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A proof of a Melrose’s trace formula

Yves Colin de Verdière*

May 13, 2022

The goal of this note is to give a proof of the wave trace formula proved by Richard Melrose in the impressive paper [Me-84]. This trace formula is an extension of the Chazarain-Duistermaat-Guillemin trace formula (denoted “CDG trace formula” in this paper) to the case of a sub-Riemannian (“sR”) Laplacian on a 3D contact closed manifold. The proof uses a normal form constructed in the papers [CHT-18, CHT-21], following the pioneering work [Me-84], in order to reduce to the case of the invariant Laplacian on the 3D-Heisenberg group. We need also the propagation of singularities results of Victor Ivrii, Bernard Lascar and Richard Melrose [Iv-76, La-82, Me-86].

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1 The CDG trace formula

The following result was proved by Chazarain [Ch-74] and refined by Duistermaat and Guillemin [DG-75] (see also [CdV-73, CdV-07] and Appendix F for the history of the trace formulae):

Theorem 1.1 *Let (M, g) be a closed connected smooth Riemannian manifold and $\lambda_1 = 0 < \lambda_2 \leq \dots$ the spectrum of the Laplace operator, then we have the following equality of Schwartz distributions:*

$$\sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}} = T_0(t) + \sum_{\gamma \in \mathcal{P}} T_{\gamma}(t) \text{ mod } C^{\infty}$$

*Université Grenoble-Alpes, Institut Fourier, Unité mixte de recherche CNRS-UGA 5582, BP 74, 38402-Saint Martin d’Hères Cedex (France); yves.colin-de-verdiere@univ-grenoble-alpes.fr

where \mathcal{P} is the set of periodic geodesics, $\text{SingSupp}(T_0) = \{0\}$ and, for γ a periodic geodesic, $\text{SingSupp}(T_\gamma) \subset \{L_\gamma\} \cup \{-L_\gamma\}$, where L_γ is the length of γ . Moreover, if γ is non degenerate,

$$T_\gamma(t) = \frac{L_0 e^{i\pi m(\gamma)/2}}{\pi \det(\text{Id} - P_\gamma)^{\frac{1}{2}}} (t + i0 - L_\gamma)^{-1} \left(1 + \sum_{j=1}^{\infty} a_j (t - L_\gamma)^j \right)$$

where

- L_0 is the length of the primitive geodesic associated to γ
- $m(\gamma)$ is the Morse index of γ
- P_γ is the linearized Poincaré map.

This result gives a nicer proof of the main result of my thesis [CdV-73] saying that, in the generic case, the Length spectrum, i.e. the set of lengths of closed geodesics, is a spectral invariant. This formula holds for any elliptic self-adjoint pseudo-differential operator P of degree 1 by replacing the periodic geodesics by the periodic orbits of the Hamiltonian flow of the principal symbol of P and the Morse index by a Maslov index. Our goal is to prove that the same statement holds for sR Laplacians on a closed contact 3-manifold.

2 Review of basic facts and notations

For more details on this section, one can look at the paper [CHT-18].

2.1 Contact 3D sR manifolds

In what follows, M is a closed (compact without boundary) connected manifold of dimension 3 equipped with a smooth volume form $|dq|$. We give also an oriented contact distribution globally defined as the kernel of a non vanishing real valued 1-form α so that $\alpha \wedge d\alpha$ is a volume form. We give also a metric g on the distribution $D = \ker \alpha$. The “co-metric” g^* is defined by $g^*(q, p) := \|p|_{D_q}\|^2$ where the norm is the dual norm of $g(q)$. To such a set of data is associated

- A geodesic flow denoted by G_t , $t \in \mathbb{R}$: the Hamiltonian flow of $\sqrt{g^*}$. The geodesics are projections of the orbits of that flow onto M and are everywhere tangent to D . We will often prefer to consider the geodesic flow as the restriction to $g^* = 1$ of the Hamiltonian flow of $\frac{1}{2}g^*$.

