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Partitions and Well-coveredness: The Graph Sandwich Problem*

Sancrey R. Alves[1], Fernanda Couto[2], Luerbio Faria[3], Sylvain Gravier[4], Sulamita Klein[5], Uéverton S. Souza[6], and Rodrigo Marinho[3]

[1]IME, Universidade Federal Fluminense, Brazil.
[1]Fundação de Apoio à Escola Técnica do Est. do Rio de Janeiro, Rio de Janeiro, Brazil
[2]Universidade Federal Rural do Rio de Janeiro, Nova Iguaçu, Brazil
[3]Universidade do Estado do Rio de Janeiro, Rio de Janeiro, Brazil
[4]CNRS, Université Grenoble Alpes, Grenoble, France
[5]Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
[6]Universidade Federal Fluminense, Niterói, Brazil

Abstract

A graph is *well-covered* if every maximal independent set is also maximum. A (k, ℓ)-*partition* of a graph G is a partition of its vertex set into k independent sets and ℓ cliques. A graph is (k, ℓ)-*well-covered* if it is well-covered and admits a (k, ℓ)-partition. The recognition of (k, ℓ)-well-covered graphs is polynomial-time solvable for the cases $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, and $(2, 0)$, and hard, otherwise. In the Graph Sandwich problem for property Π, we are given a

*An extended abstract of this work has appeared in the proceedings of LATIN 2020 [2].

pair of graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$ with $E^1 \subseteq E^2$, and asked whether there is a graph $G = (V, E)$ with $E^1 \subseteq E \subseteq E^2$, such that G satisfies the property Π. The problem of recognizing whether a graph G satisfies a property Π is equivalent to the particular graph sandwich problem where $E^1 = E^2$. In this paper, we study the Graph Sandwich problem for the property of being (k, ℓ)-well-covered. We present some structural characterizations and extending previous studies on the recognition of (k, ℓ)-well-covered graphs, we prove that Graph Sandwich for (k, ℓ)-well-coveredness is polynomial-time solvable when $(k, \ell) \in \{(0, 1), (1, 0), (1, 1), (0, 2)\}$. Besides, we show that Graph Sandwich is NP-complete for the property of being $(1, 2)$-well-covered.

1 Introduction

Well-covered graphs were first introduced by Plummer [32] in 1970 as the class of graphs in which every maximal independent set has the same cardinality, i.e., every maximal independent set is maximum. In other words, a graph is well-covered if every minimal vertex cover is also a minimum vertex cover. The graph property of being well-covered is called *well-coveredness*.

The problem of recognizing well-covered graphs, which we denote by Well-Coveredness, was proved to be coNP-complete by Chvátal and Slater [11] and, independently, by Sankaranarayana and Stewart [35]. In 2011, Brown and Hoshino proved that such a problem remains coNP-complete even when restricted to the family of circulant graphs [7]. In contrast, Well-Coveredness is in P when the input is known to be a bipartite graph [19, 34, 39], a perfect graph of bounded clique size [15], or a claw-free graph [29, 30, 37]. Structural characterizations of well-covered block graphs and well-covered unicyclic graphs have been presented in [38]. Well-covered graphs having girth at least five, and well-covered graphs having neither a C_4 nor a C_5 as a subgraph were completely described in [23] and [9, 24], respectively. Brown and Nowakowski proved that almost no random graph $G_{n,p}$ is well-covered [8], and Fradkin [25] analysed the well-coveredness of cartesian products of graphs. Prisner, Topp, and Vestergaard [33] characterize well-covered simplicial, chordal, and circular-arc graphs. Cappelle and Rautenbach [10] studied the structure and recognition of the well-covered graphs without an isolated vertex that has independence number $\frac{|V(G)|-k}{2}$ for some non-negative integer k. Pinter [31] studied well-covered graphs G

having the additional property that $G - e$ is also well-covered for every edge e in G. Besides, partial characterizations for well-covered tripartite graphs have been done in [27, 40].

Regarding parameterized complexity, FPT algorithms to recognize well-covered graphs were presented in [3, 4]. In [3], Alves et al. showed that the problem of determining whether every maximal independent set of a graph has size k is coW[2]-hard when parameterized by k. In contrast, Araújo et al. [4] showed that the problem of determining whether every minimal vertex cover of a graph has size k is FPT when parameterized by k, and it admits a polynomial kernel.

A (k, ℓ)-*partition* of a graph $G = (V, E)$ is a partition of V into k independent sets $S^1, \ldots, S^k$ and ℓ cliques $K^1, \ldots, K^\ell$. By definition, some of these sets might be empty. A graph is (k, ℓ) if it admits a (k, ℓ)-partition. The P vs NP-complete dichotomy of recognizing (k, ℓ)-graphs is well-known [6]: the problem is in P if $\max\{k, \ell\} \leq 2$, and NP-complete otherwise. (k, ℓ)-graphs and their subclasses have been extensively studied in the literature. For instance, list partitions of (k, ℓ)-graphs were studied by Feder et al. [20]. In another paper, Feder et al. [21] proved that recognizing graphs that are both chordal and (k, ℓ) is in P. Demange et al. [16] presented efficient algorithms to recognize cographs that are partitionable into k independent sets and ℓ cliques. Kolay et al. [28] and Baste et al. [5] considered the problem of removing a small number of vertices from a graph so that it becomes (k, ℓ). In 2017, Faria et al. [17] presented kernelization algorithms for Signed Max Cut on (k, ℓ)-graphs. In 2021, Alves, Nascimento, and Souza [1] studied the complexity of the problem of coloring (k, ℓ)-graphs.

Let $k, \ell \geq 0$ be two fixed integers. A graph is (k, ℓ)-*well-covered* if it is both, (k, ℓ) and well-covered. The complexity of recognizing (k, ℓ)-well-covered graphs was established in [3], and the complexity of recognizing probe (k, ℓ)-well-covered graphs was studied in [18].

