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DEGREE 3 RELATIVE INVARIANT FOR UNITARY INVOLUTIONS

DEMBA BARRY, ALEXANDRE MASQUELEIN, AND ANNE QUÉGUINER-MATHIEU

ABSTRACT. Using the Rost invariant for non split simply connected groups, we define a relative degree 3 cohomological invariant for pairs of orthogonal or unitary involutions having isomorphic Clifford or discriminant algebras. The main purpose of this paper is to study general properties of this invariant in the unitary case, that is for torsors under groups of outer type A. If the underlying algebra is split, it can be reinterpreted in terms of the Arason invariant of quadratic forms, using the trace form of a hermitian form. When the algebra with unitary involution has a symplectic or orthogonal descent, or a symplectic or orthogonal quadratic extension, we provide comparison theorems between the corresponding invariants of unitary and orthogonal or symplectic types. We also prove the relative invariant is classifying in degree 4, at least up to conjugation by the non-trivial automorphism of the underlying quadratic extension. In general, choosing a particular base point, the relative invariant also produces absolute Arason invariants, under some additional condition on the underlying algebra. Notably, if the algebra has even co-index, so that it admits a hyperbolic involution, which is unique up to isomorphism, we get a so-called *hyperbolic* Arason invariant. Assuming in addition the algebra has degree 8, we may also define a *decomposable* Arason invariant. It generally does not coincide with the hyperbolic Arason invariant, as the hyperbolic involution need not be totally decomposable.

1. INTRODUCTION AND NOTATIONS

In quadratic form theory, the Arason invariant, introduced by Arason in [1], is a cohomological invariant, with values in $H^3(F, \mu_2)$, associated to any $2m$ -dimensional quadratic form with trivial discriminant and trivial Clifford invariant. It is closely related to the Rost invariant of the algebraic group Spin_{2m} , see [13, p. 107–108 and Thm. 9.11]. Using the Rost invariant for non split Spin groups, and for absolutely almost simple simply connected groups of other types, one may try to define analogues of the Arason invariant for the underlying algebraic objects, namely hermitian forms and involutions. This was initiated by Bayer and Parimala in [2]. They defined a degree 3 invariant for hermitian forms, which plays a role in their proof of the so-called Hasse Principle Conjecture II.

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For involutions of the first kind, degree 3 cohomological invariants were investigated in several papers, such as [7] and [15] for symplectic involutions, as well as [6], [11], [33] and [34] for orthogonal involutions; see also [38] for a survey on these questions. Following a suggestion of Tignol [38, §4.3], the present paper investigates the case of unitary involutions, which correspond to groups of outer type A.

The first two sections are devoted to definitions and formal properties of analogues of the Arason invariant for unitary involutions. The results there do not depart much from what is known in the orthogonal case. In particular, the Arason invariant for quadratic forms may be used to define an invariant for unitary involutions with trivial discriminant algebra on split algebras. Nevertheless, this invariant does not extend in a functorial way to the non split case, see §2.1. Using the Rost invariant, one may still define a relative Arason invariant, satisfying formal properties established in §3. In particular, even though it is generally represented by cohomology classes of larger order, the relative invariant has order 2, and satisfies some additivity property, see Proposition 3.1 and Corollary 3.6.

An important feature compared to previous papers is that we do not restrict to involutions with trivial lower-degree invariants, but also consider pairs of unitary involutions with isomorphic discriminant algebras, and pairs of orthogonal involutions with isomorphic Clifford algebras, see §2.2 and 5.1. This leads to a notion of relative Arason invariant, with values in the quotient of $H^3(F)$ by the image under the Rost invariant of cocycles coming from the center of the relevant groups; this image is described for each type of group in [30] and [14]. The relative Arason invariant has already proved to be useful in unitary type : Merkurjev considered it in the particular case of degree 4 split algebras, and he used it to extend to groups of outer type ${}^2\mathbf{A}_3$ a theorem of Rost on R -equivalence classes, see [29, §4]. In orthogonal type, an additional condition is required for defining the relative Arason invariant. This condition may be understood via the study of outer automorphisms of algebraic groups carried out in [35], or in terms of similarity classes of hermitian forms, see Remarks 5.2(2) and 5.4. It did not occur in previous papers, since only inner forms of the underlying groups were considered there.

With those Arason invariants in hand, we can provide precise comparison theorems for degree 3 invariants of different type in several situations. Namely, when an algebra with F'/F -unitary involution (B, τ) is isomorphic to a tensor product $(C, \gamma) \otimes_F (F', \iota)$ for some F -algebra with symplectic or orthogonal involution (C, γ) , where ι denotes the unique non trivial automorphism of the quadratic extension F'/F , we call (C, γ) a symplectic or an orthogonal descent of (B, τ) . Conversely, if an algebra with symplectic or orthogonal involution over F contains a subfield isomorphic to F' with centralizer isomorphic to (B, τ) as an algebra with involution, we call it a quadratic extension of (B, τ) . In both cases, strong relations exist between the degree 3 invariants of the relevant types; see §4 and 5 for precise statements.

Finally, we study properties of the Arason invariant for algebras of small degree with unitary involutions. In particular, we prove it is classifying in degree 4. More precisely, two F'/F unitary involutions are F -isomorphic if and only if their relative Arason invariant is trivial. In degree 4 and 6, we prove that hyperbolic involutions are characterized by the vanishing of the hyperbolic Arason invariant. This is not true anymore in degree 8, where one would expect a characterization of totally

decomposable involutions. For algebras of degree 8 and index 4, we use the relative Arason invariant to define a new absolute invariant, taking an arbitrary totally decomposable involution as a base point. We prove the value of this invariant does not depend on the chosen base point, using a cohomological invariant of the underlying biquaternion algebra introduced and studied in [5]. This invariant does not coincide with the hyperbolic Arason invariant as, surprisingly, the hyperbolic involution is not totally decomposable in general, see Corollary 6.8 and Example 6.9.

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Notations and preliminary results. Let F be a field. Throughout the paper, we assume F has characteristic different from 2. This may be unnecessary, as the Arason invariant, the Rost invariant, and the image of central cocycles are known in all characteristic. Nevertheless, we wish to avoid technical complications, notably when dealing with Arason invariant for groups of orthogonal type, for which the current literature makes this assumption.

We use the notations and terminology of [24] for algebras with involution. In particular, throughout the paper, we call (B, τ) an algebra with unitary involution over F when B is a finite dimensional F -algebra, with center a quadratic étale extension $F' = F[X]/(X^2 - \delta)$ of F , such that B is either simple (if F' is a field) or a direct product of two simple algebras (if $F' \simeq F \times F$) and τ is an F'/F semi-linear involution. In particular, τ acts on F' as the unique non-trivial F -automorphism, which we denote by ι . For any field extension L/F , the pair

$$(B_L, \tau_L) = (B \otimes_F L, \tau \otimes \text{id})$$

is an algebra with unitary involution over L , with center $L' = F' \otimes_F L$.

The central simple algebra B over F' is endowed with an F'/F -unitary involution τ if and only if B has split corestriction, or equivalently, its Brauer class has trivial corestriction, see [24, (3.1) & (3.20)]. Under this condition, if D is an F' -central division algebra Brauer equivalent to B , then D is also endowed with an F'/F -unitary involution θ . Moreover, there exists a hermitian module (M, h) with values in (D, θ) , such that $(B, \tau) \simeq (\text{End}_D(M), \text{ad}_h)$, where ad_h denotes the adjoint involution. The hermitian form h is uniquely determined up to a scalar factor $\lambda \in F^\times$.

If B has even degree, $n = 2m$, we let $\mathcal{D}(\tau)$ be the discriminant algebra of (B, τ) [24, §10]. The discriminant algebra is a central simple algebra over F , and its Brauer class can be computed in some cases, as recalled below for further use. Assume (B, τ) admits an orthogonal descent, $(B, \tau) \simeq (A, \sigma) \otimes_F (F', \iota)$, for some central simple algebra with orthogonal involution (A, σ) over F . Then by [24, (10.33)], the Brauer class of the discriminant algebra is given by

$$(1) \quad \mathcal{D}(\tau) \sim \begin{cases} (\delta, \text{disc}(\sigma)) & \text{if } m \text{ is even;} \\ A \otimes (\delta, \text{disc}(\sigma)) & \text{if } m \text{ is odd,} \end{cases}$$

where $\text{disc}(\sigma)$ stands for the discriminant of the orthogonal involution σ . Moreover, a unitary involution on a quaternion algebra has a unique symplectic descent by [24, (2.22)], that is $(Q, \tau) \simeq (H, \gamma) \otimes (F', \iota)$ for some uniquely determined quaternion

algebra H over F , with canonical (symplectic) involution γ . In view of [24, (10.30)], we then have $\mathcal{D}(\tau) \sim H$. Hence, two unitary involutions on a quaternion algebra are isomorphic if and only if they have isomorphic discriminant algebras. In particular, one may check using (1) that for all $a \in F^\times$, we have

$$(2) \quad (M_2(F), \text{ad}_{\langle\langle a \rangle\rangle}) \otimes_F (F', \iota) \simeq ((\delta, a), \gamma) \otimes_F (F', \iota).$$

Assume now F' is a field and B is split, that is $B \simeq \text{End}_{F'}(V)$ for some n -dimensional vector space V over F' . The involution $\tau \simeq \text{ad}_h$ is adjoint to a hermitian form $h : V \times V \rightarrow (F', \iota)$, and again h is unique up to a scalar factor $\lambda \in F^\times$. One may associate to h a quadratic form called its Jacobson's trace and defined by $q_h(x) = h(x, x) \in F$ for all $x \in V$, see [36, Chap. 10 §1]. So q_h is a $2n$ -dimensional quadratic form over F , and by *loc. cit.* Thm 1.1, the isometry class of q_h determines the isometry class of h . If h diagonalises as $h \simeq \langle a_1, \dots, a_n \rangle$ for some $a_i \in F^\times$, then $q_h \simeq \langle 1, -\delta \rangle \otimes \langle a_1, \dots, a_n \rangle$. The invariants of q_h are computed in [36, p. 350]. Its discriminant and full Clifford algebra satisfy

$$(3) \quad d(q_h) = \begin{cases} \delta \in F^\times / F^{\times 2} & \text{if } n = 2m + 1 \text{ is odd,} \\ 1 \in F^\times / F^{\times 2} & \text{if } n = 2m \text{ is even.} \end{cases}$$

and $\mathcal{C}(q_h) \sim (\delta, d(h))$ where $d(h) = (-1)^{\frac{n(n-1)}{2}} a_1 \dots a_n \in F^\times / N_{F'/F}(F'^\times)$ is the discriminant of the hermitian form h .

In particular, if $n = 2m$ is even, by the structure theorem for Clifford algebras, see [25, V. Thm. 2.5(3)], we have $\mathcal{C}_0(q_h) \simeq \mathcal{C}_+ \times \mathcal{C}_-$ with $\mathcal{C}_+ \sim \mathcal{C}_- \sim \mathcal{C}(q_h)$. Comparing with [24, (10.35)], we get that \mathcal{C}_+ and \mathcal{C}_- are Brauer equivalent to $\mathcal{D}(\text{ad}_h)$. Assume now that $n = 2m + 1$ is odd. The even Clifford algebra $\mathcal{C}_0(q_h)$ is the centraliser of F' in $\mathcal{C}(q_h)$, hence, $\mathcal{C}_0(q_h) \sim \mathcal{C}(q_h)_{F'}$, see [36, Chap. 8, Thm. 5.4]. In view of the formula above for $\mathcal{C}(q_h)$, we have $\mathcal{C}(q_h)_{F'} \sim 0$. So we finally get

$$(4) \quad \mathcal{C}_0(q_h) \simeq \begin{cases} 0 \in \text{Br}(F') & \text{if } n = 2m + 1 \text{ is odd,} \\ \mathcal{C}_+ \times \mathcal{C}_- \in \text{Br}(F) \times \text{Br}(F), & \text{if } n = 2m \text{ is even,} \end{cases}$$

with $\mathcal{C}_+ \sim \mathcal{C}_- \sim \mathcal{D}(\text{ad}_h)$.

For any discrete torsion Galois module M , we let $H^i(F, M)$ be the Galois cohomology group $H^i(\text{Gal}(F_{\text{sep}}/F), M)$. If G is a group scheme over F , $\text{Gal}(F_{\text{sep}}/F)$ acts continuously on $G(F_{\text{sep}})$ and we let $H^1(F, G) = H^1(\text{Gal}(F_{\text{sep}}/F), G(F_{\text{sep}}))$. We also use the notation

$$H^i(F) = H^i(F, \mathbb{Q}/\mathbb{Z}(i-1)), \text{ see [13, Appendix A, p. 151].}$$

In all three cases, we denote by $\text{res}_{F/L}$, respectively $\text{cor}_{L/F}$, the restriction and corestriction maps in Galois cohomology, where L/F is an arbitrary (respectively a finite degree) field extension. For each integer $m \geq 0$, ${}_m H^i(F)$ denotes the m -torsion subgroup of $H^i(F)$. Using the norm residue isomorphism, one may check that ${}_2 H^i(F) = H^i(F, \mu_2)$ and ${}_4 H^3(F) = H^3(F, \mu_4^{\otimes 2})$. In particular, ${}_2 H^1(F) \simeq F^\times / F^{\times 2}$ and we let $(a) \in {}_2 H^1(F)$ be the square class of $a \in F^\times$ and $(a_1, \dots, a_i) \in (a_1) \cdots (a_i) \in {}_2 H^i(F)$ the cup-product of the square classes of $a_1, \dots, a_i \in F^\times$. In particular, we use the same notation (a_1, a_2) for the quaternion F -algebra and its Brauer class in ${}_2 \text{Br}(F) \simeq {}_2 H^2(F)$. Recall that the Milnor K -ring $K_*(F)$ acts

on $\sqcup_{i \geq 0} H^i(F)$. In particular, for any $\alpha \in \text{Br}(F) \simeq H^2(F)$, $F^\times \cdot \alpha \subset H^3(F)$ is a subgroup, and we let

$$M_\alpha^3(F) = H^3(F)/(F^\times \cdot \alpha).$$

Given in addition a quadratic extension F'/F and another Brauer class $\beta \in \text{Br } F'$, we define in a similar way

$$N_{\alpha,\beta}^3(F) = H^3(F)/(F^\times \cdot \alpha + \text{cor}_{F'/F}(F'^\times \cdot \beta)).$$

In particular, we have $N_{\alpha,0}^3(F) = M_\alpha^3(F)$.

We denote by $I(F)$ the fundamental ideal in the Witt ring $W(F)$. For any (non degenerate) quadratic form q over F , if the Witt class of q is in $I^3(F)$, we let $e_3(q) \in {}_2H^3(F)$ be its Arason invariant. If G/F is an absolutely almost simple simply connected algebraic group, we let ρ_G be the corresponding Rost invariant,

$$\rho_G : H^1(F, G) \rightarrow H^3(F), \text{ see [13].}$$

An explicit description of ρ_G is given in some cases in [13, p. 107–108]. In particular, if $G = \text{Spin}(q_0)$ is the Spin group of some non degenerate quadratic form q_0 , then ρ_G can be described in terms of the Arason invariant.

Let $i : G \rightarrow G'$ be a homomorphism of absolutely almost simple simply connected algebraic groups. By [13, Thm. 9.11], for all field extension L/F , we have

$$\rho_{G'_L}(i^{(1)}(\xi)) = n_i \rho_{G_L}(\xi) \in H^3(L) \text{ for all } \xi \in H^1(L, G_L),$$

where $i^{(1)} : H^1(L, G_L) \rightarrow H^1(L, G'_L)$ denotes the map induced by i , and the integer n_i is the index defined by Dynkin in [10], also called the Rost multiplier of i . We will use the following :

Lemma 1.1. *The homomorphism*

$$j : \text{SL}_n(F) \rightarrow \text{SL}_{2n}(F), \quad X \mapsto \begin{pmatrix} X & 0 \\ 0 & (X^{-1})^t \end{pmatrix}$$

has Rost multiplier $n_j = 2$.

Proof. The homomorphism j is the direct sum of the tautological representation ρ of SL_n and its dual representation ρ^* . By definition of the index, we have $n_\rho = 1$. Moreover, composition with a group automorphism does not change the index, so we also have $n_{\rho^*} = 1$. The result follows by [13, Prop. 7.9(5)]. \square

Our main interest in this paper is for outer groups of type ${}^2A_{n-1}$. So, let us assume $G = \text{SU}(B, \tau_0)$ for some degree n algebra with unitary F'/F -involution (B, τ_0) . We recall from [24, (29.18)] that the cohomology set $H^1(F, \text{SU}(B, \tau_0))$ may be identified with $\text{SSym}(B, \tau_0)^\times / \approx$, where

$$(5) \quad \text{SSym}(B, \tau_0) = \{(s, z) \in \text{Sym}(B, \tau_0)^\times \times F'^\times, \text{Nrd}_B(s) = N_{F'/F}(z)\},$$

and the equivalence relation \approx is defined by $(s, z) \approx (s', z')$ if and only if there exists $b \in B^\times$ such that $s' = bs\tau(b)$ and $z' = \text{Nrd}_B(b)z$. Let $R_{F'/F}(\mu_n)$ be the corestriction of the algebraic group μ_n of roots of unity. The center of G is isomorphic to the kernel $\mu_{n[F']}$ of the norm map $N_{F'/F} : R_{F'/F}(\mu_n) \rightarrow \mu_n$. Under (5), $H^1(F, \mu_{n[F']})$ corresponds to the classes of all pairs $(x, z) \in F^\times \times F'^\times$ such that

$x^n = N_{F'/F}(z)$. From the description of the image under ρ_G of the class of such an element given in [30, Thm. 1.10 & Thm. 1.11] and [14, §10], we get

$$(6) \quad \rho_{\mathrm{SU}(B, \tau_0)}(H^1(F, \mu_{n[F']})) \subset \begin{cases} \mathrm{cor}_{F'/F}(F'^{\times} \cdot [B]) & \text{if } n \text{ is odd,} \\ F^{\times} \cdot [\mathcal{D}(\tau_0)] + \mathrm{cor}_{F'/F}(F'^{\times} \cdot [B]) & \text{if } n \text{ is even.} \end{cases}$$

2. A DEGREE 3 INVARIANT FOR UNITARY INVOLUTIONS

The Arason invariant of quadratic forms may be used to define a degree 3 invariant for unitary involutions on split algebras, provided the underlying algebra $B = \mathrm{End}_{F'}(V)$ has even degree n and the involution has a split discriminant algebra, as we shall explain in §2.1. This invariant does not extend in a functorial way to the non-split case. Nevertheless, using the Rost invariant, we may still define a relative Arason invariant for unitary involutions, see §2.2. Choosing a particular base point, this produces absolute Arason invariants, under some additional condition on the underlying algebra. Notably, if the algebra has even co-index, so that it admits a hyperbolic involution, which is unique up to isomorphism, we get a so-called *hyperbolic* Arason invariant, see §2.3. Finally, multiplying the Rost invariant by 2, we define in §2.4 a relative invariant with values in $H^3(F)$ when B has exponent at most 2.

From now on, we let F' be a quadratic étale extension of F , $F' = F[X]/(X^2 - \delta)$ for some $\delta \in F^{\times}$, and we denote by ι the unique non-trivial F -automorphism of F' .

2.1. Absolute degree 3 invariant in the split case. Let (B, τ) be an algebra of degree n with F'/F -unitary involution such that the algebra B is split, and consider a hermitian space (V, h) over (F', ι) such that $B \simeq \mathrm{End}_{F'}(V)$ and $\tau \simeq \mathrm{ad}_h$. We assume in addition that $n = 2m$ is even and the discriminant algebra $\mathcal{D}(\tau)$ is split. As recalled in the preliminary section, it follows the Jacobson's trace q_h of h belongs to $I^3(F)$. Hence, one may consider the Arason invariant $e_3(q_h) \in {}_2H^3(F)$. Moreover, for all $\lambda \in F^{\times}$, we have $q_{\langle \lambda \rangle h} = \langle \lambda \rangle q_h$ and $e_3(\langle \lambda \rangle q_h) = e_3(q_h) \in {}_2H^3(F)$. Hence, the cohomology class $e_3(q_h)$ is a well defined invariant of the algebra with involution $(\mathrm{End}_{F'}(V), \mathrm{ad}_h)$. We will refer to it as its Arason invariant, and use the notation

$$e_3(\mathrm{ad}_h) = e_3(q_h) \in {}_2H^3(F).$$

This invariant clearly is functorial, that is for any field extension L/F , we have $e_3((\mathrm{ad}_h)_L) = \mathrm{res}_{L/F}(e_3(\mathrm{ad}_h)) \in {}_2H^3(L)$.

