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Derivatives under market impact: Disentangling cost and information

Behzad Alimoradian⁽¹⁾⁽²⁾⁽³⁾, Karim Barigou⁽¹⁾, and Anne Eyraud-Loisel⁽¹⁾

¹*Univ Lyon, Université Claude Bernard Lyon 1, Laboratoire de Sciences Actuarielle et Financière, Institut de Science Financière et d'Assurances (50 Avenue Tony Garnier, F-69007 Lyon, France)*

²*Alpha Quants, 1 Canada Square, London E14 5AB*

³*Valerian Capital Group, 30745 Pacific Coast Highway Suite 415, Malibu, CA 90265*

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Abstract

This paper studies the role of market impact in option pricing theory through the non-restrictive case study of the existence of a large trader. Under market impact, the standard Black-Scholes model and unique risk-neutral pricing theory are no more applicable. We postulate then that there are two sides to the story: i) the transaction costs and ii) the information of the large trader about its own activity. To disentangle these two factors, we define a hypothetical trader called the insider trader who has the same level of information as the large trader but does not bear any transaction costs. We show that there exist a set of probability measures, which we call *information-neutral probabilities* under which the discounted asset is a martingale for the insider trader. We then derive the optimal cost hedging program for the large trader and emphasize the importance of avoiding any market manipulation in the optimal hedge problem so that it does not become inherent to the model. We conclude with pricing models and numerical examples for both the large and insider trader.

Keywords: Option pricing, market impact, illiquid markets, transactions costs, stochastic optimal control.

1 Introduction

Standard option pricing theory is based on the assumptions of a frictionless and liquid financial market. However, having large notional derivatives on illiquid assets with market impact, one may face difficulties hedging its position. When the size of the hedge is substantial, executing shares consumes the market's liquidity, and consequently impacts the underlying spot price. In

such circumstances, the standard framework of the Black-Scholes model does not stand anymore. This paper presents a mathematical framework for hedging derivatives under such liquidity issues.

The literature predominantly refers to two types of market impact: the permanent impact, and the temporary impact which is also known as slippage or transaction cost.¹ One of the first models to take into account the temporary impact (transaction costs) in derivatives was introduced by Leland (1985). Liu & Yong (2005), on the other hand, integrated the permanent impact in their extension of the Black Scholes model and came up with a parabolic partial differential equation which can be interpreted as multiplying the volatility of the underlying asset by a sum of a series (see Glover *et al.* (2010), Said (2020)). Recent papers on the permanent impact in derivatives are also Loeper *et al.* (2018), Said (2019), Bordag & Frey (2008) and Abergel & Loeper (2017). While being in the lineage with the past literature, our paper broadens the academic debate by suggesting to consider the permanent and temporary impacts as two unrelated phenomena, different in essence and nature; the permanent impact as information and the temporary impact as costs. We do so by defining an hypothetical *insider trader* that does not bear any transaction costs but is aware of the permanent impact.

In this paper, we conceptualize the permanent price impact by the means of the auxiliary concept of *information*, meant to represent the information about trading activities of one or many traders. Knowing that such trading activities may result in liquidity (im)balance, any entity being aware of them will have a different opinion about the spot price. Such entity will apply a correction to the price. We regard the permanent impact as this correction to the price judged by the informed individual. Once the trade is executed, such information is shared among the whole market and the price impact becomes a reality; it contributes to the filtration explaining asset variation in the real world. It is worth noting that the role of information in asset pricing has been studied by Eyraud-Loisel (2005) and Eyraud-Loisel (2013) on their contribution to understanding the filtration enlargement due to some private information. Cetin *et al.* (2006) and Cetin *et al.* (2010) have also interpreted the permanent market impact through the concept of asset's supply curve. Another recent work by Brigo *et al.* (2020) studies the arbitrage pricing theory for the bond market in the presence of market impact. Our imperative standpoint is that the price impact of some information in the market is more pertinent than the nature of the information itself. In fact, information is not even measurable, however the market impact of the information is. For this reason, practitioners refer to the concept of *impact factor* when tackling information. Therefore, any theory of information results into a theory of market impact. With this imperative standpoint, there is no easily discernible example of information other than an information about one large trader's activity which is the subject of study in this paper.

One other main distinction of this paper is that our permanent impact process is stochastic and \mathcal{L}^2 -integrable, while the past literature has been using \mathcal{L}^1 -integrable price impact processes.

¹The transaction cost and temporary impact have been used interchangeably in some of the literature, see Alfonsi *et al.* (2012), or for example Huberman & Stanzl (2004) who use the concept of temporary market impact as the slippage cost. Some other studies, such as Gatheral (2010) and Bouchaud *et al.* (2003) consider a temporal price impact decaying over time.

\mathcal{L}^2 -integrability is more pertinent because, in such case, the market impact will have the same order of finite moments than the brownian motion variation, which is the same finite order of moments than the volatility. This is an important property of a price impact model because it gives the model the potential to justify/explain the market volatility. One other consideration with derivatives is the hedging policy; in general, there are two trends: the first one suggests a full hedge policy as in [Liu & Yong \(2005\)](#) and the second one suggests partial hedging, e.g. [Jonsson & Sircar \(2002\)](#). The full hedge approach generally results in option prices that are too expensive in illiquid markets since the cost of rebalancing the delta of the option can be considerable when getting closer to the strike and expiry date. A full hedge might be also not feasible because the market liquidity may be insufficient. For this reason, this paper's main focus is on partial hedging. In the case of a partial hedge, a predominant approach consists of optimizing a gain function for the large trader, e.g. [Guéant & Pu \(2017\)](#). However, with this approach the solutions may lead to the large trader manipulating the market in its own benefit as defined by its utility function. Therefore, it is important to correctly define the problem so that a market abuse does not become inherent to the model and in that regard, this paper is one of the first to introduce the issue of market manipulation in derivatives hedging. It provides a discussion on the *model risk* of market abuse in derivatives and suggests an approach to address it.

Lastly, it should be mentioned that even though at first sight the case study presented in this paper may seem too restrictive, it is not. This case example is quite common when dealing with illiquid markets.² To give more clarity about its practicality, it is worth mentioning that this research project originated from a real practical experience when a trader consulted one of the authors of the paper being concerned about the incapacity of existing models to hedge and price of a large derivative transaction. The state-of-the-art derivatives models are generally calibrated to the implied volatility surface. However if the market is blind about the existence of some new transactions it will lack a critical information, and therefore the implied volatility surface, as it stands, is biased and needs to be corrected.

Also, even though here we assume the existence of one large trader only and in reality a market may include several large traders, we should mention that it happens quite often that in the market most large traders have in general the same directional strategies, especially in activities that may result in market impact, this phenomenon has been pointed out by ([Cont & Schaanning 2017](#)) in their analysis of fire sale of large traders contingent to some critical market environment. Lastly, it should be mentioned that the results obtained in this paper could be generalized for any information process that generates market impact; to this point, we also refer to the work of ([Lee & Song 2007](#)).

This paper is organized as follows: Section 2 presents the market ecosystem, with one large and one insider trader. Section 3 describes the stock dynamics under the market impact. Section 4 derives the pricing equation for the insider trader and explains the existence of a set of probability measures called *information-neutral probabilities* under which the asset is a martingale. In Section

²The reader is referred to the introduction section of [Almgren & Li \(2016\)](#) for a lucid description that such case is common.

5, we derive an optimal hedging strategy for the large trader and discuss the important issue of market manipulation in the stochastic control problem. We present a finite difference method for solving the optimal shares of the large trader and provide numerical results that compare, in particular, strategies for both the large and insider trader in Section 6.

2 Market Ecosystem Description

We consider a market with many small active traders, but with limited liquidity. A large trader decides to enter this market with a trading policy $\{\chi_t\}_{t \geq 0}$, where χ_t is the amount of shares to be executed at time t . The large trader hides its order size from the market. Therefore the market will not know about (χ_t) until it is executed³. However, we assume that there is one insider trader who is informed about the trading policy of the large trader. Hence, we have three types of traders in this market:

- The *large trader* who has market impact and is aware of its own market impact.
- The *insider trader* who knows as much as the large trader, but has no market impact.
- The *average trader* who is too small to have a market impact and is unaware of the large trader activities. This trader is not the subject of study in this paper.

It should be noted that while the conclusions of this paper can be extended to any trading policy, here we assume that the large trade strategy is to hedge a derivative exposure.

