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► **To cite this version:**

Lina Guan, Christophe Prieur, Liguozhang. State Estimation for Heterogeneous Quasilinear Traffic Flow System with Disturbances. 2022. <hal-03668871>

HAL Id: hal-03668871

<https://hal.science/hal-03668871v1>

Preprint submitted on 16 May 2022

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State Estimation for Heterogeneous Quasilinear Traffic Flow System with Disturbances

Lina Guan* · Christophe Prieur · Liguozhang

Received: date / Accepted: date

Abstract This paper studies state estimation for a heterogeneous quasilinear traffic flow system with disturbances at the inlet of a considered road section. On the basis of the backstepping method, a quasilinear observer is designed for the quasilinear traffic flow system with only the boundary measurements at the inlet of the considered road section. The observer is constructed by duplicating the quasilinear system and adding the output injection terms to the partial differential equations and boundary conditions. Making use of the backstepping transformation, the injection gains of the observer system are derived by the computation of kernel equations, which are obtained by mapping the error system into an integral input-to-state stable target system. The applicability of the observer-based output feedback controller to stabilize the quasilinear system is finally verified by simulations of two vehicle classes.

Keywords heterogeneous quasilinear traffic flow system · disturbances · integral input-to-state stable · observer · backstepping

Mathematics Subject Classification (2010) 49N80 · 93D23 · 93B53

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The work of L. Guan is supported by a research grant from project PHC CAI YUANPEI under grant number 44029QD, and by MIAI @ Grenoble Alpes (ANR-19-P3IA-0003), and the National Natural Science Foundation of China (NSFC, Grant No. 61873007), the Beijing Natural Science Foundation (Grant No. 1182001).

1 Introduction

Input-to-state stability (ISS) is known as one of the central notions in control theory of dynamical systems since the seminal paper [18]. Indeed it allows to describe the disturbance effect on the state of nonlinear finite-dimensional systems, and provides some design methods of dynamical output feedback laws (see the survey [19]).

A closely related notion is the notion of integral input-to-state stability (iISS) as considered, e.g. in [15]. Roughly speaking, this property provides an estimation of the impact of the integral of the disturbances to the state norm. It is very well developed for nonlinear finite-dimensional systems and networks (see e.g. [11]). For infinite-dimensional systems, the theory has been recently generalized in [12].

In this paper, we exploit such notion (iISS) to a quasilinear infinite-dimensional system with boundary control and perturbation. More precisely, by providing an iISS property, we design a quasilinear observer of the heterogeneous quasilinear traffic flow model that guarantees the accurate estimation (no error induced by linearization) of the traffic state under the condition that the initial estimation is not too far from the actual state. By doing so, we succeed to design a known input U locally stabilizing the closed-loop system by combining the designed quasilinear observer with the obtained results of papers [8] and [10]. Exploiting the structure of the observer suggested in [22], this paper gives, for the first time, the theoretical proof of local H^2 stability of the quasilinear observer system, by the local H^2 Lyapunov analysis of the error system.

We use the backstepping method to pursue the quasilinear observer design of the heterogeneous quasilinear traffic flow system in this paper. The backstepping method for hyperbolic PDEs was initially introduced by [14], [13] and [17]. By using the backstepping method, a full-state feedback control law is designed to exponentially stabilize a quasilinear 2×2 first-order hyperbolic system in [5]. In [6], robust output regulation problem is solved for the linear 2×2 hyperbolic systems. The finite-time output regulation problem is solved in [7] by implementing the finite-time state feedback regulator with disturbance observers. In [20], a boundary observer is designed for nonlinear ARZ traffic flow system making use of the backstepping method. An output feedback controller is designed for the underactuated cascade network of interconnected PDEs systems using backstepping in [2]. By using the backstepping in paper [21], an observer is designed to estimate traffic states of the nonlinear ARZ traffic flow model. Paper [4] designs an output feedback boundary control to solve the stop and go traffic problem of linearized two-type AR traffic flow system. In this paper, inspired by these papers, we develop the backstepping method to design injection gains and observer dynamics so that the state feedback controller in [8] combining with the designed observer can stabilize the quasilinear traffic system.

This paper is organized as follows. Section 2 contains the preliminary for the design of the quasilinear observer. The collocated observer is designed and the theoretical proof of the iISS for the error system is done in Section 3. In Section 4, a numerical simulation is presented to verify the observer design. Finally, Section 5 contains conclusions.

Notation. $[A]_{i,j}$ denotes the entry of matrix A in the i -th row and the j -th column. Identity matrix of n -dimension is denoted by I_n . Zero matrix of l row

and n column is denoted by $0_{l,n}$. For a function $\varphi = (\varphi_1, \dots, \varphi_n)^\top : [0, L] \rightarrow \mathbb{R}^n$, $\|\varphi\|_\infty = \sup_{x \in [0, L]} |\varphi(x)|$. For a matrix $M \in \mathbb{R}^{n,n}$, $|M| = \sup\{|M\varphi| : \varphi \in \mathbb{R}^n, |\varphi| = 1\}$, $\|M\|_\infty = \sup\{|M(x, \xi)| : x, \xi \in [0, L]\}$, the definition of L^2 -norm is $\|\varphi\|_{L^2} = \sqrt{\int_0^L (\varphi_1^2(\xi) + \dots + \varphi_n^2(\xi)) d\xi}$, H^2 -norm is $\|\varphi\|_{H^2} = \|\varphi\|_{L^2} + \|\partial_x \varphi\|_{L^2} + \|\partial_{xx} \varphi\|_{L^2}$. The symbol $*$ stands for a symmetric block in a matrix.

2 Preliminary

2.1 Heterogeneous traffic flow model and problem statement

Extending the results of the papers [8] and [10], we design an observer in the H^2 space for the following heterogeneous quasilinear hyperbolic traffic flow system described as the partial differential equations in [16], under the consideration of road segment with N vehicle classes, W road width and L road length, for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\partial_t \rho_i(x, t) + \partial_x (\rho_i(x, t) v_i(x, t)) = 0, \quad (2.1)$$

$$\partial_t (v_i(x, t) + p_i(Ao)) + v_i(x, t) \partial_x (v_i(x, t) + p_i(Ao)) = \frac{V_{e,i}(Ao) - v_i(x, t)}{\tau_i}, \quad (2.2)$$

with the boundary conditions, for all $t \in [0, \infty)$,

$$(\rho_1(0, t) v_1(0, t) - \rho_1^*(0) v_1^*(0), \dots, \rho_N(0, t) v_N(0, t) - \rho_N^*(0) v_N^*(0))^\top = \bar{p}(t) + \Theta U(t), \quad (2.3)$$

$$\rho_i(L, t) = \rho_i^*(L), \quad (2.4)$$

and the initial conditions

$$\rho_i(\cdot, 0) = \rho_{0,i}(\cdot) \in H^2([0, L]; \mathbb{R}), \quad (2.5)$$

$$v_i(\cdot, 0) = v_{0,i}(\cdot) \in H^2([0, L]; \mathbb{R}), \quad (2.6)$$

where the density ρ_i and velocity v_i of vehicle class i depending on the space variable $x \in [0, L]$ and the time variable $t \in [0, \infty)$, i ($i = 1, 2, \dots, N$) is the index of vehicle class, and τ_i is the relaxation time depending on the driving behavior of vehicle class i . The area occupancy $Ao(\rho) = \frac{a^\top \rho}{W}$, with $a = (a_1, a_2, \dots, a_N)^\top$ and $\rho = (\rho_1, \rho_2, \dots, \rho_N)^\top$, a_i is the occupied surface per vehicle for class i . Area occupancy Ao describes the percentage of road space that is occupied by all the vehicle classes on the road segment, and $0 < Ao \leq 1$. The traffic pressure function $p_i(Ao)$ of vehicle class i is (see [4]) $p_i(Ao) = v_i^M \left(\frac{Ao(\rho)}{Ao_i^M} \right)^{\gamma_i}$, $i = 1, 2, \dots, N$, with the free-flow velocity v_i^M , the maximum area occupancy $0 < Ao_i^M \leq 1$, and the pressure exponent constant $\gamma_i > 1$ of class i . The equilibrium speed-Ao relationship of vehicle class i ($i = 1, 2, \dots, N$) is given by Greenshields model in [3], $V_{e,i}(Ao) = v_i^M - p_i(Ao) = v_i^M \left(1 - \left(\frac{Ao}{Ao_i^M} \right)^{\gamma_i} \right)$. The equilibrium $\rho_i^*, v_i^* \in C^1([0, L]; \mathbb{R})$

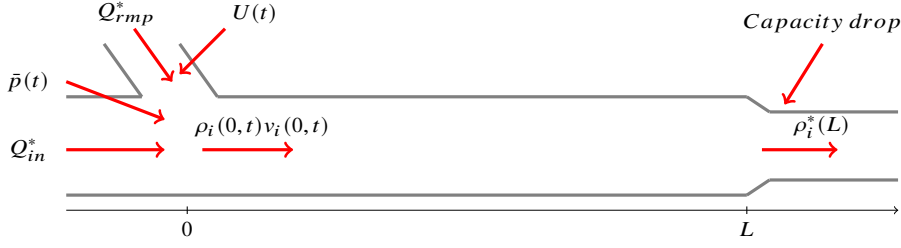


Fig. 1 Heterogeneous vehicle traffic on a road with disturbances at the inlet boundary and a capacity drop in the right boundary.

satisfies, for $i = 1, 2, \dots, N$,

$$v_i^*(x) \frac{d\rho_i^*(x)}{dx} + \rho_i^*(x) \frac{dv_i^*(x)}{dx} = 0, \quad (2.7)$$

$$v_i^*(x) \frac{dv_i^*(x)}{dx} + v_i^*(x) \frac{dp_i(x)}{dx} = \frac{V_{e,i}(A\theta(\rho^*(x))) - v_i^*(x)}{\tau_i}. \quad (2.8)$$

The diagram is presented in Figure 1. The input $U \in C^1([0, \infty); \mathbb{R}^{2N-m})$ with the coefficient matrix $\Theta \in \mathbb{R}^{N, 2N-m}$ is assumed to be known, whereas the unknown disturbance input $\bar{p} \in C^2([0, \infty); \mathbb{R}^N)$ acts as an on-ramp metering at the upstream boundary of the considered road segment. The constant inflow $Q_{in}^* \in \mathbb{R}^N$ and the nominal on-ramp flux rate $Q_{rmp}^* \in \mathbb{R}^N$ satisfy

$$Q_{in}^* + Q_{rmp}^* = (\rho_1^*(0)v_1^*(0), \rho_2^*(0)v_2^*(0), \dots, \rho_N^*(0)v_N^*(0))^\top.$$

From a practical perspective, an output is needed for the observer design to address the state estimation problem of the quasilinear traffic flow system (2.1)-(2.6) in this paper. The boundary measurements of density $\rho_i(0, t)$ and velocity $v_i(0, t)$, $t \in [0, \infty)$ are taken at the boundary $x = 0$ collocated with a known input U , then the measured output of system (2.1)-(2.6) is, for all $t \in [0, \infty)$,

$$y_1(t) = (\rho_1(0, t), v_1(0, t), \rho_2(0, t), v_2(0, t), \dots, \rho_N(0, t), v_N(0, t))^\top.$$

Defining $u^* = (\rho_1^*, v_1^*, \dots, \rho_N^*, v_N^*)^\top$, $u = (\rho_1, v_1, \dots, \rho_N, v_N)^\top$, and $\tilde{u} = u - u^*$, the system (2.1)-(2.6) is rewritten as in the paper [10], for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\partial_t \tilde{u}(x, t) + F(\tilde{u}, u^*(x)) \tilde{u}_x(x, t) = G(\tilde{u}, u^*(x)) \tilde{u}(x, t), \quad (2.9)$$

with the boundary conditions, for all $t \in [0, \infty)$,

$$A_1 \tilde{u}(0, t) = \bar{p}(t) + \Theta U(t) + w_1(t) + w_2(t) - \Pi_{NL}(\tilde{u}(0, t)), \quad (2.10)$$

$$B_1 \tilde{u}(L, t) = 0, \quad (2.11)$$

where, for all $x \in (0, L)$,

$$F(\tilde{u}, u^*(x)) = \begin{bmatrix} F_{11}(\tilde{u}, u^*(x)) & F_{12}(\tilde{u}, u^*(x)) & \cdots & F_{1N}(\tilde{u}, u^*(x)) \\ F_{21}(\tilde{u}, u^*(x)) & F_{22}(\tilde{u}, u^*(x)) & \cdots & F_{2N}(\tilde{u}, u^*(x)) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N1}(\tilde{u}, u^*(x)) & F_{N2}(\tilde{u}, u^*(x)) & \cdots & F_{NN}(\tilde{u}, u^*(x)) \end{bmatrix},$$

with, for $i, j = 1, 2, \dots, N$,

$$F_{ij}(\tilde{u}, u^*(x)) = \begin{cases} \begin{bmatrix} v_i^*(x) & \rho_i^*(x) \\ 0 & v_i^*(x) - \rho_i^*(x)\delta_{ii}(\rho) \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ (v_i^*(x) - v_j^*(x))\delta_{ij}(\rho) & -\rho_j^*(x)\delta_{ij}(\rho) \end{bmatrix}, & \text{if } j \neq i, \end{cases}$$

