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On the hydrostatic limit of stably stratified fluids with isopycnal diffusivity

Roberta Bianchini and Vincent Duchêne

ABSTRACT. This article is concerned with the rigorous justification of the hydrostatic limit for continuously stratified incompressible fluids under the influence of gravity.

The main peculiarity of this work with respect to previous studies is that no (regularizing) viscosity contribution is added to the fluid-dynamics equations and only diffusivity effects are included. Motivated by applications to oceanography, the diffusivity effects included in this work are induced by an advection term whose specific form was proposed by Gent and McWilliams in the 90's to model effective eddy correlations for non-eddy-resolving systems.

The results of this paper heavily rely on the assumption of stable stratification. We provide the well-posedness of the hydrostatic equations and of the original (non-hydrostatic) equations for stably stratified fluids, as well as their convergence in the limit of vanishing shallow-water parameter. The results are established in high but finite Sobolev regularity and keep track of the various parameters at stake.

A key ingredient of our analysis is the reformulation of the systems by means of isopycnal coordinates, which allows to provide careful energy estimates that are far from being evident in the original coordinate system.

1. Introduction

The following system describes the evolution of heterogeneous incompressible flows under the influence of gravity,

$$\begin{aligned}
 \partial_t \rho + (\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}} \rho + (w + w_*) \partial_z \rho &= 0, \\
 \rho (\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}}) \mathbf{u} + (w + w_*) \partial_z \mathbf{u}) + \nabla_{\mathbf{x}} P &= 0, \\
 \rho (\partial_t w + (\mathbf{u} + \mathbf{u}_*) \cdot \nabla_{\mathbf{x}} w + (w + w_*) \partial_z w) + \partial_z P + g \rho &= 0, \\
 \nabla_{\mathbf{x}} \cdot \mathbf{u} + \partial_z w &= 0, \\
 P|_{z=\zeta} - P_{\text{atm}} &= 0, \\
 \partial_t \zeta + (\mathbf{u} + \mathbf{u}_*)|_{z=\zeta} \cdot \nabla_{\mathbf{x}} \zeta - (w + w_*)|_{z=\zeta} &= 0, \\
 w|_{z=-H} &= 0.
 \end{aligned} \tag{1.1}$$

Here, t and (\mathbf{x}, z) are the time, and horizontal-vertical space variables, and we denote by $\nabla_{\mathbf{x}}, \Delta_{\mathbf{x}}$ the gradient and Laplacian with respect to \mathbf{x} . The vector field $(\mathbf{u}, w) \in \mathbb{R}^d \times \mathbb{R}$ is the (horizontal and vertical) velocity, $\rho > 0$ is the density, $P \in \mathbb{R}$ is the incompressible pressure, all being defined in the spatial domain

$$\Omega_t = \{(\mathbf{x}, z) : \mathbf{x} \in \mathbb{R}^d, -H < z < \zeta(t, \mathbf{x})\},$$

where $\zeta(t, \mathbf{x})$ describes the location of a free surface, and H is the depth of the layer at rest. The gravity field is assumed to be constant and vertical, and $g > 0$ is the gravity acceleration constant. Finally, $(\mathbf{u}_*, w_*) \in \mathbb{R}^d \times \mathbb{R}$ are correctors of the effective transport velocities that take into account eddy correlations in non-eddy-resolving (large-scale) models. Their specific forms were proposed by Gent & McWilliams [13] and

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read as follows

$$\mathbf{u}_\star = \kappa \partial_z \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right), \quad w_\star = -\kappa \nabla_{\mathbf{x}} \cdot \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right), \quad \kappa > 0. \quad (1.2)$$

Discarding the effective advection terms (i.e. setting $\kappa = 0$), one recovers the Euler equations for heterogeneous incompressible fluids under the influence of vertical gravity forces, where the last two lines of (1.1) model the kinematic equation at the free surface and the impermeability condition of the rigid bottom respectively.

In (1.1), the pressure P can be recovered from its (atmospheric) value at the surface, P_{atm} , by solving the elliptic boundary-value problem induced by the incompressibility constraint of divergence-free velocity fields. Yet in the shallow-water regime, where the horizontal scale of the perturbation is large compared with the depth of the layer H , formal computations (see below) suggest that vertical accelerations can be neglected and that the pressure P approximately satisfies the hydrostatic balance law, that is

$$\partial_z P + g \rho = 0. \quad (1.3)$$

Replacing the equation for the vertical velocity in (1.1) by the identity in (1.3) yields the so-called *hydrostatic equations*:

$$\begin{aligned} \partial_t \rho + \mathbf{u} \cdot \nabla_{\mathbf{x}} \rho + w \partial_z \rho &= \kappa \left[\nabla_{\mathbf{x}} \cdot \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) \partial_z \rho - \partial_z \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) \cdot \nabla_{\mathbf{x}} \rho \right], \\ \rho (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} + w \partial_z \mathbf{u}) + \nabla_{\mathbf{x}} P &= \kappa \rho \left[\nabla_{\mathbf{x}} \cdot \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) \partial_z \mathbf{u} - \partial_z \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) \cdot \nabla_{\mathbf{x}} \mathbf{u} \right], \\ P &= P_{\text{atm}} + g \int_z^\zeta \rho(z', \cdot) dz', \\ w &= - \int_{-H}^z \nabla_{\mathbf{x}} \cdot \mathbf{u}(z', \cdot) dz', \\ \partial_t \zeta + \mathbf{u}|_{z=\zeta} \cdot \nabla_{\mathbf{x}} \zeta - w|_{z=\zeta} &= -\kappa \left[\nabla_{\mathbf{x}} \cdot \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) \Big|_{z=\zeta} + \partial_z \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) \Big|_{z=\zeta} \cdot \nabla_{\mathbf{x}} \zeta \right]. \end{aligned} \quad (1.4)$$

Our aim in this work is to rigorously justify the hydrostatic equations (1.4) as an asymptotic model for the non-hydrostatic equations (1.1)-(1.2) in the shallow-water regime, for regular and stably stratified flows.

Modeling aspects. Let us now discuss the relevance and our motivation behind the introduction of the additional transport velocities \mathbf{u}_\star and w_\star defined in (1.2). While taking into account viscosity effects is standard in mathematical treatments of fluid mechanics, it should be mentioned that the aforementioned shallow-water regime where horizontal scales are larger than vertical scales produces anisotropic viscosity terms which are predominant in the vertical direction. However, it is worth pointing out that in theoretical and laboratory studies on density-stratified geophysical flows, viscosity effects do not model molecular viscosity but rather “turbulent” or “eddy” viscosities, and are widely reported to be anisotropic and only relevant in the horizontal (or more precisely isopycnal) direction; see *e.g.* [17, Section 17.6]. In this work we decide to neglect altogether viscosity effects and rather focus on diffusivity. The deterministic modeling of effective diffusivity induced by eddy correlation that we adopt in this work takes its roots in the 90’s and is due to Gent & McWilliams [13], see also [14, 15]. It adds suitable correctors, specifically suggesting (1.2), to the advective velocity field of the system of inhomogeneous incompressible fluids submitted to gravity as in (1.1). Since mesoscale eddies have an averaged dissipative effect on the large-scale flow at the macroscopic level, then it is natural to consider our unknowns (ρ, \mathbf{u}, w) as the large-scale components of the density and the velocity field respectively, according to [13].

This work is motivated by studying theoretically the interplay between (stable) stratification, shallow water limits, and diffusive effects, and leaves aside other important ingredients which are usually considered in the so-called primitive equations modeling large-scale flows (see *e.g.* [17]): typically the rotational effects, vertical boundaries, bathymetry and several tracers —say salinity and temperature— coupled by an equation of state. Many of these constituents could be easily incorporated in our study at the price of blurring

the main mechanisms at stake, while interesting singular limits (geostrophic balance, boundary layers, *etc.*) would of course deserve a specific treatment.

Let us finally mention that there exists a huge mathematical literature dedicated to the investigation of fluid-dynamics equations in the probabilistic setting, where the cumulative effect of mesoscale eddies on the large-scale flow is modeled by means of suitable (additive or multiplicative) noises. For that context, we refer to [9, 10], while our setting will be completely deterministic.

Previous mathematical results and motivation. At a technical level, the main reason to introduce viscosity or diffusivity contributions in the equations is that, without any of them, the initial-value problem for the hydrostatic equations is not known to be well-posed in finite-regularity functional spaces. In fact, restricting to homogeneous flows (that is ρ being constant), ill-posedness was established by Renardy [30] at the linear level, and Han-Kwan and Nguyen [18] at the nonlinear level. Yet if we additionally assume that the initial data satisfies the Rayleigh condition of (strict) convexity/concavity in the vertical direction, well-posedness is restored [3, 16, 25]. Now, assuming stably stratified flows, the celebrated Miles and Howard criterion [19, 26] states that the linearized equations about equilibria $(\underline{\rho}(z), \underline{\mathbf{u}}'(z))$ do not exhibit unstable modes (in dimension $d = 1$, see [12, Remark 1.3] when $d = 2$) provided that the local Richardson number is greater than $1/4$ everywhere, that is

$$\forall z \in [-H, 0], \quad |\underline{\mathbf{u}}'(z)|^2 \leq 4g \left(\frac{-\underline{\rho}'(z)}{\underline{\rho}(z)} \right).$$

Notice that the stabilizing (resp. destabilizing) effect of the stable stratification (resp. shear velocity) is clearly encoded by the above criterion. However, we underline again that the well-posedness of the (nonlinear) hydrostatic equations for initial data (strictly) satisfying the above inequality is still an open problem.

This is in sharp contrast with the available results on the non-hydrostatic equations. In this context, we mention the recent work by Desjardins, Lannes and Saut [7], which is close to our framework and provides the well-posedness of the (inviscid and non-diffusive) non-hydrostatic equations in Sobolev spaces (using the rigid-lid assumption). Even though the stabilizing effect of the stable stratification is also a key ingredient of that work, it is not powerful enough to prove that the lifespan of the solutions to the non-hydrostatic equations is uniform with respect to the shallow-water parameter measuring the ratio of vertical to horizontal lengths, without additional smallness conditions on the initial data. A more detailed comparison between [7] and our results is provided later on.

From the technical viewpoint, the reason of this discrepancy – in terms of the available results – between the non-hydrostatic and hydrostatic equations is that the vertical velocity variable w changes its role passing from prognostic (when it belongs to the set of unknowns) to diagnostic (when it is reconstructed from the unknowns), whence losing one order of regularity; see the fourth equation in (1.4).

In order to overcome the difficulties related to this loss of derivatives, without restricting the analysis to then analytic setting as already done in [21, 27], one natural approach is the introduction of (regularizing) viscosity contributions. This is the framework of most of the theoretical studies concerning the hydrostatic equations and/or the hydrostatic limit, starting with the work of Azérad and Guillén [1]. A landmark in the theory is the work of Cao and Titi [6] where the global well-posedness of the initial-value problem for the hydrostatic equations was proved in dimension $d + 1 = 3$: this striking result should be compared with the state of the art on the Navier-Stokes equations. Several mathematical studies, where partial viscosities and diffusivities and/or more physically relevant boundary conditions are investigated, were established later on. Rather than providing an extensive bibliography for this huge set of results, we limit ourselves to point out the works [4, 5], which extended the previous results to the case where only horizontal viscosity and diffusivity are added to the equations. We also mention [11, 23, 24] (in the homogeneous case) and [28, 29] (in the heterogeneous case) for recent results on the hydrostatic limit and an extended list of references (therein).

A peculiarity of our analysis with respect to the previous ones (with the exception of [7]) is that we shall crucially *use* the (stable) density stratification assumption, but we completely neglect viscosity-induced regularization and only allow for diffusivity effects. We shall also keep track of all relevant parameters in our estimates, and in particular will use diffusivity-induced regularization only when crucially needed. This allows to characterize the relevant convergence rates and timescales, and to exhibit a balance between the destabilizing effect of shear velocities and the stabilizing result of diffusivity. Moreover, to the best of our knowledge, this is the first rigorous mathematical study where the specific form of the diffusivity contributions, due to Gent and McWilliams [13] and modeling effective diffusivity induced by eddy correlation, is taken into account. It is worth highlighting that the specific form of the effective advection terms in (1.2) stems from *isopycnal diffusivity*: consistently, we will study equations (1.1) and (1.4) in the coordinate system known as *isopycnal coordinates* (hence intrinsically relying on the stable stratification assumption).

In the following paragraph, we rewrite the equations passing to isopycnal coordinates. Our main results are described and discussed thereafter, and proved in the next sections.

The model in isopycnal coordinates and non-dimensionalization. Let us consider smooth solutions to (1.1) defined on a time interval I_t . Assuming that the flow is *stably stratified*, i.e.

$$\inf(-\partial_z \rho) > 0,$$

the density $\rho : z \rightarrow \rho(\cdot, \cdot, z)$ is an invertible function of z . We denote its inverse $\eta : \varrho \rightarrow \eta(\cdot, \cdot, \varrho)$, so that

$$\rho(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)) = \varrho, \quad \eta(t, \mathbf{x}, \rho(t, \mathbf{x}, z)) = z.$$

We also assume that $\rho(t, \mathbf{x}, -H) = \rho_1$, $\rho(t, \mathbf{x}, \zeta(t, \mathbf{x})) = \rho_0$ for $(t, \mathbf{x}) \in I_t \times \mathbb{R}^d$, where $\rho_0 < \rho_1$ are two fixed and positive constant reference densities. Then we have

$$\eta : I_t \times \Omega \rightarrow \mathbb{R} \quad \text{with} \quad \Omega := \mathbb{R}^d \times (\rho_0, \rho_1) \quad \text{and} \quad h := -\partial_\varrho \eta > 0, \quad (1.5)$$

the latter inequality accounting for the stable stratification assumption. We now introduce

$$\tilde{\mathbf{u}}(t, \mathbf{x}, \varrho) = \mathbf{u}(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)), \quad \tilde{w}(t, \mathbf{x}, \varrho) = w(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)), \quad \tilde{P}(t, \mathbf{x}, \varrho) = P(t, \mathbf{x}, \eta(t, \mathbf{x}, \varrho)).$$

From the chain rule, we infer that system (1.1) in isopycnal coordinates reads

$$\begin{aligned} \partial_t \eta + \tilde{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \eta - \tilde{w} &= \kappa \Delta_{\mathbf{x}} \eta, \\ \varrho \left(\partial_t \tilde{\mathbf{u}} + \left((\tilde{\mathbf{u}} - \kappa \frac{\nabla_{\mathbf{x}} h}{h}) \cdot \nabla_{\mathbf{x}} \right) \tilde{\mathbf{u}} \right) + \nabla_{\mathbf{x}} \tilde{P} + \frac{\nabla_{\mathbf{x}} \eta}{h} \partial_\varrho \tilde{P} &= 0, \\ \varrho \left(\partial_t \tilde{w} + \left(\tilde{\mathbf{u}} - \kappa \frac{\nabla_{\mathbf{x}} h}{h} \right) \cdot \nabla_{\mathbf{x}} \tilde{w} \right) - \frac{\partial_\varrho \tilde{P}}{h} + g \varrho &= 0, \\ -h \nabla_{\mathbf{x}} \cdot \tilde{\mathbf{u}} - (\nabla_{\mathbf{x}} \eta) \cdot (\partial_\varrho \tilde{\mathbf{u}}) + \partial_\varrho \tilde{w} &= 0, \\ \tilde{P}|_{\varrho=\rho_0} = P_{\text{atm}}, \quad \tilde{w}|_{\varrho=\rho_1} &= 0. \end{aligned} \quad (1.6)$$

Notice that differentiating with respect to ϱ the first equation and using the fourth equation (stemming from the incompressibility constraint), the mass conservation reads

$$\partial_t h + \nabla_{\mathbf{x}} \cdot (h \tilde{\mathbf{u}}) = \kappa \Delta_{\mathbf{x}} h. \quad (1.7)$$

At this point, we are ready to introduce a dimensionless version of the previous system. We are interested in departures from steady solutions to the incompressible Euler equations with variable density:

$$(h_{\text{eq}}, \mathbf{u}_{\text{eq}}, w_{\text{eq}}, P_{\text{eq}}) = (\underline{h}(\varrho), \underline{\mathbf{u}}(\varrho), 0, \underline{P}(\varrho)),$$

which satisfy the equilibrium condition

$$\partial_\varrho \underline{P}(\varrho) = g \varrho \underline{h}(\varrho).$$

Therefore, we consider (non-necessarily small) fluctuations of that steady solution, so that our unknowns admit the following decomposition:

$$\begin{aligned} h(t, \mathbf{x}, \varrho) &= \underline{h}(\varrho) + h_{\text{pert}}(t, \mathbf{x}, \varrho), & \check{\mathbf{u}}(t, \mathbf{x}, \varrho) &= \underline{\mathbf{u}}(\varrho) + \mathbf{u}_{\text{pert}}(t, \mathbf{x}, \varrho), \\ \check{w}(t, \mathbf{x}, \varrho) &= 0 + w_{\text{pert}}(t, \mathbf{x}, \varrho), & \check{P}(t, \mathbf{x}, \varrho) &= \underline{P}(\varrho) + P_{\text{pert}}(t, \mathbf{x}, \varrho). \end{aligned}$$

Furthermore, we non-dimensionalize the equations through the following scaled variables: we set

$$\frac{h(t, \mathbf{x}, \varrho)}{H} = \tilde{h}(\varrho) + \tilde{h}(\tilde{t}, \tilde{\mathbf{x}}, \varrho) \quad \text{and} \quad \frac{\check{\mathbf{u}}(t, \mathbf{x}, \varrho)}{\sqrt{gH}} = \tilde{\mathbf{u}}(\varrho) + \tilde{\mathbf{u}}(\tilde{t}, \tilde{\mathbf{x}}, \varrho),$$

and ¹

$$\frac{\lambda}{H} \frac{\check{w}(t, \mathbf{x}, \varrho)}{\sqrt{gH}} = \tilde{w}(\tilde{t}, \tilde{\mathbf{x}}, \varrho), \quad \frac{\check{P}(t, \mathbf{x}, \varrho)}{gH} = \frac{P_{\text{atm}}}{gH} + \int_{\rho_0}^{\varrho} \varrho' \tilde{h}(\varrho') d\varrho' + \tilde{P}(\tilde{t}, \tilde{\mathbf{x}}, \varrho),$$

where we use the following scaled coordinates²

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{\lambda} \quad \text{and} \quad \tilde{t} = \frac{\sqrt{gH}}{\lambda} t.$$

Introducing the dimensionless diffusion parameter, $\tilde{\kappa}$ and the shallowness parameter, μ , through

$$\tilde{\kappa} = \frac{\kappa}{\lambda \sqrt{gH}} \quad \text{and} \quad \mu = \frac{H^2}{\lambda^2},$$

substituting the scaled coordinates/variables in system (1.6) and the subsequent equation *and dropping the tildes for the sake of readability* yields

$$\begin{aligned} \partial_t h + \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) &= \kappa \Delta_{\mathbf{x}} h, \\ \varrho \left(\partial_t \mathbf{u} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} \right) + \nabla_{\mathbf{x}} P + \frac{\nabla_{\mathbf{x}} \eta}{\underline{h} + h} (\varrho \underline{h} + \partial_{\varrho} P) &= 0, \\ \mu \varrho \left(\partial_t w + (\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}} w \right) - \frac{\partial_{\varrho} P}{\underline{h} + h} + \frac{\varrho h}{\underline{h} + h} &= 0, \\ -(\underline{h} + h) \nabla_{\mathbf{x}} \cdot \mathbf{u} - \nabla_{\mathbf{x}} \eta \cdot (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) + \partial_{\varrho} w &= 0, \quad (\text{div.-free cond.}) \\ \eta(\cdot, \varrho) = \int_{\varrho}^{\rho_1} h(\cdot, \varrho') d\varrho', \quad P|_{\varrho=\rho_0} = 0, \quad w|_{\varrho=\rho_1} = 0. & \quad (\text{bound. cond.}) \end{aligned} \tag{1.8}$$

The hydrostatic system is obtained by setting $\mu = 0$ in (1.8). Specifically, plugging the hydrostatic balance

$$\frac{\partial_{\varrho} P}{\underline{h} + h} = \frac{\varrho h}{\underline{h} + h} \quad \text{and} \quad P|_{\varrho=\rho_0} = 0$$

into the second equation of (1.8) yields

$$\begin{aligned} \partial_t h + \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) &= \kappa \Delta_{\mathbf{x}} h, \\ \varrho \left(\partial_t \mathbf{u} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} \right) + \nabla_{\mathbf{x}} \psi &= 0, \end{aligned} \tag{1.9a}$$

with

$$\begin{aligned} \psi(t, \mathbf{x}, \varrho) &= \int_{\rho_0}^{\varrho} \varrho' h(t, \mathbf{x}, \varrho') d\varrho' + \varrho \int_{\varrho}^{\rho_1} h(t, \mathbf{x}, \varrho') d\varrho' \\ &= \rho_0 \int_{\rho_0}^{\rho_1} h(t, \mathbf{x}, \varrho') d\varrho' + \int_{\rho_0}^{\varrho} \int_{\varrho'}^{\rho_1} h(t, \mathbf{x}, \varrho'') d\varrho'' d\varrho'. \end{aligned} \tag{1.9b}$$

¹Notice the different scaling between the horizontal and vertical velocity fields. There, λ is a reference horizontal length.

²We could scale also the ϱ -coordinate. Adjusting accordingly the other variables, this amounts to setting $\rho_0 = 1$ (say). In the following we shall not discuss the dependency with respect to ρ_1 , and in particular the physically relevant limit of small density contrast, $\frac{\rho_1 - \rho_0}{\rho_0} \ll 1$; see [8] and references therein.

We shall provide a rigorous proof of the convergence of (smooth) solutions to (1.8) towards (smooth) solutions to (1.9) as $\mu \searrow 0$, under the stable stratification assumption, $\underline{h} + h > 0$.

Our main results. Our main results are stated and commented below. Some notations, and in particular the Sobolev spaces $H^{s,k}(\Omega)$, are introduced right after. First, we prove the existence, uniqueness and control of the solutions to the hydrostatic system (1.9) for sufficiently smooth initial data. Let us point out that the existence time of our solutions encodes the aforementioned stabilizing (resp. destabilizing) effect of the stable stratification (resp. shear velocity).

THEOREM 1.1. Let $s, k \in \mathbb{N}$ be such that $s > 2 + \frac{d}{2}$, $2 \leq k \leq s$, and $\underline{M}, M, h_*, h^* > 0$ and $0 < \rho_0 < \rho_1$ be fixed. Then there exists $C > 0$ such that for any $\kappa \in (0, 1]$, any $\underline{h}, \underline{\mathbf{u}} \in W^{k,\infty}((\rho_0, \rho_1))$ satisfying

$$|\underline{h}|_{W_\varrho^{k,\infty}} + |\underline{\mathbf{u}}'|_{W_\varrho^{k-1,\infty}} \leq \underline{M}$$

and any initial data $(h_0, \mathbf{u}_0) \in H^{s,k}(\Omega)$, with $\eta_0(\cdot, \varrho) = \int_\varrho^{\rho_1} h_0(\cdot, \varrho') d\varrho'$, satisfying the following estimate

$$M_0 := \|\eta_0\|_{H^{s,k}} + \|\mathbf{u}_0\|_{H^{s,k}} + |\eta_0|_{\varrho=\rho_0}|_{H_\varrho^s} + \kappa^{1/2} \|h_0\|_{H^{s,k}} \leq M;$$

and the stable stratification assumption

$$\forall(\mathbf{x}, \varrho) \in \Omega, \quad h_* \leq \underline{h}(\varrho) + h_0(\mathbf{x}, \varrho) \leq h^*,$$

there exists a unique $(h^h, \mathbf{u}^h) \in C^0([0, T]; H^{s,k}(\Omega)^{1+d})$ solution to (1.9) and $(h^h, \mathbf{u}^h)|_{t=0} = (h_0, \mathbf{u}_0)$, where

$$T^{-1} = C(1 + \kappa^{-1}(|\underline{\mathbf{u}}'|_{L_\varrho^2}^2 + M_0^2)). \quad (1.10)$$

Moreover, $h^h \in L^2(0, T; H^{s+1,k}(\Omega))$ and one has, for any $t \in [0, T]$,

$$\forall(\mathbf{x}, \varrho) \in \Omega, \quad h_*/2 \leq \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \leq 2h^*,$$

and, denoting $\eta^h(\cdot, \varrho) = \int_\varrho^{\rho_1} h^h(\cdot, \varrho') d\varrho'$,

$$\begin{aligned} & \|\eta^h(t, \cdot)\|_{H^{s,k}} + \|\mathbf{u}^h(t, \cdot)\|_{H^{s,k}} + |\eta^h|_{\varrho=\rho_0}(t, \cdot)|_{H_\varrho^s} + \kappa^{1/2} \|h^h(t, \cdot)\|_{H^{s,k}} \\ & + \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^h\|_{L^2(0,T;H^{s,k})} + \kappa^{1/2} |\nabla_{\mathbf{x}} \eta^h|_{\varrho=\rho_0}|_{L^2(0,T;H_\varrho^s)} + \kappa \|\nabla_{\mathbf{x}} h^h\|_{L^2(0,T;H^{s,k})} \leq CM_0. \end{aligned}$$

In our second main result, we prove that within the timescale defined by (1.10), there exist solutions to the non-hydrostatic equations (1.8) for μ sufficiently small, and they converge towards the corresponding solutions of the hydrostatic equations, with the expected $\mathcal{O}(\mu)$ convergence rate.

THEOREM 1.2. There exists $p \in \mathbb{N}$ such that for any $k = s \in \mathbb{N}$, $\underline{M}, M, h_*, h^* > 0$ and $0 < \rho_0 < \rho_1$, there exists $C > 0$ such that the following holds. For any $0 < M_0 \leq M$, $0 < \kappa \leq 1$, and $\mu > 0$ such that

$$\mu \leq \kappa/(CM_0^2),$$

for any for any $(\underline{h}, \underline{\mathbf{u}}) \in W^{k+p,\infty}((\rho_0, \rho_1))^2$ satisfying

$$|\underline{h}|_{W_\varrho^{k+p,\infty}} + |\underline{\mathbf{u}}'|_{W_\varrho^{k+p-1,\infty}} \leq \underline{M};$$

for any initial data $(h_0, \mathbf{u}_0, w_0) \in H^{s+p,k+p}(\Omega)^{2+d}$ satisfying the boundary condition $w_0|_{\varrho=\rho_1} = 0$ and the incompressibility condition

$$-(\underline{h} + h_0)\nabla_{\mathbf{x}} \cdot \mathbf{u}_0 - (\nabla_{\mathbf{x}} \eta_0) \cdot (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}_0) + \partial_\varrho w_0 = 0,$$

(denoting $\eta_0(\cdot, \varrho) = \int_\varrho^{\rho_1} h_0(\cdot, \varrho') d\varrho'$), the inequality

$$\|\eta_0\|_{H^{s+p,k+p}} + \|\mathbf{u}_0\|_{H^{s+p,k+p}} + |\eta_0|_{\varrho=\rho_0}|_{H_\varrho^{s+p}} + \kappa^{1/2} \|h_0\|_{H^{s+p,k+p}} = M_0 \leq M$$

and the stable stratification assumption

$$\forall(\mathbf{x}, \varrho) \in \Omega, \quad h_* \leq \underline{h}(\varrho) + h_0(\mathbf{x}, \varrho) \leq h^*,$$

the following holds. Denoting $(h^h, \mathbf{u}^h) \in \mathcal{C}^0([0, T^h]; H^{s+p, k+p}(\Omega)^{1+d})$ the solution to the hydrostatic equations (1.9) with initial data $(h^h, \mathbf{u}^h)|_{t=0} = (h_0, \mathbf{u}_0)$ provided by Theorem 1.1, there exists a unique strong solution $(h^{\text{nh}}, \mathbf{u}^{\text{nh}}, w^{\text{nh}}) \in \mathcal{C}([0, T^h]; H^{s, k}(\Omega)^{2+d})$ to the non-hydrostatic equations (1.8) with initial data $(h^{\text{nh}}, \mathbf{u}^{\text{nh}}, w^{\text{nh}})|_{t=0} = (h_0, \mathbf{u}_0, w_0)$. Moreover, one has

$$\|h^{\text{nh}} - h^h\|_{L^\infty(0, T^h; H^{s, k})} + \|\mathbf{u}^{\text{nh}} - \mathbf{u}^h\|_{L^\infty(0, T^h; H^{s, k})} \leq C\mu.$$

Strategy of the proofs. The proofs of our results rely mainly on the energy method, exhibiting the structure of the systems of equations through well-chosen energy functionals and making use of product, commutator and composition estimates in the L^2 -based Sobolev spaces $H^{s, k}(\Omega)$ (that are summarized in the Appendix).

The natural energy functional associated with the hydrostatic equations, (1.9) involves η , \mathbf{u} as well as $\eta|_{\varrho=\rho_0}$ (that represents the free surface), and their derivatives. A key point is that we do *not* control $h = -\partial_\varrho\eta$ (see 1.5) in the same regularity class of η , unless it is multiplied by the prefactor $\kappa^{1/2}$. We crucially use the diffusivity-induced regularization in order to control terms stemming from the commutator between advection and density integration in the equation of mass conservation, *i.e.* the first equation in (1.9). This explains why the time of existence of our solution in (1.10) vanishes as $\kappa \searrow 0$, yet with a prefactor involving the shear velocity, $\underline{\mathbf{u}}'(\varrho)$ (since advection with a ϱ -independent velocity commutes with density integration). It is interesting to notice that the index of regularity with respect to the space variable, s , and the one with respect to the density variable, k , are decoupled (yet only in the hydrostatic framework). This is due to the fact that the isopycnal change of coordinate is semi-Lagrangian: the advection in isopycnal coordinates occurs only in the horizontal space directions. It would be of utmost interest (but outside of the scope of the present work) to decrease the regularity assumption with respect to the density variable, so as to admit discontinuities, representing density interfaces.

Concerning the non-hydrostatic system, (1.8), the natural energy space involves additionally $\sqrt{\mu}w$ and its derivatives (hence the control vanishes as $\mu \searrow 0$). In order to obtain suitable energy estimates, we decompose the pressure as the sum of the hydrostatic contribution and the non-hydrostatic contribution, the latter being of lower order in terms of regularity and/or smallness with respect to $\mu \ll 1$. Then we use the structure of the hydrostatic equations, which we complement with an additional symmetric structure for the non-hydrostatic contributions. There, the difficulty consists in providing controls of the energy norms that are uniform with respect to the vanishing parameter $\mu \ll 1$. Our estimates concerning the non-hydrostatic contribution of the pressure stem from elliptic estimates on a boundary-value problem. This strategy is heavily inspired by the work of Desjardins, Lannes and Saut, and it is interesting to compare our results with the analogous “large-time” well-posedness result in [7, Theorem 2]. Firstly, due to the choice of isopycnal coordinates, our boundary-value problem is no longer an anisotropic Poisson equation but involves a fully nonlinear elliptic operator. Since this operator involves h that is not controlled in the energy space, we use again the diffusivity-induced regularization at this stage. On the plus side, using isopycnal coordinates rather than Eulerian coordinates allows us to consider the free-surface framework (since isopycnal coordinates readily set the domain as a flat strip, thanks to our assumption that the density is constant at the surface and at the bottom) rather than the rigid-lid setting. We believe that our study can be extended to the rigid-lid framework with small adjustments. Incidentally, we do not employ the often-used Boussinesq approximation, since it is not useful in our context. Additionally, we do not rely on the use of strong boundary conditions on the initial density and velocities and their derivatives at the surface and the bottom, which instead are assumed in [7] (and in most of the other works, often put in a periodic framework) and rather use only the natural no-slip boundary condition at the bottom; the former allow to cancel the trace contributions resulting from vertical integration by parts. We also consider the general situation where the velocity field is a perturbation of a non-zero background current, $\underline{\mathbf{u}}$. In turn, the price to pay to handle this general framework manifests in terms of some restrictions on the length of the time of existence of our solutions, which is inversely proportional with respect to the size of the fluctuations in [7, Theorem 2].

Our strategy for the proof of the convergence is as follows. Now we describe our strategy to prove the convergence of the solutions to the non-hydrostatic system towards the hydrostatic one. First, we point out

that a direct use of energy estimates as previously allows to obtain the existence of solutions of the non-hydrostatic equations in a timescale uniform with respect to μ but not necessarily the same as the existence time of the corresponding solution to the hydrostatic equations, and (for technical reasons) restricted to sufficiently small data. To overcome these issues, we look at the hydrostatic solution as an approximate solution to the non-hydrostatic system (in the sense of consistency, as it approximately solves the non-hydrostatic equations), and deduce, using the aforementioned structure of the non-hydrostatic equations, an energy inequality that controls the difference between the solution to the non-hydrostatic system and the (respective) solution to the hydrostatic one. This strategy allows to bootstrap the control of the difference of the two solutions (and hence the control of the solution to the non-hydrostatic equations) within the timescale of existence (in a higher-regularity space) of the hydrostatic solution, provided that the parameter μ is sufficiently small. However, the rate of convergence obtained by this method is not optimal. Therefore, in a second step we implement another strategy to obtain the expected (optimal) convergence rate. It simply consists in taking the opposite viewpoint: to look at the solution to the non-hydrostatic equations as an approximate solution to the hydrostatic equations (again in the sense of consistency) and use the structure of the hydrostatic equations to infer the $\mathcal{O}(\mu)$ convergence rate. Both steps involve loss of derivatives, described by the parameter p in Theorem 1.2.

Plan of the paper. Section 2 is dedicated to the proof of Theorem 1.1 concerning the initial-value problem for the hydrostatic equations, (1.9).

In Section 3, we analyze the non-hydrostatic equations, (1.8). We first provide elliptic estimates for the boundary-value problem of the pressure reconstruction (Lemma 3.1 and Corollary 3.2), and use them to infer two partial results concerning the initial-value problem: Proposition 3.3 (restricted to small time) and Proposition 3.8 (restricted to small data).

In Section 4, we show the convergence of solutions to the non-hydrostatic equations towards corresponding solutions to the hydrostatic equations as $\mu \searrow 0$, concluding the proof of Theorem 1.2.

Finally, in Appendix A we provide product, commutator and composition estimates in the Sobolev spaces $H^{s,k}(\Omega)$.

Notation and conventions. We highlight the following conventions used throughout the paper.

- ρ_0 and ρ_1 are fixed constants such that $0 < \rho_0 < \rho_1$, and the dependency with respect to these constants is never explicitly displayed.
- For $k, s \in \mathbb{N}$ and $k \leq s$, and $\Omega = \mathbb{R}^d \times (\rho_0, \rho_1)$, we define the functional space

$$H^{s,k}(\Omega) = \left\{ f : \forall (\alpha, j) \in \mathbb{N}^{d+1}, |\alpha| + j \leq s, j \leq k, \partial_{\mathbf{x}}^{\alpha} \partial_{\rho}^j f \in L^2(\Omega) \right\}, \quad (1.11)$$

endowed with the topology of the norm

$$\|f\|_{H^{s,k}}^2 := \sum_{j=0}^k \sum_{|\alpha|=0}^{s-j} \|\partial_{\mathbf{x}}^{\alpha} \partial_{\rho}^j f\|_{L^2(\Omega)}^2. \quad (1.12)$$

When $s' \in \mathbb{R}$ (and $k \in \mathbb{N}$) we define $H^{s',k}(\Omega) = \left\{ f : \forall j \in \mathbb{N}, j \leq k, \Lambda^s \partial_{\rho}^j f \in L^2(\Omega) \right\}$ and

$$\|f\|_{H^{s',k}}^2 := \sum_{j=0}^k \|\Lambda^{s'-j} \partial_{\rho}^j f\|_{L^2(\Omega)}^2.$$

where $\Lambda = (\text{Id} - \Delta_{\mathbf{x}})^{1/2}$. Of course the two notations are consistent when $s' = s \in \mathbb{N}$, up to harmless factors in the definition of the norm.

