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Three-weight codes over rings and strongly walk regular graphs*

Minjia Shi[†], Michael Kiermaier[‡], Sascha Kurz[‡], Patrick Solé[§]

Abstract: We construct strongly walk-regular graphs as coset graphs of the duals of codes with three non-zero homogeneous weights over \mathbb{Z}_{p^m} , for p a prime, and more generally over chain rings of depth m , and with a residue field of size q , a prime power. In the case of $p = m = 2$, strong necessary conditions (diophantine equations) on the weight distribution are derived, leading to a partial classification in modest length. Infinite families of examples are built from Kerdock and generalized Teichmüller codes. As a byproduct, we give an alternative proof that the Kerdock code is nonlinear.

Keywords: strongly walk-regular graphs, three-weight codes, homogeneous weight, Kerdock codes, Teichmüller codes.

MSC (2010): Primary 05 E 30, Secondary 94 B 05

1 Introduction

Since the seminal article of Delsarte [12], there is a well-known interplay between two-weight codes and strongly regular graphs (SRG) via the coset graph of the dual code [2, 3]. Strongly walk-regular graphs (SWRG) were introduced in [11] as a generalization of strongly regular graphs. Instead of a regularity condition bearing on

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paths of length 2, the notion of SWRG demands a regularity on paths of length $s > 1$. Specifically, a graph is s -SWRG if there are three integers (λ, μ, ν) such that the number of paths of length s between any two vertices x and y is

- λ if x and y are connected;
- μ if x and y are disconnected;
- ν if $x = y$.

This is reminiscent of strongly regular graph (SRG) a regular graph on ν vertices such that the number of paths of length 2 between any two vertices x and y is

- λ if x and y are connected;
- μ if x and y are disconnected;
- κ if $x = y$.

Note that an immediate counting argument shows that strongly regular graphs are s -SWRG for all $s > 1$. In a previous paper [39], the authors connected together the two notions of triple sum sets (TSS) and SWRG over finite fields. In the present paper we extend this correspondence to a correspondence between TSS's over certain finite rings and SWRG's via the notions of syndrome graphs and homogeneous weights. This leads us to study the sum of the weights of three-weight codes. The first class of rings we consider is that of \mathbb{Z}_{p^m} , for p a prime, when the homogeneous weight can be described explicitly. Building on the study in [37], we give a sufficient condition to construct SWRGs from the dual code of a three-weight code. This condition bears on the sum of the three weights in question. In concrete examples, we focus on \mathbb{Z}_4 -codes which have been studied extensively since [20]. The Gray map, which preserves MacWilliams duality, allows us to write the four Pless power moments for the binary image code, and to classify the relevant three-weight codes in short length. Further, a class of irreducible cyclic codes [42] related to the Kerdock code [20] gives an alternative proof of the nonlinearity of the binary Kerdock code. The second class of rings considered is that of chain rings of given depth and residue field. The p -ary three-weight codes introduced in [38], and the generalized Teichmüller codes of [25] provide infinite families of three-weight codes leading to SWRG's.

The material is organized as follows. The next section collects some prerequisite on, successively, homogeneous weights, triple sum sets, and coset graphs of codes over

rings. Section 3 derives the preliminary results coming from spectral graph theory. Section 4 focuses on \mathbb{Z}_4 codes, derives strong necessary conditions, gives examples in short lengths, as well as an infinite family related to the Kerdock codes. Section 5 describes how the main results extend when the alphabet is a finite chain ring, and gives an infinite family related to generalized Teichmüller codes.

2 Background

2.1 Homogeneous weight

For simplicity, let R be a finite commutative ring. A *linear code* C over R of length n is an R -submodule of the standard R -module R^n . The *dual code* of C is defined as $C^\perp = \{x \in R^n \mid x \cdot C = 0\}$, where $x \cdot y = x_1y_1 + \cdots + x_ny_n$ denotes the standard inner product on R^n .

Definition 1. Let R be a finite ring. A function $w: R \rightarrow \mathbb{R}$ is called a *homogeneous weight*, if $w(0) = 0$ and

1. if $Rx = Ry$ then $w(x) = w(y)$ for all $x, y \in R$;
2. there exists a real number $\gamma \neq 0$ such that $\sum_{y \in Rx} w(y) = \gamma |Rx|$ for all $x \in R \setminus \{0\}$.

It is known from the work in [9, 17, 23] that a homogeneous weight exists for all finite commutative rings R , and is uniquely determined up to the normalization factor γ . In the sequel we will denote the homogeneous weight by w_{hom} , assuming that γ has been fixed. As usual we extend w_{hom} to a weight function on R^n by

$$w_{\text{hom}}(x_1, \dots, x_n) = \sum_{i=1}^n w_{\text{hom}}(x_i).$$

The set R^n then forms a distance space under the *homogeneous distance* d_{hom} defined by $d_{\text{hom}}(x, y) = w_{\text{hom}}(x - y)$. A code $C \subseteq R^n$ is called a (*homogeneous*) *three-weight code* with weights $w_1 < w_2 < w_3$ if every non-zero codeword has homogeneous weight w_1 , w_2 , or w_3 , and the three weights actually occur. If the code C is linear then on $C \times C$ the homogeneous distance takes only these three values and zero.

A matrix $G \in R^{\ell \times n}$ is called a *generator matrix* of the linear code C if the rows of G generate C as an R -module.

Definition 2. Let C have $\ell \times n$ generator matrix $G = [g_1 | \dots | g_n]$. The code C is called:

1. *regular* if $\{x \cdot g_i \mid x \in R^\ell\} = R$ for $i = 1, \dots, n$;
2. *projective* if $Rg_i \neq Rg_j$ for any pair of distinct coordinates $i, j \in \{1, \dots, n\}$.

In particular, a code is regular if every column contains at least one unit element. We extend this notion also to a single column vector.

In the sequel we shall be interested mostly in the case $R = \mathbb{Z}_{p^m}$, the ring of integers modulo a prime power p^m . In this case the homogeneous weight is given by

$$w_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0, \\ p^{m-1} & \text{if } 0 \neq x \in p^{m-1}\mathbb{Z}_{p^m}, \\ (p-1)p^{m-2} & \text{otherwise.} \end{cases}$$

The chosen normalization constant is $\gamma = (p-1)p^{m-2}$, which is the most convenient for our purposes. For $m \geq 2$ the homogeneous weight is always an integer. For general p and m , regular and projective is characterized by a metric property of the dual [31]. For $p = m = 2$, the homogeneous distance coincide with the classical Lee distance of [20]. For a \mathbb{Z}_4 -code to be both regular and projective is equivalent to $d^\perp \geq 3$, where d^\perp denotes the dual Lee distance.

2.2 Triple sum and s-sum sets

We extend the definition of triple sum sets given in [10] from finite fields to finite commutative rings R with a unit group R^\times . The set $\Omega \subseteq R^k$ is a *triple sum set* (TSS) if it is stable by scalar multiplication by units $\in R^\times$ and if there are integer constants σ_0 and σ_1 such that a non-zero $h \in R^k$ can be written as

$$h = x + y + z, \text{ with } x, y, z \in \Omega,$$

- σ_0 times if $h \in \Omega$,
- σ_1 times if $h \in R^k \setminus \Omega$.

This can be generalized immediately to the notion of an *s-sum set* by replacing the equation $h = x + y + z$ by $h = \sum_{i=1}^s x_i$ where each $x_i \in \Omega$.

2.3 Syndrome graph

An *eigenvalue* of a graph Γ (i.e., an eigenvalue of its adjacency matrix) is called a *restricted eigenvalue* if there is a corresponding eigenvector which is not a multiple of the all-one vector $\mathbf{1}$. Note that for an η -regular connected graph, the restricted eigenvalues are simply the eigenvalues different from η .

Definition 3. Let T be a finite abelian group and $S \subseteq T$ a subset satisfying $S = -S$ and $0_T \notin S$. The corresponding *Cayley graph* $C(T, S)$ has vertex set equal to T ; two vertices $g, h \in T$ are adjacent in $C(T, S)$ iff $g - h \in S$.