More general than recognizing a property of a graph is the graph sandwich problem. Golumbic, Kaplan and Shamir [26] stated the Graph Sandwich Problem for Property Π. The input is a pair of graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$ with $E^1 \subseteq E^2$, and the question is whether there is a graph $G = (V, E)$ with $E^1 \subseteq E \subseteq E^2$, such that G satisfies the property Π. For the sake of understanding we use to name the set E^1 as the *forced* edges, the set $E^0 = E^2 \setminus E^1$ as the *optional* edges, and the set $E^3 = \{e : e \notin E^2\}$ as the *forbidden* set of edges. Thus, the *optional* graph $G^0 = (V, E^0)$, and

the *forbidden* graph $G^3 = (V, E^3)$ are defined. Also, note that if $G^1 = G^2$, then the graph sandwich problem for property Π is exactly the recognition of property Π.

Graph sandwich problems are widely studied for many graph classes and also for many subclasses of (k, ℓ)-graphs [12, 13, 14, 22, 26]. More specifically, in [14], the graph sandwich problem for (k, ℓ)-graphs was settled to be NP-complete if $k + \ell \geq 3$, and polynomial-time solvable otherwise.

Motivated by the relevance of (k, ℓ) and well-covered graphs, in this paper, we are interested in exploring the complexity of Graph Sandwich for (k, ℓ)-well-coveredness, i.e., the property of being (k, ℓ)-well-covered. More precisely, in this paper, we focus on the following two decision problems restricted to (k, ℓ)-well-covered graphs.

(k, ℓ)-Well-Coveredness	
Input:	A graph G.
Question:	Is G a (k, ℓ)-well-covered graph?

Graph Sandwich for (k, ℓ)-well-coveredness	
Input:	Graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$ with $E^1 \subseteq E^2$.
Question:	Is there a graph $G = (V, E)$ with $E^1 \subseteq E \subseteq E^2$, such that G is (k, ℓ)-well-covered?

When a recognition problem for a property Π is NP-hard (resp. coNP-hard), we can consider the sets $E^1 = E^2$ to obtain that the graph sandwich problem for the property Π is also NP-hard (resp. coNP-hard). In 2018, Alves et al. [3] proved that the recognition of (k, ℓ)-well-covered graphs can be done in polynomial time for the cases $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, and $(2, 0)$ and NP-hard or coNP-hard otherwise. Therefore, the only cases where Graph Sandwich for (k, ℓ)-well-coveredness can be no longer hard are in these six polynomial cases.

We prove that Graph Sandwich for (k, ℓ)-well-coveredness is polynomial-time solvable when $(k, \ell) \in \{(0, 1), (1, 0), (1, 1), (0, 2)\}$ but it is NP-complete when $(k, \ell) = (1, 2)$ (see Table 1). Our polynomial-time algorithms generalize previous studies on (k, ℓ)-well-covered graphs' recognition, and our NP-completeness proof points out a contrast between the complexity of Recognition and Graph Sandwich problems for the polynomial-time recognizable property of being a $(1, 2)$-well-covered graph. We left the problem for the property of being $(2, 0)$-well-covered open.

k \ ℓ	0	1	2	≥ 3
0	-	P	P	NPc
1	P	P	NPc	NPc
2	?	coNPc	coNPc	(co)NPh
≥ 3	NPh	(co)NPh	(co)NPh	(co)NPh

Table 1: Complexity of Graph Sandwich for (k, ℓ)-well-coveredness. coNPc stands for coNP-complete, NPh stands for NP-hard, NPc stands for NP-complete, and (co)NPh stands for both NP-hard and coNP-hard.

2 Preliminaries

Let $G = (V, E)$ be a graph, $v \in V$, and $S \subseteq V$, we define the *neighborhood* $N(v) = \{u \in V : uv \in E\}$ of v in G, the *neighborhood* $N_S(v) = \{u \in S : uv \in E\}$ of v in S, the *degree* $d(v) = |N(v)|$ of v in G, and the *degree* $d_S(v) = |N_S(v)|$ of v in S. Also, for $W \subseteq V$ the *neighborhood* of W in S is $N_S(W) = \bigcup_{w \in W} N_S(w) \setminus W$.

Next, we present structural characterizations for (k, ℓ)-well-coveredness. Notice that every $(0, 1)$-graph, as well as $(1, 0)$-graph, is well-covered. In addition, for the case $(0, 2)$, i.e., co-bipartite graphs, it is easy to see that the following proposition holds.

Proposition 1. *A graph $G = (V, E)$ is $(0, 2)$-well-covered if and only if G is $(0, 2)$ and either G is a complete graph, or G has no universal vertex.*

In 1977, Ravindra [34] presented the following characterization of $(2, 0)$-well-covered (well-covered bipartite) graphs.

Proposition 2. *[34] A graph $G = (V, E)$ is $(2, 0)$-well-covered if and only if G is $(2, 0)$ and there is a perfect matching M of G such that for each $e = uv \in M$ the induced graph $G[N(u) \cup N(v)]$, by the union of the open neighbors of u and v, is a complete bipartite graph.*

In 2018, Alves et al. [3] provided the following characterization for $(1, 1)$-well-covered (well-covered split) graphs.

Proposition 3. *[3] A graph $G = (V, E)$ is $(1, 1)$-well-covered if and only if there is a partition $V = (K, S)$ for V where K is a clique, S is a independent*

set, and either $d_S(v) = 1$ *for each vertex* $v \in K$ *or* $d_S(v) = 0$ *for each vertex* $v \in K$.

In order to fulfill the structural characterizations for polynomial-time recognizable (k, ℓ)-well-covered graphs, next we give a structural characterization of $(1, 2)$-well-covered graphs.