- We use both the equivalent notations $H^s(\mathbb{R}^d) = H_{\mathbf{x}}^s$ (the usual L^2 -based Sobolev space on \mathbb{R}^d) and $W^{k,\infty}(\mathbb{R}^d) = W_{\mathbf{x}}^{k,\infty}$ (the L^{∞} -based Sobolev space on \mathbb{R}^d), and similarly $L^2((\rho_0, \rho_1)) = L_{\rho}^2$ and $W^{k,\infty}((\rho_0, \rho_1)) = W_{\rho}^{k,\infty}$. For functions with variables in Ω we denote for instance

$$L_{\rho}^2 L_{\mathbf{x}}^{\infty} = L^2(\rho_0, \rho_1; L^{\infty}(\mathbb{R}^d)) = \{f : \text{ess sup}_{\mathbf{x} \in \mathbb{R}^d} |f(\cdot, \mathbf{x})| \in L^2((\rho_0, \rho_1))\}.$$

Notice $L_\varrho^2 L_x^2 = L_x^2 L_\varrho^2 = L^2(\Omega)$ and $L_\varrho^\infty L_x^\infty = L_x^\infty L_\varrho^\infty = L^\infty(\Omega)$. We use similar notations for functions also depending on time. For instance, for $k \in \mathbb{N}$, and X a Banach space as above, $C^k([0, T]; X)$ is the space of functions with values in X which are continuously differentiable up to order k , and $L^p(0, T; X)$ the p -integrable X -valued functions. All these spaces are endowed with their natural norms.

- For any operator $A : f \rightarrow Af$, we denote by $[A, f]g = A(fg) - f(Ag)$ the usual commutator, while $[A; f, g] = A(fg) - f(Ag) - g(Af)$ is the symmetric commutator,
- $C(\lambda_1, \lambda_2, \dots)$ denotes a constant which depends continuously on its parameters.
- For any $a, b \in \mathbb{R}$, we use the notation $a \lesssim b$ (resp. $a \gtrsim b$) if there exists $C > 0$, independent of relevant parameters, such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \approx b$ if $a \lesssim b$ and $a \gtrsim b$.
- We put $a \vee b := \max(a, b)$. Finally,

$$\langle B_a \rangle_{a>b} = \begin{cases} 0 & \text{if } a \leq b, \\ B_a & \text{otherwise,} \end{cases} \quad \text{and} \quad \langle B_a \rangle_{a=b} = \begin{cases} 0 & \text{if } a \neq b, \\ B_a & \text{otherwise.} \end{cases}$$

2. The hydrostatic system

In this section we study the hydrostatic system in isopycnal coordinates. Specifically, we provide in this section a well-posedness result on the initial-value problem, namely Theorem 1.1. The result follows from careful *a priori* energy estimates, and the standard method of parabolic regularization. Therefore we will first study the system

$$\begin{aligned} \partial_t h + \nabla_x \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) &= \kappa \Delta_x h, \\ \partial_t \mathbf{u} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_x h}{\underline{h} + h}) \cdot \nabla_x) \mathbf{u} + \frac{1}{\varrho} \nabla_x \psi &= \nu \Delta_x \mathbf{u}, \end{aligned} \tag{2.1}$$

with

$$\nabla_x \psi(t, \mathbf{x}, \varrho) := \rho_0 \int_{\rho_0}^{\rho_1} \nabla_x h(t, \mathbf{x}, \varrho') d\varrho' + \int_{\rho_0}^{\varrho} \int_{\rho'}^{\rho_1} \nabla_x h(t, \mathbf{x}, \varrho'') d\varrho'' d\varrho',$$

and $\nu > 0$, and will rigorously establish the limit $\nu \rightarrow 0$.

2.1. Well-posedness of the regularized hydrostatic system. We start with proving the well-posedness of the initial value problem.

PROPOSITION 2.1. Let $s > \frac{3}{2} + \frac{d}{2}$, $k \in \mathbb{N}$ with $1 \leq k \leq s$, and $\underline{M}, M_0, h_\star, \nu, \kappa > 0$ and $C_0 > 1$. Then there exists $T = T(s, k, \underline{M}, M_0, h_\star, \nu, \kappa, C_0)$ such that for any $(\underline{h}, \underline{\mathbf{u}}) = (\underline{h}(\varrho), \underline{\mathbf{u}}(\varrho)) \in W^{k, \infty}((\rho_0, \rho_1))$ and for any $(h_0, \mathbf{u}_0) = (h_0(\mathbf{x}, \varrho), \mathbf{u}_0(\mathbf{x}, \varrho)) \in H^{s, k}(\Omega)$ such that

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} (\underline{h}(\varrho) + h_0(\mathbf{x}, \varrho)) \geq h_\star, \quad \|(\underline{h}, \underline{\mathbf{u}})\|_{W_\varrho^{k, \infty}} \leq \underline{M}, \quad \|(h_0, \mathbf{u}_0)\|_{H^{s, k}} \leq M_0,$$

there exists a unique solution $(h, \mathbf{u}) \in \mathcal{C}^0([0, T]; H^{s, k}(\Omega))$ to system (2.1) with $(h, \mathbf{u})|_{t=0} = (h_0, \mathbf{u}_0)$. Moreover, $(h, \mathbf{u}) \in L^2(0, T; H^{s+1, k}(\Omega))$ and, for a universal constant $c_0 > 0$, the following estimates hold

$$\begin{cases} \|h\|_{L^\infty(0, T; H^{s, k})} + c_0 \kappa^{1/2} \|\nabla_x h\|_{L^2(0, T; H^{s, k})} < C_0 \|h_0\|_{H^{s, k}}; \\ \|\mathbf{u}\|_{L^\infty(0, T; H^{s, k})} + c_0 \nu^{1/2} \|\nabla_x \mathbf{u}\|_{L^2(0, T; H^{s, k})} < C_0 \|\mathbf{u}_0\|_{H^{s, k}}; \\ \inf_{(t, \mathbf{x}, \varrho) \in (0, T) \times \Omega} (\underline{h}(\varrho) + h(t, \mathbf{x}, \varrho)) > h_\star / C_0. \end{cases} \tag{2.2}$$

PROOF. We will construct the solution as the fixed point of the Duhamel formula

$$\begin{aligned} h(t, \cdot) &= e^{\kappa t \Delta_x} h_0 + \int_0^t e^{\kappa(t-\tau) \Delta_x} f(h(\tau, \cdot), \mathbf{u}(\tau, \cdot)) d\tau, \\ \mathbf{u}(t, \cdot) &= e^{\nu t \Delta_x} \mathbf{u}_0 + \int_0^t e^{\nu(t-\tau) \Delta_x} (\mathbf{f}_1 + \mathbf{f}_2)(h(\tau, \cdot), \mathbf{u}(\tau, \cdot)) d\tau \end{aligned}$$

where $e^{\alpha t \Delta_{\mathbf{x}}}$ with $\alpha > 0$ is the heat semigroup defined by $\mathcal{F}[e^{\alpha t \Delta_{\mathbf{x}}} f](\boldsymbol{\xi}) = e^{-\alpha t |\boldsymbol{\xi}|^2} \mathcal{F}[f](\boldsymbol{\xi})$ where \mathcal{F} is the Fourier transform with respect to the variable \mathbf{x} , and

$$\begin{aligned} f(h, \mathbf{u}) &= -\nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})), \\ \mathbf{f}_1(h, \mathbf{u}) &= -((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} \\ \mathbf{f}_2(h, \mathbf{u}) &= -\frac{1}{\varrho} \left(\rho_0 \int_{\rho_0}^{\rho_1} \nabla_{\mathbf{x}} h(\cdot, \varrho') \, d\varrho' + \int_{\rho_0}^{\varrho} \int_{\rho_0}^{\rho_1} \nabla_{\mathbf{x}} h(\cdot, \varrho'') \, d\varrho'' \, d\varrho' \right). \end{aligned}$$

Let us first recall the standard regularization properties of the heat flow. For any $\nu > 0$, $T > 0$ and for any $u_0 \in H^{s,k}(\Omega)$ and $g \in L^1(0, T; H^{s,k}(\Omega))$, there exists a unique $u \in C^0([0, T]; H^{s,k}(\Omega)) \cap L^2(0, T; H^{s+1,k}(\Omega))$ solution to $\partial_t u - \nu \Delta_{\mathbf{x}} u = g$ with $u(0, \cdot) = u_0$ which reads by definition

$$u = e^{\nu t \Delta_{\mathbf{x}}} u_0 + \int_0^t e^{\nu(t-\tau) \Delta_{\mathbf{x}}} g(\tau, \cdot) \, d\tau,$$

and we have

$$\|u\|_{L^\infty(0, T; H^{s,k})} + c_0 \nu^{1/2} \|\nabla_{\mathbf{x}} u\|_{L^2(0, T; H^{s,k})} \leq \|u_0\|_{H^{s,k}} + \|g\|_{L^1(0, T; H^{s,k})},$$

where $c_0 > 0$ is a universal constant. The existence and uniqueness of the solution as well as the above estimate easily follow from solving the equation (for almost every $\varrho \in (\rho_0, \rho_1)$) in Fourier space and using Plancherel's formula, then using that ∂_{ϱ} commutes with ∂_t and $\Delta_{\mathbf{x}}$, and invoking Minkowski's integral inequality (resp. Fubini's theorem) to exchange the order of integration in the variables (t, ϱ) (resp. (\mathbf{x}, ϱ)). We also remark that, by the positivity of the heat kernel and the continuous embedding $H^{s-1,1}(\Omega) \subset L^\infty(\Omega)$ for $s > \frac{3}{2} + \frac{d}{2}$ (see Lemma A.1),

$$\inf_{\Omega} u \geq \inf_{\Omega} u_0 - \|g\|_{L^1(0, T; H^{s-1,1})}.$$

Now we consider $(f, \mathbf{f}_1 + \mathbf{f}_2)$ as a bounded operator from $H^{s+1,k}(\Omega)^{1+d}$ to $H^{s,k}(\Omega)^{1+d}$. Indeed, there exists $C_{s,k} > 0$ such that for any $(h, \mathbf{u}) \in H^{s+1,k}(\Omega)^{1+d}$,

$$\begin{aligned} \|f(h, \mathbf{u})\|_{H^{s,k}} &\leq \|\nabla_{\mathbf{x}} \cdot (\underline{h}\mathbf{u} + h\underline{\mathbf{u}} + h\mathbf{u})\|_{H^{s,k}} \\ &\leq C_{s,k} \times \left(|\underline{h}|_{W_{\varrho}^{k,\infty}} \|\mathbf{u}\|_{H^{s+1,k}} + |\mathbf{u}|_{W_{\varrho}^{k,\infty}} \|\underline{h}\|_{H^{s+1,k}} + \|\underline{h}\|_{H^{s,k}} \|\mathbf{u}\|_{H^{s+1,k}} + \|\underline{h}\|_{H^{s+1,k}} \|\mathbf{u}\|_{H^{s,k}} \right), \end{aligned}$$

where we used straightforward product estimates for the first two terms, and Lemma A.3 for the last ones. Similarly, we have

$$\begin{aligned} \|\mathbf{f}_1(h, \mathbf{u})\|_{H^{s,k}} &\leq \|((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}\|_{H^{s,k}} \\ &\leq C_{s,k} \left(|\mathbf{u}|_{W_{\varrho}^{k,\infty}} + \|\mathbf{u}\|_{H^{s,k}} \right) \|\mathbf{u}\|_{H^{s+1,k}} \\ &\quad + \kappa C_{s,k} \left(|\underline{h}^{-1}|_{W_{\varrho}^{k,\infty}} + \|\frac{h}{\underline{h}(\underline{h} + h)}\|_{H^{s,k}} \right) \|(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \mathbf{u}\|_{H^{s,k}}. \end{aligned}$$

Using the constraint $\inf_{(\rho_0, \rho_1)} \underline{h} \geq \inf_{\Omega} (\underline{h} + h_0) \geq h_{\star} > 0$ and Lemma A.6, we find that for any $h_{\star} > 0$ and $M_0, \underline{M} \geq 0$ there exists $C_{s,k}(h_{\star}, \underline{M}, M_0, C_0)$ such that for any $h \in H^{s,k}(\Omega)$ bounded by $\|h\|_{H^{s,k}} \leq C_0 M_0$ and satisfying $\inf_{(\mathbf{x}, \varrho) \in \Omega} (\underline{h}(\varrho) + h(\mathbf{x}, \varrho)) \geq h_{\star}/C_0$, one has

$$|\underline{h}^{-1}|_{W_{\varrho}^{k,\infty}} + \|\frac{h}{\underline{h}(\underline{h} + h)}\|_{H^{s,k}} \leq C_{s,k}(h_{\star}, \underline{M}, M_0, C_0).$$

Using the last estimates in Lemma A.3, since $s > \frac{3}{2} + \frac{d}{2}$, we have

$$\|(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \mathbf{u}\|_{H^{s,k}} \leq \|h\|_{H^{s,k}} \|\mathbf{u}\|_{H^{s+1,k}} + \|h\|_{H^{s+1,k}} \|\mathbf{u}\|_{H^{s,k}}.$$

Finally, from the continuous embedding $L^\infty((\rho_0, \rho_1)) \subset L^2((\rho_0, \rho_1)) \subset L^1((\rho_0, \rho_1))$ we immediately infer

$$\|\mathbf{f}_2(h, \mathbf{u})\|_{H^{s,k}} \leq C_{s,k} \|h\|_{H^{s+1,k}}.$$

Altogether, we find that for any $h_*, C_0 > 0$ and $\underline{M}, M_0 \geq 0$ there exists $C_{s,k}(h_*, \underline{M}, M_0, C_0)$ such that for any $(h, \mathbf{u}) \in H^{s+1,k}(\Omega)^{1+d}$ satisfying $\|(h, \mathbf{u})\|_{H^{s,k}} \leq C_0 M_0$ and $\inf_{(\mathbf{x}, \varrho) \in \Omega} (\underline{h}(\varrho) + h(\mathbf{x}, \varrho)) \geq h_*/C_0$, we have

$$\|(f(h, \mathbf{u}), \mathbf{f}_1(h, \mathbf{u}), \mathbf{f}_2(h, \mathbf{u}))\|_{H^{s,k}} \leq C_{s,k}(h_*, \underline{M}, M_0, C_0) (1 + \kappa) \|(h, \mathbf{u})\|_{H^{s+1,k}}.$$

By similar considerations, we find that for any $h_*, C_0 > 0$ and $\underline{M}, M_0 \geq 0$ there exists $C_{s,k}(h_*, \underline{M}, M_0, C_0)$ such that for any $(h_1, \mathbf{u}_1, h_2, \mathbf{u}_2) \in H^{s+1,k}(\Omega)^{2(1+d)}$ satisfying the bound $\|(h_i, \mathbf{u}_i)\|_{H^{s,k}} \leq C_0 M_0$ as well as $\inf_{(\mathbf{x}, \varrho) \in \Omega} (\underline{h}(\varrho) + h_i(\mathbf{x}, \varrho)) \geq h_*/C_0$ (with $i \in \{1, 2\}$), one has

$$\begin{aligned} & \|(f(h_2, \mathbf{u}_2) - f(h_1, \mathbf{u}_1), \mathbf{f}_1(h_2, \mathbf{u}_2) - \mathbf{f}_1(h_1, \mathbf{u}_1), \mathbf{f}_2(h_2, \mathbf{u}_2) - \mathbf{f}_2(h_1, \mathbf{u}_1))\|_{H^{s,k}} \\ & \leq C_{s,k}(h_*, \underline{M}, M_0, C_0) (1 + \kappa) \left(\|(h_2 - h_1, \mathbf{u}_2 - \mathbf{u}_1)\|_{H^{s+1,k}} \right. \\ & \quad \left. + \|(h_1, h_2, \mathbf{u}_1, \mathbf{u}_2)\|_{H^{s+1,k}} \|(h_2 - h_1, \mathbf{u}_2 - \mathbf{u}_1)\|_{H^{s,k}} \right). \end{aligned}$$

From the above estimates, we easily infer that for $T > 0$ sufficiently small (uniquely depending on $s, k, M_0, \underline{M}, h_*, \nu, \kappa, C_0$),

$$\mathcal{T} : \begin{pmatrix} h \\ \mathbf{u} \end{pmatrix} \mapsto \begin{pmatrix} e^{\kappa t \Delta_{\mathbf{x}}} h_0 + \int_0^t e^{\kappa(t-\tau) \Delta_{\mathbf{x}}} f(h(\tau, \cdot), \mathbf{u}(\tau, \cdot)) \, d\tau \\ e^{\nu t \Delta_{\mathbf{x}}} \mathbf{u}_0 + \int_0^t e^{\nu(t-\tau) \Delta_{\mathbf{x}}} (\mathbf{f}_1 + \mathbf{f}_2)(h(\tau, \cdot), \mathbf{u}(\tau, \cdot)) \, d\tau \end{pmatrix}$$

is a contraction mapping on

$$X = \left\{ (h, \mathbf{u}) \in \mathcal{C}^0([0, T]; H^{s,k}(\Omega)) \cap L^2(0, T; H^{s+1,k}(\Omega)) : (2.2) \text{ holds} \right\}.$$

The Banach fixed point theorem provides the existence and uniqueness of a fixed point (and hence solution to (2.1)) in X , and uniqueness in $\mathcal{C}^0([0, T]; H^{s,k}(\Omega))$ is easily checked (for instance by the energy method). \square

REMARK 2.2. It should be emphasized that the time of existence provided by Proposition 2.1 is not uniform with respect to the parameters $\kappa, \nu > 0$. More precisely, the proof provides a lower bound as

$$T \gtrsim \min(\{1, \kappa, \nu\}), \quad \text{i.e.} \quad T^{-1} \lesssim 1 + \kappa^{-1} + \nu^{-1}.$$

2.2. Quasilinearization. In the result below, we apply spatial derivatives to system (2.1) and rewrite it in such a way that the linearized equations satisfied by the highest-order terms exhibit a skew-symmetric structure, which will allow us to obtain improved energy estimates in the subsequent section.

LEMMA 2.3. Let $s, k \in \mathbb{N}$ such that $s > 2 + \frac{d}{2}$ and $2 \leq k \leq s$, and $\underline{M}, M, h_* > 0$. Then there exists $C = C(s, k, \underline{M}, M, h_*) > 0$ such that for any $\kappa \in [0, 1], \nu \geq 0$, for any $(\underline{h}, \underline{\mathbf{u}}) \in W^{k,\infty}((\rho_0, \rho_1))$ such that

$$|\underline{h}|_{W_{\varrho}^{k,\infty}} + |\underline{\mathbf{u}}'|_{W_{\varrho}^{k-1,\infty}} \leq \underline{M};$$

and any $(h, \mathbf{u}) \in L^\infty(0, T; H^{s,k}(\Omega))$ solution to (2.1) with some $T > 0$ and satisfying for almost every $t \in [0, T]$

$$\|h(t, \cdot)\|_{H^{s-1,k-1}} + \|\eta(t, \cdot)\|_{H^{s,k}} + \|\mathbf{u}(t, \cdot)\|_{H^{s,k}} + |\eta(t, \cdot)|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^s} + \kappa^{1/2} \|h(t, \cdot)\|_{H^{s,k}} \leq M$$

(where $\eta(t, \mathbf{x}, \varrho) := \int_{\varrho}^{\rho_1} h(t, \mathbf{x}, \varrho') \, d\varrho'$) and

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \geq h_*,$$

the following holds. Denote, for any multi-index $\alpha \in \mathbb{N}^d$, $\eta^{(\alpha)} = \partial_{\mathbf{x}}^\alpha \eta$, $\mathbf{u}^{(\alpha)} = \partial_{\mathbf{x}}^\alpha \mathbf{u}$.

- For any $\alpha \in \mathbb{N}^d$ with $0 \leq |\alpha| \leq s$, we have that

$$\begin{aligned} \partial_t \eta^{(\alpha)} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha)} + \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha)} d\varrho' \\ + \int_{\varrho}^{\rho_1} (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(\alpha)} d\varrho' = \kappa \Delta_{\mathbf{x}} \eta^{(\alpha)} + R_{\alpha,0}, \end{aligned} \quad (2.3a)$$

$$\begin{aligned} \partial_t \mathbf{u}^{(\alpha)} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{h+h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}^{(\alpha)} \\ + \frac{\rho_0}{\varrho} \nabla_{\mathbf{x}} \eta^{(\alpha)} \Big|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta^{(\alpha)} d\varrho' = \nu \Delta_{\mathbf{x}} \mathbf{u}^{(\alpha)} + \mathbf{R}_{\alpha,0}, \end{aligned}$$

where for almost every $t \in [0, T]$, $(R_{\alpha,0}(t, \cdot), \mathbf{R}_{\alpha,0}(t, \cdot)) \in \mathcal{C}^0([\rho_0, \rho_1]; L^2(\mathbb{R}^d)) \times L^2(\Omega)^d$ and

$$\|R_{\alpha,0}\|_{L^2(\Omega)} + \|\mathbf{R}_{\alpha,0}\|_{L^2(\Omega)} + |R_{\alpha,0}|_{\varrho=\rho_0}|_{L^2_{\mathbf{x}}} \leq C M (1 + \kappa \|\nabla_{\mathbf{x}} h\|_{H^{s,k}}). \quad (2.3b)$$

- For any $j \in \mathbb{N}$, $1 \leq j \leq k$ and any $\alpha \in \mathbb{N}^d$, $0 \leq |\alpha| \leq s - j$, it holds

$$\begin{aligned} \partial_t \partial_{\varrho}^j \eta^{(\alpha)} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \partial_{\varrho}^j \eta^{(\alpha)} = \kappa \Delta_{\mathbf{x}} \partial_{\varrho}^j \eta^{(\alpha)} + R_{\alpha,j}, \\ \partial_t \partial_{\varrho}^j \mathbf{u}^{(\alpha)} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{h+h}) \cdot \nabla_{\mathbf{x}}) \partial_{\varrho}^j \mathbf{u}^{(\alpha)} = \nu \Delta_{\mathbf{x}} \partial_{\varrho}^j \mathbf{u}^{(\alpha)} + \mathbf{R}_{\alpha,j}, \end{aligned} \quad (2.4a)$$

where for almost every $t \in [0, T]$, $(R_{\alpha,j}(t, \cdot), \mathbf{R}_{\alpha,j}(t, \cdot)) \in L^2(\Omega) \times L^2(\Omega)^d$ and

$$\|R_{\alpha,j}\|_{L^2(\Omega)} + \|\mathbf{R}_{\alpha,j}\|_{L^2(\Omega)} \leq C M (1 + \kappa \|\nabla_{\mathbf{x}} h\|_{H^{s,k}}). \quad (2.4b)$$

- For any $j \in \mathbb{N}$, $0 \leq j \leq k$ and any multi-index $\alpha \in \mathbb{N}^d$, $0 \leq |\alpha| \leq s - j$, it holds

$$\partial_t \partial_{\varrho}^j h^{(\alpha)} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \partial_{\varrho}^j h^{(\alpha)} = \kappa \Delta_{\mathbf{x}} \partial_{\varrho}^j h^{(\alpha)} + r_{\alpha,j} + \nabla_{\mathbf{x}} \cdot \mathbf{r}_{\alpha,j}, \quad (2.5a)$$

where for almost every $t \in [0, T]$, $(r_{\alpha,j}(t, \cdot), \mathbf{r}_{\alpha,j}(t, \cdot)) \in L^2(\Omega)^{1+d}$ and

$$\kappa^{1/2} \|r_{\alpha,j}\|_{L^2(\Omega)} + \|\mathbf{r}_{\alpha,j}\|_{L^2(\Omega)} \leq C M. \quad (2.5b)$$

PROOF. In this proof, we denote $s_0 = s - 2 > \frac{d}{2}$.

Estimate of $R_{\alpha,0}$. First we notice the identity by integration by parts in ϱ ,

$$(\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha)} + \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha)} d\varrho' = \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} h^{(\alpha)} d\varrho'.$$

Hence, recalling the notation $[P; u, v] = P(uv) - u(Pv) - v(Pu)$ and integrating by parts in ϱ , we get

$$\begin{aligned} R_{\alpha,0} &:= - \int_{\varrho}^{\rho_1} [\partial_{\mathbf{x}}^{\alpha}, \underline{\mathbf{u}}] \cdot \nabla_{\mathbf{x}} h + [\partial_{\mathbf{x}}^{\alpha}, h] \nabla_{\mathbf{x}} \cdot \underline{\mathbf{u}} d\varrho' \\ &= - \int_{\varrho}^{\rho_1} [\partial_{\mathbf{x}}^{\alpha}; \underline{\mathbf{u}}, \nabla_{\mathbf{x}} h] + \underline{\mathbf{u}}^{(\alpha)} \cdot \nabla_{\mathbf{x}} h + [\partial_{\mathbf{x}}^{\alpha}; h, \nabla_{\mathbf{x}} \cdot \underline{\mathbf{u}}] + h^{(\alpha)} (\nabla_{\mathbf{x}} \cdot \underline{\mathbf{u}}) d\varrho' \\ &= - [\partial_{\mathbf{x}}^{\alpha}; \underline{\mathbf{u}}, \nabla_{\mathbf{x}} \eta] - \eta^{(\alpha)} \nabla_{\mathbf{x}} \cdot \underline{\mathbf{u}} \\ &\quad - \int_{\varrho}^{\rho_1} [\partial_{\mathbf{x}}^{\alpha}; \partial_{\varrho} \mathbf{u}, \nabla_{\mathbf{x}} \eta] + \mathbf{u}^{(\alpha)} \cdot \nabla_{\mathbf{x}} h + [\partial_{\mathbf{x}}^{\alpha}; h, \nabla_{\mathbf{x}} \cdot \mathbf{u}] + \eta^{(\alpha)} \nabla_{\mathbf{x}} \cdot \partial_{\varrho} \mathbf{u} d\varrho'. \end{aligned}$$

By the standard Sobolev embedding $H^{s_0}(\mathbb{R}^d) \subset L^{\infty}(\mathbb{R}^d)$ and Lemma A.1, one gets

$$\|\eta^{(\alpha)} \nabla_{\mathbf{x}} \cdot \underline{\mathbf{u}}\|_{L^2(\Omega)} \leq \|\eta^{(\alpha)}\|_{L^2(\Omega)} \|\nabla_{\mathbf{x}} \cdot \underline{\mathbf{u}}\|_{L^{\infty}(\Omega)} \lesssim \|\eta\|_{H^{s,0}} \|\underline{\mathbf{u}}\|_{H^{s_0 + \frac{3}{2}, 1}}.$$

and

$$|(\eta^{(\alpha)} \nabla_{\mathbf{x}} \cdot \underline{\mathbf{u}})|_{\varrho=\rho_0}|_{L^2_{\mathbf{x}}} \lesssim |\eta|_{\varrho=\rho_0}|_{H^s_{\mathbf{x}}} \|\underline{\mathbf{u}}\|_{H^{s_0 + \frac{3}{2}, 1}}.$$

By Lemma A.7(3), and Lemma A.1, we have

$$\begin{aligned} \|[\partial_{\mathbf{x}}^{\alpha}; \mathbf{u}, \nabla_{\mathbf{x}} \eta]\|_{L^2(\Omega)} &\lesssim \|\mathbf{u}\|_{L_{\varrho}^{\infty} H_{\mathbf{x}}^{s-1}} \|\nabla_{\mathbf{x}} \eta\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s_0+1}} + \|\mathbf{u}\|_{L_{\varrho}^{\infty} H_{\mathbf{x}}^{s_0+1}} \|\nabla_{\mathbf{x}} \eta\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-1}} \\ &\lesssim \|\mathbf{u}\|_{H^{s-\frac{1}{2},1}} \|\eta\|_{H^{s_0+2,0}} + \|\mathbf{u}\|_{H^{s_0+\frac{3}{2},1}} \|\eta\|_{H^{s,0}}, \\ \|[\partial_{\mathbf{x}}^{\alpha}; \mathbf{u}|_{\varrho=\rho_0}, \nabla_{\mathbf{x}} \eta|_{\varrho=\rho_0}]\|_{L_{\mathbf{x}}^2} &\lesssim \|\mathbf{u}|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^{s-1}} \|\nabla_{\mathbf{x}} \eta|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^{s_0+1}} + \|\mathbf{u}|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^{s_0+1}} \|\nabla_{\mathbf{x}} \eta|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^{s-1}} \\ &\lesssim \|\mathbf{u}\|_{H^{s-\frac{1}{2},1}} \|\eta|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^{s_0+2}} + \|\mathbf{u}\|_{H^{s_0+\frac{3}{2},1}} \|\eta|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^s}, \end{aligned}$$

and using additionally the Cauchy-Schwarz inequality,

$$\begin{aligned} \|[\partial_{\mathbf{x}}^{\alpha}; \partial_{\varrho} \mathbf{u}, \nabla_{\mathbf{x}} \eta]\|_{L_{\varrho}^1 L_{\mathbf{x}}^2} &\lesssim \|\partial_{\varrho} \mathbf{u}\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-1}} \|\nabla_{\mathbf{x}} \eta\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s_0+1}} + \|\partial_{\varrho} \mathbf{u}\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s_0+1}} \|\nabla_{\mathbf{x}} \eta\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-1}} \\ &\lesssim \|\mathbf{u}\|_{H^{s,1}} \|\eta\|_{H^{s_0+2,0}} + \|\mathbf{u}\|_{H^{s_0+2,1}} \|\eta\|_{H^{s,0}}, \\ \|[\partial_{\mathbf{x}}^{\alpha}; h, \nabla_{\mathbf{x}} \cdot \mathbf{u}]\|_{L_{\varrho}^1 L_{\mathbf{x}}^2} &\lesssim \|h\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-1}} \|\nabla_{\mathbf{x}} \cdot \mathbf{u}\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s_0+1}} + \|h\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s_0+1}} \|\nabla_{\mathbf{x}} \cdot \mathbf{u}\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-1}} \\ &\lesssim \|h\|_{H^{s-1,0}} \|\mathbf{u}\|_{H^{s_0+2,0}} + \|h\|_{H^{s_0+1,0}} \|\mathbf{u}\|_{H^{s,0}}, \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{u}^{(\alpha)} \cdot \nabla_{\mathbf{x}} h\|_{L_{\varrho}^1 L_{\mathbf{x}}^2} &\lesssim \|\mathbf{u}\|_{H^{s,0}} \|h\|_{H^{s_0+1,0}}, \\ \|\eta^{(\alpha)} (\nabla_{\mathbf{x}} \cdot \partial_{\varrho} \mathbf{u})\|_{L_{\varrho}^1 L_{\mathbf{x}}^2} &\lesssim \|\eta\|_{H^{s,0}} \|\mathbf{u}\|_{H^{s_0+2,1}}. \end{aligned}$$

Altogether, using the continuous embedding $L^{\infty}((\rho_0, \rho_1)) \subset L^2((\rho_0, \rho_1)) \subset L^1((\rho_0, \rho_1))$, the Minkowski and triangle inequalities and $s \geq s_0 + 2$, we get

$$\|R_{\alpha,0}|_{\varrho=\rho_0}|_{L_{\mathbf{x}}^2} + \|R_{\alpha,0}\|_{L^2(\Omega)} \lesssim (\|\eta\|_{H^{s,0}} + \|h\|_{H^{s-1,0}} + \|\eta|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^s}) \|\mathbf{u}\|_{H^{s,1}}. \quad (2.6)$$

Estimate of $R_{\alpha,j}$ for $1 \leq j \leq k$. We have

$$\begin{aligned} R_{\alpha,j} &:= -[\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^{j-1} \nabla_{\mathbf{x}} \cdot, \underline{\mathbf{u}} + \mathbf{u}]h - \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^{j-1} \nabla_{\mathbf{x}} \cdot (h(\underline{\mathbf{u}} + \mathbf{u})) \\ &= -\sum_{i=1}^d [\partial_{\mathbf{x}}^{\alpha} \partial_{x_i} \partial_{\varrho}^{j-1}, \mathbf{u}_i]h - [\partial_{\varrho}^{j-1}, \underline{\mathbf{u}}] \cdot \partial_{\mathbf{x}}^{\alpha} \nabla_{\mathbf{x}} h - \partial_{\varrho}^{j-1} \cdot (h \partial_{\mathbf{x}}^{\alpha} \nabla_{\mathbf{x}} \mathbf{u}), \end{aligned}$$

where \mathbf{u}_i is the i^{th} component of \mathbf{u} . By Lemma A.8 and since $(|\alpha| + 1) + (j - 1) \leq s$ and $j - 1 \leq k - 1$, and $s \geq s_0 + \frac{3}{2}$, we find for $2 \leq k - 1 \leq s$

$$\|[\partial_{\mathbf{x}}^{\alpha} \partial_{x_i} \partial_{\varrho}^{j-1}, \mathbf{u}_i]h\|_{L^2(\Omega)} \lesssim \|h\|_{H^{s-1,k-1}} \|\mathbf{u}\|_{H^{s,k}}.$$

There remains to consider $1 \leq j \leq k \leq 2$. If $j = 1$ we have by Lemma A.7(2) and since $|\alpha| \leq s - 1$ and $s \geq s_0 + \frac{3}{2}$

$$\|[\partial_{\mathbf{x}}^{\alpha} \partial_{x_i}, \mathbf{u}_i]h\|_{L^2(\Omega)} \lesssim \|h\|_{L_{\varrho}^{\infty} H_{\mathbf{x}}^{s_0}} \|\mathbf{u}\|_{L_{\varrho}^2 H_{\mathbf{x}}^s} + \|h\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-1}} \|\mathbf{u}\|_{L_{\varrho}^{\infty} H_{\mathbf{x}}^{s_0+1}} \lesssim \|h\|_{H^{s-1,1}} \|\mathbf{u}\|_{H^{s,1}}.$$

If $j = k = 2$, and since $|\alpha| \leq s - 2$ and $s \geq s_0 + \frac{3}{2}$,

$$\begin{aligned} \|[\partial_{\mathbf{x}}^{\alpha} \partial_{x_i} \partial_{\varrho}^{j-1}, \mathbf{u}_i]h\|_{L^2(\Omega)} &\leq \|[\partial_{\mathbf{x}}^{\alpha} \partial_{x_i}, \mathbf{u}_i] \partial_{\varrho} h\|_{L^2(\Omega)} + \|\partial_{\mathbf{x}}^{\alpha} \partial_{x_i} (h \partial_{\varrho} \mathbf{u}_i)\|_{L^2(\Omega)} \\ &\lesssim \|\partial_{\varrho} h\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s_0}} \|\mathbf{u}\|_{L_{\varrho}^{\infty} H_{\mathbf{x}}^{s-1}} + \|\partial_{\varrho} h\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-2}} \|\mathbf{u}\|_{L_{\varrho}^{\infty} H_{\mathbf{x}}^{s_0+1}} \\ &\quad + \|h\|_{L_{\varrho}^{\infty} H_{\mathbf{x}}^{s_0}} \|\partial_{\varrho} \mathbf{u}\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-1}} + \|h\|_{L_{\varrho}^2 H_{\mathbf{x}}^{s-1}} \|\partial_{\varrho} \mathbf{u}\|_{L_{\varrho}^{\infty} H_{\mathbf{x}}^{s_0}} \\ &\lesssim \|h\|_{H^{s-1,1}} \|\mathbf{u}\|_{H^{s,2}}. \end{aligned}$$

Finally, we have immediately

$$\begin{aligned} \|[\partial_\rho^{j-1}, \underline{\mathbf{u}}] \cdot \partial_{\mathbf{x}}^\alpha \nabla_{\mathbf{x}} h\|_{L^2(\Omega)} &\lesssim |\underline{\mathbf{u}}'|_{W_\rho^{j-2, \infty}} \|h\|_{H^{s-1, j-2}}, \\ \|\partial_\rho^{j-1}(\underline{h} \partial_{\mathbf{x}}^\alpha \nabla_{\mathbf{x}} \cdot \mathbf{u})\|_{L^2(\Omega)} &\lesssim |\underline{h}|_{W_\rho^{j-1, \infty}} \|\mathbf{u}\|_{H^{s, j}}. \end{aligned}$$

