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# A proof of a trace formula by Richard Melrose

Yves Colin de Verdière\*

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The goal of this note is to give a new proof of the wave trace formula proved by Richard Melrose in the impressive paper [Me-84]. This trace formula is an extension of the Chazarain-Duistermaat-Guillemin trace formula (denoted “CDG trace formula” in this paper) to the case of a sub-Riemannian (“sR”) Laplacian on a 3D contact closed manifold. The proof uses a normal form constructed in the papers [CHT-18, CHT-21], following the pioneering work [Me-84], in order to reduce to the case of the invariant Laplacian on the 3D-Heisenberg group. We need also the propagation of singularities results of Victor Ivrii, Bernard Lascar and Richard Melrose [Iv-76, La-82, Me-86].

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## 1 The CDG trace formula

The following result was proved by Chazarain [Ch-74] and refined by Duistermaat and Guillemin [DG-75] (see also [CdV-73, CdV-07] and Appendix F for the history of the trace formulae):

**Theorem 1.1** *Let  $(M, g)$  be a closed connected smooth Riemannian manifold and  $\lambda_1 = 0 < \lambda_2 \leq \dots$  the spectrum of the Laplace operator, then we have the following equality of Schwartz distributions:*

$$\sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}} = T_0(t) + \sum_{\gamma \in \mathcal{P}} T_{\gamma}(t) \text{ mod } C^{\infty}$$

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where  $\mathcal{P}$  is the set of periodic geodesics,  $\text{SingSupp}(T_0) = \{0\}$  and, for  $\gamma$  a periodic geodesic,  $\text{SingSupp}(T_\gamma) \subset \{L_\gamma\} \cup \{-L_\gamma\}$ , where  $L_\gamma$  is the length of  $\gamma$ . Moreover, if  $\gamma$  is non degenerate,

$$T_\gamma(t) = \frac{L_0 e^{i\pi m(\gamma)/2}}{\pi \det(\text{Id} - P_\gamma)^{\frac{1}{2}}} (t + i0 - L_\gamma)^{-1} \left( 1 + \sum_{j=1}^{\infty} a_j (t - L_\gamma)^j \right)$$

where

- $L_0$  is the length of the primitive geodesic associated to  $\gamma$
- $m(\gamma)$  is the Morse index of  $\gamma$
- $P_\gamma$  is the linearized Poincaré map.

This result gives a nicer proof of the main result of my thesis [CdV-73] saying that, in the generic case, the Length spectrum, i.e. the set of lengths of closed geodesics, is a spectral invariant. This formula holds for any elliptic self-adjoint pseudo-differential operator  $P$  of degree 1 by replacing the periodic geodesics by the periodic orbits of the Hamiltonian flow of the principal symbol of  $P$  and the Morse index by a Maslov index. Our goal is to prove that the same statement holds for sR Laplacians on a closed contact 3-manifold. **What we mean here by ‘Melrose’s trace formula’ is the formulae of theorem 1, interpreted in this contact sR setting.** Richard Melrose gave a proof which I found difficult (see Theorem 6.4 in [Me-84]).

## 2 Review of basic facts and notations

For more details on this section, one can look at the paper [CHT-18].

### 2.1 Contact 3D sR manifolds

In what follows,  $M$  is a closed (compact without boundary) connected manifold of dimension 3 equipped with a smooth volume form  $|dq|$ . We consider also an oriented contact distribution globally defined as the kernel of a non vanishing real valued 1-form  $\alpha$  so that  $\alpha \wedge d\alpha$  is a volume form. Let also  $g$  be a metric on the distribution  $D = \ker \alpha$ . The “co-metric”  $g^*$  is defined by  $g^*(q, p) := \|p|_{D_q}\|^2$  where the norm is the dual norm of  $g(q)$ . To such a set of data is associated

- A geodesic flow denoted by  $G_t$ ,  $t \in \mathbb{R}$ : the Hamiltonian flow of  $\sqrt{g^*}$ . The geodesics are projections of the orbits of that flow onto  $M$  and are everywhere tangent to  $D$ . We will often prefer to consider the geodesic flow as the restriction to  $g^* = 1$  of the Hamiltonian flow of  $\frac{1}{2}g^*$ .
- A Laplacian which is locally given by  $\Delta = X_1^*X_1 + X_2^*X_2$  where  $(X_1, X_2)$  is an orthonormal frame of  $D$  and the adjoint is taken with respect to the measure  $|dq|$ .
- A canonical choice of a 1-form  $\alpha_g$  defining  $D$  by assuming that  $d\alpha_g$  restricted to  $D$  is the oriented  $g$ -volume form on  $D$ .