As for orthogonal involutions (see [3, §3.4]), we claim this invariant does not extend in a functorial way to the non-split case. Specifically, fix a non-zero integer n such that $n \equiv 4 \pmod{8}$, and assume for the sake of contradiction that there exists a degree 3 invariant satisfying the following conditions:

- (i) $e_3(\tau)$ is defined for all degree n algebra with unitary involution (B, τ) over a field extension L of the base field F such that the discriminant algebra $\mathcal{D}(\tau)$ is split;
- (ii) $e_3(\tau)$ has values in $H^3(L)/H_B$, where H_B is a subgroup of $H^3(L)$, depending on the algebra B ;
- (iii) If B is split and τ is adjoint to a hermitian form h , then the group H_B is trivial and $e_3(\tau) = e_3(\mathrm{ad}_h) = e_3(q_h) \in {}_2H^3(L)$;

- (iv) Given a field extension E of L , the restriction map in cohomology induces a map $\text{res}_{E/L} : H^3(L)/H_B \rightarrow H^3(E)/H_{B_E}$ and we have

$$e_3(\tau_E) = \text{res}_{E/L}(e_3(\tau)) \in H^3(E)/H_{B_E}.$$

Assume first that $n = 4$. In order to get a contradiction, consider a decomposable degree 4 algebra with unitary involution (B, τ) , that is

$$(B, \tau) = (Q_1, \gamma_1) \otimes (Q_2, \gamma_2) \otimes (L', \iota),$$

for some quaternion algebras Q_i with center L and canonical involution γ_i for $i = 1, 2$, and a quadratic field extension L'/L . For a suitable choice of L , Q_1 , Q_2 and L' , we may assume B is division. Moreover, $\gamma_1 \otimes \gamma_2$ is an orthogonal involution with trivial discriminant by [24, (7.3)(4)], hence the discriminant algebra $\mathcal{D}(\tau)$ is split, see (1). Therefore, we may consider $e_3(\tau) \in H^3(L)/H_B$. Pick an element $\xi \in H^3(L)$ such that $e_3(\tau) = \xi \pmod{H_B}$. We claim $\xi \in {}_4H^3(L)$. Indeed, there exists a biquadratic field extension E/L , which is a subfield of $Q_1 \otimes_F Q_2$, such that both Q_1 and Q_2 are split over E . Hence, the algebra with involution (B_E, τ_E) is split and hyperbolic. It follows τ_E is adjoint to a hyperbolic hermitian form h , and its Jacobson's trace q_h is also hyperbolic. Therefore H_{B_E} is trivial and $e_3(\tau_E) = 0 \in H^3(E)$ by condition (iii). Using condition (iv), we get that $\text{res}_{E/L}(\xi) = 0 \in H^3(E)$ and applying the corestriction map in cohomology, this leads to $4\xi = 0$ as claimed.

Now, since ξ is a sum of 2-power order cohomology classes in $H^3(F)$, there exists a family of 3-fold Pfister forms π_1, \dots, π_r such that ξ vanishes over the function field \mathcal{F} of the product of the corresponding quadrics, that is $\text{res}_{\mathcal{F}/L}(\xi) = 0$. In particular, applying again (iv), we have $e_3(\tau_{\mathcal{F}}) = 0$. On the other hand, by Merkurjev's index reduction formula [28] (see also [21]), $B_{\mathcal{F}}$ is still division, so that the involution $\tau_{\mathcal{F}}$ is anisotropic. This leads to a contradiction as follows. Let \mathcal{F}_1 be the function field of the Weil restriction $R_{\mathcal{F}'/\mathcal{F}}(\text{SB}_{B_{\mathcal{F}}})$ of the Severi-Brauer variety of $B_{\mathcal{F}}$, where $\mathcal{F}' = L' \otimes_L \mathcal{F}$ is the center of $B_{\mathcal{F}}$. By (iv), we have $e_3(\tau_{\mathcal{F}_1}) = \text{res}_{\mathcal{F}_1/\mathcal{F}}(e_3(\tau_{\mathcal{F}})) = 0$. In addition, the field \mathcal{F}_1 is a generic splitting field of $B_{\mathcal{F}}$, therefore $B_{\mathcal{F}_1}$ is split and $\tau_{\mathcal{F}_1}$ is the adjoint of a 4-dimensional hermitian form h_{τ} , whose trace form $q_{h_{\tau}}$ is an 8-dimensional quadratic form. It follows by (iii) that $q_{h_{\tau}}$ is hyperbolic, that is h_{τ} and $\tau_{\mathcal{F}_1}$ are also hyperbolic. This contradicts [23, Thm. A.2], which asserts that the anisotropic involution $\tau_{\mathcal{F}}$ remains anisotropic over \mathcal{F}_1 . Hence, there is no invariant satisfying conditions (i) to (iv) above with $n = 4$.

Assume now $n = 4 + 8k$ for some integer k , and consider the algebra $M_{2k+1}(B)$, with B as above, endowed with the involution adjoint to the hermitian form with values in (B, τ) defined by $h = \langle 1 \rangle + k\mathbb{H}$, where \mathbb{H} denotes a hyperbolic plane. Since the anisotropic kernel of this form is (B, τ) , it has the same invariants as (B, τ) . Therefore, the same argument as above leads to a contradiction, and this concludes the proof.

Remark 2.1. If B has odd degree, the quadratic form q_h has discriminant δ , hence it does not belong to $I^3(F)$. When $\deg(B) = 3$, we may nevertheless define an absolute invariant for unitary involutions, even when B is non-split. This is explained in [24, §19.B] : to any unitary involution τ of B , corresponds a uniquely defined 3-fold Pfister form $\pi(\tau)$, so that $e_3(\pi(\tau)) \in H^3(F, \mu_2)$ is a well defined invariant of τ , see [24, (30.21)]. It follows from the definition of $\pi(\tau)$ that this invariant is functorial. Moreover, it is classifying by [24, (19.6)] : two unitary involutions τ_0 and τ of B are isomorphic if and only if $e_3(\pi(\tau_0)) = e_3(\pi(\tau)) \in H^3(F, \mu_2)$.

2.2. Relative degree 3 invariant. Using the Rost invariant, one may still define a relative invariant for unitary involutions. This definition is already contained in [38, §4.3] and [27], under the more restrictive assumption that both involutions have split discriminant algebra when B has even degree.

Consider two F'/F -unitary involutions τ_0 and τ of the algebra B , and assume in addition that τ_0 and τ have isomorphic discriminant algebras when B has even degree. As explained in [24, §29.D], we may associate to τ a cohomology class $\eta \in H^1(F, \text{PGU}(B, \tau_0))$ representing the triple $[B, \tau, \text{id}_{F'}]$. The exact sequence

$$(7) \quad 1 \rightarrow \mu_{n[F']} \rightarrow \text{SU}(B, \tau_0) \rightarrow \text{PGU}(B, \tau_0) \rightarrow 1$$

induces a connecting map

$$\partial : H^1(F, \text{PGU}(B, \tau_0)) \rightarrow H^2(F, \mu_{n[F']}).$$

By a twisting argument as in [37, I.5.4] (see also [12, §1]), one may check that $\partial(\eta)$ is the difference between the Tits classes of the groups $\text{SU}(B, \tau)$ and $\text{SU}(B, \tau_0)$. If B has odd degree, $n = 2m + 1$, the restriction map identifies $H^2(F, \mu_{n[F']})$ with the kernel of the corestriction $\text{cor}_{F'/F} : {}_n \text{Br}(F') \rightarrow {}_n \text{Br}(F)$, see [8, Rem. p. 309]. By [24, (31.8)], the groups $\text{SU}(B, \tau_0)$ and $\text{SU}(B, \tau)$ both have the same Tits class, which corresponds to $[B]$ under this identification. Assume now B has even degree $n = 2m$. We may identify $H^2(F, \mu_{n[F']})$ with a subgroup of $\text{Br}(F) \times \text{Br}(F')$ as in [8, Prop. 2.10]. The Tits classes of $\text{SU}(B, \tau_0)$ and $\text{SU}(B, \tau)$ respectively correspond to the pairs $([\mathcal{D}(\tau_0)], [B])$ and $([\mathcal{D}(\tau)], [B])$ by [24, (31.8)]. Hence in both cases, we get $\partial(\eta) = 0$ so that η lifts to a class $\xi \in H^1(F, \text{SU}(B, \tau_0))$, which is uniquely defined up to the action of $H^1(F, \mu_{n[F']})$. In view of the behavior of the Rost invariant under twisting, as described in [18, Lemme 7] (see also [30, Prop. 1.7]), it follows that $\rho_{\text{SU}(B, \tau_0)}(\xi) \in H^3(F)$ is well defined up to an element which belongs to the image of $H^1(F, \mu_{n[F']})$ under $\rho_{\text{SU}(B, \tau_0)}$, hence to the subgroup of $H^3(F)$ given in (6).

This leads to the following :

Definition 2.2. Given two F'/F -unitary involutions τ_0 and τ of the degree n algebra B , with $\mathcal{D}(\tau_0) \sim \mathcal{D}(\tau)$ if n is even, we define their relative Arason invariant by

$$e_3^{\tau_0}(\tau) = \rho_{\text{SU}(B, \tau_0)}(\xi) \in N_{\alpha, \beta}^3(F),$$

where $\xi \in H^1(F, \text{SU}(B, \tau_0))$ is a cocycle with image in $H^1(F, \text{PGU}(B, \tau_0))$ corresponding to the isomorphism class of $[B, \tau, \text{id}_{F'}]$,

$$\alpha = \begin{cases} [\mathcal{D}(\tau_0)] \in \text{Br}(F) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad \text{and } \beta = [B] \in \text{Br}(F').$$

We will refer to $e_3^{\tau_0}$ as the relative Arason invariant, with τ_0 as a base point. Depending on the degree n of B , it has values in

$$\begin{cases} H^3(F, \mathbb{Q}/\mathbb{Z}(2))/\text{cor}_{F'/F}(F'^{\times} \cdot [B]) & \text{if } n \text{ is odd, and} \\ H^3(F, \mathbb{Q}/\mathbb{Z}(2))/(F^{\times} \cdot [\mathcal{D}(\tau_0)] + \text{cor}_{F'/F}(F'^{\times} \cdot [B])) & \text{if } n \text{ is even.} \end{cases}$$

If τ is isomorphic to τ_0 , then $[B, \tau, \text{id}_{F'}]$ corresponds to the base point in the cohomology set $H^1(F, \text{PGU}(B, \tau_0))$, hence $e_3^{\tau_0}(\tau) = 0$ in this case. The invariant $e_3^{\tau_0}$ is a functorial invariant. Indeed, for all field extension L/F , the restriction map in Galois cohomology induces a map

$$(8) \quad \text{res}_{L/F} : N_{\alpha, \beta}^3(F) \rightarrow N_{\alpha_L, \beta_L}^3(L),$$

where $\alpha_L = \text{res}_{L/F}(\alpha)$, and $\beta_L = [B \otimes_F L] \in \text{Br}(L')$ with $L' = F' \otimes_F L$. In particular, if n is even, we have $\alpha_L = [\mathcal{D}(\tau_0)_L] = [\mathcal{D}(\tau_0L)] \in \text{Br}(L)$. Moreover, in both cases,

$$e_3^{\tau_0L}(\tau_L) = \text{res}_{L/F}(e_3^{\tau_0}(\tau)) \in N_{\alpha_L, \beta_L}^3(L).$$

Example 2.3. If B has degree 3, then the relative Arason invariant classifies unitary involutions up to isomorphism. This follows from the description of the Rost invariant given in [24, (31.45)]. Indeed, as explained there, $\rho_{\text{SU}(B, \tau_0)}(\xi)$ belongs to $H^3(F, \mu_6^{\otimes 2})$, and its 2-primary part is equal to $e_3(\pi(\tau)) - e_3(\pi(\tau_0))$. On the other hand, $\text{cor}_{F'/F}(F'^{\times} \cdot [B])$ is contained in $H^3(F, \mu_3^{\otimes 2})$. Therefore, if $e_3^{\tau_0}(\tau) = 0 \in H^3(F)/\text{cor}_{F'/F}(F'^{\times} \cdot [B])$, the 2-primary part of $\rho_{\text{SU}(B, \tau_0)}(\xi)$ is trivial, that is $e_3(\pi(\tau)) = e_3(\pi(\tau_0)) \in H^3(F, \mu_2)$. This implies $\tau \simeq \tau_0$ by [24, (19.6)], see also Remark 2.1.

In the split case, the invariant $e_3^{\tau_0}$ can be described in terms of the the Arason invariant of quadratic forms as follows.

Lemma 2.4. *Let V be a vector space of dimension n over F' , and h_0 and h two hermitian forms on V , with the same discriminant $d(h_0) = d(h) \in F^{\times}/N_{F'/F}(F'^{\times})$. The involutions $\tau_0 = \text{ad}_{h_0}$ and $\tau = \text{ad}_h$ have isomorphic discriminant algebras when n is even, and their relative Arason invariant is given by*

$$e_3^{\tau_0}(\tau) = e_3(q_h - q_{h_0}) \in M_{\alpha}^3(F),$$

where q_h and q_{h_0} are Jacobson's traces of the hermitian forms h and h_0 , and α is 0 if n is odd and $[\mathcal{D}(\tau_0)] \in \text{Br}(F)$ if n is even.

Proof. First of all, since B is split, $\beta = [B] = 0 \in \text{Br}(F')$ and $e_3^{\tau_0}$ has values in $N_{\alpha, 0}^3(F) = M_{\alpha}^3(F)$. Pick a diagonalisation for the hermitian forms, $h_0 = \langle a_1, \dots, a_n \rangle$ and $h = \langle b_1, \dots, b_n \rangle$ where $a_i, b_i \in F^{\times}$ for $1 \leq i \leq n$. By assumption, there exists $y \in F'^{\times}$ such that $a_1 \dots a_n = b_1 \dots b_n N_{F'/F}(y)$. Therefore, we have

$$q_h - q_{h_0} = \langle 1, -\delta \rangle \langle b_1, \dots, b_n, -a_1, \dots, -a_n \rangle = q \perp \langle -\delta \rangle q,$$

with q of dimension $2n$ and discriminant $N_{F'/F}(y)$. So $q_h - q_{h_0}$ has trivial discriminant. Moreover, its Clifford invariant can be computed using [25, V (3.15)&(3.16)]

$$e_2(q_h - q_{h_0}) = e_2(q) + e_2(-\delta N_{F'/F}(y)q) = e_2(q) + e_2(q) + (-\delta N_{F'/F}(y), N_{F'/F}(y)) = 0.$$

Hence, under the conditions of the lemma, the difference $q_h - q_{h_0}$ belongs to $I^3(F)$, and we may consider $e_3(q_h - q_{h_0}) \in H^3(F)$. It remains to prove its image in $M_{\alpha}^3(F)$ coincides with $e_3^{\tau_0}(\tau)$.

By [24, (29.19)], the isometry class of h corresponds to a cohomology class

$$\xi \in H^1(F, \text{SU}(V, h_0)) = H^1(F, \text{SU}(B, \tau_0)).$$

Under (5), ξ corresponds to the class of a pair (s, z) where $s \in \text{End}_{F'}(V)$ is a τ_0 -symmetric element and $z \in F'^{\times}$ satisfies $N_{F'/F}(z) = \det(s)$. Moreover, we have $h \simeq h_s$, where h_s is the hermitian form defined by $h_s(x, y) = h_0(s^{-1}(x), y)$. Since $\tau \simeq \text{ad}_{h_s} \simeq \text{Int}(s) \circ \tau_0$, the image of ξ in $H^1(F, \text{GU}(B, \tau_0))$ corresponds to the conjugacy class of τ under the bijection described in [24, (29.16)]. Hence, ξ maps to a cohomology class corresponding to $[B, \tau, \text{id}_{F'}]$ in $H^1(F, \text{PGU}(B, \tau_0))$. By definition of the relative Arason invariant, we get

$$e_3^{\tau_0}(\tau) = \rho_{\text{SU}(B, \tau_0)}(\xi) \in M_{\alpha}^3(F).$$

On the other hand, the description of $\rho_{\text{SU}(B, \tau_0)}$ given in [24, (31.44)] gives

$$\rho_{\text{SU}(B, \tau_0)}(\xi) = e_3(q_h - q_{h_0}) \in H^3(F),$$

and this concludes the proof. \square

Remark 2.5. Given two involutions τ_0 and τ on the split algebra $\text{End}_{F'}(V)$, there exists hermitian forms h_0 and h on V such that $\tau \simeq \text{ad}_h$ and $\tau_0 \simeq \text{ad}_{h_0}$. Those hermitian forms are uniquely defined up to a scalar factor.

(i) If n is odd, we may choose h in its similarity class so that h and h_0 have the same discriminant. The above lemma applied to this particular choice for h computes $e_3^{\tau_0}(\tau) \in H^3(F)$.

(ii) Assume now n is even and τ_0 and τ have isomorphic discriminant algebras. By [24, (10.35)], the underlying hermitian forms h_0 and h have the same discriminant in $F^\times/N_{F'/F}(F'^\times)$. We claim that the value of $e_3(q_h - q_{h_0}) \in M_\alpha^3(F)$ does not depend on the choice of h_0 and h in their respective similarity classes. Indeed, given $\lambda_0, \lambda \in F^\times$, we have

$$\begin{aligned} e_3(q_{\langle \lambda \rangle h} - q_{\langle \lambda_0 \rangle h_0}) &= e_3(\langle \lambda \rangle q_h - \langle \lambda_0 \rangle q_{h_0}) \\ &= e_3(\langle \lambda \rangle (q_h - q_{h_0}) + \langle \lambda \rangle \langle \lambda^{-1} \lambda_0 \rangle q_{h_0}). \end{aligned}$$

Since both $q_h - q_{h_0}$ and $\langle \lambda^{-1} \lambda_0 \rangle q_{h_0}$ are in $I^3(F)$, we get

$$\begin{aligned} e_3(q_{\langle \lambda \rangle h} - q_{\langle \lambda_0 \rangle h_0}) &= e_3(\langle \lambda \rangle (q_h - q_{h_0})) + e_3(\langle \lambda \rangle \langle \lambda^{-1} \lambda_0 \rangle q_{h_0}) \\ &= e_3(q_h - q_{h_0}) + (\lambda^{-1} \lambda_0) \cdot [\mathcal{D}(\tau_0)], \end{aligned}$$

where the last equality follows from the fact that the Clifford invariant of q_{h_0} is $[\mathcal{D}(\tau_0)]$. Hence, replacing h and h_0 by a scalar multiple does not change the class of $e_3(q_h - q_{h_0}) \in H^3(F)/F^\times \cdot [\mathcal{D}(\tau_0)]$, as required.

(iii) Finally, assume in addition that the discriminant algebras $\mathcal{D}(\tau)$ and $\mathcal{D}(\tau_0)$ are split, so that $e_3^{\tau_0}$ has values in $H^3(F)$. Then both q_h and q_{h_0} belong to $I^3(F)$. Hence we have

$$e_3^{\tau_0}(\tau) = e_3(q_h - q_{h_0}) = e_3(q_h) - e_3(q_{h_0}) = e_3(\tau) - e_3(\tau_0) \in H^3(F).$$

In other words, the relative invariant coincides in this case with the difference of the absolute invariants of both involutions.

2.3. Hyperbolic degree 3 invariant. In this section, we assume that the algebra B has even co-index, that is $B \simeq M_{2r}(D)$ for some integer $r \geq 1$ and some central division algebra D over F' . In particular, this implies the degree n of B is even. We also assume B admits F'/F -unitary involutions, hence it may be endowed with a hyperbolic F'/F -unitary involution τ_0 , which is unique up to isomorphism. As explained in [24, Proof of (10.36)], the Brauer class of the discriminant algebra of τ_0 can be computed after extending scalars to a generic splitting field of B ; over such a field, τ_0 is adjoint to a hyperbolic hermitian form, which has trivial discriminant, hence $\mathcal{D}(\tau_0)$ is split.

