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Abstract

Let $X = \{X_t, t \geq 0\}$ be a stationary stochastic process and suppose X_0 has a probability density f. Suppose the process X is observed at jump times $\{\tau_i, i \geq 1\}$ of a point process $\{N_t, t \geq 0\}$. Nonparametric density estimation of f based on the sampled data $\{X(\tau_i), 1 \leq i \leq n\}$ is studied. Asymptotic properties of estimators of delta-family type for f based on $\{X(\tau_i), 1 \leq i \leq n\}$ are investigated.

AMS 1980 subject-classification. Primary 62G05, Secondary 62M09

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1. Introduction

Nonparametric density estimation for stationary stochastic processes has been the subject of investigation by several authors. For instance, Roussas (1969), Rosenblatt (1970), and Prakasa Rao (1977, 1978) discussed density estimation for discrete time stationary Markov processes. Banon (1977) and Prakasa Rao (1979a) studied the problem for continuous time stationary Markov processes. Results for continuous time stationary processes which are mixing type are obtained by Delecroix (1980) among others. For a comprehensive dicussion of these results, see Prakasa Rao (1983). In the case of continuous time processes, one of the basic assumptions used in all the above papers is that the process is completely observable over any given period. Estimators of density based on the complete path are constructed. In practice, it is obvious that it is impossible to observe the path continuously for various reasons, in particular, precision of measuring instruments, unavailability of observation at all time points as in medical studies etc. In otherwords, the problem of estimation of density from sampled data of the process is of extreme interest in practical problems. Sampling instants for observing the process may be of several types. They might be regularly spaced or irregularly spaced. Sampling may be done at deterministic times or random times. Random times of observation may depend on the process or may be independent of the process under observation.

Statistical inference for irregularly observed processes in time series problems has been under study for quite some time and we will not discuss this problem here. Masry (1983) discussed probability density estimation from sampled data for classes of processes satisfying various types of asymptotic independence conditions. It is not clear whether any

condition of the type of asymptotic independence for a process is inherited by the process obtained from sampling the original process at random times. Some aspects of this problem are discussed in Prakasa Rao (1987b). In general, it is also not necessary that the "information" obtained from the sampled data gives complete "information" on the process.

Suppose $X = \{X_t, t \geq 0\}$ is a stationary process and the process X is sampled at times $\{\tau_n, n \geq 0\}$ with $\tau_0 = 0$. The problem is to estimate the one-dimensional marginal density of X (assuming that it exists) from $\{X_{\tau_i}, 0 \leq i \leq n\}$. We do not assume that the values of the process $\{\tau_i\}$ are observable. Example of such a situation in laser amemometry is given in Durranti and Greated (1977). Another problem where our discussion might have potential application is in studying cardiac potentials on the chest surface (Parisetti et. al. (1983)).

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t, t \geq 0\}$ be a right continuous complete filtration defined on it. Supppose $X = \{X_t, t \geq 0\}$ is a stationary process adapted to $\{\mathcal{F}_t\}$ and $N = \{N_t, t \geq 0\}$ is a point process adapted to $\{\mathcal{F}_t\}$ with stochastic intensity $\{\lambda_t\}$. Assume that $N_0 = 0$ a.s. and $E[N_t] < \infty$ for $t \geq 0$. Let $\{\tau_i, i \geq 1\}$ be the jump times for the process N and $\tau_0 = 0$. τ_i denotes the jump time for the i-th jump of the process N. Note that

$$N_t = \sum_{i \geq 1} 1[\tau_i \leq t] .$$

Suppose the process is observed at times $\tau_0, \ldots, \tau_n$. In other words observations $X_{\tau_0}, \ldots, X_{\tau_n}$ are recorded on the process X. Note that

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega)$$

by definition. Assume that $\{X_t\}$ is a progressively measurable process. In particular, it follows that X_τ is measurable. Infact X_τ is $\mathcal{F}_\tau$-measurable where $\mathcal{F}_\tau$ is the σ-algebra generated by sets $A \in \mathcal{F}$ such that $A \cap [\tau \le t] \in \mathcal{F}_t$ for every $t \ge 0$.

For more details about these concepts, see Bremaud (1981) or Prakasa Rao (1987a).

Suppose X_0 has density f. The problem of interest is the estimation of f based on $X_{\tau_0}, \ldots, X_{\tau_n}$.

Definition 2.1. A family of functions $\{\delta_t(x), t \ge 0\}$ is said to be a family of delta-type if it satisfies the following conditions:

(i) $\int_{-\infty}^{\infty} |\delta_t(x)| dx < A < \infty$, $t \ge 0$,

(ii) $\int_{-\infty}^{\infty} \delta_t(x) dx = 1$, $t \ge 0$,

(iii) $\delta_t(x) \longrightarrow 0$ as $t \to \infty$ uniformly in $|x| > \lambda$ for any $\lambda > 0$,

and

(iv) $\int_{|t| \ge \lambda} |\delta_t(x)| dx \longrightarrow 0$ as $t \to \infty$ for any $\lambda > 0$.

Remarks 2.1. An example of a family of delta-type is

$$\delta_t(x) = h_t^{-1} K(x h_t^{-1}), \quad t \ge 0$$

where $K(\cdot)$ is a bounded probability density such that $x K(x) \to 0$ as $x \to \infty$, $h_t > 0$, $h_t \to 0$ as $t \to \infty$. Density estimators based on delta-families of the type defined above are discussed in Watson and Leadbetter (1964), Walter and Blum (1979) and Prakasa Rao (1978, 79a,b). A more general type of delta-families is studied in Földes and Révész (1974). For a complete discussion, see Prakasa Rao (1983). Assume that the following

condition holds throughout this paper.

(A_o) The family of functions $\{\delta_t(x), t\geq 0\}$ is such that the process $\{\delta_t(x-X_t),$ $0\leq t\leq T\}$ is $\mathcal{F}_t$-predictable for every x and for every $T \geq 0$.

Let

$$\hat{f}_T(x) = \frac{1}{T}\int_0^T \delta_T(x-X_s)dN_s$$

for every real x. The integral is well-defined since $\delta_T(x-X_t)$ is $\mathcal{F}_t$-predictable and N_t is $\mathcal{F}_t$-adapted. It is easy to see that

$$\hat{f}_T(x) = \frac{1}{T}\Sigma^*\delta_T(x-X_{\tau_i})$$

where Σ^* is the sum over i with jump time τ_i between [0,T].

Our aim is to study the properties of the estimator $\hat{f}_T(x)$ as an estimator of the density f(x).

The following properties are well known, for instance, see Castellana and Leadbetter (1986) and Prakasa Rao (1983).

Lemma 2.1. If $g(\cdot)$ is continuous at $x = 0$ and $\int_{-\infty}^{\infty}|g(x)|dx < \infty$, then

$$\int_{-\infty}^{\infty} g(x)\delta_t(x)dx \to g(0) \quad \text{as} \quad t\to\infty.$$

Lemma 2.2 If $\alpha_t = \int_{-\infty}^{\infty}\delta_t^2(x)dx < \infty$, then $\alpha_t \to \infty$ and

$$\delta_t^*(x) = \delta_t^2(x)\,\alpha_t^{-1}, \quad t \geq 0 \quad \text{forms a delta-family.}$$

Lemma 2.3. If $g(\cdot,\cdot)$ is a bounded and if $g(\cdot,\cdot)$ is continuous at (x,y), then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_t(u-x)\delta_t(u-y)\, g(u,v)\, du\, dv \longrightarrow g(x,y) \quad \text{as} \quad t\to\infty.$$

3. Some properties of point processes

Note that

$$\hat{f}_T(x) = \frac{1}{T}\int_0^T \delta_T(x-X_s)\, dN_s$$

where $\{N_t, t\geq 0\}$ is a point process adapted to $\{\mathcal{F}_t\}$ with stochastic intensity $\{\lambda_t\}$. By the definition of $\{N_t, t\geq 0\}$, $\{\lambda_t\}$ is a non-negative $\mathcal{F}_t$-progressively measurable process such that for all $t \geq 0$,

$$\int_0^t \lambda_s\, ds < \infty \quad \text{a.s.}[P]$$

and for all non-negative $\mathcal{F}_t$-predictable processes C_t,

$$E\Big[\int_0^\infty C_t\, d\, N_t\Big] = E\Big[\int_0^\infty C_t\, \lambda_t\, dt\Big]$$

(cf. Bremaud (1981)).

