Upper and lower bounds for the maximal Lyapunov exponent of singularly perturbed linear switching systems

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Abstract

In this paper we consider the problem of determining the stability properties, and in particular assessing the exponential stability, of a singularly perturbed linear switching system. One of the challenges of this problem arises from the intricate interplay between the small parameter of singular perturbation and the rate of switching, as both tend to zero. Our approach consists in characterizing suitable auxiliary linear systems that provide lower and upper bounds for the asymptotics of the maximal Lyapunov exponent of the linear switching system as the parameter of the singular perturbation tends to zero.

Keywords: Switching systems, Singular perturbation, Exponential stability, Maximal Lyapunov exponent, Differential inclusions.

1 Introduction

We consider in this paper a two-time-scales linear switching system, that is, a linear switching system for which some variables evolve on a much faster rate than the others. This class of systems appears in several industrial and engineering applications (see, e.g., [9, 15, 17]) where simplified models can be formulated by neglecting the effects of fast variables on the overall system. From control point of view, this allows to design a controller based on a reduced order model. However, the design based on a simplified model may not guarantee the stability of the overall system. To avoid this problem, a well-established framework developed in the mathematical and control community is that of singular perturbations [10]. The singular perturbation theory allows the separation between slow and fast variables where different controllers for different time-scale variables can be designed in order to lead the overall system to its desired performance.

In mathematical terms, we study the behavior of

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)y(t),\\ \varepsilon\dot{y}(t) &= C(t)x(t) + D(t)y(t),\end{aligned}$$

where ε denotes a small positive parameter and A, B, C, D are matrix-valued signals undergoing arbitrary switching within a prescribed bounded range. We deal, therefore, with a 1-parameter family of linear switching systems $\Sigma : \varepsilon \mapsto \Sigma_\varepsilon$. One of the main issues for such families of systems consists in understanding the time-asymptotic behavior of Σ_ε as $t \to +\infty$ in the regime where ε is small. For instance, by saying that Σ *is exponentially stable* we refer to the fact that for every $\varepsilon > 0$ sufficiently small, the corresponding linear switching system Σ_ε is exponentially stable. More refined notions, accounting for the uniform exponential

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behavior with respect to ε, are proposed in the paper (cf. Definition 4). Few stability criteria for singularly perturbed switching systems in the regime $\varepsilon \sim 0$ have been obtained in the literature: among them, let us mention [14], where conditions are obtained based on the existence of a common quadratic Lyapunov function, [7] characterizing the stability in dimension two based on the corresponding criteria in the non-singularly-perturbed case [4, 5], and [1], where stability for time-delay singularly perturbed switching systems is based on dwell-time criteria.

A major mathematical difficulty relies on the fact that we are interested in characterizing a doubly asymptotic regime, where the order in which the limits are taken is crucial. Indeed, a limit as $\varepsilon \to 0$ of the considered dynamics is well known to be given, through the Tikhonov decomposition, by

$$\dot{x}(t) = \big(A(t) - B(t)D(t)^{-1}C(t)\big)\, x(t), \tag{1}$$

with $y(t) = -D(t)^{-1}C(t)x(t)$. Heuristically, such a decomposition is obtained by saying that y tends instantaneously to $-D(t)^{-1}C(t)x(t)$ (the equilibrium of the equation for y when $x \equiv x(t)$). However, as it has already been observed in the literature, the switching system (1) may be exponentially stable even when, for every $\varepsilon > 0$, Σ_ε is unstable [14]. Notice that, in order to justify the Tikhonov decomposition, we are assuming here that each $D(t)$ is a Hurwitz matrix. Actually, a standing assumption in this work is that the fast dynamics are exponentially stable, i.e., the trajectories of

$$\dot{y}(t) = D(t)y(t) \tag{2}$$

converge to the origin with a uniform exponential rate. It should be noticed that this condition is necessary for the exponential stability of Σ_ε in the regime $\varepsilon \sim 0$, since if (2) admits trajectories diverging exponentially, then there is no hope for Σ_ε to be exponentially stable for ε small (a precise mathematical statement of this fact is proved in Proposition 6). In the limit situation where (2) is stable but not exponentially stable, we can still have exponential stability of Σ_ε for $\varepsilon \sim 0$ [7, Section V.A], but this is a rather degenerate situation that we do not consider here.

The goal of this paper is not only to give necessary or sufficient conditions ensuring that Σ has a certain time-asymptotic behavior in the regime $\varepsilon \sim 0$, but also to prescribe upper and lower bounds on the limit as $\varepsilon \to 0$ of the maximal Lyapunov exponent of Σ_ε. We recall that the maximal Lyapunov exponent of a linear switching system is the largest asymptotic exponential rate as the time goes to $+\infty$ among all trajectories of the system. Necessary or sufficient conditions for stability then follow as particular cases: indeed, a positive lower bound ensures that, for every ε small enough, system Σ_ε is unstable, while a negative upper bound guarantees that Σ_ε is exponentially stable for all ε in a right-neighborhood of zero.

The bounds on the limit as $\varepsilon \to 0$ of the maximal Lyapunov exponent of Σ_ε are obtained by identifying suitable auxiliary switching systems for the variable x (with a single time scale) that are either sub- or super-approximations of the asymptotic dynamics of Σ_ε as $\varepsilon \to 0$, in the following sense. A sub-approximation $\bar{\Sigma}$ of Σ is a system such that each of its trajectories can be approximated arbitrarily well, as $\varepsilon \to 0$, by the x-component of a trajectory of Σ_ε. Conversely, a super-approximation $\hat{\Sigma}$ of Σ is a system such that the x-component of each trajectory of Σ_ε can be approximated arbitrarily well, as $\varepsilon \to 0$, by a trajectory of $\hat{\Sigma}$.

The first and rather natural choice for $\bar{\Sigma}$ is system (1) since it corresponds to the singular perturbation approach, see for instance [12]. The heuristic explanation is that, if switching occurs "slowly" with respect to the time-scale $1/\varepsilon$, then the variable $y(t)$ converges fast enough to $-D(t)^{-1}C(t)x(t)$, and the transient phase is too short to affect the dynamics of $x(t)$. As for a choice of super-approximation $\hat{\Sigma}$, the idea is to consider the switching parameter as evolving on a time scale possibly much faster than $1/\varepsilon$. Hence the dynamics of y can be seen, on a short time interval, as a switching system with affine vector fields $y \mapsto C(t)\bar{x} + D(t)y$, where $\bar{x}$ is fixed. This kind of switching systems have been studied, for instance, in [6, 16]. The exponential convergence of the solutions of (2) implies that the trajectories of such an affine switching system converge exponentially towards a compact set $K(\bar{x})$ [6]. A natural choice for $\hat{\Sigma}$ is then the differential inclusion

$$\dot{x}(t) \in A(t)x(t) + B(t)K(x(t)).$$

We show that, indeed, this system allows to identify an upper bound for the limit of the maximal Lyapunov exponent of Σ_ε as $\varepsilon \to 0$.

As mentioned in the previous paragraph, the two choices of $\bar{\Sigma}$ and $\hat{\Sigma}$ just described correspond to signals switching at a rate much slower or much faster than $1/\varepsilon$. We also consider a third regime, namely, the situation in which the switching occurs at rate exactly $1/\varepsilon$. This corresponds to considering a signal $t \mapsto (A(t), B(t), C(t), D(t)) =: \sigma(t)$ defined on an interval $[0, T]$ and all its reparameterizations $\sigma_\varepsilon : t \mapsto \sigma(t/\varepsilon)$ defined on $[0, \varepsilon T]$. The flow of Σ_ε corresponding to σ_ε, evaluated at time εT, has an effect of order $O(\varepsilon)$ on the coordinate x (since the velocities are of order 1 and the length of the time interval is of order ε) and of order $O(1)$ on the coordinate y. Suitable computations show that the evolution of x can actually be described as

$$x \mapsto x + \varepsilon T \Lambda(T, \sigma) x + O(\varepsilon^2),$$

where the term $\Lambda(T, \sigma)$ does not depend on the initial condition of the variable y. We then propose another choice of sub-approximation of Σ, denoted $\check{\Sigma}$, which coincides with the switching system having as possible modes all the matrices of the type $\Lambda(T, \sigma)$. We provide an explicit expression for such matrices and we prove that, indeed, each trajectory of $\check{\Sigma}$ can be approximated arbitrarily well, as $\varepsilon \to 0$, by the x-component of a trajectory of Σ_ε. Moreover, we show that the modes of $\bar{\Sigma}$ are also modes of $\check{\Sigma}$, implying that the maximal Lyapunov exponents of $\bar{\Sigma}$ is upper bounded by that of $\check{\Sigma}$. The main result of our paper, therefore, consists in providing an interval containing all limits points of the maximal Lyapunov exponents of Σ_ε as ε goes to zero: the upper and lower bounds are given by the maximal Lyapunov exponents of $\hat{\Sigma}$ and $\check{\Sigma}$, respectively (see Theorem 9).

The paper is organized as follows: In Section 2 we present precise definitions of the systems $\bar{\Sigma}$, $\check{\Sigma}$, $\hat{\Sigma}$, and of their maximal Lyapunov exponents. We also provide the statement of the main result, Theorem 9, whose proof is given in the reminder of the paper. In particular, in Section 3 we compare the maximal Lyapunov exponents of $\bar{\Sigma}$ and the asymptotic values as ε goes to zero of the maximal Lyapunov exponent of Σ_ε (Proposition 11). The comparison with the maximal Lyapunov exponent of $\check{\Sigma}$ is the subject of Section 4 (Proposition 14), while Section 5 discusses the comparison with the maximal Lyapunov exponent of $\hat{\Sigma}$.

1.1 Notations

By $\mathbb{R}$ we denote the set of real numbers, by $|\cdot|$ the Euclidean norm of a real vector, and by $\|\cdot\|$ the induced matrix norm. We use $\mathbb{R}_+$ to denote the set of non-negative real numbers and we write $\lfloor x \rfloor$ to denote the evaluation of the floor function at a real number x. By $M_n(\mathbb{R})$ we denote the set of $n \times n$ real matrices. The $n \times n$ identity matrix is denoted by I_n. We use $\rho(M)$ to denote the spectral radius of a matrix $M \in M_n(\mathbb{R})$, defined as the largest modulus among the eigenvalues of M. By $B_r(x)$ we denote the closed ball of radius $r > 0$ and center $x \in \mathbb{R}^n$. The Hausdorff distance between two nonempty subsets X and Y of $\mathbb{R}^n$ is the quantity defined by

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\},$$

where $d(x, Y) = \inf_{y \in Y} |x - y|$ and $d(y, X) = \inf_{x \in X} |x - y|$. Given a subset $\mathcal{N}$ of $M_n(\mathbb{R})$, we denote by $\mathcal{S}_\mathcal{N}$ the set of all measurable functions from $\mathbb{R}_+$ to $\mathcal{N}$.