- A Laplacian which is locally given by $\Delta = X_1^*X_1 + X_2^*X_2$ where (X_1, X_2) is an orthonormal frame of D and the adjoint is taken with respect to the measure $|dq|$.
- A canonical choice of a 1-form α_g defining D by assuming that $d\alpha_g$ restricted to D is the oriented g -volume form on D .

The Laplacian is sub-elliptic and hence has a compact resolvent and a discrete spectrum $\lambda_1 = 0 < \lambda_2 \leq \dots$ with smooth eigenfunctions. It follows from the corresponding Weyl law that

$$\text{Trace}(e^{it\sqrt{\Delta}}) = \sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}}$$

is a well defined Schwartz distribution often called the *wave trace* (see Appendix A for the link with the wave equation). Our goal is to extend the CDG formula to this case.

The form α_g defines a Reeb vector field \vec{R} by the equations $\alpha_g(\vec{R}) = 1$ and $\iota(\vec{R})d\alpha_g = 0$. This vector field admits the following Hamiltonian interpretation: the cone $\Sigma = D^\perp$ is a symplectic sub-cone of $T^*M \setminus 0$ and we define the Hamiltonian $\rho : \Sigma \rightarrow \mathbb{R}$ by $\rho(\alpha) = \alpha/\alpha_g$ where $\alpha \in \Sigma$ is a covector vanishing on D . The Hamiltonian vector field of ρ is homogeneous of degree 0 and the projection of this field onto M is the Reeb vector field \vec{R} .

2.2 The 3D Heisenberg group H_3

We identify the Heisenberg group H_3 with $\mathbb{R}_{x,y,z}^3$ and the Lie algebra is generated by X, Y, Z with

$$X = \partial_x + \frac{1}{2}y\partial_z, \quad Y = \partial_y - \frac{1}{2}x\partial_z, \quad Z = \partial_z$$

We choose (D, g) by asking that (X, Y) is an oriented orthonormal basis of D for g and $|dq| = |dxdydz|$. We have $[X, Y] = -Z$ and the Reeb vector field is Z . The Laplacian is $\Delta_3 = -(X^2 + Y^2)$ and can be rewritten as

$$\Delta_3 = -|Z| \left(\left(\frac{X}{\sqrt{Z}} \right)^2 + \left(\frac{Y}{\sqrt{Z}} \right)^2 \right)$$

outside the 0-spectrum of Z . We write this as

$$\Delta_3 = |Z|\Omega$$

where Ω is an harmonic oscillator with spectrum $\{2l + 1 | l = 0, 1, \dots\}$ (see [CHT-18], prop. 3.1).

We will need the following “confining” result (see [Le-20]):

Lemma 2.1 *Given $T > 0$, $\sigma_0 \in \Sigma \setminus 0$ and U a conic neighbourhood of σ_0 , there exists a conic neighbourhood $V \subset\subset U$ of σ_0 so that $\forall t \in [-T, T]$, $G_t(V) \subset U$.*

3 Speed of propagation

First, we have the following

Theorem 3.1 *If u is a solution of an sR wave equation,*

$$\forall t \in \mathbb{R}, \text{Support}(u(t)) \subset B(\text{Support}(u(t=0)) \cup \text{Support}(du/dt(t=0)), |t|)$$

where $B(A, r)$ is the closed sR neighbourhood of radius r of A .

This result follows from the Riemannian case by passing to the limits.

We will also use the following Theorem due to Victor Ivrii, Bernard Lascar and Richard Melrose [Iv-76, La-82, Me-86] and revisited by Cyril Letrouit [Le-21]):

Theorem 3.2 *If $e(t, q, q')$ is the wave kernel of a sR Laplacian whose characteristic manifold is symplectic, i.e. e is the Schwartz kernel of $\cos(t\sqrt{\Delta})$, then*

$$WF'(e) \subset \{(q, p, q', p', t, \tau) | \tau = \pm\sqrt{g^*}, (q, p) = G_{\pm t}(q', p')\} \cup \{(q, p, q, p, t, 0)\}$$

where G_t is the geodesic flow.