Proposition 4. *Let* $G = (V, E)$ *be a graph with partition* $V = (S, K^1, K^2)$ *where* S *is maximal. Then,* $G = (V, E)$ *is a* $(1, 2)$*-well-covered graph if and only if* G *satisfies the following conditions:*

1. *If* $v \in K^1 \cup K^2$*, then* $1 \leq |N_S(v)| \leq 2$*;*
2. *If* $v \in K^i$ *and* $|N_S(v)| = 2$*, then* $\exists\ u \in K^j, i \neq j$*, with* $uv \notin E$*. In addition,* $\forall u \in K^j$ *with* $uv \notin E$*, then* $N_S(u) \subseteq N_S(v)$ *(*$i \neq j$ *and* $i, j \in \{1, 2\}$*);*
3. *If* $v \in K^i$ *and* $|N_S(v)| = 1$*, then for each* $u \in K^j$ *with* $uv \notin E$ *it holds that* $|N_S(u) \cup N_S(v)| = 2$ *(*$i \neq j$ *and* $i, j \in \{1, 2\}$*).*

Proof. Let G be a $(1, 2)$-well-covered graph with a $(1, 2)$-partition $V = (S, K^1, K^2)$, where S is a maximal independent set of G.

Note that $|N_S(v)| \geq 1, \forall v \in (K^1 \cup K^2)$, otherwise S would not be maximal. Next, suppose that there is a vertex $v \in K^1 \cup K^2$, such that $|N_S(v)| \geq 3$. We observe that $|S| - 1$ is an upper bound to the size of a maximum independent set I of G containing v, where $I = \{v\} \cup (S \setminus N_S(v)) \cup \{w\}$, such that w is a vertex of $K^1 \cup K^2$, if any, non-adjacent to any vertex in $\{v\} \cup (S \setminus N_S(v))$. Note that $|I| = |\{v\} \cup (S \setminus N_S(v)) \cup \{w\}| \leq 1 + |S| - 3 + 1 = |S| - 1$. Thus, G is not well-covered, a contradiction since S is a maximal independent set. Therefore, $\forall v \in K^1 \cup K^2$, $|N_S(v)| \leq 2$ and condition 1 holds.

Suppose there exists a vertex $v \in K^1 \cup K^2$ such that $|N_S(v)| = 2$ (say $v \in K^1$). Next, consider a maximal independent set I of G which contains the vertex v. Since G is well-covered with independence number α equals to $|S|$, then there exists a vertex u which belongs to K^2 such that $N_S(u) \subseteq N_S(v)$ and $uv \notin E$, otherwise $|I| < |S|$. Moreover, if there is a vertex $u \in K^2$ such that $uv \notin E$, then $N_S(u) \subseteq N_S(v)$, otherwise an independent set I containing u and v has size $|I| \leq 2 + |S| - 3 = |S| - 1 < |S|$, a contradiction. Therefore, condition 2 holds.

Suppose $v \in K^i$, and $u \in K^j$, $i, j \in \{1, 2\}$, $i \neq j$, with $|N_S(v)| = 1$ and $uv \notin E$. Since G is well-covered, each maximal independent set has size $|S|$.

A maximal independent set I of G which contains u and v can be described as $I = \{u, v\} \cup (S \backslash (N_S(\{u, v\}))$. Thus, $|I| = 2 + |S| - |N_S(\{u, v\})| = |S|$, and then $|N_S(u) \cup N_S(v)| = 2$ and condition 3 holds.

Now suppose the graph G satisfies the conditions 1, 2 and 3.

Notice that there are three possible configurations for a maximal independent set I of G, as listed below:

1. I has no vertex in $K^1 \cup K^2$,
2. I has one vertex in $K^1 \cup K^2$,
3. I has two vertices in $K^1 \cup K^2$.

Consider that I has no vertex in $K^1 \cup K^2$, we have $I = S$ and $|I| = |S|$.

Consider that I has exactly one vertex v in $K^1 \cup K^2$. By condition 1, either $|N_S(v)| = 1$ or $|N_S(v)| = 2$. By condition 2, if $v \in K^i$ with $|N_S(v)| = 2$, then there exists $u \in K^j$, where $i \neq j$, and $N_S(u) \subseteq N_S(v)$. Hence, u has no neighbor in I which contradicts that I is a maximal independent set of G. Thus, we may assume $|N_S(v)| = 1$, which implies that $I = \{v\} \cup (S \backslash N_S(v))$, and $|I| = 1 + |S| - 1 = |S|$.

Finally, we consider that I has two vertices u, v in $K^1 \cup K^2$. Note that $I = \{u, v\} \cup (S \backslash (N_S(u) \cup N_S(v)))$, and then $|I| = 2 + |S| - |N_S(u) \cup N_S(v)|$. Again, by condition 1 either $|N_S(u)| = 1$, or $|N_S(u)| = 2$. In both cases, by conditions 2 and 3, $|N_S(u) \cup N_S(v)| = 2$, which implies $|I| = 2 + |S| - 2 = |S|$.

Therefore, since any maximal independent set I of G has size $|S|$, it holds that G is well-covered. □

3 Polynomial-time solvable cases

In order to check whether (G^1, G^2) is a YES-instance of the GRAPH SANDWICH FOR (k, ℓ)-WELL-COVEREDNESS when $(k, \ell) = (0, 1)$ or $(1, 0)$, it is enough to check whether, respectively, either G^2 is a complete graph, or $E^1 = \emptyset$. Next, we will deal with cases $(0, 2)$ and $(1, 1)$.