2 Problem statement and main results

2.1 Stability notions for singularly perturbed linear switching systems

Fix $n, m \in \mathbb{N}$ and a compact set of matrices $\mathcal{M} \subset M_{n+m}(\mathbb{R})$. Let $\Sigma = (\Sigma_\varepsilon)_{\varepsilon > 0}$ be the family of linear switching systems

$$\Sigma_\varepsilon : \quad \begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B(t)y(t), \\ \varepsilon \dot{y}(t) & = & C(t)x(t) + D(t)y(t), \end{array}$$

where ε denotes a small positive parameter and

$$t \mapsto \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}$$

is an arbitrary element of the set $\mathcal{S}_{\mathcal{M}}$ of measurable functions from $\mathbb{R}_+$ to $\mathcal{M}$.

For a given $\varepsilon > 0$, the usual stability notions, recalled in the following definition, apply to the linear switching system Σ_ε.

Definition 1. *Let $d \in \mathbb{N}$ and $\mathcal{N}$ be a bounded subset of $M_d(\mathbb{R})$. Consider the linear switching system*

$$\Sigma_{\mathcal{N}} : \quad \dot{x}(t) = N(t)x(t), \qquad N \in \mathcal{S}_{\mathcal{N}}, \tag{3}$$

and denote by $\Phi_N(t,0)$ the flow from time 0 to time t of $\Sigma_{\mathcal{N}}$ associated with the switching signal N. Then $\Sigma_{\mathcal{N}}$ is said to be

1. exponentially stable *(ES, for short) if there exist $C > 0$ and $\delta > 0$ such that*
$$\|\Phi_N(t,0)\| \leq Ce^{-\delta t}, \qquad \forall\, t \geq 0, \forall\, N \in \mathcal{S}_{\mathcal{N}};$$
2. exponentially unstable *(EU, for short) if there exist $C > 0$, $\delta > 0$, and a nonzero trajectory $t \mapsto x(t)$ of $\Sigma_{\mathcal{N}}$ such that*
$$|x(t)| \geq Ce^{\delta t}|x(0)|, \qquad \forall t \geq 0.$$

The maximal Lyapunov exponent of $\Sigma_{\mathcal{N}}$ *is defined as*

$$\lambda(\Sigma_{\mathcal{N}}) = \limsup_{t\to+\infty} \frac{1}{t} \sup_{N\in\mathcal{S}_{\mathcal{N}}} \log \|\Phi_N(t,0)\|.$$

Remark 2. *Notice that, for every $t > 0$ and every $N \in \mathcal{S}_{\mathcal{N}}$, by Gelfand's formula,*

$$\rho(\Phi_N(t,0)) = \lim_{k\to+\infty} \|\Phi_N(t,0)^k\|^{\frac{1}{k}} = \lim_{k\to+\infty} \|\Phi_{\tilde{N}}(kt,0)\|^{\frac{1}{k}}$$

where $\tilde{N}$ denotes the t-periodic signal obtained by periodization of $N|_{[0,t]}$. As a consequence,

$$\rho(\Phi_N(t,0)) \leq e^{t\lambda(\Sigma_{\mathcal{N}})}. \tag{4}$$

Actually, it is well known (see, e.g., [19]) that an equivalent definition of the maximal Lyapunov exponent is

$$\lambda(\Sigma_{\mathcal{N}}) = \limsup_{t\to+\infty} \frac{1}{t} \sup_{N\in\mathcal{S}_{\mathcal{N}}} \log \rho(\Phi_N(t,0)). \tag{5}$$

Remark 3. *It follows from the definitions that $\Sigma_{\mathcal{N}}$ is ES if and only if $\lambda(\Sigma_{\mathcal{N}}) < 0$. Also, by (5), we have that $\Sigma_{\mathcal{N}}$ is EU if and only if $\lambda(\Sigma_{\mathcal{N}}) > 0$. Moreover, the properties of exponential stability/instability and the value of $\lambda(\Sigma_{\mathcal{N}})$ keep unchanged if we replace $\mathcal{S}_{\mathcal{N}}$ by the class of piecewise-constant functions from $\mathbb{R}_+$ to $\mathcal{N}$.*

In the following we write $\Phi^\varepsilon_M(t,0)$ to denote the flow from time 0 to time t of Σ_ε associated with a signal $M \in \mathcal{S}_{\mathcal{M}}$ and we introduce the following stability notions for the 1-parameter family Σ of switching systems.

Definition 4. *We say that the 1-parameter family of linear switching systems $\Sigma : \varepsilon \mapsto \Sigma_\varepsilon$ is*

1. ε-uniformly exponentially stable *(ε-ES, for short) if there exist $\varepsilon^\star > 0$, $C > 0$, and $\delta > 0$ such that*
$$\|\Phi^\varepsilon_M(t,0)\| \leq Ce^{-\delta t},\ \forall t \geq 0,\ \forall M \in \mathcal{S}_{\mathcal{M}}, \forall \varepsilon \in (0,\varepsilon^\star); \tag{6}$$
2. ε-uniformly exponentially unstable *(ε-EU, for short) if there exist $\delta > 0$, $C > 0$, and $\varepsilon^\star > 0$ such that for every $\varepsilon \in (0,\varepsilon^\star)$ there exist $M \in \mathcal{S}_{\mathcal{M}}$ and $z_0 \neq 0$ for which*
$$|\Phi^\varepsilon_M(t,0)z_0| \geq Ce^{\delta t}|z_0|, \qquad \forall\, t \geq 0.$$

Remark 5. *The ε-uniform exponential stability of Σ is stronger than the property of global uniform asymptotic stability introduced in [7], since it not only guarantees that $\lambda(\Sigma_\varepsilon) < 0$ for every ε small enough, but also that the stability is uniform with respect to ε, i.e., $\lambda(\Sigma_\varepsilon)$ is smaller than the negative number $-\delta$, independent of ε small, and the constant C appearing in (6) is independent of ε small. Similar considerations concern the ε-uniform exponential instability.*

2.2 Auxiliary switching systems and main result

Our stability analysis of Σ relies on the comparison with several auxiliary systems having a single time scale.

Let us first discuss the system Σ_D corresponding to the evolution of the fast variable y with $x = 0$. More precisely, let

$$\mathcal{M}_D = \{D \mid \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{M}\}$$

and consider the linear switching system

$$\Sigma_D : \quad \dot{y}(t) = D(t)y(t), \qquad D \in \mathcal{S}_{\mathcal{M}_D}.$$

We first provide a useful result for the asymptotic behavior of Σ.

Proposition 6. *With the notations above, it holds that*

$$\lim_{\varepsilon \to 0} \varepsilon\lambda(\Sigma_\varepsilon) = \max\{0, \lambda(\Sigma_D)\}.$$

Proof. Consider the family $(\Sigma_{\mathcal{N}_\varepsilon})_{\varepsilon \geq 0}$ of linear switching systems, where, for every $\varepsilon \geq 0$, the compact set of matrices $\mathcal{N}_\varepsilon \subset M_{n+m}(\mathbb{R})$ is defined by

$$\mathcal{N}_\varepsilon = \left\{ \begin{pmatrix} \varepsilon A & \varepsilon B \\ C & D \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M} \right\}.$$

Then one notices that, for every $\varepsilon > 0$, the time rescaling $t \mapsto \varepsilon t$ yields

$$\phi_{N_\varepsilon}(t, 0) = \phi^\varepsilon_M(\varepsilon t, 0),$$

where M is an arbitrary signal in $\mathcal{M}$ and the signal $N_\varepsilon \in \mathcal{N}_\varepsilon$ is defined as

$$N_\varepsilon(t) = \begin{pmatrix} \varepsilon A(t) & \varepsilon B(t) \\ C(t) & D(t) \end{pmatrix}, \quad t \geq 0.$$

This implies at once that, for every ε, $\varepsilon\lambda(\Sigma_\varepsilon) = \lambda(\Sigma_{\mathcal{N}_\varepsilon})$. Next notice that the compact sets $\mathcal{N}_\varepsilon$ converge to $\mathcal{N}_0$ as ε goes to zero for the Hausdorff topology defined on the family of compact subsets of $M_{n+m}(\mathbb{R})$. One gets the conclusion using Lemma 3.5 and Equation (19) from [19]. □

The above result yields, in the particular case where $\lambda(\Sigma_D) > 0$, that $\lambda(\Sigma_\varepsilon) \geq \mu/\varepsilon$ for $\mu \in (0, \lambda(\Sigma_D))$ and $\varepsilon > 0$ small enough, hence Σ_ε is unstable with trajectories diverging at an arbitrarily large exponential rate for ε small. This motivates the following working assumption for the rest of the paper.

Assumption 7. *The switching system Σ_D is ES, that is, $\lambda(\Sigma_D) < 0$.*

In particular, all matrices in $\mathcal{M}_D$ are Hurwitz (and then invertible). We also introduce the set

$$\bar{\mathcal{M}} := \{A - BD^{-1}C \mid \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{M}\} \subset M_n(\mathbb{R}),$$

which collects the modes of the switching system

$$\bar{\Sigma} : \dot{\bar{x}}(t) = \big(A(t) - B(t)D(t)^{-1}C(t)\big)\,\bar{x}(t), \qquad \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{S}_{\mathcal{M}}.$$

As described in the introduction, we also consider as auxiliary system another switching system, denoted by $\check{\Sigma}$, with a larger set of modes than $\bar{\Sigma}$, and which could be thought of as the slow dynamics corresponding

to signals whose switching occurs at rate exactly $1/\varepsilon$. In order to define $\check{\Sigma}$, let us associate with every $T > 0$ and every $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{S}_{\mathcal{M}}$ the matrices

$$\Lambda_0(T,\sigma) = \int_0^T \Phi_D(T,s)C(s)\ ds,$$
$$\Lambda_1(T,\sigma) = \int_0^T \left(A(s) + B(s)\Lambda_0(s,\sigma)\right)\ ds,$$
$$\Lambda_2(T,\sigma) = \int_0^T B(s)\Phi_D(s,0)\ ds,$$

and

$$\Lambda(T,\sigma) = \frac{\Lambda_1(T,\sigma) + \Lambda_2(T,\sigma)\left(I_m - \Phi_D(T,0)\right)^{-1}\Lambda_0(T,\sigma)}{T}.$$

As proved in Lemma 12 given in Section 4, the set

$$\check{\mathcal{M}} := \{\Lambda(T,\sigma) \mid T > 0,\ \sigma \in \mathcal{S}_{\mathcal{M}}\}$$

is bounded in $M_n(\mathbb{R})$. The switching system $\check{\Sigma}$ is then defined as

$$\check{\Sigma}: \quad \dot{\check{x}}(t) = M(t)\check{x}(t), \quad M \in \mathcal{S}_{\check{\mathcal{M}}}. \tag{7}$$

In the case of autonomous systems, i.e., in the absence of switching, singular perturbation theory [13] guarantees that the exponential stability of Σ is completely characterized by the associated reduced dynamics $\bar{\Sigma}$. In the switching case, however, it is well known that the stability of $\bar{\Sigma}$ is not sufficient to deduce the stability of the perturbed switching system Σ_ε for $\varepsilon \sim 0$ [14]. The main difference is that, fixed $x \in \mathbb{R}^n$, in the switching case the ω-limit set $K(x)$ of the *fast dynamics*

$$\Sigma_x: \quad \dot{y}(t) = D(t)y(t) + C(t)x, \qquad \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{S}_{\mathcal{M}}, \tag{8}$$

is in general not reduced to a single equilibrium point [6, 16]. The precise definition of the set $K(x)$ is the closure of the union of the ω-limit sets of all trajectories of Σ_x starting from the initial condition $y(0) = 0$. We will prove in Proposition 17 and Lemma 19 that $K(x)$ is compact and the set-valued map $x \mapsto K(x)$ is homogeneous of degree one and globally Lipschitz continuous for the Hausdorff distance.