3 The Market Impact Model

Let's assume that the large trader would like to execute χ_t shares at any time t . The size of χ_t being too big, there may be not enough liquidity for this order. Considering that the depth of the market varies over time and there is liquidity in some days compared to others, one can not predict in advance the impact of that execution. We assume that within an infinitesimal period $[t, t + dt)$, there would be either enough market depth or not. If there is enough market depth, then there is no market impact. On the contrary, if there is not enough depth, then there is a market impact of size $\Lambda(\chi_t)$. We assume that the chances of being in a no depth market during the period $[t, t + dt)$ are ϕdt , with ϕ a positive constant. Taking the above into consideration, we define π_t as the process for the depth of the market and we assume that it follows a Poisson process with intensity ϕdt .⁴ The dynamics of the stock price are then given by the following

³While this goes beyond the scope of this paper, there are many algorithmic methods to hide the order size, the most simple one is a price taker. Another one may be iceberg type limit orders. See [Christensen & Woodmansey \(2013\)](#) for more details.

⁴It is worth mentioning that for the sake of simplicity, we assume that the impact rate of the order (ϕ) is independent from its size (χ_t). A relationship between ϕ and χ_t is less studied in the empirical literature, because most academic literature so far investigate the average price impact rather than its variations. We make the point that while our model can be generalized by integrating this relationship, our intention is not to define a

extended Black-Scholes formulation:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sigma dW_t + \Lambda(\chi_t)d\pi_t, \\ d\pi_t &\sim \text{Poisson}(\phi dt),\end{aligned}\tag{3.1}$$

where W_t is a Brownian motion, μ is the drift and σ is the volatility of the asset. $\Lambda(\cdot)$ represents the market impact defined as a non-decreasing function of executed shares with $\Lambda(0) = 0$. Note that the market impact could be positive or negative depending on whether the shares are bought or sold. In the case of no trading activity by the large trader, Equation (3.1) reduces to the standard Black-Scholes dynamics.

For the sake of completeness, we also assume that the holder of the stock benefits from a dividend yield noted as q , and that there exists a securities lending market and a future market for the underlying stock and the holder of the stock bears a cost of carry of b . The cost b is generally the adjustment rate between the future market and the value of the stock as of today. Although these markets may be subject to stringent liquidity issues, they allow to enter into short selling positions.⁵

4 Insider Trader's Martingale Measure

The insider trader has access to the same level of information than the large trader, it knows about the trading policy $\{\chi_t\}_{t \geq 0}$ and its potential market impact. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the natural filtration generated by $\{W_t\}_{t \geq 0}$ and $\{\pi_t\}_{t \geq 0}$, and let $\{r_t\}_{t \geq 0}$ be a deterministic interest rate process, the following theorem explains that there exists a set of probability measures under which the asset is a martingale.

Theorem 1. *There exists a set of probability measures \mathcal{I} where for any measure $I \in \mathcal{I}$ the returns of the discounted asset are martingales under this measure:*

$$S_t = E^I \left[e^{-\int_t^T r_s ds} e^{(q-b)(T-t)} S_T \middle| \mathcal{F}_t \right].\tag{4.1}$$

An option \mathcal{O} bought/sold by the insider trader is also a martingale under this probability:

$$\mathcal{O}_t = E^I \left[e^{-\int_t^T r_s ds} \mathcal{O}_T \middle| \mathcal{F}_t \right].$$

We call this set of probabilities information-neutral probabilities.

very comprehensive model, but layout a justified mathematical framework for studying the role of information on market impact and arbitrage pricing theory.

⁵Even though our model does not consider this, in illiquid markets, the borrowing spread can have a very stochastic behaviour and change drastically over time. For example, if the market experiences a *short squeeze* situation in some period, the borrowing rate will increase drastically. An example was the QuantumSpace Corp on beginning of January 2021 where its lending rate reached more that 100% per year because of a lack of supply in the lending market.

Note that Theorem 1 infers that the market is not complete. In fact, it is known in the literature, see Tankov & Voltchkova (2009), that in the presence of jumps in addition to a Brownian motion, the market becomes incomplete because one cannot perfectly hedge both sources of variability with a standard hedging portfolio of the underlying asset and a risk free instrument. This also implies that there exists an infinity of martingale measures. With this consideration, the proof of Theorem 1 lays within the general framework of Martingale Pricing Theory and detailed discussions about the set of probability measures under an incomplete market can be found in Tankov (2003).

We called \mathcal{I} the set of information-neutral measures because it corresponds to the probability measure for those who hold the information and get over it. To clarify this, an example would be when χ_t is always positive as formulated in Equation (4.2); this Equation suggests that the asset is being sold at a discount. However the insider individual will still view the asset as a martingale under \mathcal{I} and gets over the discounted misprice of the asset. Despite the fact that it could benefit from the mispriced asset, it will project the asset value as a martingale in the information-neutral probability; such proposition is the basis of Equation (4.1).

$$\begin{aligned} \forall u \geq 0 : \chi_u \geq 0 &\Rightarrow \Lambda(\chi_u) \geq 0 \\ &\Rightarrow \text{Expected Discounted Asset Returns} = E^Q \left[e^{\int_0^T -r_u du} e^{(q-b)T} S_T \right] \\ &= S_0 E^Q \left[e^{\sigma W_T - \frac{1}{2} \frac{\sigma^2}{2} T} \prod_{i=0}^{\pi_T} (1 + \Lambda(\chi_{t_i})) \right] = S_0 E^Q \left[\prod_{i=0}^{\pi_T} (1 + \Lambda(\chi_{t_i})) \right] \geq S_0, \end{aligned} \quad (4.2)$$

where π_T is the number of market impact jumps occurred up to time T , and $\{t_i\}_{0 \leq i \leq \pi_T}$ represents the times of the market impact jumps. The Equation (4.2) shows that the asset returns form a sub-martingale under the risk-neutral measure Q for a filtration generated by $\{W_t\}_{t \geq 0}$ only.

Out of the set of information-neutral measures, there is one that reduces the quadratic variance between the hedging and the option portfolio, we call this measure the *minimal variance information-neutral* probability and Theorem 2 describes such measure.

Theorem 2. *Let's assume the regulatory condition formulated in Equation 4.3:*

$$\Lambda(\chi_t)(\mu + (q - b) - r_t) < \sigma^2, \quad (4.3)$$

there exists a probability measure \mathcal{I}^ that minimizes the quadratic variance of the difference between the hedging portfolio and option value, and the discounted return of the asset is a martingale:*

$$\frac{dS_t}{S_t} = (r_t - (q - b) - \Lambda(\chi_t)\omega_t) dt + \sigma dW_t^{\mathcal{I}^*} + \Lambda(\chi_t)d\theta_t. \quad (4.4)$$

Here, θ_t is a compound Poisson process with intensity ω_t , defined as

$$\omega_t = \phi \left(\frac{\sigma^2 - \Lambda(\chi_t) (\mu + (q - b) - r_t)}{\sigma^2 + \phi \Lambda(\chi_t)^2} \right).$$

The option price \mathcal{O} can be formulated via the following partial differential equation:

$$\begin{aligned} r_t \mathcal{O}(t, S_t) &= \frac{\partial \mathcal{O}(t, S_t)}{\partial t} + (r_t - (q - b) - \Lambda(\chi_t) \omega_t) S_t \frac{\partial \mathcal{O}(t, S_t)}{\partial S_t} \\ &+ \frac{1}{2} \frac{\partial^2 \mathcal{O}(t, S_t)}{\partial S_t^2} \sigma^2 S_t^2 + (\mathcal{O}(t, S_t + S_t \Lambda(\chi_t)) - \mathcal{O}(t, S_t)) \omega_t \end{aligned} \quad (4.5)$$

A detailed proof of this theorem is available in Appendix A and is based on quadratic hedging and local risk-minimization methods. It is worth mentioning that within the proof of Theorem 2, Girsanov Theorem has been applied on both the jump process and the Brownian process. The consequence of applying a change of probability on the Brownian process is a change of drift from μ to $r_t - (q - b)$, the consequence of applying the change of probability to the Poisson process is a change to the jump intensity.⁶ Note that if $\chi_t = 0$, or $\phi = 0$, then Equation (2) reduces to the standard Black & Scholes partial differential equation. We remark that Lee & Song (2007) also derived local risk-minimizing strategies for insider traders who have more information than the general public. However, the insider information is modelled via another diffusion process, which models a sequence of firm-specific information, and is only observed by insiders. We also note that the normality condition in Equation (4.3) is necessary to guarantee that the intensity process ω is positive.

5 Option Hedging for Large Trader

We define an option with a terminal payoff of cash and shares to be delivered physically⁷. If the large trader is, for instance, the seller of the option, the number of shares component, noted as $Q(S_T)$ would be negative and the terminal cash of the transaction $\mathcal{H}(S_T)$ would be positive. Here, T denotes the expiry date of the option. When hedging its position, the large trader has the amount of H_t in cash - usually negative, borrowed - and V_t in stock at time t . Contrary to the standard Black-Scholes framework, V_t is not necessarily the standard Delta (derivative of the option value to the spot price). Considering that the cost of hedging may be too high, the large trader may decide to leave some of its positions un-hedged with the hope that the market will go on its desired direction and the un-hedged portion will not become problematic, therefore it sets a delta-hedging shares trading policy $\{V_t\}_{0 \leq t \leq T}$.