for all $x \in (0, L)$,

$$G(\tilde{u}, u^*(x)) = \begin{bmatrix} G_{11}(\tilde{u}, u^*(x)) & G_{12}(\tilde{u}, u^*(x)) & \cdots & G_{1N}(\tilde{u}, u^*(x)) \\ G_{21}(\tilde{u}, u^*(x)) & G_{22}(\tilde{u}, u^*(x)) & \cdots & G_{2N}(\tilde{u}, u^*(x)) \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1}(\tilde{u}, u^*(x)) & G_{N2}(\tilde{u}, u^*(x)) & \cdots & G_{NN}(\tilde{u}, u^*(x)) \end{bmatrix},$$

with, for $i, j = 1, 2, \dots, N$,

$$G_{ij}(\tilde{u}, u^*(x)) = \begin{cases} \begin{bmatrix} \frac{dv_i^*(x)}{dx} & \frac{d\rho_i^*(x)}{dx} \\ \frac{1}{\tau_i}\delta_{ii}(\rho) + v_i^*(x)\sigma_{ii}(\rho)\frac{d\rho_i^*(x)}{dx} - \delta_{ii}(\rho)\frac{dv_i^*(x)}{dx} - \frac{1}{\tau_i} + \frac{dv_i^*(x)}{dx} + \sum_{j=1, j \neq i}^N \delta_{ij}(\rho)\frac{d\rho_j^*(x)}{dx} \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \frac{1}{\tau_i}\delta_{ij}(\rho) + v_i^*(x)\sigma_{ij}(\rho)\frac{d\rho_j^*(x)}{dx} - \delta_{ij}(\rho)\frac{dv_j^*(x)}{dx} & -\delta_{ij}(\rho)\frac{d\rho_j^*(x)}{dx} \end{bmatrix}, & \text{if } j \neq i, \end{cases}$$

and for $i, j = 1, 2, \dots, N$,

$$\delta_{ij}(\rho) = \frac{\partial p_i(Ao)}{\partial \rho_j} = \frac{v_i^M \gamma_i a_j}{Ao_i^M W} \left(\frac{Ao}{Ao_i^M} \right)^{\gamma_i - 1},$$

$$\sigma_{ij}(\rho) = \frac{\partial \delta_{ij}(\rho)}{\partial \rho_j} = \frac{v_i^M \gamma_i (\gamma_i - 1) a_j^2}{(Ao_i^M W)^2} \left(\frac{Ao}{Ao_i^M} \right)^{\gamma_i - 2}.$$

The coefficient matrices $A_1 = \text{diag}\{[v_1^*(0), \rho_1^*(0)], \dots, [v_N^*(0), \rho_N^*(0)]\} \in \mathbb{R}^{N, 2N}$,

$B_1 = \text{diag}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\} \in \mathbb{R}^{2N, 2N}$, and the nonlinear term

$$\Pi_{NL}(\tilde{u}(0, \cdot)) = \begin{bmatrix} \tilde{\rho}_1(0, \cdot)\tilde{v}_1(0, \cdot) \\ \tilde{\rho}_2(0, \cdot)\tilde{v}_2(0, \cdot) \\ \vdots \\ \tilde{\rho}_N(0, t)\tilde{v}_N(0, t) \end{bmatrix} \in C^1([0, \infty); \mathbb{R}^N).$$

$w_1, w_2 \in C^1([0, \infty); \mathbb{R}^N)$ are the solutions of the following system,

$$\begin{aligned}\dot{w}_1 &= -d_1 I_N w_1, \\ \dot{w}_2 &= -d_2 I_N w_2,\end{aligned}$$

with the constant N -dimension diagonal positive definite matrices $d_1 I_N, d_2 I_N$ ($d_1 \neq d_2$), and the initial conditions $w_1(0) = -\frac{g_2(\tilde{u}_0) + d_2 g_1(\tilde{u}_0)}{d_1 - d_2}$, $w_2(0) = \frac{d_1 g_1(\tilde{u}_0) + g_2(\tilde{u}_0)}{d_1 - d_2}$, with

$$\begin{aligned}g_1(\tilde{u}_0) &= A_1 \tilde{u}_0(0) - \bar{p}(0) - \Theta U(0) + \Pi_{NL}(\tilde{u}_0(0)), \\ g_2(\tilde{u}_0) &= A_1 (-F(\tilde{u}_0(0), u^*(0)) \tilde{u}'_0(0) + G(\tilde{u}_0(0), u^*(0)) \tilde{u}_0(0)) - \dot{\bar{p}}(0) - \Theta \dot{U}(0) \\ &\quad + \left. \frac{d\Pi_{NL}}{d\tilde{u}} \right|_{t=0} (-F(\tilde{u}_0(0), u^*(0)) \tilde{u}'_0(0) + G(\tilde{u}_0(0), u^*(0)) \tilde{u}_0(0)).\end{aligned}$$

The initial conditions

$$\tilde{u}(\cdot, 0) = \tilde{u}_0(\cdot) \in H^2([0, L]; \mathbb{R}^{2N}), \quad (2.12)$$

satisfy the second-order compatibility conditions

$$A_1 \tilde{u}_0(0) = \bar{p}(0) + \Theta U(0) + w_1(0) + w_2(0) - \Pi_{NL}(\tilde{u}_0(0)), \quad (2.13)$$

$$B_1 \tilde{u}_0(L) = 0, \quad (2.14)$$

$$\begin{aligned}A_1 (-F(\tilde{u}_0(0), u^*(0)) \tilde{u}'_0(0) + G(\tilde{u}_0(0), u^*(0)) \tilde{u}_0(0)) \\ = \dot{\bar{p}}(0) + \Theta \dot{U}(0) + (-d_1 I_N w_1(0) - d_2 I_N w_2(0)) \\ - \left. \frac{d\Pi_{NL}}{d\tilde{u}} \right|_{t=0} (-F(\tilde{u}_0(0), u^*(0)) \tilde{u}'_0(0) + G(\tilde{u}_0(0), u^*(0)) \tilde{u}_0(0)),\end{aligned} \quad (2.15)$$

$$B_1 F(\tilde{u}_0(L), u^*(L)) \tilde{u}'_0(L) = B_1 G(\tilde{u}_0(L), u^*(L)) \tilde{u}_0(L). \quad (2.16)$$

Assume that the system (2.9)-(2.16) is strictly hyperbolic, for all $u^* \in C^1([0, L]; \mathbb{R}^{2N})$, the matrix $F(0, u^*)$ has $2N$ real distinct nonzero eigenvalues. Given $2N$ eigenvalues, for all $x \in [0, L]$, $\lambda_1(x) > \lambda_2(x) > \dots > \lambda_m(x) > 0 > -\lambda_{m+1}(x) > \dots > -\lambda_{2N}(x)$, of $F(0, u^*)$ ($\lambda_i \in C^1([0, L]; \mathbb{R})$, $i = 1, \dots, 2N$, m is the number of positive eigenvalues and $0 \leq m < 2N$), and $-\lambda_{m+1}(x), \dots, -\lambda_{2N}(x) < 0$ means that the traffic wave moves backward in the congested regime. For $x \in [0, L]$, define $\Lambda(x) = \text{diag}\{\Lambda^+(x), -\Lambda^-(x)\}$, $\Lambda^+(x) = \text{diag}\{\lambda_1(x), \lambda_2(x), \dots, \lambda_m(x)\}$, $\Lambda^-(x) = \text{diag}\{\lambda_{m+1}(x), \lambda_{m+2}(x), \dots, \lambda_{2N}(x)\}$, $|\Lambda(x)| = \text{diag}\{\Lambda^+(x), \Lambda^-(x)\}$, $\Lambda'(x) = \text{diag}\{\lambda'_1(x), \dots, \lambda'_m(x), -\lambda'_{m+1}(x), \dots, -\lambda'_{2N}(x)\}$. The corresponding right eigenvectors of $2N$ eigenvalues consist of the columns of the invertible matrix $T \in C^1([0, L]; \mathbb{R}^{2N, 2N})$. The density and speed gaps $\tilde{\rho}_i(0, t), \tilde{v}_i(0, t)$ of vehicle class i , $i = 1, \dots, 2N$, between the measurements $\rho_i(0, t), v_i(0, t)$ and the corresponding equilibrium $\rho_i^*(0), v_i^*(0)$ at the inlet of the considered road section are involved in the output of system (2.9)-(2.16), for $t \in [0, \infty)$,

$$y(t) = [0_{2N-m, m} \ I_{2N-m}] T^{-1}(0) \tilde{u}(0, t) \in \mathbb{R}^{2N-m}. \quad (2.17)$$

2.2 State transformation

In order to simplify the analysis, by using the invertible transformation, for $x \in [0, L]$, $R = \Phi(x)\tilde{u}$, from \tilde{u} to the new variable $R : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^{2N}$, the system (2.9)-(2.16) is mapped into the following system in the form of characteristic values as in the paper [10], for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\partial_t R(x, t) + \Lambda(x)\partial_x R(x, t) + \Lambda_{NL}(R, x)\partial_x R(x, t) = \Sigma(x)R(x, t) + \Sigma_{NL}(R, x)R(x, t), \quad (2.18)$$

with the boundary conditions, for all $t \in [0, \infty)$,

$$R_{in}(t) = K_P R_{out}(t) + \Gamma_0(\bar{p}(t) + \Theta U(t)) + \Gamma_0(w_1(t) + w_2(t)) - \Gamma_0 \Pi_{NL}(\Phi^{-1}(0)R(0, t)), \quad (2.19)$$

and the initial conditions

$$R(\cdot, 0) = R_0(\cdot) \in H^2([0, L]; \mathbb{R}^{2N}), \quad (2.20)$$

satisfying the following second-order compatibility conditions,

$$\begin{aligned} R_{in}(0) &= K_P R_{out}(0) + \Gamma_0(\bar{p}(0) + \Theta U(0)) + \Gamma_0(w_1(0) + w_2(0)) \\ &\quad - \Gamma_0 \Pi_{NL}(\Phi^{-1}(0)R_0(0)), \\ &= ([M_i^1]_{1 \leq i \leq m}, [M_j^2]_{m+1 \leq j \leq 2N})^\top \\ &= K_P([M_i^2]_{1 \leq i \leq m}, [M_j^1]_{m+1 \leq j \leq 2N})^\top + \Gamma_0(\dot{\bar{p}}(0) + \Theta \dot{U}(0)) \\ &\quad + \Gamma_0(-d_1 I_N w_1(0) - d_2 I_N w_2(0)) - \Gamma_0 \frac{d\Pi_{NL}}{d\tilde{u}} \Big|_{t=0} \\ &\quad \Phi^{-1}(0) \left(-(\Lambda(0) + \Lambda_{NL}(R_0(0), 0)) R_0'(0) + (\Sigma(0) + \Sigma_{NL}(R_0(0), 0)) R_0(0) \right), \end{aligned} \quad (2.21)$$

with

$$\begin{aligned} M^1 &= -(\Lambda(0) + \Lambda_{NL}(R_0(0), 0)) R_0'(0) + (\Sigma(0) + \Sigma_{NL}(R_0(0), 0)) R_0(0), \\ M^2 &= -(\Lambda(L) + \Lambda_{NL}(R_0(L), L)) R_0'(L) + (\Sigma(L) + \Sigma_{NL}(R_0(L), L)) R_0(L), \end{aligned}$$

where $R = (R^+, R^-)^\top : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^{2N}$, $R_{in}(t) = (R^+(0, t), R^-(L, t))^\top$, $R_{out}(t) = (R^+(L, t), R^-(0, t))^\top$, with $R^+ : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^m$, $R^- : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^{2N-m}$,

$$\begin{aligned} \Lambda_{NL}(R, x) &= \Phi(x)F(\Phi^{-1}(x)R, u^*(x))\Phi^{-1}(x) - \Lambda(x), \\ \Sigma_{NL}(R, x) &= \Phi(x)G(\Phi^{-1}(x)R, u^*(x))\Phi^{-1}(x) - \Phi(x)F(\Phi^{-1}(x)R, u^*(x))\Phi^{-1}(x)' \\ &\quad - \Sigma(x), \end{aligned}$$

$\Sigma(x) = \begin{bmatrix} \Sigma^{++}(x) & \Sigma^{+-}(x) \\ \Sigma^{-+}(x) & \Sigma^{--}(x) \end{bmatrix}$, with $\Sigma^{++} \in C^1((0, L); \mathbb{R}^{m, m})$, $\Sigma^{+-} \in C^1((0, L); \mathbb{R}^{m, 2N-m})$, $\Sigma^{-+} \in C^1((0, L); \mathbb{R}^{2N-m, m})$, $\Sigma^{--} \in C^1((0, L); \mathbb{R}^{2N-m, 2N-m})$, the main diagonal elements of the matrix $\Sigma \in C^1((0, L); \mathbb{R}^{2N, 2N})$ are zeros, $K_P \in \mathbb{R}^{2N, 2N}$, $\Gamma_0 \in \mathbb{R}^{2N, N}$ are given gain matrices. Since for $x \in [0, L]$, $\Lambda_{NL}(0, x) = 0$, $\Sigma_{NL}(0, x) = 0$ and $\Pi_{NL}(0) = \frac{d\Pi_{NL}}{du}(0) = 0$, then the quasilinear system (2.18)-(2.22) has zero equilibrium.

