cd) \in A$ be a bitransposition and Σ_{α^\perp} the subgroup of A fixing $\{a, b\}$ pointwise and $\{c, d\}$ setwise. Then

$$r - 6 + \ell \leq \text{rk } C_\alpha \leq r + \frac{1}{2} - \sqrt{\left(2r + \frac{1}{4}\right)}$$

where ℓ is the length of any Σ_{α^\perp} -invariant subnormal series for C_α . In particular, $6 \leq r \leq 10$,

Proof of Claim. The upper bound is just Fact 2.4. For the lower bound, we apply Lemma 2.6 to the action of $\Sigma_{\alpha^\perp} \cong \text{Sym}(n-4)$ on C_α (using Claim 2) to find that $\text{rk } C_\alpha \geq (n-4) + \ell - 3 = n - 7 + \ell > r - 7 + \ell$. Then, since C_α is nonabelian, we have that $\ell \geq 2$, and now the upper and lower bounds together imply that $r \leq 10$. (Of course, $r \geq 6$ by Lemma 2.7.) \diamond

Of the low rank cases that remain, $r = 6$ stands out for two reasons that quickly arise when considering the action of A_{α^\perp} (or Σ_{α^\perp}) on C_α . First, with notation as above, $r = 6$ allows for $\ell = 3$ in Claim 4; whereas, $r \geq 7$ forces $\ell = 2$. Second, and perhaps more importantly, the $r = 6$ case must cover the possibility that A_{α^\perp} is $\text{Alt}(3)$ and hence without involutions.

Claim 5. $r = 6$.

Proof of Claim. Assume $7 \leq r \leq 10$. Set $\alpha = (12)(34)$, and let Σ_{α^\perp} and ℓ be as in Claim 4. Notice that $\ell \geq 3$ implies $r \leq 6$ by Claim 4, so in the case we are considering, every Σ_{α^\perp} -minimal subnormal series for C_α has length 2. We also find $\text{rk } C_\alpha = r - 4$, and (by Fact 2.4) C_α is of maximal possible rank among all proper definable subgroups of G .

We first show C_α is nonnilpotent. Since C_α is nonabelian, $\text{rk } C_{\alpha,\alpha'} \geq 1$ for $\alpha' = (13)(24)$; also $C_{\alpha,\alpha'} = C_K$ for $K = \langle \alpha, \alpha' \rangle$. Now consider $N := N_{C_\alpha}^\circ(C_K)$, and note that $N \neq C_\alpha$ as otherwise, by conjugacy under the action of A_α , C_K would be normal in $\langle C_\alpha, C_{\alpha'}, C_{\alpha\alpha'} \rangle = G$. Now assume C_α is nilpotent. Then we also have $N \neq C_K$. The series $1 \triangleleft C_K \triangleleft N \triangleleft C_\alpha$ is

invariant under the action of Σ_{α^\perp} (as Σ_{α^\perp} centralizes α and normalizes K), and by nilpotence of C_α , it can be refined to be subnormal and Σ_{α^\perp} -invariant (and of length at least 3), a contradiction.

Thus, we are in the case of C_α being solvable and nonnilpotent. Set $U_\alpha = F^\circ(C_\alpha)$; then $1 \leq U_\alpha \leq C_\alpha$ is Σ_{α^\perp} -minimal. Observe that Σ_{α^\perp} must act faithfully on U_α or C_α/U_α . If not, then $[C_\alpha, \beta, \beta] = 1$ for some bitransposition $\beta \in A_{\alpha^\perp}$, and this implies that the nontrivial elements of $[C_\alpha, \beta]$ (which is nontrivial by Claim 2) have order 2, a contradiction. Thus, by Fact 2.5, one of U_α or C_α/U_α must have rank at least $n - 4 - 2 \geq r - 5$, so as $\text{rk } C_\alpha = r - 4$, one of the groups has rank $r - 5$ and the other rank 1. Since $\text{rk } C_\alpha \geq 3$, properties of the Fitting subgroup in solvable groups of finite Morley rank (see [BN94, Corollary 7.15]) ensure $\text{rk } U_\alpha \neq 1$. We conclude that A_{α^\perp} acts faithfully on U_α , which has rank $r - 5$ and is an elementary abelian p -group (again by Fact 2.5).