From a famous theorem of Hörmander, we know the Laplacian is sub-elliptic and hence has a compact resolvent and a discrete spectrum  $\lambda_1 = 0 < \lambda_2 \leq \dots$  with smooth eigenfunctions. It follows from the sR Weyl law

$$\#\{j | \lambda_j \leq \lambda\} \sim C\lambda^2$$

that

$$\text{Trace}(e^{it\sqrt{\Delta}}) = \sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}}$$

is a well defined Schwartz distribution often called the *wave trace* (see Appendix A for the link with the wave equation). Our goal is to extend the CDG formula to this case.

The form  $\alpha_g$  defines a *Reeb vector field*  $\vec{R}$  by the equations  $\alpha_g(\vec{R}) = 1$  and  $\iota(\vec{R})d\alpha_g = 0$ . This vector field admits the following Hamiltonian interpretation: the cone  $\Sigma = D^\perp$  is a symplectic sub-cone of  $T^*M \setminus 0$ <sup>1</sup>. We define the Hamiltonian  $\rho : \Sigma \rightarrow \mathbb{R}$  by  $\rho(\alpha) = \alpha/\alpha_g$  where  $\alpha \in \Sigma$  is a covector vanishing on  $D$ . The Hamiltonian vector field of  $\rho$  is homogeneous of degree 0 and the projection of this field onto  $M$  is the Reeb vector field  $\vec{R}$  (see section 2.4 of [CHT-18]).

## 2.2 The 3D Heisenberg group $H_3$

For this section, the reader could look at Section 3.1 of [CHT-18]. We identify the Heisenberg group  $H_3$  with  $\mathbb{R}_{x,y,z}^3$  with the group law

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx'))$$

---

<sup>1</sup>if  $\Sigma = \{(q, t\alpha(q)) | t \in \mathbb{R} \setminus 0, q \in M\}$ , the symplectic form on  $T^*M$  restricts to  $t d\alpha + dt \wedge \alpha$  whose square is the volume form  $-tdt \wedge \alpha \wedge d\alpha$

and the Lie algebra is generated by  $X, Y, Z$  with

$$X = \partial_x + \frac{1}{2}y\partial_z, \quad Y = \partial_y - \frac{1}{2}x\partial_z, \quad Z = \partial_z$$

We choose  $(D, g)$  by asking that  $(X, Y)$  is an oriented orthonormal basis of  $D$  for  $g$  and  $|dq| = |dxdydz|$ . We have  $[X, Y] = -Z$  and the Reeb vector field is  $Z$ . The Laplacian is  $\Delta_3 = -(X^2 + Y^2)$  and can be rewritten as

$$\Delta_3 = -|Z| \left( \left( \frac{X}{\sqrt{|Z|}} \right)^2 + \left( \frac{Y}{\sqrt{|Z|}} \right)^2 \right) \quad (1)$$

on the complement of the kernel of  $Z$ . We write this as

$$\Delta_3 = |Z|\Omega$$

where  $\Omega$  is an harmonic oscillator with spectrum  $\{2l + 1 | l = 0, 1, \dots\}$  (see [CHT-18], prop. 3.1).

We will need the following ‘‘confining’’ result (see [Le-20]):

**Lemma 2.1** *Given  $T > 0$ ,  $\sigma_0 \in \Sigma \setminus 0$  and  $U$  a conic neighbourhood of  $\sigma_0$ , there exists a conic neighbourhood  $V \subset\subset U$  of  $\sigma_0$  so that  $\forall t \in [-T, T]$ ,  $G_t(V) \subset U$ .*

### 3 Speed of propagation

First, we have the following

**Theorem 3.1** *If  $u$  is a solution of an sR wave equation,*

$$\forall t \in \mathbb{R}, \text{Support}(u(t)) \subset B(\text{Support}(u(t=0)) \cup \text{Support}(du/dt(t=0)), |t|)$$

*where  $B(A, r)$  is the closed sR neighbourhood of radius  $r$  of  $A$ .*

This result follows from the Riemannian case by passing to the limits (see Section 3 of [Me-86]).

We will also use the following Theorem due to Victor Ivrii, Bernard Lascar and Richard Melrose [Iv-76, La-82, Me-86] and revisited by Cyril Letrouit [Le-21]):

**Theorem 3.2** *If  $e(t, q, q')$  is the wave kernel of a sR Laplacian whose characteristic manifold  $\Sigma$  is symplectic, i.e.  $e$  is the Schwartz kernel of  $\cos(t\sqrt{\Delta})$ , then*

$WF'(e) \subset \{(q, p, q', p', t, \tau) | \tau = \pm\sqrt{g^x}, (q, p) = G_{\pm t}(q', p')\} \cup \{(q, p, q, p, t, 0)\}$   
*where  $G_t$  is the geodesic flow.*