Let us use the set-valued map K to introduce a last auxiliary system, which is not necessarily a switching system. Namely, let us set

$$\hat{\mathcal{M}}(x) := \{Ax + By \mid \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{M},\ y \in K(x)\}, \quad x \in \mathbb{R}^n,$$

and consider the differential inclusion

$$\hat{\Sigma}: \quad \dot{x}(t) \in \hat{\mathcal{M}}(x(t)).$$

Notice that $\hat{\mathcal{M}}(x)$ is compact for every $x \in \mathbb{R}^n$, since both $\mathcal{M}$ and $K(x)$ are compact. We recall that a solution to $\hat{\Sigma}$ (in the sense of Filippov) is an absolutely continuous map $\mathbb{R}_+ \ni t \mapsto x(t) \in \mathbb{R}^n$ such that $\dot{x}(t) \in \hat{M}(x(t))$ for almost all t. The existence of solutions to $\hat{\Sigma}$ for every initial condition is a consequence of classical results on differential inclusions (see, e.g., [2, Chapter 2]).

In analogy with Definition 1, we say that $\hat{\Sigma}$ is ES (for *exponentially stable*) if there exist $C > 0$ and $\delta > 0$ such that every trajectory $x(\cdot)$ of $\hat{\Sigma}$ satisfies

$$|x(t)| \le Ce^{-\delta t}|x(0)|, \qquad \forall t \ge 0.$$

The notion of exponential instability is completely analogous to the one given for linear switching systems. As for the Lyapunov exponent of $\hat{\Sigma}$, it can be defined as

$$\lambda(\hat{\Sigma}) := \limsup_{t\to+\infty} \frac{1}{t} \sup \log|x(t)|,$$

where the sup is taken over all trajectories $x(\cdot)$ of $\hat{\Sigma}$ with $|x(0)| = 1$. Notice that the restriction to initial conditions with unit norm is justified by the fact that $x \mapsto K(x)$, and hence $x \mapsto \hat{\mathcal{M}}(x)$, is homogeneous of degree one.

The Lyapunov exponent $\lambda(\hat{\Sigma})$ satisfies the following property, which generalizes the corresponding one for switching systems recalled in Remark 3.

Lemma 8. *System $\hat{\Sigma}$ is ES if and only if $\lambda(\hat{\Sigma}) < 0$.*

Proof. The direct implication is trivial. On the other hand, by definition of $\lambda(\hat{\Sigma})$, if the latter is negative then for every $\delta \in (0, |\lambda(\hat{\Sigma})|)$ there exists $T > 0$ such that $|\hat{x}(t)| \leq e^{-\delta t}|\hat{x}(0)|$ for every solution $\hat{x}(\cdot)$ of the differential inclusion and $t \geq T$. Furthermore, $|\dot{\hat{x}}(t)| \leq \hat{C}|\hat{x}(t)|$ for some $\hat{C} > 0$, because of the compactness, continuity, and homogeneity of the map $\hat{\mathcal{M}}$. Hence $|\hat{x}(t)| \leq e^{\hat{C}t}|\hat{x}(0)| \leq \hat{K}e^{-\delta t}|\hat{x}(0)|$ on $[0, T]$, where $\hat{K} = \max_{t\in[0,T]} e^{(\hat{C}+\delta)t} = e^{(\hat{C}+\delta)T}$. We conclude that $|\hat{x}(t)| \leq \hat{K}e^{-\delta t}|\hat{x}(0)|$ for every $t > 0$. □

The following theorem summarizes the main results obtained in this paper.

Theorem 9. *Suppose that Assumption 7 holds. Then*

$$\lambda(\bar{\Sigma}) \leq \lambda(\check{\Sigma}) \leq \liminf_{\varepsilon\to 0^+} \lambda(\Sigma_\varepsilon) \leq \limsup_{\varepsilon\to 0^+} \lambda(\Sigma_\varepsilon) \leq \lambda(\hat{\Sigma}). \tag{9}$$

Moreover,

1. *if system $\check{\Sigma}$ is EU then system Σ is ε-EU;*
2. *If system $\hat{\Sigma}$ is ES then system Σ is ε-ES.*

3 Comparison between the Lyapunov exponents of Σ and $\bar{\Sigma}$

We prove in this section that $\lambda(\bar{\Sigma}) \leq \liminf_{\varepsilon\to 0^+} \lambda(\Sigma_\varepsilon)$ (Proposition 11). Strictly speaking, in view of Proposition 14 given below, this is not necessary for the proof of Theorem 9, but we prefer to provide the proof of this result since it illustrates, in a simplified framework, some of the ideas used in the proof of the inequality $\lambda(\check{\Sigma}) \leq \liminf_{\varepsilon\to 0^+} \lambda(\Sigma_\varepsilon)$.

The following lemma recalls a rather classical relation between Lyapunov exponents of linear switching systems that will be useful in the sequel. We provide its proof for completeness.

Lemma 10. *Consider a linear switching system $\Sigma_{\mathcal{N}}$ as in (3). Then, for every $\mu \in \mathbb{R}$, $\lambda(\Sigma_{\mathcal{N}+\mu I_n}) = \mu + \lambda(\Sigma_{\mathcal{N}})$.*

Proof. Let $\mu \in \mathbb{R}$. Observe that $t \mapsto x(t)$ is a trajectory of $\Sigma_{\mathcal{N}}$ if and only if $t \mapsto e^{\mu t}x(t)$ is a trajectory of $\Sigma_{\mathcal{N}+\mu I_n}$. By consequence,

$$\begin{aligned}
\lambda(\Sigma_{\mathcal{N}+\mu I_n}) &= \limsup_{t\to+\infty} \frac{1}{t} \sup_{N\in\mathcal{S}_{\mathcal{N}+\mu I_n}} \log\|\Phi_N(t,0)\| \\
&= \limsup_{t\to+\infty} \frac{1}{t} \sup_{N\in\mathcal{S}_{\mathcal{N}}} \log\|e^{\mu t}\Phi_N(t,0)\| \\
&= \mu + \lambda(\Sigma_{\mathcal{N}}),
\end{aligned}$$

concluding the proof. □

Proposition 11. *Suppose that Assumption 7 holds. Then $\lambda(\bar{\Sigma}) \leq \liminf_{\varepsilon\to 0^+} \lambda(\Sigma_\varepsilon)$.*

Proof. For $\mu \in \mathbb{R}$ and $\varepsilon > 0$ consider the switching systems

$$\Sigma_{\varepsilon,\mu}: \quad \begin{array}{rcl} \dot{x}(t) & = & A_\mu(t)x(t) + B(t)y(t), \\ \varepsilon\dot{y}(t) & = & C(t)x(t) + D_\mu(t)y(t), \end{array} \tag{10}$$

and

$$\bar{\Sigma}_\mu : \dot{\bar{x}}(t) = \big(A_\mu(t) - B(t)D^{-1}(t)C(t)\big)\,\bar{x}(t),$$

where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{S}_{\mathcal{M}}$,

$$A_\mu(t) = A(t) + \mu I_n, \quad \text{and} \quad D_\mu(t) = D(t) + \varepsilon\mu I_m.$$

It follows from Lemma 10 that

$$\lambda(\Sigma_{\varepsilon,\mu}) = \lambda(\Sigma_\varepsilon) + \mu \quad \text{and} \quad \lambda(\bar{\Sigma}_\mu) = \lambda(\bar{\Sigma}) + \mu.$$

Fix, for now, $\mu \in \mathbb{R}$, $T > 0$, and a piecewise-constant switching signal $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{S}_{\mathcal{M}}$. Denote by $t_1 < \cdots < t_N$ the switching instants of σ within the interval $[0, T]$ and set $t_0 = 0$, $t_{N+1} = T$. Following a classical approach (see e.g. [11]), we introduce the variable

$$z(t) = y(t) + D_\mu^{-1}(t)C(t)x(t) + \varepsilon P_\varepsilon(t)x(t), \tag{11}$$

where P_ε is constant on each interval $[t_k, t_{k+1})$, $k = 0, \ldots, N$, and is chosen in such a way that $\Sigma_{\varepsilon,\mu}$ is equivalently represented on each interval $[t_k, t_{k+1})$ as

$$\dot{x}(t) = \big(M_\mu(t) - \varepsilon B(t)P_\varepsilon(t)\big)\,x(t) + B(t)z(t), \tag{12}$$

$$\varepsilon\dot{z}(t) = \big(D_\mu(t) + \varepsilon Q_\varepsilon(t)B(t)\big)\,z(t), \tag{13}$$

where $M_\mu = A_\mu - BD_\mu^{-1}C$, $Q_\varepsilon = D_\mu^{-1}C + \varepsilon P_\varepsilon$, and $\|P_\varepsilon(t)\|$ is upper bounded uniformly with respect to $t \in [0, T]$ and ε small enough. System (12)-(13) is discontinuous at the instants of switching, since the variable z depends on σ. Recall that $\Phi_{\varepsilon^{-1}D_\mu}$ denotes the flow associated with $\frac{1}{\varepsilon}D_\mu$. Thanks to Assumption 7, there exist $c, \alpha > 0$ such that for ε small enough and for every $0 < s < t$

$$\|\Phi_{\varepsilon^{-1}D_\mu}(t, s)\| \leq ce^{-\frac{\alpha}{\varepsilon}(t-s)}. \tag{14}$$

Observe that there exists a positive constant K independent of ε (but possibly depending on $T, \mathcal{M}, \mu$) such that for every $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ and every $\varepsilon > 0$ small enough one has

$$|(x(t), z(t))| \leq K|(x_0, y_0)|, \qquad t \in [0, T], \tag{15}$$

where $(x(\cdot), y(\cdot))$ is the trajectory of $\Sigma_{\varepsilon,\mu}$ associated with σ and the initial condition (x_0, y_0) and $z(\cdot)$ is given by (11). By a slight abuse of notation, in what follows we still use K to denote possibly larger constants independent of ε. For $t \in [t_k, t_{k+1})$, by applying the variation of constant formula to (13) and using (11) for $t = t_k$, we have

$$z(t) = \Phi_{\varepsilon^{-1}D_\mu}(t, t_k)z(t_k) + \int_{t_k}^{t} \Phi_{\varepsilon^{-1}D_\mu}(t, s)Q_\varepsilon(s)B(s)z(s)ds.$$

As an immediate consequence of (14) and (15), one gets that

$$|z(t)| \leq (e^{-\frac{\alpha}{\varepsilon}(t-t_k)} + \varepsilon)K|(x_0, y_0)|, \; \forall t \in [t_k, t_{k+1}). \tag{16}$$

By applying the variation of constant formula to (12), one deduces that, for every $t \in [0, T]$,

$$\begin{aligned} x(t) - \bar{x}(t) = &\int_0^t \Phi_{\bar{M}_\mu}(t, s)B(s)\left(D^{-1}(s) - D_\mu^{-1}(s)\right)C(s)x(s)ds - \varepsilon\int_0^t \Phi_{\bar{M}_\mu}(t, s)B(s)P_\varepsilon(s)x(s)ds \\ &+ \int_0^t \Phi_{\bar{M}_\mu}(t, s)B(s)z(s)ds, \end{aligned}$$

where $\bar{M}_\mu := A_\mu - BD^{-1}C$. By the definition of D_μ we have that

$$\|D^{-1}(t) - D_\mu^{-1}(t)\| \leq \varepsilon K, \qquad \forall\, t \in [0,T]. \tag{17}$$

Hence, using estimates (15) and (16), one deduces that

$$|x(t) - \bar{x}(t)| \leq \varepsilon K |(x_0, y_0)|. \tag{18}$$

From (11) together with (16), (17), and (18) we obtain that

$$\begin{aligned} |y(T) + D_N^{-1} C_N \bar{x}(T)| &\leq |z(T)| + K|x(T) - \bar{x}(T)| + \varepsilon K |(x_0,y_0)| \leq (e^{-\frac{\alpha}{\varepsilon}(T-t_N)} + \varepsilon) K |(x_0,y_0)| \\ &\leq \varepsilon K |(x_0,y_0)| \end{aligned} \tag{19}$$

for ε sufficiently small, where $\left(\begin{smallmatrix} A_N & B_N \\ C_N & D_N \end{smallmatrix}\right)$ is the value of σ on $[t_N, T]$.

