



HAL
open science

An existence result for a class of nonlinear magnetorheological composites

Grigor Nika, Bogdan Vernescu

► **To cite this version:**

Grigor Nika, Bogdan Vernescu. An existence result for a class of nonlinear magnetorheological composites. 2022. hal-03668453

HAL Id: hal-03668453

<https://hal.science/hal-03668453v1>

Preprint submitted on 14 May 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 An existence result for a class of
2 nonlinear magnetorheological composites

3 Grigor Nika¹ and Bogdan Vernescu^{*2}

4 ¹Mathematics & Computer Science
5 Karlstad University
6 Universitetsgatan 2
7 65188 Karlstad, Sweden

8 grigor.nika@kau.se

9 ²Mathematical Sciences
10 Worcester Polytechnic Institute
11 100 Institute Rd.
12 Worcester, MA 01609 USA
13 vernescu@wpi.edu

14 December 24, 2021

15 **Abstract**

16 We prove existence of a weak solution for a nonlinear, multi-physics, multi-scale
17 problem of magnetorheological suspensions introduced in Nika & Vernescu (Z. Angew.
18 Math. Phys., 71(1):1–19, '20) in the three dimensional setting. The hybrid model
19 couples the Stokes' equation with the quasi-static Maxwell's equations through the
20 Lorentz force and the Maxwell stress tensor. The proof of existence is based on: i) the
21 augmented variational formulation of Maxwell's equations, ii) the definition of a new
22 function space for the magnetic induction and the proof of a Poincaré type inequality,
23 iii) the Altman-Shinbrot fixed point theorem when the magnetic Reynold's number,
24 R_m , is small.

25 **MSC 2020:** 35A15, 35J60, 74F10

26 **Keywords:** Magnetorheological fluids, Altman–Shinbrot fixed point theory, Maxwell's
27 equations, augmented variational formulation

28 **1 Introduction**

29 The use of suspensions of rigid particles as *smart materials* is of great interest, as their
30 rheological properties can be reversibly changed by the interaction with a magnetic or elec-
31 tric field. The ability of magnetorheological fluids [PV00], [Ver02], [Rab48] to modify their

*Corresponding author: vernescu@wpi.edu

32 rheology from liquid to a semi-solid state under the presence of an external magnetic field
33 in a matter of milliseconds make them desirable in many industries [LFS01, dVKHA11].

34 The modelling of magnetorheological and electrorheological fluids has been mostly ex-
35 plored from thermodynamically consistent, phenomenological point of view ([BD05, Ruz00,
36 RR01]). While this approach is well founded, it does not allow for explicit control of the
37 material properties. The theory of periodic homogenization, specifically designed to treat
38 problems for multiscale heterogeneous materials, allows to derive the effective properties of
39 the aforementioned heterogeneous materials based on the properties of the constituents at
40 the microscale, allowing thus for the design of materials with specified properties [NC19],
41 [ACN20].

42 The derivation of effective models of magnetorheological and electrorheological fluids us-
43 ing homogenization has been carried out in [L85, LH88, PV00, Ver02]. The microscale
44 problems used to derive the effective models in these works were one-way coupled systems
45 of Stokes or Navier-Stokes equations with quasi-static Maxwell's equations. In [NV20] a
46 fully coupled model between Stokes' equations and the quasi-static Maxwell's equations
47 through the Lorentz force was used to derive a class of nonlinear magnetorheological com-
48 posites. Numerical results, for this model, showed that particle-chain microstructures have
49 a non-linear contribution to the magnetorheological effect. Furthermore, in [NV21] it was
50 shown numerically that for particles of fixed volume fraction there is a decrease in the
51 strength of the magnetorheological effect as the surface-to-volume ratio increases.

52 In this work we prove existence of a weak solution to the model introduced in [NV20] in the
53 three dimensional setting, describing the stationary flow of rigid, magnetizable particles
54 in a non-conducting fluid, distributed periodically with period ϵ under the influence of
55 a magnetic field. The proof of existence is based on the Altman-Shinbrot fixed point
56 theorem [Alt57], [Shi64] and relies in the augmented variational formulation of Maxwell's
57 equations when the magnetic Reynold's number, R_m , is small. The use of an augmented
58 variational formulation is motivated by the fact that the magnetic induction does not
59 possess a full weak derivative in L^2 , rather, due to the material properties, the derivatives
60 are split into a divergence part and a skew-symmetric part that, respectively, belong in
61 L^2 . Hence, traditional fixed point arguments, like Leray-Schauder which were employed
62 by O. Ladyzhenskaya to show existence for the nonlinear stationary Navier-Stokes are not
63 available to us since the magnetic induction does not possess full weak derivatives. In
64 contrast, the Altman-Shinbrot fixed point theorem requires that the defined operator be
65 continuous only in the weak topology of the underlying space. Thus, as a consequence,
66 we introduce a new function space for the magnetic induction and prove a Poincaré type
67 inequality for this new space using the div-curl lemma.

68 The article is organized as follows: In Section 2 we introduce the model describing the
69 suspension in the two component domain. Section 3 introduces the function spaces for the
70 variational framework of the problem and certain auxiliary results regarding embeddings
71 and Poincaré's inequality while in Section 4 we write down the augmented variational
72 formulation and prove its equivalence to the strong form a.e. in the domain Ω . This is
73 done in Theorem 4.1. Moreover, we define the function space \mathcal{Y}^ϵ , where the magnetic
74 induction belongs and prove that it is a Hilbert space. Furthermore, we prove a Poincaré
75 type inequality for \mathcal{Y}^ϵ in Theorem 3.1 using the global div-curl lemma ([Tar79], [Mur78]).

76 Section 5 is dedicated to the existence proof. The proof relies on the Altman-Shinbrot
 77 fixed point theorem [Alt57], [Shi64] when R_m is small and the main result of this section is
 78 stated in Theorem 5.2. Finally, Section 6 is devoted to concluding remarks and comments.

79 Notation

80 Throughout the paper we will make use of the following notation:

i In addition to the standard Sobolev space $H^1(\Omega)$ we define the following spaces:

$$H_{\Gamma_0}^1(\Omega) = \left\{ w \in H^1(\Omega) \mid w|_{\Gamma_0} = 0 \text{ on } \Gamma_0 \right\},$$

$$H(\text{div}; \Omega) = \left\{ \mathbf{w} \in L^2(\Omega) \mid \text{div } \mathbf{w} \in L^2(\Omega) \right\},$$

$$H(\text{curl}; \Omega) = \left\{ \mathbf{w} \in L^2(\Omega) \mid \text{curl } \mathbf{w} \in L^2(\Omega; \mathbb{R}^d) \right\},$$

81 where the div and curl operators are understood in the sense of distributions and $w|_{\Gamma_0}$
 82 is the usual trace operator. Naturally, the above spaces are Hilbert spaces when they
 83 are equipped with their corresponding graph norms. Moreover, we will make use of
 84 fractional Sobolev spaces defined e.g. in [LM72].

85 ii $\chi_\Omega(\mathbf{x})$ is the indicator function over some set Ω such that,

$$\chi_\Omega(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

86 iii Throughout the article we employ the Einstein summation notation of repeated indices
 87 while the expressions “mes” and “mes_{d-1}” stand for the Lebesgue measure and for the
 88 $d - 1$ surface measure.

89 iv We restrict ourselves to the three dimensional setting, $d = 3$. Most of the results
 90 presented remain valid for two spatial dimensions as well, however, in two spatial
 91 dimensions many of the challenges encountered, particularly regarding the variational
 92 formulation of Maxwell’s equations, simplify considerably.

93 2 The model

94 Assume Ω is an open, bounded, multiply connected subset of \mathbb{R}^d lying in vacuum. Let
 95 $Y = [-1/2, 1/2)^d$ be the unit cube in \mathbb{R}^d , and \mathbb{Z}^d be the set of all d -dimensional vectors
 96 with integer components. For every positive ϵ , let $N(\epsilon)$ be the set of all points $\ell \in \mathbb{Z}^d$ such
 97 that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N(\epsilon)|$ their total number. Let Y_1
 98 be the closure of an open, connected set with sufficiently smooth boundary S , compactly
 99 included in Y and $Y_2 := Y \setminus \bar{Y}_1$. For every $\epsilon > 0$ and $\ell \in N(\epsilon)$ we consider the set
 100 $Y_{i\epsilon}^\ell \subset \subset \epsilon(\ell + Y)$, where $Y_{i\epsilon}^\ell = \epsilon(\ell + Y_i)$ for $i = 1, 2$. The set $Y_{1\epsilon}^\ell$ represents one of the rigid
 101 particles suspended in the fluid, and $S_\ell^c = \epsilon(\ell + S)$ denotes its surface (see Fig. 1).

102 We now define the following subsets of Ω : $\Omega_{1\epsilon} = \bigcup_{\ell \in N(\epsilon)} Y_{1\epsilon}^\ell$, $\Omega_{2\epsilon} = \Omega \setminus \overline{\Omega_{1\epsilon}}$. Here, $\Omega_{1\epsilon}$ is
103 the domain occupied by the rigid particles and $\Omega_{2\epsilon}$ the domain occupied by the ambient
104 surrounding fluid of viscosity $\nu \equiv 1$. We denote by $\Gamma := \partial\Omega$ the boundary of Ω . By Γ_0
105 we denote the exterior component of Γ and by S_ϵ^ℓ , $\ell = 1, \dots, N(\epsilon)$ the remaining finite
106 number of components. The vectors \mathbf{n} and \mathbf{n}^ℓ indicate the unit normal on Γ_0 and the unit
107 normal to S_ϵ^ℓ respectively with both unit normals pointing outwards. Moreover, by $[[\cdot]]$
108 we indicate the jump discontinuity between the fluid and the rigid part and by $\langle \cdot | \cdot \rangle$ the
109 duality product.