Theorem 3.1. Suppose $\{N_t, t\geq 0\}$ is a point process with stochastic intensity $\{\lambda_t\}$ such that

$$\int_0^t \lambda_s\, ds < \infty \quad \text{a.s.}[P],\ t \geq 0.$$

Then

(i) N_t is P-nonexplosive, that is $\tau_n \uparrow \infty$ a.s.[P];

(ii) $M_t = N_t - \int_0^t \lambda_s\, ds$ is an $\mathcal{F}_t$-local martingale;

(iii) if Z_t is an $\mathcal{F}_t$-predictable process such that

$$E\Big[\int_0^\infty |Z_s|\lambda_s\, ds\Big] < \infty,\quad t \geq 0,$$

then

$$\int_0^t Z_s\, dM_s \quad \text{is an } \mathcal{F}_t\text{-martingale; and}$$

(iv) if Z_t is an $\mathcal{F}_t$-predictable process such that

$$\int_0^t |Z_s|\lambda_s \, ds < \infty, \text{ a.s.}[P], \; t \geq 0,$$

then

$$\int_0^t Z_s \, dM_s \text{ is an } \mathcal{F}_t\text{-local martingale.}$$

For proof of Theorem 3.1, see Bremaud (1981). In particular, under the assumption $E(N_t) < \infty$,

$$M_t = N_t - \int_0^t \lambda_s \, ds$$

is a zero-mean $\mathcal{F}_t$-martingale. Further more, if Z_t is an $\mathcal{F}_t$-predictable process such that

$$E\left\{ \int_0^T Z_t^2 \lambda_t \, dt \right\} < \infty ,$$

then

$$I_t = \int_0^t Z_s \, dM_s \; , \; t \geq 0$$

is a zero-mean square integrable $\mathcal{F}_t$-martingale satisfying the following properties.

Theorem 3.2.

(i) $E(I_t) = 0$, equivalently, $E\left\{ \int_0^t Z_s \, dN_s \right\} = E\left\{ \int_0^t Z_s \, \lambda_s \, d_s \right\}$.

(ii) $E(I_t|\mathcal{F}_s) = I_s \; , \; 0 \leq s \leq t.$

(iii) $E(I_t^2) = E\left\{ \int_o^t Z_s^2 \, \lambda_s \, d_s \right\}$.

(iv) Define $J_t = \int_0^t W_s \, dM_s$ where W_t is $\mathcal{F}_t$-predictable such that

$E\left\{ \int_0^t W_s^2 \, \lambda_s \, d_s \right\} < \infty$. Then $E\left[I_t J_t\right] = E\left[\int_0^t Z_s \, W_s \, \lambda_s \, ds\right]$.

(v) $P\left(|I_t| > \varepsilon\right) \leq \frac{\eta}{\varepsilon^2} + P\left(\int_0^t Z_s^2 \, \lambda_s \, ds > \eta\right)$ for every $\varepsilon > 0$ and $\eta > 0$.

For proof of Theorem 3.2, see Liptcer and Siryayev (1978). A central limit theorem will be used later in this paper for obtaining asymptotic properties of estimators. We now state this result. For proof, see Kutoyants (1984).

Theorem 3.3. Let $\{N_t, t\geq 0\}$ be a point process with stochastic intensity $\{\lambda_t, t\geq 0\}$ as defined above. Let $Z_T = \{Z_T(t), \mathcal{F}_t, t\in[0,T]\}$ be $\mathcal{F}_t$-predictable and

$$I_1 = \int_0^T Z_T(t)\, dM_t,\ T \geq 0$$

where

$$M_t = N_t - \int_0^t \lambda_s\, ds\ ,\ t \geq 0.$$

Suppose that

$$\int_0^T Z_T^2(t)\, \lambda_t\, dt \xrightarrow{P} \sigma^2 \quad \text{as} \quad T \to \infty$$

and, for every $\varepsilon > 0$,

$$\int_0^T E\left[Z_T^2(s)\, \chi\left(\{|Z_T(s)| > \varepsilon\}\right) \lambda_s\right] ds \longrightarrow 0 \quad \text{as} \quad T \to \infty$$

where $\chi(A)$ denotes the indicator function of set A. Then

$$\int_0^T Z_T(s)\, dM_s \xrightarrow{\mathcal{L}} N(0,\sigma^2) \text{ as } T \to \infty.$$

4. Properties of estimators

Since

$$\hat{f}_T(x) = \frac{1}{T}\int_0^T \delta_T(x-X_s)\, dN_s\ , \tag{4.0}$$

it follows that

$$(4.1) \qquad E\left[\hat{f}_T(x)\right] = \frac{1}{T} E\left\{ \int_0^T \delta_T(x-X_s)\, dN_s \right\} .$$

Assume that

$$(A1) \qquad E\left\{ \int_0^T \delta_T^2(x-X_s)\, \lambda_s ds \right\} < \infty .$$

Note that

$$(A2) \qquad E(N_T) = E\left[\int_0^T \lambda_s\, ds\right] < \infty$$

by assumption. In view of (A1) and (A2), an application of Cauchy-Schwarz inequality implies that

$$(4.2) \qquad E\left\{ \int_0^T |\delta_T(x-X_s)|\, \lambda_s\, ds \right\} < \infty.$$

Let

$$(4.3) \qquad f_T^*(x) = \frac{1}{T} \int_0^T \delta_T(x-X_s)\lambda_s\, ds.$$

Note that

$$(4.4) \qquad E\left[\hat{f}_T(x)\right] = E\left[f_T^*(x)\right]$$

by Theorem 3.2. Further more

$$(4.5) \qquad E\left[\hat{f}_T(x)\right] = E\left[f_T^*(x)\right] - \frac{1}{T} \int_0^T E\left[\delta_T(x-X_s)\lambda_s\right] ds.$$

Let

$$(4.6) \qquad J_T(x) = \hat{f}_T(x) - f_T^*(x) = \frac{1}{T} \int_0^T \delta_T(x-X_s)\, dM_s .$$