The conditions imposed on S ensure that $C(T, S)$ is a simple graph. The graph $C(T, S)$ is regular of degree $|S|$, and it is connected iff S generates the group T . We use the same idea as in [6] to associate with any linear code C over \mathbb{Z}_{p^m} a certain Cayley graph, called the *syndrome graph* of C and denoted by $\Gamma(C)$.

Let $H = [h_1 \mid \cdots \mid h_n] \in (\mathbb{Z}_{p^m})^{\ell \times n}$ be a parity-check matrix of C . The vertex set of $\Gamma(C)$ is $V = \{Hx; x \in (\mathbb{Z}_{p^m})^n\}$, the column space of H , which is isomorphic to the dual code C^\perp . The elements of V are called *syndromes* of C . Two syndromes Hx, Hy are adjacent in $\Gamma(C)$ if they differ by a unit-multiple of a column of H :

$$Hx \sim Hy : \iff H(x - y) = uh_i \quad \text{for some } 1 \leq i \leq n \text{ and } u \in \mathbb{Z}_{p^m}^\times. \quad (1)$$

It is obvious that $\Gamma(C)$ is the Cayley graph of V corresponding to the generating $S = \{uh_i; u \in \mathbb{Z}_{p^m}^\times, 1 \leq i \leq n\}$. If C is regular then $0 \notin S$ and $-S \subseteq S$, implying that $\Gamma(C)$ is simple. Moreover, if C is projective, $\Gamma(C)$ is regular of degree $|S| = p^{m-1}(p-1)n$ and has $\frac{p^{mn}}{|C|}$ vertices. The latter follows from $|C| |C^\perp| = |(\mathbb{Z}_{p^m})^n| = p^{mn}$.

As defined, $\Gamma(C)$ depends on the particular choice of H . The terminology ‘‘syndrome graph of C ’’ is justified, however, since $\Gamma(C)$ has another representation as a ‘‘coset graph’’ of C , which shows that its isomorphism type is determined by C . It is well-known that there is a one-to-one correspondence between syndromes of a code and its cosets [30], given by $x + C \mapsto Hx$. The condition in (1) can be restated as $H(x - y - ue_i) = 0$, where e_i denotes the i th standard unit vector in $(\mathbb{Z}_{p^m})^n$. Thus $\Gamma(C)$ is isomorphic to the graph whose vertices are the cosets of C and in which two

vertices $x + C$, $y + C$ are adjacent if their difference is a coset of the form $ue_i + C$. This is the Cayley graph of the quotient group $(\mathbb{Z}_p^m)^n/C$ with generating set

$$S^* = \cup_{i=1}^n \cup_{u \in \mathbb{Z}_p^{\times m}} (ue_i + C).$$

Now, we recall the relation between the weight distribution of a linear code over \mathbb{Z}_p^m and the eigenvalues of the syndrome graph of its dual code. This extension of Lemma 3.4 in [6] was derived in [37] Theorem 4.1 and its proof is omitted here.

Theorem 1. *Suppose that C is a regular, projective linear code over \mathbb{Z}_p^m with homogeneous weights w_i and corresponding weight distribution $A_i = |\{x \in C; w_{\text{hom}}(x) = w_i\}|$. Then the eigenvalues of $\Gamma(C^\perp)$ are $n(p-1)p^{m-1} - pw_i$ with multiplicity A_i .*

The coset graph with b loops $\Gamma(C)^b$ of a code C is obtained from $\Gamma(C)$ by adding b loops around every vertex. The following result is immediate from the previous theorem. The proof is omitted. (In general, we call dual weight distribution of a code the weight distribution of the dual code).

Corollary 1. *If C is a \mathbb{Z}_p^m -code, with dual weight distribution $[\langle w_i, A_i \rangle]$, such that C^\perp is regular and projective, then the spectrum of $\Gamma(C)^b$ is $\{(b + n(p-1)p^{m-1} - pw_i)^{A_i}\}$. Thus A_i is the frequency of the homogeneous weight w_i in C^\perp and the multiplicity of the eigenvalue $b + n(p-1)p^{m-1} - pw_i$.*

3 Spectral graph theory

Let $\overline{R^k}$ denote the set of all regular vectors in R^k , i.e.,

$$\overline{R^k} = \{\omega \in R^k : \{\omega \cdot x : x \in R^k\} = R\}.$$

If $\Omega \subseteq \overline{R^k}$, we denote by $C(\Omega)$ the regular and projective code of length $n = \frac{|\Omega|}{|R^\times|}$ obtained as the kernel of the $k \times n$ matrix H with columns the projectively non-equivalent elements of Ω . Thus H is the check matrix of $C(\Omega)$. To simplify notation, let $\Gamma_{C(\Omega)} = \Gamma(\Omega)$ and $\Gamma_{C(\Omega)}^b = \Gamma(\Omega)^b$, where we also write $\Gamma(\Omega)^0$ for $\Gamma(\Omega)$.

The following result is immediate from the definitions in the introduction, and will be used repeatedly and implicitly in the rest of the paper.

Theorem 2. *Keep the above notation. Assume that $C(\Omega)$ is regular and projective. The following are equivalent*

- Ω (resp $\Omega \cup 0$) is an s -sum set,
- $\Gamma(\Omega)$ (resp. $\Gamma(\Omega)^1$) is an s -SWRG.

Proof. Observe that $\Gamma(\Omega)$ is a Cayley graph on the group $(R^k, +)$ with generators the columns of H . Thus the definition of an s -sum set can be regarded as a statement on the number of paths of length s between 0 and a linear combination of s columns of H . (By translation, we can assume that 0 is the first endpoint of the path). \square

The following result comes from the well-known connection between SRGs and two-weight codes. It is described for codes over fields in [2, 3], and for codes over rings in [5].

Theorem 3. *If $C(\Omega)^\perp$ is a two-weight code, then both $\Omega \cup 0$ and Ω are s -sum sets for all $s > 1$.*

Proof. If $C(\Omega)^\perp$ is a two-weight code, it is well-known that $\Gamma(\Omega)$ is an SRG [5]. The result follows by Theorem 2, and the observation, made in the introduction, that SRGs are s -SWRG for all $s > 1$. \square

The following result was observed in [19] Theorem III.6 for the Hamming weight.

Theorem 4. *If Ω or $\Omega \cup 0$ is an s -sum set, then $C(\Omega)^\perp$ has at most three nonzero weights. Furthermore, if s is even, then $C(\Omega)^\perp$ is a 2-weight code.*

Proof. This is a consequence of [11, Theorem 3.4] applied to the the graph $\Gamma(\Omega)$, and based on Theorem 2. \square

We come to the coding-theoretic characterization of 3-sum sets which is contained for finite fields in [10, Th. 2.1]. The consideration of syndrome graphs with loops might seem artificial but is justified by the existence of several non-trivial examples, as evidenced by Section 4.

The relation with the 3-SWRG property and the eigenvalues of a connected regular graph was obtained in [11] Proposition 4.1:

Theorem 5. *Let Γ be a connected regular graph with four distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$. Then Γ is a 3-SWRG iff $\theta_1 + \theta_2 + \theta_3 = 0$.*

In our setting the formulas for the eigenvalues in Theorem 1 and Corollary 1 give:

Theorem 6. Assume that $C(\Omega)^\perp$ is of length n and has three nonzero weights $w_1 < w_2 < w_3$. Let b be any non-negative integer.

- Ω is a TSS iff $w_1 + w_2 + w_3 = 3n(p-1)p^{m-2}$;
- $\Omega \cup 0$ is a TSS iff $w_1 + w_2 + w_3 = 3\frac{1+n(p-1)p^{m-1}}{p}$;
- $\Gamma_{C(\Omega)}^b$ is a 3-SWRG iff $w_1 + w_2 + w_3 = 3\frac{b+n(p-1)p^{m-1}}{p}$;

Proof. For the first two cases we apply Theorem 2 to conclude the equivalence to $\Gamma_{C(\Omega)}^b$ being a 3-SWRG for $b = 0$ and $b = 1$, respectively. Combining Corollary 1 with Theorem 5 gives the third case, which contains the first two cases for $b = 0$ and $b = 1$, respectively. \square

Note that if $m \geq 2$ and $p \neq 3$, then the second case of Theorem 6 is impossible. More generally, for the third case of Theorem 6 the assumptions $m \geq 2$ and $p \neq 3$ imply that b has to be divisible by p . Then, the condition may be rewritten as $w_1 + w_2 + w_3 \geq 3n(p-1)p^{m-2}$ and $w_1 + w_2 + w_3 \equiv 0 \pmod{3}$.