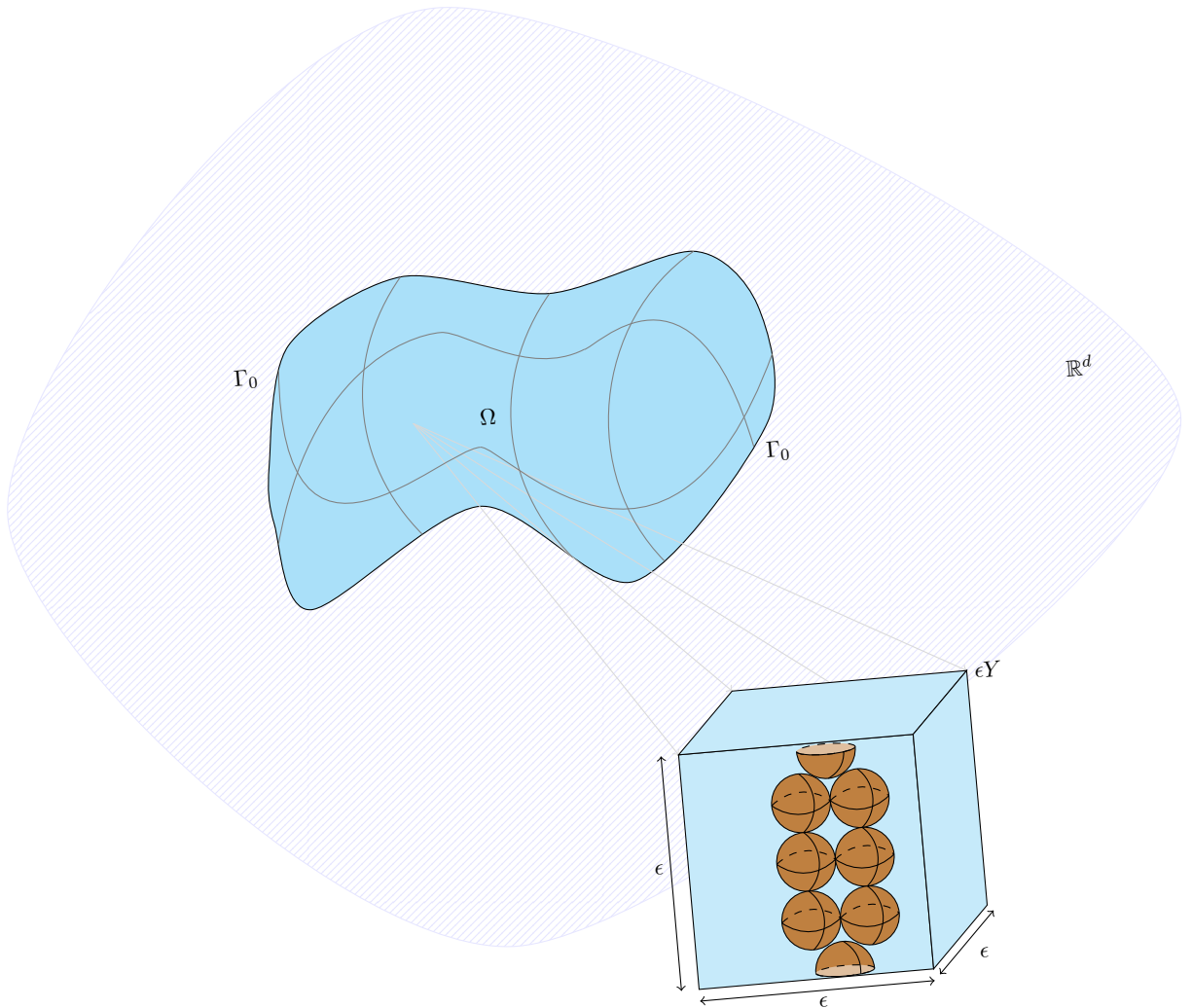


Figure 1: Schematic of the periodic suspension of rigid magnetizable particles in a non-conducting, non-magnetizable fluid. The periodic cell ϵY contains a potential geometric realization of a magnetizable, spherical, rigid particles in a chain structure.

110 The magnetorheological problem considered in [NV20] after non-dimensionalizing and as-
111 suming that the flow is at low Reynolds numbers was the following,

$$\begin{aligned}
-\operatorname{div}(\boldsymbol{\sigma}^\epsilon) &= \mathbf{0} && \text{in } \Omega_{2\epsilon}, \\
\boldsymbol{\sigma}^\epsilon &= 2e(\mathbf{v}^\epsilon) - p^\epsilon \mathbf{I} && \text{in } \Omega_{2\epsilon}, \\
\operatorname{div}(\mathbf{v}^\epsilon) &= 0 && \text{in } \Omega_{2\epsilon}, \\
e(\mathbf{v}^\epsilon) &= 0 && \text{in } \Omega_{1\epsilon},
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\operatorname{div}(\mathbf{B}^\epsilon) &= 0 && \text{in } \Omega, \\
\operatorname{curl}(\widehat{\mu}^\epsilon \mathbf{B}^\epsilon) &= \mathbf{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \chi_{\Omega_{1\epsilon}} && \text{in } \Omega.
\end{aligned} \tag{2.2}$$

112 with compatibility conditions,

$$\operatorname{div}(\mathbf{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \chi_{\Omega_{1\epsilon}}) = 0 \text{ in } \Omega, \quad \langle \mathbf{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \cdot \mathbf{n}^\epsilon \mid 1 \rangle_{H^{1/2}(S_\ell^\epsilon), H^{1/2}(S_\ell^\epsilon)} = 0, \tag{2.3}$$

113 and interface and exterior boundary conditions,

$$\llbracket \mathbf{v}^\epsilon \rrbracket = \mathbf{0} \text{ on } S_\ell^\epsilon, \quad \mathbf{v}^\epsilon = \mathbf{0}, \quad \mathbf{B}^\epsilon \cdot \mathbf{n} = \mathbf{c} \cdot \mathbf{n} \text{ on } \Gamma_0. \tag{2.4}$$

114 When the MR fluid is submitted to a magnetic field, the rigid particles are subjected to a
115 force that makes them behave like a dipole aligned in the direction of the magnetic field.
116 This force can be written in the form,

$$\mathbf{F}^\epsilon := -\frac{1}{2} |\mathbf{H}^\epsilon|^2 \nabla \mu^\epsilon,$$

117 where $|\cdot|$ represents the standard Euclidean norm. The force can also be written in terms
118 of the Maxwell stress,

$$\tau_{ij}^\epsilon = \widehat{\mu}^\epsilon B_i^\epsilon B_j^\epsilon - \frac{1}{2} \widehat{\mu}^\epsilon B_k^\epsilon B_k^\epsilon \delta_{ij}, \tag{2.5}$$

119 as $\mathbf{F}^\epsilon = \operatorname{div}(\boldsymbol{\tau}^\epsilon) - \mathbf{B}^\epsilon \times \operatorname{curl}(\widehat{\mu}^\epsilon \mathbf{B}^\epsilon)$. Since the magnetic permeability is considered constant
120 in each phase, it follows that the force is zero in each phase. Therefore, we deduce that

$$\operatorname{div}(\boldsymbol{\tau}^\epsilon) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_{2\epsilon}, \\ \mathbf{B}^\epsilon \times \operatorname{curl}(\widehat{\mu}^\epsilon \mathbf{B}^\epsilon) & \text{if } \mathbf{x} \in \Omega_{1\epsilon}. \end{cases} \tag{2.6}$$

121 Lastly, we remark that unlike the viscous stress $\boldsymbol{\sigma}^\epsilon$, the Maxwell stress is present in the
122 entire domain Ω . Hence, we can write the balance of forces and torques in each particle
123 as,

$$\begin{aligned}
0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \mathbf{n}^\epsilon ds + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^\epsilon \mathbf{n}^\epsilon \rrbracket ds - \alpha \int_{T_\ell^\epsilon} \mathbf{B}^\epsilon \times \text{curl}(\widehat{\mu}^\epsilon \mathbf{B}^\epsilon) d\mathbf{x}, \\
0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \mathbf{n}^\epsilon \times (\mathbf{x} - \mathbf{x}_c^\ell) ds + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^\epsilon \mathbf{n}^\epsilon \rrbracket \times (\mathbf{x} - \mathbf{x}_c^\ell) ds \\
&\quad - \alpha \int_{T_\ell^\epsilon} (\mathbf{B}^\epsilon \times \text{curl}(\widehat{\mu}^\epsilon \mathbf{B}^\epsilon)) \times (\mathbf{x} - \mathbf{x}_c^\ell) d\mathbf{x}.
\end{aligned} \tag{2.7}$$

124 Here \mathbf{v}^ϵ represents the fluid velocity field, p^ϵ the pressure, $e(\mathbf{v}^\epsilon)$ the strain rate, \mathbf{n}^ϵ the
125 unit normal to S_ℓ^ϵ , \mathbf{n} is the unit normal to Γ_0 , \mathbf{B}^ϵ is the magnetic induction and it is
126 related to the magnetic field \mathbf{H}^ϵ by $\mathbf{B}^\epsilon = \mu^\epsilon \mathbf{H}^\epsilon$, where $0 < \mu^\epsilon$ is the magnetic permeability
127 of the material and $\widehat{\mu}^\epsilon = (\mu^\epsilon)^{-1}$, \mathbf{x}_c^ℓ is the center of mass of the rigid particle T_ℓ^ϵ , α
128 is the Alfven number, and R_m is the magnetic Reynolds number. Moreover, $\mathbf{c} \cdot \mathbf{n}$ is a
129 transmission condition on the outer boundary indicating that a magnetic field \mathbf{c} exterior
130 to the domain Ω is present. Finally, we remark that condition $e(\mathbf{v}^\epsilon) = 0$ in $\Omega_{1\epsilon}$ means that
131 $\mathbf{v}^\epsilon = \mathbf{V}^{\ell,\epsilon} + \boldsymbol{\omega}^{\ell,\epsilon} \times (\mathbf{x} - \mathbf{x}_c^\ell)$ in $\Omega_{1\epsilon}$ where $\mathbf{V}^{\ell,\epsilon}$ is a constant translational velocity and $\boldsymbol{\omega}^{\ell,\epsilon}$
132 is a constant rotational velocity for each particle.