Note that

$$(4.7) \qquad E\left[J_T(x)\right] = 0,$$

$$(4.8)\qquad E\left[J_T(x)\right]^2 = \frac{1}{T^2}\,E\left[\int_0^T \delta_T(x-X_s)\,dM_s\right]^2$$
$$= \frac{1}{T^2}\,E\left\{\int_0^T \delta_T^2\,(x-X_s)\lambda_s\,ds\right\}$$
$$= \frac{1}{T^2}\int_0^T E\left[\delta_T^2\,(x-X_s)\lambda_s\right]ds$$

by Theorem 3.2. Further more for any $\epsilon > 0$ and $\eta > 0$,

$$(4.9)\qquad P\left(|J_T| > \epsilon\right) = P\left(\left|\frac{1}{T}\int_0^T \delta_T(x-X_s)dM_s\right| > \epsilon\right)$$
$$\le \frac{\eta}{\epsilon^2} + P\left\{\frac{1}{T^2}\int_0^T \delta_T^2\,(x-X_s)\lambda_s\,ds > \eta\right\}$$

by Theorem 3.2. For any fixed x, let

$$L_{T,x}(u) = \int_0^u \delta_T(x-X_s)dM_s,\quad 0\le u\le T.$$

Then $\{L_{T,x}(u),\ \mathcal{F}_u,\ 0\le u\le T\}$ is a zero-mean square-integrable martingale and hence

$$(4.10)\qquad E\left[\sup_{0\le u\le T} L_{T,x}^2(u)\right] \le 4\ E\ L_{T,x}^2(T).$$

Further more

$$(4.11)\qquad E\left[L_{T,x}(u)\right] = 0,$$

$$(4.12)\qquad \mathrm{Var}\left[L_{T,x}(u)\right] = E\left[\int_0^u \delta_T^2(x-X_s)\lambda_s ds\right],$$

and

$$(4.13)\qquad \mathrm{Cov}\left(L_{T,x}(u),\ L_{T,y}(u)\right) = E\left[\int_0^u \delta_T(x-X_s)\delta_T(y-X_s)\lambda_s ds\right].$$

As a consequence of the last result, we have

$$(4.14)\qquad \mathrm{Cov}\left(J_T(x),J_T(y)\right) = \frac{1}{T^2}\,E\left[\int_0^T \delta_T(x-X_s)\delta_T(y-X_s)\lambda_s ds\right].$$

Proposition 4.1. If

$$\frac{1}{T^2}\int_0^T \delta_T^2(x-X_s)\lambda_s ds \xrightarrow{p} 0 \text{ as } T \to \infty,$$

(4.16) $$\hat{f}_T(x) - f_T^*(x) \xrightarrow{p} 0 \quad \text{as} \quad T \to \infty.$$

Proof. This result follows from (4.9).

Proposition 4.2. If

(4.17) $$\frac{1}{T^2}\int_0^T E\left[\delta_T^2(x-X_s)\lambda_s\right]ds \to 0 \quad \text{as } T \to \infty,$$

then

(4.18) $$\hat{f}_T(x) - f_T^*(x) \xrightarrow{q.m} 0 \text{ as } T \to \infty.$$

Proof. This result is an application of (4.8).

Proposition 4.3. Suppose there exists $\nu_T > 0$ such that

(4.19) $$\frac{\nu_T^2}{T^2}\int_0^T \delta_T^2(x-X_s)\lambda_s \, ds \xrightarrow{p} \sigma^2 > 0 \text{ as } T \to \infty$$

and, for every $\varepsilon > 0$,

(4.20) $$\frac{\nu_T^2}{T^2}\int_0^T E\left[\delta_T^2(x-X_s)\lambda_s \, \chi\left\{|\delta_T(x-X_s)| > \varepsilon T\nu_T^{-1}\right\}\right] ds \longrightarrow 0$$

as $T \to \infty$.

Then

(4.21) $$\nu_T\left(\hat{f}_T(x)-f_T^*(x)\right) \xrightarrow{\mathcal{L}} N(0,\sigma^2) \quad \text{as} \quad T \to \infty.$$

Proof. Note that

(4.22) $$\nu_T\left(\hat{f}_T(x)-f_T^*(x)\right)$$

$$- \frac{\nu_T}{T} \int_0^T \delta_T(x-X_s)dM_s$$

$$- \int_0^T Z_T(s)dM_s$$

where

(4.23) $$Z_T(s) = \frac{\nu_T}{T} \delta_T(x-X_s), \quad 0 \le s \le T.$$

In view of conditions (4.19) and (4.20), it follows that

$$\int_0^T Z_T(s)dM_s \xrightarrow{\mathcal{L}} N(0,\sigma^2)$$

as $T \to \infty$ by Theorem 3.3.

5. $\{\lambda_s\}$ is non-random

Suppose the intensity process $\{\lambda_s\}$ is non-random function.

Note that

(5.1) $$E\left[\hat{f}_T(x)\right] = \frac{1}{T} \int_0^T E\left[\delta_T(x-X_s)\right]\lambda_s \, ds$$

$$= E\left[\delta_T(x-X_0)\right] \frac{1}{T} \int_0^T \lambda_s \, ds$$

by stationarity of the process X. In view of Lemma 2.1, the following proposition can be proved.

Proposition 5.1. If the density f is continuous at x, then

(5.2) $$E\left[\delta_T(x-X_0)\right] \longrightarrow f(x) \quad \text{as} \quad T \to \infty.$$

Proposition 5.2. $\hat{f}_T(x)$ is an asymptotically unbiased estimator of f(x) at continuity points x of f if and only if

(5.3) $\frac{1}{T}\int_0^T \lambda_s \, ds \longrightarrow 1$ as $T \to \infty$.

Proof. Follows from (5.1) and (5.2).

Proposition 5.3. Suppose the following conditions hold:

(5.4) $\overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T \lambda_s \, ds < \infty$

and

(5.5) $\alpha_t = \int_{-\infty}^{\infty} \delta_t^2(x) \, dx < \infty$.

If f is continuous at x and $\alpha_t \, T^{-1} \to 0$ as $T \to \infty$, then

(5.6) $\hat{f}_T(x) - f_T^*(x) \xrightarrow{q.m} 0$ as $T \to \infty$.

Proof. Observe that, form (4.8), we have

(5.7) $$E\left[\hat{f}_T(x) - f_T^*(x)\right]^2$$
$$= \frac{1}{T^2}\int_0^T E\left[\delta_T^2(x-X_s)\lambda_s\right] ds$$
$$= \frac{1}{T} E\left[\delta_T^2(x-X_o)\right] \frac{1}{T}\int_0^T \lambda_s \, ds$$

by stationarity. Note that

(5.8) $$E\left[\delta_T^2(x-X_o)\right] = \int_{-\infty}^{\infty} \delta_T^2(x-y) \, f(y) \, dy$$
$$= \alpha_T \int_{-\infty}^{\infty} \delta_T^*(x-y) \, f(y) \, dy$$

where

(5.9) $\delta_T^*(x) = \alpha_T^{-1} \, \delta_T^2(x)$.

Lemmas 2.1 and 2.2 and relation (5.8) imply that

(5.10) $\alpha_T^{-1} E\left[\delta_T^2(x-X_o)\right] \longrightarrow f(x)$ as $T \to \infty$

at continuity point x of f. It is clear that

(5.11) $E\left[\hat{f}_T(x)-f_T^*(x)\right]^2 = \frac{\alpha_T}{T} \alpha_T^{-1} E\left[\delta_T^2(x-X_o)\right] \frac{1}{T} \int_0^T \lambda_s \, ds$

and (5.6) follows from (5.4), (5.10) and the fact that

$$\frac{\alpha_T}{T} \longrightarrow 0 \quad \text{as} \quad T \to \infty.$$

Proposition 5.4. If

(5.12) $\frac{1}{T} \int_0^T \lambda_s \, ds \longrightarrow \lambda$ as $T \to \infty$,

and

(5.13) $\alpha_T = \int_{-\infty}^{\infty} \delta_T^2(x) \, dx < \infty$,

then

(5.14) $T \alpha_T^{-1} E\left[\hat{f}_T(x)-f_T^*(x)\right]^2 \longrightarrow \lambda f(x)$ as $T \to \infty$

at all continuity points x of f.

