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# Weighted $p$ -Laplace approximation of linear and quasi-linear elliptic problems with measure data

Robert Eymard, David Maltese and Alain Prignet\*

## Abstract

We approximate the solution to some linear and degenerate quasi-linear problem involving a linear elliptic operator (like the semi-discrete in time implicit Euler approximation of Richards and Stefan equations) with measure right-hand side and heterogeneous anisotropic diffusion matrix. This approximation is obtained through the addition of a weighted  $p$ -Laplace term. A well chosen diffeomorphism between  $\mathbb{R}$  and  $(-1, 1)$  is used for the estimates of the approximated solution, and is involved in the above weight. We show that this approximation converges to a weak sense of the problem for general right-hand-side, and to the entropy solution in the case where the right-hand-side is in  $L^1$ .

## 1 Introduction

This paper is focused on the approximation of a solution of second order linear and quasilinear elliptic equations in divergence form with coefficients in  $L^\infty(\Omega)$  ( $\Omega \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$ ,  $N \geq 2$  is an open bounded subset) and measure data. The obtained result provides the existence in the quasilinear case, and a uniqueness result is also given for  $L^1$  right-hand side. The linear problem is to find a measurable function  $u$  defined on  $\Omega$  such that, in some senses which will be given below, the following holds:

$$-\operatorname{div}(\Lambda \nabla u) = f \text{ in } \Omega, \quad (1)$$

together with the homogeneous Dirichlet boundary condition

$$u = 0 \text{ on } \partial\Omega. \quad (2)$$

The quasilinear problem consists in finding a pair of measurable functions  $(b, u)$  such that the following relations hold:

$$b - \operatorname{div}(\Lambda \nabla u) = f \text{ in } \Omega, \quad (3)$$

completed by the following relation:

$$\text{There exists } v \text{ measurable on } \Omega \text{ such that } b = \beta(v) \text{ and } u = \zeta(v) \text{ a.e. in } \Omega, \quad (4)$$

where  $\beta$  and  $\zeta$  are nonstrictly increasing functions (precise assumptions on these functions are given by (35) in Section 5). The quasilinear framework includes a semi-discrete in time version of some degenerate equations such as the Richards or the Stefan equations, as precised in Section 5. The quasilinear problem is supplemented with the boundary condition (2).

The following assumptions are made on the data  $\Lambda, f$ .

- $\Lambda \in L^\infty(\Omega)^{N \times N}$  is symmetric and there exists  $\underline{\lambda}, \bar{\lambda} > 0$  such that, for a.e.  $x \in \Omega$ ,  
and, for all  $\xi \in \mathbb{R}^N$ ,  $\underline{\lambda}|\xi|^2 \leq \Lambda(x)\xi \cdot \xi \leq \bar{\lambda}|\xi|^2$ , (5a)
- $f \in M(\Omega)$ . (5b)

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In (5b),  $M(\Omega)$  denotes the Radon measures set, defined as the dual space of the continuous functions on  $\overline{\Omega}$  with its classical norm. Note that, in the case  $N = 1$ , there holds  $M(\Omega) \subset H^{-1}(\Omega)$ ; then these problems enter into the framework of [11]; we therefore consider here only  $N \geq 2$ . We could consider as well the case where a term  $\operatorname{div}F$  with  $F \in L^2(\Omega)^N$  is added to  $f$ , since the same results as those obtained in this paper also hold in this case.

Let us recall a few results concerning the linear problem (1).

- The existence of a weak solution in the sense of Definition 4.1 for any  $N \geq 2$  is given in [22] (details of this result are given in [19]).
- Its uniqueness is proved for  $N = 2$  in [16] for general diffusion fields: the proof relies on a regularity result [18] which holds on domains  $\Omega$  with  $C^2$  boundary, extended in [17] to all domains with Lipschitz boundaries.
- In the case  $N \geq 3$ , this uniqueness result remains true if  $\Lambda$  is regular enough to apply the arguments of Agmon, Douglis and Nirenberg in the duality proof provided by [22], but it is no longer true for general diffusion fields: indeed, in [19], it is shown that, for a particular diffusion field  $\Lambda$  inspired by [21] (see [19] for more details), there exist infinitely many non-zero weak solutions  $u$  to Problem (1)-(2) (in the sense of Definition 4.1) for  $N = 3$ , even with  $f \equiv 0$ .

As in [5, 7], we consider solutions which are limit of sequences of regularised problems. Such solutions can be characterised by adding conditions in the definition of a weak solution, when the right-hand side is in  $L^1(\Omega)$ , and a uniqueness result can be proved. This is done by the notion of entropy weak sense [3], explored by several approaches in the literature (among them renormalised solutions by Lions and Murat, see [9]).

This sense is provided by Definition 4.3 in the linear case, and a straightforward adaptation in the quasilinear case is given by Definition 5.5.

The proofs of the existence of a solution in a weak sense and in an entropy weak sense are done in this paper in a different way from [5, 7, 3], where the existence is obtained through the regularization of the right-hand side. Here we keep the right-hand-side measure unchanged, hence remaining in  $(W_0^{1,p}(\Omega))'$  for any  $p > N$ . A natural idea would be to add a vanishing  $p$ -Laplace regularisation term (we show below in Section 2 that we need in fact a weighted one). The existence of a solution to the regularised problem will then be obtained through the use of a fix-point method. As in [8], a diffeomorphism between  $\mathbb{R}$  and an open bounded interval is used for deriving estimates. We follow a similar technique to [14], consisting in using the diffeomorphism  $\psi : \mathbb{R} \rightarrow (-1, 1)$ , defined by

$$\psi : \begin{cases} \mathbb{R} \longrightarrow (-1, 1) \\ s \longmapsto \frac{\ln(1 + |s|)}{1 + \ln(1 + |s|)} \operatorname{sign}(s), \end{cases} \quad (6)$$

where  $\operatorname{sign}(s) = 1$  if  $s \geq 0$  and  $-1$  if  $s < 0$ . Note that  $\psi$  is an odd strictly increasing function such that  $\psi'(s) \in (0, 1]$  for all  $s \in \mathbb{R}$ .

The advantage of this diffeomorphism over the one used in [8] is that it does not introduce a supplementary parameter which must vary in order that the weak sense be fully satisfied.

We show in Section 2 why it seems necessary to introduce a weighted dependence with respect to this function in the  $p$ -Laplace stabilisation term. We then study this regularised problem in Section 3, where we show the existence of a solution using Schaefer's theorem (which is a variant of the Leray-Schauder fix-point method). We then show some estimates on a solution to this regularised problem, enabling a convergence proof to the weak and entropy weak solutions of the linear problem (see Section 4). Similar proofs are then derived in the case of the quasilinear problem in Section 5, in which the nonlinear dependence between  $b$  and  $u$  is handled through Minty's trick.

**Notes applying to the whole paper:**

- We fix a given value  $p \in (N, +\infty)$  for the whole paper, which implies that  $W_0^{1,p}(\Omega) \subset C(\overline{\Omega}) \subset L^\infty(\Omega)$  and by duality,  $M(\Omega) \subset (W_0^{1,p}(\Omega))'$ .
- We denote for short  $\|\cdot\|_p$  instead of  $\|\cdot\|_{L^p(\Omega)}$  or  $\|\cdot\|_{L^p(\Omega)^N}$ .
- We use, for a.e.  $x \in \Omega$ ,  $\Lambda(x)$  as a linear operator from  $\mathbb{R}^N$  to  $\mathbb{R}^N$ , which applies to the element of  $\mathbb{R}^N$  which immediately follows.
- We use a few times Sobolev inequalities [1]: we denote  $C_{\text{sob}}^{(r,q)}$ , also depending on  $N$  and  $|\Omega|$ , such that

$$\|u\|_q \leq C_{\text{sob}}^{(r,q)} \|\nabla u\|_r, \text{ for any } u \in W_0^{1,r}(\Omega), \quad (7)$$

for any  $r \in [1, +\infty)$  and  $q \in [1, \frac{rN}{N-r}]$  if  $r < N$ ,  $q \in [1, +\infty)$  if  $r = N$  and  $q \in [1, +\infty]$  if  $r > N$ .

**2 Motivation for the definition of the regularised problem**

This section aims to motivate the choice of the nonlinear weight  $\alpha$  introduced in the vanishing  $p$ -Laplacian term:

$$-\text{div}(\Lambda \nabla u_\varepsilon + \varepsilon \alpha(u_\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) = f.$$

In the preceding equation,  $\varepsilon > 0$  is meant to tend to zero, and  $\alpha$  is a positive function to be chosen such that we can prove the following properties:

- there exists at least one solution  $u_\varepsilon$  to the regularised problem;
- using the function  $\psi(u_\varepsilon)$  as a test function where  $\psi$  is defined by (6), we can derive estimates on  $u_\varepsilon$  independent of  $\varepsilon$ , enabling to prove that any limit of  $u_\varepsilon$  as  $\varepsilon \rightarrow 0$  belong to the functional spaces containing the solutions to the weak or entropy weak sense of the problem;
- the vanishing term indeed vanishes as  $\varepsilon$  tends to 0.

Let us consider  $\psi(u_\varepsilon)$  as a test function in the regularised problem. Using the positivity of the term  $\varepsilon \alpha(u_\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \psi(u_\varepsilon)$ , we get in a similar way to [8] the following result:

$$\|\nabla u_\varepsilon\|_q \leq C, \text{ for all } q \in (1, \frac{N}{N-1}),$$

and therefore, from Sobolev inequalities, that

$$\|u_\varepsilon\|_{\hat{q}} \leq C, \text{ for all } \hat{q} \in I_N,$$

where  $I_N = (1, \frac{N}{N-2})$  if  $N > 2$  and  $I_N = (1, +\infty)$  if  $N = 2$ . The function  $\psi$  defined by (6) enables in particular the following estimate:

$$\left\| \frac{1}{\psi'(u_\varepsilon)} \right\|_{\hat{q}} \leq C. \quad (8)$$

We also obtain the following inequality

$$\varepsilon \int_{\Omega} \alpha(u_\varepsilon) \psi'(u_\varepsilon) |\nabla u_\varepsilon|^p \, dx \leq C.$$

Using the preceding inequalities must be sufficient to prove that, for any function  $w \in C_c^\infty(\Omega)$ ,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} \alpha(u_\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla w \, dx = 0.$$

Using a Hölder inequality, we can show that

$$\begin{aligned} \varepsilon \left| \int_{\Omega} \alpha(u_{\varepsilon}) |\nabla u_{\varepsilon}|^{p-2} \nabla u_{\varepsilon} \cdot \nabla w \, dx \right| \\ \leq \|\nabla w\|_{\infty} \varepsilon \left( \int_{\Omega} \alpha(u_{\varepsilon}) \psi'(u_{\varepsilon}) |\nabla u_{\varepsilon}|^p \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} \frac{\alpha(u_{\varepsilon})}{\psi'(u_{\varepsilon})^{p-1}} \, dx \right)^{\frac{1}{p}} \\ \leq C \varepsilon^{\frac{1}{p}} \left( \int_{\Omega} \frac{\alpha(u_{\varepsilon})}{\psi'(u_{\varepsilon})^{p-1}} \, dx \right)^{\frac{1}{p}}. \end{aligned}$$

We now need to bound the final integral in the right hand side independently of  $\varepsilon$ . Using (8), it is sufficient to choose  $\alpha$  such that

$$\frac{\alpha(u_{\varepsilon})}{\psi'(u_{\varepsilon})^{p-1}} \leq \frac{1}{\psi'(u_{\varepsilon})^{\hat{q}}},$$

for some  $\hat{q} < \frac{N}{N-2}$  (case  $N \geq 3$ ). Since  $\psi'(s) \in (0, 1]$ , taking  $\alpha(u_{\varepsilon}) = \psi'(u_{\varepsilon})^r$ , for some  $r > p - 1 - \frac{N}{N-2}$  satisfies this inequality.