For all $t \in [0, \infty)$, the boundary measurements are taken at the boundary $x = 0$ collocated with control input and the output (2.17) is equivalent to

$$y(t) = R^-(0, t). \quad (2.23)$$

3 Nonlinear Collocated Observer

3.1 Quasilinear observer design

The following observer is designed for the quasilinear system (2.18)-(2.22) by constructing the quasilinear system (2.18)-(2.22) with output injection terms, for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\begin{aligned} & \partial_t \hat{R}(x, t) + \Lambda(x) \partial_x \hat{R}(x, t) + \Lambda_{NL}(\hat{R}, x) \partial_x \hat{R}(x, t) \\ & = \Sigma(x) \hat{R}(x, t) + \Sigma_{NL}(\hat{R}, x) \hat{R}(x, t) + S(x) (y(t) - \hat{R}^-(0, t)), \end{aligned} \quad (3.1)$$

with the boundary conditions, for all $t \in [0, \infty)$,

$$\begin{aligned} \hat{R}_{in}(t) &= K_P \hat{R}_{out}(t) + \Gamma_0 (w_1(t) + w_2(t)) + \Gamma_0 (\Theta U(t) - \Pi_{NL}(T(0) \hat{R}(0, t))) \\ & - \begin{bmatrix} \Gamma_4 \\ 0_{2N-m, 2N-m} \end{bmatrix} \int_0^t (y(\tau) - \hat{R}^-(0, \tau)) d\tau, \end{aligned} \quad (3.2)$$

and the initial conditions

$$\hat{R}(\cdot, 0) = \hat{R}_0(\cdot) \in H^2([0, L]; \mathbb{R}^{2N}), \quad (3.3)$$

satisfying the following second-order compatibility conditions,

$$\begin{aligned} & \hat{R}_{in}(0) = K_P \hat{R}_{out}(0) + \Gamma_0 (w_1(0) + w_2(0)) + \Gamma_0 (\Theta U(0) - \Pi_{NL}(T(0) \hat{R}_0(0))), \quad (3.4) \\ & ([\hat{M}_i^1]_{1 \leq i \leq m}, [\hat{M}_j^2]_{m+1 \leq j \leq 2N})^\top \\ & = K_P ([\hat{M}_i^2]_{1 \leq i \leq m}, [\hat{M}_j^1]_{m+1 \leq j \leq 2N})^\top + \Gamma_0 (-d_1 I_N w_1(0) - d_2 I_N w_2(0)) \\ & + \Gamma_0 \Theta \dot{U}(0) - \Gamma_0 \frac{d\Pi_{NL}}{du} \Big|_{t=0} T(0) \left(-(\Lambda(0) + \Lambda_{NL}(\hat{R}_0(0), 0)) \hat{R}'_0(0) \right. \\ & \left. + (\Sigma(0) + \Sigma_{NL}(\hat{R}_0(0), 0)) \hat{R}_0(0) + S(0) (y(0) - \hat{R}_0^-(0)) \right) \\ & - \begin{bmatrix} \Gamma_4 \\ 0_{2N-m, 2N-m} \end{bmatrix} (y(0) - \hat{R}_0^-(0)), \end{aligned} \quad (3.5)$$

with

$$\begin{aligned}\hat{M}^1 &= -(\Lambda(0) + \Lambda_{NL}(\hat{R}_0(0), 0)) \hat{R}'_0(0) + (\Sigma(0) + \Sigma_{NL}(\hat{R}_0(0), 0)) \hat{R}_0(0) \\ &\quad + S(0) (y(0) - \hat{R}_0^-(0)), \\ \hat{M}^2 &= -(\Lambda(L) + \Lambda_{NL}(\hat{R}_0(L), L)) \hat{R}'_0(L) + (\Sigma(L) + \Sigma_{NL}(\hat{R}_0(L), L)) \hat{R}_0(L) \\ &\quad + S(L) (y(0) - \hat{R}_0^-(0)),\end{aligned}$$

where $\hat{R} = (\hat{R}^+, \hat{R}^-)^\top : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^{2N}$, $\hat{R}_{in}(t) = (\hat{R}^+(0, t), \hat{R}^-(L, t))^\top$, $\hat{R}_{out}(t) = (\hat{R}^+(L, t), \hat{R}^-(0, t))^\top$, $S = (S_1, S_2)^\top : (0, L) \rightarrow \mathbb{R}^{2N, 2N-m}$, $\Gamma_4 \in \mathbb{R}^{m, 2N-m}$. In the previous equations, $\hat{R}^+ : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^m$ and $\hat{R}^- : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^{2N-m}$ are the estimates of state variables R^+ and R^- , the terms $S_1 \in C^1((0, L); \mathbb{R}^{m, 2N-m})$ and $S_2 \in C^1((0, L); \mathbb{R}^{2N-m, 2N-m})$ are output injection gains derived from backstepping method, and for all x in $(0, L)$,

$$S_1(x) = F^1(x, 0)\Lambda^-(L), \quad (3.6)$$

$$S_2(x) = F^2(x, 0)\Lambda^-(L), \quad (3.7)$$

with $F^1 \in C^1(\mathbb{T}; \mathbb{R}^{m, 2N-m})$ and $F^2 \in C^1(\mathbb{T}; \mathbb{R}^{2N-m, 2N-m})$, which are the solutions to the following kernel equations in the triangular domain $\mathbb{T} = \{(x, \xi) \in \mathbb{R}^2 | 0 \leq \xi \leq x \leq L\}$,

$$\begin{aligned}\Lambda^+(x)\partial_x F^1(x, \xi) - \partial_\xi F^1(x, \xi)\Lambda^-(\xi) &= F^1(x, \xi)(\Lambda^-)'(\xi) + \Sigma^{++}(x)F^1(x, \xi) \\ &\quad + \Sigma^{+-}(x)F^2(x, \xi),\end{aligned} \quad (3.8)$$

$$\begin{aligned}\Lambda^-(x)\partial_x F^2(x, \xi) + \partial_\xi F^2(x, \xi)\Lambda^-(\xi) &= -F^2(x, \xi)(\Lambda^-)'(\xi) - \Sigma^{--}(x)F^2(x, \xi) \\ &\quad - \Sigma^{-+}(x)F^1(x, \xi),\end{aligned} \quad (3.9)$$

$$F^1(x, x)\Lambda^-(x) + \Lambda^+(x)F^1(x, x) = \Sigma^{+-}(x), \quad (3.10)$$

$$F^2(x, x)\Lambda^-(x) - \Lambda^-(x)F^2(x, x) = \Sigma^{--}(x), \quad (3.11)$$

$$\Gamma_3 F^1(L, \xi) = F^2(L, \xi), \quad (3.12)$$

with a given coefficient matrix $\Gamma_3 \in \mathbb{R}^{2N-m, m}$. By following a coordinate change $(x, \xi) \mapsto (L - \xi, L - x)$, the well-posedness of the solutions to kernel equations (3.8)-(3.12) follows from Theorem D.6 in [1].

3.2 Local H^2 iISS of quasilinear observer system

So as to theoretically verify the applicability of state estimation of the designed quasilinear observer, the following error system is obtained by subtracting the observer estimates (3.1)-(3.5) from the quasilinear system (2.18)-(2.22), for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\partial_t \tilde{R}(x, t) + \Lambda(x)\partial_x \tilde{R}(x, t) + F_{NL}[\tilde{R}, \hat{R}, \partial_x \tilde{R}, \partial_x \hat{R}] = \Sigma(x)\tilde{R}(x, t) - S(x)\tilde{R}^-(0, t), \quad (3.13)$$

with the boundary conditions, for all $t \in [0, \infty)$,

$$\begin{aligned} \tilde{R}_{in}(t) = & K_P \tilde{R}_{out}(t) + \begin{bmatrix} \Gamma_4 \\ 0_{2N-m, 2N-m} \end{bmatrix} \int_0^t \tilde{R}^-(0, \tau) d\tau + \Gamma_0 \tilde{p}(t) \\ & - G_{NL}[\tilde{R}(0, t), \hat{R}(0, t)], \end{aligned} \quad (3.14)$$

and the initial conditions

$$\tilde{R}(\cdot, 0) = \tilde{R}_0(\cdot) \in H^2([0, L]; \mathbb{R}^{2N}), \quad (3.15)$$

satisfying the following second-order compatibility conditions

$$\begin{aligned} \tilde{R}_{in}(0) = & K_P \tilde{R}_{out}(0) + \Gamma_0 \tilde{p}(0) - G_{NL}[\tilde{R}_0(0), \hat{R}_0(0)], \quad (3.16) \\ ([\tilde{M}_i^1]_{1 \leq i \leq m}, [\tilde{M}_j^2]_{m+1 \leq j \leq 2N})^\top \\ = & K_P([\tilde{M}_i^1]_{1 \leq i \leq m}, [\tilde{M}_j^2]_{m+1 \leq j \leq 2N})^\top + \begin{bmatrix} \Gamma_4 \\ 0_{2N-m, 2N-m} \end{bmatrix} \tilde{R}_0^-(0) + \Gamma_0 \dot{\tilde{p}}(0) \\ & - \partial_{\tilde{R}} G_{NL} \Big|_{t=0} \left(-\Lambda(0) \tilde{R}_0'(0) + \Sigma(0) \tilde{R}_0(0) - F_{NL}[\tilde{R}_0(0), \hat{R}_0(0), \tilde{R}_0'(0), \hat{R}_0'(0)] \right. \\ & \left. - S(0) \tilde{R}_0^-(0) \right) - \partial_{\hat{R}} G_{NL} \Big|_{t=0} \left(-(\Lambda(0) + \Lambda_{NL}(\hat{R}_0(0), 0)) \hat{R}_0'(0) \right. \\ & \left. + (\Sigma(0) + \Sigma_{NL}(\hat{R}_0(0), 0)) \hat{R}_0(0) + S(0) (y(0) - \hat{R}_0^-(0)) \right), \quad (3.17) \end{aligned}$$

with

$$\begin{aligned} \tilde{M}^1 = & -\Lambda(0) \tilde{R}_0'(0) + \Sigma(0) \tilde{R}_0(0) - F_{NL}[\tilde{R}_0(0), \hat{R}_0(0), \tilde{R}_0'(0), \hat{R}_0'(0)] - S(0) \tilde{R}_0^-(0), \\ \tilde{M}^2 = & -\Lambda(L) \tilde{R}_0'(L) + \Sigma(L) \tilde{R}_0(L) - F_{NL}[\tilde{R}_0(L), \hat{R}_0(L), \tilde{R}_0'(L), \hat{R}_0'(L)] - S(L) \tilde{R}_0^-(L), \end{aligned}$$

where $\tilde{R} = (\tilde{R}^+, \tilde{R}^-)^\top : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^{2N}$, $\tilde{R}_{in}(t) = (\tilde{R}^+(0, t), \tilde{R}^-(L, t))^\top$, $\tilde{R}_{out}(t) = (\tilde{R}^+(L, t), \tilde{R}^-(0, t))^\top$, and

$$\begin{aligned} F_{NL}[\tilde{R}, \hat{R}, \partial_x \tilde{R}, \partial_x \hat{R}] = & \Lambda_{NL}(\tilde{R} + \hat{R}, x) \left(\partial_x \tilde{R}(x, t) + \partial_x \hat{R}(x, t) \right) \\ & - \Sigma_{NL}(\tilde{R} + \hat{R}, x) \left(\tilde{R}(x, t) + \hat{R}(x, t) \right) - \Lambda_{NL}(\hat{R}, x) \partial_x \hat{R}(x, t) + \Sigma_{NL}(\hat{R}, x) \hat{R}(x, t), \\ G_{NL}[\tilde{R}(0, t), \hat{R}(0, t)] = & \Gamma_0 \Pi_{NL} \left(T(0) \left(\tilde{R}(0, t) + \hat{R}(0, t) \right) \right) - \Gamma_0 \Pi_{NL} \left(T(0) \hat{R}(0, t) \right). \end{aligned}$$