## 4 The local wave trace for the Heisenberg group

As a preparation, we will prove the local trace formula for the 3D-Heisenberg group  $H_3$ . The Laplacian  $\Delta_3$  commutes with  $Z$ . We can hence use a partial Fourier decomposition of  $L^2(H_3)$  identifying it with the Hilbert integral

$$L^2(\mathbb{R}^3) = \int_{\mathbb{R}}^{\oplus} \mathcal{H}_{\zeta} d\zeta$$

where  $\mathcal{H}_{\zeta} := \{f | f(x, y, z + a) = e^{ia\zeta} f(x, y, z)\}$  which is identified to  $L^2(\mathbb{R}^2)$  by looking at the value of  $f$  at  $z = 0$ . In what follows, we will omit the space  $\mathcal{H}_0$  which corresponds to a flat 2D-Euclidian Laplacian. Using this decomposition, the Laplacian rewrites as follows:

$$\Delta_3 = \sum_{l=0}^{\infty} (2l+1) \int_{\mathbb{R}}^{\oplus} |\zeta| K_{\zeta}^l d\zeta$$

where the operator  $K_{\zeta}^l$  is the projector on the  $l$ -th Landau level of  $\Delta_{\zeta}$ , the restriction of  $\Delta_3$  to  $\mathcal{H}_{\zeta}$ , which is a magnetic Schrödinger operator

$$H_{\zeta} := -((\partial_x + i\zeta y/2)^2 + (\partial_y - i\zeta x/2)^2)$$

on  $\mathbb{R}^2$  with magnetic field  $\zeta dx \wedge dy$ . The Schwartz kernel of  $K_{\zeta}^l$  satisfies

$$K_{\zeta}^l(m, m) = \frac{|\zeta|}{2\pi}$$

(see Appendix B).

Hence the half-wave operator writes

$$e^{it\sqrt{\Delta_3}} = \sum_{l=0}^{\infty} \int_{\mathbb{R}}^{\oplus} e^{it\sqrt{(2l+1)|\zeta|}} K_{\zeta}^l d\zeta$$

and the local distributional trace is given by

$$\text{Trace} \left( e^{it\sqrt{\Delta_3}} f \right) = \frac{1}{2\pi} \left( \int_{\mathbb{R}^3} f(q) |dq| \right) \sum_{l=0}^{\infty} \int_{\mathbb{R}} e^{it\sqrt{(2l+1)|\zeta|}} |\zeta| d\zeta$$

for  $f \in C_0^\infty(H_3)$ . This trace can be explicitly computed by using the distributional Fourier transform

$$\int_0^\infty e^{i\tau u} u^3 du = 6(\tau + i0)^{-4}.$$

We get, for  $t \neq 0$ ,

$$\text{Trace} \left( e^{it\sqrt{\Delta_3}} f \right) = \frac{6}{\pi t^4} \left( \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} \right) \int_{H_3} f |dq|.$$

In particular, this trace is smooth outside  $t = 0$  which is consistent with the fact that there is no periodic geodesic in  $H_3$ .

The same result holds for the trace formula microlocalized near  $\Sigma$ :

**Proposition 4.1** *Let  $P$  be a pseudo-differential operator of degree 0, which is compactly supported and so that  $WF'(P) \cap \{\zeta = 0\} = \emptyset$ . Then*

$$\text{Trace} \left( e^{it\sqrt{\Delta_3}} P \right)$$

*is smooth outside  $t = 0$ .*

*Proof.* – We will first prove that

$$\text{Trace} \left( e^{it\sqrt{\Delta_3}} \chi \left( \frac{\Delta_3}{|Z|^2}, q \right) \right),$$

with  $\chi \in C_0^\infty(\mathbb{R} \times H_3)$ , which rewrites as

$$\frac{1}{2\pi} \sum_{l=0}^{\infty} \int_{H_3} |dq| \int_{\mathbb{R}} e^{it\sqrt{(2l+1)|\zeta|}} \chi \left( \frac{2l+1}{|\zeta|}, q \right) |\zeta| d\zeta,$$

is smooth outside  $t = 0$ . Let us define

$$I_l(t) = \int_{\mathbb{R}} e^{it\sqrt{(2l+1)|\zeta|}} \chi \left( \frac{2l+1}{|\zeta|}, q \right) |\zeta| d\zeta.$$

We split the integral into  $\zeta > 0$  and  $\zeta < 0$ . Looking at the first integral and using the new variable  $s = \sqrt{\zeta/2l + 1}$ , we get

$$I_l(t) = 2(2l + 1)^2 \int_0^\infty e^{it(2l+1)s} \chi\left(\frac{1}{s^2}, q\right) s^3 ds$$

Observing that the function  $\chi\left(\frac{1}{s^2}, q\right) s^3$  is a smooth classical symbol supported away of 0, integrations by part give that  $I_l(t) = O(l^{-\infty})$  as well as its derivatives.