The question which then arises is the possibility to prove the existence of  $u_{\varepsilon}$ . The existence proof in Section 3.1 relies on the existence of  $\tilde{u}_{\varepsilon} \in W_0^{1,p}(\Omega)$  such that

$$\psi'(u_{\varepsilon})^r(u_{\varepsilon}) |\nabla u_{\varepsilon}|^{p-2} \nabla u_{\varepsilon} = |\nabla \tilde{u}_{\varepsilon}|^{p-2} \nabla \tilde{u}_{\varepsilon},$$

which yields the change of variable  $\tilde{u}_{\varepsilon} = \psi_r(u_{\varepsilon})$  with  $\psi_r(s) = \int_0^s \psi'(t)^{\frac{r}{p-1}} \, dt$ . We then recover  $u_{\varepsilon}$  using the reciprocal function  $(\psi_r)^{-1}$  of  $\psi_r$ , which requires that the domain of  $(\psi_r)^{-1}$  be equal to  $\mathbb{R}$ , and therefore that the image of  $\psi_r$  be  $\mathbb{R}$ . This does not hold for  $r \geq p - 1$ : indeed, since for any  $s \in \mathbb{R}$ ,  $0 < \psi'(s) \leq 1$ , there holds  $0 < \psi'(s)^{\frac{r}{p-1}} \leq \psi'(s)$  which yields  $|\psi_r(s)| \leq |\psi(s)| < 1$ , for such  $r$ , the image of  $\psi_r$  cannot be equal to  $\mathbb{R}$ . Hence we have to choose  $r \in (p - 1 - \frac{N}{N-2}, p - 1)$ . We choose the value  $r = p - 2$ , whose advantage is to be independent of  $N$ , and to lead to simpler expressions.

In consequence, the weighting function chosen in the remaining part of this paper is defined by

$$\forall s \in \mathbb{R}, \alpha(s) = (\psi'(s))^{p-2},$$

and we have  $\alpha(u_{\varepsilon}) |\nabla u_{\varepsilon}|^{p-2} = |\nabla \psi(u_{\varepsilon})|^{p-2}$ .

In the next sections, we prove the existence of  $u_{\varepsilon}$ , some estimates on this function, and the convergence of  $u_{\varepsilon}$  to a weak or entropy weak solution of the linear or quasilinear problems as  $\varepsilon \rightarrow 0$ .

### 3 Study of the regularised problem

In the whole section,  $\varepsilon > 0$  is given.

In view of Section 5, we introduce a nonstrictly increasing function  $\mu_{\varepsilon} \in C(\mathbb{R})$  such that  $\mu_{\varepsilon}(0) = 0$ , which covers the case  $\mu_{\varepsilon} \equiv 0$  used in the linear case. As a consequence of Section 2, we consider the following problem

$$\mu_{\varepsilon}(u_{\varepsilon}) - \operatorname{div}(\Lambda \nabla u_{\varepsilon} + \varepsilon |\nabla \psi(u_{\varepsilon})|^{p-2} \nabla u_{\varepsilon}) = f, \quad (9)$$

together with homogeneous Dirichlet boundary conditions

$$u_{\varepsilon} = 0 \text{ on } \partial\Omega. \quad (10)$$

#### 3.1 Existence of a solution to the regularised problem

Lemma 3.1 (Existence of a weak solution to Problem (9)-(10)): There exists a function  $u_{\varepsilon}$  such that

$u_{\varepsilon} \in W_0^{1,p}(\Omega)$  and for any  $w \in W_0^{1,p}(\Omega)$ ,

$$\int_{\Omega} (\mu_{\varepsilon}(u_{\varepsilon})w + \Lambda \nabla u_{\varepsilon} \cdot \nabla w) \, dx + \varepsilon \int_{\Omega} |\nabla \psi(u_{\varepsilon})|^{p-2} \nabla u_{\varepsilon} \cdot \nabla w \, dx = \int_{\Omega} w \, df. \quad (11)$$

**Proof.**

**Step 1: change of variable.**

We define the odd, strictly increasing diffeomorphism  $\psi_p : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\forall s \in \mathbb{R}, \psi_p(s) = \int_0^s \psi'(t) \frac{p-2}{p-1} dt.$$

Remarking that, for any  $\tau \in (0, 2)$ , the minimum value of the function  $s \mapsto (1 + |s|)^{1+\tau} \psi'(s)$  is attained when  $1 + \ln(1 + |s|) = \frac{2}{\tau}$ , we get

$$\forall \tau \in (0, 2), \forall s \in \mathbb{R}, \frac{\tau^2}{4(1 + |s|)^{1+\tau}} \leq \psi'(s) = \frac{1}{(1 + \ln(1 + |s|))^2 (1 + |s|)} \leq \frac{1}{1 + |s|}. \quad (12)$$

This leads, for any  $t \in (0, \frac{1}{p-1}]$ , to

$$\forall s \in [0, +\infty), \frac{(1 - t(p-1))^2}{4t(p-2)^2} ((1+s)^t - 1) \leq \psi_p(s) \leq (p-1)((1+s)^{\frac{1}{p-1}} - 1),$$

which shows that the image  $\psi_p$  is equal to  $\mathbb{R}$ .

In this proof, we are looking for the existence of  $u_\varepsilon$  solution to (11). For this purpose, we introduce the change of variable, which enables to solve by minimisation a  $p$ -Laplace problem without weight,

$$\tilde{u}_\varepsilon = \psi_p(u_\varepsilon). \quad (13)$$

This means that  $u_\varepsilon = \psi_p^{-1}(\tilde{u}_\varepsilon)$ , which can only be written using that the range of  $\psi_p$  is equal to  $\mathbb{R}$ . It leads to  $\nabla u_\varepsilon = \nabla \psi_p^{-1}(\tilde{u}_\varepsilon) = (\psi_p^{-1})'(\tilde{u}_\varepsilon) \nabla \tilde{u}_\varepsilon$ . Since  $(\psi_p^{-1})'$  is continuous, for any  $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega) \subset L^\infty(\Omega)$ , we get that  $(\psi_p^{-1})'(\tilde{u}_\varepsilon)$  remains bounded, which implies that  $u_\varepsilon \in W_0^{1,p}(\Omega)$ .

Besides, we can also write

$$|\nabla \psi(u_\varepsilon)|^{p-2} \nabla u_\varepsilon = (\psi'(u_\varepsilon))^{p-2} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon = (\psi_p'(u_\varepsilon))^{p-1} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon = |\nabla \tilde{u}_\varepsilon|^{p-2} \nabla \tilde{u}_\varepsilon,$$

and  $\mu_\varepsilon(u_\varepsilon) = \mu_\varepsilon(\psi_p^{-1}(\tilde{u}_\varepsilon))$  and  $\Lambda \nabla u_\varepsilon = (\psi_p^{-1})'(\tilde{u}_\varepsilon) \Lambda \nabla \tilde{u}_\varepsilon$ .

Hence Problem (11) is equivalent to find  $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$  such that

$$\int_\Omega (\mu_\varepsilon(\psi_p^{-1}(\tilde{u}_\varepsilon)) w + (\psi_p^{-1})'(\tilde{u}_\varepsilon) \Lambda \nabla \tilde{u}_\varepsilon \cdot \nabla w) dx + \varepsilon \int_\Omega |\nabla \tilde{u}_\varepsilon|^{p-2} \nabla \tilde{u}_\varepsilon \cdot \nabla w dx = \int_\Omega w df, \text{ for any } w \in W_0^{1,p}(\Omega). \quad (14)$$

**Step 2: existence of  $\tilde{u}_\varepsilon$ .**

In order to prove the existence of  $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$  such that (14) holds, we remark that such a solution satisfies  $\tilde{u}_\varepsilon = F(\tilde{u}_\varepsilon)$ , where the mapping  $F : W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)$  is such that, for any  $\tilde{v} \in W_0^{1,p}(\Omega)$ , the element  $\tilde{u} = F(\tilde{v})$  with  $\tilde{u} \in W_0^{1,p}(\Omega)$  and

$$\int_\Omega (\mu_\varepsilon(\psi_p^{-1}(\tilde{v})) w + (\psi_p^{-1})'(\tilde{v}) \Lambda \nabla \tilde{u} \cdot \nabla w) dx + \varepsilon \int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla w dx = \int_\Omega w df, \text{ for any } w \in W_0^{1,p}(\Omega). \quad (15)$$

We can then apply Lemma 3.3, letting  $\sigma = \mu_\varepsilon \circ \psi_p^{-1}$  and  $\rho = (\psi_p^{-1})'$ , which states that the mapping  $F$  is well defined, continuous and compact.

Let  $t \in [0, 1]$ , and let  $\tilde{u} \in W_0^{1,p}(\Omega)$  such that  $tF(\tilde{u}) = \tilde{u}$  (the existence of such  $\tilde{u}$  is not yet proved). Let us prove that  $\tilde{u}$  remains bounded. This is clear for  $t = 0$ . Let us now assume that  $t \in (0, 1]$ , and let  $\tilde{u}$  satisfy  $F(\tilde{u}) = \tilde{u}/t$ , which means that

$$\begin{aligned} \int_\Omega (t^{p-1} \mu_\varepsilon(\psi_p^{-1}(\tilde{u})) w + t^{p-2} (\psi_p^{-1})'(\tilde{u}) \Lambda \nabla \tilde{u} \cdot \nabla w) dx + \varepsilon \int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla w dx \\ = t^{p-1} \int_\Omega w df, \text{ for any } w \in W_0^{1,p}(\Omega). \end{aligned}$$

Letting  $w = \tilde{u}$ , we get, since  $\mu_\varepsilon(\psi_p^{-1}(\tilde{u}))\tilde{u} \geq 0$  and  $(\psi_p^{-1})'(\tilde{u})\Lambda\nabla\tilde{u} \cdot \nabla\tilde{u} \geq 0$ , that

$$\varepsilon\|\nabla\tilde{u}\|_p^{p-1} \leq C_{\text{sob}}^{(p,\infty)}\|f\|_{M(\Omega)},$$

which shows that  $u$  is bounded independently of  $t$ .

Hence the function  $F$ , which is continuous and compact from  $W_0^{1,p}(\Omega)$  to  $W_0^{1,p}(\Omega)$ , is such that there exists  $C$  such that, for any  $t \in [0, 1]$  and for any solution  $\tilde{u}$  to  $tF(\tilde{u}) = \tilde{u}$ , then  $\|\nabla\tilde{u}\|_p \leq C$ ; we can then apply Schaefer's fixed point theorem [20] (which is deduced from Leray-Schauder topological degree theory), which proves that there exists  $\tilde{u} \in W_0^{1,p}(\Omega)$  such that  $F(\tilde{u}) = \tilde{u}$ .  $\blacksquare$

**Remark 3.2:** In the case where  $\mu_\varepsilon = 0$ , it is possible to get directly from (15) the existence of a fix-point, by applying Leray-Schauder fix-point theorem (in this case, the norm of  $\tilde{u}$  is bounded independently of  $\tilde{v}$ ).

**Lemma 3.3 (A continuous compact operator):** Let  $\sigma \in C(\mathbb{R})$  and  $\rho \in C(\mathbb{R})$  be given, such that  $\rho(s) \geq 0$  for all  $s \in \mathbb{R}$ . Then for all  $\tilde{v} \in W_0^{1,p}(\Omega)$ , there exists one and only one function  $\tilde{u}$  such that

$$\tilde{u} \in W_0^{1,p}(\Omega) \text{ and for any } w \in W_0^{1,p}(\Omega),$$

$$\int_{\Omega} \sigma(\tilde{v})w \, dx + \int_{\Omega} \rho(\tilde{v})\Lambda\nabla\tilde{u} \cdot \nabla w \, dx + \varepsilon \int_{\Omega} |\nabla\tilde{u}|^{p-2}\nabla\tilde{u} \cdot \nabla w \, dx = \int_{\Omega} w \, df. \quad (16)$$

Moreover, denoting by  $F$  the mapping  $\tilde{v} \mapsto \tilde{u}$ , then  $F$  is continuous and compact.