Proof. This result follows from (5.11) and the relations (5.10) and (5.12).

Remarks 5.1. In view of Propositions 5.3 and 5.4, asymptotic behaviour of the estimators $\hat{f}_T(x)$ and $f_T^*(x)$ are the same if $\alpha_T/T \to 0$ as $T \to \infty$. Note that

$$J_T(x) = \hat{f}_T(x) - f_T^*(x) = \hat{f}_T(x) - E(\hat{f}_T(x)) + E(f_T^*(x)) - f_T^*(x)$$

by (4.7) and hence asymptotic behaviour of the statistic

$$\zeta_T(x) = \hat{f}_T(x) - E(\hat{f}_T(x))$$

is the same as that of

$$\eta_T(x) = f_T^*(x) - E(f_T^*(x))$$

under the conditions of Propostion 5.3. Observe that

$$\mathrm{Cov}\left(\eta_T(x),\eta_T(y)\right) = E\left[\eta_T(x),\eta_T(y)\right]$$

$$- \frac{1}{T^2} E\Big[\int_0^T \int_0^T \left\{\delta_T(x-X_s) - E\,\delta_T(x-X_s)\right\}\left\{\delta_T(y-X_t) - E\,\delta_T(y-X_t)\right\} \lambda_s\,\lambda_t\,ds\,dt\Big]$$

$$- \frac{1}{T^2} \int_0^T \int_0^T \left\{E\left[\delta_T(x-X_s)\delta_T(y-X_t)\right] - E\left[\delta_T(x-X_s)\right]E\left[\delta_T(y-X_t)\right]\right\} \lambda_s\,\lambda_t\,ds\,dt$$

$$- \frac{1}{T^2} \int_0^T \int_0^T E\left[\delta_T(x-X_s)\delta_T(y-X_t)\right] \lambda_s\,\lambda_t\,ds\,dt$$

$$- E\left[\delta_T(x-X_o)\right] E\left[\delta_T(y-X_o)\right] \frac{1}{T^2} \int_0^T \int_0^T \lambda_s\,\lambda_t\,ds\,dt$$

by stationarity of the process X. All the above equalities can be justified under assumptions (A_1) and (A_2) by Cauchy-Schwarz inequality and Fubini's Theorem. Let $f_{s,t}(\cdot,\cdot)$ denote the joint density of (X_s,X_t) and $f(\cdot)$ is the density of X_o. Hence

(5.15) $\mathrm{Cov}\left(\eta_T(x),\eta_T(y)\right)$

$$- \frac{1}{T^2} \int_0^T \int_0^T \left\{ \int_{-\infty}^{\infty} \int_{\infty}^{\infty} \delta_T(x-u)\delta_T(y-v)\, f_{s,t}(u,v)\,du\,dv \right\} \lambda_s\,\lambda_t\,ds\,dt$$

$$- \frac{1}{T^2} \int_{-\infty}^{\infty} \delta_T(x-u) f(u)\,du \int_{-\infty}^{\infty} \delta_T(y-v) f(v)\,dv \int_o^T \int_o^T \lambda_s\,\lambda_t\,ds\,dt$$

$$- \frac{1}{T^2} \int_{-\infty}^{\infty} \int_{\infty}^{\infty} \delta_T(x-u)\,\delta_T(y-v) \left[\int_o^T \int_o^T \left\{f_{s,t}(u,v) - f(u)f(v)\right\} \lambda_s\,\lambda_t\,ds\,dt \right] du\,dv$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_T(x-u)\,\delta_T(y-v)\, g_T(u,v)\,du\,dv$$

where

(5.16) $$g_T(u,v) = \frac{1}{T^2} \int_0^T \int_0^T \left[f_{s,t}(u,v) - f(u)f(v)\right] \lambda_s\,\lambda_t\,ds\,dt\,.$$

Proposition 5.5. Suppose $g_T(u,v)$ is as defined by (5.16) and there exists $\beta_T > 0$ such that

(5.17) $\beta_T^2 g_T(u,v) \to g(u,v)$ as $T \to \infty$ uniformly in u,v and $g(\cdot,\cdot)$ is bounded.

Then

(5.18) $\beta_T^2 \operatorname{Cov}\left(\eta_T(x), \eta_T(y)\right) \longrightarrow g(x,y)$ as $T \to \infty$

at all continuity points (x,y) of $g(\cdot,\cdot)$.

Proof. Note that, under the condition (5.17),

(5.19) $$\beta_T^2 \operatorname{Cov}\left(\eta_T(x), \eta_T(y)\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_T(x-u)\, \delta_T(y-v) \left[g(u,v) + o(1)\right] du\, dv$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_T(x-u)\, \delta_T(y-v) g(u,v)\, du\, dv + o(1)$$

as $T \to \infty$ since the convergence in (5.17) is uniform in u,v and the class $\{\delta_t(x), t \geq 0\}$ is a family of delta-type. Hence, as $T \to \infty$,

(5.20) $\beta_T^2 \operatorname{Cov}\left(\eta_T(x), \eta_T(y)\right) = g(x,y) + o(1)$

at all continuity points of $g(\cdot,\cdot)$ by Lemma 2.3.

Proposition 5.6. Suppose that

(5.21) $$h(u) = \lim_{T\to\infty} \frac{\beta_T^2}{T^2} \int_0^T \int_0^T \left[f_{s,t}(u,u) - f(u)f(u)\right] \lambda_s \lambda_t\, ds\, dt$$

exists for every u and the limit is uniform in u. Further suppose that $h(\cdot)$ is bounded. Then

(5.22) $\beta_T^2 \operatorname{Var}\left(\eta_T(x)\right) \longrightarrow h(x)$ as $T \to \infty$

at all continuity points of $h(\cdot)$.

Proof. This result follows by arguments similar to those given in proving Proposition 5.5.

Theorem 5.7. Suppose that (5.21) holds and

$$(5.23) \qquad \frac{1}{T}\int_0^T \lambda_s \, ds \longrightarrow 1 \quad \text{as} \quad T \to \infty.$$

Then

$$(5.24) \qquad E\left[f_T^*(x)\right] \longrightarrow f(x) \quad \text{as} \quad T \to \infty$$

and

$$(5.25) \qquad \beta_T^2 \, \mathrm{Var}\left[F_T^*(x)\right] \longrightarrow h(x) \quad \text{as} \quad T \to \infty$$

at all x which are continuity points of both h and f.

Proof. This theorem follows from Propositions 5.6 and 5.2 since

$$E\left[\hat{f}_T(x)\right] - E\left[f_T^*(x)\right] .$$

Remarks 5.2. Note that

$$f_T^*(x) - \frac{1}{T}\int_0^T \delta_T(x-X_s)\, \lambda_s \, ds$$

and

$$\beta_T\left[f_T^*(x) - Ef_T^*(x)\right] - \frac{\beta_T}{T}\int_0^T \left[\delta_T(x-X_s) - E\left(\delta_T(x-X_s)\right)\right] \lambda_s \, ds.$$

One can give sufficient conditions for asymptotic normality of the family of random variables defined above as $T \to \infty$. For instance, this was done for stationary Markov processes X when $\lambda_s = 1$ in Prakasa Rao (1979a). Castellana and Leadbetter (1986) studied the case when $\lambda_s = 1$ and X is a stationary process satisfying a mixing type of condition. We will not

discuss these results here. Our interest is to study the asymptotic behaviour of $\hat{f}_T(x)$ when compared to that of $f_T^*(x)$.