⁶Note that even though our model has the capacity to include collateral or margin, for simplicity we assumed that there is no requirement. We refer the reader to Piterbarg (2010) for more details on the impact of collateral agreements and funding on derivatives pricing.

⁷Physical delivery is the most common type of transactions on stock options, See for example Cboe Inc. (2022).

The dynamics of the portfolio are described in Equation (5.1). As presented in the equation, we assume that dV_t assets are purchased/sold at the total cost of $(S_t + dS_t)dV_t + dC_t$, where $\frac{dC_t}{dV_t}$ is a transaction cost and $S_t + dS_t$ is the realized price at the end of the execution period. We assumed that the total cost depends on the price at the end of the execution period simply because of the execution time; because of the size of dV_t it will take some time for it to be executed and this won't be instantaneous. Also there is additional transaction costs noted as one dC_t , this is the cost of purchasing dV_t shares during the interval $[t, t + dt)$. We will parametrize dC_t further in Section 6.1 but at this stage the reader shall be content to know that dC_t would depend on the size of the execution request dV_t .

$$\begin{aligned} V_{t+dt} &= V_t + dV_t \\ H_{t+dt} &= H_t(1 + r_t dt) + V_t S_t (q - b) dt - (S_t + dS_t) dV_t - dC_t \end{aligned} \quad (5.1)$$

It is worth noticing that the hedging strategy is not self-financing; there may be cash flow injection whenever it is necessary to finance any cash shortage. Taking the integral of Equation (5.1), the accounting value of hedges can be reformulated as in Equation (5.2), where $H_s + S_s V_s$ represents the ‘‘accounting’’ value of the hedging portfolio as the sum of the shares account $V_s S_s$ and the cash account H_s ⁸:

$$\begin{aligned} H_T + S_T V_T &= e^{\int_t^T r_s ds} (H_t + V_t S_t) - e^{\int_t^T r_s ds} \int_t^T e^{-\int_t^u r_s ds} S_u dC_u + P\&L_{trading}, \\ P\&L_{trading} &= e^{\int_t^T r_s ds} \int_t^T V_u d \left(e^{-\int_t^u r_s ds} S_u - (q - b) e^{-\int_t^u r_s ds} S_u du \right), \end{aligned} \quad (5.2)$$

where $P\&L_{trading}$ in Equation (5.2) is the product-sum of V_u , the number of shares owned at time u , and the discounted assets return over the time frame du . As a sum, it could be interpreted as a profit-and-loss made from the delta rebalancing trading activities. Under a risk-neutral Black & Scholes framework, the expectation value of $P\&L_{trading}$ is zero. The same is true for the information-neutral measure under the filtration \mathcal{F}_t (in mathematical terms, $E^I [P\&L_{trading} | \mathcal{F}_t] = 0$) since the discounted asset returns are martingale under such probability measure and filtration. However, this is not true under the historical measure P (in mathematical terms, $E^P [P\&L_{trading} | \mathcal{F}_t] \neq 0$). It will be explained further, one has to carefully consider this component as it may result in a *squeezing activities* that are considered as a *market abuse*.

The terminal profit and loss ($P\&L$) is defined by Equation (5.3) and has three components: the payoff, the accounting value of hedges and the terminal cost of executing the missing shares between the time of expiry and the settlement date of the derivative transactions. With a physical delivery option, any missing shares have to be executed by the time between the expiry T and the settlement date of the option denoted by $T + \tau$. Due to the size of the shares there would be

⁸See Appendix B for the proof.

some transaction costs, noted as $\mathcal{C}(Q(S_T) + V_T, \tau)$ in Equation (5.3)⁹.

$$P\&L = \overbrace{\mathcal{H}(S_T) + S_T Q(S_T)}^{\text{Payoff}} + \overbrace{H_T + S_T V_T}^{\text{Accounting value of hedges}} - \overbrace{S_T \mathcal{C}(Q(S_T) + V_T, \tau)}^{\text{Terminal execution cost of missing shares}}. \quad (5.3)$$

Combining Equations (5.2) and (5.3), one can obtain the following formulation for the terminal $P\&L$:

$$P\&L = \overbrace{\mathcal{H}(S_T) + S_T Q(S_T)}^{\text{Payoff}} - \overbrace{\left(e^{\int_t^T r_s ds} \int_t^T e^{-\int_t^u r_s ds} S_u dC_u + S_T \mathcal{C}(Q(S_T) + V_T, \tau) \right)}^{\text{Total transaction cost of missing shares}} + P\&L_{trading} + e^{\int_t^T r_s ds} (H_t + V_t S_t). \quad (5.4)$$

5.1 Market Abuse Constraints and Problem Reduction

The large trader has to be careful about manipulating the payoff through its hedging strategy. In essence, the trader's interest is to optimize the $P\&L$ of the Equation (5.4) but (s)he is bounded by two regulatory statements:

- **Manipulating the payoff:** The large trader shall not manipulate the payoff of the option through the market impact of its hedging activities. Otherwise, it will be considered as a market abuse as prescribed in article 12.1.a.ii of the EU market manipulation regulation (European Parliament and of the Council 2014) and the FCA handbook example of Market abuse (Financial Conduct Authority 2005) (MAR 1.6.15):

... entering into a transaction, placing an order to trade or any other behaviour which: ii) secures, or is likely to secure, the price of one or several financial instruments ... at an abnormal or artificial level ...

*... manipulative strategies can also extend across spot and derivatives markets. Trading in financial instruments, including commodity derivatives, can be used to manipulate related spot commodity contracts and spot commodity contracts can be used to manipulate related financial instruments. The prohibition of market manipulation should capture these inter-linkages ...*¹⁰

- **Abusive squeeze:** Equation (5.4) has a $P\&L_{trading}$ component. An expected positive value for this component means that the large trader makes profits on the valuation of its shares owned, as a result of its hedging activities. By regulation this is considered as a

⁹The amount of shares to be delivered/received is $Q(S_T)$. $\mathcal{C}(Q(S_T) + V_T, \tau)$ is a deterministic quantity at time T . At this time, we assume that we know exactly the cost of executing shares between T and $T + \tau$. Please note the sum of $Q(S_T) + V_T$ is the large trader shares account as the terminal date; to be a hedge V_T and $Q(S_T)$ do not have the same sign, in other words if $Q(S_T)$ is positive, V_T shall be negative, and vice versa.

¹⁰The statement continues with *... a trader holds a short position that will show a profit if a particular financial instrument, which is currently a component of an index, falls out of that index ... He places a large sell order in this financial instrument just before the close of trading. His purpose is to position the price of the financial instrument at a false, misleading, abnormal or artificial level so that the financial instrument will drop out of the index so as to make a profit ...*

market abuse as stated in section 1.6.4 of the Market Abuse directive of the FCA handbook ([Financial Conduct Authority 2005](#)):

...an abusive squeeze - that is, a situation in which a person ... has a significant influence over the supply of, or demand for, or delivery mechanisms for a qualifying investment or related investment or the underlying product of a derivative contract...

Considering the above, the optimization problem needs to be reduced firstly by excluding the *payoff* component of Equation (5.4) to address the *manipulating the payoff* constraint, and secondly by defining the stochastic control problem under one of the information neutral measures so that the expected value of the $P\&L_{trading}$ becomes zero and an *abusive squeeze* is not inherent to the model. Considering that the only information neutral measure we have specified in this paper is the *minimum variance information neutral measure*, we will use this measure for the stochastic control problem. Therefore the problem formulation is reduced to Equation (5.5), with function J as its solution.

$$J(t, V_t, S_t) = \inf_{\{V_s\}_{s \geq t}} E^{I^*} \left[e^{-\int_t^T r_s ds} S_T \mathcal{C}(Q(S_T) + V_T, \tau) + \int_t^T e^{-\int_t^s r_u du} S_s d\mathcal{C}_s \middle| \mathcal{F}_t \right] \quad (5.5)$$

The first component of Equation (5.5) is the transaction cost of executing missing shares between the expiry date T and the settlement date. The second part of Equation (5.5) is the transaction cost of executing shares before the expiry date. These two terms act against each other and the trader has to make a compromise between i) buying shares too early and getting closer to a full delta hedge with less missing shares to recover in the future, but paying extra costs because of executing the shares too quickly, ii) or leaving its position partially unhedged. If lucky the market will go on its direction and the partial hedge will be closer to the full hedge. Otherwise it will have some time between the expiry date T and the settlement date $T + \tau$ to execute the missing shares. Note that at the expiry date, the number of shares that is required is fixed and is no more uncertain.