**Proof. Step 1: existence of  $\tilde{u}$  solution to (16).**

Let us define the function  $\mathcal{I}_{\tilde{v}} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined for any  $w \in W_0^{1,p}(\Omega) \subset L^\infty(\Omega) \cap H_0^1(\Omega)$  by

$$\mathcal{I}_{\tilde{v}}(w) = \int_{\Omega} \sigma(\tilde{v})w \, dx + \frac{\varepsilon}{p} \int_{\Omega} |\nabla w|^p \, dx + \frac{1}{2} \int_{\Omega} \rho(\tilde{v})\Lambda\nabla w \cdot \nabla w \, dx - \int_{\Omega} w \, df.$$

We have, for any  $\alpha > 0$ , that

$$\begin{aligned} \int_{\Omega} w \, df - \int_{\Omega} \sigma(\tilde{v})w \, dx &\leq \|w\|_\infty(\|f\|_{M(\Omega)} + \|\sigma(\tilde{v})\|_1) \leq C_{\text{sob}}^{(p,\infty)}\|\nabla w\|_p(\|f\|_{M(\Omega)} + \|\sigma(\tilde{v})\|_1) \\ &\leq \alpha \frac{\|\nabla w\|_p^p}{p} + \frac{1}{\alpha} \frac{(\|f\|_{M(\Omega)} + \|\sigma(\tilde{v})\|_1)^{p'}}{p'}. \end{aligned}$$

Since  $\rho(\tilde{v}) \geq 0$ , choosing  $\alpha = \frac{\varepsilon}{2}$  shows that there exists  $c_2 \geq 0$  such that

$$\forall w \in W_0^{1,p}(\Omega), \mathcal{I}_{\tilde{v}}(w) \geq \frac{\varepsilon}{2}\|\nabla w\|_p^p - c_2.$$

We then get that  $\mathcal{I}_{\tilde{v}}(w)$  is bounded by below independently of  $w$ , and that  $\mathcal{I}_{\tilde{v}}(w) \rightarrow +\infty$  if  $\|\nabla w\|_p \rightarrow +\infty$ . Therefore there exists a bounded minimizing sequence  $(w_n)_{n \in \mathbb{N}}$ . Hence there exists a subsequence, again denoted by  $(w_n)_{n \in \mathbb{N}}$ , which is weakly converging to some  $\tilde{u} \in W_0^{1,p}(\Omega)$  (and therefore also weakly converging in  $H_0^1(\Omega)$ ). Using that the norm function is weakly lower semicontinuous and the positivity of  $\int_{\Omega} \rho(\tilde{v})\Lambda\nabla(w_n - \tilde{u}) \cdot \nabla(w_n - \tilde{u}) \, dx$ , we get that

$$\|\nabla\tilde{u}\|_p \leq \liminf_{n \rightarrow +\infty} \|\nabla w_n\|_p \text{ and } \int_{\Omega} \rho(\tilde{v})\Lambda\nabla\tilde{u} \cdot \nabla\tilde{u} \, dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \rho(\tilde{v})\Lambda\nabla w_n \cdot \nabla w_n \, dx.$$

This implies

$$\mathcal{I}_{\tilde{v}}(\tilde{u}) \leq \liminf_{n \rightarrow +\infty} \mathcal{I}_{\tilde{v}}(w_n),$$

which proves that  $\tilde{u}$  is a minimizer of  $\mathcal{I}_{\tilde{v}}(w)$ . Then, for any  $w \in W_0^{1,p}(\Omega)$ , the function defined for all  $t \in \mathbb{R}$  by  $\mathcal{I}_{\tilde{v}}(\tilde{u} + tw)$ , admits a minimum in  $t = 0$ .

Computing the derivatives of the function  $g(t) = |x + ty|^p$  for  $x, y \in \mathbb{R}^N$  and using that  $p > 2$ , we get that

$$\forall t \in [0, 1], \forall x, y \in \mathbb{R}^N, |g''(t)| \leq p(p-1)|y|^2(|x| + |y|)^{p-2}.$$

This proves the right inequality in

$$\forall x, y \in \mathbb{R}^N, 0 \leq |x + y|^p - |x|^p - p|x|^{p-2}x \cdot y \leq \frac{p(p-1)}{2}|y|^2(|x| + |y|)^{p-2},$$

and therefore, in addition to  $|x| + |y| \leq 2 \max(|x|, |y|)$  that

$$\forall t \in [-1, 1], \forall x, y \in \mathbb{R}^N, 0 \leq |x + ty|^p - |x|^p - tp|x|^{p-2}x \cdot y \leq t^2 p(p-1) 2^{p-3} \max(|x|^p, |y|^p).$$

In addition to

$$\forall t \in [-1, 1], \forall x, y \in \mathbb{R}^N, \Lambda(x + ty) \cdot (x + ty) - \Lambda x \cdot x - 2t\Lambda x \cdot y = t^2 \Lambda y \cdot y,$$

we get that the expression defined for  $t \in [-1, 1] \setminus \{0\}$  by

$$A(t) = \frac{\mathcal{I}_{\tilde{v}}(\tilde{u} + tw) - \mathcal{I}_{\tilde{v}}(\tilde{u})}{t} - \left( \int_{\Omega} \sigma(\tilde{v})w \, dx + \int_{\Omega} \rho(\tilde{v})\Lambda \nabla \tilde{u} \cdot \nabla w \, dx + \varepsilon \int_{\Omega} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla w \, dx - \int_{\Omega} w \, df \right),$$

satisfies

$$|A(t)| \leq |t| \left( \frac{\varepsilon}{p} p(p-1) 2^{p-3} (\|\tilde{u}\|_p^p + \|w\|_p^p) + \bar{\lambda} \|\rho(\tilde{v})\|_{\infty} \|w\|_2^2 \right),$$

and therefore  $\lim_{t \rightarrow 0} A(t) = 0$ . Letting  $t \rightarrow 0$  with  $t > 0$  and  $t < 0$  successively, observing that  $\frac{\mathcal{I}_{\tilde{v}}(\tilde{u} + tw) - \mathcal{I}_{\tilde{v}}(\tilde{u})}{t}$  has the sign of  $t$  since  $\mathcal{I}_{\tilde{v}}(\tilde{u})$  minimizes  $\mathcal{I}_{\tilde{v}}$ , we obtain that

$$0 = \int_{\Omega} \sigma(\tilde{v})w \, dx + \int_{\Omega} \rho(\tilde{v})\Lambda \nabla \tilde{u} \cdot \nabla w \, dx + \varepsilon \int_{\Omega} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla w \, dx - \int_{\Omega} w \, df.$$

Therefore (16) holds for the minimizer  $\tilde{u}$  of  $\mathcal{I}_{\tilde{v}}$ , which shows the existence of at least one solution to (16).

### Step 2: uniqueness.

For  $\tilde{v}_1, \tilde{v}_2 \in W_0^{1,p}(\Omega)$ , let  $\tilde{u}_1, \tilde{u}_2 \in W_0^{1,p}(\Omega)$  be respective solutions to (16). We get, for any  $w \in W_0^{1,p}(\Omega)$ ,

$$\int_{\Omega} \Lambda \rho(\tilde{v}_1) \nabla \tilde{u}_1 \cdot \nabla w \, dx + \varepsilon \int_{\Omega} |\nabla \tilde{u}_1|^{p-2} \nabla \tilde{u}_1 \cdot \nabla w \, dx = \int_{\Omega} w \, df - \int_{\Omega} \sigma(\tilde{v}_1)w \, dx, \quad (17)$$

and

$$\begin{aligned} \int_{\Omega} \Lambda \rho(\tilde{v}_1) \nabla \tilde{u}_2 \cdot \nabla w \, dx + \varepsilon \int_{\Omega} |\nabla \tilde{u}_2|^{p-2} \nabla \tilde{u}_2 \cdot \nabla w \, dx &= \int_{\Omega} w \, df - \int_{\Omega} \sigma(\tilde{v}_2)w \, dx \\ &\quad + \int_{\Omega} \Lambda(\rho(\tilde{v}_1) - \rho(\tilde{v}_2)) \nabla \tilde{u}_2 \cdot \nabla w \, dx. \end{aligned}$$

Letting  $w = \tilde{u}_1 - \tilde{u}_2$  in the first one and  $w = \tilde{u}_2 - \tilde{u}_1$  in the second one, adding both equations and using the inequality [12, Lemma 2.40], which holds since  $p \geq 2$ ,

$$\forall x, y \in \mathbb{R}^N, |x - y|^p \leq 2^{p-1}(|x|^{p-2}x - |y|^{p-2}y)(x - y),$$

we obtain

$$\begin{aligned} \int_{\Omega} \Lambda \rho(\tilde{v}_1) \nabla(\tilde{u}_2 - \tilde{u}_1) \cdot \nabla(\tilde{u}_2 - \tilde{u}_1) \, dx + \varepsilon 2^{1-p} \int_{\Omega} |\nabla(\tilde{u}_2 - \tilde{u}_1)|^p \, dx &\leq \int_{\Omega} (\sigma(\tilde{v}_1) - \sigma(\tilde{v}_2))(\tilde{u}_2 - \tilde{u}_1) \, dx \\ &+ \int_{\Omega} \Lambda(\rho(\tilde{v}_1) - \rho(\tilde{v}_2)) \nabla \tilde{u}_2 \cdot \nabla(\tilde{u}_2 - \tilde{u}_1) \, dx. \end{aligned} \quad (18)$$

The above inequality shows that, for  $\tilde{v}_1 = \tilde{v}_2$ , then  $\tilde{u}_1 = \tilde{u}_2$  (and therefore that (17) characterises the minimizer of  $\mathcal{I}_{\tilde{v}_1}$ ). Therefore the mapping  $F : \tilde{v} \mapsto \tilde{u}$  unique solution of (16) is well defined.

### Step 3: continuity and compactness of $F$ .

Letting  $w = \tilde{u}$  in (16), we get that

$$\varepsilon \|\nabla \tilde{u}\|_p^p \leq C_{\text{sob}}^{(p,\infty)} (\|f\|_{M(\Omega)} + \|\sigma(\tilde{v})\|_1). \quad (19)$$

We then obtain  $\|\tilde{u}\|_2$  and  $\|\nabla \tilde{u}\|_2$  are increasingly depending of  $(\|f\|_{M(\Omega)} + \|\sigma(\tilde{v})\|_1)$ .

Let  $(\tilde{v}_n)_{n \in \mathbb{N}}$  be a bounded sequence of  $W_0^{1,p}(\Omega)$ . We extract a subsequence, again denoted  $(\tilde{v}_n)_{n \in \mathbb{N}}$  such that  $\tilde{v}_n$  weakly converges to some  $\tilde{v} \in W_0^{1,p}(\Omega)$  and strongly in  $L^\infty(\Omega)$ . We then have the convergence in  $L^\infty(\Omega)$  of  $\sigma(\tilde{v}_n)$  and  $\rho(\tilde{v}_n)$  respectively to  $\sigma(\tilde{v})$  and  $\rho(\tilde{v})$ .