Let $\beta = (56)(78)$, and note that $W := \text{rk } C_{U_\alpha}^\circ(\beta)$ has rank at least one. Indeed, if W has rank 0, then β inverts U_α , so as A_{α^\perp} acts faithfully on U_α , this would imply that $\beta \in Z(A_{\alpha^\perp}) = 1$. Also, by p -unipotence of U_α and solvability of C_β , we find that $W = (U_\alpha \cap U_\beta)^\circ$. Consider $H := C_G^\circ(W) \geq \langle U_\alpha, U_\beta \rangle$. If $U_\alpha = U_\beta$, then $N_G(U_\alpha) \geq \langle C_\alpha, C_\beta \rangle = G$, a contradiction. Now, using Fact 2.4, we find that $r - 5 = \text{rk } U_\alpha < \text{rk } H \leq r - 4$. As U_α and U_β have corank 1 in H , $\text{rk } W = \text{rk } U_\alpha \cap U_\beta = r - 6$, so H/W has rank 2, hence is solvable. Thus, H is solvable, so $H = \langle U_\alpha, U_\beta \rangle$ is nilpotent. We now have $N_G(U_\alpha) \geq \langle C_\alpha, H \rangle = G$, a contradiction. \diamond

Claim 6. $r \neq 6$, and the proof is complete.

Proof of Claim. Assume $r = 6$; by Fact 2.4, the proper definable subgroups of G have rank at most 3.

Let $\alpha = (12)(34)$ with Σ_{α^\perp} defined as before. Set $\beta = (34)(56)$, $\gamma = (567)$, and $K := \langle \alpha, \beta \rangle$. Notice that conjugacy of α , β , and $\alpha\beta$ implies that $C_{\alpha,\beta} \neq C_\alpha$ as otherwise $C_\alpha = C_\beta = C_{\alpha\beta}$, contradicting Fact 2.3. Also notice that $C_{\alpha,\beta}$ is nontrivial since C_α is nonabelian by Claim 3. Thus, $\text{rk } C_{\alpha,\beta} \geq 1$, which (by our previous observation or Claim 4) also implies $\text{rk } C_\alpha \geq 2$.

We now claim that either C_α or $N_{\alpha,\beta} := N_G^\circ(C_{\alpha,\beta})$ is nonnilpotent. Assume C_α is nilpotent (so C_σ is nilpotent for all $\sigma \in K^*$). The normalizer condition applied in each C_σ for $\sigma \in K^*$ ensures that $\text{rk}(C_\sigma \cap N_{\alpha,\beta}) \geq 2$ and, thus, that $\text{rk } N_{\alpha,\beta}$ is at least, hence equal to, 3. Towards a contradiction, assume $N_{\alpha,\beta}$ is nilpotent. If $\text{rk } C_\alpha = 3$, then the normalizer condition would force $N_{\alpha,\beta} \cap C_\alpha$ to be normal in G . Thus, $\text{rk } C_\alpha = 2$, and now the normalizer condition ensures $C_{\alpha,\beta}$ is normal in C_σ , for each $\sigma \in K^*$, hence normal in G by Fact 2.3, a contradiction since $C_{\alpha,\beta} \neq 1$.

Now consider any proper, definable, connected, nonnilpotent, $H \leq G$. (We are of course thinking of either C_α or $N_{\alpha,\beta}$). By the structure of groups of rank 2 and 3, H must be solvable of rank 3. An H -minimal series for $U := F^\circ(H)$ must have a factor that is *not* centralized by H , and if such a factor has rank 1, we can linearize the action of H on the factor to produce a rank 1 field and thus involutions in H . Consequently, U must have rank 2 and be H -minimal. Linearizing the action of H on U , we find that $T := H/C_H(U)$ generates a field L in $\text{End}(U)$ and U is 1-dimensional over L . We now specialize this to the case when H is C_α or $N_{\alpha,\beta}$.