Theorem 3. *The cost function J can be deduced from the following partial differential equation:*

$$\frac{\partial \mathcal{J}^*}{\partial t} + rS \frac{\partial \mathcal{J}^*}{\partial S} + \frac{1}{2} \frac{\partial^2 \mathcal{J}^*}{\partial S^2} \sigma^2 S^2 + \frac{\mathcal{J}^* - J}{dt} \Bigg|_{(t, V, S)} = rJ(t, V, S), \quad (5.6)$$

with the terminal condition

$$J(T, V, S) = S_T \mathcal{C}(Q(S_T) - V_T, \tau), \quad (5.7)$$

where \mathcal{J}^* is defined as follows:

$$\begin{aligned} \mathcal{J}_\chi(t, V, S) &= J(t, V + \chi, S + S\Lambda(\chi)) \omega_t dt + J(t, V + \chi, S)(1 - \omega_t dt) + S d\mathcal{C}_t(\chi), \\ \mathcal{J}^*(t, V, S) &= \inf_{\chi} \{ \mathcal{J}_\chi(t, V, S) \}, \end{aligned} \quad (5.8)$$

and χ_t^* is the optimal number of shares to be executed during the period dt and obtained from

$$\chi^* = \underset{\chi}{\operatorname{arg\,inf}} \{ \mathcal{J}_\chi(t, V, S) \}. \quad (5.9)$$

Proof is available in Appendix D. Note that ω_t also depends on χ . Also note that in case of no market impact, $\Lambda = 0$ and Equation (5.6) reduces to the standard Black-Scholes equation.

6 Numerical Results

6.1 Transaction cost parametrization

Over the infinitesimal period dt , we define the infinitesimal cost of execution $d\mathcal{C}_t(\chi_t)$ of the trading policy χ_t by Equation (6.1) where A is the (annualized) average volume, η is an impact factor and $\alpha < 1$ is the exponential factor. Except for some differences on the denominator, this formulation corroborates with some of the past literature, e.g. Grinold & Kahn (2000) and Zarinelli *et al.* (2015) where α is often set to $1/2$, and corresponds to a square root formula.

$$d\mathcal{C}_t(\chi_t) = \eta \left(\frac{\chi_t}{\chi_t + A dt} \right)^\alpha \chi_t \quad (6.1)$$

For a longer period τ between the expiry and settlement date, one approach can be to apply an optimal purchase/liquidation algorithms which is well studied in the literature, e.g. Almgren & Chriss (1997). But here we stick into a purchase constant rate program. Then in that case the expectation of function $C(x, \tau)$ - the terminal cost of executing x at the time τ between the expiry and settlement- can be formulated as in Equation (6.2). For a formal proof, we refer to Appendix C.

$$E [C(x, \tau) | \mathcal{F}_T] = \eta \times \frac{\left(\frac{x}{\tau}\right)^\alpha}{\left(\frac{x}{\tau} + A\right)^\alpha} x \quad (6.2)$$

6.2 Solving the partial differential equation

Given the two-dimensional partial differential Equation (5.6), numerical implementation can be done via the dynamic programming approach described in Algorithm 1.

Algorithm 1 Methodology for solving the large trader’s cost function

- 1: $J(T, S, V) \leftarrow C(Q(S) - V, \tau)$;
 - 2: Discretize the time to Δt intervals;
 - 3: **for** $t = T; t \leftarrow t + \Delta t; t = 0$ **do**
 - 4: Using $J(t + \Delta t, \dots)$ apply equation (5.9) and solve χ^* ;
 - 5: Using χ^* , apply Equation (5.8) and calculate $\mathcal{J}^*(t + \Delta t, \dots)$;
 - 6: Using $\mathcal{J}^*(t + \Delta t, \dots)$, solve the Equation (5.6) and obtain $J(t, \dots, V)$, the finite difference scheme is explained in Appendix E;
 - 7: **for** $iter = 0; iter \leftarrow iter + 1; iter < numiter$ **do**
 - 8: Using $J(t, \dots)$ apply equation (5.9) and solve χ^* ;
 - 9: Using χ^* , apply Equation (5.8) and recalculate $\mathcal{J}^*(t + \Delta t, \dots)$;
 - 10: Using $\mathcal{J}^*(t + \Delta t, \dots)$, solve Equation (5.6) and obtain $J(t, \dots, V)$;
 - 11: **end for**
 - 12: Having χ^* , solve insider’s trader option price \mathcal{O} using a finite difference scheme of the partial differential Equation (2);
 - 13: **end for**
 - 14: Obtain $J(0, S_0, 0)$ and $\mathcal{O}(0, S_0)$
-

6.3 Numerical Results

Here we assume that the market follows the parameters described in Table 1 and that the large trader is the seller of the call option with parameters given in Table 2. Our overall parameter assumptions on the market impact are in line with the empirical study of Jarrow *et al.* (2018). The choice of parameters for the call option are also in line with one could expect from *over-the-counter* ("OTC") derivatives.

μ	σ	η	ϕ	λ	S_0	A	r	α	q	b
0.02	0.25	0.1	20.0	2.0	100	5e+6	0.01	2.0	0.0	0.0

Table 1: Market parameters

T	τ	Call strike	Notional
120 days	15 days	110	1e+6

Table 2: Large Trader’s option

Figure 6.1 shows the terminal cost function J for the call option. For example, the back corner of the graph shows situations when the product is in the money. The number of shares V_T is then exactly equal to the shares to be delivered and the cost function is 0. The left corner shows when V_T is 0 and the product is in the money. The large trader needs, then, to purchase the

notional amount of shares before the settlement date. In such case, the cost will be high since our model for the cost, see Equation (5.7), is linear in the spot price. As it can be viewed from Figure 6.1, even though the cost function has different values on each corners of the graph, cost function gradually flattens when we go backward in time as it can be observed in Figure 6.2.

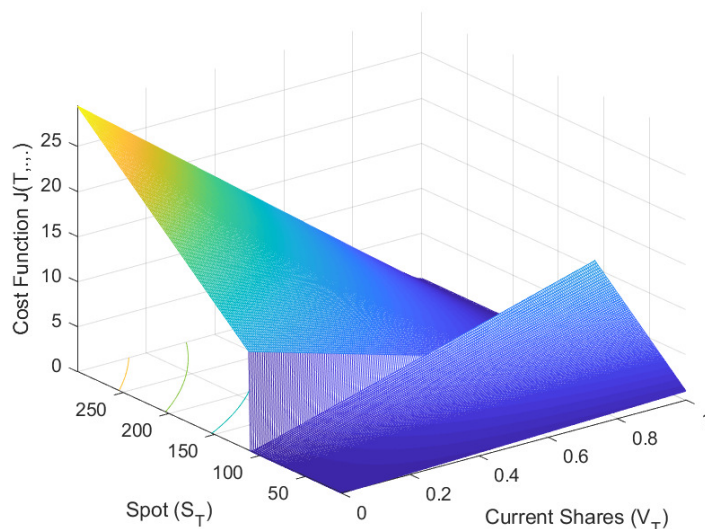


Figure 6.1: The terminal cost function J for a hedged option. V_T shall be read as a multiple of the notional.

Figure 6.4 shows the amount of shares to be executed (χ^*) at time 0, and Figure 6.3 shows this amount just before the expiry.¹¹ The graphs show that because a full hedge strategy is too costly, the optimal χ^* are much lower than the the Black-Scholes *Delta*. The difference in *Delta* are also presented in Figure 6.5; this graph suggests to purchase only few percent of χ^* at time 0 while the Black Scholes *Delta* is 100% for the same spot levels (right side of the graph). This is significantly lower than the Black Scholes *Delta*, because the large trader has time before expiry to execute the full hedge, hence it will not make a big execution at time 0, Nevertheless, the large trader strategy χ^* gets closer to the Black-Scholes *Delta* when getting closer to expiry. This is shown in Figure 6.6 illustrating the hedging strategies χ^* and Black-Scholes *Delta* just before the expiry date. We observe that the number of shares to be purchased by the large trader is still smaller than the Black Scholes *Delta* but reaches 50% of the B-S *Delta* for high spot prices. Near the expiry there is less time left for share execution and the large trader has to overcome significant stock shortage in a limited time frame compared to the inception date.

¹¹The graph shows that the number of shares to be executed in the z-axis are layered. This is mainly due to the methodology that we use in our algorithm where we determine the χ^* within the discretization points of (V) in the finite difference scheme and therefore the optimal points will be discrete. This ensures a better stability of the finite difference scheme.