Inequality (18) in which we let  $\tilde{v}_1 = \tilde{v}_n$ ,  $\tilde{v}_2 = \tilde{v}$ ,  $\tilde{u}_1 = F(\tilde{v}_n)$ ,  $\tilde{u}_2 = F(\tilde{v})$ , we get

$$\begin{aligned} \varepsilon 2^{1-p} \int_{\Omega} |\nabla(F(\tilde{v}) - F(\tilde{v}_n))|^p \, dx &\leq \int_{\Omega} (\sigma(\tilde{v}_n) - \sigma(\tilde{v}))(F(\tilde{v}) - F(\tilde{v}_n)) \, dx \\ &+ \int_{\Omega} \Lambda(\rho(\tilde{v}_n) - \rho(\tilde{v})) \nabla F(\tilde{v}) \cdot \nabla(F(\tilde{v}) - F(\tilde{v}_n)) \, dx. \end{aligned}$$

Notice that (19) implies that  $F(\tilde{v}) - F(\tilde{v}_n)$  remains bounded in  $L^1(\Omega)$  as well as  $\nabla F(\tilde{v}) \cdot \nabla(F(\tilde{v}) - F(\tilde{v}_n))$ . Therefore, using the above convergences in  $L^\infty(\Omega)$ , we get that the right hand side of the above inequality tends to 0 and therefore  $F(\tilde{v}_n)$  tends to  $F(\tilde{v})$  in  $W_0^{1,p}(\Omega)$ . This shows that  $F$  is compact and at the same time that it is continuous.  $\blacksquare$

## 3.2 Estimates on the solution of the regularised problem

We define the following function, which is an odd, strictly increasing diffeomorphism from  $\mathbb{R}$  to  $\mathbb{R}$ :

$$\tilde{\chi} : s \mapsto \int_0^s \sqrt{\psi'(t)} \, dt. \quad (20)$$

**Lemma 3.4:** Let  $u_\varepsilon$  be given such that (11) holds. Then there exists  $C_1$  only depending on  $\underline{\lambda}$  and  $\|f\|_{M(\Omega)}$  such that

$$\int_{\Omega} \psi'(u_\varepsilon) |\nabla u_\varepsilon|^2 \, dx = \|\nabla \tilde{\chi}(u_\varepsilon)\|_2^2 \leq C_1 \quad (21)$$

and there holds

$$\varepsilon \int_{\Omega} (\psi'(u_\varepsilon))^{p-1} |\nabla u_\varepsilon|^p \, dx \leq \|f\|_{M(\Omega)}. \quad (22)$$

**Proof.** Taking  $w = \psi(u_\varepsilon)$  in (11), we obtain

$$\int_{\Omega} (\mu_\varepsilon(u_\varepsilon) \psi(u_\varepsilon) + \Lambda \nabla u_\varepsilon \cdot \nabla \psi(u_\varepsilon)) \, dx + \varepsilon \int_{\Omega} (\psi'(u_\varepsilon))^{p-2} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \psi(u_\varepsilon) \, dx = \int_{\Omega} \psi(u_\varepsilon) \, df,$$

which provides, since  $\mu_\varepsilon(s) \psi(s) \geq 0$ ,

$$\int_{\Omega} \psi'(u_\varepsilon) \Lambda \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx + \varepsilon \int_{\Omega} (\psi'(u_\varepsilon))^{p-1} |\nabla u_\varepsilon|^p \, dx \leq \int_{\Omega} \psi(u_\varepsilon) \, df.$$

Using the fact that  $\|\psi(u_\varepsilon)\|_\infty \leq 1$  we then obtain (21) and (22).  $\blacksquare$

**Lemma 3.5:** Under the assumptions of Lemma 3.4, for any  $1 < q < \frac{N}{N-1}$ , there exists  $C_2$  only depending on  $q$ ,  $N$ ,  $\underline{\lambda}$  and  $\|f\|_{M(\Omega)}$  such that

$$\|\nabla u_\varepsilon\|_q \leq C_2, \quad (23)$$

and, letting  $\hat{q} = q/(2-q)$  (then  $\hat{q} \in (1, +\infty)$  if  $N = 2$  and  $\hat{q} \in (1, N/(N-2))$  if  $N \geq 3$ ),

$$\|u_\varepsilon\|_{\hat{q}} \leq C_2 \text{ and } \|1/\psi'(u_\varepsilon)\|_{\hat{q}} \leq C_2. \quad (24)$$

**Remark 3.6:** It suffices to apply [8, Lemma 2.2] for getting the proof of (23) and the left part of (24), remarking that (12) provides [8, (2.16)] for any  $m \in (0, 1)$ . In the next proof, we are essentially using the ideas issued from [6]-[8] for proving (23) and the left part of (24), with a slightly different way for applying the Sobolev inequalities. Another small difference is the use of the function  $\psi$  instead of the function  $s \mapsto (1 - (1 + |s|)^{-m})\text{sign}(s)$ .

**Proof.** Using Hölder's inequality with conjugate exponents  $\frac{2}{q} > 1$  and  $\frac{2}{2-q}$  and owing to (21) in Lemma 3.4, we obtain

$$\begin{aligned} \int_{\Omega} |\nabla u_\varepsilon|^q \, dx &= \int_{\Omega} |\nabla u_\varepsilon|^q \left( \frac{\psi'(u_\varepsilon)}{\psi'(u_\varepsilon)} \right)^{q/2} \, dx \\ &\leq \left( \int_{\Omega} \psi'(u_\varepsilon) |\nabla u_\varepsilon|^2 \, dx \right)^{q/2} \left( \int_{\Omega} \frac{1}{(\psi'(u_\varepsilon))^{q/(2-q)}} \, dx \right)^{(2-q)/2} \\ &\leq (C_1)^{q/2} \left( \int_{\Omega} \frac{1}{(\psi'(u_\varepsilon))^{q/(2-q)}} \, dx \right)^{(2-q)/2}. \end{aligned}$$

Our aim is now to bound the  $L^{q/(2-q)}$  norm of  $1/\psi'(u_\varepsilon)$ , using Sobolev inequalities and the  $L^2$  bound (21) on  $\nabla \tilde{\chi}(u_\varepsilon)$ . For this purpose, we compare, for any  $s \in \mathbb{R}$ , the expression  $1/\psi'(s)$  with powers of  $\tilde{\chi}(s)$ . Let us recall that (12) states that the main part of  $1/\psi'(s)$  is  $1 + |s|$ , up to an arbitrary small exponent  $\tau$ . In particular, the left inequality of (12) provides, for any  $\tau \in (0, 2)$  and  $s \geq 0$ ,

$$\frac{1}{\psi'(s)} \leq \frac{4(1+s)^{1+\tau}}{\tau^2}. \quad (25)$$

Recall that  $\tilde{\chi}(s)$  is a primitive of  $\sqrt{\psi'(s)}$ , and is therefore expected to behave, up to an arbitrary small exponent, as  $\sqrt{1+|s|}$ . Indeed, for any  $\tau \in (0, 2)$  and  $s > 0$ , taking the square root of the left part of (12) gives

$$\frac{\tau}{2(1+s)^{(1+\tau)/2}} \leq \sqrt{\psi'(s)}.$$

Considering only  $\tau \in (0, 1)$  and integrating the preceding relation between 0 and  $s \geq 0$  provides

$$\tau((1+s)^{\frac{1-\tau}{2}} - 1) \leq \tilde{\chi}(s). \quad (26)$$

Eliminating  $s$  between (25) and (26) yields, for any  $\tau \in (0, 1)$ ,

$$\forall s \in \mathbb{R}, \quad \frac{1}{\psi'(s)} \leq \frac{4}{\tau^2} \left( \frac{1}{\tau} |\tilde{\chi}(s)| + 1 \right)^{2\frac{1+\tau}{1-\tau}}. \quad (27)$$

We thus obtain that, defining  $\rho(\tau) = 2\frac{(1+\tau)q}{(1-\tau)(2-q)}$ , there exist  $C_3^{(q,\tau)}$  and  $C_4^{(q,\tau)}$  such that

$$\int_{\Omega} \frac{1}{(\psi'(u_\varepsilon))^{q/(2-q)}} \, dx \leq C_3^{(q,\tau)} \int_{\Omega} |\tilde{\chi}(u_\varepsilon)|^{\rho(\tau)} \, dx + C_4^{(q,\tau)} |\Omega|$$

- In the case  $N = 2$ , let us define  $\tau = \frac{1}{2}$ . Then the Sobolev inequality (7) provides

$$\|\tilde{\chi}(u_\varepsilon)\|_{\rho(\tau)} \leq C_{\text{sob}}^{(2,\rho(\tau))} \|\nabla \tilde{\chi}(u_\varepsilon)\|_2.$$

- In the case  $N > 2$ , let us select  $\tau \in (0, 1)$  such that

$$\frac{1 + \tau}{1 - \tau} = \frac{N}{N - 2} \frac{2 - q}{q}. \quad (28)$$

Indeed, since  $q \in [1, N/(N - 1))$  implies  $(2 - q)/q \in ((N - 2)/N, 1]$ , the quantity  $a_q$  such that  $a_q = \frac{N}{N-2} \frac{2-q}{q}$  is such that  $1 < a_q$ . Since (28) leads to  $\tau = \frac{a_q - 1}{a_q + 1}$ , we get that  $\tau \in (0, 1)$  and that  $\rho(\tau) = \frac{2N}{N-2}$ , and (7) holds for any such  $\rho(\tau)$ . It leads to

$$\|\tilde{\chi}(u_\varepsilon)\|_{\rho(\tau)} \leq C_{\text{sob}}^{(2, \rho(\tau))} \|\nabla \tilde{\chi}(u_\varepsilon)\|_2.$$

Gathering the preceding inequalities and applying again (21) in Lemma 3.4 provide

$$\int_{\Omega} |\nabla u_\varepsilon|^q \, dx \leq (C_1)^{q/2} \left( C_3^{(q, \tau)} (C_{\text{sob}}^{(2, \rho(\tau))}) (C_1)^{\frac{1}{2}} \right)^{\rho(\tau)} + C_4^{(q, \tau)} |\Omega|^{(2-q)/2},$$

which provides (23). A Sobolev inequality then yields the left inequality of (24). The right one is then a consequence of (25) and of the choice of  $\tau$  such that  $\hat{q}(1 + \tau) < N/(N - 2)$  if  $N > 2$ . ■

**Lemma 3.7:** Under the assumptions of Lemma 3.4, there exists  $C_5$  only depending on  $\underline{\lambda}$ ,  $k$  and  $\|f\|_{M(\Omega)}$  such that

$$\|\nabla T_k(u_\varepsilon)\|_2 \leq C_5,$$

where  $T_k$  is the truncation function defined by  $T_k(s) = \min(|s|, k) \text{sign}(s)$  for all  $s \in \mathbb{R}$  (where  $\text{sign}(s) = 1$  if  $s \geq 0$  and  $-1$  if  $s < 0$ ).

**Proof.** Using that  $T'_k(s) = 1$  for  $|s| \leq k$  and  $T'_k(s) = 0$  for  $|s| > k$ , as well as  $|\nabla u_\varepsilon|^2 = \frac{1}{\psi'(u_\varepsilon)} |\nabla \tilde{\chi}(u_\varepsilon)|^2$ , we have that

$$\int_{\Omega} |\nabla T_k u_\varepsilon|^2 \, dx = \int_{|u_\varepsilon| \leq k} |\nabla u_\varepsilon|^2 \, dx = \int_{|u_\varepsilon| \leq k} \frac{1}{\psi'(u_\varepsilon)} |\tilde{\chi}(u_\varepsilon)|^2 \leq \frac{1}{\psi'(k)} \int_{\Omega} |\nabla \tilde{\chi}(u_\varepsilon)|^2 \, dx.$$

We conclude, using (21) in Lemma 3.4. ■

## 4 Convergence of the regularized problem to the linear problem

In this section, we consider the case  $\mu_\varepsilon \equiv 0$  in Section 3, and we study how the resulting Problem (9)-(10) is an approximation of Problem (1)-(2).

### 4.1 Convergence to a weak solution

Let us provide a weak sense for a solution of Problem (1)-(2).

**Definition 4.1:** We define the space  $\mathcal{S}_N(\Omega)$  containing any solution and the space  $\mathcal{T}_N(\Omega)$  containing the test functions by

$$\mathcal{S}_N(\Omega) = \bigcap_{q \in (1, \frac{N}{N-1})} W_0^{1, q}(\Omega) \text{ and } \mathcal{T}_N(\Omega) = \bigcup_{r \in (N, +\infty)} W_0^{1, r}(\Omega) \subset C(\bar{\Omega}), \quad (29)$$

We say that a measurable function  $u$  is a weak solution to Problem (1)-(2) if

$$u \in \mathcal{S}_N(\Omega) \text{ and } \int_{\Omega} \Lambda \nabla u \cdot \nabla w \, dx = \int_{\Omega} w \, df, \text{ for any } w \in \mathcal{T}_N(\Omega). \quad (30)$$

Let us observe that  $u \in S_N(\Omega)$  implies that  $u \in L^{\hat{q}}(\Omega)$  for any  $\hat{q} \in (1, \frac{N}{N-2})$  if  $N > 2$  and for any  $\hat{q} \in (1, +\infty)$  if  $N = 2$ . Note that all  $w \in \mathcal{T}_N(\Omega)$  is an element of  $C(\overline{\Omega})$ .

We can now state a result of existence, obtained by convergence of a solution to the regularised problem to a solution of the continuous problem, which holds owing to the choice done in Section 2 for the weight in the vanishing  $p$ -Laplace term.