Let τ be a unitary F'/F involution on $B \simeq M_{2r}(D)$ with split discriminant algebra. The hyperbolic Arason invariant of τ is defined by

$$e_3^{\text{hyp}}(\tau) = e_3^{\tau_0}(\tau) \in H^3(F)/\text{cor}_{F'/F}(F'^\times \cdot [B]), \text{ where } \tau_0 \text{ is hyperbolic.}$$

The invariant e_3^{hyp} is functorial, and it vanishes for hyperbolic involutions.

Example 2.6. (See [38, §4.1]). Assume B is split and has even degree $n = 2m$. The hyperbolic degree 3 invariant coincides in this case with the absolute invariant introduced in §2.1. This follows from Remark 2.5(iii). Indeed, consider two hermitian forms h_0 and h over (F', ι) such that $\tau_0 = \text{ad}_{h_0}$ and $\tau = \text{ad}_h$. Since τ_0 is hyperbolic, h_0 and q_{h_0} also are. Therefore, $e_3(\tau_0) = e_3(q_{h_0}) = 0$ and we get $e_3^{\tau_0}(\tau) = e_3(\tau) = e_3(q_h)$.

2.4. The $f_3^{\tau_0}$ invariant. In this section, we assume that the algebra B has exponent dividing 2 and is endowed with a unitary involution τ_0 . It follows that the subgroup $\text{cor}_{F'/F}(F'^{\times} \cdot [B])$ of $H^3(F)$ consists of classes that are of order at most 2. Besides, if n is even, the discriminant algebra $\mathcal{D}(\tau_0)$ also has exponent 2. Hence, by (6), we have

$$(9) \quad 2\rho_{\text{SU}(B, \tau_0)}(H^1(F, \mu_{n[F']})) = 0.$$

Therefore, the same procedure as in §2.2, replacing the Rost invariant $\rho_{\text{SU}(B, \tau_0)}$ by $2\rho_{\text{SU}(B, \tau_0)}$, provides an invariant denoted by $f_3^{\tau_0}$ and with values in $H^3(F)$. More precisely, given two F'/F -unitary involutions τ_0 and τ of B , and assuming $\mathcal{D}(\tau_0) \sim \mathcal{D}(\tau)$ if n is even, we define their relative f_3 invariant by

$$(10) \quad f_3^{\tau_0}(\tau) = 2\rho_{\text{SU}(B, \tau_0)}(\xi) \in H^3(F),$$

where $\xi \in H^1(F, \text{SU}(B, \tau_0))$ is any element with image in $H^1(F, \text{PGU}(B, \tau_0))$ corresponding to the isomorphism class of $[B, \tau, \text{id}_{F'}]$. We thus have $f_3^{\tau_0}(\tau) = 2c$, where $c \in H^3(F)$ is an arbitrary cohomology class such that $e_3^{\tau_0}(\tau)$ is the class of c in $N_{\alpha, \beta}^3(F)$. This is a well defined invariant since any two possible values of c differ by an element of $F^{\times} \cdot \alpha + \text{cor}_{F'/F}(F'^{\times} \cdot \beta)$, which is of order at most 2.

Example 2.7. If B is split, then $f_3^{\tau_0}$ is identically zero. This follows from Lemma 2.4, since the Arason invariant for quadratic forms has values in ${}_2H^3(F)$. It is also a consequence of the fact that the Rost invariant $\rho_{\text{SU}(B, \tau_0)}$ has order 2 in this case, see [13, Thm. 12.6].

Example 2.8. If B has even co-index and exponent dividing 2, for all unitary involution τ with split discriminant algebra, we define $f_3^{\text{hyp}}(\tau) = f_3^{\tau_0}(\tau)$, where τ_0 is a hyperbolic unitary involution of B .

Example 2.9. If B has exponent 2, the Rost invariant has order 4 in general, see [13, Thm. 12.6]. Therefore, there are examples of (B, τ_0, τ) where $f_3^{\tau_0}(\tau)$ is non trivial.

3. PROPERTIES OF THE RELATIVE ARASON INVARIANT

In this section, we study properties of the relative Arason invariant. The results and proofs are largely inspired by [34], where analogous results were obtained for algebras with orthogonal involutions with trivial discriminant and trivial Clifford invariant. Most of them are contained in [27], under the additional assumption that τ_0 and τ have split discriminant algebras.

3.1. Base point change.

Proposition 3.1. *Let τ_0, τ_1 and τ_2 be three F'/F -unitary involutions on the degree n algebra B . We assume all three involutions have isomorphic discriminant algebras if n is even. We have*

$$e_3^{\tau_0}(\tau_2) = e_3^{\tau_0}(\tau_1) + e_3^{\tau_1}(\tau_2) \in N_{\alpha, \beta}^3(F),$$

where $\alpha = 0$ if n is odd, $\alpha = [\mathcal{D}(\tau_0)] = [\mathcal{D}(\tau_1)] \in \text{Br}(F)$ if n is even, and $\beta = [B] \in \text{Br}(F')$.

Proof. For $i \in \{1, 2, 3\}$, we let $G_i = \text{SU}(B, \tau_i)$ and $\bar{G}_i = \text{PGU}(B, \tau_i)$ the corresponding adjoint group. Pick a cocycle $\omega \in Z^1(F, G_0)$ such that its image in $H^1(F, \bar{G}_0)$ corresponds to the triple $[B, \tau_1, \text{id}_{F'}]$. As explained in §2.2, such a cocycle exists since G_0 and G_1 have the same Tits class. Moreover, in view of [24, §29], since any isomorphism $(B, \tau_0)_{F_{\text{sep}}} \rightarrow (B, \tau_1)_{F_{\text{sep}}}$ restricts to an isomorphism $(G_0)_{F_{\text{sep}}} \rightarrow (G_1)_{F_{\text{sep}}}$, the inner twisted form ${}_{\omega}G_0$ of G_0 is isomorphic to G_1 . We denote by ϕ_{ω} the induced isomorphism $H^1(F, G_0) \rightarrow H^1(F, G_1)$.

Now, consider a cocycle $\xi \in H^1(F, G_0)$ with image in $H^1(F, \bar{G}_0)$ corresponding to $[B, \tau_2, \text{id}_{F'}]$. Again using [24, §29], one may check that $\phi_{\omega}(\xi)$ has image in $H^1(F, \bar{G}_1)$ also corresponding to $[B, \tau_2, \text{id}_{F'}]$. Moreover, by [18, Lemme 7], we have

$$\rho_{G_1}(\phi_{\omega}(\xi)) = \rho_{G_0}(\xi) - \rho_{G_0}(\omega) \in H^3(F).$$

Therefore, by definition of the relative Arason invariant, we get

$$e_3^{\tau_1}(\tau_2) = e_3^{\tau_0}(\tau_2) - e_3^{\tau_0}(\tau_1) \in N_{\alpha, \beta}^3(F),$$

and this finishes the proof. \square

3.2. Relation with the hyperbolic Arason invariant of a sum. Consider two unitary involutions τ_0 and τ on a degree n algebra B , and pick $s \in \text{Sym}(B, \tau_0)^{\times}$ such that $\tau = \text{Int}(s) \circ \tau_0$. For any $\lambda \in F^{\times}$, we let θ_{λ} be the involution on $M_2(B)$ defined by

$$(11) \quad \theta_{\lambda} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} \tau_0(x) & -\lambda\tau_0(z)s^{-1} \\ -\lambda^{-1}s\tau_0(y) & s\tau_0(t)s^{-1} \end{pmatrix}, \text{ for all } x, y, z, t \in B.$$

Assume τ_0 is the adjoint involution with respect to some hermitian form h_0 with values in the underlying division algebra D , endowed with a fixed unitary involution. Then, the involution τ is adjoint to the hermitian form h_s defined by $h_s(u, v) = h_0(s^{-1}(u), v)$, and θ_{λ} is adjoint to $h_0 \perp \langle -\lambda \rangle h_s$. Therefore, the involution θ_{λ} is called an orthogonal sum of τ_0 and τ . Note that in general, different choices for λ may produce non-isomorphic orthogonal sums of τ_0 and τ . Nevertheless, we may use θ_{λ} to compute $e_3^{\tau_0}(\tau)$ as follows:

Proposition 3.2. *Let τ_0 and τ be two involutions of the degree n algebra B . If n is even, we assume in addition that τ_0 and τ have isomorphic discriminant algebras. For all $\lambda \in F^{\times}$, we let θ_{λ} be the orthogonal sum of τ_0 and τ defined by (11).*

- (1) *If n is odd, then there exists $\lambda \in F^{\times}$ such that θ_{λ} has split discriminant algebra. If n is even, θ_{λ} has split discriminant algebra for all $\lambda \in F^{\times}$.*
- (2) *In both cases, for all λ such that θ_{λ} has split discriminant algebra, we have*

$$e_3^{\tau_0}(\tau) = e_3^{\text{hyp}}(\theta_{\lambda}) \in N_{\alpha, \beta}^3(F),$$

where $\alpha = 0$ if n is odd, $\alpha = [\mathcal{D}(\tau_0)] \in \text{Br}(F)$ if n is even, and $\beta = [B] = [M_2(B)] \in \text{Br}(F')$.

Remark 3.3. The invariant e_3^{hyp} has values in $N_{0, \beta}^3(F)$. If n is odd, or n is even and $\mathcal{D}(\tau_0)$ is split, $e_3^{\tau_0}$ also have values in $N_{0, \beta}^3(F)$, and the proposition states that $e_3^{\tau_0}(\tau)$ and $e_3^{\text{hyp}}(\theta_{\lambda})$ are equal in this group. Whereas if n is even and $\mathcal{D}(\tau_0)$ is non-split, $e_3^{\tau_0}$ has values in $N_{\alpha, \beta}^3(F)$, with $\alpha \neq 0$. In this case, the proposition actually states

$$e_3^{\tau_0}(\tau) = e_3^{\text{hyp}}(\theta_{\lambda}) \pmod{F^{\times} \cdot [\mathcal{D}(\tau_0)]},$$

that is $e_3^{\tau_0}(\tau)$ is the image of $e_3^{\text{hyp}}(\theta_\lambda)$ under the projection $N_{0,\beta}^3(F) \rightarrow N_{\alpha,\beta}^3(F)$.

Proof. Let θ_0 be the involution on $M_2(B)$ defined by

$$\theta_0 \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} \tau_0(x) & -\tau_0(z) \\ -\tau_0(y) & \tau_0(t) \end{pmatrix}, \text{ for all } x, y, z, t \in B.$$

It is a hyperbolic involution, since $e = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is idempotent and satisfies $\theta_0(e) = 1 - e$, see [24, (6.8)]. In particular, its discriminant algebra is split, that is $[\mathcal{D}(\theta_0)] = 0 \in \text{Br } F$, and $e_3^{\theta_0} = e_3^{\text{hyp}}$. Moreover, a direct computation shows that

$$\theta_\lambda = \text{Int} \begin{pmatrix} 1 & 0 \\ 0 & \lambda s \end{pmatrix} \circ \theta_0.$$

Therefore, by [24, (10.36)], we have

$$\mathcal{D}(\theta_\lambda) \sim (\delta, \text{Nrd}_{M_2(B)} \begin{pmatrix} 1 & 0 \\ 0 & \lambda s \end{pmatrix}) \sim (\delta, \text{Nrd}_B(\lambda s)).$$

Assume $n = 2m + 1$ is odd. Since s is symmetric, $\text{Nrd}_B(s) \in F^\times$. Hence we may chose $\lambda = \text{Nrd}_B(s)$ and we get $\text{Nrd}_B(\lambda s) = \lambda^{n+1} = (\lambda^{m+1})^2$. Therefore, $\mathcal{D}(\theta_\lambda)$ is split for this particular value of λ . Assume now $n = 2m$ is even. For all $\lambda \in F^\times$, we have $[\mathcal{D}(\theta_\lambda)] = (\delta, \text{Nrd}_B(\lambda s)) = (\delta, \lambda^n \text{Nrd}_B(s)) = (\delta, \text{Nrd}_B(s))$. Since τ_0 and τ have isomorphic discriminant algebras, we get that $\mathcal{D}(\theta_\lambda)$ is split, applying again [24, (10.36)].

This proves the first assertion, and it remains to compare the relative Arason invariants. In the remaining part of the proof, we assume that $\lambda \in F^\times$ has been chosen so that $\mathcal{D}(\theta_\lambda)$ is split if n is odd. Therefore, in both cases, there exists $z \in F'^\times$ such that

$$\text{Nrd}_{M_2(B)} \begin{pmatrix} 1 & 0 \\ 0 & \lambda s \end{pmatrix} = \text{Nrd}_B(\lambda s) = N_{F'/F}(z).$$

It follows that the pair $(\lambda s, z) \in \text{Sym}(B, \tau_0) \times F'^\times$ corresponds to a cohomology class $\xi \in H^1(F, \text{SU}(B, \tau_0))$ under (5). Moreover, since $\tau = \text{Int}(s) \circ \tau_0$, the image of ξ in $H^1(F, \text{PGU}(B, \tau_0))$ corresponds to $[B, \tau, \text{id}_{F'}]$. Consider the natural embedding

$$i : \text{SU}(B, \tau_0) \rightarrow \text{SU}(M_2(B), \theta_0), \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}.$$

The induced map $i^{(1)}$ in Galois cohomology maps $(\lambda s, z)$ to $(\begin{pmatrix} 1 & 0 \\ 0 & \lambda s \end{pmatrix}, z)$. Hence, the corresponding element $i^{(1)}(\xi)$ in $H^1(F, \text{SU}(M_2(B), \theta_0))$ maps to the class of $[M_2(B), \theta_\lambda, \text{id}_{F'}]$ in $H^1(F, \text{PGU}(M_2(B), \theta_0))$. Moreover, after scalar extension to an algebraic closure of F , the map i corresponds to the standard inclusion $\text{SL}_n \rightarrow \text{SL}_{2n}$, which has Rost multiplier 1 by [13, Example 7.10]. Therefore, we have

$$\rho_{\text{SU}(M_2(B), \theta_0)}(i^{(1)}(\xi)) = \rho_{\text{SU}(B, \tau_0)}(\xi).$$

Assertion (2) follows by definition of the relative Arason invariant. \square

Remark 3.4. It follows from Proposition 3.2 that $e_3^{\text{hyp}}(\theta_{\lambda_1}) = e_3^{\text{hyp}}(\theta_{\lambda_2}) \in N_{\alpha,\beta}^3(F)$ if $\mathcal{D}(\theta_{\lambda_i})$ are split for $i = 1, 2$. If the algebra B is split, this can be checked directly as follows. With the same notations as in the beginning of this section, for $i = 1, 2$, the involution θ_{λ_i} is adjoint to the hermitian form $h_i \simeq h_0 \perp \langle -\lambda_i \rangle h_s$, where h_0 and h_s now are hermitian forms of rank n over (F', ι) . If n is odd, λ_1 and λ_2 have

been chosen so that h_1 and h_2 have trivial discriminant. If n is even, h_0 and h_s have the same discriminant, and so h_1 and h_2 both have trivial discriminant, for arbitrary values of λ_1 and λ_2 . In both cases, we have $e_3^{\text{hyp}}(\theta_{\lambda_i}) = e_3(q_{h_i}) \in H^3(F)$ for $i = 1, 2$ by Example 2.6. In addition, $h_1 - h_2$ is Witt equivalent to $\langle -\lambda_1, \lambda_2 \rangle h_s$, and its trace form $q_{h_1} - q_{h_2}$ is Witt equivalent to

$$\langle -\lambda_1, \lambda_2 \rangle q_{h_s} \simeq \langle \lambda_2 \rangle \langle 1, -\lambda_1 \lambda_2^{-1} \rangle \langle 1, -\delta \rangle \langle a_1, \dots, a_n \rangle,$$

where $\langle a_1, \dots, a_n \rangle$ is any diagonalisation of h_s .

Assume $n = 2m + 1$ is odd. For $i = 1, 2$, since h_i has trivial discriminant, $d(h_i) = \lambda_i d(h_0) d(h_s)$ is a norm for the quadratic extension F'/F . Hence, $(\delta, d(h_1) d(h_2)^{-1}) = (\delta, \lambda_1 \lambda_2^{-1}) = 0 \in \text{Br}(F)$. It follows that the quadratic forms $\langle 1, -\lambda_1 \lambda_2^{-1} \rangle \langle 1, -\delta \rangle$ and $q_{h_1} - q_{h_2}$ are hyperbolic. We get

$$e_3^{\text{hyp}}(\theta_{\lambda_1}) - e_3^{\text{hyp}}(\theta_{\lambda_2}) = e_3(q_{h_1} - q_{h_2}) = 0 \in H^3(F)$$

as expected.

Assume now $n = 2m$ is even, so that $\langle a_1, \dots, a_n \rangle$ is even dimensional. We have

$$e_3^{\text{hyp}}(\theta_{\lambda_1}) - e_3^{\text{hyp}}(\theta_{\lambda_2}) = e_3(\langle -\lambda_1, \lambda_2 \rangle q_{h_s}) = (\lambda_1 \lambda_2^{-1}, \delta, (-1)^m a_1 \dots a_n).$$

Moreover, by [24, (10.35)]

$$[\mathcal{D}(\text{ad}_{h_0})] = (\delta, d(h_0)) = (\delta, d(h_s)) = (\delta, (-1)^m a_1 \dots a_n) \in \text{Br}(F).$$

Therefore, given arbitrary $\lambda_1, \lambda_2 \in F^\times$, we get

$$e_3^{\text{hyp}}(\theta_{\lambda_1}) - e_3^{\text{hyp}}(\theta_{\lambda_2}) = (\lambda_1 \lambda_2^{-1}) \cdot [\mathcal{D}(\text{ad}_{h_0})] \in H^3(F).$$

This equality proves $e_3^{\text{hyp}}(\theta_{\lambda_1})$ and $e_3^{\text{hyp}}(\theta_{\lambda_2})$, which generally differ as elements of $H^3(F)$, have the same image in $M_\alpha^3(F)$, as required.

3.3. Unitary involution with a split rank 2 factor. Using the same argument as in the proof of Proposition 3.2, we may also compute the hyperbolic Arason invariant of an algebra with unitary involution having a rank 2 split factor.

Proposition 3.5. *Let (B, τ_0) be a degree n algebra with unitary involution. If n is odd, there exists $\lambda \in F^\times$ such that the unitary involution $\text{ad}_{\langle 1, -\lambda \rangle} \otimes \tau_0$ has trivial discriminant algebra, and for such a choice of λ , we have*

$$e_3^{\text{hyp}}(\text{ad}_{\langle 1, -\lambda \rangle} \otimes \tau_0) = 0 \in H^3(F) / \text{cor}_{F'/F}(F'^\times \cdot [B]).$$

If n is even, then $\text{ad}_{\langle 1, -\lambda \rangle} \otimes \tau_0$ has trivial discriminant algebra for all $\lambda \in F^\times$ and

$$e_3^{\text{hyp}}(\text{ad}_{\langle 1, -\lambda \rangle} \otimes \tau_0) = (\lambda) \cdot [\mathcal{D}(\tau_0)] \in H^3(F) / \text{cor}_{F'/F}(F'^\times \cdot [B]).$$

Proof. For all $\lambda \in F^\times$, consider the algebra with involution

$$(M_2(F'), \text{ad}_{\langle 1, -\lambda \rangle}) \otimes_{F'} (B, \tau_0),$$

where $\langle 1, -\lambda \rangle$ is a rank 2 hermitian form over (F', ι) . It corresponds to θ_λ as defined in the previous section, in the particular case where $\tau = \tau_0$ so that we may assume $s = 1$. Hence, Proposition 3.2 already gives the first part of both assertions, and the formula for e_3^{hyp} when n is odd. Moreover, since $s = 1$, the computation in the proof of Proposition 3.2 shows that $e_3^{\text{hyp}}(\text{ad}_{\langle 1, -\lambda \rangle} \otimes \tau_0)$ is given by $\rho_{\text{SU}(B, \tau_0)}(\xi)$, where $\xi \in H^1(F, \text{SU}(B, \tau_0))$ corresponds under (5) to the class (λ, λ^m) . Therefore, ξ actually belongs to $H^1(F, \mu_n[F'])$ and the result follows by [30, Thm. 1.11], [14, §10]. \square

3.4. Order of the relative Arason invariant. Using the previous results, we now prove that the relative Arason invariant has order 2.