If $\Phi^\varepsilon_{\sigma,\mu}$ denotes the flow of (10) associated with σ, one deduces from (18) and (19) that

$$\left\| \Phi^\varepsilon_{\sigma,\mu}(T,0) - \begin{pmatrix} \Phi_{\bar{M}_\mu}(T,0) & 0 \\ -D_N^{-1} C_N \Phi_{\bar{M}_\mu}(T,0) & 0 \end{pmatrix} \right\| \leq \varepsilon K.$$

Let μ be any constant such that $\lambda(\bar{\Sigma}) + \mu = \lambda(\bar{\Sigma}_\mu) > 0$, and choose the time $T > 0$ and the piecewise-constant switching signal σ so that the spectral radius of $\Phi_{\bar{M}_\mu}(T,0)$ is larger than one (this is possible thanks to (5)). By continuity of the spectral radius, it follows that $\rho(\Phi^\varepsilon_{\sigma,\mu}(T,0))$ is also larger than one for ε small enough. Then one deduces from (4) that $\lambda(\Sigma_\varepsilon) + \mu = \lambda(\Sigma_{\varepsilon,\mu}) > 0$. The conclusion follows by arbitrariness of $\mu > -\lambda(\bar{\Sigma})$. □

4 Comparison between the Lyapunov exponents of Σ and $\check{\Sigma}$

We have the following preliminary results.

Lemma 12. *Under Assumption 7 the set $\check{\mathcal{M}}$ is bounded.*

Proof. By Assumption 7 we have that

$$\|\Phi_D(t_2,t_1)\| \leq c e^{-\alpha(t_2-t_1)} \tag{20}$$

for every $t_2 \geq t_1$, for some constants $\alpha > 0$ and $c \geq 1$ independent of the switching law $D \in \mathcal{S}_{\mathcal{M}_D}$. As a consequence of (20) and the boundedness of $\mathcal{M}$, we have $\|\Lambda_0(T,\sigma)\| \leq C_1 \min\{1,T\}$ and $\|\Lambda_2(T,\sigma)\| \leq C_1 \min\{1,T\}$ for some $C_1 > 0$, and $\|\Lambda_1(T,\sigma)\| \leq C_2 T$ for some $C_2 > 0$. Moreover since $\rho(\Phi_D(T,0)) < 1$ for every $T > 0$ and $D \in \mathcal{S}_{\mathcal{M}_D}$ according to (4), $(I_m - \Phi_D(T,0))^{-1}$ is well defined and expandable as an absolutely convergent power series in $\Phi_D(T,0)$. Letting $\bar{T} = \frac{\log(2c)}{\alpha}$ so that $\|\Phi_D(\tau,0)\| \leq \frac{1}{2}$ for every $\tau \geq \bar{T}$, we can write

$$\begin{aligned} (I_m - \Phi_D(T,0))^{-1} &= \sum_{k\geq 0} \Phi_D(T,0)^k \\ &= \sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h + \Phi_D(T,0)^{\lfloor \frac{\bar{T}}{T} \rfloor + 1} \left(\sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h \right) \\ &\quad + \Phi_D(T,0)^{2(\lfloor \frac{\bar{T}}{T} \rfloor + 1)} \left(\sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h \right) + \dots \\ &= \left(\sum_{k\geq 0} \Phi_D(T,0)^{(\lfloor \frac{\bar{T}}{T} \rfloor + 1)k} \right) \left(\sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h \right). \end{aligned}$$

As $\Phi_D(T,0)^h$ corresponds to the flow of Σ_D for a T-periodic signal at time hT we have $\|\Phi_D(T,0)^h\| \leq c$ by (20). Hence

$$\left\| \sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h \right\| \leq \left(1 + \lfloor \frac{\bar{T}}{T} \rfloor\right) c \leq c + \frac{c\bar{T}}{T}.$$

Moreover, as $(\lfloor \frac{\bar{T}}{T} \rfloor + 1)T \geq \bar{T}$,

$$\left\| \sum_{k\geq 0} \Phi_D(T,0)^{(\lfloor \frac{\bar{T}}{T} \rfloor + 1)k} \right\| \leq \sum_{k\geq 0} \|\Phi_D(T,0)^{(\lfloor \frac{\bar{T}}{T} \rfloor + 1)}\|^k \leq \sum_{k\geq 0} \frac{1}{2^k} = 2.$$

Summing up

$$\begin{aligned} \|\Lambda(T,\sigma)\| &\leq C_2 + \frac{C_1^2 \min\{1,T\}^2 (2c + 2c\bar{T}/T)}{T} \\ &= C_2 + 2cC_1^2 \min\left\{ \frac{1}{T} + \frac{\bar{T}}{T^2}, T + \bar{T} \right\} \\ &\leq C_2 + 2cC_1^2 (1 + \bar{T}), \end{aligned}$$

concluding the proof of the lemma. □

Lemma 13. *Let $T > 0$, $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{S}_{\mathcal{M}}$, and $\mu \in \mathbb{R}$. For every $\varepsilon > 0$ denote by $\mathscr{M}(\varepsilon)$ the flow at time εT of $\Sigma_{\varepsilon,\mu}$ defined in (10) and corresponding to the signal $\sigma(\cdot/\varepsilon)$. Then there exists $P(\varepsilon)$ of the form*

$$P(\varepsilon) = \begin{pmatrix} I_n & 0 \\ Q(\varepsilon) & I_m \end{pmatrix} \tag{21}$$

such that

$$P(\varepsilon)^{-1} \mathscr{M}(\varepsilon) P(\varepsilon) = \begin{pmatrix} I_n + \varepsilon T(\Lambda(T,\sigma) + \mu I_n) + O(\varepsilon^2) & O(\varepsilon) \\ 0 & \Phi_D(T,0) + O(\varepsilon) \end{pmatrix}, \tag{22}$$

where

$$Q(\varepsilon) = \Big(I_m - \Phi_D(T,0)\Big)^{-1} \Lambda_0(T,\sigma) + O(\varepsilon) \tag{23}$$

and the functions $O(\varepsilon^k)$ are such that $\|O(\varepsilon^k)\| \leq C\varepsilon^k$ for ε small, for some positive constant C independent of ε and σ.

Proof. We first prove that $\varepsilon \mapsto \mathscr{M}(\varepsilon)$ admits a first order expansion

$$\mathscr{M}(\varepsilon) = \mathscr{M}_0 + \varepsilon \mathscr{M}_1 + O(\varepsilon^2), \tag{24}$$

for some matrices $\mathscr{M}_0, \mathscr{M}_1$ to be computed. To see that, we first apply the time rescaling $\tau = t/\varepsilon$ and we have that $\mathscr{M}(\varepsilon)$ is equal to $\Phi_{N_0+\varepsilon N_1}(T,0)$, where the signals N_0 and N_1 are defined as

$$N_0(\tau) = \begin{pmatrix} 0 & 0 \\ C(\tau) & D(\tau) \end{pmatrix}, \; N_1(\tau) = \begin{pmatrix} A(\tau) + \mu I_n & B(\tau) \\ 0 & \mu I_m \end{pmatrix}.$$

By the variation of constant formula, one has

$$\Phi_{N_0+\varepsilon N_1}(T,0) = \Phi_{N_0}(T,0) + \varepsilon \int_0^T \Phi_{N_0}(T,\tau) N_1(\tau) \Phi_{N_0+\varepsilon N_1}(\tau,0)\, d\tau.$$

One deduces that (24) holds true with

$$\mathscr{M}_0 = \Phi_{N_0}(T,0) \quad \text{and} \quad \mathscr{M}_1 = \int_0^T \Phi_{N_0}(T,\tau) N_1(\tau) \Phi_{N_0}(\tau,0)\, d\tau.$$

It is easy to get that

$$\mathscr{M}_0 = \begin{pmatrix} I_n & 0 \\ \Lambda_0(T,\sigma) & \Phi_D(T,0) \end{pmatrix}, \quad \text{and} \quad \mathscr{M}_1 = \begin{pmatrix} \Lambda_1(T,\sigma) + T\mu I_n & \Lambda_2(T,\sigma) \\ \Lambda_3(T,\sigma) & \Lambda_4(T,\sigma) \end{pmatrix}, \tag{25}$$

where

$$\Lambda_3(T,\sigma) = \Lambda_0(T,\sigma)\left(\Lambda_1(T,\sigma) + T\mu\right) - \int_0^T \Phi_D(T,\tau)\Lambda_0(\tau,\sigma)(A(\tau) + B(\tau)\Lambda_0(\tau,\sigma))d\tau$$

and

$$\Lambda_4(T,\sigma) = \mu T \Phi_D(T,0) + \Lambda_0(T,\sigma)\Lambda_2(T,\sigma) - \int_0^T \Phi_D(T,\tau)\Lambda_0(\tau,\sigma)B(\tau)\Phi_D(\tau,0)d\tau.$$

According to (24) and (25), it follows that (22) holds true with $P(\varepsilon)$ and $Q(\varepsilon)$ as in (21) and (23). This concludes the proof of the lemma. □

We can now prove the main result of this section.