4 The local wave trace for the Heisenberg group

As a preparation, we will prove the local trace formula for the 3D-Heisenberg group H_3 . The Laplacian Δ_3 commutes with Z . We can hence use a partial Fourier decomposition of $L^2(H_3)$ identifying it with the Hilbert integral

$$L^2(\mathbb{R}^3) = \int_{\mathbb{R}}^{\oplus} \mathcal{H}_{\zeta} d\zeta$$

where $\mathcal{H}_{\zeta} := \{f | f(x, y, z + a) = e^{ia\zeta} f(x, y, z)\}$ which is identified to $L^2(\mathbb{R}^2)$ by looking at the value of f at $z = 0$. In what follows, we will omit the space \mathcal{H}_0 which corresponds to a flat 2D-Euclidian Laplacian. Using this decomposition, the Laplacian rewrites as follows:

$$\Delta_3 = \sum_{l=0}^{\infty} (2l+1) \int_{\mathbb{R}}^{\oplus} |\zeta| K_{\zeta}^l d\zeta$$

where the operator K_{ζ}^l is the projector on the l -th Landau level of Δ_{ζ} , the restriction of Δ_3 to \mathcal{H}_{ζ} , which is a magnetic Schrödinger operator on \mathbb{R}^2 with magnetic field $\zeta dx \wedge dy$. The Schwartz kernel of K_{ζ}^l satisfies

$$K_{\zeta}^l(m, m) = \frac{|\zeta|}{2\pi}.$$

(see Appendix B)

Hence the half-wave operator writes

$$e^{it\sqrt{\Delta_3}} = \sum_{l=0}^{\infty} \int_{\mathbb{R}}^{\oplus} e^{it\sqrt{(2l+1)|\zeta|}} K_{\zeta}^l d\zeta$$

and the local distributional trace is given by

$$\text{Trace} \left(e^{it\sqrt{\Delta_3}} f \right) = \frac{1}{2\pi} \sum_{l=0}^{\infty} \int_{\mathbb{R}^2} |dm| \int_{\mathbb{R}} e^{it\sqrt{(2l+1)|\zeta|}} |\zeta| \hat{f}(m, \zeta - \zeta') d\zeta d\zeta'$$

for $f \in C_0^{\infty}(H_3)$ with \hat{f} the Fourier transform of f w.r. to the variable z . This trace can be explicitly computed by first computing the integral w.r. to

ζ' which gives a contribution $\int_{\mathbb{R}} f(m, z) dz$ and then using the distributional Fourier transform

$$\int_0^\infty e^{i\tau u} u^3 du = 6(\tau + i0)^{-4}.$$

We get, for $t \neq 0$,

$$\text{Trace} \left(e^{it\sqrt{\Delta_3}} f \right) = \frac{12}{\pi} \left(\sum_{l=0}^\infty (t\sqrt{2l+1})^{-4} \right) \int_{H_3} f |dq|.$$

In particular, this trace is smooth outside $t = 0$ which is consistent with the fact that there is no periodic geodesic in H_3 .

The same result holds for the trace formula microlocalized near Σ :

Proposition 4.1 *Let P be a pseudo-differential operator of degree 0, which is compactly supported and so that $WF'(P) \cap \{\zeta = 0\} = \emptyset$. Then*

$$\text{Trace} \left(e^{it\sqrt{\Delta_3}} P \right)$$

is smooth outside $t = 0$.

Proof. – We prove first by a direct computation, using integrations by parts, that

$$\text{Trace} \left(e^{it\sqrt{\Delta_3}} \chi \left(\frac{\Delta_3}{|Z|^2}, q \right) \right),$$

with $\chi \in C_0^\infty(\mathbb{R} \times H_3)$, which rewrites as

$$\frac{1}{2\pi} \sum_{l=0}^\infty \int_{H_3} |dq| \int_{\mathbb{R}} e^{it\sqrt{(2l+1)|\zeta|}} \chi \left(\frac{2l+1}{|\zeta|}, q \right) |\zeta| d\zeta,$$

is smooth outside $t = 0$.