3.1 Graph Sandwich for (0,2)-well-coveredness

First, consider the following algorithm for GRAPH SANDWICH FOR $(0, 2)$-WELL-COVEREDNESS.

```
Algorithm 1.
Input: graphs G^1 = (V, E^1) and G^2 = (V, E^2) with E^1 ⊆ E^2;
Begin
1   If (G^2 = (V, E^2) is not (0,2)) then
2     Return no;
3   Else
4     If (G^2 = (V, E^2) is a complete graph) then
5        Return yes;
6     Else
7        If (G^1 = (V, E^1) has a universal vertex) then
8           Return no;
9       Else
10         Return yes;
End.
```

Theorem 1. *Algorithm 1 correctly asserts whether there is a graph $G = (V, E)$ with $E^1 \subseteq E \subseteq E^2$ such that G is $(0, 2)$-well-covered.*

Proof. First, note that the property of being co-bipartite is closed under edge addition, thus if $G^2 = (V, E^2)$ is not $(0, 2)$ then no spanning subgraph of G^2 will be a co-bipartite graph (line 1-2). Since every $(0, 1)$-graph (complete graph) is well-covered, we can assume that G^2 is not complete, otherwise the answer of the problem is positive (line 3-5). If G^2 is $(0, 2)$ but it is not $(0, 1)$, and G^1 has a universal vertex, then every graph $G = (V, E)$ with $E^1 \subseteq E \subseteq E^2$ has a universal vertex, thus, by Proposition 1, the answer is negative (line 6-8). Finally, assume that G^2 is $(0, 2)$ but it is not $(0, 1)$, and G^1 has no universal vertex. Take a $(0, 2)$-partition, (K^1, K^2), of G^2. Notice that every edge of E^3 is crossing from K^1 to K^2. If $v \in K^i$ dominates all vertices of K^j $(i \neq j)$ in G^1, then there is a vertex $w \in K^i$ such that $vw \in E^0$, otherwise v is a universal vertex in G^1.Therefore, we can update the $(0, 2)$-partition for $(K^1 \setminus \{v\}, K^2 \cup \{v\})$. This procedure can be applied successively until obtain a $(0, 2)$-partition in which for every vertex $v \in K^i$ there is at least one vertex $w \in K^j$ $(i \neq j)$ such that $vw \in E^0 \cup E^3$, which certifies in $O(n)$ steps that there is a co-bipartite graph $G = (V, E)$ with $E^1 \subseteq E \subseteq E^2$ having no universal vertex. Thus, the answer is positive (line 9-10). $\square$

3.2 Graph Sandwich for (1,1)-well-coveredness

Lemma 1. *There is a polynomial-time algorithm that either correctly solves* Graph Sandwich for $(1,1)$-well-coveredness*, or outputs a partition* $(S', K' \cup T')$ *of* V *such that:*

1. S' *is an independent set of* G^1*;*
2. $K' \cup T'$ *induces a clique of* G^2*;*
3. $K', T' \neq \emptyset$*;*
4. *there is no edge of* G^2 *between the vertices of* T' *and of* S'*;*
5. *each vertex* $v \in K'$ *is incident to at most one edge* $vu \in E^1$ *such that* $u \in S'$*, and at least one edge* $vw \in E^2$ *such that* $w \in S'$*.*

Proof. Recall that $(1,1)$-graphs is exactly the class of split graphs. In 1995, Golumbic, Kaplan and Shamir [26] presented a polynomial-time algorithm for Graph Sandwich for split graphs. This algorithm is based on reducing the problem to an instance $I = (U, C)$ of 2Sat, which is polynomial-time solvable. The proposed construction consists of creating the set of variables

$$U = \{v_K, v_S : v \in V\}$$

and the set of clauses C equal to

$$\{(v_K \vee v_S), (\overline{v_K} \vee \overline{v_S}) : v \in V\} \cup \{(u_K \vee v_K) : uv \in E^1\} \cup \{(u_S \vee v_S) : uv \in E^3\}.$$

It is easy to see that $I = (U, C)$ is satisfiable if and only if V can be partitioned into K, S such that S induces an independent set of G^1 and K induces a clique of G^2, where $v_K = true$ (resp. $v_S = true$) means that v is set to K (resp. S).

Considering the $(1,1)$-well-coveredness, if (G^1, G^2) is an yes-instance of Graph Sandwich for $(1,1)$-well-coveredness, then (G^1, G^2) is an yes-instance of Graph Sandwich for split graphs. However, by the characterization provided in Proposition 3, we know that each vertex of the clique K must be a neighbor of at most one vertex in S, in the solution graph G. Thus, in order to attain this restriction we add the following set of clauses to $I = (U, C)$:

$$\{(\overline{v_S} \vee \overline{w_S}) : u, v, w \in V \text{ and } uv, uw \in E^1\}.$$

Observe that if $I = (U, C)$ is not satisfiable, then (G^1, G^2) is a NO-instance of GRAPH SANDWICH FOR $(1,1)$-WELL-COVEREDNESS. However, if $I = (U, C)$ is satisfiable, then V can be partitioned into K, S such that S induces an independent set of G^1, K induces a clique of G^2, and every vertex $v \in K$ has at most one neighbor in S, in the graph G^1. Therefore, we can set $S' = S$, $T' = \{v : v \in K \text{ and has no neighbor in } S, \text{ in the graph } G^2\}$, and $K' = \{K \setminus T'\}$. Now, if $T' = \emptyset$ then (G^1, G^2) is a YES-instance, since we can use the optional edges conveniently to satisfy the condition that all vertices have exactly one neighbor in S, and if $K' = \emptyset$ then (G^1, G^2) is also a YES-instance, since no vertex in K will be adjacent to a vertex in S. Therefore, we can either solve GRAPH SANDWICH FOR $(1,1)$-WELL-COVEREDNESS or output a partition $(S', K' \cup T')$ as required. □

Lemma 2. *Let (G^1, G^2) be an instance of* GRAPH SANDWICH FOR $(1,1)$-WELL-COVEREDNESS*, and $V = (S', K' \cup T')$ be a partition of V as described in Lemma 1. It can be checked in polynomial time whether there is a graph $G = (V, E)$ with $E^1 \subseteq E \subseteq E^2$, such that the set of vertices of G can be partitioned into (K, S) with $S' \subseteq S$, where K is a clique, S is an independent set, and either $d_S(v) = 1$ for each vertex $v \in K$, or $d_S(v) = 0$ for each vertex $v \in K$.*

Proof. Initially, label each vertex $v \in K' \cup T'$ with $label(v) = 0$. Now, label each vertex $v \in K'$ having a forced edge to S' with label equal to one ($label(v) = 1$). After that, for each vertex $v \in K' \cup T'$ with $label(v)$=0, such that $\exists u \in N(v)$ with $uv \in E^1$ and $label(u) = 1$, then do $label(v) = 2$.