133 3 Function spaces and auxiliary results

134 We begin with a small extension of a result in [DL72] regarding a non-homogeneous domain
135 containing sub-domains of different piece-wise constant electric permittivity, say, η_i , $i =$
136 $1, \dots, \kappa$, where κ is the number of subdomains. These domains occur naturally in problems
137 of electromagnetism (see [DL72]).

138 **Proposition 3.1.** *Let $\mathcal{O} \subset \mathbb{R}^d$ be any open, bounded, multiply connected set with boundary*
139 *$\Gamma := \partial\mathcal{O}$ of class C^2 . The exterior boundary will be denoted by Γ_0 and by Γ_j , $j =$*
140 *$1, \dots, \kappa - 1$, the other components of Γ .*

141 *Define \mathcal{Y} to be the Hilbert space of vector fields,*

$$\mathcal{Y} := \left\{ \mathbf{v} \in L^2(\mathcal{O}; \mathbb{R}^d) \mid \text{div } \mathbf{v} \in L^2(\mathcal{O}), \text{curl}(\widehat{\mu} \mathbf{v}) \in L^2(\mathcal{O}; \mathbb{R}^d), \right. \\
\left. \mathbf{v} \cdot \mathbf{n} \in H^{1/2}(\Gamma_0) \right\}, \tag{3.1}$$

142 *for the norm,*

$$\|\mathbf{v}\|_{\mathcal{Y}} := \|\mathbf{v}\|_{L^2(\mathcal{O}; \mathbb{R}^d)} + \|\text{div } \mathbf{v}\|_{L^2(\mathcal{O})} + \|\text{curl}(\widehat{\mu} \mathbf{v})\|_{L^2(\mathcal{O}; \mathbb{R}^d)} + \|\mathbf{v} \cdot \mathbf{n}\|_{H^{1/2}(\Gamma_0)}, \tag{3.2}$$

143 *then for all $\mathbf{v} \in \mathcal{Y}$ we have, $\mathbf{v}|_{\mathcal{O}_i} \in H^1(\mathcal{O}_i; \mathbb{R}^d)$ for $i = 1, \dots, \kappa$ where $\mathbf{v}|_{\mathcal{O}_i}$ is the restriction*
144 *of \mathbf{v} to \mathcal{O}_i , $0 < \mu = \widehat{\mu}^{-1}$ is constant in \mathcal{O}_i and*

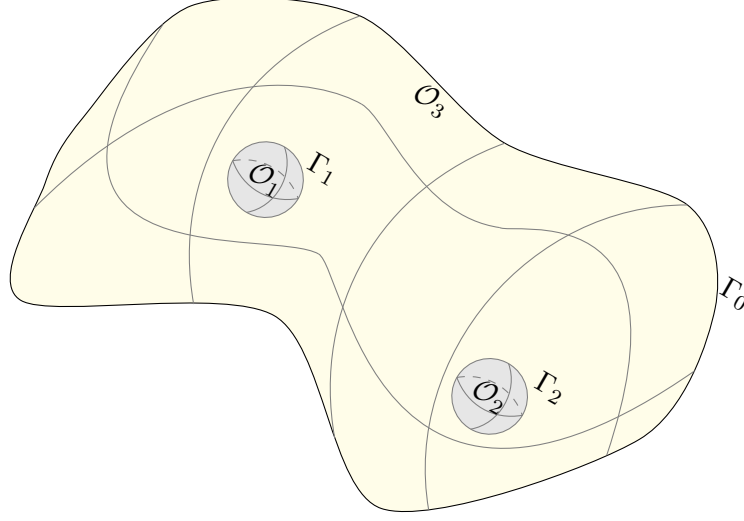


Figure 2: A schematic of a finite, multiply connected region \mathcal{O} containing two sub-regions. The open set \mathcal{O} is defined as $\mathcal{O} := \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \Gamma_1 \cup \Gamma_2$.

$$\left\| \mathbf{v}|_{\mathcal{O}_i} \right\|_{H^1(\mathcal{O}_i; \mathbb{R}^d)} \leq C_{\mathcal{O}_i} \|\mathbf{v}\|_{\mathcal{Y}}. \quad (3.3)$$

145 *Proof.* Let $\mathbf{v} \in \mathcal{Y}$ and define $\pi \in H^1(\mathcal{O})$ as the solution to the following Neumann problem,

$$\begin{aligned} \operatorname{div}(\eta \nabla \pi) &= \operatorname{div} \mathbf{v} \text{ in } \mathcal{O}, \\ \llbracket \eta \partial_n \pi \rrbracket &= 0 \text{ on } \Gamma_j, j = 1, \dots, \kappa - 1, \\ \eta \partial_n \pi &= \mathbf{v} \cdot \mathbf{n} \text{ on } \Gamma_0. \end{aligned} \quad (3.4)$$

146 Take $\mathbf{u} = \mathbf{v} - \eta \nabla \pi$ and note that $\operatorname{div} \mathbf{u} = 0$ in \mathcal{O} , $\operatorname{curl}(\widehat{\eta} \mathbf{u}) = \operatorname{curl}(\widehat{\eta} \mathbf{v}) \in L^2(\mathcal{O}; \mathbb{R}^d)$, and
147 $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_0 . Then $\mathbf{u}|_{\mathcal{O}_i} \in H^1(\mathcal{O}_i; \mathbb{R}^d)$ by [DL72, Theorem 6.2, page 355].

148 It remains to prove that $\pi|_{\mathcal{O}_i} \in H^2(\mathcal{O}_i)$ and inequality (3.3). This is the result of [LU68,
149 Chap. 3, Sec. 16, Eq. 16.12, pg. 212]. For our case, we include a sketch of the proof for
150 the higher regularity up to the interphase boundary for completion. Set $f := \operatorname{div} \mathbf{u}$ and
151 $g := \mathbf{u} \cdot \mathbf{n}$. Define a smooth function ξ such that $\xi \equiv 0$ in \mathcal{O}_i , $\xi \equiv 1$ in a neighbourhood of
152 Γ_0 , and $\xi \in [0, 1]$. Since $g \in H^{1/2}(\Gamma_0)$ there exists a $\Phi \in H^2(\mathcal{O})$ such that $\partial_n \Phi = g$ on Γ_0 .
153 Set $q := \pi - \xi \Phi$. From the variational formulation of (3.4) we have,

$$\int_{\mathcal{O}} \eta \nabla q \cdot \nabla \bar{q} \, d\mathbf{x} = \int_{\mathcal{O}} f \bar{q} \, d\mathbf{x} + \int_{\mathcal{O}} \operatorname{div}(\eta \nabla(\xi \Phi)) \bar{q} \, d\mathbf{x} \quad \forall \bar{q} \in H^1(\mathcal{O}). \quad (3.5)$$

154 Fix $V \subset\subset \mathcal{O}$ and choose W such that $V \subset\subset W \subset\subset \mathcal{O}$. Select a function $\chi \equiv 1$ in V ,

155 $\chi \equiv 0$ in $\mathbb{R}^d \setminus W$, and $\chi \in [0, 1]$. Set $\bar{q} := D_{-h}^k(\chi^2 D_h^k q)$ where D_h^k is the usual difference
 156 quotient. Using properties of difference quotients we can obtain for (3.5),

$$\begin{aligned} \int_V |D_h^k \nabla q|^2 d\mathbf{x} &\leq \int_{\mathcal{O}} \chi^2 |D_h^k \nabla q|^2 d\mathbf{x} \\ &\leq c \left(\int_{\mathcal{O}} (|f|^2 + |\nabla q|^2) d\mathbf{x} + \|\Phi\|_{H^2(\mathcal{O})}^2 \right) \end{aligned} \quad (3.6)$$

157 for $k = 1, \dots, d-1$. Passing to the limit as $h \rightarrow 0$ we obtain $\partial_{x_i x_j}^2 q \in L^2(V)$ provided
 158 that i and j are not both equal to d with the global bound in (3.6). For points belonging
 159 on the interphase Γ_i we can use local charts to straighten and rotate Γ_i so that locally it
 160 looks like a hyperplane. In doing so, since the difference quotient operator is a tangential
 161 operator, D_h^k will always remain on one side of the interphase where no jump occurs on
 162 η . Therefore, since $\eta_i \Delta q = f + \operatorname{div}(\eta \nabla(\xi \Phi))$ in \mathcal{O}_i and by assumption $\eta_i > 0$ we have
 163 $\partial_{x_d x_d}^2 q \leq c(f + \operatorname{div}(\eta \nabla(\xi \Phi)) - \sum_{i=1}^{d-1} \partial_{x_i x_i}^2 q)$. Thus, we have control over all the second
 164 derivatives of q and moreover we have the global bound,

$$\|q\|_{H^2(V \cap \mathcal{O}_i)} \leq c \left(\|f\|_{L^2(\mathcal{O})} + \|\nabla q\|_{L^2(\mathcal{O}; \mathbb{R}^d)} + \|\Phi\|_{H^2(\mathcal{O})} \right), \quad (3.7)$$

165 for $i = 1, \dots, k$.