Then we introduce

$$\bar{P} := \frac{1}{2\pi} \int_0^{2\pi} e^{it\Omega} P e^{-it\Omega} dt$$

which is again a pseudo-differential operator. We check that $P - \bar{P} = [Q, \Omega]$ for a pseudo-differential operator Q , and the corresponding part of the trace vanishes, because Ω commutes with Δ_3 . Then the full symbol of \bar{P} can be written as

$$\sum_{j=0}^\infty |\zeta|^{-j} p_j \left(\frac{I}{\zeta}, q \right)$$

with p_j compactly supported. This allows to reduce to the first case. \square

5 The trace formula for compact quotients of H_3

This section can be skipped. It contains an example with a direct derivation of the Melrose's trace formula.

Let us give a co-compact subgroup Γ of H_3 . We will prove the Melrose's trace formula for $M := \Gamma \backslash H_3$ with Laplacian Δ_M . We fix some time $T > 0$ and will look at the trace formula for $|t| \leq T$. We choose χ_D a smoothed fundamental domain, ie $\chi_D \in C_0^\infty(H_3)$ with $\sum_{\gamma \in \Gamma} \chi_D(\gamma q) = 1$. We will denote by $\int_D \cdots$ the integral $\int_{H_3} \chi_D \cdots$.

We start with

$$e_M(t, q, q') = \sum_{\gamma \in \Gamma} e_3(t, q, \gamma q')$$

where e_3 is the half-wave kernel in H_3 . Because of the finite speed of propagation and the fact that Γ is discrete, we have only to consider a finite sum for the trace:

$$\text{Trace} \left(e^{it\sqrt{\Delta_M}} \right) = \int_D e_3(t, q, q) |dq| + \sum_{\gamma \in \Gamma \setminus \text{Id}, \min_q d(q, \gamma q) \leq T} \int_D e_3(t, q, \gamma q) |dq|$$

The first term is smooth outside $t = 0$ while the second one is given by the CDG trace formula. With more details: we have $WF(e_3)(\cdot, \cdot, \gamma \cdot) \cap C_c = \emptyset$ if $C_c = \{g^* < c\zeta^2\}$ for c small enough, because there is no geodesic from q to γq of length smaller than T starting with Cauchy data in that cone. We can hence split the integrals $I_\gamma(t) = \int_D e_3(t, q, \gamma q) |dq|$ into two pieces

$$I_\gamma(t) = \text{Trace} \left(\left(e^{it\sqrt{\Delta_3}} \right) \tau_\gamma \chi_D P \right) + \text{Trace} \left(\left(e^{it\sqrt{\Delta_3}} \right) \tau_\gamma \chi_D (\text{Id} - P) \right)$$

with $\tau_\gamma(f) = f \circ \gamma^{-1}$ and $P = \psi(\Delta_3/|Z|^2)$ where ψ belongs to $C_o^\infty(\mathbb{R})$, is equal to 1 near 0 and is supported in $] -c, c[$. The first term is smooth by Theorem 3.2. The second term corresponds to the elliptic region and hence we use the parametrix for the wave equation given by "FIO"s as given in the CDG trace formula. We get then that the singularities of the wave trace locate on the length spectrum.

Note that the "heat trace" can be computed from the explicit expression of the spectrum. This is worked out in Appendix E.

6 Normal forms

In what follows, M is a closed 3D sR manifold of contact type equipped with a smooth volume. We denote by Δ the associated Laplacian. The proof of the Melrose formula will be done by using a normal form allowing a reduction to the case of Heisenberg.

6.1 Classical normal form

Theorem 6.1 *Let Σ be the characteristic manifold, i.e. the orthogonal of the distribution with respect to the duality, and let $\sigma_0 \in \Sigma \setminus 0$, then there exists a conical neighbourhood U of σ_0 and an homogeneous symplectic diffeomorphism χ of U onto a conical neighbourhood of $(0, 0, 0; 0, 0, 1)$ in T^*H_3 , so that $g_{H_3}^* \circ \chi = g_M^*$.*

We use first [Me-86] (or [CHT-21]) to reduce to ρI where ρ is the Reeb Hamiltonian and I the harmonic oscillator Hamiltonian. Then we use the normal form of Duistermaat-Hörmander [DH-72] to reduce ρ to $|\zeta|$ by a canonical transformation. We get then the normal form $|\zeta|I$ which is the canonical decomposition of $g_{H_3}^*$ used in [CHT-18].