Since $S' \subseteq S$, no vertex v with $label(v) = 1$ can be in S, otherwise S is not an independent set. Consequently, no vertex v with $label(v) = 2$ can be in S, otherwise some vertex v with $label(v) = 1$ has two neighbors in S. Therefore, if every vertex v of $K' \cup T'$ has $label(v) \neq 0$, then we can safely return NO. Moreover, if there is a vertex $v \in K' \cup T'$ such that $label(v) = 0$, we output YES, because $(S, K) = (S' \cup \{v\}, K' \cup T' \setminus \{v\})$, and $G = (V, E)$ with $E = E^1 \cup \{uw : u, w \in K' \cup T' \setminus \{v\}\} \cup \{vu : u \in K' \cup T', u \neq v, \text{ and } label(u) \neq 1\}$ is a solution. □

Lemma 3. *Let (G^1, G^2) be an instance of* GRAPH SANDWICH FOR $(1,1)$-WELL-COVEREDNESS*, and let $(S', K' \cup T')$ be a partition of V as described*

in Lemma 1. If there are three vertices a, b, c such that $ab, bc \in E^1$, $b \in K'$, and $a, c \in T'$, then we can solve (G^1, G^2) in polynomial time.

Proof. Let (S, K) be a partition as described in Proposition 3. If a and c are both in S then b has two neighbors in S, a contradiction. If, either, a or c is in K then $S' \subseteq S$, since there are only forbidden edges from a and c to the vertices of S'. Therefore, by Lemma 2, we conclude that we can solve the instance (G^1, G^2) in polynomial time. □

Theorem 2. Graph Sandwich for $(1,1)$-well-coveredness *can be solved in polynomial time.*

Proof. Let (G^1, G^2) be an instance of Graph Sandwich for $(1,1)$-well-coveredness. Without loss of generality, assume that the algorithm presented in Lemma 1 outputs a partition $(S', K' \cup T')$ of V. By Lemma 2, we can also consider that there is no sandwich graph $G = (V, E)$ such that $V(G)$ can be partitioned into (S, K) according to Proposition 3, with $S' \subseteq S$. Hence, there is no vertex b having two neighbors a and c in G^1 such that $a, c \in T'$ (see Lemma 3).

Suppose that (G^1, G^2) has a sandwich graph $G = (V, E)$ which is $(1,1)$-well-covered. Let (S, K) be a partition of $V(G)$ according to Proposition 3. Note that if $T' \cap K \neq \emptyset$, then $S' \subseteq S$, which is a contradiction. Therefore, $T' \subseteq S$. In addition, for every vertex v at a distance at most two, considering the graph G^1, of some vertex that must be in S, it holds that v must be contained in K, otherwise either S is not an independent set or some vertex of K will have two neighbors in S. Analogously, if a vertex v has a forbidden edge to a vertex that must be in K, then v must be in S.

Let A be the set of vertices of S' that must be in K, and let B be the set of vertices of $K' \cup T'$ that must be in S by the successive application of the rules described above. Note that $T' \subseteq B$, since $S' \nsubseteq S$ and each vertex of T' has a forbidden edge to a vertex of S'. Observe that sets A, B can be easily found in polynomial time using search algorithms.

If some vertex is determined to be at same time in S and in K, then (G^1, G^2) is a no-instance. Otherwise, we consider the following partition of V:

$$S'' = \{B \cup (S' \setminus A)\},$$

$$T'' = \{v : v \in A \text{ and has no neighbor in } S'' \text{ in the graph } G^2\},$$

and

$$K'' = \{(K' \setminus B) \cup (A \setminus T'')\}.$$

If (G^1, G^2) is a YES-instance then no vertex $v \in K''$ has two neighbors in S'' in the graph G^1, otherwise we have a contradiction (see Proposition 3).

If $T'' = \emptyset$ then every vertex in K'' has at least one edge of G^2 to some vertex in S''. Thus, when $T'' = \emptyset$ we can easily construct the the solution graph G for (G^1, G^2) in polynomial time. Now, suppose that $T'' \neq \emptyset$. Since $T'' \subseteq A$ it follows that S'' must be in S for a (S, K) partition of $V(G)$ according to Proposition 3, if any. Thus by Lemma 2 it holds that (G^1, G^2) can be solved in polynomial time. □

4 The NP-complete case

Next, we will deal with $(1, 2)$-well-covered graphs. The NP-completeness of such a case points out a contrast between the complexity of RECOGNITION and GRAPH SANDWICH problems for (k, ℓ)-well-covered graphs, since $(1, 2)$-well-covered graphs are polynomial-time recognizable [3].

In order to prove the NP-completeness of GRAPH SANDWICH FOR $(1, 2)$-WELL-COVEREDNESS, we present a reduction from POSITIVE 1-IN-3 SAT, a well-known NP-complete problem [36]. First, we present some auxiliary definitions and preliminary results.

Let $G = (V, E)$ be a graph and $R, T \subseteq V$. We say that T *2–dominates* R if each vertex of R has at least 2 neighbors in T. Let $I = (U, C)$ be an instance of POSITIVE 1-IN-3 SAT where U is the set of variables and C is the set of clauses. Let $T \subseteq U$, we say that T *2–dominates* C if each clause of C has at least two literals in T, in this case T is called a *2-dominating* set of variables.