Then we introduce

$$\bar{P} := \frac{1}{2\pi} \int_0^{2\pi} e^{it\Omega} P e^{-it\Omega} dt$$

which is again a pseudo-differential operator. We check that  $P - \bar{P} = [Q, \Omega]$  for a pseudo-differential operator  $Q$ : if

$$S_t := \int_0^t e^{is\Omega} (P - \bar{P}) e^{-is\Omega} dt,$$

this holds true with

$$Q = \frac{1}{2\pi i} \int_0^{2\pi} e^{-it\Omega} S_t e^{it\Omega} dt .$$

The corresponding part of the trace vanishes, because  $\Omega$  commutes with  $\Delta_3$ . Then the full symbol of  $\bar{P}$  can be written as

$$\sum_{j=0}^{\infty} |\zeta|^{-j} p_j \left( \frac{I}{\zeta}, q \right)$$

with  $p_j$  compactly supported. This allows to reduce to the first case.  $\square$

## 5 The trace formula for compact quotients of $H_3$

This section can be skipped. It contains an example with a direct derivation of Melrose's trace formula.

Let us give a co-compact subgroup  $\Gamma$  of  $H_3$ . We will prove Melrose's trace formula for  $M := \Gamma \backslash H_3$  with Laplacian  $\Delta_M$  which is  $\Delta_3$ , defined in Section



2.2, restricted to  $\Gamma$ -periodic functions. We fix some time  $T > 0$  and will look at the trace formula for  $|t| \leq T$ . We choose  $\chi_D$  a smoothed fundamental domain, ie  $\chi_D \in C_0^\infty(H_3)$  with  $\sum_{\gamma \in \Gamma} \chi_D(\gamma q) = 1$ . We will denote by  $\int_D \cdots$  the integral  $\int_{H_3} \chi_D \cdots$ .

We start with

$$e_M(t, q, q') = \sum_{\gamma \in \Gamma} e_3(t, q, \gamma q')$$

where  $e_3$  is the half-wave kernel in  $H_3$ . Because of the finite speed of propagation and the fact that  $\Gamma$  is discrete, we have only to consider a finite sum for the trace:

$$\text{Trace} \left( e^{it\sqrt{\Delta_M}} \right) = \int_D e_3(t, q, q) |dq| + \sum_{\gamma \in \Gamma \setminus \text{Id}, \min_q d(q, \gamma q) \leq T} \int_D e_3(t, q, \gamma q) |dq|$$

The first term is smooth outside  $t = 0$  while the second one is given by the CDG trace formula.

With more details: let  $c > 0$  be small enough so that the cone  $C_c = \{g^* < c\zeta^2\}$  has the property  $(\mathcal{P})$  that there is no  $\gamma \in \Gamma \setminus \text{Id}$  with a geodesic from  $q$  to  $\gamma q$  of length smaller than  $T$  starting with Cauchy data in this cone. This is possible thanks to Lemma 2.1. We can hence split the integrals  $I_\gamma(t) = \int_D e_3(t, q, \gamma q) |dq|$  into two pieces

$$I_\gamma(t) = \text{Trace} \left( \left( e^{it\sqrt{\Delta_3}} \right) \tau_\gamma \chi_D P \right) + \text{Trace} \left( \left( e^{it\sqrt{\Delta_3}} \right) \tau_\gamma \chi_D (\text{Id} - P) \right)$$

with  $\tau_\gamma(f) = f \circ \gamma^{-1}$  and  $P = \psi(\Delta_3/|Z|^2)$  where  $\psi$  belongs to  $C_0^\infty(\mathbb{R})$ , is equal to 1 near 0 and is supported in  $] -c, c[$ . The first term is smooth by Theorem 3.2 and property  $(\mathcal{P})$ . The second term corresponds to the elliptic region and hence we use the parametrix for the wave equation given by ‘‘FIO’’s as given in the CDG trace formula. We get then that the singularities of the wave trace locate on the length spectrum.

Note that the ‘‘heat trace’’ can be computed from the explicit expression of the spectrum. This is worked out in Appendix E.

## 6 Normal forms

In what follows,  $M$  is a closed 3D sR manifold of contact type equipped with a smooth volume. We denote by  $\Delta$  the associated Laplacian. The proof of the Melrose formula will be done by using a normal form allowing a reduction to the case of Heisenberg.

## 6.1 Classical normal form

**Theorem 6.1** *Let  $\Sigma$  be the characteristic manifold, i.e. the orthogonal of the distribution with respect to the duality, and let  $\sigma_0 \in \Sigma \setminus 0$ , then there exists a conical neighbourhood  $U$  of  $\sigma_0$  and an homogeneous symplectic diffeomorphism  $\chi$  of  $U$  onto a conical neighbourhood of  $(0, 0, 0; 0, 0, 1)$  in  $T^*H_3$ , so that  $g_{H_3}^* \circ \chi = g_M^*$ .*