Corollary 3.6. *Let τ_0 and τ be two unitary involutions of the degree n algebra B . If n is even, we assume in addition that τ_0 and τ have isomorphic discriminant algebras. We have*

- (a) $e_3^{\tau_0}(\tau) = e_3^\tau(\tau_0) \in N_{\alpha,\beta}^3(F)$, and
- (b) $2e_3^{\tau_0}(\tau) = 0 \in N_{\alpha,\beta}^3(F)$,

where $\alpha = 0$ if n is odd, $\alpha = [\mathcal{D}(\tau_0)] \in \text{Br } F$ if n is even, and $\beta = [B] \in \text{Br}(F')$.

Proof. By Proposition 3.1, we have $e_3^{\tau_0}(\tau) + e_3^\tau(\tau_0) = e_3^{\tau_0}(\tau_0) = 0 \in N_{\alpha,\beta}^3(F)$. Therefore, it is enough to prove assertion (a). This can be checked as follows, using Proposition 3.2. Consider the involution θ_λ defined in (3.2). If n is odd, we assume λ has been chosen so that θ_λ has split discriminant algebra. As in the beginning of §3.2, we pick a hermitian form h_0 such that $\tau_0 \simeq \text{ad}_{h_0}$ and we let h_s be defined by $h_s(u, v) = h_0(s^{-1}(u), v)$, where $s \in \text{Sym}(B, \tau_0)^\times$ is such that $\tau = \text{Int}(s) \circ \tau_0 \simeq \text{ad}_{h_s}$. The involution θ_λ is adjoint to $h_0 \perp \langle -\lambda \rangle h_s$. We have

$$e_3^{\tau_0}(\tau) = e_3^{\text{hyp}}(\theta_\lambda) = e_3^{\text{hyp}}(\text{ad}_{h_0 \perp \langle -\lambda \rangle h_s}) \in N_{\alpha,\beta}^3(F).$$

On the other hand, since $h_0 \perp \langle -\lambda \rangle h_s$ and $h_s \perp \langle -\lambda^{-1} \rangle h_0$ are similar hermitian forms, the corresponding involutions are isomorphic and have the same hyperbolic Arason invariant. Besides, $\text{ad}_{h_s \perp \langle -\lambda^{-1} \rangle h_0}$ is an orthogonal sum of τ and $\tau_0 = \text{Int}(s^{-1}) \circ \tau$. Applying again Proposition 3.2, we thus get

$$e_3^{\tau_0}(\tau) = e_3^{\text{hyp}}(\text{ad}_{h_s \perp \langle -\lambda^{-1} \rangle h_0}) = e_3^\tau(\tau_0) \in N_{\alpha,\beta}^3(F),$$

and this concludes the proof. \square

Remark 3.7. As a consequence, we get that given two unitary involutions τ_0 and τ of B , with $\mathcal{D}(\tau_0) \sim \mathcal{D}(\tau)$ if n is even, their relative f_3 invariant satisfies

$$f_3^{\tau_0}(\tau) \in \begin{cases} \text{cor}_{F'/F}(F'^\times \cdot [B]) & \text{if } n \text{ is odd,} \\ F^\times \cdot [\mathcal{D}(\tau_0)] + \text{cor}_{F'/F}(F'^\times \cdot [B]) & \text{if } n \text{ is even.} \end{cases}$$

4. COMPARISON WITH DEGREE 3 INVARIANTS OF SYMPLECTIC INVOLUTIONS

The aim of this section is to prove some formulae relating the degree 3 cohomological invariants in unitary and symplectic type for symplectic descents and quadratic symplectic extensions of a pair of unitary involutions; see Sections 4.2 and 4.3 below for precise statements. We first recall some known facts on invariants of symplectic involutions.

4.1. Degree 3-invariants for symplectic involutions. Degree 3 invariants for symplectic involutions were investigated in [7] and [15]. Let (C, γ_0, γ) be a central simple algebra over F endowed with two symplectic involutions. In particular, C has even degree $r = 2\ell$ and exponent dividing 2. If ℓ is even, that is $4 \mid r$, a relative invariant $e_3^{\gamma_0}(\gamma)$ is defined in [7]. This invariant may also be defined in terms of the Rost invariant of the symplectic group $\text{Sp}(C, \gamma_0)$, see [38, §5.3]. The process is very similar to the definition of the relative Arason invariant for unitary involutions given in Section 2.2, with the exact sequence (7) replaced by

$$1 \rightarrow \mu_2 \rightarrow \text{Sp}(C, \gamma_0) \rightarrow \text{PGSp}(C, \gamma_0) \rightarrow 1.$$

Since we will use it in the proofs of comparison theorems, we briefly recall the procedure. The isomorphism class of (C, γ) corresponds to a cohomology class $\eta \in H^1(F, \text{PGSp}(C, \gamma_0))$ by [24, (29.22)]. Since γ_0 and γ are defined on the same algebra C , this class lifts to a class $\xi \in H^1(F, \text{Sp}(C, \gamma_0))$, which is unique up to the action of $H^1(F, \mu_2)$. Moreover, by [30, Thm. 1.13], [14], we have

$$\rho_{\text{Sp}(C, \gamma_0)}(H^1(F, \mu_2)) = \begin{cases} 0 & \text{if } \ell \text{ is even,} \\ F^\times \cdot [C] & \text{if } \ell \text{ is odd.} \end{cases}$$

Hence, we may define

$$(12) \quad e_3^{\gamma_0}(\gamma) = \rho_{\text{Sp}(C, \gamma_0)}(\xi) \in M_\alpha^3(F) = H^3(F)/F^\times \cdot \alpha,$$

where α is 0 if ℓ is even and $[C]$ if ℓ is odd. On the other hand, as explained in [24, p. 440], ξ corresponds to the choice of some $s \in \text{Sym}(C, \gamma_0)^\times$ such that $\gamma = \text{Int}(s) \circ \gamma_0$, and we have

$$\rho_{\text{Sp}(C, \gamma_0)}(\xi) = (\text{Nrp}_{\gamma_0}(s)) \cdot [C],$$

where Nrp_{γ_0} denotes the Pfaffian norm map, defined on γ_0 -symmetric elements. In particular, we have

$$(13) \quad \rho_{\text{Sp}(C, \gamma_0)}(\xi) \in F^\times \cdot [C] \text{ for all } \xi \in H^1(F, \text{Sp}(C, \gamma_0)).$$

Therefore, $e_3^{\gamma_0}$ is identically zero if ℓ is odd. If ℓ is even, we recover the original definition given in [7], namely

$$(14) \quad e_3^{\gamma_0}(\gamma) = \rho_{\text{Sp}(C, \gamma_0)}(\xi) = (\text{Nrp}_{\gamma_0}(s)) \cdot [C] \in H^3(F),$$

where $s \in \text{Sym}(C, \gamma_0)^\times$ satisfies $\gamma = \text{Int}(s) \circ \gamma_0$.

If the algebra C has even co-index, it is endowed with a hyperbolic involution, and the construction above leads to a hyperbolic invariant $e_3^{\text{hyp}}(\gamma) \in H^3(F)$, which may be considered as an absolute invariant of γ . Unexpectedly, this invariant extends to a more general setting. In [15], Garibaldi, Parimala and Tignol proved there exists a degree 3 invariant $e_3(\gamma) \in H^3(F)$ for all symplectic involution γ on an algebra C of degree divisible by 8, including division algebras. This invariant vanishes on hyperbolic involutions, and it satisfies $e_3^{\gamma_0}(\gamma) = e_3(\gamma) - e_3(\gamma_0)$ for all symplectic involutions γ_0 and γ of C .

Example 4.1. Assume there exists an even degree central simple algebra A over F , endowed with two orthogonal involutions σ_0 and σ , and a quaternion algebra Q with canonical involution $\bar{}$ such that

$$(C, \gamma_0) \simeq (A, \sigma_0) \otimes_F (Q, \bar{}) \text{ and } (C, \gamma) \simeq (A, \sigma) \otimes_F (Q, \bar{}).$$

Pick $s \in \text{Sym}(A, \sigma_0)^\times$ such that $\sigma = \text{Int}(s) \circ \sigma_0$; then $s \otimes 1 \in \text{Sym}(C, \gamma_0)^\times$ and $\gamma = \text{Int}(s \otimes 1) \circ \gamma_0$. Moreover, by [7, Lemma 9(d)], and [24, (7.3)(1)]

$$\text{Nrp}_{\gamma_0}(s \otimes 1) = \text{Nrd}_A(s) = d(\sigma)d(\sigma_0)^{-1} \in F^\times/F^{\times 2}.$$

Applying the projection formula and [19, Cor. 2.6.10], we have $\text{Nrd}_A(s) \cdot [A] = 0$. Therefore, since $[C] = [A] + [Q]$, we get

$$e_3^{\gamma_0}(\gamma) = (d(\sigma)d(\sigma_0)^{-1}) \cdot [Q] \in H^3(F).$$

4.2. Unitary involutions with a symplectic descent. In this section, we consider a central simple algebra B with two F'/F -unitary involutions τ_0 and τ , and we assume they both admit a symplectic descent to the same algebra C over F , that is

$$(15) \quad (B, \tau_0) \simeq (C, \gamma_0) \otimes (F', \iota) \text{ and } (B, \tau) = (C, \gamma) \otimes (F', \iota),$$

for some symplectic involutions γ_0 and γ of C .

Remark 4.2. Condition (15) is a strong condition on the triple (B, τ_0, τ) , which does not hold in general. First of all, the existence of a symplectic descent implies that B has even degree $n = 2m$, and exponent dividing 2. Conversely, if the algebra B satisfies those conditions, it admits a descent up to Brauer equivalence since it has trivial corestriction, but need not have a descent up to isomorphism, see [5, Rem. 4.8] and Example 6.9. In addition, even assuming $B \simeq C \otimes_F F'$ for some central simple algebra C over F , a unitary involution τ of B generally does not have a symplectic descent to C . Assume for instance B is a quaternion algebra Q with trivial corestriction, and pick a particular quaternion descent Q_0 over F . The algebra Q_0 is not unique in general, and a unitary involution τ on Q has a symplectic descent to Q_0 if and only if Q_0 is the discriminant algebra of τ , see [24, (2.22)& p. 129]. By uniqueness of the symplectic involution on a quaternion algebra, our condition holds in degree 2 if and only if the unitary involutions τ_0 and τ are isomorphic.

Under condition (15), the invariants of the pairs (γ_0, γ) and (τ_0, τ) are related as follows :

Proposition 4.3. *Let (B, τ_0, τ) be a degree n central simple algebra with two F'/F -unitary involutions. We assume there exists a central simple algebra over F with two symplectic involutions (C, γ_0, γ) , such that*

$$(B, \tau_0) \simeq (C, \gamma_0) \otimes (F', \iota) \text{ and } (B, \tau) = (C, \gamma) \otimes (F', \iota).$$

Then, $n = 2m$ is even, B has exponent dividing 2, $\mathcal{D}(\tau_0) \sim \mathcal{D}(\tau) \sim C^{\otimes m}$, and

$$e_3^{\tau_0}(\tau) = \begin{cases} e_3^{\gamma_0}(\gamma) \in H^3(F)/N_{F'/F}(F'^{\times}) \cdot [C] & \text{if } m \text{ is even,} \\ 0 \in H^3(F)/F^{\times} \cdot [C] & \text{if } m \text{ is odd.} \end{cases}$$

Moreover, in both cases, we have $f_3^{\tau_0}(\tau) = 0$.

Remark 4.4. If m is even, the invariant $e_3^{\gamma_0}(\gamma)$ has values in $H^3(F)$, and the proposition says its image in the quotient $H^3(F)/N_{F'/F}(F'^{\times}) \cdot [C]$ coincides with $e_3^{\tau_0}(\tau)$.

Proof. Since C admits symplectic involutions, the algebras C and B have even degree $n = 2m$, and exponent dividing 2.

To compute the Brauer class of $\mathcal{D}(\tau_0)$, we may assume the algebra B is split, by the argument given in [24, Proof of (10.36)]. Under this assumption, C is split by F' , hence it is Brauer equivalent to a quaternion algebra $Q = (\delta, a)$ for some $a \in F^{\times}$. It follows that (C, γ_0) decomposes as $(M_m(F), \text{ad}_{\varphi_0}) \otimes (Q, -)$ for some m -dimensional quadratic form φ_0 over F , see [38, §5.1] or [32, Prop. 2.1(2)], and where $-$ denotes the canonical involution of Q . Therefore, by (2), we have

$$(B, \tau_0) \simeq (M_m(F), \text{ad}_{\varphi_0}) \otimes (Q, -) \otimes (F', \iota) \simeq (M_n(F), \text{ad}_{\varphi_0 \otimes \langle\langle a \rangle\rangle}) \otimes (F', \iota).$$

So (1) gives $\mathcal{D}(\tau_0) \sim (\delta, a^m) \sim C^{\otimes m}$, as required. The same argument also applies to the involution τ , and it remains to compare the relative e_3 invariants.

Note first that the invariant $e_3^{\tau_0}$ has values in $N_{\alpha,\beta}^3(F)$, with $\alpha = 0$ if m is even, and $\alpha = [C]$ if m is odd, and $\beta = [B]$. Moreover, by the projection formula, for all $z \in F'^{\times}$, we have $\text{cor}_{F'/F}((z) \cdot \beta) = N_{F'/F}(z) \cdot [C]$, which clearly belongs to $F^{\times} \cdot [C]$. Therefore, $e_3^{\tau_0}$ has values in $H^3(F)/N_{F'/F}(F'^{\times}) \cdot [C]$ if m is even and $H^3(F)/F^{\times} \cdot [C]$ if m is odd.

To conclude the proof, we compare the Rost invariants. Since

$$(B, \tau_0) \simeq (C, \gamma_0) \otimes (F', \iota),$$

there is a natural embedding $i : \text{Sp}(C, \gamma_0) \rightarrow \text{SU}(B, \tau_0)$. When both groups are split, identifying $\text{SU}(M_n(F) \times M_n(F)^{\text{op}}, \varepsilon)$ with $\text{SL}_n(F)$, this map corresponds to the canonical embedding $\text{Sp}_n \rightarrow \text{SL}_n$, which has Rost multiplier 1 by [13, Example 7.11]. Therefore, for all $\xi \in H^1(F, \text{Sp}(C, \gamma_0))$, we have

$$\rho_{\text{SU}(B, \tau_0)}(i^{(1)}(\xi)) = \rho_{\text{Sp}(C, \gamma_0)}(\xi).$$

Consider $s \in \text{Sym}(C, \gamma_0)^{\times}$ such that $\gamma = \text{Int}(s) \circ \gamma_0$. It determines a class $\xi \in H^1(F, \text{Sp}(C, \gamma_0))$ with image in $H^1(F, \text{PGSp}(C, \gamma_0))$ corresponding to the isomorphism class of (C, γ) . Moreover, since $\tau = \text{Int}(s \otimes 1) \circ \tau_0$, the element $i^{(1)}(\xi) \in H^1(F, \text{SU}(B, \tau_0))$ has image in $H^1(F, \text{PGU}(B, \tau_0))$ corresponding to $[B, \tau, \text{id}_{F'}]$. Therefore,

$$e_3^{\tau_0}(\tau) = \rho_{\text{SU}(B, \tau_0)}(i^{(1)}(\xi)) = \rho_{\text{Sp}(C, \gamma_0)}(\xi) \in N_{\alpha,\beta}^3(F).$$

The main result follows by (13) if m is odd and (14) if m is even. Since $e_3^{\gamma_0}(\gamma)$ has values in ${}_2H^3(F)$, we also get $f_3^{\tau_0}(\tau) = 0$. \square

Example 4.5. Assume B is split; then C is Brauer equivalent to a quaternion algebra $Q = (\delta, a)$ for some $a \in F^{\times}$. Applying as above [38, §5.1] or [32, Prop. 2.1(2)], we get

$$(C, \gamma_0) = (M_m(F), \text{ad}_{\varphi_0}) \otimes (Q, -) \text{ and } (C, \gamma) = (M_m(F), \text{ad}_{\varphi}) \otimes (Q, -).$$

Hence $e_3^{\gamma_0}(\gamma) = (d(\varphi)d(\varphi_0)^{-1}) \cdot [Q]$ by Example 4.1. Moreover, since B is split, $N_{F'/F}(F'^{\times}) \cdot [C] = \text{cor}_{F'/F}(F'^{\times} \cdot [B]) = 0$. Therefore, we have

$$e_3^{\tau_0}(\tau) = \begin{cases} (d(\varphi)d(\varphi_0)^{-1}) \cdot [Q] \in H^3(F) & \text{if } m \text{ is even,} \\ 0 \in H^3(F)/F^{\times} \cdot [Q] & \text{if } m \text{ is odd.} \end{cases}$$

Of course, this is compatible with the result given by Lemma 2.4. This can be checked using (2), which shows that the traces of the hermitian forms associated to τ_0 and τ are respectively similar to $\langle 1, -\delta \rangle \langle 1, -a \rangle \varphi_0$ and $\langle 1, -\delta \rangle \langle 1, -a \rangle \varphi$.

4.3. Symplectic quadratic extensions of unitary involutions. The algebra with symplectic involution (C, γ) is called a quadratic extension of the algebra with unitary involution (B, τ) if C contains a γ -stable quadratic étale F -algebra \tilde{F} such that

$$(\tilde{F}, \gamma|_{\tilde{F}}) \simeq (F', \iota) \text{ and } (Z_C(\tilde{F}), \gamma|_{Z_C(\tilde{F})}) \simeq (B, \tau),$$

where $Z_C(\tilde{F})$ denotes the centraliser of \tilde{F} in C . When this holds, C is a central simple F -algebra of degree $2n$, where n is the degree of B , and it satisfies $C_{F'} \sim B$. In particular, this implies B has exponent at most 2.

By the Skolem-Noether theorem, if C contains two distinct subalgebras \tilde{F}_1 and \tilde{F}_2 as above, the corresponding centralisers $Z_C(\tilde{F}_1)$ and $Z_C(\tilde{F}_2)$ are isomorphic. Nevertheless, the restrictions of γ to both centralisers are not isomorphic in general. Therefore, a given (C, γ) might be a quadratic extension of some (B, τ_1) and

(B, τ_2) for non isomorphic unitary involutions τ_1 and τ_2 . For instance, the (hyperbolic) algebra with symplectic involution $(M_4(F), \gamma)$ is a quadratic extension of $(M_2(F), -) \otimes_F (F', \iota)$ and of $((\delta, a), -) \otimes_F (F', \iota)$ by [17, Example 1.13]. In both cases, the algebra is $M_2(F') \simeq (\delta, a) \otimes_F F'$, nevertheless, the involutions are non isomorphic if (δ, a) is non split, since they have distinct discriminant algebras.

As opposed to this, the involution γ is uniquely determined by its restriction τ . More precisely, given a degree n central simple algebra with F'/F -unitary involution (B, τ) such that B has exponent at most 2, and a central simple algebra C of degree $2n$ over F such that $C_{F'} \sim B$, there exists a unique symplectic involution γ of C such that (C, γ) is a quadratic extension of (B, τ) . This is an easy consequence of the Skolem-Noether theorem, see [17, Prop. 1.9]. We call γ the symplectic extension of τ to C .