Proposition 14. *Suppose that Assumption 7 holds. Then* $\lambda(\bar{\Sigma}) \leq \lambda(\check{\Sigma}) \leq \liminf_{\varepsilon\to 0^+} \lambda(\Sigma_\varepsilon)$. *Moreover, if* $\check{\Sigma}$ *is EU then* Σ *is* ε*-EU.*

Proof. We first prove the inequality $\lambda(\bar{\Sigma}) \leq \lambda(\check{\Sigma})$ by showing that $\bar{\mathcal{M}} \subset \check{\mathcal{M}}$. To see that, let us check that $A - BD^{-1}C$ belongs to $\check{\mathcal{M}}$ for every $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\mathcal{M}$. Indeed, letting $T > 0$ and $\bar{\sigma} \in \mathcal{S}_{\mathcal{M}}$ constantly equal to M on $[0,T]$, it holds

$$\Lambda(T,\bar{\sigma}) = A - BD^{-1}C.$$

Assume that μ is chosen so that

$$\lambda(\check{\Sigma}) + \mu > 0.$$

Then, according to (5), there exist $\ell \in \mathbb{N}$, ℓ matrices $\Lambda(T_1,\sigma_1), \ldots, \Lambda(T_\ell,\sigma_\ell) \in \check{\mathcal{M}}$, and ℓ positive times $t_1, \ldots, t_\ell$ so that

$$\rho\Big(e^{t_\ell(\Lambda(T_\ell,\sigma_\ell)+\mu I_n)} \ldots e^{t_1(\Lambda(T_1,\sigma_1)+\mu I_n)}\Big) > 1. \tag{26}$$

Let ε be sufficiently small so that $\varepsilon T_k < t_k$ for every $k = 1, \ldots, \ell$, and denote by $N_k = \lfloor \frac{t_k}{\varepsilon T_k} \rfloor$ the number of intervals of length εT_k contained in $[0, t_k]$.

Then, for every $k = 1, \ldots, \ell$, consider the flow $\mathscr{M}_k(\varepsilon)$ of system $\Sigma_{\varepsilon,\mu}$ (cf. (10)) corresponding to the signal $\sigma_k(\cdot/\varepsilon)$, evaluated at time εT_k. Thanks to Lemma 13, there exists $P_k(\varepsilon)$ given by $P_k(\varepsilon) = \begin{pmatrix} I_n & 0 \\ Q_k(\varepsilon) & I_m \end{pmatrix}$ such that

$$\mathcal{T}_k(\varepsilon) = P_k(\varepsilon)^{-1} \mathscr{M}_k(\varepsilon) P_k(\varepsilon)$$

satisfies

$$\mathcal{T}_k(\varepsilon) = \begin{pmatrix} I_n + \varepsilon T_k(\Lambda(T_k,\sigma_k) + \mu I_n) + O(\varepsilon^2) & O(\varepsilon) \\ 0 & \Phi_D(T_k,0) + O(\varepsilon) \end{pmatrix}.$$

We repeat N_k times $\sigma_k(\cdot/\varepsilon)$ to get a signal on $[0, \varepsilon T_k N_k]$ and the corresponding flow of $\Sigma_{\varepsilon,\mu}$ at time $\varepsilon T_k N_k$ is given by

$$\mathscr{M}_k(\varepsilon)^{N_k} = P_k(\varepsilon)\mathcal{T}_k(\varepsilon)^{N_k} P_k(\varepsilon)^{-1}. \tag{27}$$

We claim that

$$\mathcal{T}_k(\varepsilon)^{N_k} = \begin{pmatrix} e^{t_k(\Lambda(T_k,\sigma_k)+\mu I_n)} + O(\varepsilon) & O(\varepsilon) \\ 0 & O(\varepsilon) \end{pmatrix}. \tag{28}$$

This follows from the general formula

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}^N = \begin{pmatrix} A_{11}^N & W \\ 0 & A_{22}^N \end{pmatrix}, \; W = \sum_{j=0}^{N-1} A_{11}^j A_{12} A_{22}^{N-j-1},$$

applied to $N = N_k$,

$$A_{11} = I + B_{11}/N_k + O(\varepsilon^2) = e^{\frac{B_{11}}{N_k}} + O(\varepsilon^2)$$

with $B_{11} = t_k(\Lambda(T_k, \sigma_k) + \mu I_n)$, and

$$A_{22} = \Phi_D(T_k, 0) + O(\varepsilon).$$

Then, for ε small enough one gets that

$$\begin{aligned} \|A_{11}^j\| &\leq \Big(1 + \frac{2\|B_{11}\|}{j}\Big)^j \leq e^{2\|B_{11}\|}, \quad 1 \leq j \leq N_k, \\ A_{11}^{N_k} &= e^{B_{11}} + O(\varepsilon). \end{aligned} \tag{29}$$

Moreover, since $\rho(A_{22}) < 1$ for ε small, it follows that

$$\|A_{22}^j\| \leq K\lambda^j, \quad 1 \leq j \leq N_k, \tag{30}$$

for some $K > 0$ and $\lambda \in (0, 1)$ independent of ε small enough and $k \in \{1, \dots \ell\}$. In particular,

$$\|A_{22}^{N_k}\| \leq K\lambda^{\frac{t_k}{\varepsilon T_k} - 1} = O(\varepsilon).$$

Using now (29), (30), and recalling that K can be taken so that $\|A_{12}\| \leq K\varepsilon$, one deduces that

$$\|W\| \leq \sum_{j=0}^{N_k-1} \|A_{11}^j\| \|A_{12}\| \|A_{22}^{N_k-j-1}\| \leq K^2 e^{2\|B_{11}\|} \varepsilon \sum_{j=0}^{N_k-1} \lambda^{N_k-1-j} = C\varepsilon,$$

for some $C > 0$ independent of ε small enough and $k \in \{1, \dots \ell\}$. This concludes the proof of (28).

We next use (23) and (28) in (27) to deduce that

$$\mathscr{M}_k(\varepsilon)^{N_k} = \begin{pmatrix} e^{t_k(\Lambda(T_k,\sigma_k)+\mu I_n)} & 0 \\ r_k & 0 \end{pmatrix} + O(\varepsilon), \tag{31}$$

where

$$r_k = \Big(I_m - \Phi_{D_{\sigma_k}}(T_k, 0)\Big)^{-1} \Lambda_0(T_k, \sigma_k) e^{t_k(\Lambda(T_k,\sigma_k)+\mu I_n)}.$$

Set $t_\varepsilon = \varepsilon \sum_{k=1}^{\ell} N_k T_k$ and notice that t_ε tends to $t_1 + \cdots + t_\ell$ as ε tends to zero. We concatenate the N_k times repetitions of $\sigma_k(\cdot/\varepsilon)$ for $k = 1, \dots, \ell$ to get a signal on $[0, t_\varepsilon]$ and the corresponding flow of $\Sigma_{\varepsilon,\mu}$ at time t_ε is given by the matrix product

$$\Upsilon_\varepsilon = \mathscr{M}_\ell(\varepsilon)^{N_\ell} \mathscr{M}_{\ell-1}(\varepsilon)^{N_{\ell-1}} \cdots \mathscr{M}_1(\varepsilon)^{N_1}.$$

Using (31), one deduces that

$$\Upsilon_\varepsilon = \begin{pmatrix} e^{t_\ell(\Lambda(T_\ell,\sigma_\ell)+\mu I_n)} \cdots e^{t_1(\Lambda(T_1,\sigma_1)+\mu I_n)} & 0 \\ r & 0 \end{pmatrix} + O(\varepsilon),$$

where the matrix r does not depend on ε.

Using (26), one gets that $\rho(\Upsilon_\varepsilon) > 1$ for ε small enough, yielding that

$$\lambda(\Sigma_\varepsilon) + \mu = \lambda(\Sigma_{\varepsilon,\mu}) > 0.$$

In particular $\liminf_{\varepsilon\to 0^+} \lambda(\Sigma_\varepsilon) + \mu \geq 0$. By letting μ tend to $-\lambda(\check{\Sigma})$ from above, one concludes that $\lambda(\check{\Sigma}) \leq \liminf_{\varepsilon\to 0^+} \lambda(\Sigma_\varepsilon)$ as desired.

We are left to show that if $\check{\Sigma}$ is EU then Σ is ε-EU. For this purpose, we take $\mu = 0$ in the previous calculations and we observe that the flow $\Upsilon_{\varepsilon,t}$ of Σ_ε at time $t \in [0, t_\varepsilon]$ corresponding to the signal defined above satisfies

$$\Upsilon_{\varepsilon,t} = \begin{pmatrix} e^{(t-\sum_{h=1}^{k-1} t_h)\Lambda(T_k,\sigma_k)} \prod_{\ell=1}^{k-1} e^{t_\ell \Lambda(T_\ell,\sigma_\ell)} + O(\varepsilon) & O(\varepsilon) \\ r(\varepsilon,t) & q(\varepsilon,t) \end{pmatrix}, \tag{32}$$

whenever $t \in [\sum_{h=1}^{k-1} t_h, \sum_{h=1}^{k} t_h]$, for some matrix functions r, q, where the terms $O(\varepsilon)$ are uniform with respect to $t \in [0, t_\varepsilon]$. The matrix $\Upsilon_{\varepsilon,t_\varepsilon}$ converges, as ε goes to zero, to

$$\bar{\Upsilon} = \begin{pmatrix} e^{t_\ell \Lambda(T_\ell,\sigma_\ell)} \cdots e^{t_1 \Lambda(T_1,\sigma_1)} & 0 \\ \bar{r} & 0 \end{pmatrix},$$

for some matrix $\bar{r}$. For ε small enough we construct a trajectory $z_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$ of Σ_ε satisfying $|z_\varepsilon(t)| \geq \hat{C} e^{\hat{\lambda} t} |z_\varepsilon(0)| > 0$ for every $t \geq 0$, with $\hat{\lambda} \in (0, \lambda(\check{\Sigma}))$ and $\hat{C} > 0$ independent of ε. Let π be the sum of the projectors on the generalized eigenspaces associated with the eigenvalues of $\bar{\Upsilon}$ of modulus $\rho(e^{t_\ell \Lambda(T_\ell,\sigma_\ell)} \cdots e^{t_1 \Lambda(T_1,\sigma_1)}) > 1$. Since $\Upsilon_{\varepsilon,t_\varepsilon}$ converges to $\bar{\Upsilon}$ as ε goes to zero, by classical results (see [8, Theorem 5.1, Chapter II]) there exists π_ε, a sum of projectors on generalized eigenspaces of $\Upsilon_{\varepsilon,t_\varepsilon}$, satisfying $\lim_{\varepsilon\to 0} \pi_\varepsilon = \pi$, and the corresponding eigenvalues also converge. Let v_ε be a possibly complex eigenvector of the restriction of $\Upsilon_{\varepsilon,t_\varepsilon}$ to the image of π_ε, associated with an eigenvalue α_ε. If α_ε is real then v_ε can be taken real as well, otherwise we assume without loss of generality that v_ε satisfies $|\mathrm{Re}(v_\varepsilon)| = \min_{\theta\in\mathbb{R}} |\mathrm{Re}(e^{i\theta} v_\varepsilon)|$. In particular $|\mathrm{Re}(\beta v_\varepsilon)| \geq |\beta||\mathrm{Re}(v_\varepsilon)|$ for every $\beta \in \mathbb{C}$. Note that, for every positive integer k, one has that $(\Upsilon_{\varepsilon,t_\varepsilon})^k v_\varepsilon = \alpha_\varepsilon^k v_\varepsilon$ and $(\Upsilon_{\varepsilon,t_\varepsilon})^k \bar{v}_\varepsilon = \bar{\alpha}_\varepsilon^k \bar{v}_\varepsilon$ which implies $(\Upsilon_{\varepsilon,t_\varepsilon})^k \mathrm{Re}(v_\varepsilon) = \mathrm{Re}(\alpha_\varepsilon^k v_\varepsilon)$.