We use first [Me-84] (Prop. 2.3) (or [CHT-21] (Theorem 2.1)) to reduce to  $\rho I$  where  $\rho$  is the Reeb Hamiltonian and  $I$  the harmonic oscillator Hamiltonian: this means that there is an homogeneous canonical transformation  $\chi$  from a conic neighborhood of  $\sigma_0$  into  $\Sigma_\sigma \times \mathbb{R}_{u,v}^2$  so that  $\rho I \circ \chi = g^*$  with  $I = u^2 + v^2$ . Let us denote by  $\zeta$  the principal symbol of  $Z$  in Equation (1). Then we use the normal form of Duistermaat-Hörmander [DH-72] (Prop. 6.1.3) to reduce  $\rho$  to  $|\zeta|$  by a canonical transformation. We get then the normal form  $|\zeta|I$  which is the canonical decomposition of  $g_{H_3}^*$  used in [CHT-18] (see the principal symbols in Equation (1)).

## 6.2 Quantum normal form

This is a 3-step reduction working in some conical neighbourhood  $C$  of a point of  $\Sigma$ .

1. Using FIO's associated to  $\chi$ , we first reduce the Laplacian to a pseudo-differential operator of the form  $|Z|\Omega + R_0$  where  $R_0$  is a pseudo-differential operator of degree 0. This step is worked out in Theorem 5.2 of [CHT-18].
2. We can improve the previous normal form so that  $R_0$  commutes with  $\Omega$ : the cohomological equations

$$\{|\zeta|I, a\} = b$$

(on  $T^*H_3$ ) where  $b$ , vanishing on  $\Sigma$  and homogeneous of degree  $j$  can be solved as shown in Appendix D which is an improvement of what is proved in [CHT-18]. It follows that we get a normal form  $\Delta_3 + R_0$  with  $R_0$  commuting with  $\Omega$ : the full symbol of  $R_0$  is independent of the angular part of the  $(u, v)$  variables.

3. Using the spectral decomposition of  $\Omega$ , we get a decomposition

$$\Delta \equiv \bigoplus_{l=0}^{\infty} (2l+1) \Delta_l \Pi_l$$

where the  $\Delta_l$ 's are pseudo-differential operators of the form

$$\Delta_l = \left( |Z| + \frac{1}{2l+1} R_0 \right)$$

and  $\Pi_l$  is the projector on the eigenspace of eigenvalue  $2l+1$  of  $\Omega$ . We can then use a reduction of the pseudo-differential operators  $|Z| + \frac{1}{2l+1} R_0$  to  $|Z|$  by conjugating by elliptic pseudo-differential operators  $A_l$  depending smoothly of  $\varepsilon = 1/(2l+1)$  and commuting with  $\Omega$  as in [DH-72], proposition 6.1.4. We have

$$A_l^{-1} \left( |Z| + \frac{1}{2l+1} R_0 \right) A_l \equiv |Z|$$

where  $\equiv$  means modulo smoothing operators in  $C$ .

## 7 Proof of the Melrose trace formula

Let us fix  $T > 0$  and try to prove Melrose's trace formula for times  $t \in J := [-T, T]$ . Let us fix, for each  $\sigma \in \Sigma$ , a conical neighbourhood  $U_\sigma$  of  $\sigma$  as in section 6.2. We then take  $W_\sigma \Subset V_\sigma \Subset U_\sigma$  so that, for any  $z \in V_\sigma$  and any  $t \in J$ ,  $G_t(z) \in U_\sigma$  where  $G_t$  is the geodesic flow. This is clearly possible using the classical normal form and the Lemma 2.1. We then take a finite cover of  $\Sigma$  by open cones  $W_\alpha := W_{\sigma_\alpha}$  and a finite pseudo-differential partition of unity  $(\chi_0, \chi_\alpha (\alpha \in B))$  so that  $WF'(\chi_0) \cap \Sigma = \emptyset$ ,  $\chi_\alpha = \text{Id}$  in  $W_\alpha$  and  $WF'(\chi_\alpha) \subset V_\alpha$ . We have then to compute the traces of  $(\cos t\sqrt{\Delta}) \chi_0$  and  $(\cos t\sqrt{\Delta}) \chi_\alpha$ . We will prefer to use the wave equation now because the operator  $\sqrt{\Delta}$  is not a pseudo-differential operator! We know from Theorem 3.2 that, for  $t \in J$ ,  $WF'(\cos(t\sqrt{\Delta})\chi_\alpha)$  is a subset of

$$\{(z, z, t, 0) | z \in V_\alpha\} \cup \{(z, G_{\pm t}(z), t, \tau = \pm g^*(z)) | z \in V_\alpha\}.$$