By using the backstepping transformation, for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\begin{aligned} \tilde{R}(x, t) = & \tilde{Z}(x, t) + \int_0^x \begin{bmatrix} 0_{m,m} & F^1(x, \xi) \\ 0_{2N-m,m} & F^2(x, \xi) \end{bmatrix} \tilde{Z}(\xi, t) d\xi \\ = & \mathcal{F}[\tilde{Z}], \end{aligned} \quad (3.18)$$

the error system (3.13)-(3.17) is mapped into the following quasilinear target system, for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\begin{aligned} & \partial_t \tilde{Z}(x, t) + \Lambda(x) \partial_x \tilde{Z}(x, t) + Q_{NL} [\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R}] \\ & = \Sigma_2(x) \tilde{Z}(x, t) + \int_0^x D_1(x, \xi) \tilde{Z}(\xi, t) d\xi, \end{aligned} \quad (3.19)$$

$$\dot{\tilde{X}}(t) = K_I \tilde{Z}_{out}(t) + \Gamma_0 \dot{\bar{p}}(t) - \dot{G}_{NL} [\tilde{Z}(0, t), \hat{R}(0, t)], \quad (3.20)$$

with the boundary conditions, for all $t \in [0, \infty)$,

$$\tilde{Z}_{in}(t) = K_P \tilde{Z}_{out}(t) + \tilde{X}(t), \quad (3.21)$$

$$\tilde{X}(t) = K_I \int_0^t \tilde{Z}_{out}(\sigma) d\sigma + \Gamma_0 \bar{p}(t) - G_{NL} [\tilde{Z}(0, t), \hat{R}(0, t)], \quad (3.22)$$

and the initial conditions

$$\tilde{Z}(\cdot, 0) = \tilde{Z}_0(\cdot) \in H^2([0, L]; \mathbb{R}^{2N}), \quad (3.23)$$

$$\tilde{X}(0) = \tilde{X}_0 = \Gamma_0 \bar{p}(0) - G_{NL} [\tilde{Z}_0(0), \hat{R}_0(0)] \in \mathbb{R}^{2N}, \quad (3.24)$$

satisfying the second-order compatibility conditions

$$\tilde{Z}_{in}(0) = K_P \tilde{Z}_{out}(0) + \Gamma_0 \bar{p}(0) - G_{NL} [\tilde{Z}_0(0), \hat{R}_0(0)], \quad (3.25)$$

$$\begin{aligned} & ([\tilde{N}_i^1]_{1 \leq i \leq m}, [\tilde{N}_j^2]_{m+1 \leq j \leq 2N})^\top \\ & = K_P ([\tilde{N}_i^2]_{1 \leq i \leq m}, [\tilde{N}_j^1]_{m+1 \leq j \leq 2N})^\top + K_I \tilde{Z}_{out}(0) + \Gamma_0 \dot{\bar{p}}(0) \\ & \quad - \partial_{\tilde{Z}} G_{NL} \Big|_{t=0} \left(-\Lambda(0) \tilde{Z}'_0(0) - Q_{NL} [\tilde{Z}_0(0), \hat{R}_0(0), \tilde{Z}'_0(0), \hat{R}'_0(0)] + \Sigma_2(0) \tilde{Z}_0(0) \right) \\ & \quad - \partial_{\hat{R}} G_{NL} \Big|_{t=0} \left(-(\Lambda(0) + \Lambda_{NL}(\hat{R}_0(0), 0)) \hat{R}'_0(0) \right. \\ & \quad \left. + (\Sigma(0) + \Sigma_{NL}(\hat{R}_0(0), 0)) \hat{R}_0(0) + S(0) (y(0) - \hat{R}_0^-(0)) \right), \end{aligned} \quad (3.26)$$

with

$$\begin{aligned} \tilde{N}^1 & = -\Lambda(0) \tilde{Z}'_0(0) - Q_{NL} [\tilde{Z}_0(0), \hat{R}_0(0), \tilde{Z}'_0(0), \hat{R}'_0(0)] + \Sigma_2(0) \tilde{Z}_0(0), \\ \tilde{N}^2 & = -\Lambda(L) \tilde{Z}'_0(L) - Q_{NL} [\tilde{Z}_0(L), \hat{R}_0(L), \tilde{Z}'_0(L), \hat{R}'_0(L)] + \Sigma_2(L) \tilde{Z}_0(L) \\ & \quad + \int_0^L D_1(L, \xi) \tilde{Z}_0(\xi) d\xi, \end{aligned}$$

where $\tilde{Z} = (\tilde{Z}^+, \tilde{Z}^-)^\top : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^{2N}$, $\tilde{Z}^+ : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^m$, $\tilde{Z}^- : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^{2N-m}$, $\tilde{Z}_{in}(t) = (\tilde{Z}^+(0, t), \tilde{Z}^-(L, t))^\top$, $\tilde{Z}_{out}(t) = (\tilde{Z}^+(L, t), \tilde{Z}^-(0, t))^\top$,

$$\Sigma_2 = \begin{bmatrix} \Sigma^{++} & 0_{m, 2N-m} \\ \Sigma^{-+} & 0_{2N-m, 2N-m} \end{bmatrix}, \quad D_1(x, \xi) = \begin{bmatrix} D^+(x, \xi) & 0 \\ D^-(x, \xi) & 0 \end{bmatrix},$$

$$D^+(x, \xi) = -F^1(x, \xi)\Sigma^{-+}(\xi) - \int_{\xi}^x F^1(x, s)D^-(s, \xi) ds,$$

$$D^-(x, \xi) = -F^2(x, \xi)\Sigma^{-+}(\xi) - \int_{\xi}^x F^2(x, s)D^-(s, \xi) ds,$$

defined in the triangular domain \mathbb{T} , and $K_I = \begin{bmatrix} K_I^{11} & K_I^{12} \\ 0_{2N-m, m} & 0_{2N-m, 2N-m} \end{bmatrix}$, $K_I^{11} \in \mathbb{R}^{m, m}$, $K_I^{12} \in \mathbb{R}^{m, 2N-m}$. The nonlinear terms $Q_{NL}[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R}]$ and $G_{NL}[\tilde{Z}(0, \cdot), \hat{R}(0, \cdot)]$ satisfy, for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\begin{aligned} & \left(I_{2N} + \int_0^x \begin{bmatrix} 0_{m, m} & F^1(x, \xi) \\ 0_{2N-m, m} & F^2(x, \xi) \end{bmatrix} d\xi \right) Q_{NL}[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R}] \\ &= \Lambda_{NL}(\mathcal{F}[\tilde{Z}] + \hat{R}, x) \partial_x (\mathcal{F}[\tilde{Z}] + \hat{R}) - \Sigma_{NL}(\mathcal{F}[\tilde{Z}] + \hat{R}, x) (\mathcal{F}[\tilde{Z}] + \hat{R}) \\ & \quad - \Lambda_{NL}(\hat{R}, x) \partial_x (\hat{R}) + \Sigma_{NL}(\hat{R}, x) (\hat{R}), \end{aligned} \quad (3.27)$$

$$\begin{aligned} & G_{NL}[\tilde{Z}(0, t), \hat{R}(0, t)] \\ &= \Gamma_0 \Pi_{NL} \left(T(0) (\mathcal{F}[\tilde{Z}](0, t) + \hat{R}(0, t)) \right) - \Gamma_0 \Pi_{NL} (T(0) \hat{R}(0, t)). \end{aligned} \quad (3.28)$$

Differentiating twice with respect to x in the invertible transformation $\mathcal{F}[\tilde{Z}]$, it is shown that the H^2 norm of \tilde{Z} is equivalent to the H^2 norm of \hat{R} . So local iISS of the system \hat{R} is same as local iISS of the target system \tilde{Z} . Therefore, in order to prove the state estimation performance of the quasilinear observer system (3.1)-(3.5), we need to prove the iISS of the quasilinear target system (3.19)-(3.26). By using Lyapunov method, the locally iISS of the target system (3.19)-(3.26) in the H^2 -norm is studied by analyzing the growth of $\|\tilde{Z}\|_{L^2}$, $\|\partial_t \tilde{Z}\|_{L^2}$ and $\|\partial_{tt} \tilde{Z}\|_{L^2}$ as follows.

Theorem 3.1 *If there exist positive constants $\alpha, q_1, q_2, q_3, q_4$, diagonal positive-definite matrices $P_{11}, P_3, P_4 \in \mathbb{R}^{2N, 2N}$, a symmetric positive-definite matrix $P_{22} \in \mathbb{R}^{2N, 2N}$ and a matrix $P_{12} \in \mathbb{R}^{2N, 2N}$ such that the following matrix inequalities hold, for all $x \in [0, L]$,*

$$\Omega(x) = \begin{bmatrix} \Omega_{11}(x) & \Omega_{12} & \Omega_{13}(x) & \Omega_{14} & \Omega_{15} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} \\ * & * & * & \Omega_{44} & \Omega_{45} \\ * & * & * & * & \Omega_{55} \end{bmatrix} \geq 0, \quad (3.29)$$

where

$$\begin{aligned}
\Omega_{11}(x) &= -\Lambda'(x)P_{11} - \alpha P_{11} \\
&\quad - \left(\Sigma_2^\top(x)P_{11} + P_{11}\Sigma_2(x) + q_1Lv_1^2I_{2N} + \left(\frac{L}{q_1} + \frac{L}{q_2} \right) D_1^\top(L,x)D_1(L,x) \right), \\
\Omega_{12} &= -P_{12}K_I, \\
\Omega_{13}(x) &= -\Lambda'(x)P_{12} - \alpha P_{12} - \Sigma_2^\top(x)P_{12}, \\
\Omega_{14} &= \Omega_{15} = \Omega_{25} = \Omega_{34} = \Omega_{35} = 0, \\
\Omega_{22} &= \frac{1}{L}E_2P_{11} - \frac{1}{L}K_P^\top E_1P_{11}K_P - \frac{1}{L}K_I^\top E_1P_3K_I, \\
\Omega_{23} &= -\frac{1}{L}K_P^\top E_1P_{11} - \frac{1}{L}(K_P^\top M_1 + M_2) - K_I^\top P_{22}, \\
\Omega_{24} &= -\frac{1}{L}K_I^\top E_1P_3K_P, \\
\Omega_{33} &= -\frac{1}{L}E_1P_{11} - \frac{1}{L}(M_1 + M_1^\top) - \alpha P_{22} - q_2Lv_2^2I_{2N}, \\
\Omega_{44} &= \frac{1}{L}E_2P_3 - \frac{1}{L}K_P^\top E_1P_3K_P - \frac{1}{L}K_I^\top E_1P_4K_I, \\
\Omega_{45} &= -\frac{1}{L}K_I^\top E_1P_4K_P, \\
\Omega_{55} &= \frac{1}{L}E_2P_4 - \frac{1}{L}K_P^\top E_1P_4K_P,
\end{aligned}$$

with $M_1 = \begin{bmatrix} \Lambda^+(0)P_{12}^{++} & \Lambda^+(0)P_{12}^{+-} \\ -\Lambda^-(L)P_{12}^{+-} & -\Lambda^-(L)P_{12}^{--} \end{bmatrix}$, $M_2 = \begin{bmatrix} -\Lambda^+(L)P_{12}^{++} & -\Lambda^+(L)P_{12}^{+-} \\ \Lambda^-(0)P_{12}^{+-} & \Lambda^-(0)P_{12}^{--} \end{bmatrix}$, $P_{12}^{++} \in \mathbb{R}^{m,m}$, $P_{12}^{+-} \in \mathbb{R}^{m,2N-m}$, $P_{12}^{--} \in \mathbb{R}^{2N-m,m}$, $P_{12}^{--} \in \mathbb{R}^{2N-m,2N-m}$, $E_1 = \text{diag}\{\Lambda^+(0), \Lambda^-(L)\}$, $E_2 = \text{diag}\{\Lambda^+(L), \Lambda^-(0)\}$, $v_1 = \max(\lambda(P_{11}))$, $v_2 = \max(|\lambda(P_{12})|)$, and

$$\begin{aligned}
M(x) &= (-\Lambda'(x) - \alpha I_{2N})P_3 \\
&\quad - \left(\Sigma_2^\top(x)P_3 + P_3\Sigma_2(x) + q_3Lv_3^2I_{2N} + \frac{L}{q_3}D_1^\top(L,x)D_1(L,x) \right) \geq 0, \quad (3.30)
\end{aligned}$$

$$\begin{aligned}
K(x) &= (-\Lambda'(x) - \alpha I_{2N})P_4 \\
&\quad - \left(\Sigma_2^\top(x)P_4 + P_4\Sigma_2(x) + q_4Lv_4^2I_{2N} + \frac{L}{q_4}D_1^\top(L,x)D_1(L,x) \right) \geq 0, \quad (3.31)
\end{aligned}$$

with $v_3 = \max(\lambda(P_3))$, $v_4 = \max(\lambda(P_4))$, then for every $\alpha > 0$, there exist positive constants δ , c and b such that, for any \bar{p} satisfying $\dot{\bar{p}}, \ddot{\bar{p}} \in L^2([0, \infty); \mathbb{R}^N)$, $\tilde{Z}_0 \in H^2([0, L]; \mathbb{R}^{2N})$ and $\tilde{X}_0 \in \mathbb{R}^{2N}$ satisfying $\|\tilde{Z}_0\|_{H^2} \leq \delta$ and the compatibility conditions (3.25)-(3.26), the H^2 -solution to the Cauchy problem (3.19)-(3.24) satisfies, for all $t \in [0, \infty)$,

$$\begin{aligned}
&\|\tilde{Z}(\cdot, t)\|_{H^2}^2 + |\tilde{X}(t)|^2 \\
&\leq ce^{-\alpha t} \left(\|\tilde{Z}_0\|_{H^2}^2 + |\tilde{X}_0|^2 \right) + b \int_0^t (|\dot{\bar{p}}(s)|^2 + |\ddot{\bar{p}}(s)|^2) ds. \quad (3.32)
\end{aligned}$$