Altogether, we find that for any $1 \leq j \leq k$

$$\|R_{\alpha, j}\|_{L^2(\Omega)} \lesssim (|\underline{h}|_{W_\rho^{k-1, \infty}} + |\underline{\mathbf{u}}'|_{W_\rho^{k-2, \infty}} + \|h\|_{H^{s-1, k-1}}) (\|\mathbf{u}\|_{H^{s, k}} + \|h\|_{H^{s-1, k-2}}). \quad (2.7)$$

Estimate of $\mathbf{r}_{\alpha, j}$ and $r_{\alpha, j}$ for $0 \leq j \leq k$. We have (2.5a) with

$$\begin{aligned} \mathbf{r}_{\alpha, j} &:= -[\partial_\rho^j, \underline{\mathbf{u}}] \partial_{\mathbf{x}}^\alpha h - \partial_{\mathbf{x}}^\alpha \partial_\rho^j(\underline{h} \mathbf{u}) - (\partial_{\mathbf{x}}^\alpha \partial_\rho^j \mathbf{u}) h, \\ r_{\alpha, j} &:= -[\partial_{\mathbf{x}}^\alpha \partial_\rho^j \nabla_{\mathbf{x}} \cdot; \mathbf{u}, h] + (\partial_{\mathbf{x}}^\alpha \partial_\rho^j \mathbf{u}) \cdot \nabla_{\mathbf{x}} h. \end{aligned}$$

We have immediately (since $|\alpha| + j \leq s$, $j \leq k$, and using Lemma A.1)

$$\begin{aligned} \|[\partial_\rho^j, \underline{\mathbf{u}}] \partial_{\mathbf{x}}^\alpha h\|_{L^2(\Omega)} &\lesssim |\underline{\mathbf{u}}'|_{W_\rho^{k-1, \infty}} \|h\|_{H^{s-1, k-1}}, \\ \|\partial_{\mathbf{x}}^\alpha \partial_\rho^j(\underline{h} \mathbf{u})\|_{L^2(\Omega)} &\lesssim |\underline{h}|_{W_\rho^{k, \infty}} \|\mathbf{u}\|_{H^{s, k}}, \\ \|(\partial_{\mathbf{x}}^\alpha \partial_\rho^j \mathbf{u}) h\|_{L^2(\Omega)} &\lesssim \|\mathbf{u}\|_{H^{s, k}} \|h\|_{H^{s_0 + \frac{1}{2}, 1}}. \end{aligned}$$

By Lemma A.9 and since $|\alpha| + j + 1 \leq s + 1$, $j \leq k \leq s$, $s + 1 \geq s_0 + \frac{5}{2}$, we find for $2 \leq k \leq s$

$$\|[\partial_{\mathbf{x}}^\alpha \partial_\rho^j \nabla_{\mathbf{x}} \cdot; \mathbf{u}, h]\|_{L^2(\Omega)} \lesssim \|h\|_{H^{s, k}} \|\mathbf{u}\|_{H^{s, k}},$$

and we have by Lemma A.1

$$\|(\partial_{\mathbf{x}}^\alpha \partial_\rho^j \mathbf{u}) \cdot \nabla_{\mathbf{x}} h\|_{L^2(\Omega)} \lesssim \|\mathbf{u}\|_{H^{s, k}} \|h\|_{H^{s_0 + \frac{3}{2}, 1}}.$$

Altogether, we find that for any $0 \leq j \leq k$

$$\|\mathbf{r}_{\alpha, j}\|_{L^2(\Omega)} \lesssim (|\underline{h}|_{W_\rho^{k, \infty}} + |\underline{\mathbf{u}}'|_{W_\rho^{k-1, \infty}} + \|h\|_{H^{s-1, k-1}}) (\|\mathbf{u}\|_{H^{s, k}} + \|h\|_{H^{s-1, k-1}}), \quad (2.8)$$

$$\|r_{\alpha, j}\|_{L^2(\Omega)} \lesssim \|\mathbf{u}\|_{H^{s, k}} \|h\|_{H^{s, k}}. \quad (2.9)$$

Estimate of $\mathbf{R}_{\alpha, 0}$. The precise expression of the second remainder in (2.3a) is the following:

$$\mathbf{R}_{\alpha, 0} := -([\partial_{\mathbf{x}}^\alpha, \underline{\mathbf{u}}] \cdot \nabla_{\mathbf{x}}) \mathbf{u} + \kappa [\partial_{\mathbf{x}}^\alpha, \frac{1}{\underline{h} + h}] (\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \mathbf{u} + \frac{\kappa}{\underline{h} + h} ([\partial_{\mathbf{x}}^\alpha, \nabla_{\mathbf{x}} h] \cdot \nabla_{\mathbf{x}}) \mathbf{u}.$$

By Lemma A.7(2) and Lemma A.1 we have

$$\|([\partial_{\mathbf{x}}^\alpha, \underline{\mathbf{u}}] \cdot \nabla_{\mathbf{x}}) \mathbf{u}\|_{L^2(\Omega)} \lesssim \|\mathbf{u}\|_{L_\rho^\infty H_{\mathbf{x}}^{s_0+1}} \|\mathbf{u}\|_{L_\rho^2 H_{\mathbf{x}}^s} \lesssim \|\mathbf{u}\|_{H^{s_0 + \frac{3}{2}, 1}} \|\mathbf{u}\|_{H^{s, 0}}.$$

Next, appealing again to A.7(2), we have

$$\begin{aligned} \kappa \|[\partial_{\mathbf{x}}^\alpha, \frac{1}{\underline{h} + h}] (\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \mathbf{u}\|_{L^2(\Omega)} &\lesssim \kappa \|\nabla_{\mathbf{x}} \frac{1}{\underline{h} + h}\|_{L_\rho^\infty H_{\mathbf{x}}^{s_0}} \|(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \mathbf{u}\|_{L_\rho^2 H_{\mathbf{x}}^{s-1}} \\ &\quad + \kappa \|\nabla_{\mathbf{x}} \frac{1}{\underline{h} + h}\|_{L_\rho^2 H_{\mathbf{x}}^{s-1}} \|(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \mathbf{u}\|_{L_\rho^\infty H_{\mathbf{x}}^{s_0}}. \end{aligned}$$

Now, by Lemma A.2(2) and Lemma A.4, one has for any $t \geq 0$ that

$$\begin{aligned} \|\nabla_{\mathbf{x}} \frac{1}{\underline{h} + h}\|_{H_{\mathbf{x}}^t} &= \left| \frac{\nabla_{\mathbf{x}} h}{(\underline{h} + h)^2} \right|_{H_{\mathbf{x}}^t} \leq \left| \frac{\nabla_{\mathbf{x}} h}{\underline{h}^2} \right|_{H_{\mathbf{x}}^t} + \left| \left(\frac{1}{\underline{h}^2} - \frac{1}{(\underline{h} + h)^2} \right) \nabla_{\mathbf{x}} h \right|_{H_{\mathbf{x}}^t} \\ &\lesssim (h_\star)^{-2} \|\nabla_{\mathbf{x}} h\|_{H_{\mathbf{x}}^t} + \left| \frac{1}{\underline{h}^2} - \frac{1}{(\underline{h} + h)^2} \right|_{H_{\mathbf{x}}^{s_0}} \|\nabla_{\mathbf{x}} h\|_{H_{\mathbf{x}}^t} + \left\langle \left| \frac{1}{\underline{h}^2} - \frac{1}{(\underline{h} + h)^2} \right|_{H_{\mathbf{x}}^t} \|\nabla_{\mathbf{x}} h\|_{H_{\mathbf{x}}^{s_0}} \right\rangle_{t > s_0} \\ &\leq C(h_\star, |h|_{H_{\mathbf{x}}^{s_0}}) \|\nabla_{\mathbf{x}} h\|_{H_{\mathbf{x}}^t}, \end{aligned} \quad (2.10)$$

where in the last step we used that, by Lemma A.4,

$$\left| \frac{1}{\underline{h}^2} - \frac{1}{(\underline{h} + h)^2} \right|_{H_{\mathbf{x}}^{s_0}} \leq C(h_\star, |h|_{H_{\mathbf{x}}^{s_0}})$$

and, provided that $t > s_0$,

$$\left| \frac{1}{h^2} - \frac{1}{(h+h)^2} \right|_{H_x^t} \leq \left| \frac{1}{h^2} - \frac{1}{(h+h)^2} \right|_{H_x^{s_0}} + \left| \nabla_x \frac{1}{(h+h)^2} \right|_{H_x^{t-1}},$$

and a finite induction on t , until $\left| \nabla_x \frac{1}{(h+h)^2} \right|_{L_x^2} = \left| \frac{\nabla_x h}{h+h} \right|_{L_x^2} \leq h_*^{-2} \left| \nabla_x h \right|_{L_x^2}$. Then, by Lemma A.2(2) and Lemma A.1, we have

$$\begin{aligned} \|(\nabla_x h \cdot \nabla_x) \mathbf{u}\|_{L_\rho^2 H_x^{s-1}} &\lesssim \|\nabla_x h\|_{L_\rho^2 H_x^{s-1}} \|\mathbf{u}\|_{L_\rho^\infty H_x^{s_0+1}} + \|\nabla_x h\|_{L_\rho^\infty H_x^{s_0}} \|\mathbf{u}\|_{L_\rho^2 H_x^s} \\ &\lesssim \|h\|_{H^{s,0}} \|\mathbf{u}\|_{H^{s_0+\frac{3}{2},1}} + \|h\|_{H^{s_0+\frac{3}{2},1}} \|\mathbf{u}\|_{H^{s,0}} \end{aligned}$$

and

$$\|(\nabla_x h \cdot \nabla_x) \mathbf{u}\|_{L_\rho^\infty H_x^{s_0}} \lesssim \|\nabla_x h\|_{L_\rho^\infty H_x^{s_0}} \|\mathbf{u}\|_{L_\rho^\infty H_x^{s_0+1}} \lesssim \|h\|_{H^{s_0+\frac{3}{2},1}} \|\mathbf{u}\|_{H^{s_0+\frac{3}{2},1}}.$$

Finally, we have by Lemma A.7(2) and Lemma A.1

$$\begin{aligned} \|([\partial_x^\alpha, \nabla_x h] \cdot \nabla_x) \mathbf{u}\|_{L^2(\Omega)} &\lesssim \|\nabla_x h\|_{L_\rho^\infty H_x^{s_0+1}} \|\nabla_x \mathbf{u}\|_{L_\rho^2 H_x^{s-1}} + \|\nabla_x h\|_{L_\rho^2 H_x^s} \|\nabla_x \mathbf{u}\|_{L_\rho^\infty H_x^{s_0}} \\ &\lesssim \|\nabla_x h\|_{H^{s_0+\frac{3}{2},1}} \|\mathbf{u}\|_{H^{s,0}} + \|\nabla_x h\|_{H^{s,0}} \|\mathbf{u}\|_{H^{s_0+\frac{3}{2},1}}. \end{aligned}$$

Collecting the estimates above and using that $s \geq s_0 + \frac{3}{2}$, we obtain

$$\|\mathbf{R}_{\alpha,0}\|_{L^2(\Omega)} \lesssim \|\mathbf{u}\|_{H^{s,0}} \|\mathbf{u}\|_{H^{s,1}} + \kappa C(h_*, \|h\|_{H^{s_0+\frac{1}{2},1}}) (\|h\|_{H^{s,1}}^2 + \|\nabla_x h\|_{H^{s,1}}) \|\mathbf{u}\|_{H^{s,1}}. \quad (2.11)$$

Estimate of $\mathbf{R}_{\alpha,j}$ for $1 \leq j \leq k$. The explicit expression of the second remainder in (2.4a) is the following

$$\begin{aligned} \mathbf{R}_{\alpha,j} := & -([\partial_x^\alpha \partial_\rho^j, \underline{\mathbf{u}} + \mathbf{u}] \cdot \nabla_x) \mathbf{u} + \kappa [\partial_x^\alpha \partial_\rho^j, \frac{1}{h+h}] ((\nabla_x h \cdot \nabla_x) \mathbf{u}) + \frac{\kappa}{h+h} ([\partial_x^\alpha \partial_\rho^j, \nabla_x h] \cdot \nabla_x) \mathbf{u} \\ & + \partial_\rho^j \partial_x^\alpha \left(\frac{\rho_0}{\rho} \int_{\rho_0}^{\rho_1} \nabla_x h \, d\rho' + \frac{1}{\rho} \int_{\rho_0}^{\rho} \nabla_x \eta \, d\rho' \right). \end{aligned}$$

By Lemma A.8 we have for $s \geq s_0 + \frac{3}{2}$ and since $0 \leq |\alpha| \leq s - j$ and $j \leq k$ with $k \geq 2$, that

$$\|([\partial_x^\alpha \partial_\rho^j, \underline{\mathbf{u}}] \cdot \nabla_x) \mathbf{u}\|_{L^2(\Omega)} \lesssim \|\mathbf{u}\|_{H^{s,k}} \|\nabla_x \mathbf{u}\|_{H^{s-1,k}}.$$

Then,

$$\|([\partial_x^\alpha \partial_\rho^j, \underline{\mathbf{u}}] \cdot \nabla_x) \mathbf{u}\|_{L^2(\Omega)} = \|[\partial_\rho^j, \underline{\mathbf{u}}] \cdot \nabla_x \partial_x^\alpha \mathbf{u}\|_{L^2(\Omega)} \lesssim \|\underline{\mathbf{u}}'\|_{W_\rho^{j-1,\infty}} \|\mathbf{u}\|_{H^{s,j-1}}.$$

Next, using Lemma A.3,

$$\begin{aligned} \|([\partial_\rho^j, \frac{1}{h}] \partial_x^\alpha ((\nabla_x h \cdot \nabla_x) \mathbf{u}))\|_{L^2(\Omega)} &\lesssim C(h_*) \|\underline{h}'\|_{W_\rho^{j-1,\infty}} \|(\nabla_x h \cdot \nabla_x) \mathbf{u}\|_{H^{s-1,j-1}} \\ &\lesssim C(h_*) \|\underline{h}'\|_{W_\rho^{j-1,\infty}} \|h\|_{H^{s,j}} \|\mathbf{u}\|_{H^{s,j}}, \end{aligned}$$

and by Lemma A.8, since $s \geq s_0 + \frac{3}{2}$ and $2 \leq j \leq s$, $|\alpha| + j \leq s$, Lemma A.6 and Lemma A.3,

$$\begin{aligned} \|([\partial_x^\alpha \partial_\rho^j, \frac{1}{h+h} - \frac{1}{h}] ((\nabla_x h \cdot \nabla_x) \mathbf{u}))\|_{L^2(\Omega)} &\lesssim \left\| \frac{1}{h+h} - \frac{1}{h} \right\|_{H^{s,k}} \|(\nabla_x h \cdot \nabla_x) \mathbf{u}\|_{H^{s-1,\min\{k,s-1\}}} \\ &\lesssim C(h_*) \|\underline{h}'\|_{W_\rho^{k-1,\infty}} \|h\|_{H^{s-1,k-1}} \|h\|_{H^{s,k}}^2 \|\mathbf{u}\|_{H^{s,k}}. \end{aligned}$$

By Lemma A.8 we have for $s \geq s_0 + \frac{3}{2}$ and since $|\alpha| + j \leq s$ and $2 \leq j \leq s$

$$\|([\partial_x^\alpha \partial_\rho^j, \nabla_x h] \cdot \nabla_x) \mathbf{u}\|_{L^2(\Omega)} \lesssim \|\nabla_x h\|_{H^{s,k}} \|\mathbf{u}\|_{H^{s,k}}.$$

We have immediately since $|\alpha| \leq s - j \leq s - 1$,

$$\left\| \partial_\rho^j \partial_x^\alpha \left(\frac{\rho_0}{\rho} \int_{\rho_0}^{\rho_1} \nabla_x h \, d\rho' \right) \right\|_{L^2(\Omega)} \lesssim \left| \partial_x^\alpha \nabla_x \eta \right|_{\rho=\rho_0} \Big|_{L_x^2} \lesssim \|\eta\|_{\rho=\rho_0} \Big|_{H_x^s}$$

and since $(|\alpha| + 1) + (j - 1) \leq s$,

$$\left\| \partial_\rho^j \left(\frac{1}{\rho} \int_{\rho_0}^{\rho} \partial_x^\alpha \nabla_x \eta \, d\rho' \right) \right\|_{L^2(\Omega)} \lesssim \sum_{i=0}^{j-1} \left\| \partial_\rho^i \partial_x^\alpha \nabla_x \eta \right\|_{L^2(\Omega)} \lesssim \|\eta\|_{H^{s,j-1}}.$$

Collecting the estimates above we obtain for $1 \leq j \leq k$

$$\begin{aligned} \|\mathbf{R}_{\alpha,j}\|_{L^2(\Omega)} &\lesssim |\eta|_{\varrho=\rho_0}|_{H_x^s} + \|\eta\|_{H^{s,k-1}} + (|\underline{\mathbf{u}}'|_{W_\varrho^{k-1,\infty}} + \|\mathbf{u}\|_{H^{s,k}})\|\mathbf{u}\|_{H^{s,k}} \\ &\quad + \kappa C(h_\star, |\underline{h}'|_{W_\varrho^{k-1,\infty}}, \|h\|_{H^{s-1,k-1}})(\|h\|_{H^{s,k}}^2 + \|\nabla_{\mathbf{x}}h\|_{H^{s,k}})\|\mathbf{u}\|_{H^{s,k}}. \end{aligned} \quad (2.12)$$

We infer the bound (2.3b) from (2.6) and (2.11), the bound (2.4b) from (2.7) and (2.12), and the bound (2.5b) from (2.8) and (2.9), and the proof is complete. \square

2.3. A priori energy estimates. In this section we provide *a priori* energy estimates associated with the equations featured in Lemma 2.3. We start with the transport-diffusion equations in (2.4a) and (2.5a), which we rewrite as

$$\partial_t \dot{h} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \dot{h} = \kappa \Delta_{\mathbf{x}} \dot{h} + r + \nabla_{\mathbf{x}} \cdot \mathbf{r}. \quad (2.13)$$

LEMMA 2.4. There exists a universal constant $C_0 > 0$ such that for any $\kappa > 0$ and $T > 0$, for any $\mathbf{u} \in L^\infty(0, T; L^\infty(\Omega))$ with $\nabla_{\mathbf{x}} \cdot \mathbf{u} \in L^1(0, T; L^\infty(\Omega))$, for any $(r, \mathbf{r}) \in L^2(0, T; L^2(\Omega))$ and for any $\dot{h} \in L^\infty(0, T; L^2(\Omega))$ with $\nabla_{\mathbf{x}} \dot{h} \in L^2(0, T; L^2(\Omega))$, such that (2.13) holds in $L^2(0, T; H^{1,0}(\Omega)')$, we have

$$\begin{aligned} \|\dot{h}\|_{L^\infty(0,T;L^2(\Omega))} + \kappa^{1/2} \|\nabla_{\mathbf{x}} \dot{h}\|_{L^2(0,T;L^2(\Omega))} \\ \leq C_0 (\|\dot{h}|_{t=0}\|_{L^2(\Omega)} + \|r\|_{L^1(0,T;L^2(\Omega))} + \kappa^{-1/2} \|r\|_{L^2(0,T;L^2(\Omega))}) \\ \times \exp\left(C_0 \int_0^T \|\nabla_{\mathbf{x}} \cdot \mathbf{u}(t, \cdot)\|_{L^\infty(\Omega)} dt\right). \end{aligned} \quad (2.14)$$

PROOF. Testing the equation against \dot{h} and integrating by parts (with respect to the variable \mathbf{x}) yields

$$\frac{1}{2} \frac{d}{dt} \|\dot{h}\|_{L^2(\Omega)}^2 + \kappa \|\nabla_{\mathbf{x}} \dot{h}\|_{L^2(\Omega)}^2 = \frac{1}{2} \iint_{\Omega} (\nabla_{\mathbf{x}} \cdot \mathbf{u}) \dot{h}^2 d\mathbf{x} d\varrho + \iint_{\Omega} r \dot{h} d\mathbf{x} d\varrho - \iint_{\Omega} \mathbf{r} \cdot \nabla_{\mathbf{x}} \dot{h} d\mathbf{x} d\varrho.$$

The estimate follows from the Cauchy-Schwarz inequality and Gronwall's Lemma. \square

Next, we consider system (2.3a), which we rewrite as

$$\begin{aligned} \partial_t \dot{\eta} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} + \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} d\varrho' + \int_{\varrho}^{\rho_1} (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \dot{\mathbf{u}} d\varrho' = \kappa \Delta_{\mathbf{x}} \dot{\eta} + R, \\ \varrho \left(\partial_t \dot{\mathbf{u}} + \left((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}} \right) \dot{\mathbf{u}} \right) + \rho_0 \nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0} + \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \dot{\eta} d\varrho' = \varrho \nu \Delta_{\mathbf{x}} \dot{\mathbf{u}} + \mathbf{R}. \end{aligned} \quad (2.15)$$

For the sake of readability, we introduce the following notations

$$X^0 := C^0([\rho_0, \rho_1]; L^2(\mathbb{R}^d)) \times L^2(\Omega)^d; \quad X^1 := C^0([\rho_0, \rho_1]; H^1(\mathbb{R}^d)) \times H^{1,0}(\Omega)^d. \quad (2.16)$$

LEMMA 2.5. Let $h_\star, h^\star, \underline{M} > 0$ be fixed. There exists $C(h_\star, h^\star, M) > 0$ such that for any $\kappa > 0$ and $\nu \in [0, 1]$, for any $(\underline{h}, \underline{\mathbf{u}}) \in W^{1,\infty}((\rho_0, \rho_1))$, for any $T > 0$ and $(h, \mathbf{u}) \in L^\infty(0, T; W^{1,\infty}(\Omega))$ with $\Delta_{\mathbf{x}} h \in L^1(0, T; L^\infty(\Omega))$ satisfying (2.1) and, for almost every $t \in [0, T]$, the upper bound

$$\|h(t, \cdot)\|_{L^\infty(\Omega)} + \|\nabla_{\mathbf{x}} h(t, \cdot)\|_{L_x^\infty L_\varrho^2} + \nu^{1/2} \|\nabla_{\mathbf{x}} h(t, \cdot)\|_{L^\infty(\Omega)} + \|\nabla_{\mathbf{x}} \cdot \mathbf{u}(t, \cdot)\|_{L^\infty(\Omega)} \leq M$$

and the lower and upper bounds

$$\forall (\mathbf{x}, \varrho) \in \Omega, \quad h_\star \leq \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \leq h^\star;$$

and for any $(\dot{\eta}, \dot{\mathbf{u}}) \in \mathcal{C}^0([0, T]; X^0) \cap L^2(0, T; X^1)$, with X^0, X^1 in (2.16), and $(R, \mathbf{R}) \in L^2(0, T; X^0)$ satisfying system (2.15) in $L^2(0, T; X^1)'$, the following estimate holds:

$$\begin{aligned} & \mathcal{E}(\dot{\eta}(t, \cdot), \dot{\mathbf{u}}(t, \cdot))^{1/2} + \kappa^{1/2} \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(0, t; L^2(\Omega))} + \kappa^{1/2} |\nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0}|_{L^2(0, t; L^2_{\mathbf{x}})} + \nu \|\nabla_{\mathbf{x}} \dot{\mathbf{u}}\|_{L^2(\Omega)}^2 \\ & \leq \left(\mathcal{E}(\dot{\eta}(0, \cdot), \dot{\mathbf{u}}(0, \cdot))^{1/2} + C \int_0^t \mathcal{E}(R(\tau, \cdot), \mathbf{R}(\tau, \cdot))^{1/2} d\tau \right) \\ & \quad \times \exp \left(C \int_0^t (1 + \kappa^{-1} \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}(\tau, \cdot)\|_{L^{\infty}_{\mathbf{x}} L^2_{\varrho}})^2 d\tau \right), \end{aligned}$$

where we denote

$$\mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}) := \frac{1}{2} \int_{\rho_0}^{\rho_1} \int_{\mathbb{R}^d} \dot{\eta}^2 + \varrho(\underline{h} + h) |\dot{\mathbf{u}}|^2 d\mathbf{x} d\varrho + \frac{\rho_0}{2} \int_{\mathbb{R}^d} \dot{\eta}^2|_{\varrho=\rho_0} d\mathbf{x}.$$

PROOF. We test the first equation against $\dot{\eta} \in L^2(0, T; H^{1,0}(\Omega))$, its trace on $\{(\mathbf{x}, \rho_0), \mathbf{x} \in \mathbb{R}^d\}$ against $\rho_0 \dot{\eta}|_{\varrho=\rho_0} \in L^2(0, T; H^1(\mathbb{R}^d))$, and the second equation against $(\underline{h} + h) \dot{\mathbf{u}} \in L^2(0, T; H^{1,0}(\Omega))$. This yields, after integration by parts

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}) + \kappa \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)}^2 + \rho_0 \kappa |\nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0}|_{L^2_{\mathbf{x}}}^2 + \nu \sum_{i=1}^d \int_{\Omega} \varrho(\underline{h} + h) |\partial_{x_i} \dot{\mathbf{u}}|^2 d\mathbf{x} d\varrho \\ & = - ((\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta}, \dot{\eta})_{L^2(\Omega)} - \left(\int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} d\varrho', \dot{\eta} \right)_{L^2(\Omega)} \quad (\text{i}) \\ & \quad - \left(\int_{\varrho}^{\rho_1} (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \dot{\mathbf{u}} d\varrho', \dot{\eta} \right)_{L^2(\Omega)} + (R, \dot{\eta})_{L^2(\Omega)} \quad (\text{ii}) \\ & \quad - (\varrho(\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\mathbf{u}}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} + \kappa (\varrho(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \dot{\mathbf{u}}, \dot{\mathbf{u}})_{L^2(\Omega)} \quad (\text{iii}) \\ & \quad - (\rho_0 \nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} - \left(\int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \dot{\eta} d\varrho', (\underline{h} + h) \dot{\mathbf{u}} \right)_{L^2(\Omega)} \quad (\text{iv}) \\ & \quad - \nu (\varrho(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \dot{\mathbf{u}}, \dot{\mathbf{u}})_{L^2(\Omega)} + (\varrho \mathbf{R}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} \quad (\text{v}) \\ & \quad - \rho_0 \left(((\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta})|_{\varrho=\rho_0}, \dot{\eta}|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} - \rho_0 \left(\int_{\rho_0}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} d\varrho', \dot{\eta}|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} \quad (\text{vi}) \\ & \quad - \rho_0 \left(\int_{\rho_0}^{\rho_1} (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \dot{\mathbf{u}} d\varrho', \dot{\eta}|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} + \rho_0 \left(R|_{\varrho=\rho_0}, \dot{\eta}|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} \quad (\text{vii}) \\ & \quad + \frac{1}{2} (\varrho(\partial_t h) \dot{\mathbf{u}}, \dot{\mathbf{u}})_{L^2(\Omega)}. \quad (\text{viii}) \end{aligned}$$

We consider first the second terms in (i) and (vi). We have by an immediate application of Cauchy-Schwarz inequality and the continuous embedding $L^{\infty}((\rho_0, \rho_1)) \subset L^2((\rho_0, \rho_1))$

$$\begin{aligned} & \left| \left(\int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} d\varrho', \dot{\eta} \right)_{L^2(\Omega)} \right| + \rho_0 \left| \left(\int_{\rho_0}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} d\varrho', \dot{\eta}|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} \right| \\ & \lesssim \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L^{\infty}_{\mathbf{x}} L^2_{\varrho}} \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)} \left(\|\dot{\eta}\|_{L^2(\Omega)} + |\dot{\eta}|_{\varrho=\rho_0}|_{L^2_{\mathbf{x}}} \right). \quad (2.17) \end{aligned}$$

Notice that the right-hand side (2.17) cannot be bounded by the energy functional $\mathcal{E}(\dot{\eta}, \dot{\mathbf{u}})$, and this is exactly the point where we use the assumption $\kappa > 0$. Let us now estimate all other terms.

Using integration by parts in the variable \mathbf{x} , we estimate the first addend of (i) and (vi) as follows:

$$\begin{aligned} & \left| ((\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta}, \dot{\eta})_{L^2(\Omega)} \right| + \rho_0 \left| \left(((\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta}) \Big|_{\varrho=\rho_0}, \dot{\eta} \Big|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} \right| \\ & \lesssim \|\nabla_{\mathbf{x}} \cdot \mathbf{u}\|_{L^\infty(\Omega)} \|\dot{\eta}\|_{L^2(\Omega)}^2 + \|\nabla_{\mathbf{x}} \cdot \mathbf{u}\|_{\varrho=\rho_0} \Big|_{L^\infty} \|\dot{\eta}\|_{\varrho=\rho_0} \Big|_{L^2_{\mathbf{x}}}. \end{aligned}$$

The contributions in (iii) and (viii) compensate after integration by parts in \mathbf{x} , using the first equation in (2.1). Now consider the first addend of (ii) together with the second addend of (iv). By application of Fubini's theorem we have

$$\int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \left(\int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \dot{\eta}(\varrho') \, d\varrho' \right) \cdot (\underline{h} + h)(\varrho) \dot{\mathbf{u}}(\varrho) \, d\varrho \, d\mathbf{x} = \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \left(\int_{\varrho'}^{\rho_1} (\underline{h} + h)(\varrho) \dot{\mathbf{u}}(\varrho) \, d\varrho \right) \cdot \nabla_{\mathbf{x}} \dot{\eta}(\varrho') \, d\varrho' \, d\mathbf{x}$$

and hence, integrating by parts in \mathbf{x} , we infer

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \int_{\varrho}^{\rho_1} (\underline{h} + h)(\varrho') \nabla_{\mathbf{x}} \cdot \dot{\mathbf{u}}(\varrho') \, d\varrho' \, \dot{\eta}(\varrho) \, d\varrho \, d\mathbf{x} \right. \\ & \quad \left. + \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \left(\int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \dot{\eta}(\varrho') \, d\varrho' \right) \cdot (\underline{h} + h)(\varrho) \dot{\mathbf{u}}(\varrho) \, d\varrho \, d\mathbf{x} \right| \\ & = \left| \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \int_{\varrho}^{\rho_1} (\nabla_{\mathbf{x}} h)(\varrho') \cdot \dot{\mathbf{u}}(\varrho') \dot{\eta}(\varrho) \, d\varrho' \, d\varrho \, d\mathbf{x} \right| \lesssim \|\nabla_{\mathbf{x}} h\|_{L^\infty L^2_{\varrho}} \|\dot{\mathbf{u}}\|_{L^2(\Omega)} \|\dot{\eta}\|_{L^2(\Omega)}. \end{aligned}$$

Concerning first addend of (iv) and the first addend of (vii), we have after integrating by parts with respect to the \mathbf{x} variable and using Cauchy-Schwarz inequality

$$\begin{aligned} & \left| - \left(\rho_0 \nabla_{\mathbf{x}} \dot{\eta} \Big|_{\varrho=\rho_0}, (\underline{h} + h) \dot{\mathbf{u}} \right)_{L^2(\Omega)} - \rho_0 \left(\int_{\rho_0}^{\rho_1} (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \dot{\mathbf{u}} \, d\varrho, \dot{\eta} \Big|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} \right| \\ & = \rho_0 \left| \left(\int_{\rho_0}^{\rho_1} (\nabla_{\mathbf{x}} h) \cdot \dot{\mathbf{u}} \, d\varrho, \dot{\eta} \Big|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} \right| \lesssim \|\nabla_{\mathbf{x}} h\|_{L^\infty L^2_{\varrho}} \|\dot{\mathbf{u}}\|_{L^2(\Omega)} \|\dot{\eta}\|_{\varrho=\rho_0} \Big|_{L^2_{\mathbf{x}}}. \end{aligned}$$

Concerning the first addend of (v), we have for an arbitrarily large constant $K > 0$,

$$\nu \left| (\varrho (\nabla_{\mathbf{x}} h \cdot \nabla) \dot{\mathbf{u}}, \dot{\mathbf{u}})_{L^2(\Omega)} \right| \leq \frac{1}{2K} \nu \|\nabla \dot{\mathbf{u}}\|_{L^2(\Omega)}^2 + \frac{K\rho_1^2}{2} \nu \|\nabla_{\mathbf{x}} h\|_{L^\infty(\Omega)}^2 \|\dot{\mathbf{u}}\|_{L^2(\Omega)}^2.$$

The last contributions, namely

$$\left| (R, \dot{\eta})_{L^2(\Omega)} + (\mathbf{R}, \varrho (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} + \rho_0 (R \Big|_{\varrho=\rho_0}, \dot{\eta} \Big|_{\varrho=\rho_0})_{L^2_{\mathbf{x}}} \right|,$$

are easily controlled by means of Cauchy-Schwarz inequality. Collecting all of the above, and using that

$$\mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}) \approx \|\dot{\eta}\|_{L^2(\Omega)}^2 + \|\dot{\mathbf{u}}\|_{L^2(\Omega)}^2 + \|\dot{\eta}\|_{\varrho=\rho_0} \Big|_{L^2_{\mathbf{x}}}^2$$

and

$$\nu \sum_{i=1}^d \int_{\Omega} \varrho (\underline{h} + h) |\partial_{x_i} \dot{\mathbf{u}}|^2 \, d\mathbf{x} \, d\varrho \gtrsim \nu \|\nabla_{\mathbf{x}} \dot{\mathbf{u}}\|_{L^2(\Omega)}^2$$

since $\rho_0 h_* \leq \varrho (\underline{h} + h) \leq \rho_1 h^*$, we obtain (choosing K sufficiently large)

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}) + \kappa \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)}^2 + \rho_0 \kappa \|\nabla_{\mathbf{x}} \dot{\eta}\|_{\varrho=\rho_0} \Big|_{L^2_{\mathbf{x}}}^2 \\ & \leq C \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}) + C \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L^\infty L^2_{\varrho}} \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}})^{1/2} \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)} \\ & \quad + C \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}})^{1/2} \mathcal{E}(R, \mathbf{R})^{1/2}, \end{aligned}$$

with $C = C(h_\star, h^\star, M)$. We deduce (augmenting C if necessary)

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}) + \frac{\kappa}{2} \|\nabla_{\mathbf{x}} \eta\|_{L^2(\Omega)}^2 + \rho_0 \kappa |\nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0}|_{L^2_{\mathbf{x}}}^2 \\ \leq C (1 + \kappa^{-1} \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L^{\infty}_{\mathbf{x}} L^2_{\varrho}}^2) \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}) + C \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}})^{1/2} \mathcal{E}(R, \mathbf{R})^{1/2}, \end{aligned}$$

and the desired estimate follows by Gronwall's inequality. \square

2.4. Large-time existence; proof of Theorem 1.1. We prove the large-time existence and energy estimates on solutions to the regularized system (2.1) in the following result. Compared with Proposition 2.1, we provide an existence time which is uniformly bounded (from below) with respect to the artificial regularization parameter $\nu > 0$, and specify the dependency with respect to the diffusivity parameter κ , in relation with the size of the data. It is in this sense that the existence of strong solutions to the hydrostatic system holds for *large* times. We then complete the proof of Theorem 1.1 at the end of this section.