Consider the trajectory $z_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$ of Σ_ε obtained applying the flow $\Upsilon_{\varepsilon,t}$ to the initial condition $z_\varepsilon(0) = \mathrm{Re}(v_\varepsilon)$ and repeating periodically after time t_ε. Letting Π_x be the projection of a vector of $\mathbb{R}^{n+m}$ onto its first n components, it is easy to see that $|\Pi_x v| \geq C|v|$ for every v in the image of $\bar{\Upsilon}$, where

$$C = (1 + \|\bar{r}\|^2 \|(e^{t_\ell \Lambda(T_\ell,\sigma_\ell)} \cdots e^{t_1 \Lambda(T_1,\sigma_1)})^{-1}\|^2)^{-1/2}.$$

Hence, for every nonnegative integer h and ε small enough,

$$\begin{aligned} |x_\varepsilon(ht_\varepsilon)| &= |\Pi_x z_\varepsilon(ht_\varepsilon)| \\ &= |\Pi_x \pi_\varepsilon z_\varepsilon(ht_\varepsilon)| \\ &\geq |\Pi_x \pi z_\varepsilon(ht_\varepsilon)| - \|\Pi_x\| \|\pi - \pi_\varepsilon\| \, |z_\varepsilon(ht_\varepsilon)| \\ &\geq |\Pi_x \pi z_\varepsilon(ht_\varepsilon)| - \|\pi - \pi_\varepsilon\| \, |z_\varepsilon(ht_\varepsilon)| \\ &\geq C|\pi z_\varepsilon(ht_\varepsilon)| - \|\pi - \pi_\varepsilon\| \, |z_\varepsilon(ht_\varepsilon)| \\ &\geq C|z_\varepsilon(ht_\varepsilon)| - (1 + C)\|\pi - \pi_\varepsilon\| \, |z_\varepsilon(ht_\varepsilon)| \\ &\geq \frac{C}{2} |z_\varepsilon(ht_\varepsilon)|. \end{aligned} \tag{33}$$

As $z_\varepsilon(ht_\varepsilon) = \mathrm{Re}(\alpha_\varepsilon^h v_\varepsilon)$ we also have

$$|x_\varepsilon(ht_\varepsilon)| \geq \frac{C}{2} |\alpha_\varepsilon|^h |z_\varepsilon(0)|. \tag{34}$$

By (32), (33), and setting $\kappa = \max_{\Lambda\in\check{\mathcal{M}}} \|\Lambda\|$ it follows that

$$\begin{aligned} |x_\varepsilon(ht_\varepsilon + \tau)| &\geq |x_\varepsilon(ht_\varepsilon)| e^{-\kappa\tau} - |z_\varepsilon(ht_\varepsilon)| O(\varepsilon) \\ &\geq |x_\varepsilon(ht_\varepsilon)| e^{-\kappa\tau} - \frac{2}{C} |x_\varepsilon(ht_\varepsilon)| O(\varepsilon) \\ &\geq \frac{1}{2} |x_\varepsilon(ht_\varepsilon)| e^{-\kappa\tau} \end{aligned} \tag{35}$$

for every $\tau \in [0, t_\varepsilon]$. Take ε small enough in such a way that $1 < \underline{\alpha} \leq |\alpha_\varepsilon| \leq \bar{\alpha}$ for some positive constants $\underline{\alpha}, \bar{\alpha}$, and $t_\varepsilon \leq \bar{t}$, where $\bar{t} = 2\sum_{h=1}^{\ell} t_h$. By (34) and (35) we thus get for every $t \geq 0$

$$|z_\varepsilon(t)| \geq |x_\varepsilon(t)| \geq \frac{C}{4} e^{-\kappa t_\varepsilon} |\alpha_\varepsilon|^{\lfloor \frac{t}{t_\varepsilon} \rfloor} |z_\varepsilon(0)| \geq \frac{C e^{-\kappa t_\varepsilon}}{4|\alpha_\varepsilon|} e^{\frac{\log |\alpha_\varepsilon|}{t_\varepsilon} t} |z_\varepsilon(0)| \geq \hat{C} e^{\hat{\lambda} t} |z_\varepsilon(0)|,$$

where $\hat{C} = \frac{C e^{-\kappa \bar{t}}}{4\bar{\alpha}}$ and $\hat{\lambda} = \frac{\log \underline{\alpha}}{\bar{t}}$, which shows that Σ is ε-EU. □

Example 15 ($\check{\Sigma}$ gives sharper bounds than $\bar{\Sigma}$)**.** *Consider system Σ with*

$$\mathcal{M} = \left\{ M_1 = \begin{pmatrix} -1 & 1 \\ 0 & -0.1 \end{pmatrix}, M_2 = \begin{pmatrix} -3 & 0 \\ 2 & -0.1 \end{pmatrix} \right\}.$$

The stability of singularly perturbed planar switching systems is completely characterized in [7, Theorem 2] through some necessary and sufficient conditions. Based on this characterization (cf., in particular, Item (SP5) in [7, Theorem 2]) the condition

$$\Gamma(M_1, M_2) := \frac{1}{2} \left(\mathrm{tr}(M_1)\mathrm{tr}(M_2) - \mathrm{tr}(M_1 M_2) \right) < -\sqrt{\det(M_1) \det(M_2)} \tag{36}$$

implies that Σ_ε is EU for all $\varepsilon > 0$. Condition (36) is satisfied in the case of this example with $\Gamma(M_1, M_2) = -0.8$ and $\det(M_1 M_2) = 0.03$. Look now at systems $\bar{\Sigma}$ and $\check{\Sigma}$. We have $\bar{\mathcal{M}} = \{-1, -3\}$ and then the associated system $\bar{\Sigma}$ is ES. Concerning system $\check{\Sigma}$, let us consider the switching signal

$$\sigma(t) = \alpha(t) M_1 + (1 - \alpha(t)) M_2$$

associated with the 2-periodic function

$$\alpha(t) = \begin{cases} 1 & t \in [0, 1], \\ 0 & t \in [1, 2], \end{cases}$$

and take $T = 2$. For this choice of σ and T one can easily verify that

$$\Lambda(T, \sigma) = -2 + 100(1 - e^{-0.2})^{-1}(1 - e^{-0.1})^2 > 0.$$

Then $\check{\Sigma}$ is EU, as illustrated in Figure 1.

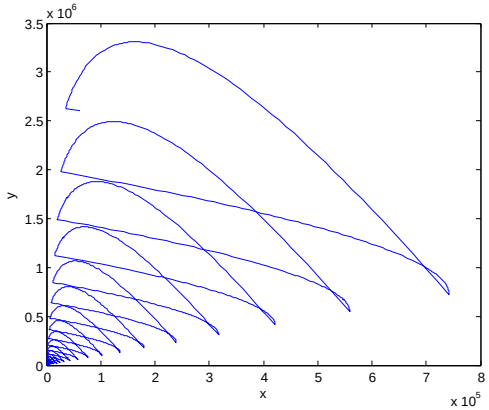

Figure 1: Fast and slow variables evolution of system Σ_ε with signal σ and $\varepsilon = 0.1$, starting from $(x_0, y_0) = (1, 1)$.

5 Comparison between the Lyapunov exponents of Σ and $\hat{\Sigma}$

5.1 Definition and structural properties of the differential inclusion $\hat{\Sigma}$

The following lemma studies the ω-limit set of the dynamics Σ_x given by (8).

Lemma 16. *Let $x \in \mathbb{R}^n$. Then, for every $y_0 \in \mathbb{R}^m$ and $\sigma \in \mathcal{S}_{\mathcal{M}}$, the ω-limit set ω_σ^x of the trajectory of Σ_x associated with σ and starting at y_0 does not depend on the initial condition y_0.*

Proof. Let $y_0, y_1 \in \mathbb{R}^m$ and let $y_0(\cdot), y_1(\cdot)$ be the trajectories of Σ_x associated with σ and starting from y_0 and y_1, respectively. Setting $z(\cdot) = y_0(\cdot) - y_1(\cdot)$, one has $\dot{z}(t) = D(t)z(t)$, and thanks to Assumption 7, there exist $c, \delta > 0$ such that

$$|y_0(t) - y_1(t)| \leq ce^{-\delta t}|y_0 - y_1|, \quad \forall\, t \geq 0. \tag{37}$$

By definition of an ω-limit set, one deduces at once that ω_σ^x does not depend on y_0. □

Let us introduce the set valued-map $K : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ defined by $K(x) = \overline{\bigcup_\sigma \omega_\sigma^x}$. We have the following proposition.

Proposition 17. *For each $x \in \mathbb{R}^n$ the set $K(x)$ is compact and forward invariant for the dynamics of Σ_x, and there exist $c, \delta > 0$ such that*

$$d(y_0(t), K(x)) \leq ce^{-\delta t} d(y_0, K(x)), \quad \forall t \geq 0,\ y_0 \in \mathbb{R}^m, \tag{38}$$

for all $\sigma \in \mathcal{S}_{\mathcal{M}}$, where $y_0(\cdot)$ is the trajectory of Σ_x associated with σ and starting from y_0.

Proof. Let c and δ be as in (37). The set $K(x)$ is closed by definition and its boundedness follows from the fact that for every $\sigma \in \mathcal{S}_{\mathcal{M}}$ the corresponding solution $y(\cdot)$ of Σ_x with initial condition $y(0) = 0$ satisfies

$$|y(t)| = \left| \int_0^t \Phi_D(t, s) C(s) x \, ds \right| \leq \frac{c|x| \max_{M \in \mathcal{M}} \|M\|}{\delta},$$

for $t \geq 0$. Hence $K(x)$ is compact.

Let now $\bar{y} \in K(x)$, i.e., $\bar{y} = \lim_{k \to +\infty} y^k$, where $y^k \in \omega_{\sigma_k}^x$, $\sigma_k \in \mathcal{S}_{\mathcal{M}}$. For $\tilde{t} > 0$ and $\tilde{\sigma}$, denote by $\Phi_{x,\tilde{\sigma}}$ the flow of Σ_x associated with the signal $\tilde{\sigma}$ and let us prove that $\tilde{y} := \Phi_{x,\tilde{\sigma}}(\tilde{t}, 0)\bar{y}$ is in $K(x)$. For $k \geq 1$, let $r_k = |\tilde{y} - \Phi_{x,\tilde{\sigma}}(\tilde{t}, 0)y^k)|$ and notice that $\lim_{k \to +\infty} r_k = 0$. Moreover, for $k \geq 1$, there exists $\rho_k > 0$ such that $\Phi_{x,\tilde{\sigma}}(\tilde{t}, 0)B_{\rho_k}(y^k) \subset B_{2r_k}(\tilde{y})$. We next define recursively the sequence $(z_k)_{k \geq 0}$ with $z_0 \in \mathbb{R}^m$ by setting

$$z_{k+1} = \Phi_{x,\tilde{\sigma}}(\tilde{t}, 0)\Phi_{x,\sigma_{k+1}}(t_k, 0)z_k, \qquad k \geq 0,$$

where the sequence of times t_k is chosen so that $\Phi_{x,\sigma_{k+1}}(t_k, 0)z_k$ is in $B_{\rho_{k+1}}(y^{k+1})$. This is possible since $y^{k+1} \in \omega_{\sigma_{k+1}}^x$ for $k \geq 0$. By construction, z_k is in $B_{2r_k}(\tilde{y})$ for every $k \geq 1$. Moreover, $z_k = \Phi_{x,\bar{\sigma}}(\tau_k, 0)z_0$ where $\tau_k \to \infty$ and $\bar{\sigma}$ is constructed by repeatedly concatenating $\sigma_k|_{[0,t_k]}$ and $\tilde{\sigma}|_{[0,\tilde{t}]}$. Then $\tilde{y}$ is in $\omega_{\bar{\sigma}}^x \subset K(x)$.