**Lemma 4.2:** Let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers which converges to zero, and let  $u_n \in W_0^{1,p}(\Omega)$  be such that (11) holds with  $\varepsilon = \varepsilon_n$ .

Then there exist a subsequence of  $(\varepsilon_n, u_n)_{n \in \mathbb{N}}$ , again denoted  $(\varepsilon_n, u_n)_{n \in \mathbb{N}}$ , and  $u \in S_N(\Omega)$ , such that the sequence  $(u_n)_{n \in \mathbb{N}}$  converges to  $u \in S_N(\Omega)$  weakly in  $W_0^{1,q}(\Omega)$  for any  $q \in (1, N/(N-1))$ , strongly in  $L^{\hat{q}}(\Omega)$  for all  $\hat{q} \in [1, +\infty)$  if  $N = 2$  and for all  $\hat{q} \in [1, N/(N-2))$  if  $N \geq 3$  and almost everywhere in  $\Omega$ . Moreover,  $u$  is a weak solution in the sense of Definition 4.1, and, for any  $1 < q < \frac{N}{N-1}$ , there exists  $C_6$  only depending on  $q, N, \underline{\lambda}$  and  $\|f\|_{M(\Omega)}$  such that

$$\|\nabla u\|_q \leq C_6.$$

**Proof.** Using Lemma 3.5, there exists a subsequence, again denoted  $(\varepsilon_n, u_n)_{n \in \mathbb{N}}$ , such that the sequence  $(u_n)_{n \in \mathbb{N}}$  converges to a function  $u \in S_N(\Omega)$  weakly in  $W_0^{1,q}(\Omega)$  for any  $q \in (1, N/(N-1))$ , strongly in  $L^{\hat{q}}(\Omega)$  for all  $\hat{q} \in [1, +\infty)$  if  $N = 2$  and for all  $\hat{q} \in [1, N/(N-2))$  if  $N \geq 3$  and almost everywhere in  $\Omega$ . Let  $\phi \in C_c^\infty(\Omega)$  and  $n \in \mathbb{N}$ . We have

$$\int_{\Omega} \Lambda \nabla u_n \cdot \nabla \phi \, dx + \varepsilon_n \int_{\Omega} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla \phi \, dx = \int_{\Omega} \phi \, df. \quad (31)$$

The weak convergence in  $W_0^{1,q}(\Omega)$  for  $q \in (1, N/(N-1))$  of the sequence  $(u_n)_{n \in \mathbb{N}}$  to  $u$  gives

$$\lim_{n \rightarrow \infty} \int_{\Omega} \Lambda \nabla u_n \cdot \nabla \phi \, dx = \int_{\Omega} \Lambda \nabla u \cdot \nabla \phi \, dx.$$

Using Hölder's inequality, we have

$$\begin{aligned} \left| \varepsilon_n \int_{\Omega} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla \phi \, dx \right| &\leq \varepsilon_n \|\nabla \phi\|_{\infty} \int_{\Omega} (\psi'(u_n))^{p-2} |\nabla u_n|^{p-1} \, dx \\ &\leq \varepsilon_n \|\nabla \phi\|_{\infty} \left( \int_{\Omega} \frac{1}{\psi'(u_n)} \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} (\psi'(u_n))^{p-1} |\nabla u_n|^p \, dx \right)^{\frac{p-1}{p}}. \end{aligned}$$

Applying Lemma 3.4 and Lemma 3.5, we therefore get

$$\left| \varepsilon_n \int_{\Omega} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla \phi \, dx \right| \leq (\varepsilon_n)^{1/p} \|\nabla \phi\|_{\infty} (C_2)^{\frac{1}{p}} \left( \|f\|_{M(\Omega)} \right)^{\frac{p-1}{p}}.$$

This shows that, letting  $n \rightarrow \infty$  in (31) with  $n \in \mathbb{N}$ , we obtain that  $u$  satisfies (30) for any  $\phi \in C_c^\infty(\Omega)$ . We then conclude the proof of the lemma by a density argument.  $\blacksquare$

## 4.2 Convergence to the entropy weak solution

As announced by the introduction, we now use the definition of the entropy solution given in [3] (which is shown to be unique). In the whole section, we consider  $f \in L^1(\Omega)$ .

**Definition 4.3:** We define the entropy solution to Problem (1)-(2) as the measurable function  $u$  such that

1.  $u \in S_N(\Omega)$  and, for all  $k > 0$ ,  $T_k(u) \in H_0^1(\Omega)$ , where we recall that  $T_k$  is the truncation function defined by  $T_k(s) = \min(|s|, k) \text{sign}(s)$  for all  $s \in \mathbb{R}$  (where  $\text{sign}(s) = 1$  if  $s \geq 0$  and  $-1$  if  $s < 0$ ),

2. the following holds

$$\int_{\Omega} \Lambda \nabla u \cdot \nabla T_k(u - \phi) \, dx \leq \int_{\Omega} f T_k(u - \phi) \, dx, \quad (32)$$

for any  $\phi \in C_c^\infty(\Omega)$  and for any  $k > 0$ .

**Remark 4.4:** Let  $k > 0$  and  $\phi \in C_c^\infty(\Omega)$ . Using the fact  $\{|u - \phi| < k\}$  is a subset of  $\{|u| < h := k + \|\phi\|_\infty\}$  we obtain

$$|\nabla T_k(u - \phi)| \leq (|\nabla u| + |\nabla \phi|) 1_{|u| < h} \leq |\nabla T_h(u)| + |\nabla \phi|.$$

The assumption that  $T_h(u) \in H_0^1(\Omega)$  for any  $h > 0$  implies that  $T_k(u - \phi) \in H_0^1(\Omega)$  for any  $k > 0$  and for any  $\phi \in C_c^\infty(\Omega)$ .

Let us now turn to the convergence to the entropy solution.

**Lemma 4.5:** Let  $u$  be given by Lemma 4.2. Then, for all  $k > 0$ , there exists  $C_7$  only depending on  $\underline{\lambda}$ ,  $k$ , and  $\|f\|_1$  such that

$$\|\nabla T_k(u)\|_2 \leq C_7.$$

Moreover,  $u$  is the entropy weak solution in the sense of Definition 4.3.

**Proof.** The sequence  $(T_k(u_n))_{n \geq 0}$  weakly converges to  $T_k(u)$  in  $H_0^1(\Omega)$  for any  $k > 0$ . Let  $\phi \in C_c^\infty(\Omega)$ ; we have  $T_k(u_n - \phi) \in W_0^{1,p}(\Omega)$ , and we replace  $\phi$  with  $T_k(u_n - \phi)$  in (31). We obtain

$$\int_{\Omega} \Lambda \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dx + \varepsilon_n \int_{\Omega} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dx = \int_{\Omega} f T_k(u_n - \phi) \, dx. \quad (33)$$

Using the fact that the sequence  $(u_n)_{n \geq 0}$  converges almost everywhere to  $u$  and the fact that  $T_k \in L^\infty(\mathbb{R})$  we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x) T_k(u_n - \phi) \, dx = \int_{\Omega} f(x) T_k(u - \phi) \, dx.$$

Applying Lemma 4.6 below, we obtain

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \Lambda \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dx \geq \int_{\Omega} \Lambda \nabla u \cdot \nabla T_k(u - \phi) \, dx.$$

For the second term of the left member, we have

$$\begin{aligned} \varepsilon_n \int_{\Omega} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dx &= \varepsilon_n \int_{\{|u_n - \phi| < k\}} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla u_n \, dx \\ &\quad - \varepsilon_n \int_{\{|u_n - \phi| < k\}} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla \phi \, dx. \end{aligned}$$

Observe that the first term of the right-hand side of the above equation is nonnegative. Applying the same estimates and Hölder's inequality as in the proof of Lemma 4.2, we get that

$$\lim_{n \rightarrow \infty} \varepsilon_n \int_{\{|u_n - \phi| < k\}} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla \phi \, dx = 0,$$

and therefore, that there holds

$$\liminf_{n \rightarrow \infty} \varepsilon_n \int_{\Omega} |\nabla \psi(u_n)|^{p-2} \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dx \geq 0.$$

This proves, letting  $n \rightarrow +\infty$  in (33) with  $n \in \mathbb{N}$ , that  $u$  satisfies (32), and is therefore the entropy weak solution in the sense of Definition 4.3.  $\blacksquare$

Lemma 4.6: For the sequence provided by Lemma 4.2, there holds, for all  $k > 0$ ,

$$\int_{\Omega} \Lambda \nabla u \cdot \nabla T_k(u - \phi) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \Lambda \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dx, \text{ for any } \phi \in C_c^\infty(\Omega). \quad (34)$$

**Proof.** We have

$$\int_{\Omega} \Lambda \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dx = \int_{\Omega} \Lambda \nabla(u_n - \phi) \cdot \nabla T_k(u_n - \phi) \, dx + \int_{\Omega} \Lambda \nabla \phi \cdot \nabla T_k(u_n - \phi) \, dx.$$

Using the fact that  $\nabla T_k(u_n - \phi) = \nabla(u_n - \phi) 1_{\{|u_n - \phi| < k\}}$ , and that  $T'_k(s) = (T'_k(s))^2$ , we have

$$\int_{\Omega} \Lambda \nabla(u_n - \phi) \cdot \nabla T_k(u_n - \phi) \, dx = \int_{\Omega} \Lambda \nabla T_k(u_n - \phi) \cdot \nabla T_k(u_n - \phi) \, dx.$$

Using the weak convergence of the sequence  $(\nabla T_k(u_n - \phi))_{n \geq 0}$  in  $L^2(\Omega)^N$  to  $T_k(u - \phi)$ , we obtain

$$\int_{\Omega} \Lambda \nabla T_k(u - \phi) \cdot \nabla T_k(u - \phi) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \Lambda \nabla T_k(u_n - \phi) \cdot \nabla T_k(u_n - \phi) \, dx.$$

We also obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} \Lambda \nabla \phi \cdot \nabla T_k(u_n - \phi) \, dx = \int_{\Omega} \Lambda \nabla \phi \cdot \nabla T_k(u - \phi) \, dx.$$

We remark that

$$\int_{\Omega} \Lambda \nabla T_k(u - \phi) \cdot \nabla T_k(u - \phi) \, dx + \int_{\Omega} \Lambda \nabla \phi \cdot \nabla T_k(u - \phi) \, dx = \int_{\Omega} \nabla T_k(u - \phi) \cdot \nabla u \, dx,$$

which gives (34). ■

## 5 The quasilinear problem

### 5.1 Origine and formulations

Two quasilinear problems, which are extensions of Problem (1), are classically involving measure data. One is the Richards problem, whose unknown is the pressure  $w$  of the water phase within a porous medium containing air and water. It reads, in a simplified version, assuming that  $\Lambda$  is the absolute permeability field,

$$\partial_t \beta(w) - \operatorname{div}(\Lambda \nabla w) = g \text{ in } \Omega,$$

where  $\beta : \mathbb{R} \rightarrow [-1, 0]$  is a nonstrictly increasing function (the quantity  $\beta(w) + 1 \in [0, 1]$  is called the “water contents”), satisfying that  $\beta(w) = 0$  for all  $w \geq 0$ . This problem is therefore a parabolic problem which degenerates into an elliptic one in the region where  $w \geq 0$ . The right-hand-side represents injection or production terms, accurately modelled using measures along lines in 3D, points in 2D [15].