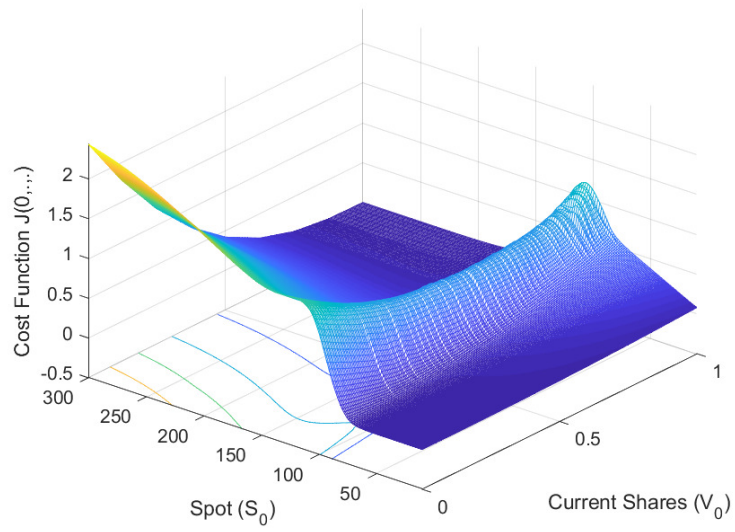


Figure 6.2: The cost function J at time 0 for a hedged call option.

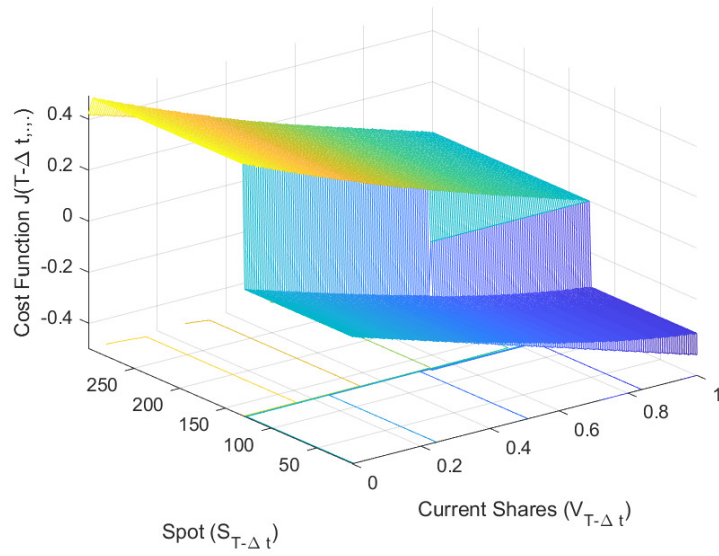


Figure 6.3: The optimal shares strategy function at time $T - dt$ for a call option.

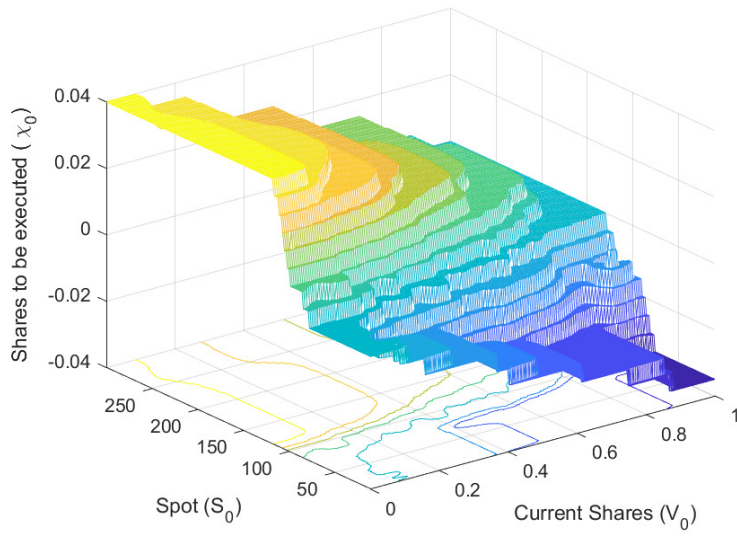


Figure 6.4: The optimal shares strategy function at time 0 for a call option

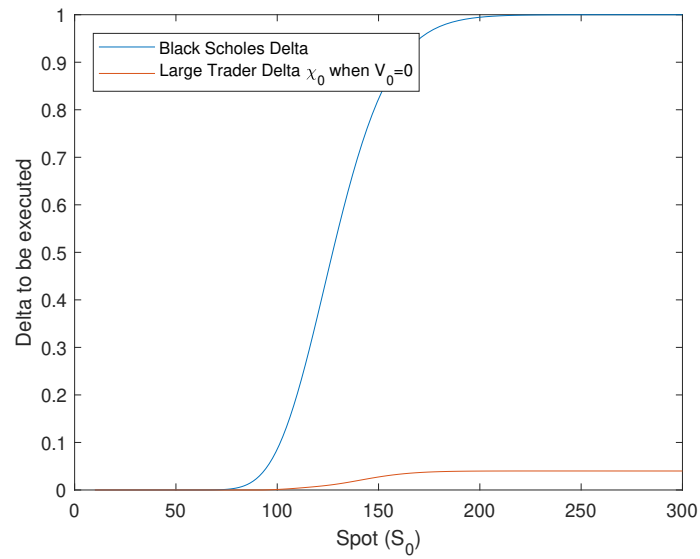


Figure 6.5: Comparison between BS delta and optimal shares for the large trader at inception, assuming $V_0 = 0$

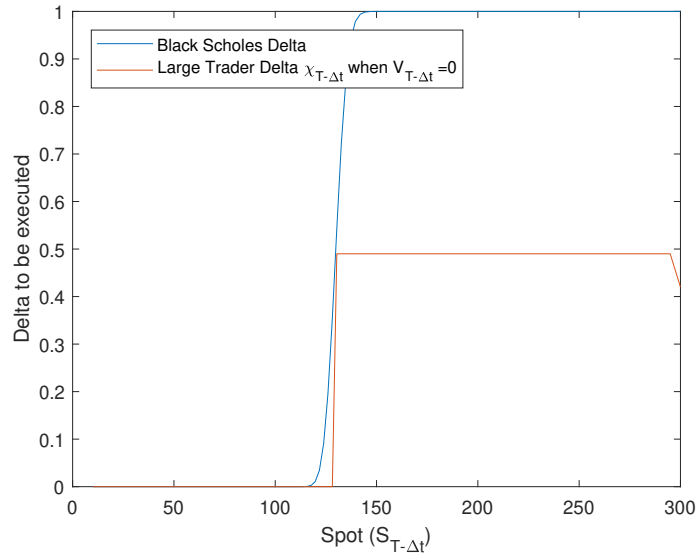


Figure 6.6: Comparison between BS delta and optimal shares for the large trader at expiry, assuming $V_T = 0$

Considering that large trader is hedging the call position defined in Table 3, it would be interesting to study how it will impact the insider trader's option price. Figure 6.7 compares the insider's trader price with the Black and Scholes model for the same call option. It shows that the option has more convexity than the Black & Scholes model around the money, it also shows that the prices are higher than the Black & Scholes when the option is in the money, and even higher than the intrinsic value. This is because of the directional market impact of the large trader. If the spot price is in-the-money, the large trader will purchase shares and therefore will make the spot price go deeper in-the-money.

T	Call strike
120 days	110

Table 3: Small Trader's option

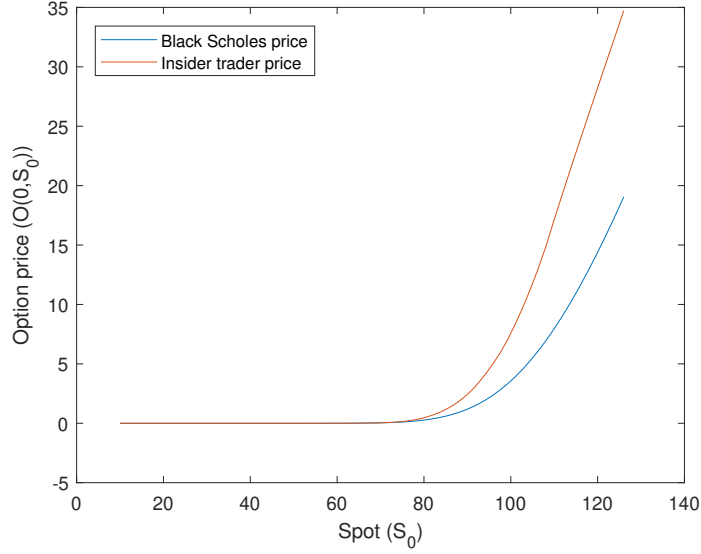


Figure 6.7: Comparison between Black and Scholes price and insider trader price for a call option.

We would like also to highlight the fact that the insider option price is sensitive to parameters ϕ , η , λ as shown in Table 4. Economically speaking, an increase in the market impact parameters ϕ and λ results in a higher call option price for the insider trader (for the option defined in Table 2).