Proof The following Lyapunov function candidate is introduced for the stability analysis of system (3.19)-(3.26), for all $x \in [0, L]$, $t \in [0, \infty)$,

$$V\left(\tilde{Z}(x, t), \tilde{X}(t), \partial_t \tilde{Z}(x, t), \partial_{tt} \tilde{Z}(x, t)\right) = V_1 + V_2 + V_3,$$

where

$$V_1 = \int_0^L \left(\tilde{Z}^\top(x, \cdot) \mathcal{P}_{11}(x) \tilde{Z}(x, \cdot) + \tilde{X}^\top(\cdot) \mathcal{P}_{22} \tilde{X}(\cdot) + \tilde{Z}^\top(x, \cdot) \mathcal{P}_{12}(x) \tilde{X}(\cdot) + \tilde{X}^\top(\cdot) \mathcal{P}_{12}^\top(x) \tilde{Z}(x, \cdot) \right) dx, \quad (3.33)$$

$$V_2 = \int_0^L \partial_t \tilde{Z}^\top(x, \cdot) \mathcal{P}_3(x) \partial_t \tilde{Z}(x, \cdot) dx, \quad (3.34)$$

$$V_3 = \int_0^L \partial_{tt} \tilde{Z}^\top(x, \cdot) \mathcal{P}_4(x) \partial_{tt} \tilde{Z}(x, \cdot) dx, \quad (3.35)$$

with $\mathcal{P}_{11}(x) = P_{11} \text{diag} \{e^{-\mu x} I_m, e^{\mu x} I_{2N-m}\}$, $\mathcal{P}_{12}(x) = P_{12} \text{diag} \{e^{-\frac{\mu}{2} x} I_m, e^{\frac{\mu}{2} x} I_{2N-m}\}$, $\mathcal{P}_3(x) = P_3 \text{diag} \{e^{-\mu x} I_m, e^{\mu x} I_{2N-m}\}$, $\mathcal{P}_4(x) = P_4 \text{diag} \{e^{-\mu x} I_m, e^{\mu x} I_{2N-m}\}$. Under the definition of V and straightforward estimations, there exists a positive real constant β such that, for every \tilde{Z} , we obtain the following inequality,

$$\begin{aligned} & \frac{1}{\beta} \int_0^L \left(|\tilde{Z}(x, \cdot)|^2 + |\tilde{X}(\cdot)|^2 + |\partial_x \tilde{Z}(x, \cdot)|^2 + |\partial_{xx} \tilde{Z}(x, \cdot)|^2 \right) dx \\ & \leq V \leq \beta \int_0^L \left(|\tilde{Z}(x, \cdot)|^2 + |\tilde{X}(\cdot)|^2 + |\partial_x \tilde{Z}(x, \cdot)|^2 + |\partial_{xx} \tilde{Z}(x, \cdot)|^2 \right) dx. \end{aligned} \quad (3.36)$$

Taking time derivative of V_1 along the solutions to (3.19)-(3.26), using integration by parts, and defining $\dot{V}_1 = \dot{V}_{1L} + \dot{V}_{1NL}$, where \dot{V}_{1L} is the time derivative of V_1 along the linear part of the quasilinear target system (3.19)-(3.26), for all $t \in [0, \infty)$, with

positive constants κ_1, κ_2 ,

$$\begin{aligned}
\dot{V}_{1L} \leq & \tilde{Z}_{out}^\top(t) \left(K_P^\top \check{E}_1 P_{11} K_P - e^{-\mu L} \check{E}_2 P_{11} \right) \tilde{Z}_{out}(t) + 2 \tilde{Z}_{out}^\top(t) K_P^\top \check{E}_1 P_{11} \tilde{X}(t) \\
& + \tilde{X}^\top(t) \check{E}_1 P_{11} \tilde{X}(t) + \int_0^L \tilde{Z}^\top(x, t) \left(\Lambda'(x) \mathcal{P}_{11}(x) - \mu |\Lambda(x)| \mathcal{P}_{11}(x) \right) \tilde{Z}(x, t) dx \\
& + 2 \int_0^L \tilde{Z}^\top(x, t) \mathcal{P}_{11}(x) \Sigma_2(x) \tilde{Z}(x, t) dx + q_1 L e^{2\mu L} v_1^2 \int_0^L \tilde{Z}^\top(x, t) \tilde{Z}(x, t) dx \\
& + \left(\frac{L}{q_1} + \frac{L}{q_2} \right) \int_0^L \left(D_1(L, x) \tilde{Z}(x, t) \right)^\top \left(D_1(L, x) \tilde{Z}(x, t) \right) dx \\
& + 2 \tilde{X}^\top(t) \check{M}_1 \tilde{X}(t) + 2 \int_0^L \tilde{Z}^\top(x, t) \left(\Lambda'(x) \mathcal{P}_{12}(x) - \frac{\mu}{2} |\Lambda(x)| \mathcal{P}_{12}(x) \right) \tilde{X}(t) dx \\
& + 2 \tilde{Z}_{out}^\top(t) \left(K_P^\top \check{M}_1 + \check{M}_2 \right) \tilde{X}(t) + 2 \int_0^L \tilde{Z}^\top(x, t) \mathcal{P}_{12}(x) K_I \tilde{Z}_{out}(t) dx \\
& + \kappa_1 \int_0^L \tilde{Z}^\top(x, t) \mathcal{P}_{12}(x) \left(\tilde{Z}^\top(x, t) \mathcal{P}_{12}(x) \right)^\top dx + \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} \right) \left(\Gamma_0 \dot{p}(t) \right)^\top \Gamma_0 \dot{p}(t) \\
& + 2 \int_0^L \tilde{X}^\top(t) \mathcal{P}_{12}^\top(x) \Sigma_2(x) \tilde{Z}(x, t) dx + q_2 L e^{\mu L} v_2^2 \int_0^L \tilde{X}^\top(t) \tilde{X}(t) dx \\
& + 2L \tilde{Z}_{out}^\top(t) K_I^\top P_{22} \tilde{X}(t) + L \kappa_2 \tilde{X}^\top(t) P_{22} \left(\tilde{X}^\top(t) P_{22} \right)^\top, \tag{3.37}
\end{aligned}$$

where $\check{E}_1 = \text{diag} \{ \Lambda^+(0), e^{\mu L} \Lambda^-(L) \}$, $\check{E}_2 = \text{diag} \{ \Lambda^+(L), e^{\mu L} \Lambda^-(0) \}$,

$$\check{M}_1 = \begin{bmatrix} \Lambda^+(0) P_{12}^{++} & \Lambda^+(0) P_{12}^{+-} \\ -e^{-\frac{\mu}{2} L} \Lambda^-(L) P_{12}^{-+} & -e^{-\frac{\mu}{2} L} \Lambda^-(L) P_{12}^{--} \end{bmatrix}, \check{M}_2 = \begin{bmatrix} -e^{-\frac{\mu}{2} L} \Lambda^+(L) P_{12}^{++} & -e^{-\frac{\mu}{2} L} \Lambda^+(L) P_{12}^{+-} \\ \Lambda^-(0) P_{12}^{-+} & \Lambda^-(0) P_{12}^{--} \end{bmatrix},$$

and where \dot{V}_{1NL} satisfies the following inequality,

$$\begin{aligned}
\dot{V}_{1NL} \leq & -2 \int_0^L Q_{NL}^\top \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right] \left(\mathcal{P}_{11}(x) \tilde{Z}(x, t) + \mathcal{P}_{12}(x) \tilde{X}(t) \right) dx \\
& + \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} \right) \left(\Gamma_0 \dot{p}(t) - \dot{G}_{NL} [\tilde{Z}(0, t), \hat{R}(0, t)] \right)^\top \left(\Gamma_0 \dot{p}(t) - \dot{G}_{NL} [\tilde{Z}(0, t), \hat{R}(0, t)] \right) \\
& - \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} \right) \left(\Gamma_0 \dot{p}(t) \right)^\top \Gamma_0 \dot{p}(t), \tag{3.38}
\end{aligned}$$

with \dot{G}_{NL} , the time derivative of G_{NL} along the solutions to (3.1)-(3.5) and (3.19)-(3.26).

By time differentiation of (3.19)-(3.20), $\partial_t \tilde{Z}$ is shown to satisfy the following equations, for all $x \in [0, L]$, $t \in [0, \infty)$,

$$\begin{aligned}
\partial_{tt} \tilde{Z}(x, t) = & -\Lambda(x) \partial_{tx} \tilde{Z}(x, t) + \Sigma_2(x) \partial_t \tilde{Z}(x, t) + \int_0^x D_1(x, \xi) \partial_t \tilde{Z}(\xi, t) d\xi \\
& - \dot{Q}_{NL} \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right], \tag{3.39}
\end{aligned}$$

$$\dot{\tilde{Z}}_{in}(t) = K_P \dot{\tilde{Z}}_{out}(t) + \dot{\tilde{X}}(t), \tag{3.40}$$

$$\dot{\tilde{X}}(t) = K_I \tilde{Z}_{out}(t) + \Gamma_0 \dot{p}(t) - \dot{G}_{NL} [\tilde{Z}(0, t), \hat{R}(0, t)], \tag{3.41}$$

where \dot{Q}_{NL} is the time derivative of Q_{NL} along the solutions to (3.1)-(3.5) and (3.19)-(3.26).

Taking time derivative of V_2 along the solutions to (3.19)-(3.26), using integration by parts, and defining $\dot{V}_2 = \dot{V}_{2L} + \dot{V}_{2NL}$, where \dot{V}_{2L} is the time derivative of V_2 along the linear part of the quasilinear target system (3.19)-(3.26), with positive constants κ_3 and κ_4 ,

$$\begin{aligned}
\dot{V}_{2L} \leq & \dot{\tilde{Z}}_{out}^\top(t) \left(K_P^\top \check{E}_1 P_3 K_P - e^{-\mu L} \check{E}_2 P_3 \right) \dot{\tilde{Z}}_{out}(t) + 2 \dot{\tilde{Z}}_{out}^\top(t) K_P^\top P_3 \check{E}_1 K_I \tilde{Z}_{out}(t) \\
& + \tilde{Z}_{out}^\top(t) K_I^\top \check{E}_1 P_3 K_I \tilde{Z}_{out}(t) + \kappa_3 \dot{\tilde{Z}}_{out}^\top(t) K_P^\top \check{E}_1 P_3 (K_P^\top \check{E}_1 P_3)^\top \dot{\tilde{Z}}_{out}(t) \\
& + \kappa_4 \tilde{Z}_{out}^\top(t) K_I^\top \check{E}_1 P_3 (K_I^\top \check{E}_1 P_3)^\top \tilde{Z}_{out}(t) \\
& + \int_0^L \partial_t \tilde{Z}^\top(x, t) (\Lambda'(x) \mathcal{P}_3(x) - \mu |\Lambda(x)| \mathcal{P}_3(x)) \partial_t \tilde{Z}(x, t) dx \\
& + 2 \int_0^L \partial_t \tilde{Z}^\top(x, t) \mathcal{P}_3(x) \Sigma_2(x) \partial_t \tilde{Z}(x, t) dx \\
& + \dot{\hat{p}}(t)^\top \Gamma_0^\top \left(\left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + \check{E}_1 P_3 \right) \Gamma_0 \dot{\hat{p}}(t) \\
& + q_3 L e^{2\mu L} \nu_3^2 \int_0^L \partial_t \tilde{Z}^\top(x, t) \partial_t \tilde{Z}(x, t) dx \\
& + \frac{L}{q_3} \int_0^L \left(D_1(L, x) \partial_t \tilde{Z}(x, t) \right)^\top \left(D_1(L, x) \partial_t \tilde{Z}(x, t) \right) dx, \tag{3.42}
\end{aligned}$$

and where \dot{V}_{2NL} satisfies the following inequality,

$$\begin{aligned}
\dot{V}_{2NL} \leq & \left(\Gamma_0 \dot{\hat{p}}(t) - \dot{G}_{NL}[\tilde{Z}(0, t), \hat{R}(0, t)] \right)^\top \left(\left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + \check{E}_1 P_3 \right) \\
& \times \left(\Gamma_0 \dot{\hat{p}}(t) - \dot{G}_{NL}[\tilde{Z}(0, t), \hat{R}(0, t)] \right) \\
& - 2 \int_0^L \dot{Q}_{NL}^\top \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right] \mathcal{P}_3(x) \partial_t \tilde{Z}(x, t) dx \\
& - \dot{\hat{p}}(t)^\top \Gamma_0^\top \left(\left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + \check{E}_1 P_3 \right) \Gamma_0 \dot{\hat{p}}(t). \tag{3.43}
\end{aligned}$$

By second time differentiation of (3.19)-(3.20), $\partial_{tt} \tilde{Z}$ is shown to satisfy the following equations, for all $x \in [0, L]$, $t \in [0, \infty)$,

$$\begin{aligned}
\partial_{ttt} \tilde{Z}(x, t) = & -\Lambda(x) \partial_{ttx} \tilde{Z}(x, t) + \Sigma_2(x) \partial_{tt} \tilde{Z}(x, t) + \int_0^x D_1(x, \xi) \partial_{tt} \tilde{Z}(\xi, t) d\xi \\
& - \ddot{Q}_{NL} \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right], \tag{3.44}
\end{aligned}$$

$$\ddot{\tilde{Z}}_{in}(t) = K_P \ddot{\tilde{Z}}_{out}(t) + \ddot{\tilde{X}}(t), \tag{3.45}$$

$$\ddot{\tilde{X}}(t) = K_I \ddot{\tilde{Z}}_{out}(t) + \Gamma_0 \ddot{\hat{p}}(t) - \ddot{G}_{NL}[\tilde{Z}(0, t), \hat{R}(0, t)], \tag{3.46}$$

where \ddot{Q}_{NL} and \ddot{G}_{NL} are respectively the second-order time derivative of Q_{NL} and G_{NL} along the solutions to (3.1)-(3.5) and (3.19)-(3.26).