6.2 Quantum normal form

This is a 3-step reduction working in some conical neighbourhood C of a point of Σ .

1. Using FIO's associated to χ , we first reduce the Laplacian to a pseudo-differential operator of the form $|Z|\Omega + R_0$ where R_0 is a pseudo-differential operator of degree 0. This step is worked out in [CHT-18].
2. We can improve the previous normal form so that R_0 commutes with Ω : the cohomological equations

$$\{|\zeta|I, a\} = b$$

where b , vanishing on Σ and homogeneous of degree j can be solved as shown in Appendix D which is an improvements of what is proved in [CHT-18]. It follows that we get a normal form $\Delta_3 + R_0$ with R_0 commuting with Ω : the full symbol of R_0 is independent of the (u, v) variables.

3. Using the spectral decomposition of Ω , we get a decomposition

$$\Delta \equiv \oplus_{l=0}^{\infty} (2l+1) \Delta_l \Pi_l$$

where the Δ_l 's are pseudo-differential operators of the form

$$\Delta_l = \left(|Z| + \frac{1}{2l+1} R_0 \right)$$

and Π_l is the projector on the eigenspace of eigenvalue $2l+1$ of Ω . We can then use a reduction of the pseudo-differential operators $|Z| + \frac{1}{2l+1} R_0$ to $|Z|$ by conjugating by elliptic pseudo-differential operators A_l depending smoothly of $\varepsilon = 1/(2l+1)$ and commuting with Ω as in [DH-72], proposition 6.1.4. We have

$$A_l^{-1} \left(|Z| + \frac{1}{2l+1} R_0 \right) A_l \equiv |Z|$$

where \equiv means modulo smoothing operators in C .

7 Proof of the Melrose trace formula

Let us fix $T > 0$ and try to prove Melrose's trace formula for $J := \{|t| \leq T\}$. Let us fix, for each $\sigma \in \Sigma$, a conical neighbourhood U_σ of σ as in the section 6.2. We then take $W_\sigma \Subset V_\sigma \Subset U_\sigma$ so that, for any $z \in V_\sigma$ and any $t \in J$, $G_t(z) \in U_\sigma$ where G_t is the geodesic flow. This is clearly possible using the classical normal form and the Lemma 2.1. We then take a finite cover of Σ by open cones $W_\alpha := W_{\sigma_\alpha}$ and a finite pseudo-differential partition of unity $(\chi_0, \chi_\alpha (\alpha \in B))$ so that $WF'(\chi_0) \cap \Sigma = \emptyset$, $\chi_\alpha = \text{Id}$ in W_α and $WF'(\chi_\alpha) \subset V_\alpha$. We have then to compute the traces of $(\cos t\sqrt{\Delta}) \chi_0$ and $(\cos t\sqrt{\Delta}) \chi_\alpha$. We will prefer to use the wave equation now because the operator $\sqrt{\Delta}$ is not a pseudo-differential operator! We know from the propagation of singularities that, for $t \in J$, $WF'(\cos(t\sqrt{\Delta})\chi_\alpha)$ is a subset of

$$\{(z, z, t, 0) | z \in V_\alpha\} \cup \{(z, G_{\pm t}(z), t, \tau = \pm g^*(z)) | z \in V_\alpha\}.$$