Example 1: If $p = m = 2$ and $n = 4$, then the possible triples of weights in the first case of Theorem 6 are, with simplified notation, and by increasing value of w_2 equal to

$$129, 138, 237, 147, 246, 345, 156, 255.$$

We will see in Subsection 4.2 that none of these possibilities can be attained by a \mathbb{Z}_4 -code and the smallest possible length for $p = m = 2$ and $b = 0$ is indeed $n = 6$.

Proposition 4.3 in [11] gives an explicit diophantine equation bearing on the eigenvalues of Γ to be an s -SWRG:

Theorem 7. Let Γ be a connected regular graph with four distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$ and $s \geq 3$. Then Γ is an s -SWRG iff

$$(\theta_2 - \theta_3)\theta_1^s + (\theta_3 - \theta_1)\theta_2^s + (\theta_1 - \theta_2)\theta_3^s = 0. \quad (2)$$

We remark that for $s = 3$ Equation (2) is equivalent to $(\theta_1 - \theta_2)(\theta_1 - \theta_3)(\theta_2 - \theta_3)(\theta_1 + \theta_2 + \theta_3) = 0$, i.e., Theorem 5 is contained as a special case. Similar as in the proof of Theorem 6, we can use the formulas for the eigenvalues in Theorem 1 and Corollary 1 to transfer [11, Proposition 4.3] to s -sum sets:

Theorem 8. *Assume that $C(\Omega)^\perp$ is of length n and has three nonzero weights $w_1 < w_2 < w_3$. Let $b \geq 0$ and $s \geq 3$ be integers.*

- Ω is an s -sum set iff $(w_3 - w_2)(n(p-1)p^{m-2} - pw_1)^s + (w_1 - w_3)(n(p-1)p^{m-2} - pw_2)^s + (w_2 - w_1)(n(p-1)p^{m-2} - pw_3)^s = 0$.
- $\Omega \cup 0$ is an s -sum set iff $(w_3 - w_2)(1 + n(p-1)p^{m-2} - pw_1)^s + (w_1 - w_3)(1 + n(p-1)p^{m-2} - pw_2)^s + (w_2 - w_1)(1 + n(p-1)p^{m-2} - pw_3)^s = 0$.
- $\Gamma_{C(\Omega)}^b$ is an s -SWRG iff $(w_3 - w_2)(b + n(p-1)p^{m-2} - pw_1)^s + (w_1 - w_3)(b + n(p-1)p^{m-2} - pw_2)^s + (w_2 - w_1)(b + n(p-1)p^{m-2} - pw_3)^s = 0$.

Example 4: If $n = 12$, $p = m = 2$ the only triples (w_1, w_2, w_3) satisfying the first condition of Theorem 8 for $s = 5$ are $\{11 - i, 12, 13 + i\}$, for $i = 0, 1, \dots, 10$.

Note that $w_2 = 12 = n$ in all cases. This can be partially explained by the fact that Equation (2) admits the solution $\theta_2 = 0$, $\theta_1 = -\theta_3$ for all odd s . In other words, a graph Γ with these eigenvalues is an s -SWRG for all odd $s > 1$, cf. [11, Proposition 4.2]. Using Theorem 2 and Theorem 1, this observation translates into a result that is new in the context of s -sum sets:

Theorem 9. *Assume that $C(\Omega)^\perp$ is of length n and has three nonzero weights $w_1 < w_2 = n(p-1)p^{m-2} < w_3$, with $w_1 + w_3 = 2n(p-1)p^{m-2}$. Then Ω is an s -sum set for all odd $s > 1$.*

In [11, Theorem 4.4] it is essentially proven that if Equation (2) admits a solution $(\theta_1, \theta_2, \theta_3)$ with $\theta_1 > \theta_2 > \theta_3$ for two different values of $s > 1$, then $\theta_2 = 0$ and $\theta_1 = -\theta_3$. This directly leads to:

Theorem 10. *Assume that $C(\Omega)^\perp$ is of length n and has three nonzero weights $w_1 < w_2 < w_3$. If $w_2 \neq n(p-1)p^{m-2}$, or $w_1 + w_3 \neq 2n(p-1)p^{m-2}$, then there is at most one $s > 1$ such that Ω is an s -sum set.*

If we are not in the situation of Theorem 9, then any given code can yield an s -sum set for at most one value of s . There are other solutions of Equation (2), e.g. $(5, \frac{1}{2}(-3 + \sqrt{281}), \frac{1}{2}(-3 - \sqrt{281}))$ for $s = 5$. However, the found solutions are non-integral and no graph that is an s -SWRG and attains these eigenvalues is known, see [11, p. 808]. Since the eigenvalues corresponding to an s -sum set are integral, we state:

Conjecture 1. Ω is an s -sum set for some $s > 3$ iff

$$(w_1, w_2, w_3) = (w_1, n(p-1)p^{m-2}, 2n(p-1)p^{m-2} - w_1),$$

for some $0 < w_1 < n(p-1)p^{m-2}$.

At least for the special case $p = m = 2$, i.e., \mathbb{Z}_4 -codes, Conjecture 1 seems to be true also for $s = 3$, see Conjecture 2 in Section 4. In [8] it has been shown that all s -sum sets of a special type also have to be 3-sum sets.

4 \mathbb{Z}_4 -codes

Here we consider \mathbb{Z}_4 linear codes and assume that the reader is familiar with [20]. When we say that C is an $[n, k]$ code, where $k = (k_1, k_2)$ is the *shape*, this means that C is a linear code over \mathbb{Z}_4 of length n , which admits a generator matrix of the form

$$G = \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2D \end{pmatrix}, \quad (3)$$

where I_{k_1} and I_{k_2} denote the $k_1 \times k_1$ and $k_2 \times k_2$ identity matrices, respectively, A and D are 0, 1 valued matrices, and B is a \mathbb{Z}_4 matrix. We assume that the stated generator matrix consists of n columns (and $k_1 + k_2$ rows). We will say that the length n is *effective* to mean that the generator matrix does not contain an all-zero vector $\mathbf{0}$ as a column. The code size, i.e., the number of codewords, is given by $\#C = 4^{k_1} \cdot 2^{k_2} = 2^{2k_1+k_2}$. For \mathbb{Z}_4 -codes we note that the homogeneous weight coincides with the Lee weight.

We note the following compatibility between a \mathbb{Z}_4 -code and its Gray image.

Proposition 1. *If C is a \mathbb{Z}_4 -code with three Lee weights satisfying the hypotheses of Theorem 6, then its binary image C^G satisfies the hypotheses of Theorem 6.*

Proof. As is well-known, the Hamming weights of C^G are the Lee weights of C . If n is the length of C then C^G is of length $2n$. The third condition of Theorem 6 becomes with $p = m = 2$,

$$w_1 + w_2 + w_3 = 3 \frac{b + n(p-1)p^{m-1}}{p} = 3 \frac{b + 2n}{2},$$

which is the same with $p = 2, m = 1$ and length $2n$. □

By A_i we denote the number of codewords of C of homogeneous weight i and by B_i we denote the number of codewords of homogeneous weight i of its dual code C^\perp . The cardinality of the dual code is $\#C^\perp = 2^{2n-2k_1-k_2}$. Since we assume that a corresponding generator matrix does not have a zero column, we have $B_1 = 0$ in our context. As mentioned at the end of Subsection 2.1, every \mathbb{Z}_4 code has minimum dual distance $d^\perp \geq 3$ iff it is regular and projective. The numbers A_i can be encoded in a weight enumerator of C :

$$\text{Hom}_C(X, Y) = \sum_{c \in C} X^{2n - w_{\text{hom}}(c)} Y^{w_{\text{hom}}(c)} = \sum_{i=0}^{2n} A_i X^{2n-i} Y^i. \quad (4)$$