Suppose C_α is nonnilpotent. With notation as above, U and T are Σ_{α^\perp} -invariant, and the image of Σ_{α^\perp} in $\text{End}(U)$ is a group of field automorphisms of L . Thus Σ_{α^\perp} acts trivially on L , hence L -linearly on U . Since U is 1-dimensional, the image of Σ_{α^\perp} in $\text{End}(U)$ lies in L^\times forcing γ (which is in $[\Sigma_\alpha^\perp, \Sigma_\alpha^\perp]$) to act trivially on U . And considering the action of Σ_{α^\perp} on the rank 1 group C_α/U , we find that $[C_\alpha, \gamma, \gamma] = 1$. This implies that $[C_\alpha, \gamma]$ is an elementary abelian 3-group. There are two cases. If $[C_\alpha, \gamma] = 1$, then $C_\alpha \leq C_\gamma$, which we have seen is a contradiction. If $[C_\alpha, \gamma] \neq 1$, then C_α -minimality of U implies that U is an elementary abelian 3-group. In particular, U is normal in every definable connected rank 3 subgroup containing it. Recall that $U \leq C_\gamma$. If equality holds, then conjugacy of α and (13)(24) under the action of $N_A(\langle \gamma \rangle)$ shows C_γ is normal in $\langle C_\alpha, C_{(13)(24)} \rangle = G$, a contradiction. And if $U \not\leq C_\gamma$, then U is normal in $\langle C_\alpha, C_\gamma \rangle = G$ (by rank considerations), again a contradiction.

Finally, suppose C_α is nilpotent; thus $N_{\alpha, \beta}$ is nonnilpotent. We use the notation U and L from above, now applied to $N_{\alpha, \beta}$. This time we find that the image of K in $\text{End}(U)$ lies in L^\times , so some nontrivial $\sigma \in K$ centralizes U . If $\text{rk } C_\sigma = 3$, then as it is nilpotent, C_σ normalizes U and is not equal to $N_{\alpha, \beta}$, forcing U to be normal in G . If $\text{rk } C_\sigma = 2$, then $U = C_\sigma \geq C_{\alpha, \beta}$, so as every element of K acts on U as an element of L^\times , it must be that K centralizes U (since every element of K centralizes the nontrivial subgroup $C_{\alpha, \beta}$). Thus $U = C_\alpha = C_\beta = C_{\alpha, \beta}$, contradicting Fact 2.3. \diamond

□

3. A CONNECTION TO PERMUTATION GROUPS

A permutation group (G, X) of finite Morley rank is said to be *generically t -transitive* if G has an orbit \mathcal{O} on X^t such that $\text{rk}(X^t \setminus \mathcal{O}) < \text{rk } X^t$. For example, the natural action of $\text{GL}_n(\mathbb{K})$ on \mathbb{K}^n (with \mathbb{K} algebraically closed to ensure finiteness of Morley rank) is generically n -transitive, with \mathcal{O} being the set of bases. Similarly, $\text{AGL}_n(\mathbb{K})$ acts generically $(n+1)$ -transitively on \mathbb{K}^n , and $\text{PGL}_{n+1}(\mathbb{K})$ acts generically $(n+2)$ -transitively on $\mathbb{P}^n(\mathbb{K})$. In fact, in all three examples, every nontrivial element of the group acts without fixing any tuple of the large orbit \mathcal{O} ; in this case, we say the action is *generically sharply t -transitive*.