PROPOSITION 2.6. Let $s, k \in \mathbb{N}$ be such that $s > 2 + \frac{d}{2}$, $2 \leq k \leq s$, and $\underline{M}, M^\star, h_\star, h^\star > 0$. Then, there exists $C > 0$ such that, for any $0 < \nu \leq \kappa \leq 1$, and

- for any $(\underline{h}, \underline{\mathbf{u}}) \in W^{k, \infty}((\rho_0, \rho_1))$ such that

$$|\underline{h}|_{W_{\varrho}^{k, \infty}} + |\underline{\mathbf{u}}'|_{W_{\varrho}^{k-1, \infty}} \leq \underline{M};$$

- for any initial data $(h_0, \mathbf{u}_0) = (h_0(\mathbf{x}, \varrho), \mathbf{u}_0(\mathbf{x}, \varrho)) \in H^{s, k}(\Omega)$ with

$$M_0 := \|\eta_0\|_{H^{s, k}} + \|\mathbf{u}_0\|_{H^{s, k}} + |\eta_0|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^s} + \kappa^{1/2} \|h_0\|_{H^{s, k}} \leq M^\star,$$

and

$$\forall(\mathbf{x}, \varrho) \in \Omega, \quad h_\star \leq \underline{h}(\varrho) + h_0(\mathbf{x}, \varrho) \leq h^\star,$$

the following holds. Denoting

$$T^{-1} = C (1 + \kappa^{-1} (|\underline{\mathbf{u}}'|_{L^2_{\varrho}}^2 + M_0^2)),$$

there exists a unique strong solution $(h, \mathbf{u}) \in \mathcal{C}([0, T]; H^{s, k}(\Omega)^{1+d})$ to the Cauchy problem associated with (2.1) and initial data $(h, \mathbf{u})|_{t=0} = (h_0, \mathbf{u}_0)$. Moreover, $h \in L^2(0, T; H^{s+1, k}(\Omega))$ and one has, for any $t \in [0, T]$, the lower and the upper bounds

$$\forall(\mathbf{x}, \varrho) \in \Omega, \quad h_\star/2 \leq \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \leq 2h^\star,$$

and the estimate

$$\begin{aligned} \mathcal{F}(t) := \|\eta(t, \cdot)\|_{H^{s, k}} + \|\mathbf{u}(t, \cdot)\|_{H^{s, k}} + |\eta|_{\varrho=\rho_0}(t, \cdot)|_{H_{\mathbf{x}}^s} + \kappa^{1/2} \|h(t, \cdot)\|_{H^{s, k}} \\ + \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta\|_{L^2(0, t; H^{s, k})} + \kappa^{1/2} |\nabla_{\mathbf{x}} \eta|_{\varrho=\rho_0}|_{L^2(0, t; H_{\mathbf{x}}^s)} + \kappa \|\nabla_{\mathbf{x}} h\|_{L^2(0, t; H^{s, k})} \leq C M_0. \end{aligned}$$

PROOF. Let us denote by $T^\star \in (0, +\infty]$ the maximal time of existence of $(h, \mathbf{u}) \in \mathcal{C}^0([0, T^\star]; H^{s, k}(\Omega))$ as provided by Proposition 2.1, and

$$T_\star = \sup \left\{ 0 < T < T^\star : \forall t \in (0, T), \quad h_\star/2 \leq \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \leq 2h^\star \quad \text{and} \quad \mathcal{F}(t) \leq C_0 M_0 \right\},$$

where $C_0 > 1$ will be determined later on. By the continuity in time of the solution, and using that the linear operator $h \mapsto \eta := \int_{\varrho}^{\rho_1} h(\cdot, \varrho') d\varrho'$ (resp. $h \mapsto \eta|_{\varrho=\rho_0}$) is well-defined and bounded from $H^{s, k}(\Omega)$ to itself (resp. $H_{\mathbf{x}}^s(\mathbb{R}^d)$) we have $T_\star > 0$. Using Lemma 2.3, 2.4 and 2.5 and, therein, the inequalities $\|h\|_{H^{s-1, k-1}} = \|\partial_{\varrho} \eta\|_{H^{s-1, k-1}} \leq \|\eta\|_{H^{s, k}}$, and (since $\nu \leq \kappa$) $\nu^{1/2} \|\nabla_{\mathbf{x}} h\|_{L^\infty(\Omega)} \leq \kappa^{1/2} \|h\|_{H^{s, k}}$, we find

that there exists $c_0 > 1$ depending only on $\rho_0 h_*, \rho_1 h^*$; and $C > 0$ depending on $\underline{M}, h_*, h^*, C_0 M_0$ such that for any $0 < t < T_*$,

$$\begin{aligned} & \|\eta(t, \cdot)\|_{H^{s,0}} + \|\mathbf{u}(t, \cdot)\|_{H^{s,0}} + |\eta|_{\varrho=\rho_0}(t, \cdot)|_{H_x^s} + \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta\|_{L^2(0,t;H^{s,0})} + \kappa^{1/2} |\nabla_{\mathbf{x}} \eta|_{\varrho=\rho_0}|_{L^2(0,t;H_x^s)} \\ & \leq c_0 \left(\|\eta_0\|_{H^{s,0}} + \|\mathbf{u}_0\|_{H^{s,0}} + |\eta_0|_{\varrho=\rho_0}|_{H_x^s} + C C_0 M_0 (t + \sqrt{t}) \right) \\ & \quad \times \exp \left(C \int_0^t (1 + \kappa^{-1} \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L_x^{\infty} L_{\varrho}^2}^2) d\tau \right); \quad (2.18) \end{aligned}$$

and (using a slightly adapted version of Lemma 2.5 which does not involve the trace of $\partial_{\varrho}^j \eta$ at the surface) for any $1 \leq j \leq k$

$$\begin{aligned} & \|\partial_{\varrho}^j \eta(t, \cdot)\|_{H^{s-j,0}} + \kappa^{1/2} \|\nabla_{\mathbf{x}} \partial_{\varrho}^j \eta\|_{L^2(0,t;H^{s-j,0})} \\ & \leq \left(\|\partial_{\varrho}^j \eta(0, \cdot)\|_{H^{s-j,0}} + C C_0 M_0 (t + \sqrt{t}) \right) \\ & \quad \times \exp \left(C \int_0^t \|\nabla_{\mathbf{x}} \cdot \mathbf{u}(\tau, \cdot)\|_{L^{\infty}(\Omega)} d\tau \right), \quad (2.19) \end{aligned}$$

and

$$\begin{aligned} & \|\partial_{\varrho}^j \mathbf{u}(t, \cdot)\|_{H^{s-j,0}} + \nu^{1/2} \|\nabla_{\mathbf{x}} \partial_{\varrho}^j \mathbf{u}\|_{L^2(0,t;H^{s-j,0})} \\ & \leq \left(\|\partial_{\varrho}^j \mathbf{u}(0, \cdot)\|_{H^{s-j,0}} + C C_0 M_0 (t + \sqrt{t}) \right) \\ & \quad \times \exp \left(C \int_0^t \|\nabla_{\mathbf{x}} \cdot (\mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{h+h})(\tau, \cdot)\|_{L^{\infty}(\Omega)} d\tau \right); \quad (2.20) \end{aligned}$$

and finally for any $0 \leq j \leq k$

$$\begin{aligned} & \kappa^{1/2} \|\partial_{\varrho}^j h(t, \cdot)\|_{H^{s-j,0}} + \kappa \|\nabla_{\mathbf{x}} \partial_{\varrho}^j h\|_{L^2(0,t;H^{s-j,0})} \\ & \leq \left(\kappa^{1/2} \|\partial_{\varrho}^j h(0, \cdot)\|_{H^{s-j,0}} + C C_0 M_0 (t + \sqrt{t}) \right) \\ & \quad \times \exp \left(C \int_0^t \|\nabla_{\mathbf{x}} \cdot \mathbf{u}(\tau, \cdot)\|_{L^{\infty}(\Omega)} d\tau \right). \quad (2.21) \end{aligned}$$

By the continuous embeddings $H^{s_0 + \frac{1}{2}, 1} \subset L_{\varrho}^{\infty} H^{s_0} \subset L^{\infty}(\Omega)$ for any $s_0 > d/2$ (see Lemma A.1) and since $k \geq 1$ and $s > \frac{3}{2} + \frac{d}{2}$, we have

$$\|\nabla_{\mathbf{x}} \cdot \mathbf{u}\|_{L^{\infty}(\Omega)} + \|\nabla_{\mathbf{x}} \cdot (\mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{h+h})\|_{L^{\infty}(\Omega)} \leq C(h_*) (\|\mathbf{u}\|_{H^{s,k}} + \kappa \|h\|_{H^{s,k}} + \kappa \|\nabla_{\mathbf{x}} h\|_{H^{s,k}}).$$

We deduce that

$$\mathcal{F}(t) \leq c \left(M_0 + C C_0 M_0 (t + \sqrt{t}) \right) \times \exp \left(C (t + \sqrt{t} + \kappa^{-1} \int_0^t \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}(\tau, \cdot)\|_{L_x^{\infty} L_{\varrho}^2}^2 d\tau) \right),$$

where we recall that $c_0 > 1$ depends only on h_* and h^* ; and $C > 0$ depends on $\underline{M}, C_0 M_0, h_*, h^*$. Hence choosing $C_0 = 2c_0$ and using that (by Lemma A.1 and since $k \geq 2$ and $s > \frac{3}{2} + \frac{d}{2}$)

$$\|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L_x^{\infty} L_{\varrho}^2} \leq \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L_{\varrho}^2 L_x^{\infty}}^2 \lesssim |\underline{\mathbf{u}}'|_{L_{\varrho}^2}^2 + \|\mathbf{u}\|_{H^{s,k}}^2 \leq |\underline{\mathbf{u}}'|_{L_{\varrho}^2}^2 + (C_0 M_0)^2,$$

we find that there exists $C_0 \geq 1$ depending only on $\underline{M}, M^*, h_*, h^*$ such that

$$t(1 + \kappa^{-1} (|\underline{\mathbf{u}}'|_{L_{\varrho}^2}^2 + M_0^2)) \leq C_0^{-1} \implies \mathcal{F}(t) \leq \frac{3}{4} C_0 M_0.$$

Now we remark that since

$$\partial_t h + \underline{\mathbf{u}} \cdot \nabla_{\mathbf{x}} h = \kappa \Delta_{\mathbf{x}} h + g \quad \text{with} \quad g = -\nabla_{\mathbf{x}} \cdot (h \underline{\mathbf{u}} + h \mathbf{u})$$

and by the positivity of the heat kernel we have

$$\inf_{\Omega} h(t, \cdot) \geq \inf_{\Omega} h_0 - \|g\|_{L^1(0,t;L^\infty(\Omega))}, \quad \sup_{\Omega} h(t, \cdot) \leq \sup_{\Omega} h_0 + \|g\|_{L^1(0,t;L^\infty(\Omega))}.$$

Now, by the continuous embedding $H^{s-1,1}(\Omega) \subset L^\infty(\Omega)$ (since $s > \frac{3}{2} + \frac{d}{2}$), we have that

$$\|g\|_{L^\infty(\Omega)} \lesssim |\underline{h}|_{W_\rho^1, \infty} \|\mathbf{u}\|_{H^{s,1}} + \|h\|_{H^{s,1}} \|\mathbf{u}\|_{H^{s,1}} \leq C(\underline{M})(1 + \kappa^{-1}M_0^2).$$

Hence augmenting C_0 if necessary we find that

$$t(1 + \kappa^{-1}M_0^2) \leq C_0^{-1} \implies \forall (\mathbf{x}, \varrho) \in \Omega, \quad \frac{2}{3}h_\star \leq \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \leq \frac{3}{2}h_\star.$$

By a continuity argument we infer $T_\star \geq \left(C(1 + \kappa^{-1}(|\underline{\mathbf{u}}'|_{L_\rho^2}^2 + M_0^2))\right)^{-1}$, and the proof is complete. \square

Completion of the proof of Theorem 1.1

In order to complete the proof of Theorem 1.1, there remains to consider vanishing viscosity limit, $\nu \searrow 0$, in Proposition 2.6. Let us briefly sketch the standard argument. By Proposition 2.6, we construct a family $(h_\nu, \mathbf{u}_\nu) \in \mathcal{C}^0([0, T]; H^{s,k}(\Omega))$ of solutions to (2.1) with $(h_\nu, \mathbf{u}_\nu)|_{t=0} = (h_0, \mathbf{u}_0)$ indexed by the parameter $\nu > 0$. Notice that the time of existence and associated bounds provided by Proposition 2.6 are uniform with respect to the parameter $\nu > 0$. Hence by the Banach-Alaoglu theorem there exists a subsequence which converges weakly towards $(h, \mathbf{u}) \in L^\infty(0, T; H^{s,k}(\Omega)^{1+d})$, satisfying the estimates of Proposition 2.6. Using the equations, we find that $(\partial_t \zeta_\nu, \partial_t \mathbf{u}_\nu)$ are uniformly bounded in $L^\infty(0, T; H^{s-2,k})$. The Aubin-Lions lemma (see [31]) implies that, up to extracting a subsequence, the convergence holds strongly in $(h, \mathbf{u}) \in \mathcal{C}^0([0, T]; H^{s',k}(B)^{1+d})$ for any $0 \leq s' < s$ for any bounded $B \subset \mathbb{R}^d \times (\rho_0, \rho_1)$. Choosing $s' > 3/2 + d/2$ and using Lemma A.1 and Sobolev embedding, we can pass to the limit in the nonlinear terms of the equation and infer that (h, \mathbf{u}) is a strong solution to (2.1) with $\nu = 0$. Moreover, since $(h, \mathbf{u}) \in \mathcal{C}^0([0, T]; H^{s-2,k}(\Omega)^{1+d})$, we have $(h, \mathbf{u}) \in \mathcal{C}^0([0, T]; H^{s',k}(\Omega)^{1+d})$ for any $0 \leq s' < s$.

Uniqueness of the solution $(h, \mathbf{u}) \in L^\infty(0, T; H^{s,k}(\Omega)^{1+d})$ follows by using Lemma 2.5 on the difference between two solutions, and Gronwall's Lemma.

There remains to prove that $(h, \mathbf{u}) \in \mathcal{C}^0([0, T]; H^{s,k}(\Omega)^{1+d})$. We prove the equivalent statement that for any $\alpha \in \mathbb{N}^d$ and $j \in \mathbb{N}$ such that $0 \leq |\alpha| + j \leq s$ and $0 \leq j \leq k$, $(\partial_\rho^j \partial_x^\alpha h, \partial_\rho^j \partial_x^\alpha \mathbf{u}) \in \mathcal{C}^0([0, T]; L^2(\Omega)^{1+d})$. By Lemma 2.3, as long as $\kappa > 0$, we can write

$$\begin{aligned} \partial_t(\partial_\rho^j \partial_x^\alpha h) - \kappa \Delta_x(\partial_\rho^j \partial_x^\alpha h) &= r_{\alpha,j} + \nabla_x \cdot \mathbf{r}_{\alpha,j}, \\ \partial_t(\partial_\rho^j \partial_x^\alpha \mathbf{u}) + (\mathbf{v} \cdot \nabla_x)(\partial_\rho^j \partial_x^\alpha \mathbf{u}) &= R_{\alpha,j}, \end{aligned}$$

with $(r_{\alpha,j}, \mathbf{r}_{\alpha,j}, R_{\alpha,j}) \in L^2(0, T; L^2(\Omega))^{1+2d}$ and $\mathbf{v}(\cdot, \varrho) := \mathbf{u}(\varrho) + (\mathbf{u} - \kappa \frac{\nabla_x h}{\underline{h} + h})(\cdot, \varrho) \in W^{k,\infty}((\rho_0, \rho_1))^d + L^2(0, T; H^{s,k}(\Omega))^d$. In other words, $\partial_\rho^j \partial_x^\alpha h$ satisfies a heat equation and continuity in time stems from the Duhamel formula, as already used in Proposition 2.1; and $\partial_\rho^j \partial_x^\alpha \mathbf{u}$ satisfies a transport equation and continuity in time is standard, see *e.g.* [2, Th. 3.19]. Let us acknowledge however that our situation is slightly different, since Ω is neither the Euclidean space or the torus, and advection occurs only in the direction \mathbf{x} (and not ϱ). It is however easy to adapt the proof of [2, Th. 3.19] to infer $\partial_\rho^j \partial_x^\alpha \mathbf{u} \in L^2(\rho_0, \rho_1; \mathcal{C}^0([0, T]; L^2(\mathbb{R}^d)))^d \subset \mathcal{C}^0([0, T]; L^2(\Omega))^d$ from the facts that $R_{\alpha,j} \in L^2(0, T; L^2(\Omega))^d \subset L^2(\rho_0, \rho_1; L^1(0, T; L^2(\mathbb{R}^d)))^d$ and $\nabla_x \mathbf{v} \in L^2(0, T; H^{s-1,k}(\Omega))^{d \times d} \subset L^1(0, T; L^\infty(\rho_0, \rho_1; H^{s-3/2}(\mathbb{R}^d)))^{d \times d}$ and $s - 3/2 > d/2$, the continuous embeddings following from Minkowski inequality and Lemma A.1.

This concludes the proof of Theorem 1.1.

3. The non-hydrostatic system

In this section we study the local well-posedness theory for the non-hydrostatic system in isopycnal coordinates, which we recall below.

$$\begin{aligned}
& \partial_t h + \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) = \kappa \Delta_{\mathbf{x}} h, \\
& \varrho \left(\partial_t \mathbf{u} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} \right) + \nabla_{\mathbf{x}} P + \frac{\nabla_{\mathbf{x}} \eta}{\underline{h} + h} (\partial_{\varrho} P + \varrho \underline{h}) = 0, \\
& \mu \varrho \left(\partial_t w + (\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}} w \right) - \frac{\partial_{\varrho} P}{\underline{h} + h} + \frac{\varrho h}{\underline{h} + h} = 0, \\
& -(\underline{h} + h) \nabla_{\mathbf{x}} \cdot \mathbf{u} - \nabla_{\mathbf{x}} \eta \cdot (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) + \partial_{\varrho} w = 0, \quad (\text{div.-free cond.}) \\
& \eta(\cdot, \varrho) = \int_{\varrho}^{\rho_1} h(\cdot, \varrho') d\varrho', \quad P|_{\varrho=\rho_0} = 0, \quad w|_{\varrho=\rho_1} = 0. \quad (\text{bound. cond.})
\end{aligned} \tag{3.1}$$

3.1. The pressure reconstruction. The first step of our analysis consists in showing how the pressure variable, P , can be uniquely reconstructed (thanks to the “divergence-free” incompressibility constraint) from prognostic variables \mathbf{u} , w and h (or, equivalently, η), through an elliptic boundary-value problem. Differentiating the “divergence-free” incompressibility constraint in (3.1) with respect to time yields

$$-(\underline{h} + h) \nabla_{\mathbf{x}} \cdot \partial_t \mathbf{u} - (\nabla_{\mathbf{x}} \eta) \cdot (\partial_{\varrho} \partial_t \mathbf{u}) + \partial_{\varrho} \partial_t w = (\partial_t h) (\nabla_{\mathbf{x}} \cdot \mathbf{u}) + (\nabla_{\mathbf{x}} \partial_t \eta) \cdot (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}).$$

We plug the expressions for $\partial_t \mathbf{u}$, $\partial_t w$, $\partial_t h$, $\partial_t \eta$ provided by (3.1) inside the above identity. Reorganizing terms, this yields the following

$$\begin{aligned}
& (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \left(\frac{1}{\varrho} \nabla_{\mathbf{x}} P + \frac{\nabla_{\mathbf{x}} \eta (\partial_{\varrho} P + \varrho \underline{h})}{\varrho (\underline{h} + h)} \right) + (\nabla_{\mathbf{x}} \eta) \cdot \left(\partial_{\varrho} \left(\frac{1}{\varrho} \nabla_{\mathbf{x}} P + \frac{\nabla_{\mathbf{x}} \eta (\partial_{\varrho} P + \varrho \underline{h})}{\varrho (\underline{h} + h)} \right) \right) + \partial_{\varrho} \left(\frac{\partial_{\varrho} P + \varrho \underline{h}}{\mu \varrho (\underline{h} + h)} \right) \\
& = -(\underline{h} + h) \nabla_{\mathbf{x}} \cdot \left(((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} \right) - (\nabla_{\mathbf{x}} \eta) \cdot \left(\partial_{\varrho} \left(((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} \right) \right) \\
& \quad + \partial_{\varrho} \left(((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}} w) \right) + \left(\kappa \Delta_{\mathbf{x}} h - \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) \right) (\nabla_{\mathbf{x}} \cdot \mathbf{u}) \\
& \quad + \left(\kappa \nabla_{\mathbf{x}} \Delta_{\mathbf{x}} \eta - \nabla_{\mathbf{x}} \int_{\varrho}^{\rho_1} \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) d\varrho' \right) \cdot (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}).
\end{aligned}$$

Using that $\partial_{\varrho} \nabla_{\mathbf{x}} \eta = -\nabla_{\mathbf{x}} h$ we can rewrite the left-hand side in a compact formulation as

$$(\text{LHS}) = \frac{1}{\mu} \begin{pmatrix} \sqrt{\mu} \nabla_{\mathbf{x}} \\ \partial_{\varrho} \end{pmatrix} \cdot \left(\begin{pmatrix} \frac{\underline{h} + h}{\varrho} \text{Id} & \frac{\sqrt{\mu} \nabla_{\mathbf{x}} \eta}{\varrho} \\ \frac{\sqrt{\mu} \nabla_{\mathbf{x}} \eta}{\varrho} & \frac{1 + \mu |\nabla_{\mathbf{x}} \eta|^2}{\varrho (\underline{h} + h)} \end{pmatrix} \begin{pmatrix} \sqrt{\mu} \nabla_{\mathbf{x}} \\ \partial_{\varrho} \end{pmatrix} (P + \underline{P}_{\text{eq}}) \right),$$

with $\underline{P}_{\text{eq}} := \int_{\rho_0}^{\varrho} \varrho' \underline{h}(\varrho') d\varrho'$. As for the right-hand side, we denote

$$\mathbf{u}_{\star} := -\kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}, \tag{3.2}$$

and we infer

$$\begin{aligned}
(\text{RHS}) & = -(\underline{h} + h) \nabla_{\mathbf{x}} \cdot \left(((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} \right) - (\nabla_{\mathbf{x}} \eta) \cdot \left(\partial_{\varrho} \left(((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} \right) \right) \\
& \quad + \partial_{\varrho} \left(((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot \nabla_{\mathbf{x}} w) \right) - \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star})) (\nabla_{\mathbf{x}} \cdot \mathbf{u}) \\
& \quad - \left(\nabla_{\mathbf{x}} \int_{\varrho}^{\rho_1} \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star})) d\varrho' \right) \cdot (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}).
\end{aligned}$$

Notice the identity (reminiscent of (1.7))

$$\int_{\varrho}^{\rho_1} \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star})) d\varrho' = (\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot \nabla_{\mathbf{x}} \eta - w - w_{\star}, \tag{3.3}$$

where

$$w_{\star} := \kappa \Delta_{\mathbf{x}} \eta - \kappa \frac{\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}} \eta}{\underline{h} + h}, \tag{3.4}$$

which is obtained by integrating with respect to ϱ the divergence-free identities

$$\begin{aligned} -(\underline{h} + h)\nabla_{\mathbf{x}} \cdot \mathbf{u} - (\nabla_{\mathbf{x}}\eta) \cdot (\underline{\mathbf{u}}' + \partial_{\varrho}\mathbf{u}) + \partial_{\varrho}w &= 0, \\ -(\underline{h} + h)\nabla_{\mathbf{x}} \cdot \mathbf{u}_{\star} - (\nabla_{\mathbf{x}}\eta) \cdot (\partial_{\varrho}\mathbf{u}_{\star}) + \partial_{\varrho}w_{\star} &= 0, \end{aligned}$$

integrating by parts with respect to ϱ , using the boundary condition $w|_{\varrho=\rho_1} = 0 = w_{\star}|_{\varrho=\rho_1}$ and $h = -\partial_{\varrho}\eta$. Hence the above can be equivalently written as

$$\begin{aligned} (\text{RHS}) &= -(\underline{h} + h)\nabla_{\mathbf{x}} \cdot ((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot \nabla_{\mathbf{x}}\mathbf{u}) - (\nabla_{\mathbf{x}}\eta) \cdot (\partial_{\varrho}((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot \nabla_{\mathbf{x}}\mathbf{u})) \\ &\quad + \partial_{\varrho}((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot \nabla_{\mathbf{x}}w) - \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}))(\nabla_{\mathbf{x}} \cdot \mathbf{u}) \\ &\quad - (\nabla_{\mathbf{x}}((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot \nabla_{\mathbf{x}}\eta - w - w_{\star})) \cdot (\underline{\mathbf{u}}' + \partial_{\varrho}\mathbf{u}). \end{aligned} \quad (3.5)$$

Taking into account the boundary conditions in (3.1), we find that the pressure satisfies the following problem (recalling $\underline{P}_{\text{eq}} := \int_{\rho_0}^{\varrho} \varrho' \underline{h}(\varrho') \, d\varrho'$):

$$\left\{ \begin{array}{l} \frac{1}{\mu} \begin{pmatrix} \sqrt{\mu}\nabla_{\mathbf{x}} \\ \partial_{\varrho} \end{pmatrix} \cdot \begin{pmatrix} \left(\begin{array}{cc} \frac{\underline{h}+h}{\varrho} \text{Id} & \frac{\sqrt{\mu}\nabla_{\mathbf{x}}\eta}{\varrho} \\ \frac{\sqrt{\mu}\nabla_{\mathbf{x}}^{\top}\eta}{\varrho} & \frac{1+\mu|\nabla_{\mathbf{x}}\eta|^2}{\varrho(\underline{h}+h)} \end{array} \right) \begin{pmatrix} \sqrt{\mu}\nabla_{\mathbf{x}} \\ \partial_{\varrho} \end{pmatrix} (P + \underline{P}_{\text{eq}}) \\ P|_{\varrho=\rho_0} = 0, \quad (\partial_{\varrho}P)|_{\varrho=\rho_1} = \rho_1 h|_{\varrho=\rho_1}. \end{array} \right. = (\text{RHS}), \quad (3.6)$$

This boundary value problem corresponds to [7, (7)] written in isopycnal coordinates, adapting the boundary conditions to the free-surface framework, and taking into account the effective transport velocities from eddy correlation. Following [7], we shall infer the existence and uniqueness as well as estimates on the pressure P from the elliptic theory applied to the above boundary value problem, as stated below.

LEMMA 3.1. Let $s_0 > d/2$, $s, k \in \mathbb{N}$ such that $s \geq s_0 + \frac{5}{2}$ and $1 \leq k \leq s$. Let $\underline{M}, M, h_{\star} > 0$. There exists $C > 0$ such that for any $\mu \in (0, 1]$, and for any \underline{h}, h, η satisfying the following bound

$$\|\underline{h}\|_{W_{\varrho}^{1 \vee k-1, \infty}} \leq \underline{M}, \quad \|h\|_{H^{s-1, 1 \vee k-1}} + \sqrt{\mu} \|\nabla_{\mathbf{x}}\eta\|_{H^{s-1, 1 \vee k-1}} \leq M;$$

(where we recall the notation $a \vee b = \max(a, b)$) and the stable stratification assumption

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h(\mathbf{x}, \varrho) \geq h_{\star};$$

and for any $(Q_0, Q_1, \mathbf{R}) \in H^{s, k-1}(\Omega)^2 \times H^{s, k}(\Omega)^{d+1}$, there exists a unique $P \in H^{s+1, k+1}(\Omega)$ solution to

$$\left\{ \begin{array}{l} \nabla_{\mathbf{x}, \varrho}^{\mu} \cdot (A^{\mu} \nabla_{\mathbf{x}, \varrho}^{\mu} P) = Q_0 + \sqrt{\mu} \Lambda Q_1 + \nabla_{\mathbf{x}, \varrho}^{\mu} \cdot \mathbf{R} \\ P|_{\varrho=\rho_0} = 0, \quad \mathbf{e}_{d+1} \cdot (A \nabla_{\mathbf{x}, \varrho}^{\mu} P)|_{\varrho=\rho_1} = \mathbf{e}_{d+1} \cdot \mathbf{R}|_{\varrho=\rho_1} \end{array} \right. \quad (3.7)$$

where we denote $\Lambda := (\text{Id} - \Delta_{\mathbf{x}})^{1/2}$,

$$\nabla_{\mathbf{x}, \varrho}^{\mu} := \begin{pmatrix} \sqrt{\mu}\nabla_{\mathbf{x}} \\ \partial_{\varrho} \end{pmatrix} \quad ; \quad A^{\mu} := \begin{pmatrix} \frac{\underline{h}+h}{\varrho} \text{Id} & \frac{\sqrt{\mu}\nabla_{\mathbf{x}}\eta}{\varrho} \\ \frac{\sqrt{\mu}\nabla_{\mathbf{x}}^{\top}\eta}{\varrho} & \frac{1+\mu|\nabla_{\mathbf{x}}\eta|^2}{\varrho(\underline{h}+h)} \end{pmatrix},$$

and one has, denoting $\|(Q_0, Q_1, \mathbf{R})\|_{r, j} := \|Q_0\|_{H^{r, j-1}} + \|Q_1\|_{H^{r, j-1}} + \|\mathbf{R}\|_{H^{r, j}}$,

$$\begin{aligned} \|P\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}, \varrho}^{\mu} P\|_{H^{s, k}} &\leq C \times \left(\|(Q_0, Q_1, \mathbf{R})\|_{s, k} \right. \\ &\quad \left. + (\|h\|_{H^{s, k}} + \sqrt{\mu} \|\nabla_{\mathbf{x}}\eta\|_{H^{s, k}}) \|(Q_0, Q_1, \mathbf{R})\|_{s-1, 1 \vee k-1} \right) \end{aligned} \quad (3.8)$$

and, when $k \geq 2$,

$$\|P\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}, \varrho}^{\mu} P\|_{H^{s-1, k-1}} \leq C \|(Q_0, Q_1, \mathbf{R})\|_{s-1, k-1}. \quad (3.9)$$

PROOF. Testing (3.7) with P , using integration by parts and the boundary conditions, we find

$$-\int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} A^\mu \nabla_{\mathbf{x}, \varrho}^\mu P \cdot \nabla_{\mathbf{x}, \varrho}^\mu P \, d\varrho \, d\mathbf{x} = \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} Q_0 P + Q_1 (\sqrt{\mu} \Lambda P) - \tilde{\mathbf{R}} \cdot \nabla_{\mathbf{x}, \varrho}^\mu P \, d\varrho \, d\mathbf{x}.$$

For $(Q_0, Q_1, \mathbf{R}) \in L^2(\Omega)^{2+d+1}$, the existence and uniqueness of a (variational) solution to (3.7) in the functional space

$$H_0^1(\Omega) := \{P \in L^2(\Omega) : \nabla_{\mathbf{x}, \varrho} P \in L^2(\Omega), P|_{\varrho=\rho_0} = 0\}$$

classically follows from the Lax-Milgram Lemma thanks to the boundedness and the coercivity of the matrix A (recall that $\underline{h} + h \geq h_\star > 0$ and the embedding of Lemma A.1), and the Poincaré inequality

$$\forall P \in H_0^1(\Omega), \quad \|P\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \left| \int_{\rho_0}^{\varrho} \partial_{\varrho'} P \, d\varrho' \right|^2 d\varrho \, d\mathbf{x} \leq (\rho_1 - \rho_0)^2 \|\partial_{\varrho} P\|_{L^2(\Omega)}^2, \quad (3.10)$$

and we have

$$\|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{L^2(\Omega)} \lesssim \|Q_0\|_{L^2(\Omega)} + \|Q_1\|_{L^2(\Omega)} + \|\mathbf{R}\|_{L^2(\Omega)}. \quad (3.11)$$

The desired regularity for $(Q_0, Q_1, \mathbf{R}) \in H^{s, k-1}(\Omega)^2 \times H^{s, k}(\Omega)^{d+1}$ is then deduced following the standard approach for elliptic equations (notice the domain is flat) from the estimates which we obtain below. For more details, we refer for instance to [22, Chapter 2] where a very similar elliptic problem is thoroughly studied. We now focus on the estimates, assuming *a priori* the needed regularity to justify the following computations.

First, we provide an estimate for $\|\nabla_{\mathbf{x}, \varrho} P\|_{H^{r, 0}(\Omega)}$ for $1 \leq r \leq s$. One readily checks that $P_r := \Lambda^r P$ with $\Lambda^r := (\text{Id} - \Delta_{\mathbf{x}})^{r/2}$ satisfies (3.7) with $Q_0 \leftarrow \Lambda^r Q_0$, $Q_1 \leftarrow \Lambda^r Q_1$ and $\mathbf{R} \leftarrow \Lambda^r \mathbf{R} - [\Lambda^r, A^\mu] \nabla_{\mathbf{x}, \varrho} P$. We focus now on the contribution of $\mathbf{P}_r := [\Lambda^r, A^\mu] \nabla_{\mathbf{x}, \varrho}^\mu P$. By continuous embedding (Lemma A.1) and commutator estimates (Lemma A.7), we have

$$\|\mathbf{P}_r\|_{L^2(\Omega)} \lesssim \|\nabla_{\mathbf{x}} A^\mu\|_{H^{s_0 + \frac{1}{2}, 1}} \|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{H^{r-1, 0}} + \left\langle \|\nabla_{\mathbf{x}} A^\mu\|_{H^{r-1, 0}} \|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{H^{s_0 + \frac{1}{2}, 1}} \right\rangle_{r > s_0 + 1}.$$

Hence, using product (Lemma A.2, A.3) and composition (Lemma A.4, A.6) estimates, we deduce

$$\|\mathbf{P}_r\|_{L^2(\Omega)} \leq C \left(\|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{H^{r-1, 0}} + \left\langle (\|h\|_{H^{r, 0}} + \sqrt{\mu} \|\nabla_{\mathbf{x}} \eta\|_{H^{r, 0}}) \|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{H^{s_0 + \frac{1}{2}, 1}} \right\rangle_{r > s_0 + 1} \right) \quad (3.12)$$

with $C = C(h_\star, \|\underline{h}\|_{W_\varrho^{1, \infty}}, \|(h, \sqrt{\mu} \nabla_{\mathbf{x}} \eta)\|_{H^{s_0 + \frac{3}{2}, 1}})$.