A second example is the Stefan problems, whose unknown is the internal energy  $w$  of a static material which is changing of state. Then the temperature is expressed as a function  $\zeta(w)$ , which is a nonstrictly increasing function, which remains constant in the range where  $0 \leq w \leq L$ , where  $L$  is the latent heat of change of state. Assuming that heat is provided by electric conductors, once again, a simplified model is

$$\partial_t w - \operatorname{div}(\nabla \zeta(w)) = g \text{ in } \Omega,$$

in which the right-hand-side is again accurately modelled using measures along lines in 3D, points in 2D. So both problems can be cast into the common following problem: find a function  $w$  such that

$$\partial_t \beta(w) - \operatorname{div}(\Lambda \nabla \zeta(w)) = g \text{ in } \Omega,$$

where  $\beta$  and  $\zeta$  are two nonstrictly increasing functions,  $\Lambda$  is a diffusion field and  $f$  is a measure. Assuming that an implicit Euler scheme is used in time, which is done in most cases, wich consists in replacing  $\partial_t \beta(w)$  by  $(\beta(w) - \beta(w_{\text{prev}}))/\delta t$  the semi-discrete problem to be solved, with respect to the at each time step is then under the form

$$\beta(w) - \operatorname{div}(\Lambda \nabla \delta t \zeta(w)) = g + \delta t \beta(w_{\text{prev}}) \text{ in } \Omega,$$

Applying the change of variable  $v = (\beta + \zeta)(w)$ , the problem becomes

$$\beta \circ (\beta + \zeta)^{-1}(v) - \operatorname{div}(\Lambda \nabla \delta t \zeta \circ (\beta + \zeta)^{-1}(v)) = g + \delta t \beta(w_{\text{prev}}) \text{ in } \Omega.$$

Then we notice that the functions  $\beta \circ (\beta + \zeta)^{-1}$  and  $\zeta \circ (\beta + \zeta)^{-1}$  are 1-Lipschitz continuous, the sum of which is equal to the identity function (see [13]). We again denote  $\beta, \zeta$  instead of  $\beta \circ (\beta + \zeta)^{-1}$  and  $\zeta \circ (\beta + \zeta)^{-1}$ , and we have  $\beta = \operatorname{Id} - \zeta$ . Letting  $\delta t = 1$ , denoting by  $f = g + \delta t \beta(w_{\text{prev}})$ ,  $b = \beta(v)$  and  $u = \zeta(v)$ , the problem is now to solve

$$b - \operatorname{div}(\Lambda \nabla u) = f,$$

which is Problem (3)-(2)-(4). We therefore make the following assumption on the functions  $\beta$  and  $\zeta$ :

- $\zeta : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and non-decreasing and 1-Lipschitz with  $\zeta(0) = 0$  and there exist  $Z_0 > 0$  and  $Z_1 > 0$  such that  $|\zeta(s)| \geq Z_1 |s| - Z_0$  for any  $s \in \mathbb{R}$ . (35a)

- $\beta = \operatorname{Id} - \zeta$  is therefore continuous, non-decreasing and 1-Lipschitz with  $\beta(0) = 0$ . (35b)

It is shown in [13] that one can also plug Problem (3)-(2)-(4) into the maximal monotone graphs framework. We define the graph  $\mathcal{G}$  and the multivalued operator  $\mathcal{T} : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  by

$$\mathcal{G} = \{(\zeta(s), \beta(s)), s \in \mathbb{R}\} \text{ and } \mathcal{T}(s) = \{t \in \mathbb{R}, (s, t) \in \mathcal{G}\}, \text{ for all } s \in \mathbb{R}. \quad (36)$$

We have the following properties (see [13]):

- $\mathcal{T}$  is a maximal monotone operator with domain  $\mathbb{R}$  such that  $0 \in \mathcal{T}(0)$ , (37a)

- there exist  $T_1, T_2, T_3, T_4 \geq 0$  such that, for all  $x \in \mathbb{R}$  and all  $y \in \mathcal{T}(x)$ , (37b)  
 $T_3 |x| - T_4 \leq |y| \leq T_1 |x| + T_2.$

It is then shown in [13] that the function  $\zeta$  can be identified as the resolvent of  $\mathcal{T}$  defined by  $(\operatorname{Id} + \mathcal{T})^{-1}$  and that (4) is equivalent to

$$b(x) \in \mathcal{T}(u(x)) \text{ for a.e. } x \in \Omega. \quad (38)$$

Note that the maximal monotone graph setting (38) is used in [4] for the study of renormalised solutions to the transient version of the problem studied in this paper. In [4], the additional assumption that the reciprocal graph  $\mathcal{T}^{-1}$  is a continuous function is used in the existence theorem for identifying the pointwise limit of solutions to regularised problems using compactness arguments (this corresponds in our setting to assume that  $\beta$ , or equivalently  $\rho$ , is strictly increasing).

## 5.2 The regularised problem in the quasilinear case

Instead of writing

$$b - \operatorname{div}(\Lambda \nabla u) = f,$$

with

$$b = \beta(v) \text{ and } u = \zeta(v),$$

we use the technique provided in [2]: in order to express  $v$  as a function of  $u$ , we introduce a given  $\varepsilon > 0$ , and we modify the problem into

$$b = \beta(v) \text{ and } u = (\varepsilon \operatorname{Id} + \zeta)(v) = \varepsilon v + \zeta(v).$$

Since the function  $\varepsilon \text{Id} + \zeta$  is continuous, strictly increasing with image  $\mathbb{R}$ , we can then deduce that

$$v = (\varepsilon \text{Id} + \zeta)^{-1}(u) \text{ and } b = \beta((\varepsilon \text{Id} + \zeta)^{-1}(u)).$$

We therefore consider the following problem: defining the function  $\mu_\varepsilon$  by

$$\forall s \in \mathbb{R}, \mu_\varepsilon(s) = \beta((\varepsilon \text{Id} + \zeta)^{-1}(s)), \quad (39)$$

find a function  $u$  defined on  $\Omega$  such that, there holds in a weak sense,

$$\mu_\varepsilon(u) - \text{div}(\Lambda \nabla u) = f.$$

We now consider the techniques introduced in the preceding sections and we consider the problem

$$\mu_\varepsilon(u_\varepsilon) - \text{div}(\Lambda \nabla u_\varepsilon + \varepsilon |\nabla \psi(u_\varepsilon)|^{p-2} \nabla u_\varepsilon) = f,$$

that is Problem (9)-(10), in which  $\mu_\varepsilon$ , which is defined by (39), is continuous and nonstrictly increasing with  $\mu_\varepsilon(0) = 0$  (this property is the only one used on  $\mu_\varepsilon$  in the whole Section 3). Observe that  $\varepsilon$  plays a double role: it is used at the same time for regularising the dependence between  $v$  and  $u$  and for regularising the equation by addition of a weighted  $p$ -Laplace term.

We can then directly apply Lemma 3.1, which provides the existence of a solution to Problem (9)-(10) in the sense of (11). The estimates provided by Lemmas 3.4, 3.5 and 4.5 hold as well. In addition, accounting from the surlinearity property (35a) of function  $\zeta$ , we obtain that the following lemma holds.

**Lemma 5.1:** Let  $\varepsilon \in (0, Z_1/2)$ . Let  $u_\varepsilon$  be given such that (11) holds with  $\mu_\varepsilon$  defined by (39). Then the function defined by

$$v_\varepsilon = (\varepsilon \text{Id} + \zeta)^{-1}(u_\varepsilon) \quad (40)$$

satisfies that there exists  $C_8$  such that, for any  $\hat{q} \in (1, +\infty)$  if  $N = 2$  and  $\hat{q} \in (1, N/(N-2))$  if  $N \geq 3$

$$\|v_\varepsilon\|_{\hat{q}} \leq C_8. \quad (41)$$

**Proof.** Since

$$u_\varepsilon = \varepsilon v_\varepsilon + \zeta(v_\varepsilon),$$

we get, applying (35a),

$$|u_\varepsilon| \geq |\zeta(v_\varepsilon)| - \varepsilon |v_\varepsilon| \geq -\varepsilon |v_\varepsilon| + Z_1 |v_\varepsilon| - Z_0 \geq \frac{Z_1}{2} |v_\varepsilon| - Z_0.$$

Applying (24), that is a bound on  $\|u_\varepsilon\|_{\hat{q}}$ , this concludes the proof. ■

### 5.3 Convergence to a weak or entropy weak solution

Let us first state the weak sense for a solution  $(b, u)$  to (4)-(2).

**Definition 5.2 (Weak solution to the quasilinear elliptic problem):** We say that pair of measurable functions  $(b, u)$  is a weak solution to Problem (3)-(2)-(4) if there exists a function  $v$  measurable on  $\Omega$  such that  $b = \beta(v)$  and  $u = \zeta(v)$  a.e. in  $\Omega$  and

$$u \in S_N(\Omega) \text{ and } \int_{\Omega} (b w + \Lambda \nabla u \cdot \nabla w) \, dx = \int_{\Omega} w \, df, \text{ for any } w \in \mathcal{T}_N(\Omega). \quad (42)$$

Let us observe that  $u \in S_N(\Omega)$  implies that  $u \in L^r(\Omega)$  for any  $r \in (1, \frac{N}{N-1})$ . Using Assumption (35a)-(35b), we deduce that  $b \in L^r(\Omega)$  for any  $r \in (1, \frac{N}{N-1})$ . Note that all  $w \in \mathcal{T}_N(\Omega)$  is an element of  $C(\overline{\Omega})$ .

We can now state a result of existence of a weak solution of the continuous problem, proved by passing to the limit on the regularised problem.

**Lemma 5.3:** Let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers which converges to zero, and let  $u_n \in W_0^{1,p}(\Omega)$  be such that (11) holds with  $\varepsilon = \varepsilon_n$  and  $\mu_{\varepsilon_n}$  given by (39).

Then there exist a subsequence of  $(\varepsilon_n, u_n)_{n \in \mathbb{N}}$ , again denoted  $(\varepsilon_n, u_n)_{n \in \mathbb{N}}$ ,  $u \in \mathcal{S}_N(\Omega)$  and  $v \in L^{\hat{q}}(\Omega)$ , such that the sequence  $(u_n)_{n \in \mathbb{N}}$  converges to  $u \in S_N(\Omega)$  weakly in  $W_0^{1,q}(\Omega)$  for any  $q \in (1, N/(N-1))$ , strongly in  $L^{\hat{q}}(\Omega)$  and almost everywhere in  $\Omega$  and  $v_n = (\varepsilon_n \text{Id} + \zeta)^{-1}(u_n)$  weakly converges to  $v$  in  $L^{\hat{q}}(\Omega)$  for all  $\hat{q} \in [1, +\infty)$  if  $N = 2$  and for all  $\hat{q} \in [1, N/(N-2))$  if  $N \geq 3$ .

Moreover, we have  $u = \zeta(v)$  and, letting  $b = \beta(v)$ , the pair  $(b, u)$  is a weak solution in the sense of Definition 5.2, and, for any  $1 < q < \frac{N}{N-1}$ , there exists  $C_9$  only depending on  $q, N, \underline{\lambda}$  and  $\|f\|_{M(\Omega)}$  such that

$$\|\nabla u\|_q \leq C_9.$$

**Proof.** Applying Lemmas 3.4 and 3.5, we construct a subsequence of  $(\varepsilon_n, u_n)_{n \in \mathbb{N}}$  of the initial sequence, that we again denote identically, and we select  $u \in \mathcal{S}_N(\Omega)$  such that the chosen sequence  $(u_n)_{n \in \mathbb{N}}$  converges to  $u \in S_N(\Omega)$  weakly in  $W_0^{1,q}(\Omega)$  for any  $q \in (1, N/(N-1))$ , strongly in  $L^{\hat{q}}(\Omega)$  for all  $\hat{q} \in [1, +\infty)$  if  $N = 2$  and for all  $\hat{q} \in [1, N/(N-2))$  if  $N \geq 3$ , and almost everywhere in  $\Omega$ .

Using Lemma 5.1, since for  $n$  large enough, we have  $\varepsilon_n \in (0, Z_1/2)$ , we can extract from this sequence another subsequence, again denoted  $(\varepsilon_n, u_n)_{n \in \mathbb{N}}$ , such that  $v_n = (\varepsilon_n \text{Id} + \zeta)^{-1}(u_n)$  (therefore  $u_n = \zeta_n(v_n)$  with  $\zeta_n = \varepsilon_n \text{Id} + \zeta$ ) converges to some  $v \in \mathcal{S}_N(\Omega)$  for the weak topology of  $L^{\hat{q}}(\Omega)$  with  $\hat{q} \in (1, +\infty)$  if  $N = 2$  and  $\hat{q} \in (1, N/(N-2))$  if  $N \geq 3$ .

We have now to check that  $u = \zeta(v)$  and that  $\mu_{\varepsilon_n}(u_n)$  converges to  $\beta(v)$  for the weak topology of  $W_0^{1,p}(\Omega)$ .