ϕ	λ	η	Option price
10.0	0.5	0.2	4.55
10.0	1.0	0.5	4.53
20.0	1.0	0.2	5.58
20.0	0.5	0	5.46
20.0	0.0	0.2	3.59
0.0	0.0	0	3.59

Table 4: Call option price for insider trader at $t = 0$

7 Conclusion

In this paper, we presented a model framework for martingale pricing theory under market impact. We considered the case of existence of a large trader. We observed that the large trader, if negligent, can cause market manipulation by the means of its hedging activities. We emphasized that the mathematical model should not inherit a market manipulative strategy, and we proposed

a hedging policy for the large trader that could deal with this issue¹². We consider the market manipulation in derivatives as an important subject for research as the problem is quite present in the industry even though there is a lack of practical models, and this paper is one of the first in the literature that addresses this issue.

We also proposed a pricing model for an insider trader that is knowledgeable of the large trader's trading policy. We derived the pricing equation for such trader and observed that there is a set of measures called information-neutral measures in which the return of the assets are a martingale for the insider. While the presented model assumes that the information the insiders holds is the large trader's share trading activity, the scope of this paper goes beyond that; in fact, the Martingale Pricing Theory that we have presented is valid for any type of insider information as long as it generates market impact. The conclusion of this article can be then extended to these cases.

We concluded the paper with numerical results for the optimal delta-hedging strategy of an out-of-the-money Call option. While the payoff of the option has significant influence on our analysis, we observed that the model suggests a more relaxed delta hedging strategy than the Black and Scholes model, and the price of the call option for the insider trader is higher under market impact than the Black and Scholes. On that note, this paper provides a practical model that could be easily implemented for pricing derivatives under market impact.

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Disclosure statement

The authors report there are no competing interests to declare.

References

Financial Conduct Authority . 2005. FCA Handbook : MAR 1.6.15 Market abuse.

Abergel, Frédéric, & Loeper, Grégoire. 2017. Option pricing and hedging with liquidity costs and market impact. *Pages 19–40 of: Econophysics and Sociophysics: Recent Progress and Future Directions*. Springer.

¹²We also would like to underline that here we are dealing with the concept of market manipulation in derivatives. Defining mathematically such concept in the case of derivatives is actually a difficult undertaking. This is already a difficult concept to formulate mathematically market manipulation for delta-one strategies. As (Curato *et al.* 2017) state in their paper *...A fully satisfactory and practical model of market impact, which is free of price manipulation and consistent with empirical observation, thus seems to be still lacking, leaving a fruitful field for future research....* A mathematical model in the case of the derivatives is even more complex

- Alfonsi, Aurélien, Schied, Alexander, & Slynko, Alla. 2012. Order book resilience, price manipulation, and the positive portfolio problem. *SIAM Journal on Financial Mathematics*, 3(1), 511–533.
- Almgren, Robert, & Chriss, Neil A. 1997. Optimal liquidation. *Available at SSRN 53501*.
- Almgren, Robert, & Li, Tianhui Michael. 2016. Option hedging with smooth market impact. *Market microstructure and liquidity*, 2(01), 1650002.
- Bordag, Ljudmila A, & Frey, Rüdiger. 2008. Pricing options in illiquid markets: symmetry reductions and exact solutions. *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, 103–129.
- Bouchaud, Jean-Philippe, Gefen, Yuval, Potters, Marc, & Wyart, Matthieu. 2003. Fluctuations and response in financial markets: the subtle nature of random price changes. *Quantitative finance*, 4(2), 176.
- Brigo, Damiano, Graceffa, Federico, & Neuman, Eyal. 2020. Price Impact on Term Structure. *Available at SSRN 3733907*.
- Cboe Inc. 2022. *Equity Options Product Specifications*.
- Cetin, Umut, Jarrow, Robert, Protter, Philip, & Warachka, Mitch. 2006. Pricing options in an extended Black Scholes economy with illiquidity: Theory and empirical evidence. *The Review of Financial Studies*, 19(2), 493–529.
- Cetin, Umut, Jarrow, Robert A, & Protter, Philip. 2010. Liquidity risk and arbitrage pricing theory. *Pages 1007–1024 of: Handbook of Quantitative Finance and Risk Management*. Springer.
- Christensen, Hugh L, & Woodmansey, Robert. 2013. Prediction of Hidden Liquidity in the Limit Order Book of GLOBEX Futures. *The Journal of Trading*, 8(3), 68–95.
- Cont, Rama, & Schaanning, Eric. 2017. Fire sales, indirect contagion and systemic stress testing. *Indirect Contagion and Systemic Stress Testing (June 13, 2017)*.
- Curato, Gianbiagio, Gatheral, Jim, & Lillo, Fabrizio. 2017. Optimal execution with non-linear transient market impact. *Quantitative Finance*, 17(1), 41–54.
- European Parliament and of the Council . 2014. Regulation (EU) No 596/2014 on market abuse (market abuse regulation).
- Eyraud-Loisel, Anne. 2005. Backward stochastic differential equations with enlarged filtration: Option hedging of an insider trader in a financial market with jumps. *Stochastic processes and their Applications*, 115(11), 1745–1763.

- Eyraud-Loisel, Anne. 2013. Quadratic hedging in an incomplete market derived by an influential informed investor. *Stochastics An International Journal of Probability and Stochastic Processes*, 85(3), 412–430.
- Gatheral, Jim. 2010. No-dynamic-arbitrage and market impact. *Quantitative finance*, 10(7), 749–759.
- Glover, Kristoffer J, Duck, Peter W, & Newton, David P. 2010. On nonlinear models of markets with finite liquidity: some cautionary notes. *SIAM Journal on Applied Mathematics*, 70(8), 3252–3271.
- Grinold, Richard C, & Kahn, Ronald N. 2000. *Active portfolio management*. McGraw Hill New York.
- Guéant, Olivier, & Pu, Jiang. 2017. Option pricing and hedging with execution costs and market impact. *Mathematical Finance*, 27(3), 803–831.
- Huberman, Gur, & Stanzl, Werner. 2004. Price manipulation and quasi-arbitrage. *Econometrica*, 72(4), 1247–1275.
- Jarrow, Robert, Fung, Scott, & Tsai, Shih-Chuan. 2018. An empirical investigation of large trader market manipulation in derivatives markets. *Review of Derivatives Research*, 21(3), 331–374.
- Jonsson, Mattias, & Sircar, K Ronnie. 2002. Partial hedging in a stochastic volatility environment. *Mathematical Finance*, 12(4), 375–409.
- Lee, Kiseop, & Song, Seongjoo. 2007. Insiders’ hedging in a jump diffusion model. *Quantitative Finance*, 7(5), 537–545.
- Leland, Hayne E. 1985. Option pricing and replication with transactions costs. *The journal of finance*, 40(5), 1283–1301.
- Liu, Hong, & Yong, Jiongmin. 2005. Option pricing with an illiquid underlying asset market. *Journal of Economic Dynamics and Control*, 29(12), 2125–2156.
- Loeper, Gregoire, *et al.* 2018. Option pricing with linear market impact and nonlinear black–scholes equations. *Annals of Applied Probability*, 28(5), 2664–2726.
- Piterbarg, Vladimir. 2010. Funding beyond discounting: collateral agreements and derivatives pricing. *Risk*, 23(2), 97.
- Said, Emilio. 2019. How Option Hedging Shapes Market Impact. *arXiv preprint arXiv:1910.05056*.
- Said, Emilio. 2020. *Market Impact in Systematic Trading and Option Pricing*. Ph.D. thesis, Université Paris-Saclay.

- Tankov, Peter. 2003. *Financial modelling with jump processes*. Chapman and Hall/CRC.
- Tankov, Peter, & Voltchkova, Ekaterina. 2009. Jump-diffusion models: a practitioner's guide. *Banque et Marchés*, 99(1), 24.
- Zarinelli, Elia, Treccani, Michele, Farmer, J Dooyne, & Lillo, Fabrizio. 2015. Beyond the square root: Evidence for logarithmic dependence of market impact on size and participation rate. *Market Microstructure and Liquidity*, 1(02), 1550004.