If $u(t) = \cos(t\sqrt{\Delta})\chi_\alpha u_0$, we have $u_{tt} + \Delta u = 0$, $u(0) = \chi_\alpha u_0$, $u_t(0) = 0$. We can hence use the normal form and denote by \equiv the equality "modulo smooth functions of $t \in J$ " to get

$$Z_\alpha(t) := \text{Trace}(\cos(t\sqrt{\Delta})\chi_\alpha) \equiv \text{Trace}(\cos(t\sqrt{\Delta_3 + R_0})\widetilde{\chi}_\alpha),$$

where $\widetilde{\chi}_\alpha$ is the OPD obtained by Egorov theorem when we take the normal form. Then, because R_0 commutes with Ω ,

$$Z_\alpha(t) \equiv \sum_{l=0}^{\infty} \text{Trace} \left(\cos \left(t \sqrt{(2l+1) \left(|Z| + \frac{1}{2l+1} R_0 \right)} \right) \Pi_l \widetilde{\chi}_\alpha \right)$$

and

$$\begin{aligned} Z_\alpha(t) &\equiv \sum_{l=0}^{\infty} \text{Trace} \left(A_l^{-1} \cos \left(t \sqrt{(2l+1)|Z|} \right) A_l \Pi_l \widetilde{\chi}_\alpha \right) \\ Z_\alpha(t) &\equiv \sum_{l=0}^{\infty} \text{Trace} \left(\cos \left(t \sqrt{(2l+1)|Z|} \right) A_l \Pi_l \widetilde{\chi}_\alpha A_l^{-1} \right). \end{aligned}$$

We can assume that A_l is invertible on $WF'(\chi_\alpha)$ and put $\widetilde{\widetilde{\chi}}_\alpha = A_l \widetilde{\chi}_\alpha A_l^{-1}$. We get

$$Z_\alpha(t) \equiv \sum_{l=0}^{\infty} \text{Trace} \left(\cos \left(t \sqrt{(2l+1)|Z|} \right) A_l \Pi_l A_l^{-1} \widetilde{\widetilde{\chi}}_\alpha \right)$$

Using the fact A_l commutes with Ω and hence with Π_l , we get finally

$$Z_\alpha(t) \equiv \sum_{l=0}^{\infty} \text{Trace} \left(\cos \left(t \sqrt{(2l+1)|Z|} \right) \Pi_l \widetilde{\widetilde{\chi}}_\alpha \Pi_l \right)$$

We can then apply a variant of the Proposition 4.1, more precisely of its proof, where P is replaced by $\bigoplus_{l=0}^{\infty} \Pi_l \widetilde{\widetilde{\chi}}_\alpha \Pi_l$ and using the fact that the A_l 's and hence the $\widetilde{\widetilde{\chi}}_\alpha$ too are uniformly bounded pseudo-differential operators.

It remains to study the part $Z_0(t) = \text{Trace} \left(\cos(t\sqrt{\Delta})\chi_0 \right)$ which involves the elliptic part of the dynamics for which we can use the FIO parametrix as in [DG-75].

Appendices

A Wave and half-wave equations

Let Δ be a self-adjoint positive sub-elliptic operator on a closed manifold. The wave equation is

$$\frac{\partial^2 u}{\partial t^2} + \Delta u = 0, u(t=0) = u_0, \frac{\partial u}{\partial t}(t=0) = v_0$$

This gives a one parameter group $U(t) = (U_0(t), U_1(t))$ on $L^2 \times L^2$. The trace of $U_0(t)$ is

$$Z_0(t) = \text{Trace} \left(\cos t\sqrt{\Delta} \right) = \sum_{j=1}^{\infty} \cos t\sqrt{\lambda_j}$$

One can introduce also the half-wave equation $\frac{\partial u}{\partial t} = i\sqrt{\Delta}u$, $u(t=0) = u_0$. The trace of the half-wave group is $Z(t) = \sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}}$. We have the relation $Z = H(Z_0)$ where H is the L^2 -projector multiplying the Fourier transform by the Heaviside function. It follows that the singularities of both distributions are easily related.

In the elliptic case, one can work directly with the half-wave group because $\sqrt{\Delta}$ is still an elliptic pseudo-differential operator (Seeley's Theorem [See-67]). This is no longer the case for sub-elliptic operators.