Notice how generic sharp t -transitivity exists (very naturally) for all t ; this stands in stark contrast to genuine sharp t -transitivity, which never exceeds $t = 3$ on an infinite set [Tit52, Hal54]. Although the degree of generic transitivity is unbounded in general, Borovik and Cherlin showed that there does exist a bound in terms of the rank of the set being acted upon [BC08, Corollary 2.2], and they have proposed the problem of showing that there is, in fact, a very natural bound given by (and characterizing!) the PGL_{n+1} example. To the best of the authors' knowledge, this problem remains open even in the algebraic case (at least in characteristic not 0).

Problem ([BC08, Problem 9]). Show that the only transitive and generically $(n+2)$ -transitive permutation group of finite Morley rank acting on a set of Morley rank n is $\text{PGL}_{n+1}(\mathbb{K})$ acting naturally on the projective n -space $\mathbb{P}^n(\mathbb{K})$ with \mathbb{K} algebraically closed.

The connection to representations of the symmetric group is as follows. Suppose (G, X) is generically t -transitive, and choose (x_1, \dots, x_t) in the generic orbit \mathcal{O} of G on X^t . Let $G_{t-1} := G_{x_1, \dots, x_{t-1}}$ be the pointwise stabilizer of x_1, \dots, x_{t-1} , and similarly define $G_t := G_{x_1, \dots, x_t}$. Now consider the subgroup S of G that stabilizes $\{x_1, \dots, x_{t-1}\}$ setwise and fixes x_t . It is easily verified that S acts *faithfully* on G_{t-1} , and S/G_t is isomorphic to $\text{Sym}(t-1)$ (see [AW18, Lemma 4.8, Lemma 4.27]). In this way, the study of generically t -transitive actions is intertwined with faithful representations (on not necessarily abelian groups) of covers of the symmetric group. Moreover, a key point of the **Problem** is that the extreme case of $t \geq \text{rk } X + 2$ should lead to the identification of (G, X) as $\text{PGL}_{n+1}(\mathbb{K})$ acting naturally on $\mathbb{P}^n(\mathbb{K})$, and in that case, G_{t-1} is precisely the maximal torus and S is the Weyl group, further underscoring importance of the subgroups G_{t-1} and S .

An important special case of the above situation is the restriction to generically *sharply* t -transitive actions. In that case, we have that $G_t = 1$ and $S \cong \text{Sym}(t-1)$. As such, our **Theorem**, yields the following modest, but nontrivial, corollary.

Corollary. *Suppose (G, X) is a generically sharply t -transitive permutation group of finite Morley rank with $\text{rk } X = n$. If $t \geq n + 2$, then the pointwise stabilizer of a generic $(t-1)$ -tuple is solvable or contains involutions.*

Proof. Assume $t \geq n + 2$; define G_{t-1} and $S \cong \text{Sym}(t-1)$ as in the previous discussion. Then S acts faithfully on G_{t-1} .

We need some additional (and straightforward) background on permutation groups of finite Morley rank. First, $t \geq 2$ implies that X is connected (i.e. of Morley degree 1) [BC08, Lemma 1.8(3)]. Also, the action of G being generically sharply t -transitive implies that G_{t-1} acts generically sharply 1-transitively on X , and this in turn implies G_{t-1} is in definable bijection with a generic subset of X . Thus G_{t-1} is connected of rank n .

Now suppose G_{t-1} is nonsolvable. As connected groups of rank 2 are solvable [Che79], $n = \text{rk } G_{t-1} \geq 3$, so $t \geq 5$. Consequently, if G_{t-1} does not contain involutions, then everything is in place to apply the **Theorem** to the action of $S \cong \text{Sym}(t-1) \geq \text{Sym}(4)$ on G_{t-1} , but this would imply that $n \geq t-1$, against our assumption that $t \geq n+2$. \square

The conclusion of the **Corollary** may be read as saying that either G_{t-1} is “close” to being a torus (as desired) or G_{t-1} is subject to further analysis via the rather well-developed theory of groups of finite Morley rank with involutions. Moreover, in the former case, existing results (e.g. Lemma 2.6 and [CDW, Theorem]) can be used to further clarify the situation. However, the remaining work needed to address the **Problem** appears to be significant and surely quite interesting.

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