Plugging (3.12) in (3.11) and using continuous embedding (Lemma A.1) and $s_0 + \frac{3}{2} \leq s - 1$ yields

$$\begin{aligned} \|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{H^{r, 0}} &\lesssim \|Q_0\|_{H^{r, 0}} + \|Q_1\|_{H^{r, 0}} + \|\mathbf{R}\|_{H^{r, 0}} + \|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{H^{r-1, 0}} \\ &\quad + C (\|h\|_{H^{r, 0}} + \|\nabla_{\mathbf{x}} \eta\|_{H^{r, 0}}) \left\langle \|\nabla_{\mathbf{x}, \varrho} P\|_{H^{s_0 + \frac{1}{2}, 1}} \right\rangle_{r > s_0 + 1}, \end{aligned} \quad (3.13)$$

where we denote, here and thereafter, $a \lesssim b$ for $a \leq Cb$ with

$$C = C(h_\star, \|\underline{h}\|_{W_\varrho^{1, \infty}}, \|h\|_{H^{s-1, 1}}, \sqrt{\mu} \|\nabla_{\mathbf{x}} \eta\|_{H^{s-1, 1}}) = C(h_\star, \underline{M}, M).$$

Next we provide an estimate for $\|\nabla_{\mathbf{x}, \varrho} P\|_{H^{r, 1}(\Omega)}$ appearing in the above right-hand side. This term involves $\partial_\varrho^2 P$, which we control by rewriting (3.7) as

$$\frac{1+\mu \|\nabla_{\mathbf{x}} \eta\|^2}{\varrho(\underline{h}+h)} \partial_\varrho^2 P = -\partial_\varrho \left(\frac{1+\mu \|\nabla_{\mathbf{x}} \eta\|^2}{\varrho(\underline{h}+h)} \right) (\partial_\varrho P) - \nabla_{\mathbf{x}, \varrho}^\mu \cdot A_0^\mu \nabla_{\mathbf{x}, \varrho}^\mu P + Q_0 + \sqrt{\mu} \Lambda Q_1 + \nabla_{\mathbf{x}, \varrho}^\mu \cdot \mathbf{R} =: \tilde{\mathbf{R}} \quad (3.14)$$

where we denote

$$\begin{aligned} \nabla_{\mathbf{x}, \varrho}^\mu \cdot A_0^\mu \nabla_{\mathbf{x}, \varrho}^\mu P &:= \nabla_{\mathbf{x}, \varrho}^\mu \cdot \left(\begin{pmatrix} \frac{\underline{h}+h}{\varrho} \text{Id} & \frac{\sqrt{\mu} \nabla_{\mathbf{x}} \eta}{\varrho} \\ \frac{\sqrt{\mu} \nabla_{\mathbf{x}}^\top \eta}{\varrho} & 0 \end{pmatrix} \nabla_{\mathbf{x}, \varrho}^\mu P \right) \\ &= \frac{\mu}{\varrho} \nabla_{\mathbf{x}} \cdot (h \nabla_{\mathbf{x}} P + (\nabla_{\mathbf{x}} \eta) (\partial_\varrho P)) + \partial_\varrho \left(\frac{\mu}{\varrho} (\nabla_{\mathbf{x}} \eta) \cdot (\nabla_{\mathbf{x}} P) \right). \end{aligned}$$

When estimating the above, we use product estimates (Lemma A.2) and then continuous embedding (Lemma A.1), treating differently terms involving $\Delta_{\mathbf{x}}P$ or $\nabla_{\mathbf{x}}\partial_{\rho}P$: for instance

$$\|\Lambda^{r-1}(h\Delta_{\mathbf{x}}P)\|_{L^2(\Omega)} \lesssim \|h\|_{H^{s_0+\frac{1}{2},1}} \|\Delta_{\mathbf{x}}P\|_{H^{r-1,0}} + \langle \|h\|_{H^{r-\frac{1}{2},1}} \|\Delta_{\mathbf{x}}P\|_{H^{s_0,0}} \rangle_{r-1>s_0};$$

and terms involving only $\nabla_{\mathbf{x}}P$ or $\partial_{\rho}P$: for instance

$$\|\Lambda^{r-1}((\Delta_{\mathbf{x}}\eta)(\partial_{\rho}P))\|_{L^2(\Omega)} \lesssim \|\Delta_{\mathbf{x}}\eta\|_{H^{s_0+\frac{1}{2},1}} \|\partial_{\rho}P\|_{H^{r-1,0}} + \langle \|\Delta_{\mathbf{x}}\eta\|_{H^{r-1,0}} \|\partial_{\rho}P\|_{H^{s_0+\frac{1}{2},1}} \rangle_{r-1>s_0}.$$

We infer, using Lemma A.1, $\mu \in (0, 1]$ and $s_0 + \frac{3}{2} \leq s - 1$, that for any $1 \leq r \leq s$,

$$\begin{aligned} \|\partial_{\rho}^2 P\|_{H^{r-1,0}} &\lesssim \|Q_0\|_{H^{r-1,0}} + \|Q_1\|_{H^{r,0}} + \|\mathbf{R}\|_{H^{r,1}} + \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{r,0}} \\ &\quad + (\|h\|_{H^{r,1}} + \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{r,1}}) \langle \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s_0+1,1}} \rangle_{r>s_0+1}. \end{aligned} \quad (3.15)$$

By combining (3.13) and (3.15) we obtain

$$\begin{aligned} \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{r,1}} &\lesssim \|Q_0\|_{H^{r,0}} + \|Q_1\|_{H^{r,0}} + \|\mathbf{R}\|_{H^{r,1}} + \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{r-1,0}} \\ &\quad + C \langle (\|h\|_{H^{r,1}} + \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{r,1}}) \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s_0+1,1}} \rangle_{r>s_0+1} \end{aligned}$$

which, after finite induction on $1 \leq r \leq s$ and using (3.11) for the initialization, yields

$$\begin{aligned} \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{r,1}} &\lesssim \|Q_0\|_{H^{r,0}} + \|Q_1\|_{H^{r,0}} + \|\mathbf{R}\|_{H^{r,1}} \\ &\quad + (\|h\|_{H^{r,1}} + \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{r,1}}) \times \langle \|Q_0\|_{H^{r-1,0}} + \|Q_1\|_{H^{r-1,0}} + \|\mathbf{R}\|_{H^{r-1,1}} \rangle_{r>s_0+1}. \end{aligned} \quad (3.16)$$

This, together with (3.10), proves (3.8) when $k = 1$.

We now proceed to estimate higher ρ -derivatives. In what follows, we denote

$$C = C(h_{\star}, \|h\|_{W_{\rho}^{k-1,\infty}}, \|h\|_{H^{s-1,k-1}}, \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{s-1,k-1}}) = C(h_{\star}, \underline{M}, M).$$

Let $2 \leq j \leq k$. By definition, and using $\mu \in (0, 1]$, we have

$$\|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s,j}} \leq \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s,j-1}} + \|\partial_{\rho} \nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s-1,j-1}} \lesssim \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s,j-1}} + \|\partial_{\rho}^2 P\|_{H^{s-1,j-1}}.$$

We shall also use, when $j \leq k - 1$, the corresponding estimate

$$\|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s-1,j}} \lesssim \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s-1,j-1}} + \|\partial_{\rho}^2 P\|_{H^{s-2,j-1}}.$$

By using (3.14) (according to which $\partial_{\rho}^2 P = \frac{\rho(h+h)}{1+\mu|\nabla_{\mathbf{x}}\eta|^2} \tilde{\mathbf{R}}$), and since $1 \leq j - 1 \leq s - 1$, using Lemma A.3 and Lemma A.5 yields

$$\|\partial_{\rho}^2 P\|_{H^{s-1,j-1}} \lesssim \left\| \frac{\rho(h+h)}{1+\mu|\nabla_{\mathbf{x}}\eta|^2} \right\|_{H^{s-1,j-1}} \|\tilde{\mathbf{R}}\|_{H^{s-1,j-1}} \leq C \|\tilde{\mathbf{R}}\|_{H^{s-1,j-1}}.$$

If moreover $j \leq k - 1 \leq s - 1$, then

$$\|\partial_{\rho}^2 P\|_{H^{s-2,j-1}} \lesssim \left\| \frac{\rho(h+h)}{1+\mu|\nabla_{\mathbf{x}}\eta|^2} \right\|_{H^{s-2,j-1}} \|\tilde{\mathbf{R}}\|_{H^{s-2,j-1}} \leq C \|\tilde{\mathbf{R}}\|_{H^{s-2,j-1}}.$$

Applying Lemma A.3 and Lemma A.5 to $\tilde{\mathbf{R}}$ defined in (3.14), we obtain

$$\begin{aligned} \|\tilde{\mathbf{R}}\|_{H^{s-1,j-1}} &\leq \|Q_0\|_{H^{s-1,j-1}} + \|Q_1\|_{H^{s,j-1}} + \|\mathbf{R}\|_{H^{s,j}} \\ &\quad + C \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s,j-1}} + C \times (\|h\|_{H^{s,j}} + \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{s,j}}) \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s-1,j-1}} \end{aligned}$$

and, if moreover $j \leq k - 1 \leq s - 1$,

$$\|\tilde{\mathbf{R}}\|_{H^{s-2,j-1}} \leq \|Q_0\|_{H^{s-2,j-1}} + \|Q_1\|_{H^{s-1,j-1}} + \|\mathbf{R}\|_{H^{s-1,j}} + C \|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s-1,j-1}}.$$

From the second set of inequalities, (3.16) with $r = s - 1$ and finite induction on $2 \leq j \leq k - 1$ we infer

$$\|\nabla_{\mathbf{x},\rho}^{\mu} P\|_{H^{s-1,j}} \leq C (\|Q_0\|_{H^{s-1,j-1}} + \|Q_1\|_{H^{s-1,j-1}} + \|\mathbf{R}\|_{H^{s-1,j}}).$$

Then, from the first set of inequalities, (3.16) with $r = s$ and the previous result, we infer by finite induction on $2 \leq j \leq k$

$$\begin{aligned} \|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{H^{s,j}} &\leq C (\|Q_0\|_{H^{s,j-1}} + \|Q_1\|_{H^{s,j-1}} + \|\mathbf{R}\|_{H^{s,j}}) \\ &\quad + C (\|h\|_{H^{s,j}} + \sqrt{\mu} \|\nabla_{\mathbf{x}} \eta\|_{H^{s,j}}) (\|Q_0\|_{H^{s-1,j-2}} + \|Q_1\|_{H^{s-1,j-2}} + \|\mathbf{R}\|_{H^{s-1,j-1}}). \end{aligned}$$

The result is proved. \square

We now apply Lemma 3.1 to obtain several estimates on the solution to (3.5)-(3.6).

COROLLARY 3.2. Let $s_0 > d/2$, $s, k \in \mathbb{N}$ such that $s \geq s_0 + \frac{5}{2}$ and $2 \leq k \leq s$. Let $\underline{M}, M, h_\star > 0$. There exists $C > 0$ such that for any $\mu \in (0, 1]$ and $\kappa \in \mathbb{R}$, for any $(\underline{h}, \mathbf{u}) \in W^{k, \infty}((\rho_0, \rho_1))^{1+d}$, and for any $(h, \mathbf{u}, w) \in H^{s+1, k}(\Omega) \times H^{s, k}(\Omega)^d \times H^{s, k-1}(\Omega)$ satisfying (denoting $\eta(\cdot, \varrho) := \int_{\varrho}^{\rho_1} h(\cdot, \varrho') d\varrho'$)

- the following bounds

$$\begin{aligned} |\underline{h}|_{W_\varrho^{k, \infty}} + |\underline{\mathbf{u}}'|_{W_\varrho^{k-1, \infty}} &\leq \underline{M}, \\ \|h\|_{H^{s-1, k-1}} + \|\nabla_{\mathbf{x}} \eta\|_{H^{s-1, k-1}} + \|\mathbf{u}\|_{H^{s, k}} + \sqrt{\mu} \|w\|_{H^{s, k-1}} &\leq M; \end{aligned}$$

- the stable stratification assumption

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h(\mathbf{x}, \varrho) \geq h_\star;$$

- the incompressibility condition

$$-(\underline{h} + h) \nabla_{\mathbf{x}} \cdot \mathbf{u} - (\nabla_{\mathbf{x}} \eta) \cdot (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}) + \partial_\varrho w = 0,$$

there exists a unique solution $P \in H^{s+1, k+1}(\Omega)$ to (3.5)-(3.6) and one has

$$\begin{aligned} \|P\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}, \varrho}^\mu P\|_{H^{s, k}} &\leq C (1 + \|h\|_{H^{s, k}} + \sqrt{\mu} \|\nabla_{\mathbf{x}} \eta\|_{H^{s, k}}) \\ &\quad \times (\|h\|_{H^{s, k}} + \sqrt{\mu} \|\nabla_{\mathbf{x}} \eta\|_{H^{s, k}} + \|(\mathbf{u}, \mathbf{u}_\star)\|_{H^{s, k}} + \sqrt{\mu} \|(w, w_\star)\|_{H^{s, k-1}}) \end{aligned} \quad (3.17)$$

where we recall the notations $\mathbf{u}_\star := -\kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}$ and $w_\star := \kappa \Delta_{\mathbf{x}} \eta - \kappa \frac{\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}} \eta}{\underline{h} + h}$.

Moreover, decomposing

$$P = P_h + P_{\text{nh}}, \quad P_h := \int_{\rho_0}^{\varrho} \varrho' h(\cdot, \varrho') d\varrho',$$

we have

$$\begin{aligned} \|P_{\text{nh}}\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}, \varrho}^\mu P_{\text{nh}}\|_{H^{s-1, k-1}} &\leq C \sqrt{\mu} \left(\|\nabla_{\mathbf{x}} \eta\|_{H^{s-1, k-1}} + |\eta|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^s} \right. \\ &\quad \left. + \|(\mathbf{u}, \mathbf{u}_\star)\|_{H^{s, k}} + \sqrt{\mu} \|(w, w_\star)\|_{H^{s, k-1}} \right), \end{aligned} \quad (3.18)$$

and, setting $\Lambda^\mu = 1 + \sqrt{\mu} |D|$,

$$\begin{aligned} \|P_{\text{nh}}\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}, \varrho}^\mu P_{\text{nh}}\|_{H^{s-1, k-1}} &\leq C \mu \left(\|(\Lambda^\mu)^{-1} \nabla_{\mathbf{x}} \eta\|_{H^{s, k-1}} + |(\Lambda^\mu)^{-1} \eta|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^{s+1}} \right. \\ &\quad \left. + (|\underline{\mathbf{u}}'|_{W_\varrho^{k-1, \infty}} + \|\mathbf{u}\|_{H^{s, k}}) (\|(\mathbf{u}, \mathbf{u}_\star)\|_{H^{s, k}} + \|(w, w_\star)\|_{H^{s, k-1}}) + \|\mathbf{u}_\star\|_{H^{s, k}} \|w\|_{H^{s, k-1}} \right). \end{aligned} \quad (3.19)$$

PROOF. In view of Lemma 3.1, we shall first estimate (RHS), defined in (3.5). We decompose

$$(\text{RHS}) = R_1 + R_2$$

where R_1 is constituted by terms involving maximum one derivative on $h, \eta, \mathbf{u}, \mathbf{u}_\star, w, w_\star$, while

$$\begin{aligned} R_2 := &-(\underline{h} + h) \left((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} (\nabla_{\mathbf{x}} \cdot \mathbf{u}) \right) - (\nabla_{\mathbf{x}} \eta) \cdot \left((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} \partial_\varrho \mathbf{u} \right) \\ &+ (\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} \partial_\varrho w - \left((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} (\nabla_{\mathbf{x}} \eta) \right) \cdot (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}). \end{aligned}$$

Appealing to the incompressibility condition

$$-(\underline{h} + h)\nabla_{\mathbf{x}} \cdot \mathbf{u} - (\nabla_{\mathbf{x}}\eta) \cdot (\underline{\mathbf{u}}' + \partial_{\varrho}\mathbf{u}) + \partial_{\varrho}w = 0,$$

we have simply

$$R_2 = ((\underline{\mathbf{u}} + \mathbf{u} + \mathbf{u}_{\star}) \cdot (\nabla_{\mathbf{x}}h)) (\nabla_{\mathbf{x}} \cdot \mathbf{u}).$$

As a matter of fact, this term compensates with the second addend of R_1 , so that contributions from h are not differentiated. By inspecting the remaining terms and using Lemma A.3, we infer that for any $r \geq s_0 + 1/2$ and $1 \leq j \leq r \leq s - 1$,

$$\begin{aligned} \|(\text{RHS})\|_{H^{r,j}} &\lesssim (1 + |\underline{h}|_{W_{\varrho}^{j,\infty}} + \|h\|_{H^{r,j}} + \|\nabla_{\mathbf{x}}\eta\|_{H^{r,j}}) \\ &\times \left((\|\underline{\mathbf{u}}'\|_{W_{\varrho}^{j,\infty}} + \|\mathbf{u}\|_{H^{r+1,j+1}}) (\|(\mathbf{u}, \mathbf{u}_{\star})\|_{H^{r+1,j+1}} + \|(w, w_{\star})\|_{H^{r+1,j}}) + \|w\|_{H^{r+1,j}} \|\mathbf{u}_{\star}\|_{H^{r+1,j+1}} \right). \end{aligned} \quad (3.20)$$

Owing to the fact that contributions from (h, η, w, w_{\star}) to (RHS) are affine, and $\mu \in (0, 1]$, we have for any $r \geq s_0 + 3/2$ and $2 \leq j \leq r \leq s$

$$\begin{aligned} \sqrt{\mu} \|(\text{RHS})\|_{H^{r-1,j-1}} &\lesssim (1 + |\underline{h}|_{W_{\varrho}^{j-1,\infty}} + \sqrt{\mu} \|h\|_{H^{r-1,j-1}} + \sqrt{\mu} \|\nabla_{\mathbf{x}}\eta\|_{H^{r-1,j-1}}) \\ &\times \left((\|\underline{\mathbf{u}}'\|_{W_{\varrho}^{j-1,\infty}} + \|\mathbf{u}\|_{H^{r,j}}) (\|(\mathbf{u}, \mathbf{u}_{\star})\|_{H^{r,j}} + \sqrt{\mu} \|(w, w_{\star})\|_{H^{r,j-1}}) + \sqrt{\mu} \|w\|_{H^{r,j-1}} \|\mathbf{u}_{\star}\|_{H^{r,j}} \right). \end{aligned} \quad (3.21)$$

For the first estimate, we write (3.6) as (3.7) with $Q_0 = 0$, $Q_1 = \sqrt{\mu}\Lambda^{-1}(\text{RHS})$ and

$$\mathbf{R} = \begin{pmatrix} -\sqrt{\mu}\underline{h}\nabla_{\mathbf{x}}\eta \\ \frac{h}{\underline{h}+h}(1 + \mu|\nabla_{\mathbf{x}}\eta|^2) - \mu|\nabla_{\mathbf{x}}\eta|^2 \end{pmatrix}$$

where we used the identities

$$\frac{1 + \mu|\nabla_{\mathbf{x}}\eta|^2}{\varrho(\underline{h} + h)} \partial_{\varrho}P_{\text{eq}} = \frac{\underline{h}(1 + \mu|\nabla_{\mathbf{x}}\eta|^2)}{\underline{h} + h} = 1 + \mu|\nabla_{\mathbf{x}}\eta|^2 - \frac{h}{\underline{h} + h}(1 + \mu|\nabla_{\mathbf{x}}\eta|^2).$$

Product (Lemma A.3) and composition (Lemma A.6) estimates yield if $r \geq s_0 + 1/2$ and $1 \leq j \leq r \leq s - 1$

$$\|\mathbf{R}\|_{H^{r,j}} \leq C(h_{\star}, |\underline{h}|_{W_{\varrho}^{j,\infty}}, \|h\|_{H^{r,j}}, \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{r,j}}) \times (\|h\|_{H^{r,j}} + \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{r,j}}); \quad (3.22)$$

and, if $r \geq s_0 + 3/2$ and $2 \leq j \leq r \leq s$, using the tame estimates, we obtain

$$\|\mathbf{R}\|_{H^{r,j}} \leq C(h_{\star}, |\underline{h}|_{W_{\varrho}^{j,\infty}}, \|h\|_{H^{r-1,j-1}}, \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{r-1,j-1}}) \times (\|h\|_{H^{r,j}} + \sqrt{\mu}\|\nabla_{\mathbf{x}}\eta\|_{H^{r,j}}). \quad (3.23)$$

Plugging in (3.8) the estimates (3.21) and (3.23) with $(r, j) = (s, k)$, and (3.22) with $(r, j) = (s - 1, k - 1)$, yields (3.17).

For the next set of estimates, we notice that, by (3.6), $P - P_h$ satisfies (3.7) with $e_{d+1} \cdot \mathbf{R}|_{\varrho=\rho_1} = 0$ and $Q_0 + \sqrt{\mu}\Lambda Q_1 + \nabla_{\mathbf{x},\varrho}^{\mu} \cdot \mathbf{R} = \mu(\text{RHS}) + \nabla_{\mathbf{x},\varrho}^{\mu} \cdot \tilde{\mathbf{R}}$ where

$$\tilde{\mathbf{R}} := - \begin{pmatrix} \sqrt{\mu}\varrho^{-1}(\underline{h} + h)\nabla_{\mathbf{x}}\psi \\ \mu\varrho^{-1}(\nabla_{\mathbf{x}}\eta) \cdot (\nabla_{\mathbf{x}}\psi) \end{pmatrix}, \quad \psi := \int_{\rho_0}^{\varrho} \eta(\cdot, \varrho') d\varrho' + \rho_0\eta|_{\varrho=\rho_0}.$$

Indeed, we have immediately $e_{d+1} \cdot \tilde{\mathbf{R}}|_{\varrho=\rho_1} = 0 = \partial_{\varrho}(P - P_h)|_{\varrho=\rho_1}$ and we infer

$$-\nabla_{\mathbf{x},\varrho}^{\mu} \cdot \tilde{\mathbf{R}} = \nabla_{\mathbf{x},\varrho}^{\mu} \cdot (A^{\mu}\nabla_{\mathbf{x},\varrho}^{\mu}(P_{\text{eq}} + P_h))$$

from (integrating by parts as in (1.9b))

$$P_h := \int_{\rho_0}^{\varrho} \varrho' h(\cdot, \varrho') d\varrho' = -\varrho\eta + \int_{\rho_0}^{\varrho} \eta(\cdot, \varrho') d\varrho' + \rho_0\eta|_{\varrho=\rho_0}. \quad (3.24)$$

By product estimates (Lemma A.3), we infer immediately for any $r \geq s_0 + 1/2$ and $1 \leq j \leq r$

$$\|\tilde{\mathbf{R}}\|_{H^{r,j}} \lesssim \left(|\underline{h}|_{W_\rho^{j,\infty}} + \|h\|_{H^{r,j}} + \sqrt{\mu} \|\nabla_{\mathbf{x}} \eta\|_{H^{r,j}} \right) \left(\sqrt{\mu} \|\nabla_{\mathbf{x}} \eta\|_{H^{r,j}} + \sqrt{\mu} |\eta|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^{r+1}} \right). \quad (3.25)$$

Moreover, using the identity

$$\nabla_{\mathbf{x},\varrho}^\mu \cdot \tilde{\mathbf{R}} = \mu \nabla_{\mathbf{x}} \eta \cdot \partial_\varrho (\varrho^{-1} \nabla_{\mathbf{x}} \psi) + \mu \varrho^{-1} (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \psi$$

we infer for any $r \geq s_0 + 1/2$ and $1 \leq j \leq r$

$$\|\nabla_{\mathbf{x},\varrho}^\mu \cdot \tilde{\mathbf{R}}\|_{H^{r,j}} \lesssim \mu \left(|\underline{h}|_{W_\rho^{j,\infty}} + \|h\|_{H^{r,j}} + \|\nabla_{\mathbf{x}} \eta\|_{H^{r,j}} \right) \left(\|\nabla_{\mathbf{x}} \eta\|_{H^{r+1,j}} + |\eta|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^{r+2}} \right). \quad (3.26)$$

Finally, recalling $\Lambda^\mu = 1 + \sqrt{\mu}|D|$ and introducing

$$\tilde{\mathbf{R}}_0 := - \left(\sqrt{\mu} \varrho^{-1} (\underline{h} + (\Lambda^\mu)^{-1} h) ((\Lambda^\mu)^{-1} \nabla_{\mathbf{x}} \psi) \right. \\ \left. \mu \varrho^{-1} ((\Lambda^\mu)^{-1} \nabla_{\mathbf{x}} \eta) \cdot ((\Lambda^\mu)^{-1} \nabla_{\mathbf{x}} \psi), \right)$$

proceeding as for (3.25) and (3.26) and using $\text{Id} - (\Lambda^\mu)^{-1} = \sqrt{\mu}|D|(\Lambda^\mu)^{-1}$ and $\|(\Lambda^\mu)^{-1}\|_{L_{\mathbf{x}}^2 \rightarrow L_{\mathbf{x}}^2} = 1$, for any $r \geq s_0 + 1/2$ and $1 \leq j \leq r$ one has

$$\|\tilde{\mathbf{R}} - \tilde{\mathbf{R}}_0\|_{H^{r,j}} + \|\nabla_{\mathbf{x},\varrho}^\mu \cdot \tilde{\mathbf{R}}_0\|_{H^{r,j}} \lesssim \mu \left(|\underline{h}|_{W_\rho^{j,\infty}} + \|h\|_{H^{r,j}} + \|\nabla_{\mathbf{x}} \eta\|_{H^{r,j}} \right) \\ \times \left(\|(\Lambda^\mu)^{-1} \nabla_{\mathbf{x}} \eta\|_{H^{r+1,j}} + |(\Lambda^\mu)^{-1} \eta|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^{r+2}} \right). \quad (3.27)$$

We obtain (3.18), by setting $Q_0 = \mu(\text{RHS})$ and $Q_1 = 0$, and plug in (3.9) the estimates (3.21) with $(r, j) = (s, k)$, and (3.25) with $(r, j) = (s-1, k-1)$. For (3.19), we set instead $Q_0 = \mu(\text{RHS}) + \nabla_{\mathbf{x},\varrho}^\mu \cdot \tilde{\mathbf{R}}_0$ and $Q_1 = 0$, $\mathbf{R} = \tilde{\mathbf{R}} - \tilde{\mathbf{R}}_0$, and plug in (3.9) the estimates (3.20) and (3.27) with $(r, j) = (s-1, k-1)$. \square

3.2. Small-time well-posedness. We infer small-time existence and uniqueness of regular solutions to the Cauchy problem associated with the non-hydrostatic problem, (3.1), proceeding as for the hydrostatic system in Section 2, that is considering the system as the combination of a transport-diffusion equation and transport equations, coupled through order-zero source terms (by the estimate (3.17) in Corollary 3.2). Specifically, we rewrite (3.1) as

$$\partial_t h + \nabla_{\mathbf{x}} \cdot ((\underline{h} + h)(\underline{\mathbf{u}} + \mathbf{u})) = \kappa \Delta_{\mathbf{x}} h, \\ \partial_t \mathbf{u} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} + \frac{1}{\varrho} \nabla_{\mathbf{x}} P + \left(1 + \frac{\partial_\varrho P - \varrho h}{\varrho(\underline{h} + h)} \right) \nabla_{\mathbf{x}} \eta = 0, \\ \partial_t w + (\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}} w - \frac{1}{\mu} \frac{\partial_\varrho P - \varrho h}{\varrho(\underline{h} + h)} = 0, \quad (3.28)$$

where $\eta(\cdot, \varrho) = \int_\varrho^{\rho_1} h(\cdot, \varrho') d\varrho'$ and P is defined by Corollary 3.2. Systems (3.1) and (3.28) are equivalent (for sufficiently regular data) by the computations of Section 3.1, and in particular regular solutions to (3.28) satisfy the boundary condition $w|_{\varrho=\rho_1} = 0$ and the incompressibility constraint

$$(\underline{h} + h) \nabla_{\mathbf{x}} \cdot \mathbf{u} + \nabla_{\mathbf{x}} \eta \cdot (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}) - \partial_\varrho w = 0 \quad (3.29)$$

provided these identities hold initially.

PROPOSITION 3.3. Let $s_0 > d/2$, $s, k \in \mathbb{N}$ such that $s \geq s_0 + \frac{5}{2}$ and $2 \leq k \leq s$. Let $h_*, \underline{M}, M, \mu, \kappa > 0$ and $C_0 > 1$. There exists $T > 0$ such that for any $(\underline{h}, \underline{\mathbf{u}}) \in W^{k,\infty}((\rho_0, \rho_1))^{1+d}$, and for any initial data $(h_0, \mathbf{u}_0, w_0) \in H^{s+1,k}(\Omega) \times H^{s,k}(\Omega)^d \times H^{s,k}(\Omega)$ satisfying

- the following bounds (where $\eta_0(\cdot, \varrho) := \int_\varrho^{\rho_1} h_0(\cdot, \varrho') d\varrho'$)

$$|\underline{h}|_{W_\rho^{k,\infty}} + |\underline{\mathbf{u}}'|_{W_\rho^{k-1,\infty}} \leq \underline{M}$$

$$M_0 := \|h_0\|_{H^{s,k}} + \|\nabla_{\mathbf{x}} \eta_0\|_{H^{s,k}} + \|\mathbf{u}_0\|_{H^{s,k}} + \|w_0\|_{H^{s,k}} \leq M,$$

- the stable stratification assumption

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h_0(\mathbf{x}, \varrho) \geq h_\star,$$

- the boundary condition $w_0|_{\varrho=\rho_1} = 0$ and the incompressibility condition

$$-(\underline{h} + h_0)\nabla_{\mathbf{x}} \cdot \mathbf{u}_0 - (\nabla_{\mathbf{x}}\eta_0) \cdot (\mathbf{u}' + \partial_{\varrho}\mathbf{u}_0) + \partial_{\varrho}w_0 = 0,$$

there exists a unique $(h, \mathbf{u}, w) \in \mathcal{C}^0(0, T; H^{s,k}(\Omega)^{2+d})$ and $P \in L^2(0, T; H^{s+1, k+1}(\Omega))$ solution to (3.28) satisfying the initial data $(h, \mathbf{u}, w)|_{t=0} = (h_0, \mathbf{u}_0, w_0)$. Moreover, the conditions $w|_{\varrho=\rho_1} = P|_{\varrho=\rho_0} = 0$ and the incompressibility condition (3.29) hold on $[0, T]$ (and hence the solution satisfies (3.1)) and one has $\eta \in L^\infty(0, T; H^{s+1, k}(\Omega))$ and $(h, \nabla_{\mathbf{x}}\eta) \in L^2(0, T; H^{s+1, k}(\Omega))$ and

$$\begin{aligned} \mathcal{F}_{s,k,T} := & \|h\|_{L^\infty(0,T;H^{s,k})} + \|\nabla_{\mathbf{x}}\eta\|_{L^\infty(0,T;H^{s,k})} + \|\mathbf{u}\|_{L^\infty(0,T;H^{s,k})} + \|w\|_{L^\infty(0,T;H^{s,k})} \\ & + c_0\kappa^{1/2}\|h\|_{L^2(0,T;H^{s,k})} + c_0\kappa^{1/2}\|\nabla_{\mathbf{x}}^2\eta\|_{L^2(0,T;H^{s,k})} \leq C_0M_0 \end{aligned}$$

with c_0 a universal constant.

PROOF. Since a very similar proof has been detailed in the hydrostatic framework in Section 2, we will only briefly sketch the main arguments. As aforementioned, thanks to Corollary 3.2, we may consider the contributions of the pressure as zero-order source terms in the energy space displayed in the statement, and (3.28) is then interpreted as a standard set of evolution equations. We now explain how to infer the necessary bounds on all contributions to $\mathcal{F}_{s,k,T}$, assuming enough regularity.

The desired control of $\|h\|_{L^\infty(0,T;H^{s,k})} + c_0\kappa^{1/2}\|\nabla_{\mathbf{x}}h\|_{L^2(0,T;H^{s,k})}$ is a direct consequence of the first equation of (3.28), and the regularization properties of the heat semigroup already summoned in Proposition 2.1. The corresponding control of $\|\nabla_{\mathbf{x}}\eta\|_{L^\infty(0,T;H^{s,k})} + c_0\kappa^{1/2}\|\nabla_{\mathbf{x}}^2\eta\|_{L^2(0,T;H^{s,k})}$ demands an additional structure. We recall (see (1.7) or (3.3)) that by the identity (3.29) and integrating the first equation of (3.28), one has

$$\partial_t\eta + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}}\eta - w = \kappa\Delta_{\mathbf{x}}\eta. \quad (3.30)$$

By the regularization properties of the heat semigroup, we infer (with c_0 a universal constant)

$$\|\nabla_{\mathbf{x}}\eta\|_{L^\infty(0,T;H^{s,k})} + c_0\kappa^{1/2}\|\nabla_{\mathbf{x}}^2\eta\|_{L^2(0,T;H^{s,k})} \leq \|\nabla_{\mathbf{x}}\eta_0\|_{H^{s,k}} + \frac{1}{c_0\kappa^{1/2}}\|(\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}}\eta - w\|_{L^2(0,T;H^{s,k})},$$

and the right-hand side is estimated by product estimates (Lemma A.3). Finally, the desired *a priori* estimates on $\|\mathbf{u}\|_{L^\infty(0,T;H^{s,k})}$ and $\|w\|_{L^\infty(0,T;H^{s,k})}$ for sufficiently regular solutions follow by the energy method (that is integrating by parts in the variable \mathbf{x}) on the second and third equations of (3.28), which can be seen as transport equations with source terms. More precisely, by (3.17) in Corollary 3.2, we have the existence and uniqueness of $P \in L^2(0, T; H^{s+1, k+1}(\Omega))$, satisfying the bound

$$\|P\|_{L^2(0,T;H^{s+1,k+1})} \leq C(h_\star, \mu, \kappa, \underline{M}, \mathcal{F}_{s,k,T})\mathcal{F}_{s,k,T}.$$

Moreover, the advection velocity is controlled (using Lemma A.1, $s-2 \geq s_0 + \frac{1}{2}$, $k \geq 1$) by

$$\|\nabla \cdot (\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}}h}{\underline{h}+h})\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C(h_\star, \kappa, \mathcal{F}_{s,k,T}),$$

and using commutator (Lemma A.8) and composition (Lemma A.6) estimates, one has for any $f \in H^{s,k}(\Omega)$, and any $\alpha \in \mathbb{N}^d$, $j \in \mathbb{N}$ with $0 \leq j \leq k$ and $|\alpha| + j \leq s$,

$$\|[\partial_{\mathbf{x}}^\alpha \partial_{\varrho}^j, \underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}}h}{\underline{h}+h}]\nabla_{\mathbf{x}}f\|_{L^2(0,T;L^2(\Omega))} \leq C(h_\star, \kappa, \underline{M}, \mathcal{F}_{s,k,T})\|f\|_{H^{s,k}}.$$

It follows

$$\|\mathbf{u}\|_{H^{s,k}} + \|w\|_{H^{s,k}} \leq \left(\|\mathbf{u}_0\|_{H^{s,k}} + \|w_0\|_{H^{s,k}} + C\sqrt{T} \right) \exp(CT),$$

with $C = C(h_\star, \mu, \kappa, \underline{M}, \mathcal{F}_{s,k,T})$.

Altogether, and using standard continuity arguments, we find that for any $C_0 > 1$ we can restrict the time $T = T(h_\star, \mu, \kappa, \underline{M}, C_0M_0) > 0$ so that all sufficiently regular solutions to (3.28) satisfy the bound

$\mathcal{F}_{s,k,T} \leq C_0 M_0$. We may infer the existence of solutions using for instance the parabolic regularization approach (see the closing paragraph of Section 2), and uniqueness is straightforward. This concludes the proof. \square

REMARK 3.4. Proposition 3.3 does not provide any lower bound on the time of existence (and control) of solutions with respect to either $\mu \ll 1$ or $\kappa \ll 1$, hence the ‘‘small-time’’ terminology.

3.3. Quasi-linearization of the non-hydrostatic system. In this section we extract the leading-order terms of the equations satisfied by the spatial derivatives of the solutions to system (3.28). This will allow us to obtain improved energy estimates in the subsequent section. Notice that starting from here, our study is restricted to the situation $k = s$.

LEMMA 3.5. Let $s, k \in \mathbb{N}$ such that $k = s > \frac{5}{2} + \frac{d}{2}$ and $\underline{M}, M, h_\star > 0$. Then there exists $C > 0$ such that for any $\mu, \kappa \in (0, 1]$, and for any $(\underline{h}, \underline{u}) \in W^{k,\infty}((\rho_0, \rho_1)) \times W^{k+1,\infty}((\rho_0, \rho_1))$ satisfying

$$|\underline{h}|_{W_\varrho^{k,\infty}} + |\underline{u}'|_{W_\varrho^{k,\infty}} \leq \underline{M};$$

and any $(h, \mathbf{u}, w) \in L^\infty(0, T; H^{s,k}(\Omega)^{d+2})$ solution to (3.28) (with P defined by Corollary 3.2) with any $T > 0$ and satisfying for almost every $t \in [0, T]$,

$$\|h(t, \cdot)\|_{H^{s-1,k-1}} + \|\eta(t, \cdot)\|_{H^{s,k}} + \|\eta|_{\varrho=\rho_0}(t, \cdot)\|_{H_{\mathbf{x}}^s} + \|\mathbf{u}(t, \cdot)\|_{H^{s,k}} + \sqrt{\mu} \|w(t, \cdot)\|_{H^{s,k}} + \kappa^{1/2} \|h(t, \cdot)\|_{H^{s,k}} \leq M$$

(where $\eta(t, \mathbf{x}, \varrho) := \int_{\varrho}^{\rho_1} h(t, \mathbf{x}, \varrho') d\varrho'$) and

$$\inf_{(t, \mathbf{x}, \varrho) \in (0, T) \times \Omega} \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \geq h_\star,$$

the following results hold.