B The value of $K_{\zeta}^l(m, m)$

Recall that K_{ζ}^l is the orthogonal projector on the l -th Landau level with a magnetic field in \mathbb{R}^2 equal to $\zeta dx \wedge dy$. An easy rescaling shows that $K_{\zeta}^l(m, m) = |\zeta| K_1^l(m, m)$. We know from the Mehler formula ([Si-79], p. 168), that the heat kernel is given on the diagonal by

$$e(t, m, m) = \frac{1}{4\pi \sinh t}$$

On the other hand, we have

$$e(t, m, m) = \sum_{l=0}^{\infty} e^{-(2l+1)t} K_1^l(m, m)$$

and

$$\frac{1}{4\pi \sinh t} = \frac{1}{2\pi} \sum_{l=0}^{\infty} e^{-(2l+1)t}$$

Identifying both sums as Taylor series in $x = e^{-t}$ gives

$$K_1^l(m, m) = \frac{1}{2\pi}.$$

C Toeplitz operators

Let Σ be a symplectic cone with a compact basis. Louis Boutet de Montvel and Victor Guillemin associate in [Bo-80, BG-81] to such a cone an Hilbert space and an algebra of operators called the Toeplitz operators with the same properties as the classical pseudo-differential operators. The latter case corresponds to the cone which is a cotangent cone. For an introduction, one can look at [CdV-94].

Two examples are implicitly present in this paper:

1. *Harmonic oscillator*: the harmonic oscillator $\Omega = -d_x^2 + x^2$ is an elliptic self-adjoint Toeplitz operator; the cone Σ is $\mathbb{R}_{u,v}^2 \setminus 0$ with the symplectic form $du \wedge dv$ and the dilations $\lambda.(u, v) = (\sqrt{\lambda}u, \sqrt{\lambda}v)$. The symbol of Ω is $u^2 + v^2$.
2. *Quantization of the Reeb flow*: if $\Sigma \subset T^*X \setminus 0$ is the characteristic cone of our sR Laplacian, one can quantize the Reeb Hamiltonian ρ as a first order elliptic Toeplitz operator of degree 1.

D A cohomological equation

The following proposition is a global formulation of the formal cohomological equations discussed in [CHT-18] (section 5.1 and Appendix C) with a simple proof:

Proposition D.1 *We consider the cohomological equation*

$$\{|\zeta|I, A\} = B \tag{1}$$

where A, B are smooth homogeneous functions in the cone $C := \{I < c|\zeta|\}$ with compact support in $q \in H_3$. If B is homogeneous of degree j and vanishes

on $\Sigma := \{I = 0\}$, Equation (1) admits a solution A homogeneous of degree $j - 1$.

Restricting to $\zeta = 1$, reduces to prove the following Lemma.

Lemma D.1 *Let us consider the differential equation*

$$\frac{\partial a}{\partial \theta} + \frac{1}{2}I \frac{\partial a}{\partial z} = b(z, w) \quad (2)$$

with $(z, w) \in \mathbb{R} \times \{|w| < c\}$, b smooth, compactly supported in z and $I = |w|^2$. We assume that $b(z, 0) = 0$.

Then Equation (2) admits a smooth solution a depending smoothly of b .

Recall that a smooth function f in some disk in \mathbb{C} admits a Fourier expansion

$$f(w) = \sum_{n=0}^{\infty} f_n(|w|^2)w^n + \sum_{n=1}^{\infty} g_n(|w|^2)\bar{w}^n$$

where the f_n 's and the g_n 's are smooth. We consider only the first sum. The second can be worked out in a similar way. We expand $a = \sum_{n=1}^{\infty} a_n(z, I)w^n + a_0(z, I)$ and $b = \sum_{n=1}^{\infty} b_n(z, I)w^n + Ib_0(z, I)$. We can take $a_0(z, I) = 2 \int_{-\infty}^z b_0(s, I)ds$. The equation for a_n , with $n \geq 1$, writes

$$ina_n + \frac{1}{2}I \frac{\partial a_n}{\partial z} = b_n$$

We can solve it, for $I \neq 0$, by

$$a_n(z, I) = \frac{2}{I} \int_{-\infty}^0 b_n(z + u)e^{in2u/I} du = \int_{-\infty}^0 b_n(z + Iw/2)e^{inw} dw$$

and $a_n(z, 0) = b_n(z, 0)/in$. It follows, using integrations by parts, that all derivatives of a_n admit a full expansion in positive powers of I . Moreover, the solutions for $I \neq 0$ admit the limit $b_n(z, 0)/in$ as $I \rightarrow 0$.