Applying Lemma 5.4, which states a consequence of Minty's trick, we indeed prove that  $u = \zeta(v)$ , since  $\zeta_n = \zeta + \varepsilon_n \text{Id}$  satisfies the hypotheses of the lemma.

Turning to  $\mu_{\varepsilon_n}(u_n)$ , the relation

$$\forall s \in \mathbb{R}, \mu_{\varepsilon}(s) = \beta((\varepsilon \text{Id} + \zeta)^{-1}(s)) = (\varepsilon \text{Id} + \zeta)^{-1}(s) - \zeta((\varepsilon \text{Id} + \zeta)^{-1}(s)) = (1 + \varepsilon)(\varepsilon \text{Id} + \zeta)^{-1}(s) - s.$$

issued from (39), leads to

$$\mu_{\varepsilon_n}(u_n) = \beta(v_n) = (1 + \varepsilon_n)v_n - u_n,$$

which proves the convergence of  $\mu_{\varepsilon_n}(u_n)$  to  $v - \zeta(v) = \beta(v) = b$  for the weak topology of  $W_0^{1,p}(\Omega)$ . ■

**Lemma 5.4 (Minty in  $L^p$  space):** Let  $(\zeta_n)_{n \in \mathbb{N}} \subset C(\mathbb{R})$  be a sequence of (non strictly) increasing Lipschitz-continuous function with the same Lipschitz constant  $L_{\zeta}$ , with  $\zeta_n(0) = 0$  for all  $n \geq 0$ , which simply converges to a function  $\zeta$  (which is therefore Lipschitz-continuous function with Lipschitz constant  $L_{\zeta}$ ). Let  $q_1, q_2 \in (1, 2)$  be given, and let  $(v_n)_{n \in \mathbb{N}}$  be a sequence of elements of  $L^{q_2}(\Omega)$  such that

- the sequence  $(v_n)_{n \in \mathbb{N}}$  weakly converges to  $v$  in  $L^{q_2}(\Omega)$ ,
- the sequence  $(\zeta_n(v_n))_{n \in \mathbb{N}}$  converges (strongly) to  $u$  in  $L^{q_1}(\Omega)$ .

Then  $u = \zeta(v)$  a.e. in  $\Omega$ .

**Proof.** We first extract a subsequence of  $(v_n, \zeta_n(v_n))_{n \in \mathbb{N}}$  a subsequence, again denoted  $(v_n, \zeta_n(v_n))_{n \in \mathbb{N}}$ , such that  $\zeta_n(v_n)$  converges to  $u$  a.e. and such that  $|\zeta_n(v_n)|^{q_1}$  is dominated in  $L^1(\Omega)$ . Let  $\theta$  be such that  $1 < \theta \frac{q_2}{q_2-1} \leq \min(q_1, q_2)$ , for any  $s \in \mathbb{R}$ , and let us denote by  $P_{\theta}(s) = s^{\theta}$  if  $s \geq 0$  and  $P_{\theta}(s) = -(-s)^{\theta}$  if  $s < 0$ . This choice of  $\theta$  ensures that:

1.  $|P_\theta(\zeta_n(v_n))|^{q_2/(q_2-1)} \leq \max(1, |\zeta_n(v_n)|)^{q_1}$  a.e., which implies that by dominated convergence  $(P_\theta(\zeta_n(v_n)))_{n \in \mathbb{N}}$  converges in  $L^{q_2/(q_2-1)}(\Omega)$  to  $P_\theta(u)$ ,
2. for any  $w \in L^{q_2}(\Omega)$ , using  $|\zeta_n(w)| \leq L_\zeta|w|$  a.e., we have  $|P_\theta(\zeta_n(w))|^{q_2/(q_2-1)} \leq \max(1, (L_\zeta|w|)^{q_2})$  a.e., and, using the simple convergence of  $\zeta_n$  to  $\zeta$ , we then get by dominated convergence that  $(P_\theta(\zeta_n(w)))_{n \in \mathbb{N}}$  converges in  $L^{q_2/(q_2-1)}(\Omega)$  to  $P_\theta(\zeta(w))$ .

Using the fact that, for any  $n \geq 0$ , the function  $P_\theta \circ \zeta_n$  is (nonstrictly) increasing, we obtain for any  $w \in L^{q_2}(\Omega)$ ,

$$\int_{\Omega} (P_\theta(\zeta_n(v_n)) - P_\theta(\zeta_n(w)))(v_n - w) \, dx \geq 0.$$

Notice that, in the above expression,  $P_\theta(\zeta_n(v_n)) - P_\theta(\zeta_n(w)) \in L^{q_2/(q_2-1)}(\Omega)$ . It is then possible to let  $n \rightarrow \infty$  in the previous inequality, which leads, by strong/weak convergence, to

$$\int_{\Omega} (P_\theta(u) - P_\theta(\zeta(w)))(v - w) \, dx \geq 0.$$

We let  $w = v + t\varphi$  where  $\varphi \in C_c^\infty(\Omega)$  and  $t \in (0, 1)$ , and we obtain

$$t \int_{\Omega} (P_\theta(u) - P_\theta(\zeta(v + t\varphi)))\varphi \, dx \geq 0.$$

Dividing by  $t$  and using that  $|P_\theta(\zeta(v + t\varphi))|$  is dominated in  $L^1(\Omega)$  by  $P_\theta(L_\zeta(|v| + |\varphi|))$ , we obtain by letting  $t \rightarrow 0$  and using dominated convergence

$$\int_{\Omega} (P_\theta(u) - P_\theta(\zeta(v)))\varphi \, dx \geq 0.$$

Since the above inequality also holds changing  $\varphi$  in  $-\varphi$ , it is therefore an equality, which leads, since  $\varphi$  is arbitrary, to

$$P_\theta(u) - P_\theta(\zeta(v)) = 0 \text{ a.e in } \Omega.$$

The previous identity thus gives

$$u = \zeta(v) \text{ a.e in } \Omega.$$

■

Let us now give the sense for the entropy solution  $(b, u)$  to (4)-(2) in the case  $f \in L^1(\Omega)$ . The proof of uniqueness of this solution is done in Section 6.

**Definition 5.5 (Entropy solution to the quasilinear elliptic problem with particular right-hand sides):** We assume that  $f \in L^1(\Omega)$ . We define an entropy solution of Problem (3)-(2)-(4) as a pair of measurable functions  $(b, u)$  if there exists a function  $v$  measurable on  $\Omega$  such that  $b = \beta(v)$  and  $u = \zeta(v)$  a.e. in  $\Omega$  and

1.  $u \in S_N(\Omega)$  and, for all  $k > 0$ ,  $T_k(u) \in H_0^1(\Omega)$ , where  $T_k$  is the truncation function defined by  $T_k(s) = \min(|s|, k)\text{sign}(s)$  for all  $s \in \mathbb{R}$  (where  $\text{sign}(s) = 1$  if  $s \geq 0$  and  $-1$  if  $s < 0$ ),
2. the following holds

$$\int_{\Omega} b T_k(u - \phi) \, dx + \int_{\Omega} \Lambda \nabla u \cdot \nabla T_k(u - \phi) \, dx \leq \int_{\Omega} f T_k(u - \phi) \, dx, \quad (43)$$

for any  $\phi \in C_c^\infty(\Omega)$  and for any  $k > 0$ .

We now turn to the existence result, the proof of which is again using the limit of the regularised problem.

**Lemma 5.6:** Let  $(b, u)$  be given by Lemma 5.3. Then, for all  $k > 0$ , there exists  $C_{10}$  only depending on  $\underline{\lambda}$ ,  $k$  and  $\|f\|_1$  such that

$$\|\nabla T_k(u)\|_2 \leq C_{10}.$$

Moreover, the pair  $(b, u)$  is the entropy weak solution in the sense of Definition 5.5.

**Proof.** We follow the proof of Lemma 4.5. The only difference is the convergence of the first term. This convergence is a consequence of the convergence of  $\int_{\Omega} \mu_{\varepsilon}(u_{\varepsilon}) T_k(u_{\varepsilon} - \phi)$  to  $\int_{\Omega} b T_k(u - \phi)$ , owing to Lemma 5.1, and to the fact that  $\mu_{\varepsilon}(u_{\varepsilon})$  weakly converges to  $b$  in  $L^2(\Omega)$ . ■

## 6 Uniqueness of the entropy solution for the quasilinear problem

The following lemma enables the use of a larger test function space in the entropy weak sense.

**Lemma 6.1 (Test functions in  $H_0^1(\Omega) \cap L^\infty(\Omega)$ ):** We assume that Assumptions (5) and (35) hold. Let us assume that  $f \in L^1(\Omega)$ . Let  $u$  be an entropy solution in the sense of Definition 5.5. Then (43) holds for any  $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$  and for any  $k > 0$ .

**Proof.** The proof follows the technique of [3, Lemma 3.3]. From a sequence  $(\phi_n)_{n \geq 0} \in C_c^\infty(\Omega)$  converging to  $\phi$  in  $H_0^1(\Omega)$ , one constructs a sequence, again denoted by  $(\phi_n)_{n \geq 0} \in C_c^\infty(\Omega)$  such that  $(\phi_n)_{n \geq 0}$  is uniformly bounded by  $M$ , converges almost everywhere in  $\Omega$  to  $\phi$  and  $|\nabla \phi_n|$  is dominated in  $L^2(\Omega)$ . Then  $T_k(u - \phi_n)$  converges in  $L^s(\Omega)$  to  $T_k(u - \phi)$  for all  $s \in [1, +\infty)$ ,  $\nabla T_k(u - \phi_n)$  weakly converges in  $L^2(\Omega)^N$  to  $\nabla T_k(u - \phi)$ . We then remark that (43) yields

$$\int_{\Omega} \beta(u) T_k(u - \phi_n) \, dx + \int_{\Omega} \Lambda \nabla T_h(u) \cdot \nabla T_k(u - \phi_n) \, dx \leq \int_{\Omega} (f T_k(u - \phi_n) + F \cdot \nabla T_k(u - \phi_n)) \, dx,$$

with  $h = k + M$  (recall that  $\nabla T_k(u - \phi_n) = 0$  on the set  $u > h$ ). We then let  $n \rightarrow \infty$  in the above inequality, which gives (43) for any  $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ . ■

**Lemma 6.2 (An entropy weak solution is a weak solution):** We assume that Assumptions (5) and (35) hold. Let us assume that  $f \in L^1(\Omega)$ . Let  $(b, u)$  be an entropy solution in the sense of Definition 5.5. Then  $(b, u)$  is a weak solution in the sense of Definition 4.1.