Denote, for any multi-index $\alpha \in \mathbb{N}^d$ and any $j \in \mathbb{N}$ such that $|\alpha| + j \leq s$, $h^{(\alpha,j)} = \partial_\varrho^j \partial_{\mathbf{x}}^\alpha h$, $\eta^{(\alpha,j)} = \partial_\varrho^j \partial_{\mathbf{x}}^\alpha \eta$, $\mathbf{u}^{(\alpha,j)} = \partial_\varrho^j \partial_{\mathbf{x}}^\alpha \mathbf{u}$, $w^{(\alpha,j)} = \partial_\varrho^j \partial_{\mathbf{x}}^\alpha w$, and $P_{\text{nh}}^{(\alpha)} = \partial_\varrho^j \partial_{\mathbf{x}}^\alpha P_{\text{nh}}$, with

$$P_{\text{nh}} := P - P_{\text{h}}, \quad P_{\text{h}} := \int_{\rho_0}^{\varrho} \varrho' h(\cdot, \varrho') d\varrho'.$$

We have

$$\begin{aligned} \partial_t \eta^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} - w^{(\alpha,j)} - \kappa \Delta_{\mathbf{x}} \eta^{(\alpha,j)} &= \tilde{R}_{\alpha,j}, \\ \partial_t \eta^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} + \left\langle \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} d\varrho' \right. \\ &\quad \left. + \int_{\varrho}^{\rho_1} (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(\alpha,j)} d\varrho' \right\rangle_{j=0} - \kappa \Delta_{\mathbf{x}} \eta^{(\alpha,j)} = R_{\alpha,j}, \\ \partial_t \mathbf{u}^{(\alpha,j)} + \left((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}} \right) \mathbf{u}^{(\alpha,j)} + \left\langle \frac{\rho_0}{\varrho} \nabla_{\mathbf{x}} \eta^{(\alpha,j)} \Big|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta^{(\alpha,j)} d\varrho' \right\rangle_{j=0} \\ &\quad + \frac{1}{\varrho} \nabla_{\mathbf{x}} P_{\text{nh}}^{(\alpha,j)} + \frac{\nabla_{\mathbf{x}} \eta}{\varrho(\underline{h} + h)} \partial_\varrho P_{\text{nh}}^{(\alpha,j)} = \mathbf{R}_{\alpha,j}^{\text{nh}}, \\ \sqrt{\mu} \left(\partial_t w^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}} w^{(\alpha,j)} \right) - \frac{1}{\sqrt{\mu}} \frac{\partial_\varrho P_{\text{nh}}^{(\alpha,j)}}{\varrho(\underline{h} + h)} &= R_{\alpha,j}^{\text{nh}}, \\ -\partial_\varrho w^{(\alpha)} + (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(\alpha,j)} + (\nabla_{\mathbf{x}} \eta) \cdot (\partial_\varrho \mathbf{u}^{(\alpha,j)}) \\ &\quad + (\nabla_{\mathbf{x}} \cdot \mathbf{u}) h^{(\alpha,j)} + (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} = R_{\alpha,j}^{\text{div}}, \end{aligned} \tag{3.31a}$$

where for almost every $t \in [0, T]$, one has $(R_{\alpha,j}(t, \cdot), \mathbf{R}_{\alpha,j}^{\text{nh}}(t, \cdot), R_{\alpha,j}^{\text{nh}}(t, \cdot), R_{\alpha,j}^{\text{div}}(t, \cdot)) \in L^2(\Omega)^{d+3}$ and $R_{\alpha,0}(t, \cdot) \in \mathcal{C}((\rho_0, \rho_1); L^2(\mathbb{R}^d))$, and

$$\begin{aligned} & \|R_{\alpha,j}\|_{L^2(\Omega)} + |R_{\alpha,0}|_{\varrho=\rho_0}|_{L_{\mathbb{x}}^2} + \|\tilde{R}_{\alpha,j}\|_{L^2(\Omega)} + \|R_{\alpha,j}^{\text{div}}\|_{L^2(\Omega)} \leq C M, \\ & \|\mathbf{R}_{\alpha,j}^{\text{nh}}\|_{L^2(\Omega)} + \|R_{\alpha,j}^{\text{nh}}\|_{L^2(\Omega)} \leq C M (1 + \kappa \|\nabla_{\mathbf{x}} h\|_{H^{s,k}}) \\ & \quad + C (\|h\|_{H^{s,k}} + \sqrt{\mu} \|\nabla_{\mathbf{x}} \eta\|_{H^{s,k}}) (M + \|\mathbf{u}_\star\|_{H^{s,k}} + \sqrt{\mu} \|w_\star\|_{H^{s,k-1}}), \end{aligned} \quad (3.31b)$$

and

$$\partial_t h^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} h^{(\alpha,j)} = \kappa \Delta_{\mathbf{x}} h^{(\alpha,j)} + r_{\alpha,j} + \nabla_{\mathbf{x}} \cdot \mathbf{r}_{\alpha,j}, \quad (3.32a)$$

where for almost every $t \in [0, T]$, one has $(r_{\alpha,j}(t, \cdot), \mathbf{r}_{\alpha,j}(t, \cdot)) \in L^2(\Omega)^{1+d}$ and

$$\kappa^{1/2} \|r_{\alpha,j}\|_{L^2(\Omega)} + \|\mathbf{r}_{\alpha,j}\|_{L^2(\Omega)} \leq C M. \quad (3.32b)$$

PROOF. Let us first point out that the estimates for $|R_{\alpha,0}|_{\varrho=\rho_0}|_{L_{\mathbb{x}}^2}$, $\|R_{\alpha,j}\|_{L^2(\Omega)}$, $\|r_{\alpha,j}\|_{L^2(\Omega)}$ and $\|\mathbf{r}_{\alpha,j}\|_{L^2(\Omega)}$ have been stated and proved in Lemma 2.3. Thus we only need to focus on the other terms. In the following, we denote $s_0 = s - \frac{5}{2} > \frac{d}{2}$.

Using the identity (already pointed out in (3.30))

$$\partial_t \eta + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta - w = \kappa \Delta_{\mathbf{x}} \eta,$$

and the commutator estimate in Lemma A.8, we find immediately

$$\tilde{R}_{\alpha,j} = [\partial_{\varrho}^j \partial_{\mathbf{x}}^{\alpha}, \underline{\mathbf{u}} + \mathbf{u}] \cdot \nabla_{\mathbf{x}} \eta, \quad \|\tilde{R}_{\alpha,j}\|_{L^2(\Omega)} \lesssim (|\underline{\mathbf{u}}'|_{W_{\varrho}^{k-1,\infty}} + \|\mathbf{u}\|_{H^{s,k}}) \|\eta\|_{H^{s,k}}.$$

Using the decomposition $P = P_{\text{h}} + P_{\text{nh}}$ as in Corollary 3.2 and the identity (3.24) we have

$$\nabla_{\mathbf{x}} P_{\text{h}} := \int_{\rho_0}^{\varrho} \varrho' \nabla_{\mathbf{x}} h(\cdot, \varrho') \, d\varrho' = -\varrho \nabla_{\mathbf{x}} \eta + \rho_0 \nabla_{\mathbf{x}} \eta|_{\varrho=\rho_0} + \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta(\cdot, \varrho') \, d\varrho',$$

and hence the evolution equation for \mathbf{u} reads

$$\begin{aligned} \partial_t \mathbf{u} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\mathbf{x}} h}{\underline{h} + h}) \cdot \nabla_{\mathbf{x}}) \mathbf{u} + \frac{\rho_0}{\varrho} \nabla_{\mathbf{x}} \eta|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta(\cdot, \varrho') \, d\varrho' \\ + \frac{1}{\varrho} \nabla_{\mathbf{x}} P_{\text{nh}} + \frac{\nabla_{\mathbf{x}} \eta}{\varrho(\underline{h} + h)} \partial_{\varrho} P_{\text{nh}} = 0. \end{aligned}$$

Differentiating α times with respect to \mathbf{x} and j times with respect to ϱ yields the corresponding equations in (3.31a), with remainder terms

$$\mathbf{R}_{\alpha,j}^{\text{nh}} := \mathbf{R}_{\alpha,j} - [\partial_{\varrho}^j \partial_{\mathbf{x}}^{\alpha}, \frac{\nabla_{\mathbf{x}} \eta}{\varrho(\underline{h} + h)}] \partial_{\varrho} P_{\text{nh}},$$

using the notation $\mathbf{R}_{\alpha,j}$ for the hydrostatic contributions introduced in Lemma 2.3. The first addends have been estimated in Lemma 2.3, (2.3b) (when $j = 0$) and (2.4b) (when $j \geq 1$). We now estimate the second addend as follows. By the commutator estimate in Lemma A.8 with $k = s \geq s_0 + 3/2$, we have

$$\|[\partial_{\varrho}^j \partial_{\mathbf{x}}^{\alpha}, \frac{\nabla_{\mathbf{x}} \eta}{\varrho(\underline{h} + h)}] \partial_{\varrho} P_{\text{nh}}\|_{L^2(\Omega)} \lesssim \|\frac{\nabla_{\mathbf{x}} \eta}{\varrho(\underline{h} + h)}\|_{H^{s,k}} \|\partial_{\varrho} P_{\text{nh}}\|_{H^{s-1,k-1}}.$$

Then by tame product estimate Lemma A.3 and composition estimates in Lemma A.6, we have

$$\|\frac{\nabla_{\mathbf{x}} \eta}{\varrho(\underline{h} + h)}\|_{H^{s,k}} \leq C(h_\star, |h|_{W^{k,\infty}}, \|h\|_{H^{s-1,k-1}}) (\|h\|_{H^{s,k}} \|\nabla_{\mathbf{x}} \eta\|_{H^{s-1,k-1}} + \|\nabla_{\mathbf{x}} \eta\|_{H^{s,k}})$$

and there remains to use estimate (3.18) in Corollary 3.2 to infer

$$\begin{aligned} \|\mathbf{R}_{\alpha,j}^{\text{nh}}\|_{L^2(\Omega)} & \leq C(h_\star, \underline{M}, M) M (1 + \kappa \|\nabla_{\mathbf{x}} h\|_{H^{s,k}}) \\ & \quad + C(h_\star, \underline{M}, M) \sqrt{\mu} (\|h\|_{H^{s,k}} + \|\nabla_{\mathbf{x}} \eta\|_{H^{s,k}}) (M + \|\mathbf{u}_\star\|_{H^{s,k}} + \sqrt{\mu} \|w_\star\|_{H^{s,k-1}}). \end{aligned}$$

Now consider

$$R_{\alpha,j}^{\text{nh}} := -\sqrt{\mu}[\partial_\rho^j \partial_{\underline{x}}^\alpha, \underline{\mathbf{u}} + \mathbf{u}] \cdot \nabla_{\underline{x}} w + \kappa \sqrt{\mu} \left[\partial_\rho^j \partial_{\underline{x}}^\alpha, \frac{\nabla_{\underline{x}} h}{\underline{h}+h} \right] \cdot \nabla_{\underline{x}} w + \frac{1}{\sqrt{\mu}} [\partial_\rho^j \partial_{\underline{x}}^\alpha, \frac{1}{\rho(\underline{h}+h)}] \partial_\rho P.$$

We have, by Lemma A.8 with $k = s \geq s_0 + 3/2$,

$$\sqrt{\mu} \| [\partial_\rho^j \partial_{\underline{x}}^\alpha, \underline{\mathbf{u}} + \mathbf{u}] \cdot \nabla_{\underline{x}} w \|_{L^2(\Omega)} \lesssim \sqrt{\mu} (\| \underline{\mathbf{u}}' \|_{W_\rho^{k-1,\infty}} + \| \mathbf{u} \|_{H^{s,k}}) \| \nabla_{\underline{x}} w \|_{H^{s-1,k-1}},$$

and similarly, using tame product estimate Lemma A.3 and composition estimates in Lemma A.6 as above,

$$\kappa \sqrt{\mu} \| [\partial_\rho^j \partial_{\underline{x}}^\alpha, \frac{\nabla_{\underline{x}} h}{\underline{h}+h}] \cdot \nabla_{\underline{x}} w \|_{L^2(\Omega)} \leq \kappa \sqrt{\mu} C(h_\star, \underline{M}, M) (\| h \|_{H^{s,k}}^2 + \| \nabla_{\underline{x}} h \|_{H^{s,k}}) \| w \|_{H^{s,k-1}}.$$

and

$$\frac{1}{\sqrt{\mu}} \| [\partial_\rho^j \partial_{\underline{x}}^\alpha, \frac{1}{\rho(\underline{h}+h)}] \partial_\rho P_{\text{nh}} \|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\mu}} C(h_\star, \underline{M}, M) \| h \|_{H^{s,k}} \| \partial_\rho P_{\text{nh}} \|_{H^{s-1,k-1}}.$$

Collecting the above and using estimate (3.18) in Corollary 3.2 yields

$$\begin{aligned} \| R_{\alpha,j}^{\text{nh}} \|_{L^2(\Omega)} &\leq C(h_\star, \underline{M}, M) M (1 + \kappa \| \nabla_{\underline{x}} h \|_{H^{s,k}}) \\ &\quad + C(h_\star, \underline{M}, M) \| h \|_{H^{s,k}} (M + \| \mathbf{u}_\star \|_{H^{s,k}} + \sqrt{\mu} \| w_\star \|_{H^{s,k-1}}). \end{aligned}$$

Finally, we consider the remainder (stemming from differentiating the incompressibility condition (3.29))

$$\begin{aligned} R_{\alpha,j}^{\text{div}} &= (\partial_\rho^j \partial_{\underline{x}}^\alpha (h \nabla_{\underline{x}} \cdot \mathbf{u}) - (h^{(\alpha,j)}) \nabla_{\underline{x}} \cdot \mathbf{u} - h \nabla_{\underline{x}} \cdot \mathbf{u}^{(\alpha,j)}) \\ &\quad + (\partial_\rho^j \partial_{\underline{x}}^\alpha ((\partial_\rho \mathbf{u}) \cdot (\nabla_{\underline{x}} \eta)) - (\partial_\rho \mathbf{u}^{(\alpha,j)}) \cdot (\nabla_{\underline{x}} \eta) - (\partial_\rho \mathbf{u}) \cdot (\nabla_{\underline{x}} \eta^{(\alpha,j)})) \\ &\quad + (\partial_\rho^j \partial_{\underline{x}}^\alpha (h \nabla_{\underline{x}} \cdot \mathbf{u}) - h \nabla_{\underline{x}} \cdot \mathbf{u}^{(\alpha,j)}) + (\partial_\rho^j \partial_{\underline{x}}^\alpha (\underline{\mathbf{u}}' \cdot \nabla_{\underline{x}} \eta) - \underline{\mathbf{u}}' \cdot \nabla_{\underline{x}} \eta^{(\alpha,j)}). \end{aligned}$$

Using Lemma A.9 for the first two terms and direct estimates for the last two terms, and $k = s \geq s_0 + 5/2$,

$$\begin{aligned} \| R_{\alpha,j}^{\text{div}} \| &\lesssim \| h \|_{H^{s-1,k-1}} \| \nabla_{\underline{x}} \cdot \mathbf{u} \|_{H^{s-1,k-1}} + \| \partial_\rho \mathbf{u} \|_{H^{s-1,k-1}} \| \nabla_{\underline{x}} \eta \|_{H^{s-1,k-1}} \\ &\quad + \| h \|_{W_\rho^{k,\infty}} \| \nabla_{\underline{x}} \cdot \mathbf{u} \|_{H^{s-1,k-1}} + \| \underline{\mathbf{u}}' \|_{W_\rho^{k,\infty}} \| \nabla_{\underline{x}} \eta \|_{H^{s-1,k-1}} \\ &\lesssim (\| h \|_{W_\rho^{k,\infty}} + \| \underline{\mathbf{u}}' \|_{W_\rho^{k,\infty}} + \| h \|_{H^{s-1,k-1}} + \| \mathbf{u} \|_{H^{s,k}}) (\| \eta \|_{H^{s,k-1}} + \| \mathbf{u} \|_{H^{s,k-1}}). \end{aligned}$$

This concludes the proof. \square

3.4. A priori energy estimates. In this section we provide *a priori* energy estimates associated with the equations featured in Lemma 3.5. We point out that such estimates concerning $h^{(\alpha,j)}$ solving the transport-diffusion equation (3.32a) have been provided in Lemma 2.4. Corresponding estimates for $\nabla_{\underline{x}} \eta$ stemming from the first equation of (3.31a) are easily obtained. Hence we consider the remaining equations in (3.31a). Specifically, recalling the notation $\dot{\eta} = \eta^{(\alpha,j)}$, $\dot{h} = h^{(\alpha,j)}$, $\dot{\mathbf{u}} = \mathbf{u}^{(\alpha,j)}$, $\dot{w} = w^{(\alpha,j)}$, $\dot{P}_{\text{nh}} = P_{\text{nh}}^{(\alpha,j)}$, we consider the following linearized system:

$$\begin{aligned} \partial_t \dot{\eta} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\underline{x}} \dot{\eta} + \int_\rho^{\rho_1} ((\underline{\mathbf{u}}' + \partial_\rho \mathbf{u}) \cdot \nabla_{\underline{x}} \dot{\eta} + (\underline{h} + h) \nabla_{\underline{x}} \cdot \dot{\mathbf{u}}) d\rho' - \kappa \Delta_{\underline{x}} \dot{\eta} &= R, \\ \rho (\partial_t \dot{\mathbf{u}} + ((\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\underline{x}} h}{\underline{h}+h}) \cdot \nabla_{\underline{x}}) \dot{\mathbf{u}}) + \rho_0 \nabla_{\underline{x}} \dot{\eta}|_{\rho=\rho_0} + \int_{\rho_0}^\rho \nabla_{\underline{x}} \dot{\eta} d\rho' + \nabla_{\underline{x}} \dot{P}_{\text{nh}} + \frac{\nabla_{\underline{x}} \eta}{\underline{h}+h} \partial_\rho \dot{P}_{\text{nh}} &= \mathbf{R}^{\text{nh}}, \\ \sqrt{\mu} \rho (\partial_t \dot{w} + \mu (\underline{\mathbf{u}} + \mathbf{u} - \kappa \frac{\nabla_{\underline{x}} h}{\underline{h}+h}) \cdot \nabla_{\underline{x}} \dot{w}) - \frac{1}{\sqrt{\mu}} \frac{\partial_\rho \dot{P}_{\text{nh}}}{\underline{h}+h} &= R^{\text{nh}}, \\ -\partial_\rho \dot{w} + (\underline{h} + h) \nabla_{\underline{x}} \cdot \dot{\mathbf{u}} + \nabla_{\underline{x}} \eta \cdot \partial_\rho \dot{\mathbf{u}} - (\partial_\rho \dot{\eta}) \nabla_{\underline{x}} \cdot \mathbf{u} + \nabla_{\underline{x}} \eta \cdot (\underline{\mathbf{u}}' + \partial_\rho \mathbf{u}) &= R^{\text{div}}, \end{aligned} \tag{3.33}$$

where we denote as always $\eta(\cdot, \rho) = \int_\rho^{\rho_1} h(\cdot, \rho) d\rho$.

We shall use the following definitions of the spaces Y^0 and Y^1

$$\begin{aligned} Y^0 &:= C^0([\rho_0, \rho_1]; L^2(\mathbb{R}^d)) \times L^2(\Omega)^d \times L^2(\Omega) \times L^2(\Omega), \quad \text{and} \\ Y^1 &:= \left\{ (\eta, \mathbf{u}, w, P) \in H^{1,1}(\Omega)^{d+3} : \eta|_{\rho=\rho_0} \in H^1(\mathbb{R}^d), w|_{\rho=\rho_1} = 0, P|_{\rho=\rho_0} = 0 \right\}. \end{aligned} \tag{3.34}$$

LEMMA 3.6. Let $M, h_*, h^* > 0$ be fixed. There exists $C(M, h_*, h^*) > 0$ such that for any $\kappa > 0$ and $\mu > 0$, and for any $(\underline{h}, \underline{\mathbf{u}}) \in W^{1,\infty}((\rho_0, \rho_1))^{1+d}$ and any $T > 0$ and $(h, \mathbf{u}, w) \in L^\infty(0, T; W^{1,\infty}(\Omega))$ with $\Delta_{\mathbf{x}} h \in L^1(0, T; L^\infty(\Omega))$ satisfying (3.28) and, for almost any $t \in [0, T]$, the estimate

$$\|h(t, \cdot)\|_{L^\infty(\Omega)} + \|\nabla_{\mathbf{x}} h(t, \cdot)\|_{L^\infty_x L^2_\varrho} + \|\nabla_{\mathbf{x}} \cdot \mathbf{u}(t, \cdot)\|_{L^\infty(\Omega)} \leq M$$

and the upper and lower bounds

$$\forall(\mathbf{x}, \varrho) \in \Omega, \quad h_* \leq \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \leq h^*;$$

and for any $(\dot{\eta}, \dot{\mathbf{u}}, \dot{w}, \dot{P}_{\text{nh}}) \in C^0([0, T]; Y^0) \cap L^2(0, T; Y^1)$ and $(R, \mathbf{R}^{\text{nh}}, R^{\text{nh}}, R^{\text{div}}) \in L^2(0, T; Y^0)$ satisfying system (3.33) in $L^2(0, T; Y^1)'$, the following inequality holds:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, \dot{w}) + \frac{\kappa}{2} \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)}^2 + \rho_0 \kappa |\nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0}|_{L^2_{\mathbf{x}}}^2 \\ \leq C(1 + \kappa^{-1} \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L^\infty_x L^2_\varrho}^2) \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, \dot{w}) \\ + C(M + \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L^\infty_x L^2_\varrho}) \|\dot{P}_{\text{nh}}\|_{L^2(\Omega)} (\|\partial_{\varrho} \dot{\eta}\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)}) \\ + \|\dot{P}_{\text{nh}}\|_{L^2(\Omega)} \|R^{\text{div}}\|_{L^2(\Omega)} + C \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, \dot{w})^{1/2} \mathcal{E}(R, \mathbf{R}^{\text{nh}}, R^{\text{nh}})^{1/2}, \end{aligned}$$

where we denote

$$\mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, \dot{w}) = \frac{1}{2} \int_{\rho_0}^{\rho_1} \int_{\mathbb{R}^d} \dot{\eta}^2 + \varrho(\underline{h} + h) |\dot{\mathbf{u}}|^2 + \mu \varrho(\underline{h} + h) \dot{w}^2 \, d\mathbf{x} \, d\varrho + \frac{1}{2} \int_{\mathbb{R}^d} \dot{\eta}^2|_{\varrho=\rho_0} \, d\mathbf{x}. \quad (3.35)$$

PROOF. We test the first equation against $\dot{\eta} \in L^2(0, T; H^{1,1}(\Omega))$ and its trace on $\{(\mathbf{x}, \rho_0), \mathbf{x} \in \mathbb{R}^d\}$ against $\rho_0 \dot{\eta}|_{\varrho=\rho_0} \in L^2(0, T; H^1_{\mathbf{x}}(\mathbb{R}^d))$, the second equation against $(\underline{h} + h) \dot{\mathbf{u}} \in L^2(0, T; H^{1,1}(\Omega)^d)$ and the third equation against $\sqrt{\mu}(\underline{h} + h) \dot{w} \in L^2(0, T; H^{1,1}(\Omega))$. This yields:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, \dot{w}) + \kappa \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)}^2 + \kappa |\nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0}|_{L^2_{\mathbf{x}}}^2 \\ = -((\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta}, \dot{\eta})_{L^2(\Omega)} - \left(\int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} \, d\varrho', \dot{\eta} \right)_{L^2(\Omega)} \quad (i) \\ - \left(\int_{\varrho}^{\rho_1} (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \dot{\mathbf{u}} \, d\varrho', \dot{\eta} \right)_{L^2(\Omega)} + (R, \dot{\eta})_{L^2(\Omega)} \quad (ii) \\ - (\varrho(\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\mathbf{u}}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} + \kappa (\varrho(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \dot{\mathbf{u}}, \dot{\mathbf{u}})_{L^2(\Omega)} \quad (iii) \\ - (\rho_0 \nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} - \left(\int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \dot{\eta} \, d\varrho', (\underline{h} + h) \dot{\mathbf{u}} \right)_{L^2(\Omega)} \quad (iv) \\ - (\nabla_{\mathbf{x}} \dot{P}_{\text{nh}}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} - ((\partial_{\varrho} \dot{P}_{\text{nh}}) \nabla_{\mathbf{x}} \dot{\eta}, \dot{\mathbf{u}})_{L^2(\Omega)} + (\mathbf{R}^{\text{nh}}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} \quad (v) \\ - \mu (\varrho(\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{w}, (\underline{h} + h) \dot{w})_{L^2(\Omega)} + \mu \kappa (\varrho(\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}}) \dot{w}, \dot{w})_{L^2(\Omega)} \quad (vi) \\ + (\partial_{\varrho} \dot{P}_{\text{nh}}, \dot{w})_{L^2(\Omega)} + \sqrt{\mu} (R^{\text{nh}}, (\underline{h} + h) \dot{w})_{L^2(\Omega)} \quad (vii) \\ - \rho_0 ((\underline{\mathbf{u}} + \mathbf{u}) (\nabla_{\mathbf{x}} \dot{\eta}|_{\varrho=\rho_0}), \dot{\eta}|_{\varrho=\rho_0})_{L^2_{\mathbf{x}}} - \rho_0 \left(\int_{\rho_0}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} \, d\varrho', \dot{\eta}|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} \quad (viii) \\ - \rho_0 \left(\int_{\rho_0}^{\rho_1} (h + \underline{h}) (\nabla_{\mathbf{x}} \cdot \dot{\mathbf{u}}) \, d\varrho', \dot{\eta}|_{\varrho=\rho_0} \right)_{L^2_{\mathbf{x}}} + (R|_{\varrho=\rho_0}, \dot{\eta}|_{\varrho=\rho_0})_{L^2_{\mathbf{x}}} \quad (ix) \\ + \frac{1}{2} (\varrho(\partial_t h) \dot{\mathbf{u}}, \dot{\mathbf{u}})_{L^2(\Omega)} + \frac{\mu}{2} (\varrho(\partial_t h) \dot{w}, \dot{w})_{L^2(\Omega)}. \quad (x) \end{aligned}$$

Some terms have already been treated in the course of the proof of Lemma 2.5: the second term in (i) and the second term in (viii) require $\kappa > 0$; the first terms in (i), (viii) are advection terms; the first addend of

(ii) together with the second term in (iv) after integration by parts; the first addend of (iv) with the first term in (ix). The contributions in (iii) compensate with the first addend of (x), using the first equation of (3.28) and, in the same way, the contributions in (vi) compensate with the second addend of (x). It remains only to deal with the contribution from the non-hydrostatic pressure terms in (v) and (vii), and remainder terms.

Consider the sum of the first two terms in (v) and the first term in (vii). We integrate by parts in \mathbf{x} the first term and in ϱ in the last two terms. Thus we have

$$\begin{aligned} & - (\nabla_{\mathbf{x}} \dot{P}_{\text{nh}}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)} - ((\partial_{\varrho} \dot{P}_{\text{nh}}) \nabla_{\mathbf{x}} \eta, \dot{\mathbf{u}})_{L^2(\Omega)} + (\partial_{\varrho} \dot{P}_{\text{nh}}, \dot{w})_{L^2(\Omega)} \\ & = (\dot{P}_{\text{nh}}, (\underline{h} + h) \nabla_{\mathbf{x}} \cdot \dot{\mathbf{u}})_{L^2(\Omega)} + (\dot{P}_{\text{nh}} \nabla_{\mathbf{x}} \eta, \partial_{\varrho} \dot{\mathbf{u}})_{L^2(\Omega)} - (\dot{P}_{\text{nh}}, \partial_{\varrho} \dot{w})_{L^2(\Omega)}, \end{aligned}$$

where we used the identity $h = -\partial_{\varrho} \eta$ and the boundary conditions $\dot{P}_{\text{nh}}|_{\varrho=\rho_0} = \dot{\eta}|_{\varrho=\rho_1} = \dot{w}|_{\varrho=\rho_1} = 0$ when integrating by parts with respect to ϱ . Using the last equation in (3.33) (stemming from the incompressibility condition), the above term reads

$$(\dot{P}_{\text{nh}}, (\nabla_{\mathbf{x}} \cdot \mathbf{u})(\partial_{\varrho} \dot{\eta}) - (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}) \cdot \nabla_{\mathbf{x}} \dot{\eta} + R^{\text{div}})_{L^2(\Omega)}.$$

These terms, alike remainder terms

$$|(R, \dot{\eta})_{L^2(\Omega)}| + |(R|_{\varrho=\rho_0}, \dot{\eta}|_{\varrho=\rho_0})_{L^2_{\mathbf{x}}}| + |(\mathbf{R}^{\text{nh}}, (\underline{h} + h) \dot{\mathbf{u}})_{L^2(\Omega)}| + \sqrt{\mu} |(R^{\text{nh}}, (\underline{h} + h) \dot{w})_{L^2(\Omega)}|,$$

are bounded by Cauchy-Schwarz inequality and using $\rho_0 h_{\star} \leq \varrho(\underline{h} + h) \leq \rho_1 h^{\star}$.

Altogether, we obtain the differential inequality

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, \dot{w}) + \kappa \|\nabla_{\mathbf{x}} \eta\|_{L^2(\Omega)}^2 + \rho_0 \kappa \|\nabla_{\mathbf{x}} \eta|_{\varrho=\rho_0}\|_{L^2_{\mathbf{x}}}^2 \\ & \leq C \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, 0) + C \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L^2_{\varrho} L^{\infty}_{\mathbf{x}}} \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, 0)^{1/2} \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)} \\ & \quad + C (M + \|\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}\|_{L^{\infty}_{\varrho} L^{\infty}_{\mathbf{x}}}) \|\dot{P}_{\text{nh}}\|_{L^2(\Omega)} (\|\partial_{\varrho} \dot{\eta}\|_{L^2(\Omega)} + \|\nabla_{\mathbf{x}} \dot{\eta}\|_{L^2(\Omega)}) \\ & \quad + \|\dot{P}_{\text{nh}}\|_{L^2(\Omega)} \|R^{\text{div}}\|_{L^2(\Omega)} + C \mathcal{E}(\dot{\eta}, \dot{\mathbf{u}}, \dot{w})^{1/2} \mathcal{E}(R, \mathbf{R}^{\text{nh}}, R^{\text{nh}})^{1/2} \end{aligned}$$

with $C = C(h_{\star}, h^{\star}, M)$, and the desired estimate follows straightforwardly. \square

REMARK 3.7. Lemma 3.6 will be applied to the system (3.31a)-(3.31b) appearing in Lemma 3.5, when $j = 0$. A similar result holds for the simplified system when $j \neq 0$. The main difference is that the result does not require nor provide the control of the trace $\partial_{\varrho}^j \eta|_{\varrho=\rho_0}$.

3.5. Large-time well-posedness. We prove the large-time existence of strong solutions to system (3.1). As for the hydrostatic system, *large time* underlines the fact that the existence time that is provided by the following result is uniformly bounded (from below) with respect to the vanishing parameter $\mu \in (0, 1]$. Besides, the result below keeps track of the dependency of this large time-scale on the diffusivity parameter $\kappa \in [\mu, 1]$.

PROPOSITION 3.8. Let $s, k \in \mathbb{N}$ be such that $k = s > \frac{5}{2} + \frac{d}{2}$ and $\underline{M}, M, h_{\star}, h^{\star} > 0$. Then, there exists $C > 0$ such that, for any $0 < \mu \leq \kappa \leq 1$, and any $(\underline{h}, \underline{\mathbf{u}}) \in W^{k, \infty}((\rho_0, \rho_1)) \times W^{k+1, \infty}((\rho_0, \rho_1))^d$ such that

$$\|\underline{h}\|_{W^{k, \infty}_{\varrho}} + \|\underline{\mathbf{u}}'\|_{W^{k, \infty}_{\varrho}} \leq \underline{M};$$

for any initial data $(h_0, \mathbf{u}_0, w_0) \in H^{s, k}(\Omega)^{d+2}$ with

$$M_0 := \|\eta_0\|_{H^{s, k}} + \|\mathbf{u}_0\|_{H^{s, k}} + \sqrt{\mu} \|w_0\|_{H^{s, k}} + \|\eta_0|_{\varrho=\rho_0}\|_{H^s_{\mathbf{x}}} + \kappa^{1/2} \|h_0\|_{H^{s, k}} + \mu^{1/2} \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta_0\|_{H^{s, k}} \leq M,$$

and satisfying the boundary condition $w_0|_{\varrho=\rho_1} = 0$ and the incompressibility condition

$$-(\underline{h} + h_0) \nabla_{\mathbf{x}} \cdot \mathbf{u}_0 - (\nabla_{\mathbf{x}} \eta_0) \cdot (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}_0) + \partial_{\varrho} w_0 = 0,$$

the lower and upper bounds

$$\forall (\mathbf{x}, \varrho) \in \Omega, \quad h_{\star} \leq \underline{h}(\varrho) + h_0(\mathbf{x}, \varrho) \leq h^{\star},$$

and the smallness assumption

$$C\kappa^{-1} \left(\|\underline{\mathbf{u}}'\|_{L^\infty}^2 + M_0^2 \right) \leq 1,$$

the following holds. Denoting by

$$T^{-1} = C \left(1 + \kappa^{-1} \left(\|\underline{\mathbf{u}}'\|_{L^2}^2 + M_0^2 \right) \right),$$

there exists a unique $(h, \mathbf{u}, w) \in \mathcal{C}^0([0, T]; H^{s,k}(\Omega)^{2+d})$ and $P \in L^2(0, T; H^{s+1,k+1}(\Omega))$ strong solution to (3.28) with initial data $(h, \mathbf{u}, w)|_{t=0} = (h_0, \mathbf{u}_0, w_0)$. Moreover, one has $\eta \in L^\infty(0, T; H^{s+1,k}(\Omega))$ and $(h, \nabla_{\mathbf{x}}\eta) \in L^2(0, T; H^{s+1,k}(\Omega))$ and, for any $t \in [0, T]$, the lower and the upper bounds hold

$$\forall(\mathbf{x}, \varrho) \in \Omega, \quad h_\star/2 \leq \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \leq 2h^\star,$$

and the estimate below holds true

$$\begin{aligned} \mathcal{F}(t) &:= \|\eta(t, \cdot)\|_{H^{s,k}} + \|\mathbf{u}(t, \cdot)\|_{H^{s,k}} + \mu^{1/2} \|w(t, \cdot)\|_{H^{s,k}} + \|\eta|_{\varrho=\rho_0}(t, \cdot)\|_{H_x^s} \\ &\quad + \kappa^{1/2} \|h(t, \cdot)\|_{H^{s,k}} + \mu^{1/2} \kappa^{1/2} \|\nabla_{\mathbf{x}}\eta(t, \cdot)\|_{H^{s,k}} \\ &\quad + \kappa^{1/2} \|\nabla_{\mathbf{x}}\eta\|_{L^2(0,t;H^{s,k})} + \kappa^{1/2} \|\nabla_{\mathbf{x}}\eta|_{\varrho=\rho_0}\|_{L^2(0,t;H_x^s)} \\ &\quad + \kappa \|\nabla_{\mathbf{x}}h\|_{L^2(0,t;H^{s,k})} + \mu^{1/2} \kappa \|\nabla_{\mathbf{x}}^2\eta\|_{L^2(0,t;H^{s,k})} \leq C M_0. \end{aligned} \quad (3.36)$$

PROOF. As for the large-time existence for the hydrostatic system (see Proposition 2.6), the proof is based on a bootstrap argument on the functional \mathcal{F} . Recalling that the (short-time) existence and uniqueness of the solution has been provided in Proposition 3.3, we denote by T^\star the maximal existence time, and set

$$T_\star = \sup\{0 < T < T^\star : \forall t \in (0, T), h_\star/2 \leq \underline{h}(\varrho) + h(t, \mathbf{x}, \varrho) \leq 2h^\star \text{ and } \mathcal{F}(t) \leq C_0 M_0\}, \quad (3.37)$$

with $C_0 = C(h_\star, h^\star, \underline{M}, M)$ sufficiently large (to be determined). Henceforth, we restrain to $0 < T < T_\star$, and denote by C any positive constant depending uniquely on $\underline{M}, h_\star, h^\star, C_0 M_0$ and s, k .