The corresponding MacWilliams identity is given by $\text{Hom}_C(X, Y) = \frac{1}{\#C^\perp} \cdot \text{Hom}_{C^\perp}(X+Y, X-Y)$, see e.g. [20]. Note that this formula equals the Hamming weight enumerator for a linear code over \mathbb{F}_2 with cardinality $\#C = 2^k$ if we replace n by $2n$ and k by $2k_1 + k_2$. As mentioned in the introduction and used in the proof of Lemma 1, the same conclusion can be obtained via the Gray map. Thus, we can easily rewrite the classical *MacWilliams identities* [34] or the *Pless power moments* [35] for the Hamming weight over \mathbb{F}_2 . Using $A_0 = B_0 = 1$ and $B_1 = 0$ the first four (*Pless*) *power moments* for the w_{hom} are given by:

$$\sum_{i>0} A_i = 2^{2k_1+k_2} - 1, \quad (5)$$

$$\sum_{i \geq 0} i A_i = 2^{2k_1+k_2} n, \quad (6)$$

$$\sum_{i \geq 0} i^2 A_i = 2^{2k_1+k_2-1} (B_2 + n(2n+1)), \quad (7)$$

$$\sum_{i \geq 0} i^3 A_i = 2^{2k_1+k_2-2} (3(B_2 n - B_3) + 2n^2(2n+3)). \quad (8)$$

We remark, that the first three equations (including B_1) can also be found in [41, Theorem 3.1].

In this section we are interested in structural results for l -weight codes where l is small. Mostly we will restrict our attention to projective codes only. In some cases we will require additional constraints on the sum $\sum_{i=1}^l w_i$ of weights (which are motivated by strongly walk regular graphs, see e.g. [11] and see also [39], which studies the field case).

4.1 Theoretical results for \mathbb{Z}_4 -codes with few weights

In this section we give only very few results that are of interest for the remaining part of this paper. For more information we refer the interested reader to e.g. [41, 40], where one-Lee weight codes and projective two-Lee weight codes are studied. For every pair (k_1, k_2) of non-negative integers there exists exactly one non-isomorphic 1-weight $[n, (k_1, k_2)]$ code (for a unique length n depending on k_1 and k_2), see [7].

Lemma 1. *Let C be an $[n, (k_1, k_2)]$ code. By C_2 we denote the subcode of C spanned by the codewords of even weight. The cardinality of C_2 is either $2^{2k_1+k_2-1}$ or $2^{2k_1+k_2}$ and all codewords of C_2 have an even weight.*

Proof. Let E be the code consisting of codewords of C of even Lee weight. This code is linear as the dual of the code spanned by the all-2 vector, so that $C_2 = E \cap C$. Thus C_2 is linear and its size depends on whether $2 \cdot \mathbf{1} \in C^\perp$ or not. \square

We also call C_2 the even-weight subcode of C .

Lemma 2. *Let C be an $[n, (k_1, k_2)]$ code with $t \geq 2$ different nonzero weights. Then, at most $t - 1$ weights can be odd.*

Proof. Let $c \in C$ be a codeword of odd weight, then $2c \in C$ is a codeword of even weight. \square

Lemma 3. *Let C be an $[n, (k_1, k_2)]$ three-weight code with $d^\perp \geq 3$ and weights w_1 , w_2 , and w_3 that occur A_1 , A_2 , and A_3 times, respectively. Using the abbreviation $y = 2^{2k_1+k_2-1}$ we have*

$$A_1 = \frac{y \cdot (2n^2 - 2nw_2 - 2nw_3 + 2w_2w_3 + n) - w_2w_3}{(w_2 - w_1)(w_3 - w_1)} \quad (9)$$

$$A_2 = \frac{y \cdot (2n^2 - 2nw_1 - 2nw_3 + 2w_1w_3 + n) - w_1w_3}{(w_2 - w_3)(w_2 - w_1)} \quad (10)$$

$$A_3 = \frac{y \cdot (2n^2 - 2nw_1 - 2nw_2 + 2w_1w_2 + n) - w_1w_2}{(w_3 - w_1)(w_3 - w_2)} \quad (11)$$

and

$$\begin{aligned} 3B_3 &= 2n^2(2n + 3) - (w_1 + w_2 + w_3)2n(2n + 1) - 4w_1w_2w_3 \\ &\quad + 2(w_1w_2 + w_1w_3 + w_2w_3)2n + \frac{2w_1w_2w_3}{y} \end{aligned}$$

for the number of dual codewords of weight 3.

Proof. Solving the first three power moment equations (5)-(7) for A_1 , A_2 , and A_3 gives the first statement. Plugging into the fourth power moment equation (8) and solving for $3B_3$ gives the last equation of the statement. \square

We remark that the integrality of B_3 implies that the product of the three weights $w_1w_2w_3$ is divisible by $y/2 = 2^{2k_1-k_2-2}$. Now let us look at the first case of Theorem 6, which is $w_1 + w_2 + w_3 = 3n$ in our situation. Our first observation is that the length n has to be even.

Lemma 4. *Let C be an $[n, (k_1, k_2)]$ 3-weight code with $d^\perp \geq 3$ and weights w_1 , w_2 , and w_3 satisfying $w_1 + w_2 + w_3 = 3n$. Then, $n \equiv 0 \pmod{2}$.*

Proof. Assume that n is odd. Then, also $w_1 + w_2 + w_3 = 3n$ is odd. Using Lemma 2 we conclude that exactly one weight – say w_2 – is odd. We consider the even weight subcode C_2 of C . Due to Lemma 1 C_2 only contains codewords of even weight and has cardinality $\#C_2 = 2^{2k_1+k_2-1}$, i.e., $A_2 = 2^{2k_1+k_2-1}$. Since C is projective C_2 has effective length $n' \in \{n-1, n\}$. Equation (6) gives

$$w_1A_1 + w_2A_2 + w_3A_3 = 2^{2k_1+k_2}n$$

for C and

$$w_1A_1 + w_3A_3 = 2^{2k_1+k_2-1}n'$$

for C_2 . Taking the difference and dividing by $A_2 = 2^{2k_1+k_2-1}$ yields $w_2 = n + (n - n')$. Since w_2, n are odd and $n - n' \in \{0, 1\}$, we have $n' = n$ and $w_2 = n$. So, let us write $w_1 = n - t$ and $w_3 = n + t$ for some odd integer t (observe that w_1 and w_3 have to be even). With this, Equation (10) and $A_2 = 2^{2k_1+k_2-1} = y$ gives $(y-1)t^2 = (y-n)n$. Note that $t = 1$ gives $y = n+1$ or $n = 1$. In the first case we have $A_1 = 0$ and $w_1 = 0$ in the second, so that we can assume $t \geq 3$. Solving the above equations for n gives

$$n = \frac{y}{2} \pm \frac{\sqrt{-4t^2y + 4t^2 + y^2}}{2}.$$

So, we consider the diophantine equation

$$x^2 = -4t^2y + 4t^2 + y^2, \tag{12}$$

where y is a power of 2 and all integers are positive.¹ In our situation we have $t \geq 3$, so that $y \geq 64 = 2^6$, since x^2 would be negative for $t \geq 3$ and $y \leq 32$. Now let us set

¹Note that Equation (12) has infinitely many solutions for $t \in \{-1, 0, 1\}$. Also if y is not a power of 2 there are lots of solutions, e.g. $(x, t, y) \in \{(1, 2, 15), (11, 2, 21), (14, 3, 40), (17, 6, 145)\}$.

$x = 2z$ and $y = 2^{r+1}$, i.e., $r \geq 5$. With this we can rewrite Equation (12) to

$$(z - t)(z + t) = 2^{2r} - 2^{r+1}t^2. \quad (13)$$

Since t is odd, z has to be odd too. We conclude that either $z - t$ or $z + t$ has to be divisible by 2^r from the fact that the greatest common divisor of $z - t$ and $z + t$ divides $2t$ and $r \geq 5$.