Taking time derivative of V_3 along the solutions to (3.19)-(3.26), using integration by parts, and defining $\dot{V}_3 = \dot{V}_{3L} + \dot{V}_{3NL}$, where \dot{V}_{3L} is the time derivative of V_3 along the linear part of the quasilinear target system (3.19)-(3.26), with positive constants κ_5 and κ_6 ,

$$\begin{aligned}
\dot{V}_{3L} \leq & \ddot{Z}_{out}^\top(t) \left(K_P^\top \check{E}_1 P_4 K_P - e^{-\mu L} \check{E}_2 P_4 \right) \ddot{Z}_{out}(t) + 2 \ddot{Z}_{out}^\top(t) K_P^\top P_4 \check{E}_1 K_I \ddot{Z}_{out}(t) \\
& + \ddot{Z}_{out}^\top(t) K_I^\top \check{E}_1 P_4 K_I \ddot{Z}_{out}(t) + \kappa_5 \ddot{Z}_{out}^\top(t) K_P^\top \check{E}_1 P_4 \left(K_P^\top \check{E}_1 P_4 \right)^\top \ddot{Z}_{out}(t) \\
& + \kappa_6 \ddot{Z}_{out}^\top(t) K_I^\top \check{E}_1 P_4 \left(K_I^\top \check{E}_1 P_4 \right)^\top \ddot{Z}_{out}(t) \\
& + \int_0^L \partial_{tt} \tilde{Z}^\top(x, t) (\Lambda'(x) \mathcal{P}_4(x) - \mu |\Lambda(x)| \mathcal{P}_4(x)) \partial_{tt} \tilde{Z}(x, t) dx \\
& + 2 \int_0^L \partial_{tt} \tilde{Z}^\top(x, t) \mathcal{P}_4(x) \Sigma_2(x) \partial_{tt} \tilde{Z}(x, t) dx \\
& + q_4 L e^{2\mu L} \nu_4^2 \int_0^L \partial_{tt} \tilde{Z}^\top(x, t) \partial_{tt} \tilde{Z}(x, t) dx + \ddot{p}^\top(t) \Gamma_0^\top \left(\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} \right) I_{2N} + P_4 \check{E}_1 \right) \Gamma_0 \ddot{p}(t) \\
& + \frac{L}{q_4} \int_0^L \left(D_1(L, x) \partial_{tt} \tilde{Z}(x, t) \right)^\top \left(D_1(L, x) \partial_{tt} \tilde{Z}(x, t) \right) dx, \tag{3.47}
\end{aligned}$$

and where \dot{V}_{3NL} satisfies the following inequality,

$$\begin{aligned}
\dot{V}_{3NL} \leq & \left(\Gamma_0 \ddot{p}(t) - \ddot{G}_{NL}[\tilde{Z}(0, t), \hat{R}(0, t)] \right)^\top \left(\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} \right) I_{2N} + P_4 \check{E}_1 \right) \\
& \times \left(\Gamma_0 \ddot{p}(t) - \ddot{G}_{NL}[\tilde{Z}(0, t), \hat{R}(0, t)] \right) \\
& - 2 \int_0^L \ddot{Q}_{NL}^\top \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right] \mathcal{P}_4(x) \partial_{tt} \tilde{Z}(x, t) dx \\
& - \ddot{p}^\top(t) \Gamma_0^\top \left(\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} \right) I_{2N} + P_4 \check{E}_1 \right) \Gamma_0 \ddot{p}(t). \tag{3.48}
\end{aligned}$$

For the linear term $\dot{V}_{1L} + \dot{V}_{2L} + \dot{V}_{3L}$, by using (3.37), (3.42) and (3.47), there exists a constant $\alpha > 0$ such that,

$$\begin{aligned}
& \dot{V}_{1L} + \dot{V}_{2L} + \dot{V}_{3L} \\
& \leq -\alpha V - \int_0^L \begin{bmatrix} \bar{Z}(x, \cdot) \\ \bar{Z}_{out}(\cdot) \\ \bar{X}(\cdot) \\ \check{Z}_{out}(\cdot) \\ \check{Z}_{out}(\cdot) \end{bmatrix}^\top \check{\Omega}(x) \begin{bmatrix} \bar{Z}(x, \cdot) \\ \bar{Z}_{out}(\cdot) \\ \bar{X}(\cdot) \\ \check{Z}_{out}(\cdot) \\ \check{Z}_{out}(\cdot) \end{bmatrix} dx \\
& \quad - \int_0^L \partial_t \bar{Z}^\top(x, t) \check{M}(x) \partial_t \bar{Z}(x, t) dx - \int_0^L \partial_{tt} \bar{Z}^\top(x, t) \check{K}(x) \partial_{tt} \bar{Z}(x, t) dx \\
& \quad + \dot{\bar{p}}^\top(t) \Gamma_0^\top \left(\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + P_3 \check{E}_1 \right) \Gamma_0 \dot{\bar{p}}(t) \\
& \quad + \ddot{\bar{p}}^\top(t) \Gamma_0^\top \left(\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} \right) I_{2N} + P_4 \check{E}_1 \right) \Gamma_0 \ddot{\bar{p}}(t), \tag{3.49}
\end{aligned}$$

$$\text{where, for } x \in [0, L], \check{\Omega}(x) = \begin{bmatrix} \check{\Omega}_{11}(x) & \check{\Omega}_{12}(x) & \check{\Omega}_{13}(x) & \check{\Omega}_{14} & \check{\Omega}_{15} \\ * & \check{\Omega}_{22} & \check{\Omega}_{23} & \check{\Omega}_{24} & \check{\Omega}_{25} \\ * & * & \check{\Omega}_{33} & \check{\Omega}_{34} & \check{\Omega}_{35} \\ * & * & * & \check{\Omega}_{44} & \check{\Omega}_{45} \\ * & * & * & * & \check{\Omega}_{55} \end{bmatrix}, \text{ with}$$

$$\begin{aligned} \check{\Omega}_{11}(x) &= \mu|\Lambda(x)|\mathcal{P}_{11}(x) - \Lambda'(x)\mathcal{P}_{11}(x) - \alpha\mathcal{P}_{11}(x) - \kappa_1\mathcal{P}_{12}(x) (\mathcal{P}_{12}(x))^\top \\ &\quad - \left(\Sigma_2^\top(x)\mathcal{P}_{11}(x) + \mathcal{P}_{11}(x)\Sigma_2(x) + q_1Le^{2\mu L}v_1^2I_{2N} + \left(\frac{L}{q_1} + \frac{L}{q_2} \right) D_1^\top(L, x)D_1(L, x) \right), \end{aligned}$$

$$\check{\Omega}_{12}(x) = -\mathcal{P}_{12}(x)K_I,$$

$$\check{\Omega}_{13}(x) = \frac{\mu}{2}|\Lambda(x)|\mathcal{P}_{12}(x) - \Lambda'(x)\mathcal{P}_{12}(x) - \alpha\mathcal{P}_{12}(x) - \Sigma_2^\top(x)\mathcal{P}_{12}(x),$$

$$\check{\Omega}_{14} = \check{\Omega}_{15} = \check{\Omega}_{25} = \check{\Omega}_{34} = \check{\Omega}_{35} = 0,$$

$$\check{\Omega}_{22} = \frac{e^{-\mu L}}{L}\check{E}_2P_{11} - \frac{1}{L}K_P^\top\check{E}_1P_{11}K_P - \frac{1}{L}K_I^\top\check{E}_1P_3K_I - \frac{\kappa_4}{L}K_I^\top\check{E}_1P_3(K_I^\top\check{E}_1P_3)^\top,$$

$$\check{\Omega}_{23} = -\frac{1}{L}K_P^\top\check{E}_1P_{11} - \frac{1}{L}(K_P^\top\check{M}_1 + \check{M}_2) - K_I^\top P_{22},$$

$$\check{\Omega}_{24} = -\frac{1}{L}K_I^\top\check{E}_1P_3K_P,$$

$$\check{\Omega}_{33} = -\frac{1}{L}\check{E}_1P_{11} - \frac{1}{L}(\check{M}_1 + \check{M}_1^\top) - \kappa_2P_{22}(P_{22})^\top - \alpha P_{22} - q_2Le^{\mu L}v_2^2I_{2N},$$

$$\begin{aligned} \check{\Omega}_{44} &= \frac{e^{-\mu L}}{L}\check{E}_2P_3 - \frac{1}{L}K_P^\top\check{E}_1P_3K_P - \frac{\kappa_3}{L}K_P^\top\check{E}_1P_3(K_P^\top\check{E}_1P_3)^\top - \frac{1}{L}K_I^\top\check{E}_1P_4K_I \\ &\quad - \frac{\kappa_6}{L}K_I^\top\check{E}_1P_4(K_I^\top\check{E}_1P_4)^\top, \end{aligned}$$

$$\check{\Omega}_{45} = -\frac{1}{L}K_I^\top\check{E}_1P_4K_P,$$

$$\check{\Omega}_{55} = \frac{e^{-\mu L}}{L}\check{E}_2P_4 - \frac{1}{L}K_P^\top\check{E}_1P_4K_P - \frac{\kappa_5}{L}K_P^\top\check{E}_1P_4(K_P^\top\check{E}_1P_4)^\top,$$

$$\begin{aligned} \check{M}(x) &= (-\Lambda'(x) + \mu|\Lambda(x)| - \alpha I_{2N})\mathcal{P}_3(x) \\ &\quad - \left(\Sigma_2^\top(x)\mathcal{P}_3(x) + \mathcal{P}_3(x)\Sigma_2(x) + q_3Le^{2\mu L}v_3^2I_{2N} + \frac{L}{q_3}D_1^\top(L, x)D_1(L, x) \right), \end{aligned}$$

$$\begin{aligned} \check{K}(x) &= (-\Lambda'(x) + \mu|\Lambda(x)| - \alpha I_{2N})\mathcal{P}_4(x) \\ &\quad - \left(\Sigma_2^\top(x)\mathcal{P}_4(x) + \mathcal{P}_4(x)\Sigma_2(x) + q_4Le^{2\mu L}v_4^2I_{2N} + \frac{L}{q_4}D_1^\top(L, x)D_1(L, x) \right). \end{aligned}$$

Under the conditions (3.29), (3.30), (3.31), there exist constants $\mu, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6 > 0$ small enough, such that $\check{\Omega} \geq 0$ and $\check{M}, \check{K} \geq 0$, thus there exist positive constants α_1, α_2 such that, with $\alpha_1 = \max \left\{ \lambda \left(\Gamma_0^\top \left(\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + P_3\check{E}_1 \right) \Gamma_0 \right) \right\}$ and

$$\alpha_2 = \max \left\{ \lambda \left(\Gamma_0^\top \left(\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} \right) I_{2N} + P_4\check{E}_1 \right) \Gamma_0 \right) \right\},$$

the linear term $\dot{V}_{1L} + \dot{V}_{2L} + \dot{V}_{3L}$ satisfies the following inequality,

$$\dot{V}_{1L} + \dot{V}_{2L} + \dot{V}_{3L} \leq -\alpha V + \alpha_1 \dot{\bar{p}}^\top \dot{\bar{p}} + \alpha_2 \ddot{\bar{p}}^\top \ddot{\bar{p}}. \quad (3.50)$$

Now we analyze the nonlinear term $\dot{V}_{1NL} + \dot{V}_{2NL} + \dot{V}_{3NL}$. From (3.38), (3.43) and (3.48), there exist positive constants h_1, h_2, h_3, h_4, h_5 and h_6 such that

$$\begin{aligned} \dot{V}_{1NL} \leq & 2h_1 \int_0^L \left| Q_{NL}^\top \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right] \right| \left(|\mathcal{P}_{11}(x)| |\tilde{Z}(x,t)| + |\mathcal{P}_{12}(x)| |\tilde{X}(t)| \right) dx \\ & + h_2 \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} \right) \left(2 \left| \dot{G}_{NL}[\tilde{Z}(0,t), \hat{R}(0,t)] \right|^2 + \dot{\bar{p}}^\top(t) \Gamma_0^\top \Gamma_0 \dot{\bar{p}}(t) \right), \end{aligned} \quad (3.51)$$

$$\begin{aligned} \dot{V}_{2NL} \leq & h_3 \left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} + |\check{E}_1 P_3| \right) \left(2 \left| \dot{G}_{NL}[\tilde{Z}(0,t), \hat{R}(0,t)] \right|^2 + \dot{\bar{p}}^\top(t) \Gamma_0^\top \Gamma_0 \dot{\bar{p}}(t) \right) \\ & + 2h_4 \int_0^L \left| \dot{Q}_{NL}^\top \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right] \right| |\mathcal{P}_3(x)| \left| \partial_t \tilde{Z}(x,t) \right| dx, \end{aligned} \quad (3.52)$$

$$\begin{aligned} \dot{V}_{3NL} \leq & h_5 \left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} + |P_4 \check{E}_1| \right) \left(2 \left| \ddot{G}_{NL}[\tilde{Z}(0,t), \hat{R}(0,t)] \right|^2 + \ddot{\bar{p}}^\top(t) \Gamma_0^\top \Gamma_0 \ddot{\bar{p}}(t) \right) \\ & + 2h_6 \int_0^L \left| \ddot{Q}_{NL}^\top \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right] \right| |\mathcal{P}_4(x)| \left| \partial_{tt} \tilde{Z}(x,t) \right| dx. \end{aligned} \quad (3.53)$$