Theorem 5.8. Suppose conditions (5.4) and (5.5) hold. Then, for every $\epsilon > 0$, as $T \to \infty$,

$$(5.26) \qquad P\left(\beta_T |f_T^*(x) - \hat{f}_T(x)| > \epsilon\right) = O\left(\epsilon^{-1} \alpha_T^{1/2} T^{-1/2} \beta_T\right) .$$

Proof. Observe that, for any $\epsilon > 0$ and $\eta > 0$,

$$(5.27) \qquad P\left(\beta_T |\hat{f}_T(x) - f_T^*(x)| > \epsilon\right)$$

$$= P\left(\left|\int_0^T \delta_T(x - X_s)\, dM_s\right| > \epsilon\, \beta_T^{-1}\, T\right)$$

$$\leq \frac{\eta}{\epsilon^2} \frac{\beta_T^2}{T^2} + P\left(\left|\int_0^T \delta_T^2(x - X_s)\lambda_s\, ds\right| > \eta\right)$$

(by Theorem 3.2)

$$\leq \frac{\eta}{\epsilon^2} \frac{\beta_T^2}{T^2} + \frac{1}{\eta} E\left\{\int_0^T \delta_T^2(x - X_s)\lambda_s\, ds\right\}$$

$$= \frac{\eta}{\epsilon^2} \frac{\beta_T^2}{T^2} + \frac{1}{\eta} E\left(\delta_T^2(x - X_0)\right) \int_0^T \lambda_s\, ds$$

(by stationarity of X)

Under the conditions (5.4) and (5.5), the second term on the right side of (5.27) is of the order

$$\frac{1}{\eta}\, \alpha_T\, f(x)\, T$$

and hence

$$(5.28) \qquad P\left(\beta_T |\hat{f}_T(x) - f_T^*(x)| > \epsilon\right) = O\left(\frac{\eta}{\epsilon^2} \frac{\beta_T^2}{T^2} + \frac{1}{\eta}\, \alpha_T\, T\right)$$

for every $\epsilon > 0$ and $\eta > 0$. Choosing $\eta = \epsilon\, \alpha_T^{1/2}\, T^{3/2}\, \beta_T^{-1}$, we have

$$(5.29)\qquad P\left(\beta_T|\hat{f}_T(x)-f_T^*(x)| > \varepsilon\right) = O\left(\varepsilon^{-1}\ \alpha_T^{1/2}\ T^{-1/2}\ \beta_T\right).$$

Corollary 5.9. Suppose that the conditions (5.4) and (5.5) hold and $\beta_T > 0$ is such that

$$(5.30)\qquad \alpha_T^{1/2}\ T^{-1/2}\ \beta_T \longrightarrow 0 \quad \text{as} \quad T \to \infty.$$

Then

$$(5.31)\qquad \beta_T\left(\hat{f}_T(x)-f_T^*(x)\right) \xrightarrow{P} 0 \quad \text{as} \quad T \to \infty.$$

at all continuity points x of f.

Proof. This result follows from Theorem 5.8.

As a consequence of Corollary 5.9 and Proposition 5.6, we have the following theorem. Note that $E\left(\hat{f}_T(x)\right) = E\left(f_T^*(x)\right)$.

Theorem 5.10. Suppose the conditions (5.4), (5.5) and (5.38) hold. Further suppose that

$$(5.32)\qquad \beta_T\left(f_T^*(x)-E(f_T^*(x))\right) \xrightarrow{\mathcal{L}} N(o,h(x)) \quad \text{as} \quad T \to \infty.$$

Then

$$(5.33)\qquad \beta_T\left(\hat{f}_T(x)-E(\hat{f}_T(x))\right) \xrightarrow{\mathcal{L}} N(o,h(x)) \quad \text{as} \quad T \to \infty.$$

Remarks 5.3. Observe that h(x) is given by (5.21) under some conditions as stated earlier. Further, for the validity of the results in this section, we do not need the fact that $\{X_t, t\geq 0\}$ is a stationary process but that the finite-dimensional marginal densities of $\{X_t\}$ exist and the one-dimensional marginal density of X_t is the same for every $t \geq 0$.

6. $\{\lambda_s\}$ is a random process independent of $\{X_t\}$

Suppose the intensity process $\{\lambda_s, s\geq 0\}$ is random but independent of $\{X_s, s\geq 0\}$. Note the λ_s is $\mathcal{F}_s$-measurable for every $s \geq 0$. Further more

$$(6.1) \qquad E\left[\hat{f}_T(x)\right] = \frac{1}{T}\int_0^T E\left[\delta_T(x-X_s)\lambda_s\right] ds$$

$$= E\left[\delta_T(x-X_o)\right] \frac{1}{T}\int_0^T E(\lambda_s)\, ds$$

by independence of $\{\lambda_s\}$ and $\{X_t\}$ and by stationarity of X process.

Proposition 6.1. $\hat{f}_T(x)$ is an asymptotically unbiased estimator of $f(x)$ at continuity points x of f if and only if

$$(6.2) \qquad \frac{1}{T}\int_o^T E(\lambda_s)\, ds \longrightarrow 1 \quad \text{as} \quad T \to \infty.$$

Proof. This result follows from (6.1) and Proposition 5.1.

Proposition 6.2. Suppose the following conditions hold:

$$(6.4) \qquad \overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T E(\lambda_s)ds < \infty \ ;$$

and

$$(6.5) \qquad \alpha_t = \int_{-\infty}^{\infty} \delta_t^2(x)dx < \infty \ .$$

If f is continuous at x and $\alpha_T\, T^{-1} \to 0$ as $T \to \infty$, then

$$(6.6) \qquad \hat{f}_T(x) - f_T^*(x) \xrightarrow{q.m} 0 \quad \text{as} \quad T \to \infty.$$

Proof. Note that, from (4.8), we have

$$(6.7) \qquad E\left[\hat{f}_T(x) - f_T^*(x)\right]^2$$

$$= \frac{1}{T^2}\int_0^T E\left[\delta_T^2(x-X_s)\lambda_s\right] ds$$

$$= \frac{1}{T} E\left[\delta_T^2(x-X_o)\right] \frac{1}{T}\int_0^T E(\lambda_s)ds$$

by independence of $\{X_t\}$ and $\{\lambda_s\}$ and by stationarity of $\{X_t, t\geq 0\}$. The rest of the proof is similar to that of Proposition 5.3 (by replacing λ_s by $E(\lambda_s)$). We omit the details.

Proposition 6.3. If

$$\frac{1}{T}\int_0^T E(\lambda_s)ds \longrightarrow \lambda \quad \text{as } T \to \infty \tag{6.8}$$

and

$$\alpha_t = \int_{-\infty}^{\infty} \delta_t^2(x)dx < \infty , \tag{6.9}$$

then

$$T\,\alpha_T^{-1}\, E\left[\hat{f}_T(x) - f_T^*(x)\right]^2 \longrightarrow \lambda\, f(x) \quad \text{as} \quad T \to \infty \tag{6.10}$$

at all continuity points x of f.

Proof. Similar to that of Proposition 5.4.