Assume that $z = s2^r - t$ for some nonzero integer s (this is the case that 2^r divides $z+t$). Plugging in into Equation (13), dividing by 2^{r+1} , and simplifying gives $(2^{r-1}s - t) \cdot s = 2^{r-1} - t^2$, so that

$$2^{r-1} \cdot (s^2 - 1) = t \cdot (s - t).$$

Note that $s^2 \geq 1$. If $s^2 = 1$, then $s = t$ or $t = 0$, which contradicts $t \geq 3$. Thus, the left hand side is positive and $s > t \geq 3$. This implies $t(s - t) \leq s(s - t) < 2^{r-1}(s^2 - 1)$ – contradiction.

Assume that $z = s2^r + t$ for some nonzero integer s (this is the case that 2^r divides $z-t$). Plugging in into Equation (13), dividing by 2^{r+1} , and simplifying gives $(2^{r-1}s + t) \cdot s = 2^{r-1} - t^2$, so that

$$2^{r-1} \cdot (s^2 - 1) = t \cdot (-s - t).$$

Note that $s^2 \geq 1$. If $s^2 = 1$, then $s = -t$ or $t = 0$, which contradicts $t \geq 3$. Thus, the left hand side is positive and $-s > t \geq 3$. This implies $t(s - t) \leq s(s - t) < 2^{r-1}(s^2 - 1)$ – contradiction. \square

We remark that in the case where also C_2 is projective the proof can be shortened significantly since [4, Corollary 16] implies that $2t$ is a power of two. An example where Lemma 4 excludes parameters of a code is given by $n = 29$, $2k_1 + k_2 = 8$, $w = (24, 31, 32)$, and weight distribution $0^1 24^{76} 31^{128} 32^{51}$. Applying the MacWilliams transform gives $B_0 = 1$, $B_1 = B_2 = 0$, $B_3 = 164$, and indeed all B_i are non-negative integers. Moreover, the number of even-weight codewords is 128, i.e., half the size of the code.

Next we show that under the assumption of Theorem 10 the length and the three weights all are divisible by large powers of 2 if the length is suitably large.

Proposition 2. *Let C be an $[n, (k_1, k_2)]$ 3-weight code with $d^\perp \geq 3$ and weights satisfying $w_1 + w_2 + w_3 = 3n$ and $w_2 = n$. For each positive integer r there exists an integer $N(r)$ such that $n \geq N(r)$ implies that 2^r divides n and 2^{r-1} divides all three weights w_1, w_2, w_3 .*

Proof. Let t be a positive integer with $w_1 = n - t$ and $w_3 = n + t$. With this (and $y = 2^{2k_1+k_2-1}$) the equations of Lemma 3 are equivalent to

$$A_1 = \frac{n(y - n - t)}{2t^2} \quad (14)$$

$$A_2 = \frac{t^2(2y - 1) - n(y - n)}{t^2} \quad (15)$$

$$A_3 = \frac{n(y - n + t)}{2t^2} \quad (16)$$

and

$$3B_3 = \frac{2n(n - t)(n + t)}{y}. \quad (17)$$

Since $A_3 - A_1 = \frac{n}{t}$ the effective length n has to be divisible by t . From $A_2 \in \mathbb{N}$ we conclude that t^2 divides $n(n - y)$. So, if p^l divides t for some odd prime p , then p^{2l} has to divide n since y is a power of 2. As an abbreviation we set $k = 2k_1 + k_2$. Now let us try to parameterize $t = 2^u \cdot v$ and $n = 2^x \cdot v^2 \cdot z$ for odd positive integers v, z and non-negative integers u, x . Plugging in and simplifying gives

$$A_1 = \frac{z \cdot (2^{k-u-1} - 2^{x-u}v^2z - v)}{2^{u+1-x}} \quad (18)$$

$$A_2 = 2^{2(x-u)}v^2z^2 + 2^k - 2^{x+k-2u-1}z - 1 \quad (19)$$

$$A_3 = \frac{z \cdot (2^{k-u-1} - 2^{x-u}v^2z + v)}{2^{u+1-x}} \quad (20)$$

$$3B_3 = \frac{v^4z \cdot (2^{x-u}vz - 1) \cdot (2^{x-u}vz + 1)}{2^{k-x-2u-2}}, \quad (21)$$

where $u \leq x$ (t divides n) and $x \leq k - 1$ ($n \leq 2^k - 1$ since C is projective).

If $k - x - 2u - 2 \geq 1$ then $B_3 \in \mathbb{N}$ and $v, z \equiv 1 \pmod{2}$ imply $u = x$. Since $\gcd(vz - 1, vz + 1) = 2$, we have that 2^{k-3u-3} either divides $vz - 1$ or $vz + 1$. So, we use the parameterization $vz = s \cdot (2^{k-3u-3}) + \alpha$ for some positive integer s and $\alpha \in \{-1, 1\}$. With this $A_1 > 0$ gives

$$2^{k-u-1} - v(s \cdot (2^{k-3u-3}) + \alpha + 1) > 0,$$

so that $vs < 2^{2u+2}$, i.e., $sv \leq 2^{2u+2} - 1$. Now $A_2 > 0$ gives $v^2z^2 + 2^k > 2^{k-u-1}z$, which is equivalent to

$$sv(vz)^2 + 2^k sv > s2^{k-u-1}vz = s^22^{2k-4u-4} + \alpha s2^{k-u-1} > s^22^{2k-4u-4} - s2^{k+2u+2}. \quad (22)$$

Since $sv \leq 2^{2u+2} - 1$ the left hand side is at most

$$\begin{aligned}
& s^2 2^{2k-4u-4} - s^2 2^{2k-6u-6} + s\alpha 2^{k-u} - s\alpha 2^{k-3u-2} \\
& + \alpha^2 2^{2u+2} - \alpha^2 + 2^{k+2u+2} - 2^k \\
\leq & s^2 2^{2k-4u-4} - s^2 2^{2k-6u-6} + 2s 2^{k-u} + 2^{2u+2} + 2^{k+2u+2} \\
\leq & s^2 2^{2k-4u-4} - s^2 2^{2k-6u-6} + 3s 2^{k+2u+2}.
\end{aligned}$$

Thus

$$4s \cdot 2^{k+2u+2} > s^2 2^{2k-6u-6} \quad (23)$$

has to be satisfied, so that $k \leq 8u + 9$ and

$$x = u \geq \frac{k-9}{8}. \quad (24)$$

Otherwise we have $k \leq x + 2u + 2$. From $A_1 > 0$ we conclude $y - n > 0$. Since $y - n$ is an integer and both y and n are divisible by 2^x we have $y - n \geq 2^x$. Now $A_2 > 0$, Equation (15), and $2y - 1 \leq 2^k$ imply

$$v^2 2^{2u} \cdot 2^k - 2^x v^2 z \cdot 2^x > 0,$$

so that $k + 2u > 2x$, i.e.,

$$k \geq 2x - 2u + 1. \quad (25)$$

Combined with $k \leq x + 2u + 2$ we obtain $x \leq 4u + 1$ and $k \leq 6u + 3$, i.e.,

$$x \geq u \geq \frac{k-3}{6}. \quad (26)$$

In both cases we can conclude $x \geq u \geq \frac{k-9}{8}$, so that the result follows from $n < 2^k$. \square

Conjecture 2. *Let C be an $[n, (k_1, k_2)]$ 3-weight code with $d^\perp \geq 3$ and weights satisfying $w_1 + w_2 + w_3 = 3n$ and $w_1 < w_2 < w_3$. Then, $w_2 = n$.*

We remark that in order to prove Conjecture 2 the Diophantine conditions of Lemma 3, i.e., $A_1, A_2, A_3 \in \mathbb{N}_{>0}$ and $B_3 \in \mathbb{N}_{\geq 0}$ are not sufficient. Up to $n = 50$ we have the following exceptional tuples $(n, w_1, w_2, w_3, y, A_1, A_2, A_3, B_3)$:

- (29, 24, 31, 32, 64, 76, 128, 51, 164)
- (33, 29, 32, 38, 64, 64, 111, 80, 157)

- (34, 30, 32, 40, 128, 64, 299, 148, 36)
- (50, 46, 48, 56, 64, 32, 145, 78, 580)

So, from Lemma 4 we can conclude that Conjecture 2 is true for all $n < 34$. We remark that the examples for $n \in \{29, 34, 50\}$ satisfy $B_i \in \mathbb{N}_{\geq 0}$ for all $1 \leq i \leq n$, i.e., the MacWilliams identities are not sufficient in order to conclude the non-existence of the corresponding code.