166 One can extend (3.7) up to the boundary Γ_0 , since it is of class C^2 , using standard elliptic
 167 regularity theory and therefore obtain

$$\|q\|_{H^2(\mathcal{O}_i)} \leq c \left(\|f\|_{L^2(\mathcal{O})} + \|\nabla q\|_{L^2(\mathcal{O}; \mathbb{R}^d)} + \|\Phi\|_{H^2(\mathcal{O})} \right) \text{ for } i = 1, \dots, k. \quad (3.8)$$

168 Using (3.9) we can obtain a bound on π for $i = 1, \dots, \kappa$,

$$\begin{aligned} \|\pi\|_{H^2(\mathcal{O}_i)} &\leq \|q\|_{H^2(\mathcal{O}_i)} + \|\xi \Phi\|_{H^2(\mathcal{O}_i)} \\ &\leq c \left(\|f\|_{L^2(\mathcal{O})} + \|\nabla q\|_{L^2(\mathcal{O}; \mathbb{R}^d)} + \|\Phi\|_{H^2(\mathcal{O})} \right) \\ &\leq c \left(\|f\|_{L^2(\mathcal{O})} + \|g\|_{H^{1/2}(\Gamma_0)} \right) \end{aligned} \quad (3.9)$$

169 where the last inequality is a result of standard estimates of q in $H^1(\mathcal{O})$ resulting from
 170 (3.5) and the trace theorem (see [Gal11, Thm. II.4.4, pg. 68]). Hence, inequality (3.3)
 171 follows. \square

172 **Lemma 3.1.** *Let $\mathbf{v} \in \{\mathbf{w} \in H(\operatorname{div}, \mathcal{O}) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \mathcal{O}\}$ and define $\mathcal{Y}_{\Gamma_0} = \{\mathbf{w} \in \mathcal{Y} \mid$
 173 $\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_0\}$, $\mathcal{Y}_{\Gamma_0}^0 = \{\mathbf{w} \in \mathcal{Y}_{\Gamma_0} \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \mathcal{O}\}$. There exists a vector potential
 174 $\mathbf{w} \in \mathcal{Y}_{\Gamma_0}^0$ such that*

$$\operatorname{curl}(\hat{\mu} \mathbf{w}) = \mathbf{v}, \quad \|\hat{\mu} \mathbf{w}\|_{\mathcal{Y}_{\Gamma_0}^0} \leq c \|\mathbf{v}\|_{L^2(\mathcal{O}; \mathbb{R}^d)}, \quad (3.10)$$

175 where $c := c(\mathcal{O})$. Moreover, there exists $\xi \in (3, 6]$ such that

$$\widehat{\mu} \mathbf{w} \in L^\xi(\mathcal{O}; \mathbb{R}^d) \text{ and } \|\widehat{\mu} \mathbf{w}\|_{L^\xi(\mathcal{O}; \mathbb{R}^d)} \leq c \|\mathbf{v}\|_{L^2(\mathcal{O}; \mathbb{R}^d)}. \quad (3.11)$$

176 *Proof.* This is [Dru, Prop. 2.2]. \square

177 **Proposition 3.2.** *The space \mathcal{Y} is embedded into $L^q(\mathcal{O}, \mathbb{R}^d)$ for $q \in [1, 6]$ with the embed-*
 178 *ding being continuous.*

179 *Proof.* Assume $\mathbf{z} \in \mathcal{Y}$ and define $\mathbf{g} := \text{curl}(\widehat{\eta} \mathbf{z})$. Then $\text{div } \mathbf{g} = 0$ in the sense of distributions
 180 in \mathcal{O} and by Lemma 3.1 there exists a $\mathbf{w} \in \mathcal{Y}_{\Gamma_0}^0$ such that $\text{curl}(\widehat{\eta} \mathbf{w}) = \mathbf{g}$. Hence, we have
 181 $\widehat{\eta} \mathbf{z} - \widehat{\eta} \mathbf{w} = \nabla \pi$ where π is the solution to the following problem:

182 Find $\pi \in H^1(\mathcal{O})$ such that

$$\int_{\mathcal{O}} \eta \nabla \pi \cdot \nabla \phi \, d\mathbf{x} = F(\phi) \quad \forall \phi \in H^1(\mathcal{O}) \quad (3.12)$$

183 where $F(\phi) := \int_{\mathcal{O}} (\text{div } \mathbf{z}) \phi \, d\mathbf{x} + \int_{\Gamma_0} (\mathbf{z} \cdot \mathbf{n}) \phi \, ds$. Evidently, $F \in H^{-1}(\mathcal{O})$. Hence, by [ERS07],
 184 there exists $\xi \in (3, 6]$ such that $\pi \in W^{1, \xi}(\mathcal{O})$. Combining this result with Lemma 3.1 the
 185 result follows. Additionally, one obtains the bound

$$\|\pi\|_{W^{1, \xi}(\mathcal{O})} \leq c \left(\|\text{div } \mathbf{z}\|_{L^2(\mathcal{O})} + \|\mathbf{z} \cdot \mathbf{n}\|_{H^{1/2}(\Gamma_0)} \right). \quad (3.13)$$

186 Recalling that $\widehat{\eta} \mathbf{z} - \widehat{\eta} \mathbf{w} = \nabla \pi$ and using (3.13) and then (3.11) we obtain,

$$\begin{aligned} \|\mathbf{z}\|_{L^\xi(\mathcal{O}, \mathbb{R}^d)} &\leq \|\nabla \pi\|_{L^\xi(\mathcal{O}; \mathbb{R}^d)} + \|\widehat{\eta} \mathbf{w}\|_{L^\xi(\mathcal{O}; \mathbb{R}^d)} \\ &\leq c \left(\|\text{div } \mathbf{z}\|_{L^2(\mathcal{O})} + \|\mathbf{z} \cdot \mathbf{n}\|_{H^{1/2}(\Gamma_0)} + \|\text{curl}(\widehat{\eta} \mathbf{z})\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \right) \\ &\leq c \|\mathbf{z}\|_{\mathcal{Y}}. \end{aligned} \quad (3.14)$$

187 It is possible to pick $\xi = 6$ as an exponent due to the fact that we have assumed Γ_0 has C^2
 188 regularity. For more details the reader can consult the works in [Dru, Prop. 2.6 (2)]. \square

189 **Proposition 3.3.** *Define a new norm on \mathcal{Y} by*

$$[\mathbf{v}]_{\mathcal{Y}} := \|\text{div } \mathbf{v}\|_{L^2(\mathcal{O})} + \|\text{curl}(\widehat{\mu} \mathbf{v})\|_{L^2(\mathcal{O}; \mathbb{R}^d)} + \|\mathbf{v} \cdot \mathbf{n}\|_{H^{1/2}(\Gamma_0)}, \quad (3.15)$$

190 *then \mathcal{Y} is also a Hilbert space with norm $[\cdot]_{\mathcal{Y}}$.*

191 *Proof.* It is evident that if $\mathbf{v} = \mathbf{0}$ then $[\mathbf{v}]_{\mathcal{Y}} = 0$. For the other direction we have, $[\mathbf{v}]_{\mathcal{Y}} = 0$
 192 implies that $\text{curl}(\widehat{\mu} \mathbf{v}) = \mathbf{0}$ in Ω . Hence, \mathbf{v} can be written as, $\mathbf{v} = -\mu \nabla \theta$. Thus, we get
 193 that θ satisfies the following elliptic problem,

$$\begin{aligned} -\operatorname{div}(\mu \nabla \theta) &= 0 \text{ in } \mathcal{O}, \\ \mu \nabla \theta \cdot \mathbf{n} &= 0 \text{ on } \Gamma_0. \end{aligned} \tag{3.16}$$

194 The above problem has a unique solution, $\theta = 0$ (if we fix constants) and the result
 195 follows. We remark that \mathcal{Y} is complete which follows by similar arguments used to show
 196 the completeness of the classical $H(\operatorname{div}; \mathcal{O})$ or $H(\operatorname{curl}; \mathcal{O})$ spaces (see [Tem84]). \square

197 **Theorem 3.1** (Poincaré type inequality for $(\mathcal{Y}, [\cdot]_{\mathcal{Y}})$). *There exists a constant, $c := c(\mathcal{O})$,*
 198 *such that*

$$\|\mathbf{w}\|_{L^2(\mathcal{O}; \mathbb{R}^d)} \leq c [\mathbf{w}]_{\mathcal{Y}}, \tag{3.17}$$

199 for all $\mathbf{w} \in \mathcal{Y}$.

200 *Proof.* We proceed by contradiction. If (3.17) is false then there exists a sequence $\mathbf{w}_n \in \mathcal{Y}$,
 201 such that

$$\|\mathbf{w}_n\|_{L^2(\mathcal{O}; \mathbb{R}^d)} > n [\mathbf{w}_n]_{\mathcal{Y}} \text{ for all } n \in \mathbb{N}. \tag{3.18}$$

202 We can suppose that $\|\mathbf{w}_n\|_{L^2(\mathcal{O}; \mathbb{R}^d)} = 1$. Then as $n \rightarrow \infty$ and up to a, non-relabeled,
 203 subsequence we have:

$$\mathbf{w}_n \rightharpoonup \mathbf{w} \text{ in } H(\operatorname{div}; \mathcal{O}), \quad \widehat{\mu} \mathbf{w}_n \rightharpoonup \widehat{\mu} \mathbf{w} \text{ in } H(\operatorname{curl}; \mathcal{O}), \tag{3.19}$$

$$\begin{aligned} \operatorname{div} \mathbf{w}_n &\rightarrow 0 \text{ in } L^2(\mathcal{O}), \\ \operatorname{curl}(\widehat{\mu} \mathbf{w}_n) &\rightarrow 0 \text{ in } L^2(\mathcal{O}; \mathbb{R}^d), \\ \mathbf{w}_n \cdot \mathbf{n} &\rightarrow 0 \text{ in } H^{1/2}(\Gamma_0). \end{aligned} \tag{3.20}$$

204 Decompose $\widehat{\mu} \mathbf{w}_n$ using Helmholtz decomposition as $\widehat{\mu} \mathbf{w}_n = \nabla p_n + \operatorname{curl} \mathbf{f}_n$. Denote by
 205 $\mathbf{g}_n := \operatorname{curl} \mathbf{f}_n$, then $\mathbf{g}_n \in L^2(\mathcal{O}; \mathbb{R}^d)$, $\operatorname{div} \mathbf{g}_n = 0$, $\operatorname{curl} \mathbf{g}_n \in L^2(\mathcal{O}; \mathbb{R}^d)$, and $\mathbf{g}_n \cdot \mathbf{n} = 0$ on Γ_0
 206 in the sense of distributions. By theorem, [FT78, Prop. 1.4, pg. 41] or [DL72, Thm 6.1,
 207 pg. 354] $\mathbf{g}_n \in H^1(\mathcal{O}; \mathbb{R}^d)$ and is bounded uniformly. By the compact embedding of H^1
 208 into L^2 , $\mathbf{g}_n \rightarrow \mathbf{g}$ in $L^2(\mathcal{O}; \mathbb{R}^d)$ or $\operatorname{curl} \mathbf{f}_n \rightarrow \operatorname{curl} \mathbf{f}$ in $L^2(\mathcal{O}; \mathbb{R}^d)$. Hence, if $\widehat{\mu}_0 := \min_i \widehat{\mu}_i$
 209 denotes the the smallest of the $\widehat{\mu}_i$, we have

$$\begin{aligned} 0 < \widehat{\mu}_0 &\leq \int_{\mathcal{O}} \widehat{\mu} \mathbf{w}_n \cdot \mathbf{w}_n \, d\mathbf{x} \\ &= \int_{\mathcal{O}} \nabla p_n \cdot \mathbf{w}_n \, d\mathbf{x} + \int_{\mathcal{O}} \operatorname{curl} \mathbf{f}_n \cdot \mathbf{w}_n \, d\mathbf{x} \\ &= \int_{\Gamma_0} p_n \mathbf{w}_n \cdot \mathbf{n} \, ds - \int_{\mathcal{O}} p_n \cdot \operatorname{div} \mathbf{w}_n \, d\mathbf{x} + \int_{\mathcal{O}} \operatorname{curl} \mathbf{f}_n \cdot \mathbf{w}_n \, d\mathbf{x}. \end{aligned} \tag{3.21}$$