Now we want to add up the series $\sum_n a_n$. I was not able to do that directly and will proceed as follows: the sum $\sum_n a_n$ is convergent as a formal series along Σ , because $a_n = O(I^{n/2})$. Using Borel procedure, we need only to solve our cohomological equation with a flat righthandside. This follows clearly from the expression

$$a(z, w) = \frac{2}{I} \int_{-\infty}^0 b(z + t, e^{2itz/I}w) dt .$$

E The heat trace for a compact quotient of H_3

Let us consider the discrete subgroup $\Gamma = \left(\sqrt{2\pi\mathbb{Z}}\right)^2 \times \pi\mathbb{Z}$ of H_3 . The spectrum of the Laplace operator on $M = H_3/\Gamma$ is the union of the spectrum of the flat torus $\mathbb{R}^2 / \left(\sqrt{2\pi\mathbb{Z}}\right)^2$ and the eigenvalues $2m(2l+1)$, $m \geq 1$, $l \geq 0$, with multiplicities $2m$. The corresponding part of the heat trace is hence

$$Z_0(t) = \sum_{m=1}^{\infty} 2m \sum_{l=0}^{\infty} e^{-2m(2l+1)t}$$

This sum can be evaluated using the Poisson summation formula. After some calculations, we get

$$Z_0(t) = \frac{\pi^2}{16t^2} - \frac{1}{2t} + \frac{\pi^2}{4t^2} \sum_{n=1}^{\infty} \frac{1}{1 + \cosh \pi^2 n/t}$$

The first term gives the Weyl law. Each term in the sum w.r. to n is equivalent to

$$\frac{\pi^2}{2t^2} e^{-4\pi^2 n/4t}$$

We observe that $2\pi\sqrt{n}$ is the length of a periodic geodesic of M . Hence we recover also the length spectrum giving contributions of the order of $\exp(-L^2/4t)$. Similarly, the heat trace for the Riemannian Laplacian on M was computed by Hubert Pesce [Pe-94].

F A short history of the trace formulae

The trace formulae were first discovered independently by two groups of physicists: Martin Gutzwiller [Gu-71] for a semi-classical Schrödinger operator and Roger Balian & Claude Bloch in a very impressive series of papers for Laplacians in Euclidean domains [Ba-BI-70, Ba-BI-71, Ba-BI-72]. In [Ba-BI-72], page 154, the authors suggested already a possible application to the inverse spectral problems¹, an industry which just started at the end of

¹They wrote “The analysis of the eigenvalue density as a sum of oscillating terms gives a new insight into the problem of “hearing the shape of a drum” [Kac paper]. ... It is

the sixties. From the point of view of mathematics, the Poisson summation formula can be interpreted as a trace formula for the Euclidian Laplacian on flat tori. Similarly, the famous Selberg trace formula [Se-56] (see also Heinz Huber [Hu-59]) is a trace formula for the Laplacian on hyperbolic surfaces. Then my thesis [CdV-73], inspired by the work of Balian and Bloch and the Selberg trace formula, uses the complex heat equation for general closed Riemannian manifold. The definitive version, the CDG formula, using wave equation, was discovered by Jacques Chazarain and the tandem Hans Duistermaat & Victor Guillemin in [Ch-74, DG-75]. They use the power of the Fourier Integral Operators calculus [Ho-71, DH-72]. See [CdV-07] for a review paper. Later results recover the cases of manifolds with boundaries and semi-classical versions.

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convenient, for the discussion to start from the fact that the knowledge of eigenvalues determines uniquely the path generating function ... Thus ... the lengths of the closed stationary polygons are determined”.

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