Lemma 4. *Let $I = (U, C)$ be an instance of* POSITIVE 1-IN-3 SAT*, where $U = \{u_1, u_2, u_3, \ldots, u_n\}$ and $C = \{c_1, c_2, c_3, \ldots, c_m\}$, and let $T \subseteq U$ such that T 2-dominates C. Then, I can be solved in time $O(2^{|T|}m)$.*

Proof. We can check each of the $2^{|T|}$ truth assignments of T. For a given truth assignment of T, each clause contains at most one variable with undefined value. Thus, it is easy to see that, in linear time, one can check whether such an assignment can be extended into a satisfiable assignment for I. □

From Lemma 4, we know that the hard instances of Posivite 1-in-3 Sat have no *2-dominating* set of variables of bounded size. Also, we assume that there is no variable occurring in all clauses, otherwise the instance is trivial.

Theorem 3. Graph Sandwich for $(1,2)$-well-coveredness *is* NP-*complete.*

Proof. Graph Sandwich for $(1,2)$-well-coveredness is in NP, since given a graph G and a partition $V = (S, K^1, K^2)$, we can check in polynomial time the conditions (1), (2), (3) of the characterization presented in Proposition 4.

Let $I = (U, C)$ be an instance of Posivite 1-in-3 Sat. Without loss of generality, we assume that I does not have a 2-dominating set of variables of size smaller than 8, and that every satisfying truth assignment of I, if any, requires at least 2 literals set as true. In addition, we consider that every variable occurs in at least one clause.

We construct an instance (G^1, G^2) of Graph Sandwich for $(1,2)$-well-coveredness, such that I is satisfiable if and only if there is a $(1,2)$-well-covered sandwich graph G for $G^1 = (V, E^1), G^2 = (V, E^2)$, as follows:

1. First, initialize $V = \{a, b, c\}$, and $E^1, E^2, E^3 = \emptyset$;
2. For each $u_i \in U$ add a vertex (called variable vertex) u_i to V.
3. For each $c_j \in C$ add a vertex (called clause vertex) c_j to V.
4. For each $c_j = (u_x \vee u_y \vee u_z) \in C$, add $u_i c_j, \forall\ i \in \{x, y, z\}$, to E^1.
5. For each pair $c_j, c_\ell \in C, j \neq \ell$, add $c_j c_\ell$ to E^1.
6. For each $c_j \in C$, add $c_j a, c_j b, c_j c$ to E^1.
7. For each $u_i \in U$ add $u_i a, u_i b, u_i c$ to E^0. (Remark $E^0 = E^2 \setminus E^1$)
8. Add every possible edge $u_i u_j$ to E^0.
9. Set $E^3 = (\{uv : u \neq v \text{ and } u, v \in V\} \setminus E^2)$

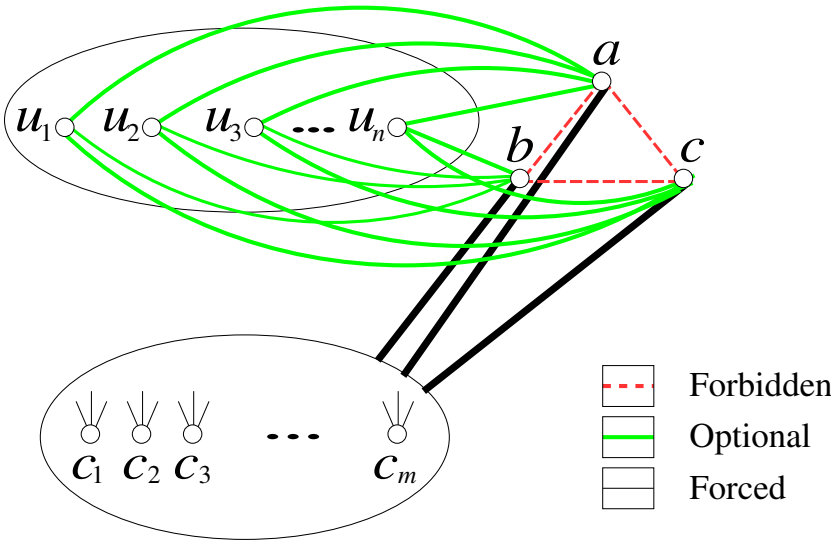

Figure 1: Graph Sandwich for $(1,2)$-well-coveredness instance, (G^1, G^2), obtained from an instance $I = (U, C)$ of Positive 1-in-3 Sat. Here are depicted all optional edges in thick green lines, and all forced edges in thin black straight lines of $(G^1 = (V, E^1), G^2 = (V, E^2))$. We depict only the forbidden edges ab, ac, and bc in dashed red straight lines, omitting the forbidden edges between a variable vertex x and a clause vertex c_j such that x does not occur in c_j.

This completes the construction of $(G^1 = (V, E^1), G^2 = (V, E^2))$.

For the sake of reader's convenience, we offer in Figure 1 a framework for the construction.

Now, suppose that $I = (U, C)$ is satisfiable. Let $\eta : U \to \{T, F\}$ be a satisfiable truth assignment for I. We build a $(1, 2)$-well-covered sandwich graph $G = (V, E)$ for $(G^1 = (V, E^1), G^2 = (V, E^2))$ with partition (S, K^1, K^2) from η as follows (see Figure 2):

- first, set $E = E^1$;
- set $u \in S$ if and only if $\eta(u) = T$;
- set $u \in K^2$ if and only if $\eta(u) = F$;
- set all vertices c_j in K^1;
- set vertices a in S; b in K^1; and c in K^2;
- if $\eta(u_i) = F$ then add to E the edges $u_i a, u_i b, u_i c$;
- add to E two edges, $u_i b$, $u_j c$, where $u_i \neq u_j$ and $\eta(u_i) = \eta(u_j) = T$;
- finally, add to E the edges $u_i u_j$ whether $\eta(u_i) = \eta(u_j) = F$.