210 Since p_n remains bounded in $H^1(\mathcal{O})$, $\text{curl } \mathbf{f}_n \rightarrow \text{curl } \mathbf{f}$ in $L^2(\mathcal{O}; \mathbb{R}^d)$, and using (3.19) and
 211 (3.20) we can pass to the limit as $n \rightarrow \infty$. Noting further from (3.20) that the $\text{curl } \mathbf{f} = \mathbf{0}$
 212 in \mathcal{O} we obtain that $0 < \widehat{\mu}_0 = 0$, which is a contradiction.

213 Sometimes this is referred to as the global div-curl lemma (see [Sch18]). □

214 **Corollary 3.1.** *The norms $\|\cdot\|_{\mathcal{Y}}$ and $[\cdot]_{\mathcal{Y}}$ are equivalent norms on \mathcal{Y}*

215 *Proof.* It is a consequence of Theorem 3.1. □

216 4 Augmented variational formulation

217 4.1 Assumptions

218 We frame the magnetorheological model (2.1)–(2.2) under the following general **Assump-**
 219 **tions (A)**.

- 220 • We assume Ω is a bounded, multiply connected domain such that $\text{mes}_{d-1}(\Gamma) > 0$
 221 and $\text{mes}_{d-1}(S_\ell^\epsilon) > 0$ for $\ell = 1, \dots, N(\epsilon)$.
- 222 • Γ_0 and S_ℓ^ϵ are surfaces of class C^2 , $S_p^\epsilon \cap S_q^\epsilon = \emptyset$ for $p, q = 1, \dots, N(\epsilon)$ with $p \neq q$, and
 223 $\Gamma_0 \cap S_\ell^\epsilon = \emptyset$ for every $\ell = 1, \dots, N(\epsilon)$.
- 224 • The magnetic permeability of the magnetorheological fluid, μ^ϵ , is assumed be a piece-
 225 wise constant function with values $\mu^\epsilon(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_{1\epsilon}$ and $\mu^\epsilon(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_{2\epsilon}$
 226 with $0 < \mu_2 < \mu_1 < +\infty$.

227 4.2 Variational formulation

228 To properly establish a weak solution to the system of equations (2.1), (2.2), (2.3), (2.4),
 229 and (2.7) we need appropriate variational formulations and function spaces. We begin by
 230 defining the following function spaces,

$$\mathcal{V}^\epsilon = \left\{ \mathbf{v} \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d) \mid \text{div}(\mathbf{v}) = 0 \text{ in } \Omega, e(\mathbf{v}) = 0 \text{ in } \Omega_{1\epsilon} \right\}. \quad (4.1)$$

231 It is clear that \mathcal{V}^ϵ is a closed subspace of $H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$ and thus a Hilbert space with the
 232 induced $H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$ inner product which by Korn's inequality is equivalent to

$$(\mathbf{v} \mid \boldsymbol{\phi})_{\mathcal{V}^\epsilon} = \int_{\Omega_{2\epsilon}} 2 e(\mathbf{v}) : e(\boldsymbol{\phi}) \, d\mathbf{x}. \quad (4.2)$$

233 The corresponding norm will be denoted by $\|\cdot\|_{\mathcal{V}^\epsilon}$. Furthermore, we define the function
 234 space,

$$\mathcal{Y}^\epsilon = \left\{ \mathbf{w} \in L^2(\Omega; \mathbb{R}^d) \mid \operatorname{div} \mathbf{w} \in L^2(\Omega), \operatorname{curl}(\widehat{\mu}^\epsilon \mathbf{w}) \in L^2(\Omega; \mathbb{R}^d), \right. \\ \left. \mathbf{w} \cdot \mathbf{n} \in H^{1/2}(\Gamma_0) \right\}, \quad (4.3)$$

235 equipped with the inner product,

$$(\mathbf{h} \mid \boldsymbol{\psi})_{\mathcal{Y}^\epsilon} = \int_{\Omega} \operatorname{div}(\mathbf{h}) \operatorname{div}(\boldsymbol{\psi}) \, d\mathbf{x} + \int_{\Omega} \operatorname{curl}(\widehat{\mu}^\epsilon \mathbf{h}) \cdot \operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{\psi}) \, d\mathbf{x} \\ + \int_{\Gamma_0} (\mathbf{h} \cdot \mathbf{n})(\boldsymbol{\psi} \cdot \mathbf{n}) \, ds, \quad (4.4)$$

236 while the corresponding norm will be denoted by $[\cdot]_{\mathcal{Y}^\epsilon}$. It is evident that $(\mathcal{Y}^\epsilon, [\cdot]_{\mathcal{Y}^\epsilon})$ is a
237 Hilbert space from Proposition 3.3, since \mathcal{Y}^ϵ is the Hilbert space \mathcal{Y} with $\widehat{\mu} := \widehat{\mu}^\epsilon$ and \mathcal{O} is
238 now the domain Ω .

239 The variational formulation of (2.1), (2.2), (2.3), (2.4) and (2.7) is written in two steps.
240 First, we write down the variational formulation of the Stokes' equations and the Maxwell
241 equations separately and then add the resulting variational problems. The variational
242 formulation of the Stokes' equation reads: Find $\mathbf{u}^\epsilon \in \mathcal{V}^\epsilon$ such that,

$$(\mathbf{u}^\epsilon \mid \boldsymbol{\phi})_{\mathcal{V}^\epsilon} + \alpha \int_{\Omega_{2\epsilon}} \tau^\epsilon : e(\boldsymbol{\phi}) \, d\mathbf{x} = \mathbf{0} \quad \text{for all } \boldsymbol{\phi} \in \mathcal{V}^\epsilon. \quad (4.5)$$

243 For the quasi-static Maxwell's equations, we consider an augmented variational formula-
244 tion in \mathcal{Y}^ϵ [Jr05]. Find $\mathbf{B}^\epsilon \in \mathcal{Y}^\epsilon$ such that

$$\frac{\alpha}{R_m} (\mathbf{B}^\epsilon \mid \boldsymbol{\psi})_{\mathcal{Y}^\epsilon} = \alpha \int_{\Omega_{1\epsilon}} \mathbf{u}^\epsilon \times \mathbf{B}^\epsilon \cdot \operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{\psi}) \, d\mathbf{x} + \frac{\alpha}{R_m} \int_{\Gamma_0} (\mathbf{c} \cdot \mathbf{n})(\boldsymbol{\psi} \cdot \mathbf{n}) \, ds, \quad (4.6)$$

245 for all $\boldsymbol{\psi} \in \mathcal{Y}^\epsilon$.

246 Hence, the variational formulation of (2.1), (2.2), (2.3), (2.4) and (2.7) reads: Find
247 $(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$ such that

$$(\mathbf{u}^\epsilon \mid \boldsymbol{\phi})_{\mathcal{V}^\epsilon} + \frac{\alpha}{R_m} (\mathbf{B}^\epsilon \mid \boldsymbol{\psi})_{\mathcal{Y}^\epsilon} = -\alpha \int_{\Omega_{2\epsilon}} \tau^\epsilon : e(\boldsymbol{\phi}) \, d\mathbf{x} \\ + \alpha \int_{\Omega_{1\epsilon}} \mathbf{u}^\epsilon \times \mathbf{B}^\epsilon \cdot \operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{\psi}) \, d\mathbf{x} \\ + \frac{\alpha}{R_m} \int_{\Gamma_0} (\mathbf{c} \cdot \mathbf{n})(\boldsymbol{\psi} \cdot \mathbf{n}) \, ds, \quad (4.7)$$

248 for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$.

249 **Theorem 4.1.** *The pair $(\mathbf{v}^\epsilon, \mathbf{B}^\epsilon)$ satisfies (2.1), (2.2), (2.3), (2.4) and (2.7) if and only*
 250 *if it is a weak solution to (4.7).*

251 *Proof.* It is clear that if $(\mathbf{v}^\epsilon, \mathbf{B}^\epsilon)$ satisfies (2.1), (2.2), (2.3), (2.4) and (2.7) then it is a
 252 solution to (4.7). To see this, multiply the Stokes' equations by a test function in $\boldsymbol{\phi} \in \mathcal{V}^\epsilon$
 253 and carry out the variational formulation as in [Tem84], [GMV14], [NV16, Appendix].
 254 For Maxwell's equations multiply the divergence part by $\frac{\alpha}{\mathbb{R}_m} \operatorname{div} \boldsymbol{\psi}$, the rotational part by
 255 $\frac{\alpha}{\mathbb{R}_m} \operatorname{curl} \boldsymbol{\psi}$, and the exterior boundary condition by $\frac{\alpha}{\mathbb{R}_m} \boldsymbol{\psi} \cdot \mathbf{n}$, respectively.