4.2 Classification of 3-weights \mathbb{Z}_4 -codes with $d^\perp \geq 3$ and short length

Having Theorem 6 in mind we aim to classify 3-weights \mathbb{Z}_4 -codes with $d^\perp \geq 3$ and short length. Optimal linear codes over \mathbb{Z}_4 with length $n \leq 7$ were classified in [43]. Without the restriction to optimal codes, all linear codes over \mathbb{Z}_4 with length $n \leq 10$ and $2k_1 + k_2 \leq 8$ were classified in [15] for the homogeneous weight. The corresponding tables at http://www.algorithm.uni-bayreuth.de/en/research/Coding_Theory/CanonicalForm/Classification/index.html contain the complete counts for $n \leq 10$. E.g. for $n = 10$ and $(k_1, k_2) = (5, 0)$ there are exactly 8 848 026 non-isomorphic codes. For each given n our approach is to loop over all $1 \leq w_1 < w_2 < w_3 \leq 2n$ such that A_1, A_2, A_3, B_3 , as specified in Lemma 3, are integers with $A_1, A_2, A_3 \geq 1$ and $B_3 \geq 0$. Since $2^{2k_1+k_2-2}$ has to divide $w_1 w_2 w_3$, which is due to $B_3 \in \mathbb{N}$, we obtain a finite list of possibilities for each length n . Note that $d^\perp \geq 3$ implies $k_1 \geq 1$. With respect to Theorem 6 we consider all remaining tuples where $w_1 + w_2 + w_3 \geq 3n$ and $w_1 + w_2 + w_3 \equiv 0 \pmod{3}$. In Table 1 we list the parameters that can be attained by a \mathbb{Z}_4 codes and refer to a corresponding generator matrix. As abbreviations we use $w = (w_1, w_2, w_3)$, $S = w_1 + w_2 + w_3$, and $A = (A_1, A_2, A_3)$. The parameters of computationally excluded codes are listed in Table 2 and Table 3. For the latter we perform an exhaustive search using the given restrictions on the weights w_i , the length n , and the shape (k_1, k_2) .

$$G_{6,1}^1 = \begin{pmatrix} 101112 \\ 010331 \\ 002200 \end{pmatrix}, G_{6,2}^1 = \begin{pmatrix} 100122 \\ 010212 \\ 001331 \end{pmatrix}, G_{6,3}^1 = \begin{pmatrix} 101112 \\ 011121 \\ 002002 \\ 000220 \end{pmatrix}$$

n	w	A	$(k_1, k_2) \rightarrow G$
6	(4,6,8)	(6,16,9)	$(2, 1) \rightarrow G_{6,1}^1$
6	(4,6,8)	(18,24,21)	$(3, 0) \rightarrow G_{6,2}^1, (2, 2) \rightarrow G_{6,3}^1$
8	(4,8,12)	(1,27,3)	$(2, 1) \rightarrow G_{8,1}^1$
8	(4,8,12)	(5,51,7)	$(3, 0) \rightarrow G_{8,2}^1, (2, 2) \rightarrow G_{8,3}^1$
3	(2,4,6)	(15,15,1)	$(2, 1) \rightarrow G_{3,1}^2$
5	(4,6,8)	(16,12,3)	$(2, 1) \rightarrow G_{5,1}^2$
7	(6,8,10)	(42,7,14)	$(3, 0) \rightarrow G_{7,1}^2$
9	(8,10,12)	(15,12,4)	$(2, 1) \rightarrow G_{9,1}^2$
10	(8,12,16)	(62,64,1)	$(3, 1) \rightarrow G_{10,1}^2$
10	(8,12,16)	(130,120,5)	$(4, 0) \rightarrow G_{10,2}^2$

Table 1: Attainable parameters for $S \equiv 0 \pmod{3}$

$$\begin{aligned}
G_{8,1}^1 &= \begin{pmatrix} 10101122 \\ 01112311 \\ 00220202 \end{pmatrix}, G_{8,2}^1 = \begin{pmatrix} 10001222 \\ 01022111 \\ 00110013 \end{pmatrix}, G_{8,3}^1 = \begin{pmatrix} 10111112 \\ 01112331 \\ 00200020 \\ 00020020 \end{pmatrix} \\
G_{3,1}^2 &= \begin{pmatrix} 101 \\ 011 \\ 002 \end{pmatrix}, G_{5,1}^2 = \begin{pmatrix} 10122 \\ 01111 \\ 00202 \end{pmatrix}, G_{7,1}^2 = \begin{pmatrix} 1001112 \\ 0101231 \\ 0012133 \end{pmatrix} \\
G_{9,1}^2 &= \begin{pmatrix} 101111122 \\ 011223311 \\ 002020202 \end{pmatrix}, G_{10,1}^2 = \begin{pmatrix} 1000211111 \\ 0101223010 \\ 0010111233 \\ 0002220222 \end{pmatrix}, G_{10,2}^2 = \begin{pmatrix} 1000212110 \\ 0100031232 \\ 0010132012 \\ 0001210231 \end{pmatrix}
\end{aligned}$$

We remark that more examples can be found for the parameters:

- $n = 15, k = 9, w = (12, 16, 20), A = (190, 255, 66)$;
- $n = 18, k = 8, w = (16, 20, 24), A = (153, 72, 30)$;
- $n = 22, k = 7, w = (20, 24, 28), A = (71, 43, 13)$.

n	$2k_1 + k_2$	w	A	(k_1, k_2)
2	3	(1,2,3)	(1,3,3)	(1,1)
4	4	(2,4,6)	(1,11,3)	(2,0), (1,2)
4	5	(2,4,6)	(5,19,7)	(2,1), (1,3)
4	6	(2,4,6)	(13,35,15)	(3,0), (2,2), (1,4)
6	5	(4,6,8)	(6,16,9)	(1,3)
6	6	(4,6,8)	(18,24,21)	(1,4)
8	5	(6,8,10)	(6,15,10)	(2,1), (1,3)
8	6	(6,8,10)	(22,15,26)	(3,0), (2,2), (1,4)
8	7	(6,8,10)	(54,15,58)	(3,1), (2,3), (1,5)
8	5	(4,8,12)	(1,27,3)	(1,3)
8	6	(4,8,12)	(5,51,7)	(1,4)
8	7	(4,8,12)	(13,99,15)	(3,1), (2,3), (1,5)
10	5	(8,10,12)	(5,16,10)	(2,1), (1,3)
10	6	(8,10,12)	(25,8,30)	(3,0), (2,2), (1,4)

Table 2: Computationally excluded cases for $S = 3n$

n	$2k_1 + k_2$	w	A	(k_1, k_2)
3	5	(2,4,6)	(15,15,1)	(1,3)
5	5	(4,6,8)	(16,12,3)	(1,3)
7	5	(6,8,10)	(16,11,4)	(2,1), (1,3)
7	6	(6,8,10)	(42,7,14)	(2,2), (1,4)
7	7	(4,8,12)	(31,95,1)	(3,1), (2,3), (1,5)
7	8	(4,8,12)	(65,187,3)	(4,0), (3,2), (2,4), (1,6)
9	5	(8,10,12)	(15,12,4)	(1,3)
9	6	(8,11,14)	(43,16,4)	(3,0), (2,2), (1,4)
10	7	(8,12,16)	(62,64,1)	(2,3), (1,5)
10	8	(8,12,16)	(130,120,5)	(3,2), (2,4), (1,6)

Table 3: Computationally excluded cases for $S > 3n$, $S \equiv 0 \pmod{3}$

4.3 Cyclic Kerdock code

The Kerdock code K of length 2^s with $s \geq 3$ odd over \mathbb{Z}_4 can be expressed as

$$K = \mathbb{Z}_4 \mathbf{j} \oplus Q,$$

where \mathbf{j} denotes the all-one vector. The code Q is a free code of size 4^s with weight spectrum

$$2^s, 2^s \pm 2^{\frac{s-1}{2}}.$$

By puncturing on the first coordinate we obtain a cyclic code of length $n = 2^s - 1$ that was used in [42] to generate low correlation quadriphase sequences. This code is called K^- in [20] and has the same weights as Q . It can be shown to be projective by the same type of arguments as in [20, §3.3]. The sum of these weights being $3 \cdot 2^s = 3(n + 1)$, we see that, by Theorem 6 with $b = 2$, the syndrome graph of K^- with a double loop at each vertex is a 3-SWRG. It can be easily checked that the parameters of K^- go in line with our results the previous subsection.