Remarks 6.1. Let

$$\zeta_T(x) = \hat{f}_T(x) - E(\hat{f}_T(x)) , \tag{6.11}$$

and

$$\eta_T(x) = f_T^*(x) - E(f_T^*(x)) . \tag{6.12}$$

Since

$$E\left(\hat{f}_T(x)\right) = E\left(f_T^*(x)\right) , \tag{6.13}$$

it follows that

$$\begin{aligned} J_T(x) &= \hat{f}_T(x) - f_T^*(x) \\ &= \zeta_T(x) - \eta_T(x) \end{aligned} \tag{6.14}$$

and the asymptotic behaviour of $\zeta_T(x)$ and $\eta_T(x)$ are the same under the conditions (6.4) and (6.5). Observe that

$$(6.15)\qquad \mathrm{Cov}\big(\eta_T(x),\eta_T(y)\big) = E\big(\eta_T(x)\eta_T(y)\big)$$

$$- \frac{1}{T^2}\, E\Big[\int_0^T \Big\{\delta_T(x-X_s)\lambda_s - E\Big\{\delta_T(x-X_s)\lambda_s\Big\}\Big\}ds \int_0^T \Big\{\delta_T(y-X_t)\lambda_t - E\Big\{\delta_T(y-X_t)\lambda_t\Big\}ds\ dt\Big]$$

$$- \frac{1}{T^2}\int_0^T\int_0^T \Big\{E\Big[\delta_T(x-X_s)\delta_T(y-X_t)\lambda_s\lambda_t\Big] - E\Big[\delta_T(x-X_s)\lambda_s\Big]E\Big[\delta_T(y-X_t)\lambda_t\Big]\Big\}ds\ dt$$

$$- \frac{1}{T^2}\int_0^T\int_0^T \Big\{E\Big[\delta_T(x-X_s)\delta_T(y-X_t)\Big]\ E\Big[\lambda_s\lambda_t\Big] - E\Big[\delta_T(x-X_s)\Big]E\Big[\delta_T(y-X_t)\lambda_t\Big]E[\lambda_s]E[\lambda_t]\Big\}ds\ dt$$

$$- \frac{1}{T^2}\int_0^T\int_0^T \Big\{E\Big[\delta_T(x-X_s)\delta_T(y-X_t)\Big] - E\Big[\delta_T(x-X_s)\Big]\ E\Big[\delta_T(y-X_t)\Big]\Big\}\ E\Big[\lambda_s\lambda_t\Big]ds\ dt$$

$$+ \frac{1}{T^2}\int_0^T\int_0^T E\Big[\delta_T(x-X_s)\Big]E\Big[\delta_T(y-X_t)\Big]\ \Big\{E\Big[\lambda_s\lambda_t\Big] - E[\lambda_s]E[\lambda_t]\Big\}ds\ dt$$

$$- \frac{1}{T^2}\int_0^T\int_0^T \Big\{E\Big[\delta_T(x-X_s)\delta_T(y-X_t)\Big] - E\Big[\delta_T(x-X_s)\Big]\ E\Big[\delta_T(y-X_t)\Big]\Big\}\ E\Big[\lambda_s\lambda_t\Big]ds\ dt$$

$$+ E\Big[\delta_T(x-X_o)\Big]E\Big[\delta_T(y-X_o)\Big]\ \frac{1}{T^2}\int_0^T\int_0^T \mathrm{Cov}\big(\lambda_s,\lambda_t\big)ds\ dt$$

$$- \frac{1}{T^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_T(x-u)\delta_T(y-v)\Big[\int_0^T\int_0^T \Big\{f_{s,t}(u,v) - f(u)f(v)\Big\}\ E\Big[\lambda_s\lambda_t\Big]ds\ dt\Big]\ du\ dv$$

$$+ E\Big[\delta_T(x-X_o)\Big]\ E\Big[\delta_T(y-X_o)\Big]\ \frac{1}{T^2}\int_0^T\int_0^T \mathrm{Cov}\big(\lambda_s,\lambda_t\big)ds\ dt$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_T(x-u)\delta_T(y-v)\tilde{g}_T(u,v)\ du\ dv$$

$$+ E\Big[\delta_T(x-X_o)\Big]\ E\Big[\delta_T(y-X_o)\Big]\ \frac{1}{T^2}\int_0^T\int_0^T \mathrm{Cov}\big(\lambda_s,\lambda_t\big)ds\ dt$$

where

$$(6.16)\qquad \tilde{g}_T(u,v) = \frac{1}{T^2}\int_0^T\int_0^T \Big[f_{s,t}(u,v) - f(u)f(v)\Big]\ E\Big[\lambda_s\lambda_t\Big]ds\ dt.$$

Proposition 6.4. Suppose $\tilde{g}_T(u,v)$ is as defined by (6.16) and there exists $\beta_T > 0$ such that

(6.17) $\beta_T^2 \, \tilde{g}_T(u,v) \longrightarrow g(u,v)$ as $T \to \infty$ uniformly in u,v and $g(\cdot,\cdot)$ is bounded; and

(6.18) $\frac{\beta_T^2}{T^2} \int_0^T \int_0^T \mathrm{Cov}(\lambda_s,\lambda_t)\,ds\,dt \longrightarrow \beta$ as $T \to \infty$.

Then

(6.19) $\beta_T^2 \, \mathrm{Cov}\left(\eta_T(x),\eta_T(y)\right) \longrightarrow g(x,y) + \beta\, f(x)f(y)$ as $T \to \infty$

at all continuity points (x,y) of $g(\cdot,\cdot)$ where x and y are continuity points of f.

Proof. This result follows from (6.15), (6.16) by arguments similar to those given in proving Proposition 5.5.

Proposition 6.5. Suppose that

(6.20) $h(u) = \lim_{T\to\infty} \frac{\beta_T^2}{T^2} \int_0^T \int_0^T \left[f_{s,t}(u,u) - f(u)f(u)\right] E\left[\lambda_s\lambda_t\right] ds\,dt$

exists for every u and the limit is uniform in u. Further suppose that $h(\cdot)$ is bounded and

(6.21) $\frac{\beta_T^2}{T^2} \int_0^T \int_0^T \mathrm{Cov}\left(\lambda_s,\lambda_t\right) ds\,dt \longrightarrow \beta$ as $T \to \infty$.

Then

(6.22) $\beta_T^2 \, \mathrm{Var}\left(\eta_T(x)\right) \longrightarrow h(x) + \beta\, f^2(x)$ as $T \to \infty$

at all x which are continuity points of h and f.

Proof. This result follows from Proposition 6.4.

Proposition 6.6. Suppose the following conditions hold

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T E(\lambda_s)ds < \infty \; ; \tag{6.23}$$

and

$$\alpha_t = \int_{-\infty}^{\infty} \delta_t^2(x)dx < \infty \; . \tag{6.24}$$

Let x be a continuity point of f. Then, for every $\epsilon > 0$,

$$P\left(\beta_T |f_T^*(x) - \hat{f}_T(x)| > \epsilon\right) = O\left(\epsilon^{-1}\, \alpha_T^{1/2}\, T^{-1/2}\, \beta_T\right) \quad \text{as} \quad T \to \infty. \tag{6.25}$$

Proof. Similar to that of Theorem 5.8.

Proposition 6.7. Suppose that the conditions (6.23) and (6.24) hold and

$$\alpha_T^{1/2}\, T^{-1/2}\, \beta_T \longrightarrow 0 \quad \text{as} \quad T \to \infty. \tag{6.26}$$

Then

$$\beta_T\left(\hat{f}_T(x) - f_T^*(x)\right) \xrightarrow{P} 0 \quad \text{as} \quad T \to \infty \tag{6.27}$$

at all continuity points x of f.

Proof. This result follows from Proposition 6.6.