If  $u(t) = \cos(t\sqrt{\Delta})\chi_\alpha u_0$ , we have  $u_{tt} + \Delta u = 0$ ,  $u(0) = \chi_\alpha u_0$ ,  $u_t(0) = 0$ . We can hence use the normal form and denote by  $\equiv$  the equality "modulo smooth functions of  $t \in J$ " to get

$$Z_\alpha(t) := \text{Trace}(\cos(t\sqrt{\Delta})\chi_\alpha) \equiv \text{Trace}(\cos(t\sqrt{\Delta_3 + R_0})\widetilde{\chi}_\alpha),$$

where  $\widetilde{\chi}_\alpha$  is the PDO obtained by Egorov theorem when we take the normal form. Then, because  $R_0$  commutes with  $\Omega$ ,

$$Z_\alpha(t) \equiv \sum_{l=0}^{\infty} \text{Trace} \left( \cos \left( t \sqrt{(2l+1) \left( |Z| + \frac{1}{2l+1} R_0 \right)} \right) \Pi_l \widetilde{\chi}_\alpha \right)$$

and

$$\begin{aligned} Z_\alpha(t) &\equiv \sum_{l=0}^{\infty} \text{Trace} \left( A_l^{-1} \cos \left( t \sqrt{(2l+1)|Z|} \right) A_l \Pi_l \widetilde{\chi}_\alpha \right) \\ Z_\alpha(t) &\equiv \sum_{l=0}^{\infty} \text{Trace} \left( \cos \left( t \sqrt{(2l+1)|Z|} \right) A_l \Pi_l \widetilde{\chi}_\alpha A_l^{-1} \right). \end{aligned}$$

We can assume that  $A_l$  is invertible on  $WF'(\chi_\alpha)$  and put  $\widetilde{\widetilde{\chi}}_\alpha^l = A_l \widetilde{\chi}_\alpha A_l^{-1}$ . We get

$$Z_\alpha(t) \equiv \sum_{l=0}^{\infty} \text{Trace} \left( \cos \left( t \sqrt{(2l+1)|Z|} \right) A_l \Pi_l A_l^{-1} \widetilde{\widetilde{\chi}}_\alpha^l \right)$$

Using the fact  $A_l$  commutes with  $\Omega$  and hence with  $\Pi_l$ , we get finally

$$Z_\alpha(t) \equiv \sum_{l=0}^{\infty} \text{Trace} \left( \cos \left( t \sqrt{(2l+1)|Z|} \right) \Pi_l \widetilde{\widetilde{\chi}}_\alpha^l \Pi_l \right)$$

We can then apply a variant of the Proposition 4.1, more precisely of its proof, where  $P$  is replaced by  $\bigoplus_{l=0}^{\infty} \Pi_l \widetilde{\widetilde{\chi}}_\alpha^l \Pi_l$  and using the fact that the  $A_l$ 's and hence the  $\widetilde{\widetilde{\chi}}_\alpha^l$  too are uniformly bounded pseudo-differential operators.

It remains to study the part  $Z_0(t) = \text{Trace} \left( \cos(t\sqrt{\Delta})\chi_0 \right)$  which involves the elliptic part of the dynamics for which we can use the FIO parametrix as in [DG-75]. More precisely, for  $t \in J$ , the geodesic flow maps the microsupport of  $\chi_0$  away of  $\Sigma$ , therefore there exists a parametrix for  $U(t)\chi_0$  given by Fourier integral operators as in the paper [DG-75] and the calculation of the trace therefore follows the same path.

# Appendices

## A Wave and half-wave equations

Let  $\Delta$  be a self-adjoint positive sub-elliptic operator on a closed manifold. The wave equation is

$$\frac{\partial^2 u}{\partial t^2} + \Delta u = 0, u(t=0) = u_0, \frac{\partial u}{\partial t}(t=0) = v_0$$

This gives a one parameter group  $U(t) = (U_0(t), U_1(t))$  on  $L^2 \times L^2$ . The trace of  $U_0(t)$  is

$$Z_0(t) = \text{Trace} \left( \cos t\sqrt{\Delta} \right) = \sum_{j=1}^{\infty} \cos t\sqrt{\lambda_j}$$

One can introduce also the half-wave equation  $\frac{\partial u}{\partial t} = i\sqrt{\Delta}u$ ,  $u(t=0) = u_0$ . The trace of the half-wave group is  $Z(t) = \sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}}$ . We have the relation  $Z = 2H(Z_0)$  where  $H$  is the  $L^2$ -projector multiplying the Fourier transform by the Heaviside function. It follows that the singularities of both distributions are easily related.

In the elliptic case, one can work directly with the half-wave group because  $\sqrt{\Delta}$  is still an elliptic pseudo-differential operator (Seeley's Theorem [See-67]). This is no longer the case for sub-elliptic operators.