In order to prove that G is a $(1, 2)$-well-covered we consider the characterization of Proposition 4.

Notice first that S is a maximal independent set and that K^1, and K^2 are cliques. Also note that, because η is a satisfiable truth assignment for Positive 1-in-3 Sat, it holds that every vertex of $K^1 \cup K^2$ has 1 or 2 neighbors in S. Since each clause vertex c_j has $N_S(c_j) = \{u_i, a\}$ where u_i is the only true literal of c_j in η, each false literal vertex u_i satisfies $N_S(u_i) = \{a\}$, and, by construction, both, b and c, have exactly one neighbor in S. This satisfies condition (1) of Proposition 4.

Now, observe that edges of E^3 between cliques K^1 and K^2 are: the edge bc, and the edges between c_j and u_ℓ, where $\eta(u_\ell) = F$ and u_ℓ does not occur in c_j. If $\eta(u_\ell) = F$ and u_ℓ does not occur in c_j, then $|N_S(c_j) \cup N_S(u_\ell)| = |\{a, u_i\} \cup \{a\}| = 2$. Since, by construction, b and c have no common neighbor in S, then $|N_S(b) \cup N_S(c)| = |\{u_i\} \cup \{u_j\}| = 2$. Thus, conditions (2), (3) of Proposition 4 are satisfied, and G is a $(1, 2)$-well-covered graph.

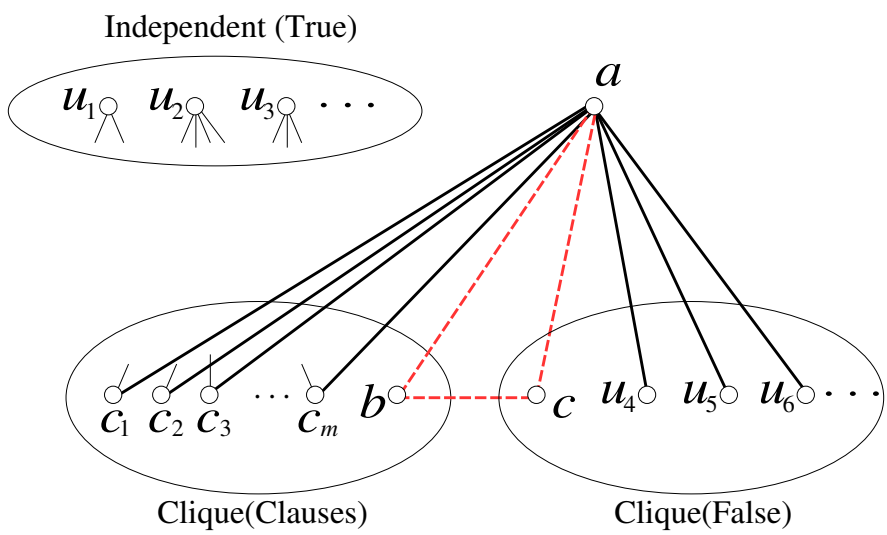

Figure 2: Diagram showing a $(1,2)$-well-covered sandwich graph G with partition (S, K^1, K^2) obtained from a satisfiable 1-IN-3 SAT truth assignment $\eta : U \to \{T, F\}$, where some edges are omitted for simplicity.

Conversely, suppose that there is a $(1,2)$-well-covered sandwich graph $G = (V, E)$ for (G^1, G^2). Since $G = (V, E)$ is a $(1,2)$-well-covered graph, there is a partition of V into a maximal independent set S and cliques K^1, K^2, as described in Proposition 4 . We proceed by defining a satisfiable POSITIVE 1-IN-3SAT assignment $\eta : U \to \{T, F\}$ for $I = (U, C)$, where the boolean variable $u_i = T$ if and only if vertex $u_i \in S$.

It remains to show that η is satisfiable.

Claim 1. *Every vertex clause c_j belongs to $K^1 \cup K^2$.*

Proof. Since every clause vertex c_j satisfies that ac_j, bc_j and $cc_j \in E^1$, if $c_j \in S$, then vertices a, b, c belong together to $K^1 \cup K^2$, what is a contradiction, since $ab, ac, bc \in E^3$. Therefore, $c_j \in K^1 \cup K^2$. □

Claim 2. *There is only one vertex x in $\{a, b, c\}$, such that $x \in S$.*

Proof. Since $ab, ac, bc \in E^3$, then a, b, c cannot be together in $K^1 \cup K^2$. By Claim 1, we have that $c_j \in K^1 \cup K^2$. Hence, a, b, c cannot be together in S, otherwise $|N_S(c_j)| \geq 3$, contradicting Proposition 4. Suppose there are exactly two vertices of a, b, c in S, say $a, b \in S$. Then, all the variable vertices belong to $K^1 \cup K^2$, otherwise $|N_S(c_j)| \geq 3$ for some c_j, because every

clause vertex belong to $K^1 \cup K^2$, and c_j is adjacent to a, and b. Thus $S = \{a, b\}$, which contradicts the maximality of S, since $S \subsetneq \{a, b, c\}$. Therefore, $|S \cap \{a, b, c\}| = 1$. □

From previous claims, and since a, b, c have the same open neighborhood in G^1 (also in G^2), we can assume that: $a \in S$; $c_j \in K^1 \cup K^2$ (for all c_j); $b \in K^1$; $c \in K^2$.

Claim 3. *Each clause vertex c_j lies at a same clique, say K^1.*

Proof. Suppose that there are two clause vertices lying at distinct cliques, say $c_j \in K^1$ and $c_\ell \in K^2$. Since $a \in S$, each clause vertex belongs to $K^1 \cup K^2$, and each clause vertex c_j is adjacent to a, then, by Proposition 4, it follows that each vertex clause c_j is adjacent to at most one variable vertex in S, and the other two variable neighbors are in $K^1 \cup K^2$. Notice now that the set of variable vertices in $K^1 \cup K^2$ forms a 2-dominating set of variables for I. Since each clause vertex has forbidden edges for all but three variables, we have at most three variable vertices in each clique, which implies that $I = (U, C)$ has a 2-dominating set of variables of size at most six, a contradiction, since we are assuming that I does not have a 2-dominating set of variables of size smaller than 8.