Proposition 4.6. *Let (B, τ_0, τ) be a degree n algebra with two F'/F unitary involutions. We assume B has exponent dividing 2, and if n is even, we assume τ_0 and τ have isomorphic discriminant algebras. Let C be a degree $2n$ algebra over F such that $C_{F'} \sim B$, and denote by γ_0 , respectively γ , the symplectic extension of τ_0 , respectively τ , to C . Their relative Arason invariant is given by*

$$e_3^{\gamma_0}(\gamma) = f_3^{\tau_0}(\tau) \in H^3(F).$$

Proof. The proof is similar to the proof of Proposition 4.3, with the role of the symplectic and unitary involutions reversed. Since (C, γ_0) is a quadratic symplectic extension of (B, τ_0) , there is a canonical embedding $i : \mathrm{SU}(B, \tau_0) \rightarrow \mathrm{Sp}(C, \gamma_0)$. Pick a cohomology class $\xi \in H^1(F, \mathrm{SU}(B, \tau_0))$ with image in $H^1(F, \mathrm{PGU}(B, \tau_0))$ corresponding to $[B, \tau, \mathrm{id}_{F'}]$. Such a class exists since τ_0 and τ have isomorphic discriminant algebras if n is even. By (5), ξ corresponds to the class of a pair $(s, z) \in \mathrm{SSym}(B, \tau_0)$, where $s \in \mathrm{Sym}(B, \tau_0)^\times$ satisfies $\tau = \mathrm{Int}(s) \circ \tau_0$. Clearly, $\mathrm{Sym}(B, \tau_0) \subset \mathrm{Sym}(C, \gamma_0)$. Therefore, $\mathrm{Int}(s) \circ \gamma_0$ is a symplectic involution on C , acting as τ on B . By uniqueness of the quadratic symplectic extension, we thus have $\gamma = \mathrm{Int}(s) \circ \gamma_0$. Hence, we get

$$f_3^{\tau_0}(\tau) = 2\rho_{\mathrm{SU}(B, \tau_0)}(\xi) \in H^3(F) \text{ and } e_3^{\gamma_0}(\gamma) = \rho_{\mathrm{Sp}(C, \gamma_0)}(i^{(1)}(\xi)) \in H^3(F).$$

Therefore, the following lemma finishes the proof :

Lemma 4.7. *The natural embedding $i : \mathrm{SU}(B, \tau_0) \rightarrow \mathrm{Sp}(C, \gamma_0)$ has Rost multiplier 2, that is for all $\xi \in H^1(F, \mathrm{SU}(B, \tau_0))$, we have $\rho_{\mathrm{Sp}(C, \gamma_0)}(i^{(1)}(\xi)) = 2\rho_{\mathrm{SU}(B, \tau_0)}(\xi)$.*

To compute the Rost multiplier, we use the same strategy as for the examples given in [13, §7]. First of all, by [13, Prop. 7.9(4)], we may assume both groups are split. In other words, viewing matrices in $M_{2n}(F)$ as 2×2 block matrices, we may identify (C, γ_0) with $M_{2n}(F)$ endowed with the involution defined by

$$\gamma_0 \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}, \text{ for all } A, B, C, D \in M_n(F).$$

Therefore, the map

$$\psi : (M_n(F) \times M_n(F)^{\mathrm{op}}, \varepsilon) \rightarrow (M_{2n}(F), \gamma_0), \quad (X, Y^{\mathrm{op}}) \mapsto \begin{pmatrix} X & 0 \\ 0 & Y^t \end{pmatrix}$$

preserves the involutions. Moreover, $\tilde{F} = F[\Delta]$, where $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, satisfies $(\tilde{F}, \gamma_{|\tilde{F}}) \simeq (F \times F, \iota)$, and the centraliser, endowed with the restriction of γ_0 , coincides with the image of ψ . Hence, γ_0 is the symplectic quadratic extension of ε to $M_{2n}(F)$ and the map i we are interested in corresponds in the split case to the restriction of ψ to the group $\mathrm{SU}(M_n(F) \times M_n(F)^{\mathrm{op}}, \varepsilon)$. On the other hand, this group is isomorphic to $\mathrm{SL}_n(F)$ by $X \mapsto (X, (X^{-1})^{\mathrm{op}})$. So we are computing the Rost multiplier of the embedding

$$\mathrm{SL}_n(F) \rightarrow \mathrm{Sp}_{2n}(F), \quad X \mapsto \begin{pmatrix} X & 0 \\ 0 & (X^{-1})^t \end{pmatrix}.$$

By [13, Example 7.11], the standard embedding $\mathrm{Sp}_{2n} \rightarrow \mathrm{SL}_{2n}$ has Rost multiplier 1, therefore, we may replace Sp_{2n} by SL_{2n} by [13, Prop. 7.9(1)]. The result now follows from Lemma 1.1. \square

5. COMPARISON WITH DEGREE 3 INVARIANTS OF ORTHOGONAL INVOLUTIONS

We now prove some formulae relating the degree 3 cohomological invariants of orthogonal descents and quadratic orthogonal extensions of pairs of unitary involutions; see Sections 5.2 and 5.3 for precise statements. First, we extend the definition of the relative Arason invariant for orthogonal involutions to involutions with isomorphic non split Clifford algebras, see Section 5.1 below. The definition of this relative invariant follows the same lines as in the unitary and symplectic cases, but there are some complications if the algebra has even degree, related to the fact that given two involutions with the same discriminant, there are two different ways of identifying the centers of their Clifford algebras. Therefore, an additional condition is required for the relative Arason invariant to be defined, see (18). This condition can also be interpreted in terms of outer automorphisms of groups of type D, following [35], and in terms of similarity factors of hermitian forms, see Remarks 5.2(2) and 5.4.

5.1. Degree 3-invariants for orthogonal involutions. Using the Rost invariant, one may define absolute and relative Arason invariants for some orthogonal involutions, see [38] and [34] for precise definitions and some properties. The purpose of this subsection is to slightly expand this definition, by considering pairs of involutions having isomorphic (and generally non trivial) Clifford algebras. In particular, if the underlying algebra has even degree, this implies that the involutions have the same discriminant. Throughout, we assume σ_0 and σ are two orthogonal involutions of the central simple algebra A .

5.1.1. Odd degree. Assume first that A has odd degree $r = 2\ell + 1$, with $\ell \geq 2$, so that the underlying algebraic groups have type B_ℓ . Then the algebra A is split, and there exists two quadratic forms φ_0 and φ , unique up to a scalar factor, such that $\sigma_0 = \mathrm{ad}_{\varphi_0}$ and $\sigma = \mathrm{ad}_{\varphi}$. We assume in addition that φ_0 and φ have isomorphic even Clifford algebras. Under this condition, we have

$$\varphi_0 - \langle \mathrm{disc}(\varphi_0) \mathrm{disc}(\varphi)^{-1} \rangle \varphi \in I^3(F),$$

and we may define a relative Arason invariant by

$$(16) \quad e_3^{\sigma_0}(\sigma) = e_3(\varphi_0 - \langle \mathrm{disc}(\varphi_0) \mathrm{disc}(\varphi)^{-1} \rangle \varphi) \in {}_2H^3(F).$$

Indeed, the quadratic form $\varphi_0 - \langle \text{disc}(\varphi_0) \text{disc}(\varphi)^{-1} \rangle \varphi$ has dimension $2r$ and trivial discriminant. Moreover, by [25, Chap. V (3.13)&(3.16)], its Clifford invariant is $[\mathcal{C}_0(\varphi_0)] + [\mathcal{C}_0(\varphi)] \in \text{Br}(F)$, which vanishes if and only if $\mathcal{C}_0(\varphi_0)$ and $\mathcal{C}_0(\varphi)$ are isomorphic, since those algebras have the same degree and exponent at most 2. Therefore, we may consider the Arason invariant of this quadratic form

$$e_3(\varphi_0 - \langle \text{disc}(\varphi_0) \text{disc}(\varphi)^{-1} \rangle \varphi) \in H^3(F).$$

The value of this invariant does not depend on the particular choices we made for φ_0 and φ in their respective similarity classes. Indeed, for all $\lambda_0, \lambda \in F^\times$, we have

$$\langle \lambda_0 \rangle \varphi_0 - \langle \text{disc}(\langle \lambda_0 \rangle \varphi_0) \text{disc}(\langle \lambda \rangle \varphi)^{-1} \rangle \langle \lambda \rangle \varphi = \langle \lambda_0 \rangle (\varphi_0 - \langle \text{disc}(\varphi_0) \text{disc}(\varphi)^{-1} \rangle \varphi),$$

and similar forms in $I^3(F)$ have the same Arason invariant.

Remark 5.1. (1) Alternately, one may chose φ_0 and φ of discriminant 1, in which case they are uniquely defined up to isometry, and we have

$$e_3^{\sigma_0}(\sigma) = e_3(\varphi_0 - \varphi) \in H^3(F).$$

(2) In view of the description of the Rost invariant for the group $\text{Spin}(\varphi_0)$ given in [24, p. 437], we also have

$$e_3^{\sigma_0}(\sigma) = \rho_{\text{Spin}(\varphi_0)}(\xi),$$

where $\xi \in H^1(F, \text{Spin}(\varphi_0))$ is any cocycle with image in $H^1(F, \text{O}(\varphi_0))$ corresponding under [24, (29.28)] to the class of a quadratic form φ such that $\sigma \simeq \text{ad}_\varphi$. Rephrasing this in terms of involutions, ξ is any cocycle with image in $H^1(F, \text{O}(A, \sigma_0))$ corresponding under [24, (29.26)] to the class of a symmetric element $s \in \text{Sym}(A, \sigma_0)$ such that $\sigma = \text{Int}(s) \circ \sigma_0$, or equivalently any cocycle with image in $H^1(F, \text{GO}(A, \sigma_0))$ corresponding under the bijection described in [24, p. 405] to the conjugacy class of σ .

5.1.2. *Even degree.* Assume now A has even degree $r = 2\ell$, with $\ell \geq 3$, so that the underlying groups have type D_ℓ . Throughout this subsection, we assume in addition that the Clifford algebras $\mathcal{C}(A, \sigma_0)$ and $\mathcal{C}(A, \sigma)$ are F -isomorphic. In particular, they have isomorphic center $Z_0 \simeq Z$, so that σ_0 and σ have the same discriminant.

Pick an F -isomorphism

$$\Psi : \mathcal{C}(A, \sigma) \rightarrow \mathcal{C}(A, \sigma_0),$$

and denote by $\psi : Z \rightarrow Z_0$ the induced isomorphism of the centers. The triple (A, σ, ψ) corresponds to an element $\eta \in H^1(F, \text{PGO}^+(A, \sigma_0))$ by [24, §29.F]. Let μ_0 and μ denote the respective centers of the groups $\text{Spin}(A, \sigma_0)$ and $\text{Spin}(A, \sigma)$. We have

$$\mu_0 = \begin{cases} \mu_{4[Z_0]} & \text{if } \ell \text{ is odd,} \\ R_{Z_0/F}(\mu_2) & \text{if } \ell \text{ is even,} \end{cases}$$

and similarly for μ . Therefore, the isomorphism ψ induces an isomorphism between μ and μ_0 , hence also an isomorphism $\psi^{(2)} : H^2(F, \mu) \rightarrow H^2(F, \mu_0)$. We may use $\psi^{(2)}$ to compare the Tits classes of the groups $\text{Spin}(A, \sigma)$ and $\text{Spin}(A, \sigma_0)$.

If ℓ is even, so that $H^2(F, \mu_0) \simeq {}_2\text{Br}(Z_0)$, we have $t_{\text{Spin}(A, \sigma_0)} = [\mathcal{C}(A, \sigma_0)]$ by [24, (31.13)]. Assume now ℓ is odd. We may identify $H^2(F, \mu_0)$ with a subgroup of $\text{Br}(F) \times \text{Br}(Z_0)$ by [8, Prop. 2.10], and it follows from [24, (31.11)] that the Tits class of $\text{Spin}(A, \sigma_0)$ corresponds to the pair $([A], [\mathcal{C}(A, \sigma_0)])$. In both cases, the F -linear isomorphism Ψ induces a Z_0 -linear isomorphism between

$$\mathcal{C}(A, \sigma) \otimes_{Z, \psi} Z_0 \rightarrow \mathcal{C}(A, \sigma_0),$$

where the tensor product structure is given by $\psi : Z \rightarrow Z_0$. Therefore, we have

$$\psi^{(2)}([\mathcal{C}(A, \sigma)]) = [\mathcal{C}(A, \sigma) \otimes_{Z, \psi} Z_0] = [\mathcal{C}(A, \sigma_0)] \in \text{Br}(Z_0).$$

Hence, independently of the parity of ℓ , the Tits classes of $\text{Spin}(A, \sigma_0)$ and $\text{Spin}(A, \sigma)$ coincide under the identification $\psi^{(2)}$.

With this in hand, we aim at defining a relative Arason invariant $e_3^{\sigma_0}(\sigma)$ as we did in the unitary case, replacing (7) by

$$1 \rightarrow \mu_0 \rightarrow \text{Spin}(A, \sigma_0) \rightarrow \text{PGO}^+(A, \sigma_0) \rightarrow 1.$$

This sequence induces a connecting map

$$\partial : H^1(F, \text{PGO}^+(A, \sigma_0)) \rightarrow H^2(F, \mu_0).$$

Applying again a twisting argument as in [37, I.5.4] (see also [12, §1]), one may check that $\partial(\eta)$ is the difference between the Tits classes of the groups $\text{Spin}(A, \sigma)$ and $\text{Spin}(A, \sigma_0)$, where the first Tits class, which belongs to $H^2(F, \mu)$ is viewed as an element of $H^2(F, \mu_0)$ via the isomorphism $\psi^{(2)}$. Hence, as we have just checked, we have $\partial(\eta) = 0$, so that η lifts to a class $\xi \in H^1(F, \text{Spin}(A, \sigma_0))$, which is unique up to the action of $H^1(F, \mu_0)$. In view of the behavior of the Rost invariant under twisting as described in [18, Lemme 7] (see also [30, Prop. 1.7]), it follows that the class η determines $\rho_{\text{Spin}(A, \sigma_0)}(\xi) \in H^3(F)$ up to an element which belongs to the image of $H^1(F, \mu_0)$ under $\rho_{\text{Spin}(A, \sigma_0)}$. This image is described in [30, Thm. 1.15], [14], and we have

$$\rho_{\text{Spin}(A, \sigma_0)}(H^1(F, \mu_0)) \subset \begin{cases} F^\times \cdot [A] + \text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)]) & \text{if } \ell \text{ is odd,} \\ \text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)]) & \text{if } \ell \text{ is even.} \end{cases}$$

Note that if ℓ is even, then by [24, (9.14)], we have $\text{cor}_{Z_0/F}([\mathcal{C}(A, \sigma_0)]) = [A]$. Therefore, by the projection formula, for all $\lambda \in F^\times$, we have

$$(\lambda) \cdot [A] = \text{cor}_{Z_0/F}((\lambda) \cdot [\mathcal{C}(A, \sigma_0)]).$$

Hence, independently of the parity of ℓ , we actually have

$$(17) \quad \rho_{\text{Spin}(A, \sigma_0)}(H^1(F, \mu_0)) \subset F^\times \cdot [A] + \text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)]).$$

We still have to check that the value of $\rho_{\text{Spin}(A, \sigma_0)}(\xi)$ in the quotient of $H^3(F)$ by this subgroup depends only on the involution σ of A , and not of the cocycle η , which corresponds to the triple (A, σ, ψ) . An additional condition is required to guarantee this fact. More precisely, assume there exists an F -isomorphism

$$\Psi' : \mathcal{C}(A, \sigma) \rightarrow \mathcal{C}(A, \sigma_0),$$

such that the induced isomorphism on the centers is $\psi \circ \iota : Z \rightarrow Z_0$, where ι denotes here the non-trivial F -automorphism of Z . Composing Ψ' with Ψ^{-1} , we get that the algebra $\mathcal{C}(A, \sigma)$ is isomorphic to its conjugate ${}^t\mathcal{C}(A, \sigma)$. Under this condition, the class $\eta' \in H^1(F, \text{PGO}^+(A, \sigma_0))$ corresponding to the triple $(A, \sigma, \iota \circ \psi)$ also lifts to a $\xi' \in H^1(F, \text{Spin}(A, \sigma_0))$, and it is not known in general how the values of the Rost invariant compare for ξ and ξ' . Hence we have to stay away from this situation. So, from now on, we assume

- (18) Either $\mathcal{C}(A, \sigma)$ and its conjugate ${}^t\mathcal{C}(A, \sigma)$ are not isomorphic,
or (A, σ) admits improper similitudes.

In the first case, the class η' satisfies $\partial(\eta') \neq 0$, so η' does not lift to an element $\xi' \in H^1(F, \text{Spin}(A, \sigma_0))$. In the second case, any improper similitude g for (A, σ)

induces an automorphism $\text{Int}(g)$ of (A, σ) and an automorphism $\mathcal{C}(g)$ of $\mathcal{C}(A, \sigma)$ which acts non trivially on Z by [24, (13.2)]. Therefore, the triples (A, σ, ψ) and $(A, \sigma, \iota \circ \psi)$ are F -isomorphic, and it follows that $\eta = \eta' \in H^1(F, \text{PGO}^+(A, \sigma_0))$. Hence, in both cases, there is a unique $\eta \in H^1(F, \text{PGO}^+(A, \sigma_0))$ which is associated to σ and which lifts to a $\xi \in H^1(F, \text{Spin}(A, \sigma_0))$. So we have a well defined invariant

$$e_3^{\sigma_0}(\sigma) = \rho_{\text{Spin}(A, \sigma_0)}(\xi) \in N_{\alpha, \beta}^3(F),$$

where $\alpha = [A]$, $\beta = [\mathcal{C}(A, \sigma_0)] \in \text{Br}(Z_0)$, and the element $\xi \in H^1(F, \text{Spin}(A, \sigma_0))$ has image in $H^1(F, \text{PGO}^+(A, \sigma_0))$ corresponding to the triple (A, σ, ψ) . We will refer to $e_3^{\sigma_0}$ as the relative Arason invariant with σ_0 as a base point. It has values in

$$N_{\alpha, \beta}^3(F) = H^3(F)/(F^\times \cdot [A] + \text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)])),$$

which coincides with $N_{0, \beta}^3(F) = H^3(F)/\text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)])$ when ℓ is even.

- Remark 5.2.* (1) Using the so-called fundamental relations [24, (9.12)], one may check that $\mathcal{C}(A, \sigma)$ is isomorphic to its conjugate ${}^t\mathcal{C}(A, \sigma)$ if and only if the algebra A is split by the quadratic extension Z corresponding to the discriminant of σ .
- (2) Condition (18) can be rephrased in the language of [35]; it is equivalent to saying that either the group $G = \text{PGO}^+(A, \sigma)$ admits an outer automorphism defined over F , or its Tits class is not fixed under the nontrivial automorphism of its Dynkin diagram. We need to avoid groups for which there is no Tits class obstruction, and yet, there is no outer automorphism defined over F , see loc. cit. Prop. 2.5 and Thm 1.1(2).
- (3) Assume σ_0 and σ have trivial discriminant, so that $Z \simeq F \times F \simeq Z_0$. In this case, either A_Z is non-split, or A is split, in which case it does admit improper similitudes. Therefore, the additional condition (18) is always satisfied when the involutions have trivial discriminant. This explains why it does not occur in previous papers such as [38] and [34].

Example 5.3. Let (D, θ) be a division algebra with orthogonal involution, and consider two hermitian forms h and h_0 on a finite dimensional D -module M with values in (D, θ) . If the hermitian form $h \perp (-h_0)$ has trivial discriminant and trivial Clifford invariant, we may consider the relative Arason invariant

$$e_3(h/h_0) = e_3(h \perp (-h_0)) \in H^3(F)/F^\times \cdot [D],$$

as defined by Bayer and Parimala in [2], see also [34, Def. 2.11]. We claim that if the relative Arason invariant of the corresponding adjoint involutions is well defined, then we have

$$e_3^{\text{ad}_{h_0}}(\text{ad}_h) = e_3(h/h_0) \in N_{\alpha, \beta}^3(F).$$

In other words, $e_3^{\text{ad}_{h_0}}(\text{ad}_h)$ is the image of $e_3(h/h_0)$ under the natural projection

$$H^3(F)/F^\times \cdot [D] \rightarrow H^3(F)/(F^\times \cdot [D] + \text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(\text{End}_D(M), \text{ad}_{h_0})])).$$

This is an easy consequence of the definition of both invariants, see notably [34, Rem. 2.12].