## B The value of $K_{\zeta}^l(m, m)$

Recall that  $K_{\zeta}^l$  is the orthogonal projector on the  $l$ -th Landau level with a magnetic field in  $\mathbb{R}^2$  equal to  $\zeta dx \wedge dy$ . An easy rescaling shows that  $K_{\zeta}^l(m, m) = |\zeta| K_1^l(m, m)$ . We know from the Mehler formula (see [Si-79], p. 168) that the heat kernel of the magnetic Schrödinger operator with constant magnetic field equal to 1 is given on the diagonal by

$$e(t, m, m) = \frac{1}{4\pi \sinh t}$$

On the other hand, we have

$$e(t, m, m) = \sum_{l=0}^{\infty} e^{-(2l+1)t} K_1^l(m, m)$$

and

$$\frac{1}{4\pi \sinh t} = \frac{1}{2\pi} \sum_{l=0}^{\infty} e^{-(2l+1)t}$$

Identifying both sums as Taylor series in  $x = e^{-t}$  gives

$$K_1^l(m, m) = \frac{1}{2\pi}.$$

## C Toeplitz operators

Let  $\Sigma$  be a symplectic cone with a compact basis. Louis Boutet de Monvel and Victor Guillemin associate in [Bo-80, BG-81] to such a cone an Hilbert space and an algebra of operators called the Toeplitz operators with the same properties as the classical pseudo-differential operators. The latter case corresponds to the cone which is a cotangent cone. For an introduction, one can look at [CdV-94].

Two examples are implicitly present in this paper:

1. *Harmonic oscillator*: the harmonic oscillator  $\Omega = -d_x^2 + x^2$  is an elliptic self-adjoint Toeplitz operator; the cone  $\Sigma$  is  $\mathbb{R}_{u,v}^2 \setminus 0$  with the symplectic form  $du \wedge dv$  and the dilations  $\lambda.(u, v) = (\sqrt{\lambda}u, \sqrt{\lambda}v)$ . The symbol of  $\Omega$  is  $u^2 + v^2$ .
2. *Quantization of the Reeb flow*: if  $\Sigma \subset T^*X \setminus 0$  is the characteristic cone of our sR Laplacian, one can quantize the Reeb Hamiltonian  $\rho$  as a first order elliptic Toeplitz operator of degree 1.

## D A cohomological equation

The following proposition is a global formulation of the formal cohomological equations discussed in [CHT-18] (section 5.1 and Appendix C) with a simple proof:

**Proposition D.1** *We consider the cohomological equation*

$$\{|\zeta|I, A\} = B \tag{2}$$

where  $A, B$  are smooth homogeneous functions in the cone  $C := \{I < c|\zeta|\}$  with compact support in  $q \in H_3$ . If  $B$  is homogeneous of degree  $j$  and vanishes

on  $\Sigma := \{I = 0\}$ , Equation (2) admits a solution  $A$  homogeneous of degree  $j - 1$ .

Restricting to  $\zeta = 1$ , reduces to prove the following Lemma.

**Lemma D.1** *Let us consider the differential equation*

$$\frac{\partial a}{\partial \theta} + \frac{1}{2}I \frac{\partial a}{\partial z} = b(z, w) \quad (3)$$

with  $(z, w) \in \mathbb{R} \times \{|w| < c\}$ ,  $w = |w|e^{i\theta}$ ,  $b$  smooth, compactly supported in  $z$  and  $I = |w|^2$ . We assume that  $b(z, 0) = 0$ .

Then Equation (3) admits a smooth solution  $a$  depending smoothly of  $b$ .

Any smooth function  $f$  in some disk in  $\mathbb{C}$  admits a Fourier expansion

$$f(w) = \sum_{n=0}^{\infty} f_n(|w|^2)w^n + \sum_{n=1}^{\infty} g_n(|w|^2)\bar{w}^n$$

where the  $f_n$ 's and the  $g_n$ 's are smooth<sup>2</sup>. We can use this expansion with  $f = a(z, \cdot)$  and  $g = b(z, \cdot)$ . We consider only the sum of powers of  $w^n$ . The second can be worked out in a similar way. We then put  $a = \sum_{n=1}^{\infty} a_n(z, I)w^n + a_0(z, I)$  and  $b = \sum_{n=1}^{\infty} b_n(z, I)w^n + Ic_0(z, I)$ . The factorization  $b_0(z, I) = Ic_0(z, I)$  follows from the assumption on  $b$ . We can take  $a_0(z, I) = 2 \int_{-\infty}^z b_0(s, I)ds$ . The equation for  $a_n$ , with  $n \geq 1$ , writes

$$ina_n + \frac{1}{2}I \frac{\partial a_n}{\partial z} = b_n$$

We can solve it, for  $I \neq 0$ , by

$$a_n(z, I) = \frac{2}{I} \int_{-\infty}^0 b_n(z + u, I)e^{in2u/I} du = \int_{-\infty}^0 b_n(z + Is/2, I)e^{ins} ds$$

and  $a_n(z, 0) = b_n(z, 0)/in$ . We need to prove that  $a_n$  is smooth: we check that  $a_n$  is continuous and then the derivatives are given by the same kind of integrals with derivatives of  $b_n$  as shown by the last expression of  $a_n$ . The continuity of  $a_n$  follows from an integrations by parts in the first expression of  $a_n$ .