Finally, using (37) with $y_1 \in K(x)$ and the forward invariance of $K(x)$ for the dynamics of Σ_x, we have

$$d(y_0(t), K(x)) \leq |y_0(t) - y_1(t)| \leq ce^{-\delta t}|y_0 - y_1|, \quad \forall\, t \geq 0,$$

and one gets (38) by arbitrariness of $y_1 \in K(x)$. □

Remark 18. *The contents of the above proposition are essentially contained in the preprint [6] (Theorem 1 and Proposition 2), where the authors study general switching affine systems and the role of $K(x)$ is played by the set $\mathcal{K}_\infty$.*

Lemma 19. *The set-valued map K is globally Lipschitz continuous for the Hausdorff distance and homogeneous of degree one.*

Proof. Notice that, given $x \in \mathbb{R}^n$ and a nonzero $\lambda \in \mathbb{R}$, $y(\cdot)$ is a trajectory of Σ_x if and only if $\lambda y(\cdot)$ is a trajectory of $\Sigma_{\lambda x}$. We deduce that $z \in \omega_\sigma^x$ if and only if $\lambda z \in \omega_\sigma^{\lambda x}$ and hence that $K(\lambda x) = \lambda K(x)$.

As for the Lipschitz continuity of K, let $x_1, x_2 \in \mathbb{R}^n$. It is easy to deduce from the variation of constant formula that there exists $L_K > 0$ independent of x_1, x_2 such that

$$|y_1(t) - y_2(t)| \leq L_K |x_1 - x_2|, \qquad \forall\, t \geq 0, \forall\, \sigma \in \mathcal{S}_{\mathcal{M}},$$

where $y_1(\cdot)$ and $y_2(\cdot)$ are the trajectories of Σ_{x_1} and Σ_{x_2}, respectively, associated with σ and starting from the same initial condition y_0. By consequence, we have

$$\max\{d(y_1, \omega_\sigma^{x_2}), d(y_2, \omega_\sigma^{x_1})\} \leq L_K |x_1 - x_2|,$$

for $y_1 \in \omega_\sigma^{x_1}$ and $y_2 \in \omega_\sigma^{x_2}$, where $\omega_\sigma^{x_1}$ and $\omega_\sigma^{x_2}$ are the ω-limit sets of Σ_{x_1} and Σ_{x_2}, respectively, which, thanks to Lemma 16, do not depend on the initial condition y_0. From the previous inequality we obtain that,

$$\max\{d(y_1, \cup_\sigma \omega_\sigma^{x_2}), d(y_2, \cup_\sigma \omega_\sigma^{x_1})\} \leq L_K |x_1 - x_2|,$$

for $y_1 \in \cup_\sigma \omega_\sigma^{x_1}$ and $y_2 \in \cup_\sigma \omega_\sigma^{x_2}$. By a standard density argument,

$$\max\{d(y_1, K(x_2)), d(y_2, K(x_1))\} \leq L_K |x_1 - x_2|,$$

for $y_1 \in K(x_1)$ and $y_2 \in K(x_2)$, from which we obtain, by the arbitrariness of y_1 and y_2, that

$$d_H(K(x_1), K(x_2)) \leq L_K |x_1 - x_2|,$$

where we recall that d_H denotes the Hausdorff distance in $\mathbb{R}^m$. □

5.2 Asymptotic estimates by converse Lyapunov arguments

The argument provided below bears similarities with proofs given in [18], where more general dynamics are considered.

We consider the μ-shifted differential inclusion

$$\hat{\Sigma}_\mu : \quad \dot{\hat{x}} \in \{A_\mu \hat{x} + By \mid M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{M},\ y \in K(\hat{x})\},$$

where $A_\mu = A + \mu I_n$ with $\mu \in \mathbb{R}$. By homogeneity of $K(\cdot)$ it follows that for every solution $\hat{x}(\cdot)$ of $\hat{\Sigma}$ the trajectory $t \mapsto \hat{x}_\mu(t) = e^{\mu t}\hat{x}(t)$ is a solution of $\hat{\Sigma}_\mu$. As a consequence $\lambda(\hat{\Sigma}_\mu) = \lambda(\hat{\Sigma}) + \mu$. Hence, recalling that $\lambda(\Sigma_{\varepsilon,\mu}) = \lambda(\Sigma_\varepsilon) + \mu$, in order to prove the right-hand side of inequality (9) it is enough to show that, under Assumption 7, $\lambda(\hat{\Sigma}_\mu) < 0$ implies $\lambda(\Sigma_{\varepsilon,\mu}) < 0$ for ε small enough. In order to prove the latter statement we will construct a common Lyapunov function for the systems $\Sigma_{\varepsilon,\mu}$ as the sum of Lyapunov functions for $\hat{\Sigma}_\mu$ and for Σ_x defined next. Assume that $\lambda(\hat{\Sigma}_\mu) < 0$. Then, by Lemma 8, $|\hat{x}(t)| \leq ce^{-\gamma t}|\hat{x}(0)|$ for some $c \geq 1$ and $\gamma > 0$, for every trajectory $\hat{x}(\cdot)$ of $\hat{\Sigma}_\mu$. Define

$$V_1(x) = \sup_{\hat{x}(\cdot), t \in [0, \hat{t}\,]} e^{\gamma t}|\hat{x}(t)|^2, \tag{39}$$

where the supremum is computed among all trajectories $\hat{x}(\cdot)$ of $\hat{\Sigma}_\mu$ starting from $x \in \mathbb{R}^n$, and $\hat{t} = \frac{\log(c)}{2\gamma}$.

Furthermore, let $\delta \in (0, |\lambda(\Sigma_D)|)$ and $\bar{c} \geq 1$ be such that $|y(t)| \leq \bar{c}e^{-\delta t}|y(0)|$ for every trajectory $y(\cdot)$ of Σ_D. By Proposition 17 one has

$$d(y_x(t), K(x)) \leq \bar{c}e^{-\delta t} d(y_x(0), K(x))$$

for every trajectory y_x of Σ_x, for every $x \in \mathbb{R}^n$ and $t > 0$. Define

$$V_2(x, y) = \sup_{y_x(\cdot), t \in [0, \bar{t}\,]} e^{\delta t} d(y_x(t), K(x))^2, \tag{40}$$

where the supremum is computed among all trajectories $y_x(\cdot)$ of Σ_x starting from $y \in \mathbb{R}^m$ and $\bar{t} = \frac{\log(\bar{c})}{2\delta}$.

In the next lemma, we summarize the main properties of V_1 and V_2.

Lemma 20. *The positive definite functions V_1 and V_2 introduced in* (39) *and* (40) *are homogeneous of degree two and locally Lipschitz continuous. Moreover, V_1 and V_2 are nonincreasing along every trajectory of $\hat{\Sigma}_\mu$ and Σ_x respectively and satisfy the following estimates:*

$$V_1(\hat{x}(t)) \leq e^{-\gamma t} V_1(\hat{x}(0)), \tag{41}$$

$$V_2(x, \bar{y}_x(t)) \leq e^{-\delta t} V_2(x, \bar{y}_x(0)), \tag{42}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $\hat{x}(\cdot)$ is an arbitrary trajectory of $\hat{\Sigma}_\mu$, and $\bar{y}_x(\cdot)$ is an arbitrary trajectory of Σ_x.

Proof. It is clear that both V_1 and V_2 are homogeneous of degree two. We give the proof of the remaining properties for V_2, the corresponding arguments for V_1 being completely analogous.

Let us next show that V_2 is locally Lipschitz continuous. For every bounded set $\mathcal{B} \subset \mathbb{R}^m$ there exists a compact set of $\mathbb{R}^m$ containing every trajectory of Σ_x starting from $\mathcal{B}$. Take two points y_1, y_2 in $\mathcal{B}$ and consider the trajectories y_x^1, y_x^2 of Σ_x corresponding to the same switching law starting respectively from y_1 and y_2. Then $|y_x^2(t) - y_x^1(t)| \leq \bar{c} e^{-\delta t} |y_2 - y_1|$ for every $t \geq 0$. We deduce that the function $y \mapsto e^{\delta t} d(y_x(t), K(x))^2$, where y_x is the trajectory starting from $y \in \mathcal{B}$ corresponding to a fixed switching law $\sigma \in \mathcal{S}_\mathcal{M}$, is Lipschitz continuous, and the Lipschitz constant does not depend on t nor on the switching law. Since the supremum among a family of uniformly Lipschitz continuous functions is Lipschitz continuous, we deduce that V_2 is locally Lipschitz continuous in the variable y. Similarly, local Lipschitz continuity of the map $x \mapsto e^{\delta t} d(y_x(t), K(x))^2$ for a fixed switching law follows from the fact that $y_x(\cdot)$ is affine with respect to the variable x and $K(\cdot)$ is Lipschitz continuous. Furthermore, the corresponding Lipschitz constant (locally) does not depend on t nor on the switching law.

Consider now a trajectory $\bar{y}_x(\cdot)$ of Σ_x and let us prove that $V_2(x, \cdot)$ is nonincreasing along it. For $h > 0$ one has

$$\begin{aligned}
V_2(x, \bar{y}_x(h)) &= \sup_{\substack{y_x(\cdot),\ y_x|_{[0,h]} = \bar{y}_x|_{[0,h]} \\ t \in [h, \bar{t}+h]}} e^{\delta(t-h)} d(y_x(t), K(x))^2 \\
&\leq \sup_{\substack{y_x(\cdot),\ y_x(0) = \bar{y}_x(0) \\ t \in [h, \bar{t}+h]}} e^{\delta(t-h)} d(y_x(t), K(x))^2 \\
&\leq \sup_{\substack{y_x(\cdot),\ y_x(0) = \bar{y}_x(0) \\ t \in [0, \bar{t}+h]}} e^{\delta(t-h)} d(y_x(t), K(x))^2 \\
&= e^{-\delta h} \sup_{\substack{y_x(\cdot),\ y_x(0) = \bar{y}_x(0) \\ t \in [0, \bar{t}]}} e^{\delta t} d(y_x(t), K(x))^2 \\
&= e^{-\delta h} V_2(x, \bar{y}_x(0)),
\end{aligned}$$

where we have used the fact that

$$e^{\delta t} d(y_x(t), K(x))^2 \leq \bar{c}^2 e^{-\delta t} d(y_x(0), K(x))^2 < d(y_x(0), K(x))^2$$

for every $t > \bar{t}$ along every trajectory of Σ_x. □

Based on the above construction of V_1 and V_2, we next show the existence of a common Lyapunov function allowing us to prove that $(\Sigma_{\varepsilon,\mu})_{\varepsilon>0}$ is ε-ES.