Remark 5.4. Reversing the viewpoint, the previous example shed new light on condition (18). Indeed, consider two hermitian forms h and h_0 on M with values in (D, θ) . Let $A = \text{End}_D(M)$ and assume that the Clifford algebras of ad_h and ad_{h_0}

are F -isomorphic. By [26, Lemma 2], there exists $\lambda \in F^\times$ such that $\langle \lambda \rangle h \perp (-h_0)$ has trivial Clifford invariant, so that $e_3(\langle \lambda \rangle h \perp (-h_0)) \in H^3(F)/F^\times \cdot [D]$ is well defined. To define a relative invariant $e_3^{\text{ad}_{h_0}}(\text{ad}_h)$ from this element, one has to check it does not depend on the choice of the scalar λ . Hence, consider $\mu \in F^\times$ such that $\langle \mu \rangle h \perp (-h_0)$ has trivial Clifford invariant. The difference between $\langle \lambda \rangle h \perp (-h_0)$ and $\langle \mu \rangle h \perp (-h_0)$ also has trivial Clifford invariant, and is Witt equivalent to $\langle \lambda \rangle \langle 1, -\lambda \mu \rangle h$. Therefore, by Tao's formula for the Clifford algebra of a tensor product, we have either $(\lambda \mu, \text{disc}(h)) = 0$ or $(\lambda \mu, \text{disc}(h)) \sim D$. In the first case, there exists $y \in Z^\times$ such that $\lambda \mu = N_{Z/F}(y)$. Therefore, by [34, Prop. 2.6], we get

$$e_3(\langle \lambda \rangle h \perp (-h_0)) = e_3(\langle \mu \rangle h \perp (-h_0)) + \text{cor}_{Z/F}(y \cdot \mathcal{C}(A, \text{ad}_h)) \in H^3(F)/F^\times \cdot [D].$$

Assume now that $(\lambda \mu, \text{disc}(h)) \sim D$. In particular, D_Z is split. Hence, assuming (18) holds, we get that the algebra with involution (A, ad_h) admits improper similitudes. Pick such a similitude and denote by α its multiplier. We have $h \simeq \langle \alpha \rangle h$ and $D \sim (\alpha, \text{disc}(h))$ see [24, (12.20)&(13.38)]. Therefore, $(\lambda \mu \alpha, \text{disc}(h)) = 0$, and there exists $y \in Z^\times$ such that $\lambda \mu = \alpha N_{Z/F}(y)$. Hence, we have

$$\langle 1, -\lambda \mu \rangle h \simeq \langle 1, -\alpha N_{Z/F}(y) \rangle h \simeq \langle 1, -N_{Z/F}(y) \rangle h,$$

and applying again [34, Prop. 2.6], we get

$$e_3(\langle \lambda \rangle h \perp (-h_0)) = e_3(\langle \mu \rangle h \perp (-h_0)) + \text{cor}_{Z/F}(y \cdot \mathcal{C}(A, \text{ad}_h)) \in H^3(F)/F^\times \cdot [D].$$

Hence in both cases, under condition (18), the image of $e_3(\langle \lambda \rangle h \perp (-h_0))$ in $N_{\alpha, \beta}^3(F)$ does not depend on the choice of λ such that $\langle \lambda \rangle h \perp (-h_0)$ has trivial Clifford invariant. This provides as an alternate definition of the relative Arason invariant $e_3^{\text{ad}_{h_0}}(\text{ad}_h)$.

The definition of the relative f_3 invariant given in [34, Def. 2.15] also extends to this broader setting. We proceed as in Section 2.4. From the fundamental relations given in [24, (9.12)], one may easily check that for all

$$y \in F^\times \cdot [A] + \text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)]),$$

we have

$$2y = 0 \text{ if } \ell \text{ is even, and } 2y \in N_{Z_0/F}(Z_0^\times) \cdot [A] \text{ if } \ell \text{ is odd.}$$

Therefore, for any algebra A endowed with two orthogonal involutions σ_0 and σ for which $e_3^{\sigma_0}(\sigma)$ is well-defined, the invariant

$$f_3^{\sigma_0}(\sigma) = 2c \in \begin{cases} H^3(F) & \text{if } \ell \text{ is even,} \\ H^3(F)/N_{Z_0/F}(Z_0^\times) \cdot [A] & \text{if } \ell \text{ is odd,} \end{cases}$$

also is well defined, where $c \in H^3(F)$ is any cohomology class such that

$$e_3^{\sigma_0}(\sigma) = c \pmod{F^\times \cdot [A] + \text{cor}_{Z_0/Z}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)])}.$$

5.2. Orthogonal quadratic extensions of unitary involutions. Orthogonal quadratic extensions of a unitary algebra with involution are defined as symplectic ones, see Section 4.3. Namely, the algebra with orthogonal involution (A, σ) is called a quadratic extension of the algebra with unitary involution (B, τ) if A contains a σ -stable quadratic étale F -algebra \tilde{F} such that

$$(\tilde{F}, \sigma|_{\tilde{F}}) \simeq (F', \iota) \text{ and } (Z_A(\tilde{F}), \sigma|_{Z_A(\tilde{F})}) \simeq (B, \tau),$$

where $Z_A(\tilde{F})$ denotes the centraliser of \tilde{F} in A . When this holds, A is a central simple F -algebra of degree $2n$, where n is the degree of B , and it satisfies $A_{F'} \sim B$. In particular, this implies B has exponent at most 2.

As in the symplectic case, the involution σ is uniquely determined by its restriction τ . More precisely, given a degree n central simple algebra with F'/F -unitary involution (B, τ) such that B has exponent at most 2, and a central simple algebra A of degree $2n$ over F such that $A_{F'} \sim B$, there exists a unique orthogonal involution σ of A such that (A, σ) is a quadratic extension of (B, τ) . This is an easy consequence of the Skolem-Noether theorem, see [17, Prop. 1.9]. We call σ the orthogonal extension of τ to A .

Example 5.5. Assume $F' = F(\sqrt{\delta})$ is a field and B is split, that is $B \simeq \text{End}_{F'}(V)$ for some n -dimensional vector space V over F' . The involution $\tau \simeq \text{ad}_h$ is adjoint to a hermitian form $h : V \times V \rightarrow (F', \iota)$, and we denote by q_h the corresponding trace, defined on the F vector-space V by $q_h(x) = h(x, x)$. The underlying bilinear form satisfies

$$b_{q_h}(x, y) = \frac{1}{2}(h(x, y) + h(y, x)), \text{ for all } x, y \in V.$$

From this, one may easily check that ad_{q_h} is the orthogonal extension of ad_h to $A = \text{End}_F(V)$. Indeed, multiplication by $\sqrt{\delta}$ is an element of $\text{End}_F(V)$ with centraliser $B = \text{End}_K(V)$, and it generates a subfield $\tilde{F} \subset \text{End}_F(V)$ which satisfies the required conditions.

The discriminant and the Clifford algebra of the orthogonal quadratic extension (A, σ) of an algebra with unitary involution (B, τ) may be computed as follows :

Lemma 5.6. *Let (A, σ) be a quadratic orthogonal extension of the algebra with F'/F -unitary involution (B, τ) . Recall $F' = F[X]/(X^2 - \delta)$ and denote by n the degree of B , so that A has degree $2n$. The invariants of the involution σ are given by*

$$d(\sigma) = \begin{cases} \delta \in F^\times/F^{\times 2} & \text{if } n = 2m + 1 \text{ is odd,} \\ 1 \in F^\times/F^{\times 2} & \text{if } n = 2m \text{ is even.} \end{cases}$$

$$\text{and } \mathcal{C}(A, \sigma) = \begin{cases} 0 \in \text{Br}(F') & \text{if } n = 2m + 1 \text{ is odd,} \\ \mathcal{C}_+ \times \mathcal{C}_- \in \text{Br}(F) \times \text{Br}(F) & \text{if } n = 2m \text{ is even,} \end{cases}$$

with either \mathcal{C}_+ or \mathcal{C}_- Brauer-equivalent to the discriminant algebra $\mathcal{D}(\tau)$.

Proof. If $F' \simeq F \times F$, that is $\delta = 1 \in F^\times/F^{\times 2}$, then both τ and σ are hyperbolic, see [17, Example 1.12]. Therefore all invariants are trivial in this case, and the lemma holds. Hence we may assume $F' = F(\sqrt{\delta})$ is a field. Assume in addition that the algebra A is split, so that $B \sim A_{F'}$ also is. The result then follows from Example 5.5 together with (3) and (4). In the general case, consider the function field F_A of the Severi-Brauer variety of A . The result holds after extending scalars to F_A , since A is split over this field. Moreover, F is quadratically closed in F_A , and by Amitsur's theorem, the kernel of the map $\text{Br}(F) \rightarrow \text{Br}(F_A)$ is $\{0, [A]\}$. In view of the fundamental relations [24, (9.12)], this proves the lemma when $n = 2m$ is even. Assume now n is odd. Since $B \sim A_{F'}$, B has exponent 2 and odd degree, hence it is split. Again by Amitsur's theorem, $\text{Br}(F') \rightarrow \text{Br}(F' \otimes_F F_A)$ has trivial kernel, and this concludes the proof. \square

With this in hand, we now prove the following :

Proposition 5.7. *Let (B, τ_0, τ) be a degree n algebra with two F'/F unitary involutions. We assume B has exponent dividing 2, and if n is even, we assume τ_0 and τ have isomorphic discriminant algebras. Let A be a degree $2n$ algebra over F such that $A_{F'} \sim B$, and denote by σ_0 , respectively σ , the orthogonal extension of τ_0 , respectively τ , to the algebra A . If n is odd, we assume in addition that (A, σ) admits improper similitudes. Then both relative Arason invariants are defined and we have*

$$e_3^{\sigma_0}(\sigma) = e_3^{\tau_0}(\tau) \pmod{F^\times \cdot [A]}.$$

Remark 5.8. (1) By Lemma 5.6, the involutions σ_0 and σ have discriminant 1 if n is even and δ if n is odd. It follows that condition (18) is always satisfied if n is even, see Remark 5.2 (3). If n is odd, then $B \sim A_{F'}$ is split, since it has exponent 2 and odd degree. Therefore, condition (18) is satisfied in this case if and only if (A, σ) admits improper similitudes, see Remark 5.2 (1).

(2) If n is odd, then B is split and $e_3^{\tau_0}$ has values in $N_{0,0}^3(F) = H^3(F)$. Moreover, since $\mathcal{C}(A, \sigma) = 0 \in \text{Br}(F')$, $e_3^{\sigma_0}$ has values in $H^3(F)/F^\times \cdot [A]$. The result in this case means that $e_3^{\tau_0}(\tau)$ has image $e_3^{\sigma_0}(\sigma)$ under the natural map

$$H^3(F) \rightarrow H^3(F)/F^\times \cdot [A].$$

Assume now that n is even, so that A has degree divisible by 4. Since $A_{F'} \sim B$, by the projection formula, we have $\text{cor}_{F'/F}(F'^\times \cdot [B]) = N_{F'/F}(F'^\times) \cdot [A]$. Therefore, the invariant $e_3^{\tau_0}$ has values in

$$H^3(F)/(N_{F'/F}(F'^\times) \cdot [A] + F^\times \cdot [\mathcal{D}(\tau_0)]).$$

On the other hand, combining Lemma 5.6 and [24, (9.12)], we have

$$\text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)]) = F^\times \cdot [\mathcal{C}_+] + F^\times \cdot [\mathcal{C}_-] = F^\times \cdot [\mathcal{D}(\tau_0)] + F^\times \cdot [A].$$

Therefore, the invariant $e_3^{\sigma_0}$ has values in

$$H^3(F)/(F^\times \cdot [A] + F^\times \cdot [\mathcal{D}(\tau_0)]).$$

The result in this case means that $e_3^{\tau_0}(\tau)$ has image $e_3^{\sigma_0}(\sigma)$ under the natural map

$$H^3(F)/(N_{F'/F}(F'^\times) \cdot [A] + F^\times \cdot [\mathcal{D}(\tau_0)]) \rightarrow H^3(F)/(F^\times \cdot [A] + F^\times \cdot [\mathcal{D}(\tau_0)]).$$

Proof of Proposition 5.7. The proof is similar to the proof of Proposition 4.6. Since (A, σ_0) is a quadratic orthogonal extension of (B, τ_0) , there is a canonical embedding

$$i : SU(B, \tau_0) \rightarrow O^+(A, \sigma_0),$$

which factors through a homomorphism $i' : SU(B, \tau_0) \rightarrow \text{Spin}(A, \sigma_0)$. Pick a cohomology class $\xi \in H^1(F, SU(B, \tau_0))$ with image in $H^1(F, \text{PGU}(B, \tau_0))$ corresponding to $[B, \tau, \text{id}_{F'}]$. Such a class exists since τ_0 and τ have isomorphic discriminant algebras if n is even. By (5), ξ corresponds to the class of a pair $(s, z) \in \text{SSym}(B, \tau_0)$, where $s \in \text{Sym}(B, \tau_0)^\times$ satisfies $\tau = \text{Int}(s) \circ \tau_0$. Clearly, $\text{Sym}(B, \tau_0) \subset \text{Sym}(A, \sigma_0)$. Therefore, $\text{Int}(s) \circ \sigma_0$ is an orthogonal involution on A , acting as τ on B . By uniqueness of the orthogonal extension of τ to A , we thus have $\sigma = \text{Int}(s) \circ \sigma_0$. Hence, it follows from the definition of the relative Arason invariants in the unitary and orthogonal cases that

$$e_3^{\tau_0}(\tau) = \rho_{\text{SU}(B, \tau_0)}(\xi) \text{ and } e_3^{\sigma_0}(\sigma) = \rho_{\text{Spin}(A, \sigma_0)}(i'^{(1)}(\xi)),$$

where each equality holds in the relevant quotient of $H^3(F)$, see Remark 5.8(2). The following lemma finishes the proof :

Lemma 5.9. *The homomorphism $i' : \mathrm{SU}(B, \tau_0) \rightarrow \mathrm{Spin}(A, \sigma_0)$ has Rost multiplier $n_{i'} = 1$, that is for all $\xi \in H^1(F, \mathrm{SU}(B, \tau_0))$, we have*

$$\rho_{\mathrm{Spin}(A, \sigma_0)}(i'^{(1)}(\xi)) = \rho_{\mathrm{SU}(B, \tau_0)}(\xi).$$

By [13, Prop. 7.9(4) and Example 7.15], the composition

$$\mathrm{Spin}(A, \sigma_0) \rightarrow \mathrm{O}^+(A, \sigma_0) \rightarrow \mathrm{SL}_1(A).$$

has Rost multiplier 2. Therefore, composing further on the left with i' , we get a map

$$j : \mathrm{SU}(B, \tau_0) \rightarrow \mathrm{SL}_1(A)$$

with Rost multiplier $n_j = 2n_{i'}$ by [13, Prop. 7.9(1)]. On the other hand, we may also compute the Rost multiplier of j directly, assuming both groups are split. Under this assumption, the involutions are hyperbolic, and we may identify (A, σ_0) with $M_{2n}(F)$ endowed with the involution defined by

$$\sigma_0 \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix}, \text{ for all } A, B, C, D \in M_n(F).$$

Therefore, as in the symplectic case, the map

$$\psi : (M_n(F) \times M_n(F)^{\mathrm{op}}, \varepsilon) \rightarrow (M_{2n}(F), \sigma_0), \quad (X, Y^{\mathrm{op}}) \mapsto \begin{pmatrix} X & 0 \\ 0 & Y^t \end{pmatrix}$$

preserves the involutions. Moreover, $\tilde{F} = F[\Delta]$, where $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, satisfies $(\tilde{F}, \sigma_0|_{\tilde{F}}) \simeq (F \times F, \iota)$, and the centraliser, endowed with the restriction of σ_0 , coincides with the image of ψ . Hence, σ_0 is the orthogonal extension of ε to $M_{2n}(F)$. Identifying $\mathrm{SU}(M_n(F) \times M_n(F)^{\mathrm{op}}, \varepsilon)$ with $\mathrm{SL}_n(F)$ by $X \mapsto (X, (X^{-1})^{\mathrm{op}})$, we get that j corresponds in the split case to the map

$$\mathrm{SL}_n(F) \rightarrow \mathrm{SL}_{2n}(F), \quad X \mapsto \begin{pmatrix} X & 0 \\ 0 & (X^{-1})^t \end{pmatrix}.$$

By Lemma 1.1, we have $n_j = 2n_{i'} = 2$ and this concludes the proof. \square

Remark 5.10. If n is odd, Proposition 5.7 has a much easier proof. Under this assumption, the algebra B is split, and A is similar to a quaternion algebra Q . By [31, Prop. 2.2], $H^3(F)/F^\times \cdot [Q]$ maps injectively to $H^3(F_Q)$, where F_Q is the function field of the Severi-Brauer variety of Q . Therefore, since $e_3^{\sigma_0}$ has values in $H^3(F)/F^\times \cdot [Q]$ and F_Q is a generic splitting field for Q , it is enough to check the result when Q is split, in which case it is a direct consequence of Example 5.5 and Lemma 2.4.

5.3. Unitary involutions with an orthogonal descent. In this section, we consider a central simple algebra B with two F'/F -unitary involutions τ_0 and τ , and we assume they both admit an orthogonal descent to the same algebra A over F , that is

$$(19) \quad (B, \tau_0) \simeq (A, \sigma_0) \otimes_F (F', \iota) \text{ and } (B, \tau) \simeq (A, \sigma) \otimes_F (F', \iota),$$

for some orthogonal involutions σ and σ_0 of A . As in the symplectic case, this does not hold in general, see Remark 4.2. When it is satisfied, we aim at comparing the invariants of the pairs (τ_0, τ) and (σ_0, σ) of unitary and orthogonal involutions.

We first prove the following:

Proposition 5.11. *Let τ_0 and τ be two unitary involutions of the degree n split algebra $B = \text{End}_{F'}(V) \simeq \text{End}_F(V) \otimes_F F'$. If n is even, we assume in addition that the discriminant algebras $\mathcal{D}(\tau_0)$ and $\mathcal{D}(\tau)$ are isomorphic. Then, there exists two quadratic forms q_0 and q over F , having the same discriminant $d \in F^\times/F^{\times 2}$, and such that $\tau_0 \simeq \text{ad}_{q_0} \otimes \iota$ and $\tau \simeq \text{ad}_q \otimes \iota$. Moreover, the relative Arason invariant of the unitary involutions τ_0 and τ is given by*

$$e_3^{\tau_0}(\tau) = (\delta) \cdot [\mathcal{C}(q - q_0)] \in \begin{cases} H^3(F) & \text{if } n \text{ is odd,} \\ H^3(F)/F^\times \cdot (\delta, d) & \text{if } n \text{ is even,} \end{cases}$$

where $[\mathcal{C}(q - q_0)] \in \text{Br}(F)$ denotes the Brauer class of the full Clifford algebra of $q - q_0$.

Proof. Let h_0 be a hermitian form defined on V and with values in (F', ι) such that $\tau_0 \simeq \text{ad}_{h_0}$. Pick a diagonalisation $h_0 \simeq \langle a_1, \dots, a_n \rangle$. Then the elements $a_i \in F'$ are fixed under ι , hence they belong to F . So, one may consider the n dimensional quadratic form $q_0 \simeq \langle a_1, \dots, a_n \rangle$, and a direct computation shows $\tau_0 \simeq \text{ad}_{q_0} \otimes \iota$. Define similarly a quadratic forms q such that $\tau \simeq \text{ad}_q \otimes \iota$. If n is odd, replacing q by $\langle d(q_0)d(q)^{-1} \rangle q$, we may assume that q and q_0 have the same discriminant, denoted by d . Assume now that n is even. Since $\mathcal{D}(\tau_0)$ and $\mathcal{D}(\tau)$ are isomorphic, (1) shows there exists $z \in F'^\times$ such that $d(q_0) = d(q)N_{F'/F}(z)$. The one dimensional hermitian forms $\langle a_1 \rangle$ and $\langle \iota(z)a_1z \rangle \simeq \langle N_{F'/F}(z)a_1 \rangle$ are isomorphic, so we may replace q_0 by $\langle a_1N_{F'/F}(z), a_2, \dots, a_n \rangle$ and we get $d(q) = d(q_0) \in F^\times/F^{\times 2}$ as required. Hence in both cases, we have $d(q_0) = d(q)$, so that $q - q_0$ has trivial discriminant. We may now compute $e_3^{\tau_0}(\tau)$ using Lemma 2.4. Since the trace forms of h_0 and h respectively are $q_{h_0} = \langle \langle \delta \rangle \rangle \otimes q_0$ and $q_h = \langle \langle \delta \rangle \rangle \otimes q$, we get

$$e_3^{\tau_0}(\tau) = e_3(\langle \langle \delta \rangle \rangle \otimes (q - q_0)) = (\delta) \cdot e_2(q - q_0) = (\delta) \cdot [\mathcal{C}(q - q_0)],$$

and this concludes the proof. \square

Remark 5.12. If n is odd, and if τ_0 admits an orthogonal descent, then the algebra B has exponent dividing 2, hence it is split. This case is covered by the previous proposition. Therefore, we will assume in the following that n is even.