**Proof.** The proof follows the technique of [3, Corollary 4.3]. Owing to Lemma 6.1, for given  $\psi \in C_c^\infty(\Omega)$ ,  $k > \|\psi\|_{L^\infty(\Omega)}$  and  $h > 0$ , we can let  $\phi = T_h(u) - \psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$  in (43). This gives  $A_1(h) + A_2(h) \leq A_3(h)$  with

$$A_1(h) = \int_{\Omega} b T_k(u - T_h(u) + \psi) \, dx,$$

$$A_2(h) = \int_{\Omega} \Lambda \nabla u \cdot \nabla T_k(u - T_h(u) + \psi) \, dx,$$

$$A_3(h) = \int_{\Omega} T_k(u - T_h(u) + \psi) f(x) \, dx.$$

We observe that, defining

$$\chi_h(x) = 1 \text{ if } |u - T_h(u) + \psi| < k \text{ and } 0 \text{ otherwise,}$$

we have

$$A_2(h) = \int_{\Omega} \chi_h \Lambda \nabla u \cdot \nabla (u - T_h(u) + \psi) \, dx = A_{21}(h) + A_{21}(h),$$

with

$$A_{21}(h) = \int_{\Omega} \chi_h \Lambda \nabla u \cdot \nabla(u - T_h(u)) \, dx \text{ and } A_{22}(h) = \int_{\Omega} \chi_h \Lambda \nabla u \cdot \nabla \psi \, dx.$$

Using the fact that the function  $s \mapsto s - T_h(s)$  is (nonstrictly) increasing, we get that  $\nabla u \cdot \nabla(u - T_h(u)) \geq 0$  a.e., and therefore  $A_{21}(h) \geq 0$ . We therefore obtain

$$A_1(h) + A_{22}(h) \leq A_3(h) + A_4(h) \text{ for all } h > 0. \quad (44)$$

We now study the limit of (44), letting  $h \rightarrow +\infty$ . Since  $\chi_h(x)$  converges to 1 for a.e.  $x \in \Omega$  as  $h \rightarrow +\infty$  (recall that  $k > \|\psi\|_{L^\infty(\Omega)}$ ), by dominated convergence, we get

$$\lim_{h \rightarrow +\infty} A_{22}(h) = \int_{\Omega} \Lambda \nabla u \cdot \nabla \psi \, dx.$$

Using the fact that the sequence  $(T_k(u - T_h(u) + \psi))_{h \geq 0}$  is bounded in  $H_0^1(\Omega)$  and that  $T_h(u)$  converges to  $u$  almost everywhere in  $\Omega$ , we obtain the weak convergence in  $H_0^1(\Omega)$  of the sequence  $(T_k(u - T_h(u) + \psi))_{h \geq 0}$  to  $\psi$ . This leads to

$$\lim_{h \rightarrow +\infty} A_1(h) = \int_{\Omega} b\psi \, dx \text{ and } \lim_{h \rightarrow +\infty} A_3(h) = \int_{\Omega} \psi f(x) \, dx,$$

which enables to conclude that

$$\int_{\Omega} (b\psi + \Lambda \nabla u \cdot \nabla \psi) \, dx \leq \int_{\Omega} f\psi \, dx.$$

Replacing  $\psi$  by  $-\psi$ , we get that the above inequality is in fact an equality, which provides (42) for  $w = \psi$ . We then get (42) for any  $w \in \mathcal{T}_N(\Omega)$  by the density of  $C_c^\infty(\Omega)$  in any  $W_0^{1,r}(\Omega)$  for  $r \in [1, +\infty)$ . ■

**Lemma 6.3:** We assume that Assumptions (5) and (35) hold. Let  $(b, u)$  be an entropy solution in the sense of Definition 5.5. Then, for all  $k > 0$ , there holds

$$\lim_{h \rightarrow +\infty} \int_{h-k < |u| \leq h+k} |\nabla u|^2 \, dx = 0. \quad (45)$$

**Proof.** Letting, for given  $k, h > 0$ ,  $\phi = T_h(u)$  in (43) (this is possible thanks to Lemma 6.1), we get

$$\int_{\Omega} b T_k(u - T_h(u)) \, dx + \int_{\Omega} \Lambda \nabla u \cdot \nabla T_k(u - T_h(u)) \, dx \leq \int_{\Omega} f T_k(u - T_h(u)) \, dx.$$

Using  $\nabla u = \nabla T_k(u - T_h(u))$  for a.e.  $x$  such that  $\nabla T_k(u - T_h(u))(x) \neq 0$ , we get, denoting by  $E_h = \{x \in \Omega, h < |u(x)| \leq h + k\}$ ,

$$\Delta \|\nabla u\|_{L^2(E_h)}^2 \leq \int_{\Omega} (f - b) T_k(u - T_h(u)) \, dx,$$

which gives

$$\Delta \|\nabla u\|_{L^2(E_h)}^2 \leq \int_{E_h} k(|f| + |b|) \, dx.$$

By dominated convergence, since  $\chi_{E_h}(x)$  tends to 0 a.e. as  $h \rightarrow \infty$ , we get

$$\lim_{h \rightarrow +\infty} \int_{E_h} k(|f| + |b|) \, dx = 0,$$

and therefore we obtain

$$\lim_{h \rightarrow +\infty} \Delta \|\nabla u\|_{L^2(E_h)}^2 = 0. \quad (46)$$

Note that (46) implies that

$$\lim_{h \rightarrow +\infty} \int_{h < |u| \leq h+k} |\nabla u|^2 dx = \lim_{h \rightarrow +\infty} \int_{h-k < |u| \leq h} |\nabla u|^2 dx = 0,$$

hence providing (45). ■

**Lemma 6.4** ([10]): We assume that Assumptions (5) and (35) hold. Let  $(b, u)$  be an entropy solution in the sense of Definition 5.5. Then, for any  $k > 0$  and for any  $\phi \in C_c^\infty(\Omega)$ , there holds

$$\int_{\Omega} \left( bT_k(u - \phi) + \Lambda(x)\nabla u \cdot \nabla T_k(u - \phi) \right) dx = \int_{\Omega} T_k(u - \phi)f(x) dx. \quad (47)$$

**Proof.** Let  $\phi = 2T_h(u) - \tilde{\phi}$ , for given  $h > 0$  and  $\tilde{\phi} \in C_c^\infty(\Omega)$ . Let  $M = k + \|\tilde{\phi}\|_\infty$ . For  $h > M$ , we have:

- $T_k(u - 2T_h(u) + \tilde{\phi}) = u + 2h + \tilde{\phi}$  for  $|u + 2h + \tilde{\phi}| \leq k$ ,
- $T_k(u - 2T_h(u) + \tilde{\phi}) = -u + \tilde{\phi}$  for  $|-u + \tilde{\phi}| \leq k$ ,
- $T_k(u - 2T_h(u) + \tilde{\phi}) = u - 2h + \tilde{\phi}$  for  $|u - 2h + \tilde{\phi}| \leq k$ ,
- otherwise  $T_k(u - 2T_h(u) + \tilde{\phi}) = \pm k$ ,

and we also have

$$T_k(u - 2T_h(u) + \tilde{\phi}) = T_k(-u + \tilde{\phi}) \text{ if } |u| \leq 2h - M. \quad (48)$$

This proves that

$$\int_{\Omega} \Lambda(x)\nabla u \cdot \nabla T_k(u - \phi) dx = \int_{\Omega} \Lambda(x)\nabla u \cdot \nabla T_k(-u + \tilde{\phi}) dx + R_h,$$

with

$$|R_h| \leq \bar{\lambda} \int_{2h-M < |u| < 2h+M} |\nabla u| (|\nabla u| + |\nabla \tilde{\phi}|) dx.$$

Applying Lemma (6.3), we get that

$$\lim_{h \rightarrow \infty} R_h = 0.$$

Besides, we get from (48) that

$$\left| \int_{\Omega} T_k(u - \phi)f(x) dx - \int_{\Omega} T_k(-u + \tilde{\phi})f(x) dx \right| \leq 2k \int_{|u| \geq 2h-M} |f(x) - b(x)| dx.$$

By dominated convergence, we get that

$$\lim_{h \rightarrow \infty} \int_{|u| \geq 2h-M} |f(x) - b(x)| dx = 0.$$

We consider  $\phi = 2T_h(u) - \tilde{\phi}$  in (43) (this is possible owing to Lemma 6.1), and we obtain by letting  $h \rightarrow \infty$ ,

$$\int_{\Omega} \left( bT_k(-u + \tilde{\phi}) + \Lambda(x)\nabla u \cdot \nabla T_k(-u + \tilde{\phi}) \right) dx \leq \int_{\Omega} T_k(-u + \tilde{\phi})f(x) dx,$$

which, in addition to (43) with  $\phi = \tilde{\phi}$ , provides (47). ■

**Theorem 6.5:** We assume that Assumptions (5) and (35) hold. Then there exists an unique entropy solution to Problem (3)-(2)-(4) in the sense of Definition 5.5.

**Proof.** The proof follows that of [3, Theorem 5.1]. Let  $(b_1, u_1)$  and  $(b_2, u_2)$  be two entropy solutions in the sense of Definition 5.5. We let, for  $h, k > 0$  be given,  $\phi = T_h(u_i)$  in (43) written for  $(b_j, z_j)$  (one more time, this is possible thanks to Lemma 6.1) and we add the resulting inequalities. We get  $A_1(h) + A_3(h) \leq A_2(h)$  with

$$\begin{aligned} A_1(h) &= \int_{\Omega} (b_1 T_k(u_1 - T_h(u_2)) + b_2 T_k(u_2 - T_h(u_1))) \, dx, \\ A_2(h) &= \int_{\Omega} (T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))) f \, dx, \\ A_3(h) &= \int_{\Omega} (\Lambda \nabla u_1 \cdot \nabla T_k(u_1 - T_h(u_2)) + \Lambda \nabla u_2 \cdot \nabla T_k(u_2 - T_h(u_1))) \, dx. \end{aligned}$$

Let us first study  $A_1(h)$ . We can write

$$\begin{aligned} & \int_{\Omega} b_1 T_k(u_1 - T_h(u_2)) \, dx \\ &= \int_{|u_2| \leq h} b_1 T_k(u_1 - u_2) \, dx + \int_{h < |u_2|} b_1 T_k(u_1 - T_h(u_2)) \, dx \\ &= \int_{|u_2| \leq h, |u_1| \leq h} b_1 T_k(u_1 - u_2) \, dx \\ & \quad + \int_{|u_2| \leq h, h < |u_1|} b_1 T_k(u_1 - u_2) \, dx + \int_{h < |u_2|} b_1 T_k(u_1 - T_h(u_2)) \, dx \\ &\geq \int_{|u_2| \leq h, |u_1| \leq h} b_1 T_k(u_1 - u_2) \, dx - \int_{|u_2| \leq h, h < |u_1|} |b_1| k \, dx - \int_{h < |u_2|} |b_1| k \, dx \\ &\geq \int_{|u_2| \leq h, |u_1| \leq h} b_1 T_k(u_1 - u_2) \, dx - \int_{h < |u_1|} |b_1| k \, dx - \int_{h < |u_2|} |b_1| k \, dx. \end{aligned}$$

Hence, applying the preceding computation to  $(b_1, u_1)$  and  $(b_2, u_2)$ , and defining

$$\chi_h(x) = 1 \text{ if } (h < |u_1|, h < |u_2|) \text{ and } 0 \text{ otherwise,}$$

we get

$$\begin{aligned} & \int_{\Omega} (b_1 T_k(u_1 - T_h(u_2)) + b_2 T_k(u_2 - T_h(u_1))) \, dx \\ & \geq \int_{|u_2| \leq h, |u_1| \leq h} (b_1 - b_2) T_k(u_1 - u_2) \, dx - \int_{\Omega} \chi_h (|b_1| + |b_2|) k \, dx. \end{aligned}$$

Using  $(b_1 - b_2) T_k(u_1 - u_2) \geq 0$  which is a consequence of Assumption (35), we conclude that

$$A_1(h) \geq - \int_{\Omega} \chi_h (|b_1| + |b_2|) k \, dx,$$

which shows, by dominated convergence, that

$$\liminf_{h \rightarrow +\infty} A_1(h) \geq 0. \quad (49)$$

Similar computations show that

$$A_2(h) \leq 2 \int_{\Omega} \chi_h |f| k \, dx,$$

and therefore that

$$\limsup_{h \rightarrow +\infty} A_2(h) \leq 0. \quad (50)$$

Following the analysis in [3], we obtain that

$$\liminf_{h \rightarrow +\infty} A_3(h) \geq \liminf_{h \rightarrow +\infty} \int_{|u_2| \leq h, |u_1| \leq h} \nabla(u_1 - u_2) \nabla T_k(u_1 - u_2) \, dx \geq \int_{\Omega} \nabla(u_1 - u_2) \nabla T_k(u_1 - u_2) \, dx. \quad (51)$$

Gathering (49), (50) and (51), we conclude that

$$\int_{\Omega} \nabla(u_1 - u_2) \nabla T_k(u_1 - u_2) \, dx = 0.$$

Since the above relation holds for all  $k > 0$ , we thus obtain that  $\nabla(u_1 - u_2) = 0$  a.e. Using that  $u_1$  and  $u_2$  belong to  $\mathcal{S}_N(\Omega)$ , we conclude that  $u_1 = u_2$  a.e.

Applying Lemma 6.2 for  $(b_1, u_1)$  and  $(b_2, u_2)$ , we get that, for all  $\phi \in C_c^\infty(\Omega)$ ,

$$\int_{\Omega} b_1 \phi \, dx = \int_{\Omega} b_2 \phi \, dx,$$

which implies that  $b_1 = b_2$  a.e. and concludes the proof of the uniqueness of the entropy solution. ■

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