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<sup>2</sup>This is proved first in formal power series in  $(w, \bar{w})$ . This reduces the problem to functions which are flat at the origin. A Fourier expansion in  $\theta$  allows to finish the proof.

Now we want to add up the series  $\sum_n a_n w^n$ . I was not able to do that directly and will proceed as follows: the sum  $\sum_n a_n w^n$  is convergent as a formal series along  $\Sigma$ , because  $a_n w^n = O(I^{n/2})$ . Using Borel procedure, we need only to solve our cohomological equation with a flat righthandside. This follows clearly from the expression

$$a(z, w) = \int_{-\infty}^0 b(z + It/2, e^{it}w) dt .$$

In fact only the behaviour as  $I \rightarrow 0$  could be a problem, but we can divide  $b$  by any power of  $I$ .

## E The heat trace for a compact quotient of $H_3$

Let us consider the discrete subgroup  $\Gamma = (\sqrt{2\pi}\mathbb{Z})^2 \times \pi\mathbb{Z}$  of  $H_3$  identified with  $\mathbb{R}_{x,y,z}^3$  as in Section 2.2. The spectrum of the sub-Laplacian defined in section 2.2 on  $M = H_3/\Gamma$  is the union of the spectrum of the flat torus  $\mathbb{R}^2/(\sqrt{2\pi}\mathbb{Z})^2$  and the eigenvalues  $2m(2l+1)$ ,  $m \geq 1$ ,  $l \geq 0$ , with multiplicities  $2m$ . The corresponding part of the complexified heat trace is hence

$$Z_o(z) = \sum_{m=1}^{\infty} 2m \sum_{l=0}^{\infty} e^{-2m(2l+1)z}$$

with  $\Re(z) > 0$ . Summing with respect to  $l$  gives

$$Z_o(z) = \sum_{m=1}^{\infty} \frac{m}{\sinh 2mz}$$

which we rewrite as

$$Z_o(z) = \frac{1}{4z} \sum_{m \in \mathbb{Z}} \frac{2mz}{\sinh 2mz} - \frac{1}{4z}$$

The Fourier transform of  $\frac{x}{\sinh x}$  is  $\frac{\pi^2}{1 + \cosh \pi\xi}$ . Applying Poisson summation formula to the last expression of  $Z_o$ , we get

$$Z_o(z) = \frac{\pi^2}{16z^2} - \frac{1}{4z} + \frac{\pi^2}{4z^2} \sum_{n=1}^{\infty} \frac{1}{1 + \cosh \pi^2 n/z}$$



The first term gives the Weyl law. Each term in the sum w.r. to  $n$  is equivalent to

$$\frac{\pi^2}{2z^2} e^{-\pi^2 n/z}$$

We observe that the lengths of the periodic geodesic of  $M$  are the numbers  $2\pi\sqrt{n}$ . Hence we recover also the length spectrum giving contributions of the order of  $\exp(-L^2/4z)$  as in the Riemannian case as proved in [CdV-73]. It could be nice to derive an exact formula for the wave trace from our expression of the heat trace. Similarly, the heat trace for the Riemannian Laplacian on  $M$  was computed by Hubert Pesce [Pe-94].

## F A short history of the trace formulae

The trace formulae were first discovered independently by two groups of physicists: Martin Gutzwiller [Gu-71] for a semi-classical Schrödinger operator and Roger Balian & Claude Bloch in a very impressive series of papers for Laplacians in Euclidean domains [Ba-Bl-70, Ba-Bl-71, Ba-Bl-72]. In [Ba-Bl-72], page 154, the authors suggested already a possible application to the inverse spectral problems<sup>3</sup>, an industry which just started at the end of the sixties. From the point of view of mathematics, the Poisson summation formula can be interpreted as a trace formula for the Euclidian Laplacian on flat tori. Similarly, the famous Selberg trace formula [Se-56] (see also Heinz Huber [Hu-59]) is a trace formula for the Laplacian on hyperbolic surfaces. Then my thesis [CdV-73], inspired by the work of Balian and Bloch and the Selberg trace formula, uses the complex heat equation for general closed Riemannian manifold. The definitive version, the CDG formula, using wave equation, was discovered by Jacques Chazarain and the tandem Hans Duistermaat & Victor Guillemin in [Ch-74, DG-75]. They use the power of the Fourier Integral Operators calculus [Ho-71, DH-72]. See [CdV-07] for a review paper. Later results cover the cases of manifolds with boundaries and semi-classical versions.

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<sup>3</sup>They wrote “The analysis of the eigenvalue density as a sum of oscillating terms gives a new insight into the problem of “hearing the shape of a drum” [Kac paper]. ...It is convenient, for the discussion to start from the fact that the knowledge of eigenvalues determines uniquely the path generating function ... Thus ... the lengths of the closed stationary polygons are determined”.

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