If the algebra B is non split, we assume a stronger condition on the orthogonal descent, namely that their relative Arason invariant is well defined; we get :

Proposition 5.13. *Let $n = 2m$ be an even integer, and let B be a degree n algebra endowed with two unitary F'/F involutions τ_0 and τ . We assume τ_0 and τ admit orthogonal descents to the same algebra, that is*

$$B \simeq A \otimes_F F', \quad \tau_0 \simeq \sigma_0 \otimes \iota \text{ and } \tau \simeq \sigma \otimes \iota,$$

for some central simple algebra A over F , and some orthogonal involutions σ_0 and σ of A . We assume in addition that the Clifford algebras $\mathcal{C}(A, \sigma_0)$ and $\mathcal{C}(A, \sigma)$ are F -isomorphic, and (A, σ) satisfies condition (18), so that $e_3^{\sigma_0}(\sigma)$ and $f_3^{\sigma_0}(\sigma)$ are well defined. Then, the involutions τ_0 and τ have isomorphic discriminant algebras, and their relative Arason invariant is given by

$$e_3^{\tau_0}(\tau) = f_3^{\sigma_0}(\sigma) \in H^3(F)/N_{F'/F}(F'^\times) \cdot [A] + F^\times \cdot [\mathcal{D}(\tau_0)].$$

Remark 5.14. (1) In the situation of the proposition above, the computation of the discriminant algebra recalled in (1) shows that $e_3^{\tau_0}$ has values in

$$\begin{cases} H^3(F)/N_{F'/F}(F'^\times) \cdot [A] + F^\times \cdot (\delta, d(\sigma_0)) & \text{if } m \text{ is even, and} \\ H^3(F)/F^\times \cdot [A] + F^\times \cdot (\delta, d(\sigma_0)) & \text{if } m \text{ is odd.} \end{cases}$$

On the other hand, $f_3^{\sigma_0}(\sigma)$ belongs to the group $H^3(F)$ if m is even, and the quotient $H^3(F)/N_{Z_0/F}(Z_0^\times) \cdot [A]$ if m is odd, see the end of Section 5.1. The statement of the proposition means that $e_3^{\tau_0}(\tau)$ is the image of $f_3^{\sigma_0}(\sigma)$ under the relevant projection.

(2) Note that Propositions 5.11 and 5.13 are compatible when they both apply. Indeed, assume $n = 2m$ even, B is split, $\tau_0 \simeq \text{ad}_{q_0} \otimes \iota$ and $\tau \simeq \text{ad}_q \otimes \iota$, where q_0 and q have the same discriminant d and isomorphic even Clifford algebras $\mathcal{C}_0(q_0)$ and $\mathcal{C}_0(q)$. By the structure theorem for Clifford algebras [25, Thm. 2.5], $\mathcal{C}_0(q_0)$ is the centralizer in $\mathcal{C}(q_0)$ of the quadratic extension $Z = F[X]/(X^2 - d)$. Therefore, by [36, Chap. 8, Thm. 5.4], $\mathcal{C}(q_{0Z})$ is Brauer equivalent to $\mathcal{C}_0(q_0)$. The same argument also applies to q . Hence, $\mathcal{C}(q_{0Z})$ and $\mathcal{C}(q_Z)$ are isomorphic. So there exists $a \in F^\times$ such that $[\mathcal{C}(q - q_0)] = (d, a)$, and we get $(\delta) \cdot [\mathcal{C}(q - q_0)] = (\delta, d, a) \in H^3(F)$, and $e_3^{\tau_0}(\tau) = 0 \pmod{F^\times \cdot (\delta, d)}$. On the other hand, since the Arason invariant for quadratic forms takes values in ${}_2H^3(F)$, we also have $f_3^{\sigma_0}(\sigma) = 0$ in this situation.

Proof of Proposition 5.13. By assumption, the algebras $\mathcal{C}(A, \sigma_0)$ and $\mathcal{C}(A, \sigma)$ are F -isomorphic, so they have isomorphic centers. Therefore, σ_0 and σ have the same discriminant, and it follows that $\mathcal{D}(\tau_0)$ and $\mathcal{D}(\tau)$ are isomorphic. Hence $e_3^{\tau_0}(\tau)$ is well defined.

Moreover, since $(B, \tau_0) \simeq (A, \sigma_0) \otimes_F (F', \iota)$, there is a natural inclusion

$$\text{O}^+(A, \sigma_0) \subset \text{SU}(B, \tau_0),$$

which induces a map $j : \text{Spin}(A, \sigma_0) \rightarrow \text{SU}(B, \tau_0)$. When both groups are split, identifying $\text{SU}(M_n(F) \times M_n(F)^{\text{op}}, \varepsilon)$ with $\text{SL}_n(F)$, this map corresponds to the canonical composition

$$\text{Spin}_n \rightarrow \text{O}_n^+ \rightarrow \text{SL}_n.$$

Since $n = 2m$ is even, it has Rost multiplier 2 by [13, Example 7.15], hence for all $\xi \in H^1(F, \text{Spin}(A, \sigma_0))$, we have

$$\rho_{\text{SU}(B, \tau_0)}(j^{(1)}(\xi)) = 2\rho_{\text{Spin}(A, \sigma_0)}(\xi) \in H^3(F).$$

Let $\xi \in H^1(F, \text{Spin}(A, \sigma_0))$ be a cohomology class with image in $H^1(F, \text{PGO}^+(A, \sigma_0))$ corresponding to the triple (A, σ, ψ) , so that

$$e_3^{\sigma_0}(\sigma) = \rho_{\text{Spin}(A, \sigma_0)}(\xi) \in H^3(F)/(F^\times \cdot [A] + \text{cor}_{Z_0/F}(Z_0^\times \cdot [\mathcal{C}(A, \sigma_0)])),$$

and

$$f_3^{\sigma_0}(\sigma) = 2\rho_{\text{Spin}(A, \sigma_0)}(\xi) \in \begin{cases} H^3(F) & \text{if } m \text{ is even,} \\ H^3(F)/N_{Z_0/F}(Z_0^\times) \cdot [A] & \text{if } m \text{ is odd.} \end{cases}$$

The image of ξ in $H^1(F, \text{O}^+(A, \sigma_0))$ corresponds under the canonical bijection described in [24, (29.27)] to the class of a pair (s, z) , where $s \in \text{Sym}(A, \sigma_0)$ satisfies $\sigma = \text{Int}(s) \circ \sigma_0$. We thus have $\tau = \text{Int}(s \otimes 1) \circ \tau_0$, and it follows that $j^{(1)}(\xi) \in H^1(F, \text{SU}(B, \tau_0))$ has image in $H^1(F, \text{PGU}(B, \tau_0))$ corresponding to $(B, \tau, \text{id}_{F'})$. Therefore, $e_3^{\tau_0}(\tau)$ coincides with $\rho_{\text{SU}(B, \tau_0)}(j^{(1)}(\xi))$ in the relevant quotient of $H^3(F)$, and this concludes the proof. \square

6. SPECIFIC RESULTS IN SMALL DEGREE

In this section, we explore the relative Arason invariant for unitary involutions on small degree algebras. In particular, we prove it is classifying in degree 4 under some additional assumption. In degree 4 and 6, the hyperbolic Arason invariant characterizes hyperbolic involutions. This is not true anymore if the underlying

algebra has degree 8 and index at most 4. In this situation, we define a new absolute Arason invariant, taking a totally decomposable involution as a base point instead of the hyperbolic one, and we compare it with the hyperbolic Arason invariant. It is not known whether or not this invariant detects totally decomposable involutions, but partial results in this direction are established.

6.1. Unitary involutions on degree 4 algebras. Throughout this section, we let B be a degree 4 central simple algebra over F' endowed with two F'/F -unitary involutions τ_0 and τ . We assume in addition that τ_0 and τ have isomorphic discriminant algebra, denoted by A . Hence, the relative Arason invariant $e_3^{\tau_0}(\tau)$ is well defined, and it belongs to $N_{\alpha,\beta}^3(F)$, where $\alpha = [A] \in \text{Br}(F)$ and $\beta = [B] \in \text{Br}(F')$. The main result in this subsection is the following :

Theorem 6.1. *Let (B, τ_0, τ) be a degree 4 algebra with two F'/F -unitary involutions with isomorphic discriminant algebras.*

- (a) *If $e_3^{\tau_0}(\tau) = 0 \in N_{\alpha,\beta}^3(F)$ then (B, τ) and (B, τ_0) are F -isomorphic, that is (B, τ) is F' -isomorphic either to (B, τ_0) or to its conjugate $({}^t B, {}^t \tau_0)$.*
- (b) *Assume in addition that either B and ${}^t B$ are not isomorphic, or (B, τ_0) and $({}^t B, {}^t \tau_0)$ are F' -isomorphic. Then $e_3^{\tau_0}(\tau) = 0 \in N_{\alpha,\beta}^3(F)$ if and only if τ and τ_0 are isomorphic.*

Remark 6.2. If B is split and has degree 4, Merkurjev introduced a degree 3 invariant denoted by $e_3(B, \tau)$ in [29, §4]. It can be interpreted as a relative invariant as follows. Since B is split, $\tau = \text{ad}_h$ for some rank 4 hermitian form h with discriminant $d(h) \in F^\times / N_{F'/F}(F'^\times)$. Consider the isotropic hermitian form $h_0 = \langle 1, -d(h) \rangle + \mathbb{H}$. Both involutions τ_0 and τ have discriminant algebra Brauer equivalent to the quaternion algebra $(\delta, d(h))$ and we have $e_3(B, \tau) = e_3^{\text{ad}_{h_0}}(\text{ad}_h)$. In particular, it follows from our theorem that this invariant vanishes if and only if τ is isotropic, which is Lemma 4.1 in Merkurjev's paper.

Proof of Theorem 6.1. Let (B, τ_0, τ) be as in the statement, and assume the relative Arason invariant $e_3^{\tau_0}(\tau) = 0 \in N_{\alpha,\beta}^3(F)$. As explained in §2.2, $e_3^{\tau_0}(\tau)$ is the class in $N_{\alpha,\beta}^3(F)$ of $\rho_{\text{SU}(B, \tau_0)}(\xi)$, where $\xi \in H^1(F, \text{SU}(B, \tau_0))$ has image $\eta \in H^1(F, \text{PGU}(B, \tau_0))$ corresponding to the class of the triple $[B, \tau, \text{id}_{F'}]$. Hence, under our assumption, there exists $\mu \in F'^\times$ such that

$$(20) \quad \rho_{\text{SU}(B, \tau_0)}(\xi) = \text{cor}_{F'/F}((\mu) \cdot [B]) \quad \text{mod } F^\times \cdot [A].$$

In order to prove assertion (a), we use the equivalence of groupoids $\mathbf{A}_3 \equiv \mathbf{D}_3$ established in [24, (15.D)], and we interpret the element of $H^3(F)/F^\times \cdot [A]$ above as the Arason invariant of some hermitian form of orthogonal type.

Specifically, denote by σ_0 the canonical involution of the discriminant algebra A of τ_0 . Thus, A is a degree 6 central simple algebra over F , σ_0 is an orthogonal involution of A , and by [24, (15.D)], the Clifford algebra of (A, σ_0) is F -isomorphic to (B, τ_0) . Similarly, the canonical involution of the discriminant algebra of τ is an orthogonal involution σ of A , and the Clifford algebra of (A, σ) is F -isomorphic to (B, τ) . The algebra A is Brauer equivalent to a division algebra D , which is either F or an F -quaternion algebra. Pick an orthogonal involution θ of D and a hermitian module (M, h_0) over (D, θ) such that $(A, \sigma_0) \simeq (\text{End}_D(M), \text{ad}_{h_0})$.

By [24, (15.26)], we have $\mathrm{PGU}(B, \tau_0) \simeq \mathrm{PGO}^+(A, \sigma_0) \simeq \mathrm{PGO}^+(h_0)$, hence we may view ξ and η as elements of $H^1(F, \mathrm{Spin}(h_0))$ and $H^1(F, \mathrm{PGO}^+(h_0))$, respectively. The morphism $\mathrm{Spin}(h_0) \rightarrow \mathrm{PGO}^+(h_0)$ factors through $\mathrm{O}^+(h_0)$. Let $\zeta \in H^1(F, \mathrm{O}^+(h_0))$ be the image of ξ under the induced map. The cocycle ζ corresponds to a hermitian form h over M , with image η in $H^1(F, \mathrm{PGO}^+(h_0))$, so that $\mathrm{ad}_h \simeq \sigma$. Moreover, since ζ is the image of $\xi \in H^1(F, \mathrm{Spin}(h_0))$, we may consider the relative Arason invariant

$$e_3(h/h_0) = \rho_{\mathrm{Spin}(A, \sigma_0)}(\xi) \pmod{F^\times \cdot [A]},$$

which coincides with $e_3(h \perp (-h_0))$, see Example 5.3. Hence (20) gives

$$e_3(h \perp (-h_0)) = \mathrm{cor}_{F'/F}((\mu) \cdot [B]) \pmod{F^\times \cdot [A]}.$$

On the other hand, the Clifford algebra of $(\mathrm{ad}_{h_0})_{F'}$ is $B \otimes_F F' \simeq B \times B$. By [34, Prop. 2.6], it follows that

$$e_3(\langle 1, -N_{F'/F}(\mu) \rangle h_0) = \mathrm{cor}_{F'/F}((\mu) \cdot [B]) \pmod{F^\times \cdot [A]}.$$

Hence, adding these two equalities, and using [34, Lemma 2.2], we get

$$e_3(h \perp \langle -N_{F'/F}(\mu) \rangle h_0) = 2 \mathrm{cor}_{F'/F}((\mu) \cdot [B]) \pmod{F^\times \cdot [A]}.$$

By the fundamental relations given in [24, (9.12)], we have $2[B] = [A_{F'}]$, hence

$$2 \mathrm{cor}_{F'/F}((\mu) \cdot [B]) = \mathrm{cor}_{F'/F}((\mu) \cdot [A_{F'}]) = N_{F'/F}(\mu) \cdot [A],$$

and this proves $e_3(h \perp \langle -N_{F'/F}(\mu) \rangle h_0) = 0 \in H^3(F)/F^\times \cdot [A]$. Hence the corresponding adjoint involution of $\mathrm{End}_D(M \oplus M)$ also has trivial Arason invariant. The algebra $\mathrm{End}_D(M \oplus M)$ has degree 12 and index at most 2. So we may apply [34, Thm. 4.1], and we get that the involution $\mathrm{ad}_{h \perp \langle -N_{F'/F}(\mu) \rangle h_0}$ is hyperbolic, and so is the hermitian form. Consequently, the forms h_0 and h are similar, so σ_0 and σ are isomorphic, and their Clifford algebras (B, τ_0) and (B, τ) are F -isomorphic. This proves (a), and assertion (b) follows immediately. \square

Remark 6.3. (1) As observed by Garibaldi-Petersson in [16], the additional condition in (b) is satisfied if and only if the group $\mathrm{SU}(B, \tau)$ admits an outer automorphism defined over F as soon as there is no Tits class obstruction, see also [35, Prop. 2.1]. By Remark 5.2(2), this holds if and only if (A, σ_0) satisfies (18). Specifically, since A has degree 6, the canonical homomorphism

$$c : \mathrm{PGO}(A, \sigma_0) \rightarrow \mathrm{Aut}_F(B, \tau_0)$$

is an isomorphism, see [24, (15.1)]. Moreover, by [24, (13.2)], improper similitudes of (A, σ_0) correspond to automorphisms of (B, τ_0) that act as ι on F' . Hence, (A, σ_0) admits improper similitudes if and only if (B, τ_0) and $({}^t B, {}^t \tau_0)$ are F' -isomorphic.

(2) When (18) is satisfied, the Arason invariant $e_3^{\sigma_0}(\sigma)$ is well defined. Moreover, identifying the groups $\mathrm{PGU}(B, \tau_0)$ and $\mathrm{PGO}^+(A, \sigma_0)$, one may check that the cocycle η corresponding to the class of $[B, \tau, \mathrm{id}_{F'}]$ also corresponds to $[A, \sigma, \varphi]$ for some isomorphism φ between the center of the Clifford algebras of (A, σ_0) and (A, σ) , which are both isomorphic to F' . It follows that

$$e_3^{\tau_0}(\tau) = e_3^{\sigma_0}(\sigma) \in N_{\alpha, \beta}^3(F).$$

Therefore, the argument in the proof of Theorem 6.1 also shows that if A is a degree 6 algebra with two orthogonal involutions σ_0 and σ with F -isomorphic

Clifford algebras and such that (18) holds, then $e_3^{\sigma_0}(\sigma) = 0 \in N_{\alpha,\beta}^3(F)$ if and only if σ and σ_0 are isomorphic.

In particular, the theorem shows that e_3^{hyp} detects hyperbolic involutions :

Corollary 6.4. *Assume B has degree 4 and index at most 2, and τ is an F'/F -unitary involution. Then $e_3^{\text{hyp}}(\tau)$ vanishes if and only if τ is hyperbolic.*

Proof. Let τ_0 be the hyperbolic involution of B ; then ${}^t\tau_0$ also is hyperbolic. Hence the additional condition in Theorem 6.1(b) is automatically satisfied and this proves the corollary. \square

6.2. Unitary involutions on degree 8 and index 4 algebras. The main tool in this section is the cohomological invariant introduced in [5]. Let D be a biquaternion algebra over F' with trivial F'/F -corestriction. The algebra D always has a descent up to Brauer equivalence, but it generally does not have a descent up to isomorphism, see [5, Rem. 4.8] and Example 6.9. Specifically, the Brauer class of D always comes from a Brauer class of F , but the underlying division algebra might be of degree 8. In [5, Def. 4.2, Prop. 4.4], a cohomological invariant

$$\delta_{F'/F}(D) \in H^3(F, \mu_2) / \text{cor}_{F'/F}(F'^{\times} \cdot [D])$$

is associated to D , and it is proved that $\delta_{F'/F}(D) = 0$ if and only if $D = D_0 \otimes_F F'$ for some biquaternion algebra D_0 defined over F . In other words, the invariant $\delta_{F'/F}(D)$ detects whether or not D has a descent to F up to isomorphism. Since $H^3(F, \mu_2)$ embeds canonically into $H^3(F)$, we may as well consider $\delta_{F'/F}(D)$ as an element of $H^3(F) / \text{cor}_{F'/F}(F'^{\times} \cdot [D]) = N_{0,\beta}(F)$, with $\beta = [D] \in \text{Br}(F')$.

Throughout this section, (B, τ) denotes a central simple algebra over F' of degree 8 and index at most 4, endowed with an F'/F -unitary involution. Hence, $B \simeq M_2(D)$ for some biquaternion algebra D over F' . Moreover, B and D have trivial corestriction, so that $\delta_{F'/F}(D)$ is well defined. We first prove the following :

Proposition 6.5. *Let (B, τ) be an algebra with F'/F -unitary involution. We assume $B = M_2(D)$ for some biquaternion algebra D . If (B, τ) is totally decomposable, then*

$$\mathcal{D}(\tau) \sim 0 \text{ and } e_3^{\text{hyp}}(\tau) = \delta_{F'/F}(D) \in H^3(F) / \text{cor}_{F'/F}(F'^{\times} \cdot [B]).$$

Proof. Assume (B, τ) satisfies the conditions of the Proposition. In particular, it is totally decomposable, that is there exist quaternion algebras with unitary involutions (Q_i, τ_i) over F' such that

$$(B, \tau) \simeq \otimes_{i=1}^3 (Q_i, \tau_i).$$

By [24, (2.22)], each (Q_i, τ_i) decomposes as $(H_i, \gamma_i) \otimes (F', \iota)$, where H_i is a quaternion algebra over F , and γ_i its canonical involution. Hence (B, τ) admits the algebra with symplectic involution

$$(C, \gamma) \simeq (H_1, \gamma_1) \otimes (H_2, \gamma_2) \otimes (H_3, \gamma_3)$$

as a symplectic descent. In particular, its discriminant algebra is split by Proposition 4.3. Moreover, we may use the same proposition to compute $e_3^{\text{hyp}}(\tau)$. Indeed, we have :

Lemma 6.6. *The hyperbolic unitary involution τ_0 of B admits a symplectic descent to C .*

This can be checked as follows. If C is not division, it has a symplectic hyperbolic involution γ_0 and by uniqueness of the hyperbolic involution of a given type on a given algebra, we have $\tau_0 \simeq \gamma_0 \otimes \iota$. Assume now C is division. Since $C \otimes_F F' \simeq B \simeq M_2(D)$ is not division, C contains a quadratic subfield \tilde{F} which is isomorphic to F' , and the centralizer of \tilde{F} in C is isomorphic to D . In particular, it has degree 4. Applying [24, (4.14)] to the subalgebra $(\tilde{F}, \text{id}_{\tilde{F}})$ of (C, γ) , we get a symplectic involution γ_0 of C which acts as $\text{id}_{\tilde{F}}$ on \tilde{F} . Therefore there exists $u \in C$ such that $u^2 = \delta$ and $\gamma_0(u) = u$. Consider the F'/F -unitary involution $\gamma_0 \otimes \iota$ on B and the idempotent $e = \frac{1}{2}(1 + \delta^{-2}u \otimes \sqrt{\delta}) \in B$. A direct computation shows that $(\gamma_0 \otimes \iota)(e) = 1 - e$. Hence $\gamma_0 \otimes \iota$ is hyperbolic, and so $\tau_0 \simeq \gamma_0 \otimes \iota$.