256 For the other direction we have: Take $(\boldsymbol{\phi}, \mathbf{0})$ as a test function in (4.7) and obtain the vari-
 257 ational formulation of Stokes' equation (4.5) from which we can recover Stokes' equation,
 258 boundary conditions, and balance of forces and torques in the distributional sense as usual
 259 (see [Tem84], [GMV14]). On the other hand if we take $(\mathbf{0}, \boldsymbol{\psi})$ as a test function in (4.7)
 260 we obtain (4.6). In order to recover Maxwell's equations we need to introduce appropriate
 261 test functions on each domain $\Omega_{1\epsilon}$ and $\Omega_{2\epsilon}$. To this end if we let $\zeta^\delta : \mathbb{R}^d \rightarrow [0, 1]$ be a
 262 smooth cut-off function defined by

$$\zeta^\delta(\mathbf{x}) = \begin{cases} 1 & \text{if } d(\mathbf{x}, \Gamma_0) < \delta, \\ 0 & \text{if } d(\mathbf{x}, \Gamma_0) > 2\delta, \end{cases} \quad (4.8)$$

263 where δ is chosen in a way that the inner most neighbourhood does not intersect the
 264 rigid particles. Following [Lad14], define $\boldsymbol{\theta}(\mathbf{x}) := (c_2x_3, c_3x_1, c_1x_2)$ where the vector field
 265 $\mathbf{c} = (c_1, c_2, c_3)$ is the constant vector field from the outer transmission condition on the
 266 boundary Γ_0 . Set $\mathbf{a}^\delta(\mathbf{x}) := \operatorname{curl}(\zeta^\delta(\mathbf{x})\boldsymbol{\theta}(\mathbf{x}))$ then $\mathbf{a}^\delta(\mathbf{x})$ is a divergence free vector field that
 267 is zero in the domain $\Omega^\delta := \{\mathbf{x} \in \Omega \mid d(\mathbf{x}, \Gamma_0) < 2\delta\}$ and equals \mathbf{c} in the δ neighbourhood
 268 of Γ_0 .

269 Moreover, by Proposition 3.1 we have that $\mathbf{B}^\epsilon \in H^1(\Omega_{i\epsilon})$, $i = 1, 2$ and by the classical
 270 Sobolev embedding of H^1 into L^q for $1 \leq q < 6$ we have that $\mathbf{B}^\epsilon \in L^4(\Omega; \mathbb{R}^d)$. Likewise, for
 271 \mathbf{v}^ϵ , namely, $\mathbf{v}^\epsilon \in L^4(\Omega; \mathbb{R}^d)$. Thus, $\mathbb{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \chi_{\Omega_{1\epsilon}} \in L^2(\Omega; \mathbb{R}^d)$ by the Cauchy–Schwartz
 272 inequality. Using (2.3) we also have that $\operatorname{div}(\mathbb{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \chi_{\Omega_{1\epsilon}}) = 0$ in Ω . Therefore,
 273 $\mathbb{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \chi_{\Omega_{1\epsilon}} \in \{\mathbf{v} \in H(\operatorname{div}, \Omega) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}$. By Lemma 3.1 there exists a $\mathbf{w} \in \mathcal{Y}_{\Gamma_0}^0$
 274 such that,

$$\operatorname{curl}(\widehat{\mu}_\epsilon \mathbf{w}) = \mathbb{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \chi_{\Omega_{1\epsilon}}. \quad (4.9)$$

275 Setting $\boldsymbol{\psi} := \mathbf{B}^\epsilon - \mathbf{w} - \mathbf{a}^\delta \in \mathcal{Y}^\epsilon$ we reduce (4.6) to the following,

$$\begin{aligned} & \int_{\Omega} \operatorname{div} \mathbf{B}^\epsilon \operatorname{div}(\mathbf{B}^\epsilon - \mathbf{w} - \mathbf{a}^\delta) \, d\mathbf{x} \\ & + \int_{\Omega} (\operatorname{curl}(\widehat{\mu}_\epsilon \mathbf{B}^\epsilon) - \mathbb{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \chi_{\Omega_{1\epsilon}}) \cdot \operatorname{curl}(\widehat{\mu}_\epsilon (\mathbf{B}^\epsilon - \mathbf{w} - \mathbf{a}^\delta)) \, d\mathbf{x} \\ & + \int_{\Gamma_0} ((\mathbf{B}^\epsilon - \mathbf{c}) \cdot \mathbf{n})((\mathbf{B}^\epsilon - \mathbf{w} - \mathbf{a}^\delta) \cdot \mathbf{n}) \, ds = 0 \end{aligned} \quad (4.10)$$

276 Using the properties of the vector fields \mathbf{w} and \mathbf{a}^δ we obtain:

$$\begin{aligned}
& \int_{\Omega} |\operatorname{div} \mathbf{B}^\epsilon|^2 d\mathbf{x} + \int_{\Omega} |\operatorname{curl}(\widehat{\mu}^\epsilon \mathbf{B}^\epsilon) - \mathbf{R}_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \chi_{\Omega_{1\epsilon}}|^2 d\mathbf{x} \\
& \quad + \int_{\Gamma_0} |(\mathbf{B}^\epsilon - \mathbf{c}) \cdot \mathbf{n}|^2 ds = 0
\end{aligned} \tag{4.11}$$

277 Since the expression above is a sum of squares that is equal to zero, each integral must be
278 equal to zero and the claim follows. \square

279 5 Existence of a weak solution via the Altman-Shinbrot 280 fixed point theorem

281 5.1 M. Shinbrot's fixed point argument

282 To prove existence we employ the fixed point argument of Altman-Shinbrot [Shi64], [Alt57].
283 For the readers convenience, we recall the main theorem and corollaries as formulated by
284 M. Shinbrot and whose proofs can be found in [Shi64].

285 In what follows, \mathcal{H} denotes a real or complex Hilbert space, and \mathcal{S}_r and \mathcal{B}_r will denote the
286 sphere and the closed unit ball of radius r centered at zero:

$$\mathcal{S}_r = \{x \in \mathcal{H} \mid \|x\|_{\mathcal{H}} = r\}, \quad \mathcal{B}_r = \{x \in \mathcal{H} \mid \|x\|_{\mathcal{H}} \leq r\}.$$

287 **Theorem 5.1.** *Let H be an operator on the separable Hilbert space \mathcal{H} , continuous in the
288 weak topology on \mathcal{H} . If there is a positive constant r such that*

$$\Re(Hx, x) \leq \|x\|_{\mathcal{H}}^2 \text{ for all } x \in \mathcal{B}_r,$$

289 *then H has a fixed point in \mathcal{B}_r .*

290 **Corollary 5.1.** *Let G be an operator on the separable Hilbert space \mathcal{H} , continuous in the
291 weak topology on \mathcal{H} . Let y be an element of \mathcal{H} . Let y be an element of \mathcal{H} . If there exists
292 a positive r such that either,*

$$\Re(Gx - y, x) \geq 0 \text{ for all } x \in \mathcal{S}_r,$$

293 *or*

$$\Re(Gx - y, x) \leq 0 \text{ for all } x \in \mathcal{S}_r,$$

294 *then y is in the range of G .*

295 **Corollary 5.2.** *Let G be an operator on the separable Hilbert space \mathcal{H} , continuous in the
296 weak topology on \mathcal{H} . Then, zero is in the range of G if (Gx, x) is of one sign on some
297 sphere \mathcal{S}_r .*

298 **5.2 Existence**

299 For all $\mathbf{u}^\epsilon, \mathbf{B}^\epsilon, \boldsymbol{\phi}, \boldsymbol{\psi}$ we define the following expression \mathcal{Q} by,

$$\begin{aligned} \mathcal{Q}[(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})] := & -\alpha \int_{\Omega_{2\epsilon}} \widehat{\mu}_2 \mathbf{B}^\epsilon \otimes \mathbf{B}^\epsilon : e(\boldsymbol{\phi}) \, d\mathbf{x} \\ & + \alpha \int_{\Omega_{1\epsilon}} \mathbf{u}^\epsilon \times \mathbf{B}^\epsilon \cdot \text{curl}(\widehat{\mu}_1 \boldsymbol{\psi}) \, d\mathbf{x}. \end{aligned} \quad (5.1)$$

300 We can immediately see that by combining the results of Proposition 3.1 and Theorem 3.1
301 with classical Sobolev embedding theorems of H^1 into L^q , $q \in [1, 2d/(d-2))$ we obtain,

$$|\mathcal{Q}[(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})]| \leq c \|\mathbf{u}^\epsilon, \mathbf{B}^\epsilon\|^2 \|(\boldsymbol{\phi}, \boldsymbol{\psi})\|, \quad (5.2)$$

302 where $\|(-, \cdot)\| := \|\cdot\|_{\mathcal{V}^\epsilon} + \frac{\alpha}{R_m} [\cdot]_{\mathcal{W}^\epsilon}$ and c is a generic constant depending on $\Omega_{i\epsilon}, \alpha, \widehat{\mu}_i$ for
303 $i = 1, 2$.

304 Thus, we can write (4.7) as: Find $(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$ such that,

$$(\mathbf{u}^\epsilon | \boldsymbol{\phi})_{\mathcal{V}^\epsilon} + \frac{\alpha}{R_m} (\mathbf{B}^\epsilon | \boldsymbol{\psi})_{\mathcal{Y}^\epsilon} - \mathcal{Q}[(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})] = \frac{\alpha}{R_m} \int_{\Gamma_0} (\mathbf{c} \cdot \mathbf{n})(\boldsymbol{\phi} \cdot \mathbf{n}) \, ds, \quad (5.3)$$

305 for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$.

306 The Cauchy-Schwartz inequality and the definition of the norm $\frac{\alpha}{R_m} [\cdot]_{\mathcal{Y}^\epsilon}$ make the right
307 hand side of equation (5.3) a bounded linear functional of $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$. Using Riesz's
308 theorem, we can express the right hand side of (5.3) as the scalar product of a well
309 determined element $(\mathbf{f}, \mathbf{g}) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$ by $(\boldsymbol{\phi}, \boldsymbol{\psi})$.

310 Likewise, if we fix $(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$ and take into account the estimate (5.2), we can write
311 the left hand side as a product of an element in $\mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$, denoted by $\mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)$ that depends
312 nonlinearly on $(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)$, by $(\boldsymbol{\phi}, \boldsymbol{\psi})$.