We observe that the binary images have a length congruent to 2 (mod 4), which is enough to show they are not linear, since by [27, Corollary 1], we should have a length multiple of 4. This is an alternative proof that Kerdock codes cannot be linear. The classical proof builds on the fact that the Preparata code, or any code with the same weight distribution is nonlinear [16].

5 Extension to chain rings

In this section, we briefly indicate how the preceding work extends when replacing \mathbb{Z}_p^m by a finite chain ring R of depth $r \geq 2$, say, and residue field \mathbb{F}_q , for some prime power q . Recall that such a ring is a local ring whose lattice of ideals form a chain of length r . In particular, the maximal ideal is principal, generated by an element γ , a nilpotent element of nilpotency index r [36, Chap. 3]. An example is \mathbb{Z}_p^m , when $r = m$, and $q = p$. The definition of the homogeneous weight becomes

$$w_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0, \\ q^{r-1} & \text{if } 0 \neq x \in (\gamma^{e-1}), \\ (q-1)q^{r-2} & \text{otherwise.} \end{cases}$$

The definition of the syndrome graph in Subsection 2.3 to this generalized situation only needs the obvious modifications. Also the proof of [37, Theorem 4.1] can be generalized to conclude the connection between coset graph eigenvalues and weights:

Theorem 11. *Suppose that C is a regular, projective linear code over R with homogeneous weights w_i and corresponding weight distribution $A_i = |\{x \in C; w_{\text{hom}}(x) = w_i\}|$. Then the eigenvalues of $\Gamma(C^\perp)$ are $n(q-1)q^{r-1} - qw_i$ with multiplicity A_i .*

In particular, for the coset graph with loops, we have the following result.

Corollary 2. *If C is an R -code, of minimum distance at least three, with dual weight distribution $[\langle w_i, A_i \rangle]$, then the spectrum of $\Gamma(C)^b$ is $\{(b + n(q-1)q^{r-1} - qw_i)^{A_i}\}$. Thus A_i is the frequency of the homogeneous weight w_i in C^\perp and the multiplicity of the eigenvalue $b + n(q-1)q^{r-1} - qw_i$.*

From there, upon observing that Theorem 2 is valid for any finite commutative ring, the main condition for the existence of a TSS becomes

Theorem 12. *Assume that $C(\Omega)^\perp$ is of length n and has three nonzero weights $w_1 < w_2 < w_3$. Let b be any integer.*

- Ω is a TSS iff $w_1 + w_2 + w_3 = 3n(q-1)q^{r-2}$;
- $\Omega \cup 0$ is a TSS iff $w_1 + w_2 + w_3 = 3 \frac{1+n(q-1)q^{r-1}}{q}$;
- $\Gamma_{C(\Omega)}^b$ is a 3-SWRG iff $w_1 + w_2 + w_3 = 3 \frac{b+n(q-1)q^{r-1}}{q}$;

We remark that $r \geq 2$ and q not being divisible by 3 the second case is impossible and b has to be divisible by q in the third case.

5.1 Trace codes over chain rings

Consider the chain ring $\mathbb{F}_p + u\mathbb{F}_p$, with $u^2 = 0$. This is a chain ring with $r = 2$ and $q = p$. The codes $C(m, p)$ of length $\frac{p^{2m}-p^m}{2}$ with m singly even in [38, §5.1] are defined as

$$\{(Tr(ax))_{x \in L} \mid a \in \mathcal{R}\},$$

where $L = Q + u\mathbb{F}_p^m$, Q is the set of squares of \mathbb{F}_{p^m} , and $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. Note that, since m is even, we have $\mathbb{F}_p^\times \subseteq Q$. This implies that these codes are replications by a

factor $p - 1$ of projective codes $P(m, p)$ say of length $n = \frac{p^{2m} - p^m}{2(p-1)}$. These latter codes have weights

$$p^{2m-1} - p^{m-1}, p^{2m-1} - p^{m-1} - p^{m-1}(p^{m/2} + 1), p^{2m-1} - p^{m-1} + p^{m-1}(p^{m/2} - 1),$$

and so satisfy the relation

$$w_1 + w_2 + w_3 = 3(1 - 1/p)n,$$

for the weight of the Gray map in [38], normalized by a factor 2. This shows, by the first condition of Theorem 12, that the coset graph $\Gamma(C)$ is a 3-SWRG, or, equivalently, that the Ω such that $C(\Omega) = C$ is a TSS.

5.2 Generalized Teichmüller codes

Let $q = 2^r$ and $R = \text{GR}(4, r)$ the Galois ring of characteristic 4 and residue field \mathbb{F}_{2^r} . The order of R is q^2 . For integers $k \geq 2$ and

$$s \in \begin{cases} \{0, 2, 4, \dots, (k-1)r\} & \text{if } k \text{ odd;} \\ \{r, r+2, r+4, \dots, (k-1)r\} & \text{if } k \text{ even,} \end{cases}$$

the *generalized Teichmüller codes* $\mathcal{T}_{q,k,s}$ are constructed in [25, Sec. 3.1], [26], generalizing the *Teichmüller codes* in [28]. They are regular projective R -linear codes of length $n = 2^s \cdot \frac{q^k - 1}{q - 1}$ and with three non-zero homogeneous weights, scaled by the factor $1/q^{r-2}$,²

$$w = (2^s q^k - 2^{s/2} q^{(k-1)/2}, 2^s q^k, 2^s q^k + 2^{s/2} q^{(k-1)/2})$$

of frequencies

$$A = \left(\frac{1}{2}(q^k - 1)(q^k + 2^{s/2} q^{(k+1)/2}), q^k - 1, \frac{1}{2}(q^k - 1)(q^k - 2^{s/2} q^{(k+1)/2}) \right).$$

From the length we get $2^s q^k = n(q - 1)$. Thus, the sum of the weights is $S = 3 \cdot 2^s q^k q^{r-2} = 3(n(q - 1) + 2^s) q^{r-2} = \frac{3}{q}(b + n(q - 1) q^{r-1})$ for $b = 2^s q^{r-1}$. By Theorem 12 we see that the coset graph with b loops on each vertices is a 3-SWRG.

We remark that in [28] and [25], more projective three-weight-codes over Galois rings are found, among them the dualized extended Kerdock codes $\hat{\mathcal{K}}_{k+1}^*$ and the dualized generalized Teichmüller codes $\mathcal{T}_{q,k,s}^*$. However, they are either not regular or the sum of the weights does not satisfy the divisibility conditions needed in the context of this paper.

²The scaling of the homogeneous weight in [25] and [28] differs by ours by the factor q^{r-2} .

5.3 Classification of 3-weights codes over $\mathbb{F}_2 + u\mathbb{F}_2$ with $d^\perp \geq 3$ and short length

In a similar manner as what we did over \mathbb{Z}_4 , we classify 3-weight codes with dual weight $d^\perp \geq 3$ over the other proper chain ring of order 4, which is the ring $R = \mathbb{F}_2 + u\mathbb{F}_2$ of dual numbers over \mathbb{F}_2 . The Gray image of such a code is always a linear binary code, and a linear binary code is the Gray image of an R -linear code if and only if it has a fixed-point free automorphism of order 2 [22]. We classify the codes in the same range of parameters as in the \mathbb{Z}_4 -case, that is $n \leq 10$, $S \geq 3n$ and $S \equiv 0 \pmod{3}$. Since the MacWilliams identities for R -linear codes are the same as for \mathbb{Z}_4 -linear codes (and up to doubling the length, the same as the MacWilliams identities for the linear Gray image), we can start with the same list of feasible parameters we computed for \mathbb{Z}_4 -linear codes.

The feasible parameters, together with the weight distribution and the value $S - 3n \geq 0$, are listed in Table 4. The column (k_1, k_2) lists the shapes of the realizable codes. An entry “–” indicates that no R -linear code with the given parameters does exist.