With this in hand, Proposition 4.3 applies to (B, τ_0, τ) and we get :

$$e_3^{\text{hyp}}(\tau) = e_3^{\tau_0}(\tau) = e_3^{\tau_0}(\gamma) = e_3(\gamma_0) - e_3(\gamma) \in H^3(F)/\text{cor}_{F'/F}(F'^{\times} \cdot [B]).$$

Moreover, since (C, γ) is totally decomposable, $e_3(\gamma) = 0$ by [15, Thm. B], and it remains to prove that $e_3(\gamma_0) = \delta_{F'/F}(D) \in H^3(F)/\text{cor}_{F'/F}(F'^{\times} \cdot [B])$. If C is not division, then D has a descent up to isomorphism and γ_0 is hyperbolic so $\delta(D) = 0$ and $e_3(\gamma_0) = 0$. If C is division, the definition of γ_0 above shows it contains a subfield \tilde{F} isomorphic to F' over which γ_0 acts as identity. The result follows in this case from [5, Rem. 4.3], and this finishes the proof. \square

Example 6.7. When D has a descent up to isomorphism, all totally decomposable involutions have trivial e_3^{hyp} invariant, even the non hyperbolic ones. Specifically, for any quaternion algebras H_1 and H_2 over F , we have

$$\delta_{F'/F}((H_1 \otimes_F H_2) \otimes_F F') = 0.$$

Hence, given F -linear involutions ρ_i of H_i for $i = 1, 2$, and $\lambda \in F^{\times}$, we get that

$$(B, \tau) = (M_2(F), \text{ad}_{\langle \lambda \rangle}) \otimes_F (H_1, \rho_1) \otimes_F (H_2, \rho_2) \otimes_F (F', \iota)$$

satisfies $e_3^{\text{hyp}}(\tau) = 0$. Nevertheless, τ is not hyperbolic in general, so the invariant e_3^{hyp} does not characterize hyperbolic involutions in degree 8. To construct an explicit example, let F be a purely transcendental extension $F = k(t)$ of a field k , $H_1 \otimes H_2$ a biquaternion algebra over k which is division over $k' = k(\sqrt{\delta})$, and pick $\lambda = t$. The involution τ is adjoint to the hermitian form $\langle 1, -t \rangle$ with values in $(H_1 \otimes H_2 \otimes F', \rho_1 \otimes \rho_2 \otimes \iota)$, which is anisotropic. Note that the algebra with involution considered here is a 'generic sum' of two copies of $\gamma_1 \otimes \gamma_2 \otimes \iota$ in the sense of [35].

As a consequence of the previous proposition, we have :

Corollary 6.8. *Let $B = M_2(D)$ for some biquaternion algebra D over F' with trivial F'/F -corestriction. The hyperbolic unitary involution τ_0 is totally decomposable if and only if the invariant $\delta_{F'/F}(D)$ vanishes.*

Proof. Assume first that τ_0 is totally decomposable. By the proposition above, we have $e_3^{\text{hyp}}(\tau_0) = \delta_{F'/F}(D)$. On the other hand, since τ_0 is hyperbolic, $e_3^{\text{hyp}}(\tau_0) = 0$, and this proves the first implication.

To prove the converse, assume $\delta_{F'/F}(D) = 0 \in H^3(F)/\text{cor}_{F'/F}(F'^{\times} \cdot [D])$. By [5, Prop. 4.4], there exists a biquaternion algebra D_0 over F such that $D \simeq D_0 \otimes_F F'$. By uniqueness of the hyperbolic unitary involution of B , we have

$$(B, \tau_0) \simeq ((M_2(F), \gamma) \otimes_F (D_0, \rho)) \otimes_F (F', \iota),$$

where γ is the canonical involution of $M_2(F)$, which is hyperbolic, and ρ is any involution of the first kind on D_0 . Since D_0 is a biquaternion algebra, it admits decomposable involutions, and the result follows immediately. \square

Example 6.9. Let $C \simeq H_1 \otimes H_2 \otimes H_3$ be a totally decomposable algebra over F such that $F' \subset C$ and there is no quaternion subalgebra of C containing F' . Explicit examples are given in [4, Cor. 4.5]. Since F' is a subfield of C , $C_{F'}$ is isomorphic to $M_2(D)$ for some biquaternion algebra D over F' . Assume $D = D_0 \otimes_F F'$ for some biquaternion algebra D_0 over F . Then $C \simeq D_0 \otimes H$ for some quaternion algebra H containing F' . Hence, by construction of C , such a D_0 does not exist and it follows that $\delta_{F'/F}(D) \neq 0$. As a consequence, the hyperbolic unitary involution on $M_2(D)$ is not totally decomposable and the totally decomposable involution $\tau = \gamma_1 \otimes \gamma_2 \otimes \gamma_3 \otimes \iota$ of $C_{F'} \simeq M_2(D)$ satisfies $e_3^{\text{hyp}}(\tau) = \delta_{F'/F}(D) \neq 0$.

In view of this example, in order to characterize totally decomposable involutions using a cohomological invariant, it seems relevant to work with a totally decomposable base point rather than the hyperbolic one. The following lemma explains how this can be done :

Lemma 6.10. *Let τ_1 be a totally decomposable unitary involution of the degree 8 algebra $M_2(D)$. The discriminant algebra of τ_1 is split, and for all unitary involutions τ of B with split discriminant algebra, we have :*

$$e_3^{\tau_1}(\tau) = e_3^{\text{hyp}}(\tau) + \delta_{F'/F}(D) \in H^3(F)/\text{cor}_{F'/F}(F'^{\times} \cdot [B]).$$

In particular, $e_3^{\tau_1}(\tau)$ does not depend on the choice of a totally decomposable involution τ_1 , and will be denoted by $e_3^{\text{td}}(\tau)$.

Proof. The discriminant algebra of τ_1 is split by Proposition 6.5. Let τ_0 be the hyperbolic unitary involution of B . By Proposition 3.1 and Corollary 3.6, we have

$$e_3^{\tau_1}(\tau) = e_3^{\tau_1}(\tau_0) + e_3^{\tau_0}(\tau) = e_3^{\text{hyp}}(\tau_1) + e_3^{\text{hyp}}(\tau).$$

This finishes the proof since $e_3^{\text{hyp}}(\tau_1) = \delta_{F'/F}(D)$ by Proposition 6.5. \square

Hence, we have defined another degree 3 invariant for unitary involutions with split discriminant algebra by

$$e_3^{\text{td}}(\tau) = e_3^{\tau_1}(\tau) \in H^3(F)/\text{cor}_{F'/F}(F'^{\times} \cdot [B]),$$

where τ_1 is an arbitrary totally decomposable involution of B . This invariant is related to the hyperbolic Arason invariant by

$$e_3^{\text{td}}(\tau) = e_3^{\text{hyp}}(\tau) + \delta_{F'/F}(D),$$

and the two coincide if and only if $\delta_{F'/F}(D) = 0$. Moreover, since any totally decomposable involution can be chosen as a base point, we have :

Lemma 6.11. *Let τ be a totally decomposable unitary involution of $M_2(D)$, where D is a biquaternion algebra over F' . Then τ has split discriminant algebra and*

$$e_3^{\text{td}}(\tau) = 0.$$

It is not known whether or not the converse holds in general. In the remaining part of this section, we prove partial results in this direction. The first one deals with the split case :

Proposition 6.12. *Let τ be an F'/F -unitary involution on a split degree 8 algebra $B = M_8(F')$. The involution τ is totally decomposable if and only if*

$$\mathcal{D}(\tau) \text{ is split and } e_3^{\text{td}}(\tau) = 0 \in H^3(F).$$

Remark 6.13. If B is split, so is D . Hence, $\delta_{F'/F}(D) = 0$ and $e_3^{\text{td}} = e_3^{\text{hyp}}$. Therefore we could replace e_3^{td} by e_3^{hyp} in this case. Moreover, both invariants have values in $H^3(F)/\text{cor}(F'^{\times} \cdot [B]) = H^3(F)$.

Proof of Proposition 6.12. One implication is given by Lemma 6.11, so we only need to prove the converse. Assume $e_3^{\text{hyp}}(\tau) = 0 \in H^3(F)$. Let h be a hermitian forms over (F', ι) such that $\tau = \text{ad}_h$. As observed in Example 2.6, we have $e_3^{\text{hyp}}(\tau) = e_3(q_h)$. Since q_h is a 16-dimensional quadratic form, the equality $e_3(q_h) = 0$ implies that q_h is similar to a 4-fold Pfister form φ over F . Moreover, since q_h contains the subform $\langle\langle \delta \rangle\rangle$, the form $\varphi_{F'}$ is isotropic, hence hyperbolic, since it is a Pfister form. It follows that the pure subform of φ represents $-\delta$, see [25, X. Thm. 4.5]. Hence, by [25, X. Thm. 1.5], φ is of the form $\langle\langle \delta, a, b, c \rangle\rangle$ for some $a, b, c \in F^{\times}$. Therefore h is similar to $\langle\langle a, b, c \rangle\rangle$, now viewed as a hermitian form over (F', ι) and $\tau \simeq \text{ad}_{\langle\langle a, b, c \rangle\rangle}$ which is totally decomposable. \square

Another situation where involutions with trivial e_3^{td} are totally decomposable is the following. Using orthogonal decompositions of the underlying hermitian forms, one may always view a unitary involution τ of $M_2(D)$ as an orthogonal sum of two involutions ρ_1 and ρ_2 of D . In other words, $\tau = \theta_1$, given by

$$(21) \quad \theta_1 \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} \rho_1(x) & -\rho_1(z)s^{-1} \\ -s\rho_1(y) & s\rho_1(t)s^{-1} \end{pmatrix}, \text{ for all } x, y, z, t \in D,$$

for some well chosen $s \in \text{Sym}(D, \rho_1)^{\times}$ such that $\rho_2 = \text{Int}(s) \circ \rho_1$. This also means τ is the the adjoint involution with respect to the hermitian form $\langle 1, -s \rangle$ with values in (D, ρ_1) . This hermitian form represents 1; this can only be achieved for certain choices of ρ_1 . The next result deals with unitary involutions for which we may find such a decomposition with ρ_1 having split discriminant algebra.

Proposition 6.14. *Let $B = M_2(D)$, where D is a biquaternion algebra over F' and assume D is endowed with a unitary involution ρ_1 with split discriminant algebra. For all $s \in \text{Sym}(D, \rho_1)^{\times}$, the involution θ_1 of $M_2(D)$ given by (21) is totally decomposable if and only if*

$$\mathcal{D}(\theta_1) \text{ is split and } e_3^{\text{td}}(\theta_1) = 0 \in H^3(F)/\text{cor}_{F'/F}(F'^{\times} \cdot [B]).$$

Proof. One implication is given by Proposition 6.5; let us prove the converse. Since D has degree 4 and ρ_1 has split discriminant algebra, there exists quaternion algebras H_1 and H_2 over F with canonical involutions γ_1 and γ_2 , respectively, such that

$$(D, \rho_1) \simeq (H_1, \gamma_1) \otimes_F (H_2, \gamma_2) \otimes_F (F', \iota),$$

see [22, thm. 3.1] and [24, (2.22)]. In particular, D has a descent up to isomorphism, so $\delta_{F'/F}(D) = 0$ and $e_3^{\text{td}}(\theta_1) = e_3^{\text{hyp}}(\theta_1)$. Moreover, $\text{id}_{H_1 \otimes H_2} \otimes \iota$ is an F automorphism of (D, ρ_1) acting as ι on F' , so that $(D, \rho_1) \simeq ({}^t D, {}^t \rho_1)$.

By assumption, θ_1 has split discriminant algebra. Hence, the same computation as in the proof of Proposition 3.2 shows $\mathcal{D}(\theta_1) \sim (\delta, \text{Nrd}_D(s)) \sim \mathcal{D}(\rho_2)$. Therefore ρ_2 also has split discriminant algebra. Hence, Proposition 3.2 applies to (D, ρ_1, ρ_2) and we get that $e_3^{\text{td}}(\theta_1) = e_3^{\text{hyp}}(\theta_1) = e_3^{\rho_1}(\rho_2) = 0$. By Theorem 6.1(b), we get that

ρ_2 is isomorphic to ρ_1 . So, there exists $u \in D^\times$ such that $\rho_2 = \text{Int}(u) \circ \rho_1 \circ \text{Int}(u^{-1}) = \text{Int}(u\rho_1(u)) \circ \rho_1$. It follows that $s = \lambda u\rho_1(u)$ for some $\lambda \in F^\times$. So the hermitian form $\langle 1, -s \rangle$ with values in (D, ρ_1) is isomorphic to $\langle 1, -\lambda \rangle$ and a direct computation shows

$$(M_2(D), \theta_1) \simeq (M_2(F), \text{ad}_{\langle\langle\lambda\rangle\rangle}) \otimes_F (H_1, \gamma_1) \otimes_F (H_2, \gamma_2) \otimes_F (F', \iota).$$

In particular, it is totally decomposable and this finishes the proof. \square

6.3. Unitary involutions on degree 6 algebras. In this section, we assume that the algebra B has degree 6 and is endowed with a hyperbolic F'/F -unitary involution, so that $B = M_2(D)$ where D has degree 3. We first prove that e_3^{hyp} detects hyperbolicity in this case :

Proposition 6.15. *Let B be a degree 6 and index dividing 3 central simple F -algebra with center F' and let τ be an F'/F -unitary involution on B . The involution τ is hyperbolic if and only if $\mathcal{D}(\tau)$ is split and $e_3^{\text{hyp}}(\tau) = 0 \in N_{0,\beta}^3(F)$.*

Proof. One implication follows from 2.3, so we only need to prove the converse. Assume $\mathcal{D}(\tau)$ is split and $e_3^{\text{hyp}}(\tau) = 0 \in N_{0,\beta}^3(F)$, and let us prove τ is hyperbolic. As we did in degree 8, using a diagonalisation of the underlying hermitian form, we may view τ as an orthogonal sum of two unitary involutions ρ_0 and ρ of D . By Proposition 3.2, we have $e_3^{\text{hyp}}(\tau) = e_3^{\rho_0}(\rho) = 0$. Therefore, since D has degree 3, $\rho_0 \simeq \rho$ by Example 2.3. Hence we may assume $s = 1$ in (11), so that τ is adjoint to the hermitian form $\langle 1, -\lambda \rangle$ with values in (D, ρ_0) , where $\lambda \in F^\times$. Moreover, by the same computation as in the proof of Proposition 3.2, the discriminant algebra of τ is $(\delta, \lambda^3) = 0$. So, there exists $x \in F'^\times$ such that $\lambda = N_{F'/F}(x) = x\iota(x) = x\rho_0(x)$, and this proves that the hermitian form $\langle 1, -\lambda \rangle$ over (D, ρ_0) is isomorphic to $\langle 1, -1 \rangle$. We get that the involution τ is hyperbolic as expected. \square

The relative Arason invariant also is classifying under some additional conditions. Let us first assume the underlying algebra B is split.

Proposition 6.16. *Let $B = M_6(F')$ and consider two F'/F -unitary involutions τ_0 and τ of B . We assume that $\mathcal{D}(\tau_0) = \mathcal{D}(\tau) = 0$. The involutions τ_0 and τ are isomorphic if and only if $e_3^{\tau_0}(\tau) = 0 \in H^3(F)$.*

Proof. Assume $e_3^{\tau_0}(\tau) = 0 \in H^3(F)$. Since B is split, there are 6-dimensional hermitian forms h_0 and h with values in (F', ι) and with trivial discriminant such that $\tau_0 = \text{ad}_{h_0}$ and $\tau = \text{ad}_h$. So, we may find $p, q, r, s, t \in F$ and $u \in F'^\times$ such that

$$h = \langle p, q, r, s, t, pqrstN_{F'/F}(u) \rangle = \langle pqr \rangle \langle pq, pr, qr, pqrs, pqrt, stN_{F'/F}(u) \rangle.$$

The 3-dimensional quadratic forms $\langle pq, pr, qr \rangle$ and $\langle pqrs, pqrt, stN_{F'/F}(u) \rangle$ have respective determinant 1 and $N_{F'/F}(u)$. Hence, h is similar to the form

$$\langle a, b, ab \rangle - \langle N_{F'/F}(u) \rangle \langle c, d, cd \rangle$$

for some $a, b, c, d \in F^\times$. Hence, the Jacobson trace of h is

$$q_h \simeq \langle\langle \delta \rangle\rangle (\langle a, b, ab \rangle - \langle N_{F'/F}(u) \rangle \langle c, d, cd \rangle) \simeq \langle\langle \delta \rangle\rangle \langle a, b, ab, -c, -d, -cd \rangle,$$

which is the product of the 1-fold Pfister form $\langle\langle \delta \rangle\rangle$ with an Albert form. Likewise, we may find $a_0, b_0, c_0, d_0 \in F^\times$ such that

$$q_{h_0} \simeq \langle\langle \delta \rangle\rangle \langle a_0, b_0, a_0b_0, -c_0, -d_0, -c_0d_0 \rangle.$$

Since $e_3^{\tau_0}(\tau) = e_3(q_h - q_{h_0}) \in H^3(F)$ by Lemma 2.4, the assumption implies that $e_3(q_h - q_{h_0}) = 0$. Hence, $q_h = q_{h_0} \bmod I^4(F)$. By [20] this implies that the two forms q_h and q_{h_0} are similar. Consequently, the corresponding hermitian forms h and h_0 also are similar, and the involutions $\tau_0 \simeq \text{ad}_{h_0}$ and $\tau \simeq \text{ad}_h$ are isomorphic. \square

The next result extends slightly Proposition 6.15. More precisely, we assume $B = M_2(D)$ for some degree 3 algebra D , and we consider two involutions having orthogonal sum decompositions with a common factor. In other words, both are adjoint to hermitian forms representing 1 with values in (D, ρ_0) for some F'/F -unitary involution ρ_0 of D .

Proposition 6.17. *Let (D, ρ_0) be a degree 3 central simple F -algebra with an F'/F -unitary involution, and let $s_1, s_2 \in \text{Sym}(D, \rho_0)^\times$. Consider the unitary involutions τ_1 and τ_2 respectively adjoint to the hermitian forms $\langle\langle s_1 \rangle\rangle$ and $\langle\langle s_2 \rangle\rangle$ with values in (D, ρ_0) . We assume in addition that $\mathcal{D}(\tau_1)$ and $\mathcal{D}(\tau_2)$ are both split. Then τ_1 and τ_2 are isomorphic if and only if $e_3^{\tau_1}(\tau_2) = 0 \in N_{0,\beta}^3(F)$.*

Proof. The argument is pretty similar to the proof of Proposition 6.15; the main difference is that we need to assume here that the two involutions admit orthogonal sum decompositions with a common factor ρ_0 , while this is always the case when one of them is hyperbolic. More precisely, consider τ_1 and τ_2 as in the statement, and assume $e_3^{\tau_1}(\tau_2) = 0$. Consider the unitary involutions of D defined by $\rho_i = \text{Int}(s_i) \circ \rho$ for $i = 1, 2$. By Proposition 3.1, Corollary 3.6 and Proposition 3.2, we have the following equalities in $H^3(F)/N_{F'/F}(F'^\times \cdot [B]) = H^3(F)/N_{F'/F}(F'^\times \cdot [D])$:

$$0 = e_3^{\tau_1}(\tau_2) = e_3^{\text{hyp}}(\tau_1) + e_3^{\text{hyp}}(\tau_2) = e_3^{\rho_0}(\rho_1) + e_3^{\rho_0}(\rho_2) = e_3^{\rho_1}(\rho_2).$$

Therefore, by Example 2.3, the involutions ρ_1 and ρ_2 are isomorphic. Hence, there exists $\lambda \in F^\times$ such that $s_2 = \lambda s_1$. Comparing the discriminant algebras of τ_1 and τ_2 , the same kind of computation as in the proof of Proposition 6.15 shows $\lambda = x\rho_0(x)$ for some x and this concludes the proof. \square

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