As derived in paper [10], since $\Lambda_{NL}(\tilde{Z}, \cdot)$ is twice differentiable with respect to \tilde{Z} and x , and $\Lambda_{NL}(0, \cdot) = 0$, there exist positive constants δ_Λ and s_1, s_2, s_3 such that for any $w_1, v_1 \in \mathbb{R}^{2N}$, if $\|\tilde{Z}\|_\infty \leq \delta_\Lambda$, it holds that

$$\left\| \Lambda_{NL}(\tilde{Z}, \cdot) \right\|_\infty \leq s_1 \|\tilde{Z}\|_\infty, \quad (3.54)$$

$$\left\| \partial_{\tilde{Z}} \Lambda_{NL}(\tilde{Z}, \cdot) w_1 \right\|_\infty + \left\| \partial_x \Lambda_{NL}(\tilde{Z}, \cdot) \right\|_\infty \leq s_2 \|w_1\|_\infty, \quad (3.55)$$

$$\left\| \partial_{\tilde{Z}}^2 \Lambda_{NL}(\tilde{Z}, \cdot) v_1 \right\|_\infty \leq s_3 \|v_1\|_\infty. \quad (3.56)$$

Similarly, since $\Sigma_{NL}(\tilde{Z}, \cdot)$ is twice differentiable with respect to \tilde{Z} and x , and $\Sigma_{NL}(0, \cdot) = 0$, there exist positive constants δ_Σ and s_4, s_5, s_6 such that for any $w_2, v_2 \in \mathbb{R}^{2N}$, if $\|\tilde{Z}\|_\infty \leq \delta_\Sigma$, it holds that

$$\left\| \Sigma_{NL}(\tilde{Z}, \cdot) \right\|_\infty \leq s_4 \|\tilde{Z}\|_\infty, \quad (3.57)$$

$$\left\| \partial_{\tilde{Z}} \Sigma_{NL}(\tilde{Z}, \cdot) w_2 \right\|_\infty + \left\| \partial_x \Sigma_{NL}(\tilde{Z}, \cdot) \right\|_\infty \leq s_5 \|w_2\|_\infty, \quad (3.58)$$

$$\left\| \partial_{\tilde{Z}}^2 \Sigma_{NL}(\tilde{Z}, \cdot) v_2 \right\|_\infty \leq s_6 \|v_2\|_\infty. \quad (3.59)$$

Because Π_{NL} is once differentiable with respect to \tilde{Z} , and $\Pi_{NL}(0) = 0$, there exist positive constants δ_Π and s_7, s_8 such that for any $w_3 \in \mathbb{R}^{2N}$, if $\|\tilde{Z}\|_\infty \leq \delta_\Pi$,

$$\left| \Gamma_0 \Pi_{NL}(\tilde{Z}(0, \cdot)) \right| \leq s_7 \|\tilde{Z}(0, \cdot)\|_\infty, \quad (3.60)$$

$$\left| \Gamma_0 \partial_{\tilde{Z}} \Pi_{NL}(\tilde{Z}(0, \cdot)) w_3 \right| \leq s_8 \|w_3\|_\infty. \quad (3.61)$$

Note that the nonlinear terms Q_{NL} and G_{NL} in (3.27)-(3.28) are dependent on the variables \tilde{Z} , $\partial_x \tilde{Z}$, \hat{R} and $\partial_x \hat{R}$, and there exists $\delta_{\hat{R}} > 0$ such that $\|\hat{R}\|_{L^2} + \|\partial_t \hat{R}\|_{L^2} + \|\partial_{tt} \hat{R}\|_{L^2} \leq \delta_{\hat{R}}$ in the H^2 space. Therefore, from (3.18), (3.27)-(3.28) and inequalities (3.54)-(3.61), there exist positive constants $\delta_1, h_7, h_8, h_9, h_{10}, h_{11}$ such that for all \tilde{Z} satisfying $\|\tilde{Z}\|_\infty \leq \delta_1 \leq \min\{\delta_\Lambda, \delta_\Sigma, \delta_\Pi\}$, it holds $\|\mathcal{F}[\tilde{Z}]\|_{L^2} \leq h_7 \|\tilde{Z}\|_{L^2}, \|\tilde{Z}\|_{L^2}^2 + |\tilde{X}|^2 \leq h_8 V_1, \|\partial_t \tilde{Z}\|_{L^2}^2 \leq h_9 V_2$, and

$$\left| Q_{NL} \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right] \right| \leq h_{10} \left(\|\partial_t \tilde{Z}\|_\infty^2 + \|\tilde{Z}\|_\infty^2 \right),$$

$$\left| \dot{G}_{NL}[\tilde{Z}(0, \cdot), \hat{R}(0, \cdot)] \right| \leq h_{11} |\partial_t \tilde{Z}(0, \cdot)|.$$

For all \tilde{Z} satisfying $\|\tilde{Z}\|_\infty \leq \delta_1$, the following inequality is deduced from (3.51),

$$\begin{aligned} \dot{V}_{1NL} &\leq 2h_1 h_{10} \left(\|\partial_t \tilde{Z}\|_\infty + \delta_1 \right) (V_1 + V_2) \\ &\quad + h_2 \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} \right) \left(2h_{11}^2 \|\partial_t \tilde{Z}\|_\infty V_2 + \dot{\tilde{p}}^\top(t) \Gamma_0^\top \Gamma_0 \dot{\tilde{p}}(t) \right). \end{aligned} \quad (3.62)$$

From (3.27)-(3.28) and inequalities (3.54)-(3.61), there exist positive constants $\delta_2 \leq \delta_1, h_{12}$ such that for all \tilde{Z} satisfying $\|\tilde{Z}\|_\infty + \|\partial_t \tilde{Z}\|_\infty \leq \delta_2$, it holds $\left| \dot{Q}_{NL} \left[\tilde{Z}, \hat{R}, \partial_x \tilde{Z}, \partial_x \hat{R} \right] \right| \leq h_{12} \left(\|\partial_t \tilde{Z}\|_\infty^2 + \|\tilde{Z}\|_\infty^2 \right)$, so from (3.52),

$$\begin{aligned} \dot{V}_{2NL} &\leq h_3 \left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} + \max \{ \lambda (P_3 \check{E}_1) \} \right) \left(2h_{11}^2 \delta_2 V_2 + \dot{\tilde{p}}^\top(t) \Gamma_0^\top \Gamma_0 \dot{\tilde{p}}(t) \right) \\ &\quad + 2h_4 h_{12} \delta_2 (V_1 + V_2). \end{aligned} \quad (3.63)$$

From (3.27)-(3.28) and inequalities (3.54)-(3.61), there exist positive constants $\delta_3 \leq \delta_2, h_{13}, h_{14}$ such that for all \tilde{Z} satisfying $\|\tilde{Z}\|_\infty + \|\partial_t \tilde{Z}\|_\infty + \|\partial_{tt} \tilde{Z}\|_\infty \leq \delta_3$, it holds $\left| \ddot{Q}_{NL} \left[\tilde{Z}, \hat{R}, \partial_t \tilde{Z}, \partial_x \hat{R} \right] \right| \leq h_{13} \left(\|\partial_{tt} \tilde{Z}\|_\infty^2 + \|\partial_t \tilde{Z}\|_\infty^2 + \|\tilde{Z}\|_\infty^2 \right), \left| \ddot{G}_{NL}[\tilde{Z}(0, \cdot), \hat{R}(0, \cdot)] \right| \leq h_{14} \|\partial_{tt} \tilde{Z}(0, \cdot)\|_\infty$ and we deduce from (3.53),

$$\begin{aligned} \dot{V}_{3NL} &\leq h_5 \left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} + \max \{ \lambda (P_4 \check{E}_1) \} \right) \left(2h_{14}^2 \delta_3 V_2 + \ddot{\tilde{p}}^\top(t) \Gamma_0^\top \Gamma_0 \ddot{\tilde{p}}(t) \right) \\ &\quad + 2h_6 h_{13} \delta_3 (V_1 + V_2 + V_3). \end{aligned} \quad (3.64)$$

Therefore, the nonlinear term $\dot{V}_{1NL} + \dot{V}_{2NL} + \dot{V}_{3NL}$, by using (3.62), (3.63) and (3.64) with $\alpha_3 = \max \left\{ \lambda \left(\Gamma_0^\top \left(h_2 \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} \right) I_{2N} + h_3 \left(\frac{1}{\kappa_3} I_{2N} + \frac{1}{\kappa_4} I_{2N} + P_3 \check{E}_1 \right) \right) \Gamma_0 \right) \right\}$ and

$$\alpha_4 = \max \left\{ \lambda \left(\Gamma_0^\top \left(h_5 \left(\frac{1}{\kappa_5} I_{2N} + \frac{1}{\kappa_6} I_{2N} + P_4 \check{E}_1 \right) \right) \Gamma_0 \right) \right\},$$

satisfies, for $\|\bar{Z}\|_\infty + \|\partial_t \bar{Z}\|_\infty + \|\partial_{tt} \bar{Z}\|_\infty \leq \delta_3$,

$$\begin{aligned} & \dot{V}_{1NL} + \dot{V}_{2NL} + \dot{V}_{3NL} \\ & \leq (2h_1 h_{10}(\delta_2 + \delta_1) + 2h_4 h_{12} \delta_2)(V_1 + V_2) + 2h_6 h_{13} \delta_3(V_1 + V_2 + V_3) \\ & \quad + \left(h_2 \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} \right) + h_3 \left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} + \max \{ \lambda(P_3 \check{E}_1) \} \right) \right) 2h_{11}^2 \delta_2 V_2 \\ & \quad + h_5 \left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} + \max \{ \lambda(P_4 \check{E}_1) \} \right) 2h_{14}^2 \delta_2 V_2 + \alpha_3 \bar{p}^\top(t) \dot{\bar{p}}(t) + \alpha_4 \bar{p}^\top(t) \ddot{\bar{p}}(t). \end{aligned} \quad (3.65)$$

Thus combining the inequalities (3.50) with (3.65), along the solutions to the system (3.19)-(3.26), for all $t \in [0, \infty)$, we get the existence of a positive constant $\delta_4 \leq \delta_3$ such that, for all \bar{Z} satisfying $\|\bar{Z}\|_\infty + \|\partial_t \bar{Z}\|_\infty + \|\partial_{tt} \bar{Z}\|_\infty \leq \delta_4$,

$$\begin{aligned} V & \leq V(0)e^{-\alpha t/2} + \alpha_5 e^{-\alpha t/2} \int_0^t (|\dot{\bar{p}}(s)|^2 + |\ddot{\bar{p}}(s)|^2) e^{\alpha s/2} ds \\ & \leq V(0)e^{-\alpha t/2} + \alpha_5 \int_0^t (|\dot{\bar{p}}(s)|^2 + |\ddot{\bar{p}}(s)|^2) ds, \end{aligned} \quad (3.66)$$

with $\alpha_5 = \max \{ \alpha_1 + \alpha_3, \alpha_2 + \alpha_4 \}$ and such that

$$\begin{aligned} & 4h_1 h_{10} \delta_4 + 2h_4 h_{12} \delta_4 + 2h_6 h_{13} \delta_4 \\ & \quad + \left(h_2 \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} \right) + h_3 \left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} + \max \{ \lambda(P_3 \check{E}_1) \} \right) \right) 2h_{11}^2 \delta_4 \\ & \quad + h_5 \left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} + \max \{ \lambda(P_4 \check{E}_1) \} \right) 2h_{14}^2 \delta_4 < \alpha/2. \end{aligned}$$

Combining this relation with (3.36), there exist positive constants

$$c = \beta^2, \quad b = \beta \alpha_5,$$

such that, for all $t \geq 0$,

$$\begin{aligned} & \int_0^L (|\bar{Z}(x, t)|^2 + |\bar{X}(x, t)|^2 + |\partial_x \bar{Z}(x, t)|^2 + |\partial_{xx} \bar{Z}(x, t)|^2) dx \\ & \leq \beta V(t) \\ & \leq \beta \left(\bar{V}(0)e^{-\alpha t/2} + \alpha_5 \int_0^t (|\dot{\bar{p}}(s)|^2 + |\ddot{\bar{p}}(s)|^2) ds \right) \\ & \leq \beta^2 e^{-\alpha t/2} \left(\int_0^L (|\bar{Z}_0(x)|^2 + |\bar{X}_0|^2 + |\partial_x \bar{Z}(x, 0)|^2 + |\partial_{xx} \bar{Z}(x, 0)|^2) dx \right) \\ & \quad + \beta \left(\alpha_5 \int_0^t (|\dot{\bar{p}}(s)|^2 + |\ddot{\bar{p}}(s)|^2) ds \right) \\ & = c e^{-\alpha t/2} \left(\int_0^L (|\bar{Z}_0(x)|^2 + |\bar{X}_0|^2 + |\partial_x \bar{Z}(x, 0)|^2 + |\partial_{xx} \bar{Z}(x, 0)|^2) dx \right) \\ & \quad + b \int_0^t (|\dot{\bar{p}}(s)|^2 + |\ddot{\bar{p}}(s)|^2) ds, \end{aligned} \quad (3.67)$$

completing the proof of Theorem 3.1. \square

Remark 3.1 Based on the reversibility of backstepping transformation, it is straightforward to deduce the iISS of error system (3.13)-(3.17) in the H^2 sense by studying the stability of the target system (3.19)-(3.26) under the assumptions of Theorem 3.1. The iISS of the error system (3.13)-(3.17) implies that the state estimates go to the real values as time goes on.