APPENDIX

A Differential Equation for insider trader

The insider trader is aware of the large trader position. It knows the amount $\lambda\chi_t$ and can be hedged against it. We assume that the insider trader is aware of the hedges χ_t . Let's assume that the insider trader has entered into a transaction with a specific payoff $\mathcal{O}(S_T)$, we call $\mathcal{O}(t, S)$, the price of this option at time t and spot level S . The insider trader portfolio at time t is given by

$$P_t = \delta_t S_t + \overbrace{(P_t - \delta_t S_t)}^{\text{Cash Account}}$$

The insider trader has δ_t shares and $P_t - \delta_t S_t$ in cash amount. Due to the fact that the notional is small, the insider trader, contrary to the large trader, can hedge its position without additional cost of execution. The dynamics of the portfolio P_t are as follows:

$$dP_t = \delta_t (dS_t + S_t q dt - S_t b dt - S_t r_t dt) + r_t P_t dt$$

In our incomplete market, the option cannot be perfectly hedged by a self-financing portfolio. Therefore, we search for a hedging strategy that satisfies the following properties:

- $P_t = \mathcal{O}_t$: As this is not a self financing portfolio, at each time step there will be cash injection/depletion so that the hedging portfolio value is equal to the theoretical value of the option.
- $E[dP_t] = E[d\mathcal{O}_t]$: The average (first order) option price movement is fully hedged by the portfolio.
- $\delta_t = \arg \inf_{\delta} d \langle P_t - \mathcal{O}(t, S_t) \rangle$: The second order option price movement is reduced and optimally minimized by the hedging portfolio.

On the other hand, by Itô formula, the option dynamics are given by

$$d\mathcal{O} = \frac{\partial \mathcal{O}}{\partial t} dt + \frac{\partial \mathcal{O}}{\partial S} dS^C + \frac{1}{2} \frac{\partial^2 \mathcal{O}}{\partial S^2} \sigma^2 S^2 dt + [\mathcal{O}(t, S + S\Lambda(\chi_t)) - \mathcal{O}(t, S)] d\pi_t,$$

where $\mathcal{O}(t, S + S\lambda\chi) - \mathcal{O}(t, S)$ is the jump in the price of the option due to the number of shares purchased, dS^C is the continuous part of the spot movement. Therefore the stochastic part of $\mathcal{O}_t - P_t$ will be:

$$\left(\frac{\partial \mathcal{O}}{\partial S} - \delta_t \right) S \sigma dW_t + [\mathcal{O}(t, S + S\Lambda(\chi_t)) - \mathcal{O}(t, S) - \delta_t S_t \Lambda(\chi_t)] d\pi_t.$$

If the market was complete, it is possible to find δ_t such that the quadratic variation $d \langle P - \mathcal{O} \rangle_t$ is always zero. This would have been the case for the Black-Scholes model without any market

impact. However, it is impossible due to the jump component of the market impact. One can only find a strategy that minimizes the quadratic variation but it will not neutralize it. This means that the market is incomplete and one cannot perfectly hedge a call option with the underlying stock only, in the presence of market impact. The quadratic variance of $\mathcal{O}_t - P_t$ reads as follows:

$$d\langle P_t - \mathcal{O}(t, S_t) \rangle = \left(\frac{\partial \mathcal{O}}{\partial S} - \delta_t \right)^2 S^2 \sigma^2 dt + \left[\delta_t - \left[\frac{\mathcal{O}(t, S + S\Lambda(\chi_t)) - \mathcal{O}(t, S)}{S\Lambda(\chi_t)} \right] \right]^2 \phi S^2 \Lambda(\chi_t)^2 dt.$$

The insider trader minimizing the quadratic variance will obtain the following formula:

$$\delta_t S_t = \frac{\sigma^2}{\sigma^2 + \phi \Lambda(\chi_t)^2} \frac{\partial \mathcal{O}}{\partial S} S + \frac{\phi \Lambda(\chi_t)}{\sigma^2 + \phi \Lambda(\chi_t)^2} (\mathcal{O}(t, S + S\Lambda(\chi_t)) - \mathcal{O}(t, S))$$

Having the expectation of $dP_t - d\mathcal{O}_t$ to be equal to 0 knowing \mathcal{F}_t . We can derive the following partial differential equation:

$$\begin{aligned} \frac{E[dP_t]}{dt} &= \delta_t S_t (\mu + q - b - r_t + \Lambda(\chi_t)\phi) + r_t E[P_t] \\ &= r_t P_t + (\mu + q - b - r_t + \Lambda(\chi_t)\phi) \left(\frac{\sigma^2}{\sigma^2 + \phi \Lambda(\chi_t)^2} \frac{\partial \mathcal{O}}{\partial S} S + \frac{\phi \Lambda(\chi_t) (\mathcal{O}(t, S + S\Lambda(\chi_t)) - \mathcal{O}(t, S))}{\sigma^2 + \phi \Lambda(\chi_t)^2} \right) \end{aligned}$$

On the other hand, we have

$$\frac{E[d\mathcal{O}_t]}{dt} = \frac{\partial \mathcal{O}}{\partial t} + \frac{\partial \mathcal{O}}{\partial S} S_t \mu + \frac{1}{2} \frac{\partial^2 \mathcal{O}}{\partial S^2} \sigma^2 S^2 + [\mathcal{O}(t, S + S\Lambda(\chi_t)) - \mathcal{O}(t, S)] \phi$$

We conclude that:

$$\begin{aligned} r_t P_t &= \frac{\partial \mathcal{O}}{\partial t} + \left(\mu - \frac{\sigma^2 (\mu + q - b - r_t + \Lambda(\chi_t)\phi)}{\sigma^2 + \phi \Lambda(\chi_t)^2} \right) S_t \frac{\partial \mathcal{O}}{\partial S} + \frac{1}{2} \frac{\partial^2 \mathcal{O}}{\partial S^2} \sigma^2 S^2 \\ &\quad + \phi \left(1 - \frac{\Lambda(\chi_t)}{\sigma^2 + \phi \Lambda(\chi_t)^2} (\mu + q - b - r_t + \Lambda(\chi_t)\phi) \right) (\mathcal{O}(t, S + S\Lambda(\chi_t)) - \mathcal{O}(t, S)) \end{aligned} \quad (\text{A.1})$$

We simplify Equation (A.1) by defining ω_t as

$$\omega_t = \phi \left(\frac{\sigma^2 - \Lambda(\chi_t) (\mu + q - b - r_t)}{\sigma^2 + \phi \Lambda(\chi_t)^2} \right). \quad (\text{A.2})$$

One can easily verify that based on the definition of ω_t and based on the assumption that we have made on the value of the hedging portfolio being cash injected/depleted to satisfy $P_t = \mathcal{O}_t$, we obtain:

$$r_t \mathcal{O} = \frac{\partial \mathcal{O}}{\partial t} + (r_t - q + b - \Lambda(\chi_t)\omega_t) S_t \frac{\partial \mathcal{O}}{\partial S} + \frac{1}{2} \frac{\partial^2 \mathcal{O}}{\partial S^2} \sigma^2 S^2 + \omega_t (\mathcal{O}(t, S + S\Lambda(\chi_t)) - \mathcal{O}(t, S))$$

This is the partial differential equation for insider trader. We now use the Feynman-Kac formula, the equivalent stochastic equation for the spot price S_t under a probability measure that we call minimal-variance information neutral probability and note it as \mathcal{I}^* :

$$\frac{dS_t}{S_t} = (r + b - q - \chi_t \Lambda(\chi_t) \omega_t) dt + \sigma dW_t + \Lambda(\chi_t) d\theta_t,$$

where $d\theta_t$ is a compensated jump process with intensity equal to ω_t . Also $e^{-\int_0^t r_s ds} S_t$ and $e^{-\int_0^t r_s ds} \mathcal{O}$ are martingales under the \mathcal{I}^* probability probability:

$$\begin{aligned} \mathcal{O}(t, S_t) &= E^{\mathcal{I}^*} \left[e^{-\int_t^T r_s ds} \mathcal{O}(T, S_T) \middle| \mathcal{F}_t \right], \\ S_t &= E^{\mathcal{I}^*} \left[e^{-\int_t^T r_s ds} S_T \middle| \mathcal{F}_t \right]. \end{aligned}$$

B The P&L formula

Let $b_u = e^{\int_t^u -r_s ds}$ denote the discount factor or the zero coupon bond price, \bar{H}_u the discounted cash $\bar{H}_u = b_u H_u$, and $\bar{S}_u = b_u S_u$ the discounted asset. Therefore:

$$\begin{aligned} d\bar{H}_u &= -r_u \bar{H}_u du + b_u dH_u = V_u \bar{S}_u (q - b) du - \bar{S}_u dV_u - b_u dS_u dV_u - b_u S_u d\mathcal{C}_u(\chi_u) \\ d\bar{S}_u &= -r_u \bar{S}_u du + b_u dS_u \end{aligned} \quad (\text{B.1})$$

On the other hand:

$$d(V_u \bar{S}_u) = -r_u \bar{S}_u dV_u du + V_u d\bar{S}_u + \bar{S}_u dV_u = b_u dV_u dS_u + V_u d\bar{S}_u + \bar{S}_u dV_u \quad (\text{B.2})$$

Therefore we have:

$$d(V_u \bar{S}_u + \bar{H}_u) = (V_u + dV_u) d\bar{S}_u + V_u \bar{S}_u (q - b) du - b_u d\mathcal{C}_u(\chi_u) \quad (\text{B.3})$$