313 Therefore, we can re-write (5.3) using the operator \mathcal{F} as,

$$(\mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})) = ((\mathbf{f}, \mathbf{g}); (\boldsymbol{\phi}, \boldsymbol{\psi})), \quad (5.4)$$

314 for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$ where,

$$(\mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})) := (\mathbf{u}^\epsilon | \boldsymbol{\phi}) + \frac{\alpha}{R_m} (\mathbf{B}^\epsilon | \boldsymbol{\psi}) - \mathcal{Q}[(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})], \quad (5.5)$$

315 and

$$((\mathbf{f}, \mathbf{g}); (\boldsymbol{\phi}, \boldsymbol{\psi})) := \frac{\alpha}{R_m} \int_{\Gamma_0} (\mathbf{c} \cdot \mathbf{n})(\boldsymbol{\phi} \cdot \mathbf{n}) \, ds. \quad (5.6)$$

316 Hence, searching for a solution to (4.7) reduces to showing that at least one solution exists
317 to the above nonlinear operator equation.

318 **Lemma 5.1.** *The nonlinear operator $\mathcal{F} : (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon) \mapsto \mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)$ is continuous in the weak*
 319 *topology of the product space $\mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$.*

320 *Proof.* Assume that $(\mathbf{u}_\kappa^\epsilon, \mathbf{B}_\kappa^\epsilon)$ is a weakly convergent sequence in $\mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$ to $(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)$ as
 321 $\kappa \rightarrow +\infty$ then,

$$\begin{aligned} & |(\mathcal{F}(\mathbf{u}_\kappa^\epsilon - \mathbf{u}^\epsilon, \mathbf{B}_\kappa^\epsilon - \mathbf{B}^\epsilon); (\boldsymbol{\psi}, \boldsymbol{\psi}))| \\ &= \left| (\mathbf{u}_\kappa^\epsilon - \mathbf{u}^\epsilon \mid \boldsymbol{\phi}) + \frac{\alpha}{R_m} (\mathbf{B}_\kappa^\epsilon - \mathbf{B}^\epsilon \mid \boldsymbol{\psi}) - \mathcal{Q}[(\mathbf{u}_\kappa^\epsilon - \mathbf{u}, \mathbf{B}_\kappa^\epsilon - \mathbf{B}); (\boldsymbol{\phi}, \boldsymbol{\psi})] \right|. \end{aligned} \quad (5.7)$$

322 By Hölder's inequality and the embedding of $H^1(\Omega_{i\epsilon}; \mathbb{R}^d)$ into $L^q(\Omega_{i\epsilon}; \mathbb{R}^d)$, $i = 1, 2$ with
 323 $1 \leq q < 6$ we have,

$$\begin{aligned} & |\mathcal{Q}[(\mathbf{u}_\kappa^\epsilon - \mathbf{u}, \mathbf{B}_\kappa^\epsilon - \mathbf{B}); (\boldsymbol{\phi}, \boldsymbol{\psi})]| \\ & \leq c \|\mathbf{B}_\kappa^\epsilon - \mathbf{B}^\epsilon\|_{L^4(\Omega_{1\epsilon}; \mathbb{R}^d)}^2 \|\boldsymbol{\phi}\|_{L^2(\Omega_{2\epsilon}; \mathbb{R}^{d \times d})} \\ & \leq c \|\mathbf{u}_\kappa^\epsilon - \mathbf{u}^\epsilon\|_{L^4(\Omega_{1\epsilon}; \mathbb{R}^d)} \|\mathbf{B}_\kappa^\epsilon - \mathbf{B}^\epsilon\|_{L^4(\Omega_{1\epsilon}; \mathbb{R}^d)} \|\operatorname{curl}(\widehat{\boldsymbol{\mu}}_1 \boldsymbol{\psi})\|_{L^2(\Omega_{2\epsilon}; \mathbb{R}^d)}, \end{aligned} \quad (5.8)$$

324 for generic constant $c := c(\Omega_{i\epsilon}, a, R_m, \widehat{\boldsymbol{\mu}}_i)$, $i = 1, 2$. Moreover, since the above embedding of
 325 $H^1(\Omega_{i\epsilon}; \mathbb{R}^d)$ into $L^q(\Omega_{i\epsilon}; \mathbb{R}^d)$ is compact we can extract strongly κ convergent subsequences
 326 (not relabelled) in $L^4(\Omega_{i\epsilon}; \mathbb{R}^d)$ of $\mathbf{u}_\kappa^\epsilon$ and $\mathbf{B}_\kappa^\epsilon$ to \mathbf{u}^ϵ and \mathbf{B}^ϵ , respectively.

327 Passing to the limit as $\kappa \rightarrow +\infty$ in (5.7) we have,

$$\lim_{\kappa \rightarrow +\infty} (\mathcal{F}(\mathbf{u}_\kappa^\epsilon - \mathbf{u}^\epsilon, \mathbf{B}_\kappa^\epsilon - \mathbf{B}^\epsilon); (\boldsymbol{\psi}, \boldsymbol{\psi})) = 0. \quad (5.9)$$

328

□

329 **Lemma 5.2.** *If the magnetic Reynolds number, R_m , is small then*

$$(\mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)) \geq \frac{1}{2} \|(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)\|^2, \quad (5.10)$$

330 for all $(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon) \in \mathcal{V}^\epsilon \times \mathcal{Y}^\epsilon$.

331 *Proof.* We prove the lemma in two steps. In step 1 we obtain an estimate of the magnetic
 332 induction in terms of R_m using Proposition 3.2. In step 2. we obtain an estimate for \mathcal{Q} .
 333 Combining both steps gives bounds on R_m for the existence of solutions.

334 Step 1: We begin with a bound on \mathbf{B}^ϵ in $L^q(\Omega; \mathbb{R}^d)$ for $q \in (1, 2d/(d-2)]$. By Proposition 3.2
 335 we have that

$$\begin{aligned}
\|\mathbf{B}^\epsilon\|_{L^q(\Omega;\mathbb{R}^d)} &\leq c[\mathbf{B}^\epsilon]_{\mathcal{Y}^\epsilon} \\
&= c(\|\operatorname{curl}(\widehat{\mu}^\epsilon \mathbf{B}^\epsilon)\|_{L^2(\Omega;\mathbb{R}^d)} + \|\mathbf{B}^\epsilon \cdot \mathbf{n}\|_{H^{1/2}(\Gamma_0)}) \\
&\leq c(\|\mathbf{R}_m \mathbf{u}^\epsilon \times \mathbf{B}^\epsilon\|_{L^2(\Omega_{1\epsilon};\mathbb{R}^d)} + |\mathbf{c}| \operatorname{mes}_{d-1}(\Gamma_0)).
\end{aligned} \tag{5.11}$$

336 On the rigid particles the velocity takes the form $\mathbf{u}^\epsilon = \mathbf{V}^{\ell,\epsilon} + \boldsymbol{\omega}^{\ell,\epsilon} \times (\mathbf{x} - \mathbf{x}_c^\ell)$, $\ell = 1, \dots, N(\epsilon)$
337 with the translational and rotational velocity $\mathbf{V}^{\ell,\epsilon}$ and $\boldsymbol{\omega}^{\ell,\epsilon}$, respectively, being constant.
338 Additionally, the term $|\mathbf{x} - \mathbf{x}_c^\ell|$ is such that $|\mathbf{x} - \mathbf{x}_c^\ell| < \operatorname{diam}(T_\ell^\epsilon) < \epsilon \ll 1$. Hence,

$$\begin{aligned}
c \|\mathbf{R}_m \mathbf{u}^\epsilon \times \mathbf{B}^\epsilon\|_{L^2(\Omega_{1\epsilon};\mathbb{R}^d)} &\leq c_\epsilon \mathbf{R}_m \|\mathbf{B}^\epsilon\|_{L^2(\Omega_{1\epsilon};\mathbb{R}^d)} \\
&\leq c_\epsilon \mathbf{R}_m \|\mathbf{B}^\epsilon\|_{L^q(\Omega;\mathbb{R}^d)},
\end{aligned} \tag{5.12}$$

339 for $q \in [2, 2d/(d-2)]$.

340 Combining (5.11) and (5.12) we obtain the following L^q , $q \in [2, 2d/(d-2)]$, bound for \mathbf{B}^ϵ ,

$$\|\mathbf{B}^\epsilon\|_{L^q(\Omega;\mathbb{R}^d)} \leq \frac{c|\mathbf{c}| \operatorname{mes}_{d-1}(\Gamma_0)}{1 - c_\epsilon \mathbf{R}_m}, \tag{5.13}$$

341 if $\mathbf{R}_m < 1/c_\epsilon$.