A list of suitable generator matrices for the realizable parameters is given in Table 5.

We remark that we found further examples of R -linear three-weight codes with $d^\perp \geq 3$, $S - 3n \geq 0$ and $3 \mid S$. Their parameters are displayed in Table 6

6 Conclusion and open problems

In this work we have generalized from fields to rings the constructions of SWRGs from [39]. The rings considered here are \mathbb{Z}_{p^m} , and more generally finite chain rings. The weight playing the role of the Hamming in this generalization is the homogeneous weight. This weight can be defined in general over finite Frobenius rings. It would be of theoretical interest to consider three-weight codes over those rings.

We have classified short length three-weight \mathbb{Z}_4 -codes leading to SWRGs. To undertake such a classification for higher lengths and other rings is a challenging open problem, both theoretically and computationally. The polynomial analogues of \mathbb{Z}_p^m , namely the rings $\mathbb{F}_p[x]/(x^m)$, are worth considering.

n	$2k_1 + k_2$	weights	$S - 3n$	(k_1, k_2)
2	3	$1^1 2^3 3^3$	0	—
4	4	$2^1 4^{11} 6^3$	0	—
4	5	$2^5 4^{19} 6^7$	0	—
4	6	$2^{13} 4^{35} 6^{15}$	0	—
6	5	$4^6 6^{16} 8^9$	0	(2, 1)
6	6	$4^{18} 6^{24} 8^{21}$	0	(3, 0), (2, 2)
8	5	$6^6 8^{15} 10^{10}$	0	—
8	6	$6^{22} 8^{15} 10^{26}$	0	—
8	7	$6^{54} 8^{15} 10^{58}$	0	—
8	5	$4^1 8^{27} 12^3$	0	(2, 1)
8	6	$4^5 8^{51} 12^7$	0	(3, 0), (2, 2)
8	7	$4^{13} 8^{99} 12^{15}$	0	—
10	5	$8^5 10^{16} 12^{10}$	0	—
10	6	$8^{25} 10^8 12^{30}$	0	—
3	5	$2^{15} 4^{15} 6^1$	3	(2, 1)
5	5	$4^{16} 6^{12} 8^3$	3	(2, 1)
7	5	$6^{16} 8^{11} 10^4$	3	—
7	6	$6^{42} 8^7 10^{14}$	3	—
7	7	$4^{31} 8^{95} 12^1$	3	—
7	8	$4^{65} 8^{187} 12^3$	3	—
9	5	$8^{15} 10^{12} 12^4$	3	(2, 1)
9	6	$8^{43} 11^{16} 14^4$	6	—
10	7	$8^{62} 12^{64} 16^1$	6	(3, 1)
10	8	$8^{130} 12^{120} 16^5$	6	(4, 0)

Table 4: Classification for $R = \mathbb{F}_2[X]/(X^2)$, $S \geq 3n$, $n \leq 10$

n	k	weights	$S - 3n$	(k_1, k_2)	G
3	5	$2^{15}4^{15}6^1$	3	(2, 1)	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & X \end{pmatrix}$
5	5	$4^{16}6^{12}8^3$	3	(2, 1)	$\begin{pmatrix} 1 & 0 & 1 & 0 & X \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & X & X \end{pmatrix}$
6	5	$4^66^{16}8^9$	0	(2, 1)	$\begin{pmatrix} 1 & 0 & X & X+1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & X \\ 0 & 0 & 0 & 0 & X & X \\ 1 & 0 & 0 & 1 & 1 & X+1 \\ 0 & 1 & 0 & 1 & X & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$
6	6	$4^{18}6^{24}8^{21}$	0	(3, 0)	$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & X & 1 & X+1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & X & 1 & X+1 \\ 0 & 0 & X & X & 0 & 0 \\ 0 & 0 & 0 & 0 & X & X \end{pmatrix}$
6	6	$4^{18}6^{24}8^{21}$	0	(2, 2)	$\begin{pmatrix} 1 & 0 & X & 1 & 0 & 1 & X \\ 0 & 1 & 1 & X & 1 & 1 & X+1 \\ 0 & 0 & X & X & 0 & 0 \\ 0 & 0 & 0 & 0 & X & X \\ 1 & 0 & X & 1 & 0 & 1 & X \\ 0 & 1 & 1 & X & 1 & 1 & X+1 \\ 0 & 0 & 0 & 0 & X & X & X \\ 1 & 0 & 0 & X & X & 0 & 1 & X \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & X & X & X+1 \end{pmatrix}$
8	5	$4^{18}8^{27}12^3$	0	(2, 1)	$\begin{pmatrix} 1 & 0 & X & 1 & 0 & 1 & X \\ 0 & 1 & 1 & X & 1 & 1 & 1 & X+1 \\ 0 & 0 & 0 & 0 & X & X & X & X \\ 1 & 0 & 0 & X & X & 0 & 1 & X \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & X & X & X+1 \end{pmatrix}$
8	6	$4^58^{51}12^7$	0	(3, 0)	$\begin{pmatrix} 1 & 0 & X & 1 & 1 & 1 & 1 & X+1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & X & X & 0 & X & X \\ 0 & 0 & 0 & 0 & 0 & X & X & 0 \\ 1 & 0 & 1 & X & 1 & X & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & X & X & X+1 \end{pmatrix}$
8	6	$4^58^{51}12^7$	0	(2, 2)	$\begin{pmatrix} 1 & 0 & X & 1 & 1 & 1 & 1 & X+1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & X & X & 0 & X & X \\ 0 & 0 & 0 & 0 & 0 & X & X & 0 \\ 1 & 0 & 1 & X & 0 & 1 & 1 & 1 & X+1 \\ 0 & 1 & 1 & 1 & 1 & X & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & X & X & X & X & 0 \\ 1 & 0 & 0 & 1 & X & 1 & X & 0 & 1 \\ 0 & 1 & 0 & 1 & X & 0 & 1 & 1 & X+1 & X+1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & X & 1 & 1 \\ 0 & 0 & 0 & X & X & X & X & X & 0 & X \end{pmatrix}$
9	5	$8^{15}10^{12}12^4$	3	(2, 1)	$\begin{pmatrix} 1 & 0 & 1 & X & 0 & 1 & 1 & 1 & X+1 \\ 0 & 1 & 1 & 1 & 1 & X & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & X & X & X & X & 0 \\ 1 & 0 & 0 & 1 & X & 1 & X & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & X \\ 0 & 0 & 1 & 1 & 1 & X & X & X & X+1 \end{pmatrix}$
10	7	$8^{62}12^{64}16^1$	6	(3, 1)	$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & X & X & X & X+1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & X & X & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & X & 1 & X \\ 0 & 0 & 0 & X & X & X & X & X & 0 & X \\ 1 & 0 & 0 & 0 & 1 & 0 & X & X & X+1 & 1 \\ 0 & 1 & 0 & 0 & X & X & 0 & 1 & X+1 & 1 \\ 0 & 0 & 1 & 0 & 0 & X & 1 & X & 1 & 1 \\ 0 & 0 & 0 & 1 & X & 1 & X & 0 & 1 & 1 \end{pmatrix}$

Table 5: Generator matrices for $R = \mathbb{F}_2[X]/(X^2)$, $S \geq 3n$, $n \leq 10$

n	k	weights	$S - 3n$
11	9	$8^{162}12^{312}16^{37}$	3
11	10	$8^{330}12^{616}16^{77}$	3
13	6	$12^{45}16^{17}20^1$	9
15	9	$12^{190}16^{255}20^{66}$	3
18	8	$16^{153}20^{72}24^{30}$	6
26	8	$24^{172}28^{32}32^{51}$	6
42	8	$40^{189}48^{63}56^3$	18
49	7	$48^{91}52^{28}56^8$	9
54	11	$48^{724}56^{1104}64^{219}$	6
61	11	$56^{980}64^{847}72^{220}$	9
63	13	$56^{2556}64^{4095}72^{1540}$	3

Table 6: Further projective three-weight-codes over $R = \mathbb{F}_2[X]/(X^2)$ with $S \geq 3n$

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