If we are in one of the information neutral probability spaces, we can deduce that $d\bar{S}_u - \bar{S}_u (q - b) du$ is a martingale. In the case of the minimum variance probability \mathcal{I}^* , we can deduce that:

$$d\bar{S}_u - \bar{S}_u (q - b) du = \bar{S}_u \sigma dW_u + \bar{S}_u \Lambda(\chi_u) \theta_u - \bar{S}_u \Lambda(\chi_u) \omega_u du \quad (\text{B.4})$$

Now taking the integration we obtain:

$$\begin{aligned} e^{-\int_t^T r_s ds} (S_T V_T + H_T) &= H_t + S_t V_t - \int_t^T V_u d \left(e^{-\int_t^u r_s ds} S_u - (q - b) e^{-\int_t^u r_s ds} S_u du \right) \\ &\quad - \int_t^T e^{-\int_t^u r_s ds} d\mathcal{C}_u(\chi_u) \end{aligned}$$

C Transaction cost at option expiry

For a longer purchase period, the optimal purchase/liquidation algorithms problem is a well studied topic, e.g. [Almgren & Chriss \(1997\)](#), but here we intend to consider a constant rate program. There are good reasons why we should apply a constant purchase program rather than a more complicated one since most derivatives desks do not optimise their cost of execution and they are more worried about the volatility. Therefore even though optimal algorithms exist, it will not be applied in derivatives. Here, we do not intend to make a suggestion on how the purchase should be done, but how things are behaving in the most practicable scenario. This is why we will use a constant rate program. In the case of having the Equation 6.1 for the short term cost, and with the strong assumptions that all shares requested will be executed, the long term cost of execution of X shares in τ time is

$$\mathcal{C}(X, \tau) = \int_0^\tau S_t d\mathcal{C}_t \left(\frac{X}{\tau} \right).$$

Note that in our case, $\frac{X}{\tau}$ is the constant execution rate. In case we have Equation (6.1) for the short term cost, the long term cost of execution is as follows:

$$\begin{aligned} \int_T^{T+\tau} S_t d\mathcal{C}_t &= \eta \times \frac{\left(\frac{N}{\tau}\right)^{\alpha+1}}{\left(\frac{N}{\tau} + A\right)^\alpha} \int_T^{T+\tau} e^{-rt} \left(1 + \Lambda\left(\frac{N}{\tau}\right)\right)^t S_t dt, \\ E^{\mathcal{I}} \left[\int_T^{T+\tau} S_t d\mathcal{C}_t \right] &= \eta \frac{\left(\frac{N}{\tau}\right)^{\alpha+1}}{\left(\frac{N}{\tau} + A\right)^\alpha} S_T \tau. \end{aligned}$$

D Hamilton-Jacobi-Bellman Equation for the Large Trader

The trader wants to follow a delta hedge program that minimizes its cost. Under the information neutral measure the large trader wants to optimize the following equation:

$$J(t, V, S_t) = e^{-rt} \inf_{\{V_s\}_{s \geq t}} E [J(t+dt, V+dV, S_{t+dt}) + S_t d\mathcal{C}_t | \mathcal{F}_t]$$

This equation can also be written as follows:

$$J(t, V, S) + rJ(t, V, S) dt = \inf_{dV} E \left[(dV_t) J(t+dt, V+dV, S+dS) + S d\mathcal{C}_t(dV_t) \right] \quad (\text{D.1})$$

Let's define \mathcal{J}_χ as the solution to the following equation:

$$\mathcal{J}_\chi(t, V, S) = J(t, V + \chi, S(1 + \Lambda(\chi)d\pi_t)) + d\mathcal{C}(\chi)$$

Then applying Itô formula with jumps leads to the following equation where dS^C represent the non-jump part of the asset return:

$$\mathcal{J}_\chi(t+dt, V, S+dS^C) = \mathcal{J}_\chi + \frac{\partial \mathcal{J}_\chi}{\partial t} dt + \frac{\partial \mathcal{J}_\chi}{\partial S} dS^C + \frac{1}{2} \frac{\partial^2 \mathcal{J}_\chi}{\partial S^2} (dS^C)^2 \Big|_{(t, V, S)} \quad (\text{D.2})$$

And let's also define the optimal χ^* such that:

$$\chi^* = \operatorname{arginf}_\chi E[J(t, V + \chi, S(1 + \Lambda(\chi)d\pi_t)) + d\mathcal{C}(\chi)]$$

In such case, the equation (D.1) becomes as such:

$$rJdt = \mathcal{J}_{\chi^*} - J + \frac{\partial \mathcal{J}_{\chi^*}}{\partial t} dt + \frac{\partial \mathcal{J}_{\chi^*}}{\partial S} E[dS^C] + \frac{1}{2} \frac{\partial^2 \mathcal{J}_{\chi^*}}{\partial S^2} E[(dS^C)^2] \Big|_{(t, V, S)} \quad (\text{D.3})$$

Hence, expanding the expectation of (D.3), we obtain the partial differential equations as formulated in (5.6) and (5.7).

E Numerical implementation of the large trader PDE

In terms of log returns, and assuming that χ^* is already obtained, Equation (5.6) can be written as

$$\begin{aligned} x &= \log(S), \\ \Upsilon(t, x, V) &= J(t, V, S), \\ \Upsilon^*(t, x, V) &= \mathcal{J}^*(t, V, S), \\ \bar{\Lambda}(\cdot) &= \ln(1 + \Lambda(\cdot)), \\ rJ &= \frac{\partial \Upsilon^*}{\partial t} + \left(r - (q - b) - \frac{1}{2} \sigma^2 \right) \frac{\partial \Upsilon^*}{\partial x} + \frac{1}{2} \frac{\partial^2 \Upsilon^*}{\partial x^2} \sigma^2 \\ &\quad + (\Upsilon^*(t, V, x + \bar{\Lambda}(\chi^*)) - \Upsilon(t, V, x)(\chi^*)) \frac{1}{dt} \end{aligned}$$

We define the boundary levels x_{min}, x_{max} for the log spot and d_{min}, d_{max} for the shares. We define the discretization $\{(t_i, x_l, d_k)\}$ with

$$\begin{aligned} (0, 0, 0) &\leq (i, l, k) \leq (I, L, K), \\ (0, x_{min}, d_{min}) &\leq (0, x_k, d_l) \leq (T, x_{max}, d_{max}). \end{aligned}$$

Here we assume a uniform grid, with intervals of triplet $(\Delta t, \Delta x, \Delta d)$. At the maturity level, we define the following equation:

$$\Upsilon_{I, l, k} = C(d_k - Q(e^{x_l}), \tau)$$

At t_{i+1} , we first calculate the optimal value in discretize terms. Assuming that we have the

optimal χ^* we define $\iota^* = \frac{\chi^*}{\Delta d}$ and $L_{i,l,k}^*$ as below equation for the gridpoint (t_i, x_l, d_k) .

$$L_{i,l,k}^* = (\mathcal{Y}_{i+1,l,k+\iota^*}^* - \mathcal{Y}_{i+1,l,k}) (1 - \omega^* \Delta t) + \left(\mathcal{Y}_{i+1,l+\lfloor \frac{\Delta(\iota^* \Delta d)}{\Delta x} \rfloor, k+\iota^*}^* - \mathcal{Y}_{i+1,l,k} \right) \omega^* \Delta t + e^{x_l} d\mathcal{C}(\iota \Delta d) \quad (\text{E.1})$$

We apply a fully implicit scheme and the discretization will be as of below:

$$\begin{aligned} & \left(1 + r\Delta t + \frac{\Delta t}{(\Delta x)^2} \sigma^2 \right) \mathcal{Y}_{i,l,k} + \left((r - (q - b) - \frac{1}{2} \sigma^2) \frac{\Delta t}{2\Delta x} - \frac{1}{2} \frac{\Delta t}{(\Delta x)^2} \sigma^2 \right) \mathcal{Y}_{i,l-1,k} \\ & + \left(-(r - (q - b) - \frac{1}{2} \sigma^2) \frac{\Delta t}{2\Delta x} - \frac{1}{2} \frac{\Delta t}{(\Delta x)^2} \sigma^2 \right) \mathcal{Y}_{i,l+1,k} = \mathcal{Y}_{i+1,l,k}^* + L_{i,l,k}^* \end{aligned}$$

With Neumann boundary condition, we have:

$$\begin{aligned} \mathcal{Y}_{i,l,k} = \mathcal{Y}_{i,0,k} &= C(N_T - Q(e^{x_{min}}), \tau) \approx -C(d_k - Q(e^{x_{min}}), \tau + T - t_i) \\ \mathcal{Y}_{i,L-1,k} = \mathcal{Y}_{i,L,k} &\approx C(d_k - Q(e^{x_{max}}), \tau + T - t_i) \end{aligned}$$