Proposition 21. *There exists $\chi > 0$, $0 < \alpha_- < \alpha_+$, $\eta > 0$, and $\varepsilon_* > 0$ such that, setting $V = V_1 + \chi V_2$, one has*

$$\alpha_- |(x, y)|^2 \leq V(x, y) \leq \alpha_+ |(x, y)|^2, \qquad \forall x \in \mathbb{R}^n,\ \forall y \in \mathbb{R}^m, \tag{43}$$

$$V(x(t), y(t)) \leq V(x(0), y(0)) e^{-\eta t}, \qquad \forall t \geq 0, \tag{44}$$

where (44) *holds true for every solution $(x(\cdot), y(\cdot))$ of $\Sigma_{\varepsilon,\mu}$ for $\varepsilon < \varepsilon_*$. As a consequence $\limsup_{\varepsilon \to 0^+} \lambda(\Sigma_\varepsilon) \leq \lambda(\hat{\Sigma})$ and if $\lambda(\hat{\Sigma}) < 0$ then Σ is ε-ES.*

Proof. The right inequality in (43) follows from the bounds

$$V_1(x) \leq c^2|x|^2$$

and

$$V_2(x,y) \leq \bar{c}^2\, d(y,K(x))^2 \leq \bar{c}^2\, (|y| + d(0,K(x)))^2 \leq \bar{c}^2(|y| + L_K|x|)^2,$$

where L_K is the Lipschitz constant for $K(\cdot)$. Concerning the left inequality in (43), note that $d(y,K(x)) \geq |y| - L_K|x|$, where L_K is the Lipschitz constant for $K(\cdot)$. Then, either $|y| > 2L_K|x|$, in which case $d(y,K(x)) > \frac{1}{2}|y|$ and

$$V(x,y) \geq |x|^2 + \chi d(y,K(x))^2 > |x|^2 + \frac{\chi}{4}|y|^2,$$

or $|y| \leq 2L_K|x|$, in which case

$$V(x,y) \geq |x|^2 \geq \frac{1}{1+4L_K^2}|(x,y)|^2.$$

The desired inequality holds true with

$$\alpha_- = \min\left\{\frac{1}{1+4L_K^2}, \frac{\chi}{4}\right\}.$$

Consider now a trajectory $(x(\cdot), y(\cdot))$ of $\Sigma_{\varepsilon,\mu}$ corresponding to the switching law σ. In order to prove (44) we first estimate the difference $V(x(\varepsilon h), y(\varepsilon h)) - V(x(0), y(0))$ for small h. Note that

$$\begin{aligned} |x(\varepsilon h) - x(0)| &= \varepsilon |(x(0), y(0))|\, O(h), \\ |y(\varepsilon h) - y(0)| &= |(x(0), y(0))|\, O(h), \end{aligned}$$

where $O(h)$ denotes a function bounded in absolute value by Ch, where the constant C does not depend on $x(0)$, $y(0)$, h, nor ε, for ε, h in a small right-neighborhood of zero. Knowing that

$$t \mapsto d(\dot{x}(t), A_\mu(t)x(t) + B(t)K(x(t))) \leq \|B\| d(y(t), K(x(t)))$$

is integrable, we deduce from Theorem 10.4.1 in [3] that there exists a solution $\tilde{x}(\cdot)$ of $\hat{\Sigma}_\mu$ such that $\tilde{x}(0) = x(0)$ and

$$\begin{aligned} |x(\varepsilon h) - \tilde{x}(\varepsilon h)| &\leq \|B\| e^{(\|A_\mu\| + L_K\|B\|)\varepsilon h} \int_0^{\varepsilon h} d(y(s), K(x(s)))ds \\ &\leq C_1 \varepsilon h\, d(y(0), K(x(0))) + \varepsilon |(x(0), y(0))|\, O(h^2), \end{aligned}$$

for some $C_1 > 0$, where we have used the fact that

$$d(p,P) \leq |p-q| + d(q,Q) + d_H(Q,P),$$

for $p, q \in \mathbb{R}^m$ and $P, Q \subset \mathbb{R}^m$. Hence

$$V_1(x(\varepsilon h)) - V_1(\tilde{x}(h)) \leq \varepsilon |(x(0), y(0))|\, d(y(0), K(x(0)))\, O(h) + \varepsilon |(x(0), y(0))|^2\, O(h^2),$$

as it follows from the fact that the Lipschitz constant of V_1 on a ball of radius r is of order r. By using (41),

$$\begin{aligned} V_1(x(\varepsilon h)) - V_1(x(0)) &= V_1(x(\varepsilon h)) - V_1(\tilde{x}(\varepsilon h)) + V_1(\tilde{x}(\varepsilon h)) - V_1(x(0)) \\ &\leq V_1(x(0))(e^{-\gamma \varepsilon h} - 1) + \varepsilon |(x(0), y(0))|^2\, O(h^2) + \varepsilon |(x(0), y(0))|\, d(y(0), K(x(0)))\, O(h). \end{aligned}$$

Moreover,

$$\begin{aligned} V_2(x(\varepsilon h), y(\varepsilon h)) - V_2(x(0), y(0)) \leq\, & |V_2(x(\varepsilon h), y(\varepsilon h)) - V_2(x(0), y(\varepsilon h))| + |V_2(x(0), y(\varepsilon h)) - V_2(x(0), \tilde{y}(\varepsilon h))| \\ & + V_2(x(0), \tilde{y}(\varepsilon h)) - V_2(x(0), y(0)), \end{aligned}$$

where $\tilde{y}(\cdot/\varepsilon)$ is the solution of $\Sigma_{x(0)}$ starting at $y(0)$ and corresponding to the signal $\sigma(\cdot/\varepsilon)$. The first term is of order $\varepsilon|(x(0),y(0))|^2 O(h)$, while, by (42),

$$V_2(x(0),\tilde{y}(\varepsilon h)) - V_2(x(0),y(0)) \leq (e^{-\delta h}-1)V_2(x(0),y(0)).$$

Furthermore

$$\frac{d}{dt}(y(t)-\tilde{y}(t)) = \frac{1}{\varepsilon}D(t)(y(t)-\tilde{y}(t)) + \frac{1}{\varepsilon}C(t)(x(t)-x(0)) + \mu y(t),$$

from which one gets that

$$|y(\varepsilon h)-\tilde{y}(\varepsilon h)| = \varepsilon|(x(0),y(0))|\, O(h),$$

so that

$$|V_2(x(0),y(\varepsilon h)) - V_2(x(0),\tilde{y}(\varepsilon h))| = \varepsilon|(x(0),y(0))|^2\, O(h).$$

Summing up,

$$V_2(x(\varepsilon h),y(\varepsilon h)) - V_2(x(0),y(0)) \leq (e^{-\delta h}-1)V_2(x(0),y(0)) + \varepsilon|(x(0),y(0))|^2\, O(h)$$

and

$$\begin{aligned} V(x(\varepsilon h),y(\varepsilon h)) - V(x(0),y(0)) \leq{}& (e^{-\gamma\varepsilon h}-1)V_1(x(0)) + \chi(e^{-\delta h}-1)V_2(x(0),y(0)) \\ &+ C_2\varepsilon h|(x(0),y(0))|\, d(y(0),K(x(0))) \\ &+ C_3\chi\varepsilon h|(x(0),y(0))|^2 + C_4\varepsilon h^2|(x(0),y(0))|^2, \end{aligned}$$

where C_2, C_3, C_4 do not depend on $x(0)$, $y(0)$, h, nor ε, for h, ε in a small right-neighborhood of zero. We have

$$\begin{aligned} \limsup_{h\to 0^+} \frac{V(x(\varepsilon h),y(\varepsilon h)) - V(x(0),y(0))}{h} \leq{}& -\gamma\varepsilon V_1(x(0)) - \delta\chi V_2(x(0),y(0)) + C_2\varepsilon|(x(0),y(0))|\, d(y(0),K(x(0))) \\ &+ C_3\chi\varepsilon|(x(0),y(0))|^2. \end{aligned}$$

We write the right-hand side as the sum of the three terms

$$\begin{aligned} W_1 &= -\frac{\gamma\varepsilon}{2}(V_1(x(0)) + \chi V_2(x(0),y(0))) = -\frac{\gamma\varepsilon}{2}V(x(0),y(0)), \\ W_2 &= -\frac{\chi}{4}(2\delta-\gamma\varepsilon)V_2(x(0),y(0)) - C_3\chi\varepsilon|(x(0),y(0))|^2 - C_2\varepsilon|(x(0),y(0))|\, d(y(0),K(x(0))) \\ &\leq -\frac{\chi}{4}(2\delta-\gamma\varepsilon)d(y(0),K(x(0))^2 - C_3\chi\varepsilon|(x(0),y(0))|^2 - C_2\varepsilon|(x(0),y(0))|\, d(y(0),K(x(0))), \\ W_3 &= -\frac{\gamma\varepsilon}{2}V(x(0)) - \frac{\chi}{4}(2\delta-\gamma\varepsilon)V_2(x(0),y(0)) + 2C_3\chi\varepsilon|(x(0),y(0))|^2 \\ &\leq -\frac{\gamma\varepsilon}{2}|x(0)|^2 - \frac{\chi}{4}(2\delta-\gamma\varepsilon)d(y(0),K(x(0))^2 + 2C_3\chi\varepsilon|(x(0),y(0))|^2, \end{aligned}$$

where the inequalities in W_2, W_3 are obtained assuming $\varepsilon \leq \frac{2\delta}{\gamma}$. For any given $\chi > 0$, it is easy to see that $W_2 \leq 0$ if ε is small enough. Since

$$|y(0)| \leq L_K|x(0)| + d(y(0),K(x(0)))$$

we get

$$|(x(0),y(0))|^2 \leq (1+2L_K^2)|x(0)|^2 + 2d(y(0),K(x(0)))^2$$

so that $W_3 \leq 0$, provided that χ is chosen so that

$$-\frac{\gamma}{2} + 2C_3\chi(1+2L_K^2) \leq 0$$

and ε is small enough. By a time-shift we obtain, for $t \geq 0$,

$$\limsup_{\tau \to 0^+} \frac{V(x(t+\tau), y(t+\tau)) - V(x(t), y(t))}{\tau} \leq -\frac{\gamma}{2} V(x(t), y(t)).$$

Since $V(x(\cdot), y(\cdot))$ is absolutely continuous we deduce that

$$\frac{d}{dt} V(x(t), y(t)) \leq -\frac{\gamma}{2} V(x(t), y(t)), \quad \text{a.e. } t \geq 0,$$

and

$$V(x(t), y(t)) \leq V(x(0), y(0)) e^{-\frac{\gamma}{2} t}, \quad \forall\, t \geq 0,$$

concluding the proof. □

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