Thus, all the clause vertices belong to just one clique, that we will assume to be K^1. □

Claim 4. *If a variable vertex $u_i \in K^1 \cup K^2$, then $u_i \in K^2$.*

Proof. Since each clause vertex belongs to K^1, if $u_i \in K^1$ then every clause has u_i as literal. Thus, by setting u_i as true and the other literals as false, we obtain a satisfiable assignment for $I = (U, C)$, contradicting the fact that every satisfying truth assignment of I, if any, requires at least 2 literals set as true. □

In order to conclude the proof of Theorem 3, it remains to prove the following claim.

Claim 5. *For any clause vertex c_j of G, it holds that there is exactly one variable vertex u_r adjacent to c_j such that $u_r \in S$.*

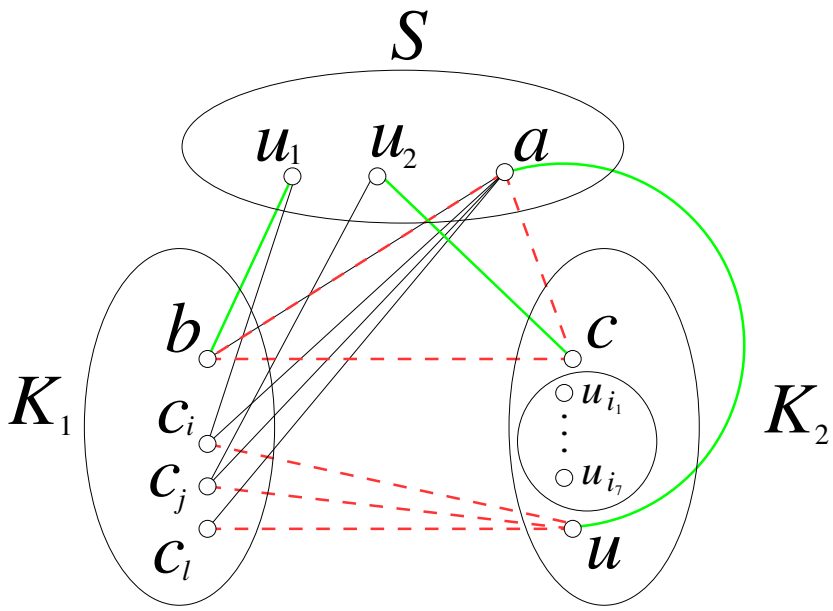

Figure 3: Diagram showing a $(1,2)$-well-covered sandwich graph G with partition (S, K^1, K^2), where some edges are omitted for simplicity.

Proof. Since each clause vertex c_j is adjacent to a, by Proposition 4(1), there is at most one additional neighbor of c_j in S. Hence, in order to prove the claim, it is enough to prove that each clause vertex c_j has one additional neighbor in S. From previous claims we know that $b \in K^1$ and that $c \in K^2$. By Proposition 4, it follows that $|N_S(b) \cup N_S(c)| = 2$. Hence, there are two variable vertices, say u_1 and u_2 in S, such that, in G, it holds that $\{u_1, u_2\} = (N_S(b) \cup N_S(c))$. Since there are clauses c_i and c_j containing, respectively, variables u_1 and u_2, clause vertices c_i and c_j have two neighbors in S, satisfying that $N_S(c_i) = \{a, u_1\}$ and $N_S(c_j) = \{a, u_2\}$.

Suppose there is a clause vertex c_ℓ with $N_S(c_\ell) = \{a\}$. Since each clause has at most one neighbor in $S \setminus \{a\}$, we have that the variable vertices in K^2 correspond to a 2-dominating set of variables. Therefore, by assumption, $|K^2 \cap U| \geq 8$. Hence, there is a vertex $u \in K^2 \cap U$ such that variable u does not occur in clauses c_i, c_j and c_ℓ. Therefore, $c_i u$, $c_j u$, $c_\ell u \in E^3$ (see Fig. 3).

Since $N_S(c_i) = \{u_1, a\}$, from Proposition 4(2) $N_S(u) \subseteq \{u_1, a\}$. Moreover, since $N_S(c_j) = \{u_2, a\}$, then, from Proposition 4(2), $N_S(u) \subseteq \{u_2, a\}$. Hence, $N_S(u) = \{a\}$. But, since $N_S(c_\ell) = \{a\}$ and $c_\ell u \in E^3$, from Proposition 4(3)

$|N_S(c_\ell) \cup N_S(u)|$ should be equal to two, a contradiction. Thus, every clause vertex has a variable neighbor in S, and the claim holds. □

Therefore, the defined assignment η is a satisfiable assignment for Positive 1-in-3 Sat, and this concludes the proof of Theorem 3. □

5 Concluding remarks

In this paper, it was proved that the Graph Sandwich problem for the property of being well-covered and partitionable into k independent sets and ℓ cliques is in P for $(k, \ell) \in \{(0, 1), (1, 0), (1, 1), (0, 2)\}$ but it is NP-complete when $(k, \ell) = (1, 2)$ (see Table 1). The tractable results generalize studies on (k, ℓ)-well-covered graphs' recognition, and the NP-completeness case points out a contrast between the complexity of Recognition and Graph Sandwich problems for the property of being a $(1, 2)$-well-covered graph.

We left the Graph Sandwich problem for the property of being $(2, 0)$-well-covered (well-covered bipartite) open. Besides, the study of Graph Sandwich for other polynomial-time recognizable subclasses of well-covered graphs, such as well-covered claw-free graphs, also seems interesting.

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