By means of (3.32a)-(3.32b) in Lemma 3.5 and Lemma 2.4, we infer as in the proof of Proposition 2.6 the control

$$\kappa^{1/2} \|h\|_{L^\infty(0,T;H^{s,k})} + \kappa \|\nabla_{\mathbf{x}}h\|_{L^2(0,T;H^{s,k})} \leq \left(c_0 M_0 + C C_0 M_0 (T + \sqrt{T}) \right) \times \exp \left(C C_0 M_0 T \right) \quad (3.38)$$

with the same notations as above and c_0 a universal constant. In the non-hydrostatic situation, additional controls can be inferred on η . Indeed, from the first equation in Lemma 3.5, (3.31a)-(3.31b), we find that

$$\partial_t \eta^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} = \kappa \Delta_{\mathbf{x}} \eta^{(\alpha)} + \tilde{R}_{\alpha,j} + w^{(\alpha,j)}$$

with

$$\sqrt{\mu} \|\tilde{R}_{\alpha,j} + w^{(\alpha,j)}\|_{L^2(\Omega)} \leq C C_0 M_0.$$

Differentiating once with respect to the space variables and proceeding as in Lemma 2.4, we infer

$$\begin{aligned} \mu^{1/2} \kappa^{1/2} \|\nabla_{\mathbf{x}}\eta\|_{L^\infty(0,T;H^{s,k})} + \mu^{1/2} \kappa \|\nabla_{\mathbf{x}}^2\eta\|_{L^2(0,T;H^{s,k})} \\ \leq \left(c_0 M_0 + C C_0 M_0 (T + \sqrt{T}) \right) \times \exp \left(C C_0 M_0 T \right). \end{aligned} \quad (3.39)$$

Next we use again Lemma 3.5, (3.31a)-(3.31b), together with Lemma 3.6 (see also Remark 3.7) to obtain that the functional

$$\mathcal{E}^{s,k} := \frac{1}{2} \sum_{j=0}^k \sum_{|\alpha|=0}^{s-j} \iint_{\Omega} \left(\partial_{\varrho}^j \partial_{\mathbf{x}}^{\alpha} \eta \right)^2 + \varrho (\underline{h} + h) |\partial_{\varrho}^j \partial_{\mathbf{x}}^{\alpha} \mathbf{u}|^2 + \mu \varrho (\underline{h} + h) \left(\partial_{\varrho}^j \partial_{\mathbf{x}}^{\alpha} w \right)^2 \, d\mathbf{x} \, d\varrho + \frac{1}{2} \sum_{|\alpha|=0}^s \int_{\mathbb{R}^d} \left(\partial_{\mathbf{x}}^{\alpha} \eta|_{\varrho=\rho_0} \right)^2 \, d\mathbf{x},$$

satisfies the differential inequality

$$\frac{d}{dt} \mathcal{E}^{s,k} + \frac{\kappa}{2} \|\nabla_{\mathbf{x}}\eta\|_{H^{s,k}}^2 + \rho_0 \kappa \|\nabla_{\mathbf{x}}\eta|_{\varrho=\rho_0}\|_{H_x^s}^2 \leq C (R_1 + R_2 + R_3); \quad (3.40)$$

with

$$\begin{aligned} R_1 &:= (1 + \kappa^{-1} \|\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}\|_{L^\infty_\varrho L^2_\varrho}^2) \mathcal{E}^{s,k}, \\ R_2 &:= (C_0 M_0 + \|\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}\|_{L^\infty_\varrho L^\infty_\varrho}) \|P_{\text{nh}}\|_{H^{s,k}} (\|h\|_{H^{s,k}} + \|\nabla_{\mathbf{x}} \eta\|_{H^{s,k}}), \\ R_3 &:= \|P_{\text{nh}}\|_{H^{s,k}} \|R_{s,k}^{\text{div}}\|_{L^2(\Omega)} + (\mathcal{E}^{s,k})^{1/2} \|\mathcal{R}_{s,k}\|_{L^2(\Omega)}, \end{aligned}$$

and

$$\|R_{s,k}^{\text{div}}\|_{L^2(\Omega)} \leq C C_0 M_0, \quad (3.41)$$

$$\begin{aligned} \|\mathcal{R}_{s,k}\|_{L^2(\Omega)} &\leq C C_0 M_0 (1 + \kappa \|\nabla_{\mathbf{x}} h\|_{H^{s,k}}) \\ &\quad + C (\|h\|_{H^{s,k}} + \mu^{1/2} \|\nabla_{\mathbf{x}} \eta\|_{H^{s,k}}) (C_0 M_0 + \|\mathbf{u}_\star\|_{H^{s,k}} + \mu^{1/2} \|w_\star\|_{H^{s,k}}). \end{aligned} \quad (3.42)$$

By (3.37), we have obviously for any $0 < t < T_\star$,

$$\frac{1}{2\rho_1 h_\star} \mathcal{E}^{s,k}(t) \leq \|\eta(t, \cdot)\|_{H^{s,k}}^2 + \|\mathbf{u}(t, \cdot)\|_{H^{s,k}}^2 + \mu \|w(t, \cdot)\|_{H^{s,k}}^2 + |\eta|_{\varrho=\rho_0}(t, \cdot)|_{H^s_\varrho}^2 \leq \frac{2}{\rho_0 h_\star} \mathcal{E}^{s,k}(t).$$

Moreover, we have the following control on $\mathbf{u}_\star := -\kappa \frac{\nabla_{\mathbf{x}} h}{h+h}$ and $w_\star := \kappa \Delta_{\mathbf{x}} \eta - \kappa \frac{\nabla_{\mathbf{x}} h \cdot \nabla_{\mathbf{x}} \eta}{h+h}$ stemming from (tame) product and composition estimates (Lemma A.3 and A.6), and using that $\mu \leq \kappa \leq 1$:

$$\|\mathbf{u}_\star\|_{L^2(0,T;H^{s,k})} + \mu^{1/2} \|w_\star\|_{L^2(0,T;H^{s,k})} \leq C C_0 M_0 (1 + \sqrt{T}). \quad (3.43)$$

Finally, using estimate (3.18) in Corollary 3.2 yields

$$\begin{aligned} \|P_{\text{nh}}\|_{H^{s,k}} &\leq \|P_{\text{nh}}\|_{L^2} + \|\nabla_{\mathbf{x},\varrho} P_{\text{nh}}\|_{H^{s-1,k-1}} \leq \|P_{\text{nh}}\|_{L^2} + \mu^{-1/2} \|\nabla_{\mathbf{x},\varrho}^\mu P_{\text{nh}}\|_{H^{s-1,k-1}} \\ &\leq C \left(\|\nabla_{\mathbf{x}} \eta\|_{H^{s-1,k-1}} + |\eta|_{\varrho=\rho_0}|_{H^s_\varrho} + \|(\mathbf{u}, \mathbf{u}_\star)\|_{H^{s,k}} + \mu^{1/2} \|(w, w_\star)\|_{H^{s,k-1}} \right), \end{aligned}$$

from which we infer, using the controls (3.37) and (3.43), that

$$\|P_{\text{nh}}\|_{L^2(0,T;H^{s,k})} \leq C C_0 M_0 (1 + \sqrt{T}). \quad (3.44)$$

From (3.37) and (3.41)-(3.42)-(3.43)-(3.44) we infer

$$\begin{aligned} \int_0^T R_1(t) dt &\leq C (C_0 M_0)^2 (1 + \kappa^{-1} \|\underline{\mathbf{u}}'\|_{L^2_\varrho}^2 + \kappa^{-1} (C_0 M_0)^2) T, \\ \int_0^T R_2(t) dt &\leq C \kappa^{-1/2} (C_0 M_0 + \|\underline{\mathbf{u}}'\|_{L^\infty_\varrho}) (C_0 M_0)^2 (1 + \sqrt{T})^2, \\ \int_0^T R_3(t) dt &\leq C (C_0 M_0)^2 (T + \sqrt{T}) + C (C_0 M_0)^2 (T + C_0 M_0 \sqrt{T} + \kappa^{-1/2} (C_0 M_0) (T + \sqrt{T})). \end{aligned}$$

Hence there exists $C > 0$, depending on $\underline{M}, h_\star, h^\star, C_0, M_0$ (and s, k), such that if

$$C T (1 + \kappa^{-1} (\|\underline{\mathbf{u}}'\|_{L^2_\varrho}^2 + M_0^2)) \leq 1,$$

and imposing additionally³ that

$$C \kappa^{-1/2} (C_0 M_0 + \|\underline{\mathbf{u}}'\|_{L^\infty_\varrho}) \leq \frac{1}{16} \rho_0 h_\star \quad (3.45)$$

we have, when integrating the differential inequality (3.40) and combine with (3.38) and (3.39),

$$\mathcal{E}^{s,k}(t) \leq \mathcal{E}^{s,k}(0) + \frac{1}{8} (\rho_0 h_\star) (C_0 M_0)^2.$$

Now, setting $C_0 = \max(\{4(\frac{\rho_1 h^\star}{\rho_0 h_\star})^{1/2}, 8c_0\})$, and C accordingly, one has $\mathcal{F}(t) \leq C_0 M_0 / 2$ for all $0 < t < T$. We obtain as in the proof of Proposition 2.6 the lower and upper bounds $2h_\star/3 \leq \underline{h}(\varrho) +$

³We point out that the only term requiring the above smallness condition (3.45) on the initial data is (the time integral of) R_2 , and more precisely the product $\|P_{\text{nh}}\|_{H^{s,k}} \|\nabla_{\mathbf{x}} \eta\|_{H^{s,k}}$, where both terms are only square-integrable in time.

$h(t, \mathbf{x}, \varrho) \leq 3h^*/2$, augmenting C if necessary, and the standard continuity argument allows to conclude the proof. \square

4. Convergence

This section is devoted to the proof of the convergence of regular solutions to the non-hydrostatic equations (1.8) towards the corresponding solutions to the limit hydrostatic equations (1.9), namely Theorem 1.2. Our convergence result holds in the strong sense and “with loss of derivatives”: we prove that the solutions to the approximating (non-hydrostatic) equations converge towards the solutions to the limit (hydrostatic) equations in a suitable strong topology that is strictly weaker than the one measuring the size of the initial data.

For a given set of initial data, we use the apex h to refer to the solution to the hydrostatic equations (provided by the analysis of Section 2 culminating with Theorem 1.1), and the apex nh for the corresponding solution to the non-hydrostatic equations (provided by the analysis of Section 3, specifically Proposition 3.3). The apex d denotes the difference between the non-hydrostatic solution and the hydrostatic one, whose size will be controlled in the limit $\mu \searrow 0$.

While we can appeal to Theorem 1.1 to obtain the existence, uniqueness and control of solutions to the hydrostatic equations over a large time interval, Proposition 3.3 provides only a time interval which *a priori* vanishes as $\mu \searrow 0$, and Proposition 3.8 only applies to sufficiently small initial data. The standard strategy (used for instance in [20] in the context of weakly compressible flows) that we apply here relies on a bootstrap argument to control the difference between the non-hydrostatic solution and the hydrostatic one in the time-interval provided by the hydrostatic solution, from which the existence and control of the non-hydrostatic solution (again, with loss of derivatives) can be inferred. We perform this analysis in Sections 4.1 to 4.3, where we first provide a consistency result (Lemma 4.1), then exhibit the (non-hydrostatic) quasilinear structure of the equations satisfied by the difference (Lemma 4.2), and finally infer the uniform control of the non-hydrostatic solution and the strong convergence towards the corresponding hydrostatic solution (Proposition 4.3). In a last step, in Section 4.4, we use this uniform control to offer an improved convergence rate based this time on the structure of the hydrostatic equations (Proposition 4.4). Propositions 4.3 and 4.4 immediately yield Theorem 1.2.

4.1. Consistency. In the following result we prove that solutions to the hydrostatic equations (1.9) emerging from smooth initial data satisfy (suitably defining the horizontal velocity and pressure variables) the non-hydrostatic equations (1.8), up to small remainder terms.

LEMMA 4.1. There exists $p \in \mathbb{N}$ such that for any $s, k \in \mathbb{N}$ with $0 \leq k \leq s$, the following holds. Let $\underline{M}, M, h_*, h^* > 0$ be fixed. Then there exists $C_0 > 0$ and $C_1 > 0$ such that for any $\kappa \in (0, 1]$, any $(\underline{h}, \underline{\mathbf{u}}) \in W^{k+p, \infty}((\rho_0, \rho_1))^{1+d}$ satisfying

$$|\underline{h}|_{W_\varrho^{k+p, \infty}} + |\underline{\mathbf{u}}'|_{W_\varrho^{k+p-1, \infty}} \leq \underline{M},$$

and any initial data $(h_0, \mathbf{u}_0) \in H^{s+p, k+p}(\Omega)$ satisfying the following estimate

$$M_0 := \|\eta_0\|_{H^{s+p, k+p}} + \|\mathbf{u}_0\|_{H^{s+p, k+p}} + |\eta_0|_{\varrho=\rho_0}|_{H_x^{s+p}} + \kappa^{1/2} \|h_0\|_{H^{s+p, k+p}} \leq M$$

(where we denote $\eta_0(\cdot, \varrho) := \int_\varrho^{\rho_1} h_0(\cdot, \varrho') d\varrho'$) and the stable stratification assumption

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} h_* \leq \underline{h}(\varrho) + h_0(\mathbf{x}, \varrho) \leq h^*,$$

there exists a unique $(h^h, \mathbf{u}^h) \in C^0([0, T]; H^{s+p, k+p}(\Omega)^{1+d})$ strong solution to (1.9) with initial data $(h^h, \mathbf{u}^h)|_{t=0} = (h_0, \mathbf{u}_0)$, where

$$T^{-1} = C_0 (1 + \kappa^{-1} (|\underline{\mathbf{u}}'|_{L_\varrho^2}^2 + M_0^2)).$$

Moreover, one has for all $t \in [0, T]$,

$$\forall(\mathbf{x}, \varrho) \in \Omega, \quad h_\star/2 \leq \underline{h}(\varrho) + h^h(t, \mathbf{x}, \varrho) \leq 2h^\star,$$

and, denoting $\eta^h(\cdot, \varrho) := \int_{\varrho}^{\rho_1} h^h(\cdot, \varrho') d\varrho'$ and

$$w^h(\cdot, \varrho) := - \int_{\varrho}^{\rho_1} (\underline{h}(\varrho') + h^h(\cdot, \varrho')) \nabla_{\mathbf{x}} \cdot \mathbf{u}^h(\cdot, \varrho') + \nabla_{\mathbf{x}} \eta^h(\cdot, \varrho') \cdot (\underline{\mathbf{u}}'(\varrho') + \partial_{\varrho} \mathbf{u}^h(\cdot, \varrho')) d\varrho', \quad (4.1)$$

$$P^h(\cdot, \varrho) := \int_{\rho_0}^{\varrho} \varrho' h^h(\cdot, \varrho') d\varrho', \quad (4.2)$$

one has for any $t \in [0, T]$,

$$\| (h^h(t, \cdot), \eta^h(t, \cdot), \mathbf{u}^h(t, \cdot), w^h(t, \cdot), P^h(t, \cdot)) \|_{H^{s+1, k+1}} \leq C_1 M_0, \quad (4.3)$$

and

$$\begin{aligned} \partial_t h^h + \nabla_{\mathbf{x}} \cdot ((\underline{h} + h^h)(\underline{\mathbf{u}} + \mathbf{u}^h)) &= \kappa \Delta_{\mathbf{x}} h^h, \\ \varrho \left(\partial_t \mathbf{u}^h + \left((\underline{\mathbf{u}} + \mathbf{u}^h - \kappa \frac{\nabla_{\mathbf{x}} h^h}{\underline{h} + h^h}) \cdot \nabla_{\mathbf{x}} \right) \mathbf{u}^h \right) + \nabla_{\mathbf{x}} P^h + \frac{\nabla_{\mathbf{x}} \eta^h}{\underline{h} + h^h} (\partial_{\varrho} P^h + \varrho \underline{h}^h) &= 0, \\ \mu \varrho \left(\partial_t w^h + \left(\underline{\mathbf{u}} + \mathbf{u}^h - \kappa \frac{\nabla_{\mathbf{x}} h^h}{\underline{h} + h^h} \right) \cdot \nabla_{\mathbf{x}} w^h \right) - \frac{\partial_{\varrho} P^h}{\underline{h} + h^h} + \frac{\varrho \underline{h}^h}{\underline{h} + h^h} &= \mu R^h, \\ -(\underline{h} + h^h) \nabla_{\mathbf{x}} \cdot \mathbf{u}^h - \nabla_{\mathbf{x}} \eta^h \cdot (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}^h) + \partial_{\varrho} w^h &= 0, \end{aligned} \quad (4.4a)$$

with $R^h(t, \cdot) \in \mathcal{C}^0([0, T]; H^{s, k}(\Omega))$ and satisfying for any $t \in [0, T]$,

$$\| R^h(t, \cdot) \|_{H^{s, k}} \leq C_1 M_0. \quad (4.4b)$$

PROOF. From Theorem 1.1 we infer immediately (for $p > 2 + d/2$) the existence, uniqueness and control of the hydrostatic solution $(h^h, \mathbf{u}^h) \in \mathcal{C}^0([0, T]; H^{s+p, k+p}(\Omega)^{1+d})$, and $C_0 > 0$. From the formula (4.1), (4.2) and product estimates (Lemma A.3) in the space $H^{s+p', k+p'}(\Omega)$ (for $1 \leq p' \leq p$ sufficiently large) we infer the estimate (4.3).

We obtain similarly the desired consistency estimate, (4.4a)-(4.4b), using the identity (recall (3.24))

$$P^h + \varrho \eta^h = \int_{\rho_0}^{\varrho} \eta^h(\cdot, \varrho') d\varrho' + \rho_0 \eta^h|_{\varrho=\rho_0},$$

and denoting

$$R^h := \varrho \left(\partial_t w^h + \left(\underline{\mathbf{u}} + \mathbf{u}^h - \kappa \frac{\nabla_{\mathbf{x}} h^h}{\underline{h} + h^h} \right) \cdot \nabla_{\mathbf{x}} w^h \right),$$

differentiating with respect to time the identity (4.1), and using (1.9) to infer the control of $\partial_t \mathbf{u}^h$ and $\partial_t w^h$. \square

As a corollary to the above, we can write the equations satisfied by the difference between (h^h, \mathbf{u}^h, w^h) , i.e. the maximal solution to the hydrostatic equations emerging from given regular, well-prepared initial data, and $(h^{nh}, \mathbf{u}^{nh}, w^{nh})$, i.e. the maximal solution to the non-hydrostatic with the same data (see Proposition 3.3). Specifically, under the assumptions and using the notations of Lemma 4.1, we have that

$$h^d := h^{nh} - h^h; \quad \mathbf{u}^d := \mathbf{u}^{nh} - \mathbf{u}^h; \quad w^d := w^{nh} - w^h;$$

satisfies $(h^d, \mathbf{u}^d, w^d)|_{t=0} = (0, 0, 0)$ and

$$\begin{aligned}
& \partial_t h^d + \nabla_{\mathbf{x}} \cdot ((\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}})h^d + (\underline{h} + h^h)\mathbf{u}^d) = \kappa \Delta_{\mathbf{x}} h^d, \\
& \partial_t \eta^d + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^d + \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^d \, d\varrho' + \int_{\varrho}^{\rho_1} (\underline{h} + h^{\text{nh}}) \nabla_{\mathbf{x}} \cdot \mathbf{u}^d \, d\varrho' \\
& \quad + \int_{\varrho}^{\rho_1} \mathbf{u}^d \cdot \nabla_{\mathbf{x}} h^h + h^d \nabla_{\mathbf{x}} \cdot \mathbf{u}^h \, d\varrho' = \kappa \Delta_{\mathbf{x}} \eta^d, \\
& \partial_t \mathbf{u}^d + ((\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}} - \kappa \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}^d + \frac{\rho_0}{\varrho} \nabla_{\mathbf{x}} \eta^d|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta^d \, d\varrho' \\
& \quad + ((\mathbf{u}^d - \kappa (\frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} - \frac{\nabla_{\mathbf{x}} h^h}{\underline{h} + h^h})) \cdot \nabla_{\mathbf{x}}) \mathbf{u}^h + \frac{\nabla_{\mathbf{x}} P_{\text{nh}}}{\varrho} + \frac{\nabla_{\mathbf{x}} \eta^{\text{nh}}}{\varrho(\underline{h} + h^{\text{nh}})} \partial_{\varrho} P_{\text{nh}} = 0, \\
& \mu (\partial_t w^d + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}} - \kappa \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}) \cdot \nabla_{\mathbf{x}} w^d + (\mathbf{u}^d - \kappa (\frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} - \frac{\nabla_{\mathbf{x}} h^h}{\underline{h} + h^h})) \cdot \nabla_{\mathbf{x}} w^h) - \frac{\partial_{\varrho} P_{\text{nh}}}{\varrho(\underline{h} + h^{\text{nh}})} = -\mu R^h, \\
& -(\underline{h} + h^{\text{nh}}) \nabla_{\mathbf{x}} \cdot \mathbf{u}^d - h^d \nabla_{\mathbf{x}} \cdot \mathbf{u}^h - \nabla_{\mathbf{x}} \eta^d \cdot (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}^{\text{nh}}) - \nabla_{\mathbf{x}} \eta^h \cdot (\partial_{\varrho} \mathbf{u}^d) + \partial_{\varrho} w^d = 0,
\end{aligned} \tag{4.5}$$

where we denote as usual $\eta^h(\cdot, \varrho) = \int_{\varrho}^{\rho_1} h^h(\cdot, \varrho') \, d\varrho'$ (and analogously η^{nh}, η^d), and define the non-hydrostatic pressure $P_{\text{nh}}(\cdot, \varrho) := P^{\text{nh}}(\cdot, \varrho) - \int_{\rho_0}^{\varrho} \varrho' h^{\text{nh}}(\cdot, \varrho') \, d\varrho'$ where P^{nh} is defined by Corollary 3.2.

4.2. Quasi-linearization. In this section we extract the leading order terms of the system (4.5), in the spirit of Lemma 3.5.

LEMMA 4.2. There exists $p \in \mathbb{N}$ such that for any $s, k \in \mathbb{N}$ such that $k = s > \frac{5}{2} + \frac{d}{2}$ and $\underline{M}, M, h_{\star} > 0$, there exists $C > 0$ and $C_1 > 0$ such that the following holds. For any $0 < \mu \leq \kappa \leq 1$, and for any $(\underline{h}, \underline{\mathbf{u}}) \in W^{k+p, \infty}((\rho_0, \rho_1))^{1+d}$ satisfying

$$|\underline{h}|_{W_{\varrho}^{k+p, \infty}} + |\underline{\mathbf{u}}'|_{W_{\varrho}^{k+p-1, \infty}} \leq \underline{M};$$

and any $(h^{\text{nh}}, \mathbf{u}^{\text{nh}}, w^{\text{nh}}) \in \mathcal{C}^0([0, T^{\text{nh}}]; H^{s, k}(\Omega)^{d+2})$ and $P^{\text{nh}} \in L^2(0, T^{\text{nh}}; H^{s+1, k+1}(\Omega))$ solution to (3.28) with some $T^{\text{nh}} > 0$ and satisfying for any $t \in [0, T^{\text{nh}}]$

$$\begin{aligned}
& \|h^{\text{nh}}(t, \cdot)\|_{H^{s-1, k-1}} + \|\eta^{\text{nh}}(t, \cdot)\|_{H^{s, k}} + |\eta^{\text{nh}}(t, \cdot)|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^s} + \|\mathbf{u}^{\text{nh}}(t, \cdot)\|_{H^{s, k}} + \mu^{1/2} \|w^{\text{nh}}(t, \cdot)\|_{H^{s, k}} \\
& \quad + \kappa^{1/2} \|h^{\text{nh}}(t, \cdot)\|_{H^{s, k}} + \mu^{1/2} \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^{\text{nh}}(t, \cdot)\|_{H^{s, k}} \leq M
\end{aligned}$$

(where $\eta^{\text{nh}}(t, \mathbf{x}, \varrho) := \int_{\varrho}^{\rho_1} h^{\text{nh}}(t, \mathbf{x}, \varrho') \, d\varrho'$), the stable stratification assumption

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h^{\text{nh}}(t, \mathbf{x}, \varrho) \geq h_{\star},$$

and the initial bound

$$M_0 := \|\eta^{\text{nh}}|_{t=0}\|_{H^{s+p, k+p}} + \|\mathbf{u}^{\text{nh}}|_{t=0}\|_{H^{s+p, k+p}} + |(\eta^{\text{nh}}|_{\varrho=\rho_0})|_{t=0}|_{H_{\mathbf{x}}^{s+p}} + \kappa^{1/2} \|h^{\text{nh}}|_{t=0}\|_{H^{s+p, k+p}} \leq M,$$

we have the following.

Denote $(h^h, \mathbf{u}^h, w^h) \in \mathcal{C}^0([0, T^h]; H^{s+1, k+1}(\Omega)^{2+d})$ the corresponding strong solution to the hydrostatic equations (2.1) (see Lemma 4.1) satisfying

$$\|h^h(t, \cdot)\|_{H^{s+1, k+1}} + \|\mathbf{u}^h(t, \cdot)\|_{H^{s+1, k+1}} + \|\eta^h(t, \cdot)\|_{H^{s+1, k+1}} + \|w^h(t, \cdot)\|_{H^{s+1, k+1}} \leq C_1 M_0$$

and, for any multi-index $\alpha \in \mathbb{N}^d$ and $j \in \mathbb{N}$ such that $0 \leq |\alpha| + j \leq s$,

$$\eta^{(\alpha, j)} := \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j \eta^{\text{nh}} - \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j \eta^h; \quad \mathbf{u}^{(\alpha, j)} := \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j \mathbf{u}^{\text{nh}} - \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j \mathbf{u}^h; \quad w^{(\alpha, j)} := \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j w^{\text{nh}} - \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j w^h;$$

and $P_{\text{nh}}^{(\alpha, j)}(\cdot, \varrho) = \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j (P^{\text{nh}}(\cdot, \varrho) - \int_{\rho_0}^{\varrho} \varrho' h^{\text{nh}}(\cdot, \varrho') \, d\varrho')$.

Then restricting to $t \in [0, \min(T^h, T^{\text{nh}})]$ and such that

$$\begin{aligned} \mathcal{F}_{s,k} := & \|h^d\|_{H^{s-1,k-1}} + \|\eta^d\|_{H^{s,k}} + |\eta^d|_{\varrho=\rho_0}|_{H^s_x} + \|\mathbf{u}^d\|_{H^{s,k}} + \mu^{1/2}\|w^d\|_{H^{s,k}} \\ & + \kappa^{1/2}\|h^d\|_{H^{s,k}} + \mu^{1/2}\kappa^{1/2}\|\nabla_{\mathbf{x}}\eta^d\|_{H^{s,k}} \leq \kappa^{1/2}M \end{aligned}$$

we have

$$\begin{aligned} & \partial_t \eta^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} + w^{(\alpha,j)} - \kappa \Delta_{\mathbf{x}} \eta^{(\alpha)} = \tilde{R}_{\alpha,j}, \\ & \partial_t \eta^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} + \left\langle \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} \, d\varrho' \right. \\ & \quad \left. + \int_{\varrho}^{\rho_1} (\underline{h} + h^{\text{nh}}) \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(\alpha,j)} \, d\varrho' \right\rangle_{j=0} - \kappa \Delta_{\mathbf{x}} \eta^{(\alpha,j)} = R_{\alpha,j}, \\ & \partial_t \mathbf{u}^{(\alpha,j)} + \left((\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}} - \kappa \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}) \cdot \nabla_{\mathbf{x}} \right) \mathbf{u}^{(\alpha,j)} + \left\langle \frac{\rho_0}{\varrho} \nabla_{\mathbf{x}} \eta^{(\alpha,j)} \Big|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta^{(\alpha,j)} \, d\varrho' \right\rangle_{j=0} \\ & \quad + \frac{1}{\varrho} \nabla_{\mathbf{x}} P_{\text{nh}}^{(\alpha,j)} + \frac{\nabla_{\mathbf{x}} \eta^{\text{nh}}}{\varrho(\underline{h} + h^{\text{nh}})} \partial_{\varrho} P_{\text{nh}}^{(\alpha,j)} = \mathbf{R}_{\alpha,j}^{\text{nh}}, \\ & \mu^{1/2} \left(\partial_t w^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}} - \kappa \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}) \cdot \nabla_{\mathbf{x}} w^{(\alpha,j)} \right) - \frac{1}{\mu^{1/2}} \frac{\partial_{\varrho} P_{\text{nh}}^{(\alpha,j)}}{\varrho(\underline{h} + h^{\text{nh}})} = R_{\alpha,j}^{\text{nh}}, \\ & -\partial_{\varrho} w^{(\alpha,j)} + (\underline{h} + h^{\text{nh}}) \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(\alpha,j)} + (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} \\ & \quad + (\nabla_{\mathbf{x}} \cdot \mathbf{u}^{\text{nh}}) h^{(\alpha,j)} + (\nabla_{\mathbf{x}} \eta^{\text{nh}}) \cdot (\partial_{\varrho} \mathbf{u}^{(\alpha,j)}) = R_{\alpha,j}^{\text{div}}, \end{aligned} \tag{4.6a}$$

where $(R_{\alpha,j}(t, \cdot), \mathbf{R}_{\alpha,j}^{\text{nh}}(t, \cdot), R_{\alpha,j}^{\text{nh}}(t, \cdot), R_{\alpha,j}^{\text{div}}) \in L^2(\Omega)^{d+3}$, $R_{\alpha,0}(t, \cdot) \in \mathcal{C}((\rho_0, \rho_1); L^2(\mathbb{R}^d))$ and

$$\|R_{\alpha,j}\|_{L^2(\Omega)} + |R_{\alpha,0}|_{\varrho=\rho_0}|_{L^2_x} + \|\tilde{R}_{\alpha,j}\|_{L^2(\Omega)} + \|R_{\alpha,j}^{\text{div}}\|_{L^2(\Omega)} \leq C \mathcal{F}_{s,k}, \tag{4.6b}$$

$$\|\mathbf{R}_{\alpha,j}^{\text{nh}}\|_{L^2(\Omega)} + \|R_{\alpha,j}^{\text{nh}}\|_{L^2(\Omega)} \leq C (\mathcal{F}_{s,k} + \kappa \|\nabla_{\mathbf{x}} h^d\|_{H^{s,k}} + \mu^{1/2} \kappa \|\Delta_{\mathbf{x}} \eta^d\|_{H^{s,k}}) + C \mu^{1/2} M, \tag{4.6c}$$

and

$$\partial_t h^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} h^{(\alpha,j)} - \kappa \Delta_{\mathbf{x}} h^{(\alpha,j)} = r_{\alpha,j} + \nabla_{\mathbf{x}} \cdot \mathbf{r}_{\alpha,j}, \tag{4.7a}$$

where $(r_{\alpha,j}(t, \cdot), \mathbf{r}_{\alpha,j}(t, \cdot)) \in L^2(\Omega)^{1+d}$ and

$$\kappa^{1/2} \|r_{\alpha,j}\|_{L^2(\Omega)} + \|\mathbf{r}_{\alpha,j}\|_{L^2(\Omega)} \leq C \mathcal{F}_{s,k}. \tag{4.7b}$$

PROOF. Explicit expressions for the remainder terms follow from (4.5). Specifically, the following equation is obtained by combining the second and last equation (recall (3.30))

$$\partial_t \eta^d + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^d + \mathbf{u}^d \cdot \nabla_{\mathbf{x}} \eta^h - w^d = \kappa \Delta_{\mathbf{x}} \eta^d$$

and hence

$$\tilde{R}_{\alpha,j} := -[\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, \underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}] \cdot \nabla_{\mathbf{x}} \eta^d - \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j (\mathbf{u}^d \cdot \nabla_{\mathbf{x}} \eta^h),$$

and it follows from product (Lemma A.3) and commutator (Lemma A.8) estimates

$$\|\tilde{R}_{\alpha,j}\|_{L^2(\Omega)} \lesssim (\|\underline{\mathbf{u}}'\|_{W_{\varrho}^{k-1,\infty}} + \|\mathbf{u}^{\text{nh}}\|_{H^{s,k}} + \|\eta^h\|_{H^{s+1,k}}) (\|\eta^d\|_{H^{s,k-1}} + \|\mathbf{u}^d\|_{H^{s,k}}).$$

Then, from the second equation we have

$$R_{\alpha,j} := R_{\alpha,j}^{(i)} + R_{\alpha,j}^{(ii)}$$

with $R_{\alpha,j}^{(i)} := -[\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, \underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}] \cdot \nabla_{\mathbf{x}} \eta^d$ and

$$R_{\alpha,j}^{(ii)} := \begin{cases} -\int_{\varrho}^{\rho_1} [\partial_{\mathbf{x}}^{\alpha}, \partial_{\varrho} \mathbf{u}^{\text{nh}}] \cdot \nabla_{\mathbf{x}} \eta^d + [\partial_{\mathbf{x}}^{\alpha}, h^{\text{nh}}] \nabla_{\mathbf{x}} \cdot \mathbf{u}^d + \partial_{\mathbf{x}}^{\alpha} (\mathbf{u}^d \cdot \nabla_{\mathbf{x}} h^h + h^d \nabla_{\mathbf{x}} \cdot \mathbf{u}^h) \, d\varrho' & \text{if } j = 0, \\ \partial_{\varrho}^{j-1} \partial_{\mathbf{x}}^{\alpha} ((\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^d + (\underline{h} + h^{\text{nh}}) \nabla_{\mathbf{x}} \cdot \mathbf{u}^d + \mathbf{u}^d \cdot \nabla_{\mathbf{x}} h^h + h^d \nabla_{\mathbf{x}} \cdot \mathbf{u}^h) & \text{if } j \geq 1. \end{cases}$$

Using Lemma A.3, Lemma A.8 and the continuous embedding $L^\infty((\rho_0, \rho_1)) \subset L^2((\rho_0, \rho_1)) \subset L^1((\rho_0, \rho_1))$ we find

$$\begin{aligned} \|R_{\alpha,j}\|_{L^2(\Omega)} &\lesssim (\|\underline{\mathbf{u}}'\|_{W_\rho^{k-1,\infty}} + \|\mathbf{u}^{\text{nh}}\|_{H^{s,k}} + \|h^{\text{nh}}\|_{H^{s-1,k-1}} + \|\eta^{\text{nh}}\|_{H^{s,k-1}} + \|h^{\text{h}}\|_{H^{s+1,k-1}} + \|\mathbf{u}^{\text{h}}\|_{H^{s+1,k-1}}) \\ &\quad \times (\|\eta^{\text{d}}\|_{H^{s,k-1}} + \|\mathbf{u}^{\text{d}}\|_{H^{s,k-1}} + \|h^{\text{d}}\|_{H^{s-1,k-1}}) \end{aligned}$$

where for $j = 0$ we used the identities (and Lemma A.1 and Lemma A.7,(3) and (3))

$$\begin{aligned} \int_\rho^{\rho_1} [\partial_x^\alpha, \partial_\rho \mathbf{u}^{\text{nh}}] \cdot \nabla_x \eta^{\text{d}} + [\partial_x^\alpha, h^{\text{nh}}] \nabla_x \cdot \mathbf{u}^{\text{d}} \, d\rho' &= \int_\rho^{\rho_1} [\partial_x^\alpha; \partial_\rho \mathbf{u}^{\text{nh}}, \nabla_x \eta^{\text{d}}] + [\partial_x^\alpha; h^{\text{nh}}, \nabla_x \cdot \mathbf{u}^{\text{d}}] \, d\rho' \\ &\quad + \int_\rho^{\rho_1} \partial_x^\alpha \mathbf{u}^{\text{nh}} \cdot \nabla_x h^{\text{d}} + (\partial_x^\alpha \eta^{\text{nh}})(\nabla_x \cdot \partial_\rho \mathbf{u}^{\text{d}}) \, d\rho' + \partial_x^\alpha \mathbf{u}^{\text{nh}} \cdot \nabla_x \eta^{\text{d}} + (\partial_x^\alpha \eta^{\text{nh}})(\nabla_x \cdot \mathbf{u}^{\text{d}}) \end{aligned}$$

and

$$\int_\rho^{\rho_1} \partial_x^\alpha (h^{\text{d}} \nabla_x \cdot \mathbf{u}^{\text{h}}) \, d\rho' = \int_\rho^{\rho_1} [\partial_x^\alpha, \nabla_x \cdot \mathbf{u}^{\text{h}}] h^{\text{d}} + (\partial_x^\alpha \eta^{\text{d}})(\nabla_x \cdot \partial_\rho \mathbf{u}^{\text{h}}) \, d\rho' + (\partial_x^\alpha \eta^{\text{d}})(\nabla_x \cdot \mathbf{u}^{\text{h}}).$$

This yields the desired estimate for $\|R_{\alpha,j}\|_{L^2(\Omega)}$ and the corresponding estimate for $|R_{\alpha,0}|_{\ell=\rho_0}|_{L_x^2}$ relies on the additional estimate (stemming from Lemma A.7(2) and Lemma A.1)

$$|(R_{\alpha,0}^{(i)} + \partial_x^\alpha \mathbf{u}^{\text{nh}} \cdot \nabla_x \eta^{\text{d}})|_{\ell=\rho_0}|_{L_x^2} = |[\partial_x^\alpha; \mathbf{u}^{\text{nh}}|_{\ell=\rho_0}, \nabla_x \eta^{\text{d}}|_{\ell=\rho_0}]|_{L_x^2} \lesssim \|\mathbf{u}^{\text{nh}}\|_{H^{s,1}} |\eta^{\text{d}}|_{\ell=\rho_0}|_{H_x^s}.$$

Then, we have

$$\begin{aligned} R_{\alpha,j}^{\text{div}} &:= [\partial_x^\alpha \partial_\rho^j, \underline{h}] \nabla_x \cdot \mathbf{u}^{\text{d}} + [\partial_x^\alpha \partial_\rho^j, \underline{\mathbf{u}}'] \cdot \nabla_x \eta^{\text{d}} \\ &\quad + [\partial_x^\alpha \partial_\rho^j, h^{\text{nh}}] \nabla_x \cdot \mathbf{u}^{\text{d}} + [\partial_x^\alpha \partial_\rho^j, \nabla_x \cdot \mathbf{u}^{\text{h}}] h^{\text{d}} + \nabla_x \cdot (\mathbf{u}^{\text{h}} - \mathbf{u}^{\text{nh}}) \partial_x^\alpha \partial_\rho^j h^{\text{d}} \\ &\quad + [\partial_x^\alpha \partial_\rho^j, \partial_\rho \mathbf{u}^{\text{nh}}] \cdot \nabla_x \eta^{\text{d}} + [\partial_x^\alpha \partial_\rho^j, \nabla_x \eta^{\text{h}}] \cdot \partial_\rho \mathbf{u}^{\text{d}} + \nabla_x (\eta^{\text{h}} - \eta^{\text{nh}}) \cdot \partial_x^\alpha \partial_\rho^j \partial_\rho \mathbf{u}^{\text{d}}. \end{aligned}$$

Decomposing $h^{\text{nh}} = h^{\text{h}} + h^{\text{d}}$, $\partial_\rho \mathbf{u}^{\text{nh}} = \partial_\rho \mathbf{u}^{\text{h}} + \partial_\rho \mathbf{u}^{\text{d}}$, some manipulations of the terms to exhibit symmetric commutators and the use of Lemma A.8 and A.9 lead to

$$\begin{aligned} \|R_{\alpha,j}^{\text{div}}\|_{L^2(\Omega)} &\lesssim (\|\underline{h}\|_{W_\rho^{k,\infty}} + \|\underline{\mathbf{u}}'\|_{W_\rho^{k,\infty}} + \|h^{\text{d}}\|_{H^{s-1,k-1}} + \|\partial_\rho \mathbf{u}^{\text{d}}\|_{H^{s-1,k-1}} \\ &\quad + \|h^{\text{h}}\|_{H^{s,k}} + \|\nabla_x \eta^{\text{h}}\|_{H^{s,k}} + \|\mathbf{u}^{\text{h}}\|_{H^{s+1,k+1}}) \times (\|\mathbf{u}^{\text{d}}\|_{H^{s,k}} + \|\eta^{\text{d}}\|_{H^{s,k-1}} + \|h^{\text{d}}\|_{H^{s-1,k-1}}) \end{aligned}$$

which concludes the estimate (4.6b).