Theorem 6.8. Suppose the condition (6.23), (6.24) and (6.26) hold. Further suppose that

$$\beta_T\left(f_T^*(x) - E(f_T^*(x)\right) \xrightarrow{\mathcal{L}} N(0, \bar{h}(x)) \quad \text{as} \quad T \to \infty.$$

Then

$$\beta_T\left(\hat{f}_T(x) - E(\hat{f}_T(x)\right) \xrightarrow{\mathcal{L}} N(0, \bar{h}(x)) \quad \text{as} \quad T \to \infty.$$

Proof. This result follows from Proposition 6.7 since $E(\hat{f}_T(x)) = E(f_T^*(x))$.

Remarks 6.2. Note that

$$\tilde{h}(x) - h(x) + \beta f^2(x)$$

under some conditions where $h(\cdot)$ and β are defined by (6.20) and (6.21). If $\mathrm{Cov}(\lambda_s, \lambda_t) = 0$ for all $s \neq t$, then $\beta = 0$ from (6.21) and $\tilde{h}(x) = h(x)$. In fact

$$h(x) = \lim_{T\to\infty} \frac{\beta_T^2}{T^2} \int_0^T \int_0^T \left[g_{s,t}(x,x) - f(x)f(x)\right] E(\lambda_s)E(\lambda_t)ds\, dt .$$

The expression on the right side of the above equation is the same as the asymptotic variance of the estimator $\hat{f}_T(x)$ obtained from sampling at random times with nonrandom intensity process $\{E(\lambda_t), t\geq 0\}$. This can be seen from (5.21). Observe that, if $\beta < 0$ then $\tilde{h}(x) < h(x)$ and if $\beta > 0$ then $\tilde{h}(x) > h(x)$.

7. $\{\lambda_s\}$ is a random process asymptotically uncorrelated with $\{X_t\}$

Suppose the intensity process $\{\lambda_s, s\geq 0\}$ is asymptotically uncorrelated with the process $\{X_s, s\geq 0\}$ in the sense that for every delta-family $\{\delta_t(x), t\geq 0\}$.

$$(7.1) \qquad \sup_s \left| E\left[\delta_T(x-X_s)\lambda_s\right] - E\left[\delta_T(x-X_s)\right] E\left[\lambda_s\right] \right| \longrightarrow 0 \quad \text{as } T \to \infty.$$

Note that

$$(7.2) \qquad E\left[\hat{f}_T(x)\right] = \frac{1}{T}\int_0^T E\left[\delta_T(x-X_s)\lambda_s\right] ds - \frac{1}{T}\int_0^T E\left[\delta_T(x-X_s)\right] E\left[\lambda_s\right] ds + \frac{1}{T}\int_0^T \psi_T(s)\, ds$$

where

$$(7.3) \qquad |\psi_T(x)| \leq \gamma_T \longrightarrow 0 \quad \text{as } T \to \infty.$$

Hence

$$(7.4)\qquad E\left[\hat{f}_T(x)\right] - \frac{1}{T}\int_0^T E\left[\delta_T(x-X_s)\right] E\left[\lambda_s\right] ds + o(1)$$

$$- E\left[\delta_T(x-X_o)\right] \frac{1}{T}\int_0^T E\left[\lambda_s\right] ds + o(1)$$

by stationarity of the process X.

Proposition 7.1. $\hat{f}_T(x)$ is asymptotically unbiased estimator of f(x) at continuity points x of f if and only if

$$(7.5)\qquad \frac{1}{T}\int_0^T E(\lambda_s)\, ds \longrightarrow 1 \quad \text{as} \quad T \to \infty.$$

Proof. This result follows from Proposition 5.1 and (7.4).

Proposition 7.2. Suppose the following conditions hold:

$$(7.6)\qquad \overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T E(\lambda_s)ds < \infty \ ;$$

and

$$(7.7)\qquad \alpha_t = \int_{-\infty}^{\infty} \delta_t^2(x)dx < \infty \ .$$

If f is continuous at x and $\alpha_T\, T^{-1} \to 0$ as $T \to \infty$, then

$$(7.8)\qquad \hat{f}_T(x) - f_T^*(x) \xrightarrow{q.m} 0 \quad \text{as } T \to \infty.$$

Proof. Note that

$$\delta_T^*(x) = \frac{\delta_T^2(x)}{\alpha_T}, \quad T \geq 0$$

is a delta-family under (7.7). Hence

$$\sup_s \left| E\left[\delta_T^*(x-X_s)\lambda_s\right] - E\left[\delta_T^*(x-X_s)\right] E[\lambda_s] \right| \longrightarrow 0 \quad \text{as } T \to \infty.$$

Furthermore

$$E\left[\hat{f}_T(x) - f_T^*(x)\right]^2$$

$$- \frac{1}{T^2} \int_0^T E\left[\delta_T^2(x-X_s)\lambda_s\right] ds$$

$$- \frac{\alpha_T}{T^2} \int_0^T E\left[\delta_T^*(x-X_s)\lambda_s\right] ds$$

$$- \frac{\alpha_T}{T^2} \int_0^T E\left[\delta_T^*(x-X_s)\right] E[\lambda_s] ds + \frac{\alpha_T}{T^2} \int_0^T \psi_T^*(s) ds$$

where $|\psi_T^*(s)| \leq \gamma_T^*$ and $\gamma_T^* \to 0$ as $T \to \infty$. Hence

$$E\left[\hat{f}_T(x) - f_T^*(x)\right]^2 - \frac{\alpha_T}{T} \left\{ E\left[\delta_T^*(x-X_o)\right] \frac{1}{T} \int_0^T E(\lambda_s) ds) + \frac{1}{T} \int_0^T \psi_T^*(s) ds \right\} .$$

Under the conditions (7.6) and (7.7), it follows by arguments similar to those given in proving Proposition 7.1, that

$$\overline{\lim_{T\to\infty}} \left[E\left[\delta_T^*(x-X_o)\right] \left\{ \frac{1}{T} \int_0^T E(\lambda_s) ds \right\} + \frac{1}{T} \int_0^T \psi_T^*(s) ds \right] < \infty.$$

Hence

$$E\left[\hat{f}_T(x) - f_T^*(x)\right]^2 \longrightarrow 0 \quad \text{as } T \to \infty$$

if

$$\frac{\alpha_T}{T} \longrightarrow 0 \quad \text{as } T \to \infty \quad \text{and } f \text{ is continuous at } x.$$

Remarks 7.1. Under the conditions (6.8) and (6.9), if f is continuous at x, then

$$(7.9) \qquad \frac{T}{\alpha_T} E\left[\hat{f}_T(x) - f_T^*(x)\right]^2 \longrightarrow \lambda f(x) \quad \text{as} \quad T \to \infty.$$

This can be seen by arguments similar to those given in proving Proposition 5.4.

Remark 7.2. For obtaining analogues of Propositions 6.4 to 6.7 we need a stronger condition than that given by (7.1). We will not discuss details here.

8. General $\{\lambda_t\}$

In the general caseof intensity process $\{\lambda_t\}$, no specific results can be obtained. However, we can still obtain information on the difference between the estimator $\hat{f}_T(x)$ obtained from the sampled data and the estimator $f_T^*(x)$ as defined by (4.0) and (4.3). Let

$$J_T(X) = \hat{f}_T(x) - f_T^*(x) = \frac{1}{T}\int_0^T \delta_T(x-X_s)dM_s$$

be as defined in (4.6). The following proposition is a restatement of results in Section 4.

Proposition 8.1.