342 Step 2: By Korn's inequality, Hölder's inequality, and (5.13) we can bound \mathcal{Q} by,

$$\begin{aligned}
|\mathcal{Q}[(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)]| &\leq c \|\mathbf{B}^\epsilon\|_{L^4(\Omega_{2\epsilon};\mathbb{R}^d)} \|\mathbf{B}^\epsilon\|_{L^4(\Omega_{2\epsilon};\mathbb{R}^d)} \|e(\mathbf{u}^\epsilon)\|_{L^2(\Omega_{2\epsilon};\mathbb{R}^{d \times d})} \\
&\quad + c \|\mathbf{u}^\epsilon\|_{L^4(\Omega_{1\epsilon};\mathbb{R}^d)} \|\mathbf{B}^\epsilon\|_{L^4(\Omega_{1\epsilon};\mathbb{R}^d)} \|\operatorname{curl}(\widehat{\mu}^\epsilon \mathbf{u}^\epsilon)\|_{L^2(\Omega_{1\epsilon};\mathbb{R}^d)} \\
&\leq \frac{c|\mathbf{c}| \operatorname{mes}_{d-1}(\Gamma_0)}{1 - c_\epsilon \mathbf{R}_m} |||(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)|||^2.
\end{aligned} \tag{5.14}$$

343 Hence we obtain,

$$(\mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)) = |||(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)|||^2 - \frac{c|\mathbf{c}| \operatorname{mes}_{d-1}(\Gamma_0)}{1 - c_\epsilon \mathbf{R}_m} |||(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)|||^2, \tag{5.15}$$

344 if $\mathbf{R}_m > (1 - 2c|\mathbf{c}| \operatorname{mes}_{d-1}(\Gamma_0))/c_\epsilon$ the result follows. \square

345 **Theorem 5.2.** *Given the assumptions in Subsection 4.1, if the magnetic Reynolds num-*
346 *ber, \mathbf{R}_m , is small then problem (5.4) admits at least one weak solution.*

347 *Proof.* According to [Shi64, Corollary 2] (see also [Fin65], [SP68]) if we can show that
348 there exists a number r such that

$$(\mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon) - (\mathbf{f}, \mathbf{g}); (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)) \geq 0 \text{ for all } (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon) \text{ with } |||(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)||| = r \tag{5.16}$$

349 then equation (5.4) has at least one solution. Hence, by Lemma 5.2 we have,

$$\begin{aligned}
(\mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon) - (\mathbf{f}, \mathbf{g}); (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)) &= (\mathcal{F}(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon); (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)) - ((\mathbf{f}, \mathbf{g}); (\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)) \\
&\geq \frac{1}{2} \|\|(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)\|\|^2 - \|\|(\mathbf{f}, \mathbf{g})\|\| \|\|(\mathbf{u}^\epsilon, \mathbf{B}^\epsilon)\|\| \quad (5.17) \\
&\geq 0,
\end{aligned}$$

350 if we select $r = 2 \|\|(\mathbf{f}, \mathbf{g})\|\|$. □

351 6 Conclusions

352 We proved existence of a weak solution in three spatial dimensions to a coupled sys-
353 tem of Stokes' equations and quasi-static Maxwell's equations under moderate magnetic
354 field strength using the Altman–Shinbrot fixed point theorem and the augmented vari-
355 ational formulation of Maxwell's equations when the magnetic Reynold's number, R_m ,
356 is small. The novelty of our approach lies in the fact that the magnetic induction that
357 does not possess full weak derivatives in L^2 due to material inhomogeneities. Classical
358 fixed point arguments, like Leray-Schauder, require that the defined operator be com-
359 pletely continuous which is a consequence of the Sobolev embedding of H^1 into L^2 for
360 three dimensions. In contrast, the Altman–Shinbrot fixed point argument requires that
361 the operator constructed need only be continuous in the weak topology of the underlying
362 function space (and not completely continuous as is required by the fixed point theorem
363 of Leray-Schauder). Moreover, in order to apply the Altman–Shinbrot theory, we prove
364 higher regularity of the non-linear term present due to the Lorentz force and define a new
365 function space for the magnetic induction. Additionally, we prove using the celebrated
366 div–curl lemma that this new space is compactly embedded into L^2 .

367 By and large, the existence result holds true when the magnetic Reynold's number, R_m
368 is small. The case of $R_m \equiv 0$ can be thought off as a limit case of the above model.
369 When $R_m \equiv 0$ the system becomes weakly coupled and, existence and uniqueness follow
370 by invoking the Lax-Milgram lemma once higher integrability of the magnetic induction
371 is established.

372 Acknowledgments

373 GN gratefully acknowledges the funding by the Deutsche Forschungsgemeinschaft (DFG,
374 German Research Foundation) under Germany's Excellence Strategy – The Berlin Mathe-
375 matics Research Center MATH+ (EXC-2046/1, project ID: 390685689) in project AA2-1.

376 References

- 377 [ACN20] F. Agnelli, A. Constantinescu, and G. Nika. Design and testing of 3D-printed microar-
378 chitected polymer materials exhibiting a negative Poisson's ratio. *Cont. Mechanics*
379 *& Thermodyn.*, 32(2):433–449, 2020.

- 380 [Alt57] M. Altman. A fixed point theorem in Hilbert space. *Bull. Acad. Polon. Sci.*, 5:19–22,
381 1957.
- 382 [BD05] I. A. Brigadnov and A. Dorfmann. Mathematical modelling of magnetorheological
383 fluids. *Continuum Mech. Therm.*, pages 29–42, 2005.
- 384 [DL72] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique*. Travaux
385 Recherches Math. Dunod, Paris, 1972.
- 386 [Dru] P.-E. Druet. Higher integrability of the Lorentz force for weak solutions to Maxwell’s
387 equations in complex geometries. WIAS pre-print 2007. [https://doi.org/10.
388 20347/WIAS.PREPRINT.1270](https://doi.org/10.20347/WIAS.PREPRINT.1270).
- 389 [dVKHA11] J. de Vicente, D. J. Klingenberg, and R. Hidalgo-Alvarez. Magnetorheological fluids:
390 a review. *Soft Matter*, 7:3701–3710, 2011.
- 391 [ERS07] J. Elschner, J. Rehberg, and G. Schmidt. Optimal regularity for elliptic transmission
392 problems including C^1 interfaces. *Interface Free Bound.*, 9:233–252, 2007.
- 393 [Fin65] R. Finn. Stationary solutions of the Navier-Stokes equations. *Proc. Symp. Appl.
394 Math.*, 17:121–153, 1965.
- 395 [FT78] C. Foias and R. Temam. Remarques sur les équations de Navier-Stokes stationnaires
396 et les phénomènes successifs de bifurcation. *Ann. Sc. Norm. Sup. Pisa*, 4,5:29–63,
397 1978.
- 398 [Gal11] G.P. Galdi. *An introduction to the mathematical theory of the Navier–Stokes equa-
399 tions: Steady-state problems*. Springer Science & Business Media, 2011.
- 400 [GMV14] Y. Gorb, F. Maris, and B. Vernescu. Homogenization for rigid suspensions with
401 random velocity-dependent interfacial forces. *J. Math. Anal. Appl.*, 420:632–668,
402 2014.
- 403 [Jr05] P. Ciarlet Jr. Augmented formulations for solving Maxwell’s equations. *Comput.
404 Methods Appl. Mech. Engrg.*, 194:559–586, 2005.
- 405 [L85] T. Lévy. Suspension de particules solides soumises à des couples. *J. Méch. Théor.
406 App.*, (Numéro Spécial):53–71, 1985.
- 407 [Lad14] O. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow (translated
408 from Russian, revised English edition)*. Martino Publishing, Mansfield Centre, CT,
409 2014.
- 410 [LFS01] J. Liu, G. A. Flores, and R. Sheng. In-vitro investigation of blood embolization in
411 cancer treatment using magnetorheological fluids. *J. Magn. Magn. Mater.*, 225(1-
412 2):209–217, 2001.
- 413 [LH88] T. Lévy and R. K. T. Hsieh. Homogenization mechanics of a non-dilute suspension
414 of magnetic particles. *Int. J. Engng. Sci.*, 26:1087–1097, 1988.
- 415 [LM72] J.-L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Ap-
416 plications I*. Springer-Verlag Berlin Heidelberg, 1972.
- 417 [LU68] O. A. Ladyzhenskaya and N. N. Uraltseva. *Linear and Quasilinear Elliptic Equations*.
418 Academic Press, New York/London, 1968.
- 419 [Mur78] F. Murat. Compacité par compensation. *Ann. Sc. Norm. Sup. Pisa Sc1. Fis. Mat.*,
420 5:489–507, 1978.
- 421 [NC19] G. Nika and A. Constantinescu. Design of multi-layer materials using inverse homog-
422 enization and a level set method. *Comput. Methods Appl. Mech. Engrg.*, 346:388–409,
423 2019.

- 424 [NV16] G. Nika and B. Vernescu. Dilute emulsions with surface tension. *Quart. Applied*
425 *Math.*, 1(1):89–111, 2016.
- 426 [NV20] G. Nika and B. Vernescu. Multiscale modeling of magnetorheological suspensions. *Z.*
427 *Angew. Math. Phys.*, 71(1):1–19, 2020.
- 428 [NV21] G. Nika and B. Vernescu. Micro-geometry effects on the nonlinear effective yield
429 strength response of magnetorheological fluids. In P. Donato and M. Luna-Laynez,
430 editors, *Emerging problems in the Homogenization of Partial Differential Equations*,
431 pages 1–16. SEMA SIMAI Springer series, 2021.
- 432 [PV00] J. Perlak and B. Vernescu. Constitutive equations for electrorheological fluids. *Rev.*
433 *Roumaine Math. Pures Appl.*, 45:287–297, 2000.
- 434 [Rab48] J. Rabinow. The magnetic fluid clutch. *AIEE Trans.*, 67(17-18):1308, 1948.
- 435 [RR01] K.R. Rajagopal and M. Ruzicka. Mathematical modeling of electrorheological mate-
436 rials. *Continuum Mech. Therm.*, 13(1):59–78, 2001.
- 437 [Ruz00] M. Ruzicka. *Electrorheological Fluids: Modeling and Mathematical Theory*. Lecture
438 Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2000.
- 439 [Sch18] B. Schweizer. On Friedrichs Inequality, Helmholtz Decomposition, Vector Potentials,
440 and the div-curl Lemma. In E. Rocca, U. Stefanelli, L. Truskinovsky, and A. Visintin,
441 editors, *Trends in Applications of Mathematics to Mechanics*, pages 65–79. Springer
442 INdAM Series 27, 2018.
- 443 [Shi64] M. Shinbrot. A fixed point theorem, and some applications. *Arch. Rational Mech.*
444 *Anal.*, 17(4):255–271, 1964.
- 445 [SP68] E. Sanchez-Palencia. Existence des solutions de certains problèmes aux limites en
446 magnétohydrodynamique. *J. de Mécanique*, 7(3):405–426, 1968.
- 447 [Tar79] L.C. Tartar. Compensated compactness and applications to partial differential equa-
448 tions. *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, IV:136–212, 1979.
- 449 [Tem84] R. Temam. *Navier-Stokes equations: Theory and Numerical analysis (2nd edition)*.
450 North-Holland publishing company, 1984.
- 451 [Ver02] B. Vernescu. Multiscale analysis of electrorheological fluids. *Inter. J. Modern Physics*
452 *B*, 16(1):2643–2648, 2002.