4 Simulations

In order to validate the observer design for the heterogeneous congested traffic with a high traffic demand and a capacity drop respectively at the inlet and outlet boundaries of the considered road segment, the traffic parameters of two vehicle classes on a road section of 1km length and 6.5m width are chosen as in the papers [8] and [10], see Table 1. The Lax-Wendroff scheme is applied to discretize the traffic model in the spatio-

Name	Symbol	Value	Unit
Relaxation time	τ_1	30	s
	τ_2	60	s
Pressure exponent	γ_1	2.5	1
	γ_2	2	1
Free-flow velocity	v_1^M	80	$\frac{km}{h}$
	v_2^M	60	$\frac{km}{h}$
Maximum AO	Ao_1^M	0.9	1
	Ao_2^M	0.85	1
Occupied surface per vehicle	a_1	10	m^2
	a_2	42	m^2
equilibrium density at the inlet	$\rho_1^*(0)$	110	$\frac{veh}{km}$
	$\rho_2^*(0)$	70	$\frac{veh}{km}$
equilibrium velocity at the inlet	$v_1^*(0)$	50	$\frac{km}{h}$
	$v_2^*(0)$	25	$\frac{km}{h}$

Table 1 Selected values of parameters.

temporal domain. The equilibrium $u^*(x) = (\rho_1^*(x), v_1^*(x), \rho_2^*(x), v_2^*(x))^T, x \in [0, 1]$, $\rho_i^* : \text{veh./km}$, $v_i^* : \text{km/h}$, ($i = 1, 2$) is computed by solving the original differential equations (2.7)-(2.8) on MATLAB as in the papers [8] and [10], see Figure 2.

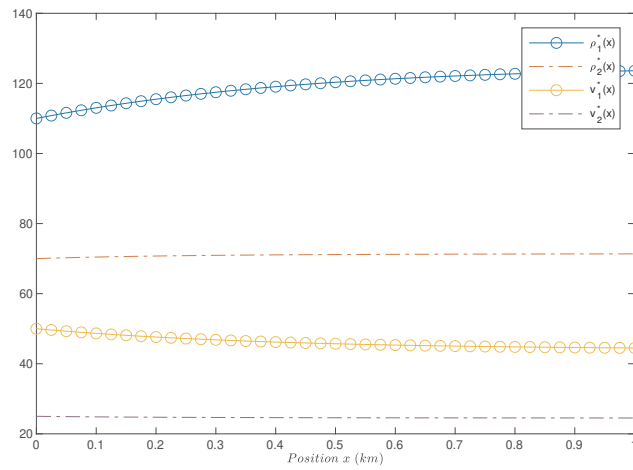


Fig. 2 Relation between spacial variable x and the nonuniform equilibrium $u^*(x) = (\rho_1^*(x), v_1^*(x), \rho_2^*(x), v_2^*(x))^T$.

The quasilinear system (2.18)-(2.22) is first simulated in open loop (namely $U = 0$) until $t = 0.06$ hour, and then it is simulated by the known input U , which is the combination of the designed quasilinear observer (3.1)-(3.5) in this paper with the control law in paper [10], see Figure 3.

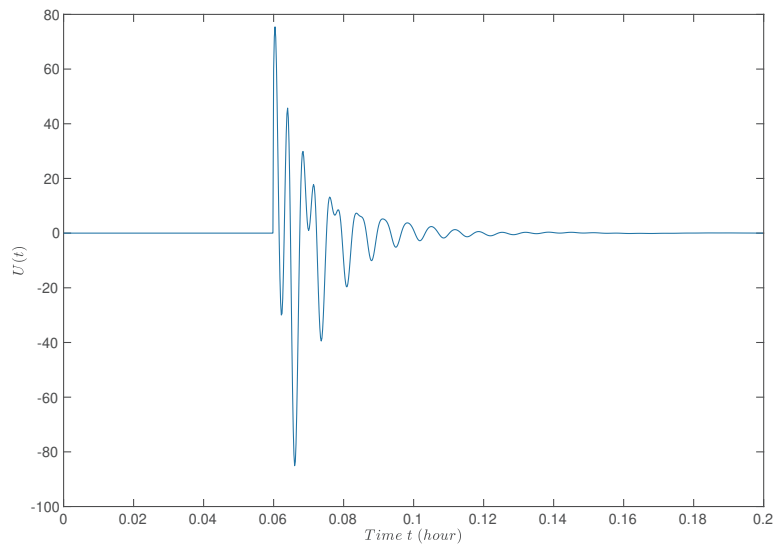


Fig. 3 Known input U of quasilinear system R in (2.18)-(2.22), this input is zero before $t = 0.06$ hour, and as in [10] afterwards.

The values of parameters $K_P, K_I, \Gamma_0, \Gamma_3$ derived from seeking the optimal known input U to minimize the likelihood of congested traffic in paper [9], and the coefficient matrices Θ and $\bar{\Theta}$ of the known input U are given as in paper [10] for simulations,

$$K_P = \begin{bmatrix} 0 & 0 & 0 & -7.85 \\ 0 & 0 & 0 & 10.47 \\ 0 & 0 & 0 & -42.04 \\ -5.67 & 5.09 & -7.15 & 0 \end{bmatrix} \times 10^{-5},$$

$$K_I = \begin{bmatrix} -20 & 30 & 30 & 60 \\ -24 & -7 & 26 & 30 \\ -10 & 20 & -30 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^{-5}, \quad \Gamma_0 = \begin{bmatrix} 0 & 0.0469 \\ 0.0156 & -0.0625 \\ 0.0332 & 0.2041 \\ 0 & 0 \end{bmatrix},$$

$$\Gamma_3 = [-5.67 \ 5.09 \ -7.15] \times 10^{-5}, \quad \Theta = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} \times 10^{-5}, \quad \bar{\Theta} = [1 \ 1 \ 1].$$

Given the same initial condition to demonstrate the congestion phenomenon as in paper [10], the disturbances $\bar{p}(t)$ to demonstrate the rush-hour, and the value of parameter Γ_4 , for $x \in (0, L)$,

$$u(x, 0) = \begin{bmatrix} \rho_1^*(x) + 0.05\rho_1^*(x) \cos 12\pi x \\ v_1^*(x) + 0.05v_1^*(x) \sin 12\pi x \\ \rho_2^*(x) + 0.05\rho_2^*(x) \cos 12\pi x \\ v_2^*(x) + 0.05v_2^*(x) \sin 12\pi x \end{bmatrix}, \quad \Gamma_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{p}(t) = \begin{bmatrix} 275e^{-30t} \cos 45\pi t \\ 87.5e^{-20t} \cos 45\pi t \end{bmatrix} \text{ (veh./h)}.$$

The observer injection gain S is obtained by using the code attached in [1] to numerically compute the equations of kernels F^1, F^2 in (3.8)-(3.12). By solving the linear matrix inequalities (LMIs) conditions, we derive the values of the variables $P_{11}, P_{12}, P_{22}, P_3, P_4$. Therefore, the satisfied conditions of Theorem 3.1 verify the iISS of the quasilinear target system.

$$P_{11} = \text{diag}\{1.7182, 2.2218, 4.3172, 2.4498\} \times 10^3,$$

$$P_{12} = \begin{bmatrix} -10.3592 & -0.0253 & -0.0142 & -0.0073 \\ 0.0289 & -13.1271 & -0.0041 & -0.0853 \\ 0.0284 & 0.0073 & -25.1226 & 0.6777 \\ -0.0080 & -0.0819 & 0.3717 & 14.1881 \end{bmatrix},$$

$$P_{22} = \begin{bmatrix} 5.1241 & 0 & 0 & 0 \\ * & 5.1241 & 0 & 0 \\ * & * & 5.1246 & 0 \\ * & * & * & 5.1241 \end{bmatrix} \times 10^3,$$

$$P_3 = \begin{bmatrix} 2.4190 & 0 & 0 & 0 \\ * & 2.8226 & 0 & 0 \\ * & * & 4.4099 & 0 \\ * & * & * & 2.8187 \end{bmatrix} \times 10^3,$$

$$P_4 = \text{diag} \{2.419, 2.8226, 4.4099, 2.8187\} \times 10^3.$$

State estimation of the quasilinear system (2.18)-(2.22) is shown in Figure 4. The iISS of the error system (3.13)-(3.17) is seen in Figure 5 as discussed in Remark 3.1. It is obvious that the quasilinear observer is capable of estimating the state of the quasilinear system in open loop and closed loop only by the boundary measurements, and it maintains the exponential convergence performance of the closed-loop system. Figure 6 gives the numerical simulations of the heterogeneous quasilinear traffic model in open loop $U = 0$ (until the time instant $t = 0.06$ hour) and in closed loop with the known input U . It is deduced that the observer converges in both open loop and closed loop cases. Moreover, it assumes that the observer-based known input U guarantees the iISS of the plant system (2.1)-(2.6). This observer-based input is obtained applying (formally) the separation principle between the control in [10] and estimation problem (3.1)-(3.5).

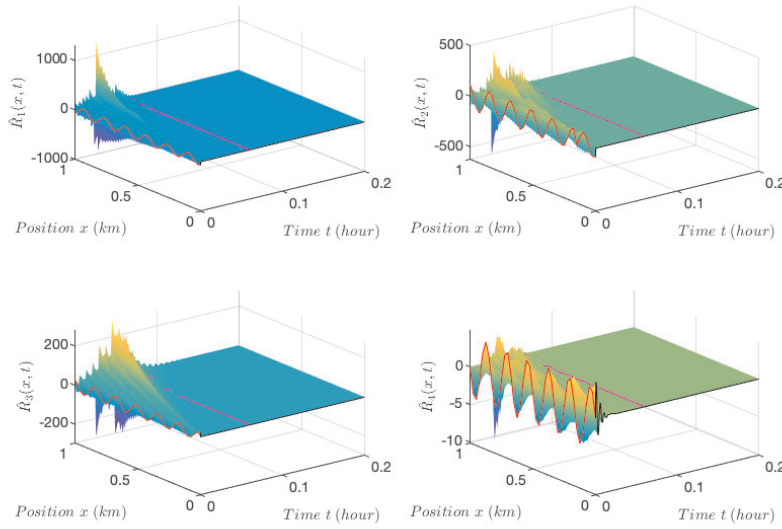


Fig. 4 Evolutions of the state variables of observer system $\hat{R} = (\hat{R}_1, \hat{R}_2, \hat{R}_3, \hat{R}_4)^\top$ in (3.1)-(3.5) with respect to the spacial variable x and the time variable t .

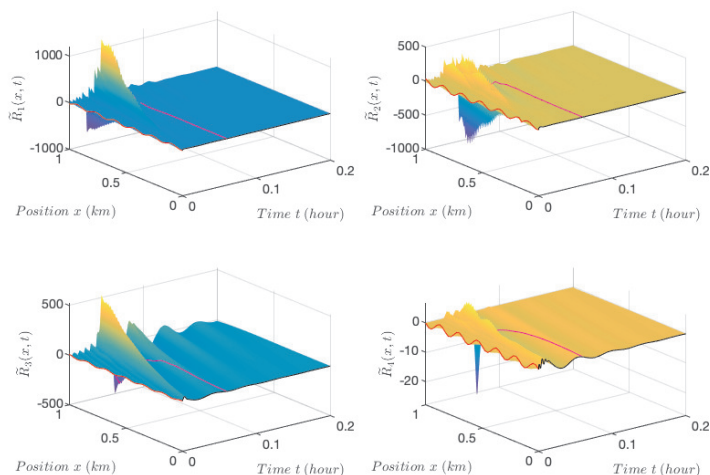


Fig. 5 Evolutions of the state variables of error system $\tilde{R} = (\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_4)^\top$ in (3.13)-(3.17) with respect to the spacial variable x and the time variable t .

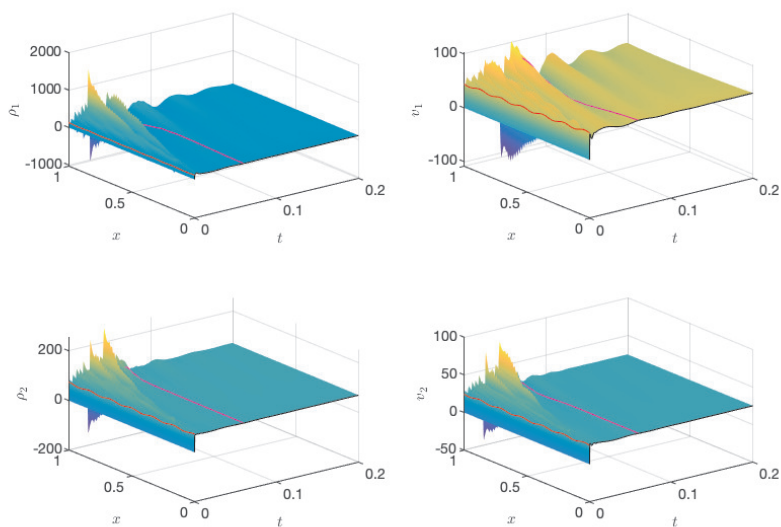


Fig. 6 Evolutions of the state variables of plant system $u = (\rho_1, v_1, \rho_2, v_2)^\top$ in (2.1)-(2.6) with respect to the spacial variable x and the time variable t .

5 Conclusion

This paper has solved the collocated observer design and stabilization for a heterogeneous PDEs traffic flow system by making use of the backstepping method. The observer gains were obtained by solving the kernel equations. The simulations of the systems were dealt with to verify the performance of known input U which was based on the nonlinear designed observer. It would be of interest to extend the results of controller design, for output feedback stabilization or for regulation problems to nonlinear heterogeneous traffic flow network systems.

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