(i) $E\left[J_T(x)\right] = 0.$

(ii) $E\left[J_T(x)\right]^2 = \frac{1}{T^2}\int_0^T E\left[\delta_T^2(x-X_s)\lambda_s\right] ds .$

(iii) For any $\epsilon > 0$ and $\eta > 0$,

$$P\left(|J_T(x)|>\epsilon\right) \le \frac{\eta}{\epsilon^2} + P\left\{\frac{1}{T^2}\int_0^T \delta_T^2(x-X_s)\lambda_s\, ds > \eta\right\} .$$

(iv) Suppose there exists $\nu_T > 0$ such that (4.19) and (4.20) hold. Then

$$\nu_T\, J_T(x) \xrightarrow{\mathcal{L}} N(0,\sigma^2) \quad \text{as} \quad T \to \infty.$$

Remarks 8.1. Suppose $p_{X_s,\lambda_s}(\cdot,\cdot)$ is the joint density of (X_s,λ_s) assuming that it exists. Assume that the conditional densities in the following computation are well-defined. Note that

$$(8.1) \qquad E\left[\delta_T(x-X_s)\lambda_s\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_T(x-u)\, v\, p_{X_s,\lambda_2}(u,v)\, du\, dv$$

$$- \int_{-\infty}^{\infty} \Big\{ \int_0^{\infty} \delta_T(x-u) \; v \; p_{X_s|\lambda_s}(u|v) \; du \Big\} \; v \; p_{\lambda_s}(v) dv \; .$$

From the property of delta-families, it follows that

(8.2) $$\int_0^{\infty} \delta_T(x-u) p_{X_s|\lambda_s}(u|v) du \longrightarrow p_{X_s|\lambda_s}(x|v) \quad \text{as} \quad T \to \infty$$

at all continuity points x of $p_{X_s|\lambda_s}(\cdot|v)$ by Lemma 2.1.

(A_3) Suppose the limit exists in (8.2) for every x and is uniform in v. Then

(8.3) $$E\left[\delta_T(x-X_s)\lambda_s\right] - \int_0^{\infty} \left[p_{X_s|\lambda_s}(x|v) + o(1)\right] \; v \; p_{\lambda_s}(v) \; dv$$

$$- \int_0^{\infty} p_{X_s,\lambda_s}(x,v) v \; dv + \left\{\int_0^{\infty} v \; p_{\lambda_s}(v) dv\right\} o(1)$$

$$- \int_0^{\infty} p_{X_s}(x) \; p_{X_s|\lambda_s}(v|x) v \; dv + o(1)$$

$$- p_{X_s}(x) \; E\left[\lambda_s | X_s = x\right] + o(1) \; .$$

Hence

(8.4) $$E\left[\hat{f}_T(x)\right] - \frac{1}{T} \int_0^T E\left[\delta_T(x-X_s)\right] ds$$

$$- \frac{1}{T} \int_0^T \left\{p_{X_o}(x) \; E[\lambda_s | X_s = x] + o(1)\right\} ds$$

$$- p_{X_o}(x) \left\{ \frac{1}{T} \int_0^T E[\lambda_s | X_s = x]\right\} + o(1)$$

provided

(A_4) $$\sup_s E(\lambda_s) < \infty.$$

Proposition 8.2. Under the conditions (A_3) and (A_4), $\hat{f}_T(x)$ is an asymptotically unbiased estimator of f(x) where x is a cotinuity point of $p_{X_s,\lambda_s}(\cdot|v)$ for every v if and only if

(8.6) $$\frac{1}{T} \int_0^T E[\lambda_s | X_s = x] ds \longrightarrow 1 \quad \text{as} \quad T \to \infty.$$

We will not discuss other properties of $\hat{f}_T(x)$ under the general set up given here.

9. Remarks

Remarks9.1. Suppose $\{X_t\}$ is a stationary process such that, for any x,

$$(9.1)\qquad \mathrm{Cov}\left[\delta_T(x-X_s),\delta_T(x-X_t)\right] = r_T(|t-s|,x)$$
$$= r_T(o,x) + \psi_T(o,x)|t-s|^{\alpha}$$
$$+ \gamma_T(x,|t-s|)$$

where $0 < \alpha \le 2$. Further suppose that $\{\lambda_s\}$ is non-random as in Setion 5.

Then, for $\eta_T(x) = f_T^*(x) - E\, f_T^*(x)$, we have

$$(9.2)\qquad \mathrm{Var}\left[\eta_T(x)\right] = \frac{1}{T^2}\int_0^T\int_0^T \mathrm{Cov}\left(\delta_T(x-X_s),\delta_T(x-X_t)\right)\lambda_s\lambda_t\, ds\, dt$$
$$= \frac{1}{T^2}\int_0^T\int_0^T r_T(x)|t-s|)\lambda_s\lambda_t\, ds\, dt$$
$$= \frac{1}{T^2}\int_0^T\int_0^T \left\{r_T(o,x) + \psi_T(o,x)|t-s|^{\alpha} + \gamma_T(x,|t-s|)\right\}\lambda_s\lambda_t\, ds\, dt$$
$$= r_T(o,x)\,\frac{1}{T^2}\int_0^T\int_0^T \lambda_s\,\lambda_t\, ds\, dt$$
$$+ \frac{\psi_T(o,x)}{T^2}\int_0^T\int_0^T |t-s|^{\alpha}\,\lambda_s\lambda_t\, ds\, dt$$
$$+ \frac{1}{T^2}\int_0^T\int_0^T \gamma_T(x,|t-s|)\lambda_s\lambda_t\, ds\, dt.$$

If $\lambda_s = 1$ (Poisson Sampling of the process)

$$(9.3)\qquad \mathrm{Var}\left[\eta_T(x)\right] = r_T(o,x) + \psi_T(o,x)\,\frac{1}{T^2}\int_0^T\int_0^T |t-s|^{\alpha}\, ds\, dt$$
$$+ \frac{1}{T^2}\int_0^T\int_0^T \gamma_T(x,|t-s|)ds\, dt$$
$$= r_T(o,x) + \psi_T(o,x)\,\frac{1}{T^2}\,2\,\frac{T^{\alpha+2}}{(\alpha+1)(\alpha+2)}$$

$$+ \frac{1}{T^2} \int_0^T \int_0^T \gamma_T(x,|t-s|)ds\ dt$$

$$- r_T(o,x) + 2\ \psi_T(o,x)\ \frac{T^\alpha}{(\alpha+1)(\alpha+2)}$$

$$+ \frac{1}{T^2} \int_0^T \int_0^T \gamma_T(x,|t-s|)ds\ dt.$$

Note that

$$(9.4) \qquad r_T(o,x) = E\left[\delta_T^2(x-X_o)\right] - \left\{E\left[\delta_T(x-X_o)\right]\right\}^2 \simeq \alpha_T\ f(x)$$

as $T \to \infty$. Relations (9.3) and (9.4) show that

$$(9.5) \qquad \alpha_T^{-1}\ \mathrm{Var}\left[\eta_T(x)\right] \simeq f(x)$$

provided

$$(9.6) \qquad \alpha_T^{-1}\ \psi_T(o,x)T^\alpha \longrightarrow 0 \quad \text{as} \quad T \to \infty$$

and

$$(9.7) \qquad \frac{\alpha_T^{-1}}{T^2} \int_0^T \int_0^T \gamma_T(x,|t-s|)ds\ dt \longrightarrow 0 \quad \text{as} \quad T \to \infty.$$

Remarks 9.2. One can specialize the results obtained in Sections 3 to 8 for kernel type estimators by choosing

$$\delta_t(x) = h_t^{-1}K\left(h_t^{-1}x\right)$$

where $K(\cdot)$ is a bounded probability density such that $t\ K(t) \to 0$ as $t \to \infty$, $h_t > 0$, $h_t \to 0$ as $t \to \infty$. We will not discuss this case here.

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