We focus now on $\|R_{\alpha,j}^{\text{nh}}\|_{L^2(\Omega)}$ and $\|R_{\alpha,j}^{\text{nh}}\|_{L^2(\Omega)}$. We have

$$\begin{aligned} R_{\alpha,j}^{\text{nh}} &:= [\partial_x^\alpha \partial_\rho^j, (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}} - \kappa \frac{\nabla_x h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}) \cdot \nabla_x] \mathbf{u}^{\text{d}} + \langle \partial_x^\alpha \partial_\rho^j (\frac{\rho_0}{\rho} \nabla_x \eta^{\text{d}}|_{\ell=\rho_0} + \frac{1}{\rho} \int_\rho^\rho \nabla_x \eta^{\text{d}} \, d\rho') \rangle_{j \geq 1} \\ &\quad + \partial_x^\alpha \partial_\rho^j (((\mathbf{u}^{\text{d}} - \kappa (\frac{\nabla_x h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} - \frac{\nabla_x h^{\text{h}}}{\underline{h} + h^{\text{h}}})) \cdot \nabla_x) \mathbf{u}^{\text{h}}) + [\partial_x^\alpha \partial_\rho^j, \frac{1}{\rho}] \nabla_x P_{\text{nh}} + [\partial_x^\alpha \partial_\rho^j, \frac{\nabla_x \eta^{\text{nh}}}{\rho(\underline{h} + h^{\text{nh}})}] \partial_\rho P_{\text{nh}}, \\ R_{\alpha,j}^{\text{nh}} &:= \mu^{1/2} [\partial_x^\alpha \partial_\rho^j, (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}} - \kappa \frac{\nabla_x h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}) \cdot \nabla_x] w^{\text{d}} + \mu^{1/2} \partial_x^\alpha \partial_\rho^j ((\mathbf{u}^{\text{d}} - \kappa (\frac{\nabla_x h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} - \frac{\nabla_x h^{\text{h}}}{\underline{h} + h^{\text{h}}})) \cdot \nabla_x w^{\text{h}}) \\ &\quad - \frac{1}{\mu^{1/2}} [\partial_x^\alpha \partial_\rho^j, \frac{1}{\rho(\underline{h} + h^{\text{nh}})}] \partial_\rho P_{\text{nh}} - \mu^{1/2} \partial_x^\alpha \partial_\rho^j R^{\text{h}}, \end{aligned}$$

where R^{h} is the consistency remainder introduced in Lemma 4.1, (4.4a) and estimated in (4.4b), namely

$$\|\partial_x^\alpha \partial_\rho^j R^{\text{h}}\|_{L^2(\Omega)} \lesssim M_0 \leq M.$$

Let us estimate each contribution. In the following, we shall use repeatedly that $\mathcal{F}^{s,k} \leq \kappa^{1/2} M$ and hence $\|h^{\text{d}}\|_{H^{s,k}} \leq M$. As a consequence, by Lemma A.6 and triangular inequality,

$$\|\frac{h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}\|_{H^{s,k}} \leq C(h_\star, \|\underline{h}\|_{W_\rho^{k,\infty}} \|h^{\text{nh}}\|_{H^{s-1,k-1}}) \|h^{\text{nh}}\|_{H^{s,k}} \leq C(h_\star, \underline{M}, M) M.$$

By Lemma A.8, we have

$$\begin{aligned} & \left\| [\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}}] \mathbf{u}^{\text{d}} \right\|_{L^2(\Omega)} + \mu^{1/2} \left\| [\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}}] w^{\text{d}} \right\|_{L^2(\Omega)} \\ & \lesssim \left(\|\underline{\mathbf{u}}'\|_{W_{\varrho}^{k-1, \infty}} + \|\mathbf{u}^{\text{nh}}\|_{H^{s,k}} \right) \left(\|\nabla_{\mathbf{x}} \mathbf{u}^{\text{d}}\|_{H^{s-1, k-1}} + \mu^{1/2} \|\nabla_{\mathbf{x}} w^{\text{d}}\|_{H^{s-1, k-1}} \right) \leq C(\underline{M}, M) \mathcal{F}_{s,k}. \end{aligned}$$

By Lemma A.9, Lemma A.3 and Lemma A.6

$$\begin{aligned} \left\| [\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, \left(\frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} \cdot \nabla_{\mathbf{x}} \right)] \mathbf{u}^{\text{d}} \right\|_{L^2(\Omega)} & \lesssim \left\| \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} \right\|_{H^{s,k}} \|\nabla_{\mathbf{x}} \mathbf{u}^{\text{d}}\|_{H^{s-1, k-1}} \\ & \lesssim \left\| \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h}} \right\|_{H^{s,k}} \left(1 + \left\| \frac{h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} \right\|_{H^{s,k}} \right) \mathcal{F}_{s,k} \\ & \leq C(h_{\star}, \underline{M}, M) (M + \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k}}) \mathcal{F}_{s,k}. \end{aligned}$$

In the same way, we have

$$\mu^{1/2} \left\| [\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, \left(\frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} \cdot \nabla_{\mathbf{x}} \right)] w^{\text{d}} \right\|_{L^2(\Omega)} \leq C(h_{\star}, \underline{M}, M) (M + \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k}}) \mathcal{F}_{s,k}.$$

When $j \geq 1$,

$$\begin{aligned} & \left\| \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j \left(\frac{\rho_0}{\varrho} \nabla_{\mathbf{x}} \eta^{\text{d}} \Big|_{\varrho=\rho_0} \right) \right\|_{L^2(\Omega)} + \left\| \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j \left(\frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta^{\text{d}} \, d\varrho' \right) \right\|_{L^2(\Omega)} \\ & \lesssim \|\nabla_{\mathbf{x}} \eta^{\text{d}} \Big|_{\varrho=\rho_0}\|_{H_{\mathbf{x}}^{s-1}} + \|\nabla_{\mathbf{x}} \eta^{\text{d}}\|_{H^{s-1, k-1}} \leq \mathcal{F}_{s,k}. \end{aligned}$$

By Lemma A.3, we have

$$\begin{aligned} & \left\| \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j ((\mathbf{u}^{\text{d}} \cdot \nabla_{\mathbf{x}}) \mathbf{u}^{\text{h}}) \right\|_{L^2(\Omega)} + \mu^{1/2} \left\| \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j ((\mathbf{u}^{\text{d}} \cdot \nabla_{\mathbf{x}}) w^{\text{h}}) \right\|_{L^2(\Omega)} \\ & \lesssim \|\mathbf{u}^{\text{d}}\|_{H^{s,k}} \left(\|\nabla_{\mathbf{x}} \mathbf{u}^{\text{h}}\|_{H^{s,k}} + \mu^{1/2} \|\nabla_{\mathbf{x}} w^{\text{d}}\|_{H^{s,k}} \right) \leq C(\underline{M}, M) \mathcal{F}_{s,k}. \end{aligned}$$

By repeated use of tame estimates in Lemma A.3 and Lemma A.5, we find

$$\begin{aligned} \left\| \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j \left(\left(\frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} - \frac{\nabla_{\mathbf{x}} h^{\text{h}}}{\underline{h} + h^{\text{h}}} \right) \cdot \nabla_{\mathbf{x}} \right) \mathbf{u}^{\text{h}} \right\|_{L^2(\Omega)} & \lesssim \left\| \frac{\nabla_{\mathbf{x}} h^{\text{d}}}{\underline{h} + h^{\text{h}}} + \frac{h^{\text{d}} \nabla_{\mathbf{x}} h^{\text{nh}}}{(\underline{h} + h^{\text{nh}})(\underline{h} + h^{\text{h}})} \right\|_{H^{s,k}} \|\nabla_{\mathbf{x}} \mathbf{u}^{\text{h}}\|_{H^{s,k}} \\ & \leq C(h_{\star}, \underline{M}, M) M \left(\|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k}} + M \|h^{\text{d}}\|_{H^{s,k}} \right), \end{aligned}$$

and similarly

$$\mu^{1/2} \left\| \partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j \left(\left(\frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} - \frac{\nabla_{\mathbf{x}} h^{\text{h}}}{\underline{h} + h^{\text{h}}} \right) \cdot \nabla_{\mathbf{x}} \right) w^{\text{h}} \right\|_{L^2(\Omega)} \leq C(h_{\star}, \underline{M}, M) M \left(\|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k}} + M \|h^{\text{d}}\|_{H^{s,k}} \right).$$

Contributions from the pressure remain. By direct inspection, and since $|\alpha| + j - 1 \leq s - 1$,

$$\left\| [\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, \frac{1}{\varrho}] \nabla_{\mathbf{x}} P_{\text{nh}} \right\|_{L^2(\Omega)} \lesssim \|\nabla_{\mathbf{x}} P_{\text{nh}}\|_{H^{s-1, k-1}}.$$

By Lemma A.8 and since $s = k > \frac{5}{2} + \frac{d}{2}$, using the above and Lemma A.1

$$\begin{aligned} \left\| [\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, \frac{\nabla_{\mathbf{x}} \eta^{\text{nh}}}{\varrho(\underline{h} + h^{\text{nh}})}] \partial_{\varrho} P_{\text{nh}} \right\|_{L^2(\Omega)} & \lesssim \left\| \frac{\nabla_{\mathbf{x}} \eta^{\text{nh}}}{\varrho(\underline{h} + h^{\text{nh}})} \right\|_{H^{s,k}} \|\partial_{\varrho} P_{\text{nh}}\|_{H^{s-1, k-1}} \\ & \leq C(h_{\star}, \underline{M}, M) (M + \|\nabla_{\mathbf{x}} \eta^{\text{d}}\|_{H^{s,k}}) \|\partial_{\varrho} P_{\text{nh}}\|_{H^{s-1, k-1}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\| [\partial_{\mathbf{x}}^{\alpha} \partial_{\varrho}^j, \frac{1}{\varrho(\underline{h} + h^{\text{nh}})}] \partial_{\varrho} P_{\text{nh}} \right\|_{L^2(\Omega)} & \lesssim \left(\left\| \frac{1}{\varrho \underline{h}} \right\|_{W_{\varrho}^{k, \infty}} + \left\| \frac{h^{\text{nh}}}{\varrho(\underline{h} + h^{\text{nh}})} \right\|_{H^{s,k}} \right) \|\partial_{\varrho} P_{\text{nh}}\|_{H^{s-1, k-1}} \\ & \leq C(h_{\star}, \underline{M}, M) \|\partial_{\varrho} P_{\text{nh}}\|_{H^{s-1, k-1}}. \end{aligned}$$

Altogether, and using $\mathcal{F}_{s,k} \leq \kappa^{1/2} M$ and $\mu \leq \kappa$, we find

$$\left\| \mathbf{R}_{\alpha, j}^{\text{nh}} \right\|_{L^2(\Omega)} + \left\| \mathbf{R}_{\alpha, j}^{\text{h}} \right\|_{L^2(\Omega)} \leq C(h_{\star}, \underline{M}, M) (\mathcal{F}_{s,k} + \kappa \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k}} + \mu^{-1/2} \|\nabla_{\mathbf{x}, \varrho}^{\mu} P_{\text{nh}}\|_{H^{s-1, k-1}}). \quad (4.8)$$

Now, we use Corollary 3.2, specifically (3.19):

$$\begin{aligned} \|\nabla_{\mathbf{x},\varrho}^\mu P_{\text{nh}}\|_{H^{s-1,k-1}} &\leq C(h_\star, \underline{M}, M) \mu \left(\|(\Lambda^\mu)^{-1} \nabla_{\mathbf{x}} \eta^{\text{nh}}\|_{H^{s,k}} + |(\Lambda^\mu)^{-1} \eta^{\text{nh}}|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^{s+1}} \right. \\ &\quad \left. + \|(\mathbf{u}^{\text{nh}}, \mathbf{u}_\star^{\text{nh}})\|_{H^{s,k}} + \|(w^{\text{nh}}, w_\star^{\text{nh}})\|_{H^{s,k-1}} + \|\mathbf{u}_\star^{\text{nh}}\|_{H^{s,k}} \|w^{\text{nh}}\|_{H^{s,k-1}} \right). \end{aligned}$$

where we recall the notations $\Lambda^\mu := 1 + \sqrt{\mu}|D|$, $\mathbf{u}_\star^{\text{nh}} := -\kappa \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}$ and $w_\star^{\text{nh}} := \kappa \Delta_{\mathbf{x}} \eta^{\text{nh}} - \kappa \frac{\nabla_{\mathbf{x}} h^{\text{nh}} \cdot \nabla_{\mathbf{x}} \eta^{\text{nh}}}{\underline{h} + h^{\text{nh}}}$.

Then we use on one hand that

$$\|(\Lambda^\mu)^{-1} h^{\text{nh}}\|_{H^{s,k-1}} \leq \|h^{\text{h}}\|_{H^{s,k-1}} + \mu^{-1/2} \|h^{\text{d}}\|_{H^{s-1,k-1}} \lesssim M + \mu^{-1/2} \mathcal{F}_{s,k},$$

and, similarly,

$$\begin{aligned} \|(\Lambda^\mu)^{-1} \nabla_{\mathbf{x}} \eta^{\text{nh}}\|_{H^{s,k}} &\leq \|\nabla_{\mathbf{x}} \eta^{\text{h}}\|_{H^{s,k}} + \mu^{-1/2} \|\nabla_{\mathbf{x}} \eta^{\text{d}}\|_{H^{s-1,k}} \lesssim M + \mu^{-1/2} \mathcal{F}_{s,k}, \\ \|(\Lambda^\mu)^{-1} \eta^{\text{nh}}|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^{s+1}} &\leq |\eta^{\text{h}}|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^{s+1}} + \mu^{-1/2} |\eta^{\text{d}}|_{\varrho=\rho_0} \Big|_{H_{\mathbf{x}}^s} \lesssim M + \mu^{-1/2} \mathcal{F}_{s,k}. \end{aligned}$$

On the other hand,

$$\|w^{\text{nh}}\|_{H^{s,k-1}} \leq \|w^{\text{h}}\|_{H^{s,k-1}} + \|w^{\text{d}}\|_{H^{s,k-1}} \lesssim C(\underline{M}, M) M + \mu^{-1/2} \mathcal{F}_{s,k}$$

where, for the first contribution, we applied the product estimates to the expression in (4.1). Then, we have

$$\begin{aligned} \|\mathbf{u}_\star^{\text{nh}}\|_{H^{s,k}} &\leq \kappa \left\| \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} \right\|_{H^{s,k}} \leq \kappa \left(\|\nabla_{\mathbf{x}} h^{\text{nh}}\|_{H^{s,k}} + \|h^{\text{nh}}\|_{H^{s,k}}^2 \right) \\ &\leq C(h_\star, \underline{M}, M) (M + \kappa \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k}}), \\ \|w_\star^{\text{nh}}\|_{H^{s,k-1}} &\leq \kappa \|\Delta_{\mathbf{x}} \eta^{\text{nh}}\|_{H^{s,k-1}} + C(h_\star, \underline{M}, M) \kappa \|\nabla_{\mathbf{x}} h^{\text{nh}}\|_{H^{s,k-1}} \|\nabla_{\mathbf{x}} \eta^{\text{nh}}\|_{H^{s,k-1}} \\ &\leq C(h_\star, \underline{M}, M) (M + \kappa \|\Delta_{\mathbf{x}} \eta^{\text{d}}\|_{H^{s,k-1}} + \kappa^{1/2} M \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k-1}}). \end{aligned}$$

Altogether, this yields

$$\begin{aligned} \mu^{-1/2} \|\nabla_{\mathbf{x},\varrho}^\mu P_{\text{nh}}\|_{H^{s-1,k-1}} &\leq C_0 \left(\mu^{1/2} M + \mathcal{F}_{s,k} + \mu^{1/2} \kappa^{1/2} \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k}} + \mu^{1/2} \kappa \|\Delta_{\mathbf{x}} \eta^{\text{d}}\|_{H^{s,k}} \right. \\ &\quad \left. + (M + \kappa \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{H^{s,k}}) (\mu^{1/2} M + \mathcal{F}_{s,k}) \right). \end{aligned}$$

Plugging this estimate in (4.8), using $\mathcal{F}_{s,k} \leq \kappa^{1/2} M$ and $\mu \leq \kappa$, we obtain (4.6c).

Finally, we set

$$\mathbf{r}_{\alpha,j} := -[\partial_{\mathbf{x}}^\alpha \partial_{\varrho}^j \mathbf{u} + \mathbf{u}^{\text{nh}}] h^{\text{d}} - \partial_{\mathbf{x}}^\alpha \partial_{\varrho}^j ((\underline{h} + h^{\text{h}}) \mathbf{u}^{\text{d}}), \quad r_{\alpha,j} := -(\partial_{\mathbf{x}}^\alpha \partial_{\varrho}^j h^{\text{d}}) \nabla_{\mathbf{x}} \cdot \mathbf{u}^{\text{nh}}.$$

By Lemma A.3 and Lemma A.8 and since $s \geq s_0 + \frac{3}{2}$ and $2 \leq k = s$, we have

$$\|\mathbf{r}_{\alpha,j}\|_{L^2(\Omega)} \lesssim (\|\mathbf{u}'\|_{W_{\varrho}^{k-1,\infty}} + \|\mathbf{u}^{\text{nh}}\|_{H^{s,k}}) \|h^{\text{d}}\|_{H^{s-1,k-1}} + (\|\underline{h}\|_{W_{\varrho}^{k,\infty}} + \|h^{\text{h}}\|_{H^{s,k}}) \|\mathbf{u}^{\text{d}}\|_{H^{s,k}}$$

and (by Lemma A.1)

$$\|\mathbf{r}_{\alpha,j}\|_{L^2(\Omega)} \lesssim \|h^{\text{d}}\|_{H^{s,k}} \|\mathbf{u}^{\text{nh}}\|_{H^{s,k}}.$$

This yields immediately (4.7b). The proof is complete. \square

4.3. Strong convergence. In this section, we prove that for μ sufficiently small and starting from regular and well-prepared initial data, the solution to the non-hydrostatic equations exists at least within the existence time of the solution to the hydrostatic equation. We also prove the strong convergence of the non-hydrostatic system to the hydrostatic one as $\mu \searrow 0$.

PROPOSITION 4.3. There exists $p \in \mathbb{N}$ such that for any $s, k \in \mathbb{N}$ such that $k = s > \frac{5}{2} + \frac{d}{2}$ and any $\underline{M}, M, h_\star, h^\star > 0$, there exists $C = C(s, k, \underline{M}, M, h_\star, h^\star) > 0$ such that the following holds. For any $0 < M_0 \leq M$, $0 < \kappa \leq 1$, and $\mu > 0$ such that

$$\mu \leq \kappa / (CM_0^2),$$

for any $(\underline{h}, \underline{\mathbf{u}}) \in W^{k+p, \infty}((\rho_0, \rho_1))^{1+d}$ satisfying

$$|\underline{h}|_{W_\varrho^{k+p, \infty}} + |\underline{\mathbf{u}}'|_{W_\varrho^{k+p-1, \infty}} \leq \underline{M};$$

for any initial data $(h_0, \mathbf{u}_0, w_0) \in H^{s, k}(\Omega)^{2+d}$ satisfying the boundary condition $w_0|_{\varrho=\rho_1} = 0$ and the incompressibility condition

$$-(\underline{h} + h_0) \nabla_{\mathbf{x}} \cdot \mathbf{u}_0 - (\nabla_{\mathbf{x}} \eta_0) \cdot (\underline{\mathbf{u}}' + \partial_\varrho \mathbf{u}_0) + \partial_\varrho w_0 = 0,$$

(denoting $\eta_0(\cdot, \varrho) = \int_\varrho^{\rho_1} h_0(\cdot, \varrho') d\varrho'$), the bounds

$$\|\eta_0\|_{H^{s+p, k+p}} + \|\mathbf{u}_0\|_{H^{s+p, k+p}} + |\eta_0|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^{s+p}} + \kappa^{1/2} \|h_0\|_{H^{s+p, k+p}} = M_0 \leq M$$

and the stable stratification assumption

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} h_\star \leq \underline{h}(\varrho) + h_0(\mathbf{x}, \varrho) \leq h^\star,$$

the following holds. Denoting

$$(T^{\text{h}})^{-1} = C^{\text{h}} (1 + \kappa^{-1} (|\underline{\mathbf{u}}'|_{L_\varrho^2}^2 + M_0^2)),$$

as in Lemma 4.1 there exists a unique strong solution $(h^{\text{nh}}, \mathbf{u}^{\text{nh}}, w^{\text{nh}}) \in C^0([0, T^{\text{h}}]; H^{s, k}(\Omega)^{1+d})$ to the non-hydrostatic equations (1.8) with initial data $(h^{\text{nh}}, \mathbf{u}^{\text{nh}}, w^{\text{nh}})|_{t=0} = (h_0, \mathbf{u}_0, w_0)$. Moreover, one has $h^{\text{nh}} \in L^2(0, T^{\text{h}}; H^{s+1, k}(\Omega))$, $\eta^{\text{nh}} \in L^2(0, T^{\text{h}}; H^{s+2, k}(\Omega))$ and, for any $t \in [0, T^{\text{h}}]$, the lower and the upper bounds hold

$$\inf_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h^{\text{nh}}(t, \mathbf{x}, \varrho) \geq h_\star/3, \quad \sup_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h^{\text{nh}}(t, \mathbf{x}, \varrho) \leq 3h^\star,$$

and the estimate below holds true

$$\begin{aligned} & \|\eta^{\text{nh}}(t, \cdot)\|_{H^{s, k}} + \|\mathbf{u}^{\text{nh}}(t, \cdot)\|_{H^{s, k}} + \mu^{1/2} \|w^{\text{nh}}(t, \cdot)\|_{H^{s, k}} + |\eta^{\text{nh}}|_{\varrho=\rho_0}(t, \cdot)|_{H_{\mathbf{x}}^s} \\ & + \kappa^{1/2} \|h^{\text{nh}}(t, \cdot)\|_{H^{s, k}} + \mu^{1/2} \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^{\text{nh}}(t, \cdot)\|_{H^{s, k}} \\ & + \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^{\text{nh}}\|_{L^2(0, t; H^{s, k})} + \kappa^{1/2} |\nabla_{\mathbf{x}} \eta^{\text{nh}}|_{\varrho=\rho_0}|_{L^2(0, t; H_{\mathbf{x}}^s)} \\ & + \kappa \|\nabla_{\mathbf{x}} h^{\text{nh}}\|_{L^2(0, t; H^{s, k})} + \mu^{1/2} \kappa \|\nabla_{\mathbf{x}}^2 \eta^{\text{nh}}\|_{L^2(0, t; H^{s, k})} \leq C M_0, \end{aligned} \quad (4.9)$$

and $(h^{\text{nh}}, \mathbf{u}^{\text{nh}})$ converges strongly in $L^\infty(0, T; H^{s, k}(\Omega)^{1+d})$ towards $(h^{\text{h}}, \mathbf{u}^{\text{h}})$ the corresponding solution to the hydrostatic equations (1.9), as $\mu \searrow 0$.

PROOF. We closely follow the proof of Proposition 3.8 and exhibit a bootstrap argument on the functional

$$\begin{aligned} \mathcal{F}(t) & := \|\eta^{\text{d}}(t, \cdot)\|_{H^{s, k}} + \|\mathbf{u}^{\text{d}}(t, \cdot)\|_{H^{s, k}} + \mu^{1/2} \|w^{\text{d}}(t, \cdot)\|_{H^{s, k}} + |\eta^{\text{d}}|_{\varrho=\rho_0}(t, \cdot)|_{H_{\mathbf{x}}^s} \\ & + \kappa^{1/2} \|h^{\text{d}}(t, \cdot)\|_{H^{s, k}} + \mu^{1/2} \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^{\text{d}}(t, \cdot)\|_{H^{s, k}} \\ & + \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^{\text{d}}\|_{L^2(0, t; H^{s, k})} + \kappa^{1/2} |\nabla_{\mathbf{x}} \eta^{\text{d}}|_{\varrho=\rho_0}|_{L^2(0, t; H_{\mathbf{x}}^s)} \\ & + \kappa \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{L^2(0, t; H^{s, k})} + \mu^{1/2} \kappa \|\nabla_{\mathbf{x}}^2 \eta^{\text{d}}\|_{L^2(0, t; H^{s, k})} \end{aligned}$$

where we denote

$$h^{\text{d}} := h^{\text{nh}} - h^{\text{h}}; \quad \eta^{\text{d}} := \eta^{\text{nh}} - \eta^{\text{h}}; \quad \mathbf{u}^{\text{d}} := \mathbf{u}^{\text{nh}} - \mathbf{u}^{\text{h}}; \quad w^{\text{d}} := w^{\text{nh}} - w^{\text{h}}$$

with the usual notation for η^{nh} , η^{h} , and w^{h} is defined by (4.1). Denoting by T^* the maximal existence time of the non-hydrostatic solution provided by Proposition 3.3, we set

$$T^{\text{nh}} = \sup \left\{ 0 < T < \min(T^*, T^{\text{h}}) : \forall t \in (0, T), h_*/3 \leq \underline{h}(\varrho) + h^{\text{nh}}(t, \mathbf{x}, \varrho) \leq 3h^* \right. \\ \left. \text{and } \mathcal{F}(t) \leq \mu^{1/2} M_0 \exp(C_0 t), \quad \mathcal{F}(t) \leq \kappa^{1/2} M_0 \right\}, \quad (4.10)$$

with C_0 sufficiently large (to be determined later on). We will show by the standard continuity argument that $T^{\text{nh}} = \min(T^*, T^{\text{h}})$, which in turns yields $T^* > T^{\text{h}}$ and shows the result. Indeed, the converse inequality $T^{\text{nh}} = T^* \leq T^{\text{h}}$ yields a contradiction by Proposition 3.3 and the desired estimates immediately follow from the control of \mathcal{F} , the bound

$$\|h^{\text{h}}(t, \cdot)\|_{H^{s+1, k+1}} + \|\mathbf{u}^{\text{h}}(t, \cdot)\|_{H^{s+1, k+1}} + \|\eta^{\text{h}}(t, \cdot)\|_{H^{s+1, k+1}} + \|w^{\text{h}}(t, \cdot)\|_{H^{s+1, k+1}} \leq C^{\text{h}} M_0 \quad (4.11)$$

provided by Lemma 4.1, and triangular inequality (when C is chosen sufficiently large).

Let us now derive from Lemma 4.2 the necessary estimates for the bootstrap argument. In the following we repeatedly use the triangular inequality to infer from (4.10) and (4.11) the corresponding control (4.9) with C depending only C^{h} , T^{h} (and $\kappa \leq 1$). We shall denote by C a constant depending uniquely on $s, k, \underline{M}, M, h_*, h^*$ and $C^{\text{h}}, T^{\text{h}}$, but not on C_0 , and which may change from line to line.

By means of (4.7a)-(4.7b) and Lemma 2.4, we infer from (4.10)-(4.11)

$$\kappa^{1/2} \|h^{\text{d}}\|_{L^\infty(0, T; H^{s, k})} + \kappa \|\nabla_{\mathbf{x}} h^{\text{d}}\|_{L^2(0, T; H^{s, k})} \leq C (|\mathcal{F}|_{L_T^1} + |\mathcal{F}|_{L_T^2}).$$

Next, by differentiating with respect to space the first equation of (4.6a) using (4.6b) and Lemma 2.4, we infer

$$\mu^{1/2} \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^{\text{d}}\|_{L^\infty(0, T; H^{s, k})} + \mu^{1/2} \kappa \|\nabla_{\mathbf{x}}^2 \eta^{\text{d}}\|_{L^2(0, T; H^{s, k})} \leq C (|\mathcal{F}|_{L_T^1} + |\mathcal{F}|_{L_T^2}).$$

Now, we use (4.6a)-(4.6b)-(4.6c) and proceeding as in the proof of Proposition 3.8 (together with the above estimates) we infer that for any $t \in (0, T)$,

$$\mathcal{F}(t) \leq C_1 |\mathcal{F}|_{L_t^1} + C_2 |\mathcal{F}|_{L_t^2} + C_3 \mu^{1/2} M_0 t$$

with C_i ($i \in \{1, 2, 3\}$) depending uniquely on $s, k, \underline{M}, M, h_*, h^*$ and $C^{\text{h}}, T^{\text{h}}$. By using the inequality $\mathcal{F}(t) \leq \mu^{1/2} M_0 \exp(C_0 t)$ from (4.10) and the inequality $\tau \leq \exp(\tau)$ (for $\tau \geq 0$), we deduce

$$\mathcal{F}(t) \leq C_1 \mu^{1/2} M_0 C_0^{-1} \exp(C_0 t) + C_2 \mu^{1/2} M_0 (2C_0)^{-1/2} \exp(C_0 t) + C_3 \mu^{1/2} M_0 C_0^{-1} \exp(C_0 t).$$

There remains to choose C_0 sufficiently large so that $C_1 C_0^{-1} + C_2 (2C_0)^{-1/2} + C_3 C_0^{-1} < 1$, and restrict to μ sufficiently small so that $\mu^{1/2} M_0 \exp(C_0 T^{\text{h}}) \leq \mu^{1/2} M_0 C^{1/2} / 2 \leq \kappa^{1/2} / 2$. The upper and lower bounds for $\underline{h} + h^{\text{nh}}$ follow immediately from the corresponding ones for $\underline{h} + h^{\text{h}}$ provided by Lemma 4.1 and triangular inequality, augmenting C if necessary. Then the usual continuity argument yields, as desired, $T^{\text{nh}} = \min(T^*, T^{\text{h}})$. \square

4.4. Improved convergence rate. Proposition 4.3 established the strong convergence for regular well-prepared initial data of the solution to the non-hydrostatic equations, (1.8), towards the corresponding solution to the hydrostatic equations, (1.9), as $\mu \searrow 0$. The convergence rate displayed in the proof is $\mathcal{O}(\mu^{1/2})$. The aim of this section is to provide an improved and *optimal* convergence rate $\mathcal{O}(\mu)$. The strategy is based on the interpretation of the non-hydrostatic solution as an approximate solution to the hydrostatic equations (in the sense of consistency) and the use of the uniform control obtained in Proposition 4.3.

PROPOSITION 4.4. There exists $p \in \mathbb{N}$ such that for any $s, k \in \mathbb{N}$ with $k = s > \frac{5}{2} + \frac{d}{2}$ and $\underline{M}, M, h_*, h^* > 0$, there exists $C = C(s, k, \underline{M}, M, h_*, h^*) > 0$ such that under the assumptions of Proposition 4.3 and using the notations therein,

$$\|h^{\text{nh}} - h^{\text{h}}\|_{L^\infty(0, T^{\text{h}}; H^{s-1, k-1})} + \|\eta^{\text{nh}} - \eta^{\text{h}}\|_{L^\infty(0, T^{\text{h}}; H^{s, k})} + \|\mathbf{u}^{\text{nh}} - \mathbf{u}^{\text{h}}\|_{L^\infty(0, T^{\text{h}}; H^{s, k})} \leq C \mu.$$

COROLLARY 4.5. Incrementing $p \in \mathbb{N}$, we find that for any $s, k \in \mathbb{N}$ such that $k = s > \frac{3}{2} + \frac{d}{2}$,

$$\|h^{\text{nh}} - h^{\text{h}}\|_{L^\infty(0, T^{\text{h}}; H^{s, k})} + \|\eta^{\text{nh}} - \eta^{\text{h}}\|_{L^\infty(0, T^{\text{h}}; H^{s+1, k+1})} + \|\mathbf{u}^{\text{nh}} - \mathbf{u}^{\text{h}}\|_{L^\infty(0, T^{\text{h}}; H^{s+1, k+1})} \leq C \mu$$

with $C = C(s + 1, k + 1, \underline{M}, M, h_*, h^*) > 0$.

PROOF. Since all arguments of the proof have been already used in slightly different contexts, we only quickly sketch the argument.

For any $p' \in \mathbb{N}$, we may use Proposition 4.3 with indices $s + p'$ and $k + p'$ to infer the existence of the non-hydrostatic solution $(h^{\text{nh}}, \mathbf{u}^{\text{nh}}, w^{\text{nh}}) \in \mathcal{C}([0, T^{\text{h}}]; H^{s+p', k+p'}(\Omega)^{2+d})$ and the control

$$\sup_{t \in [0, T^{\text{h}}]} \left(\|\eta^{\text{nh}}(t, \cdot)\|_{H^{s+p', k+p'}} + \|\mathbf{u}^{\text{nh}}(t, \cdot)\|_{H^{s+p', k+p'}} + |\eta^{\text{nh}}|_{\varrho=\rho_0}(t, \cdot)|_{H_{\mathbf{x}}^{s+p'}} \right) \leq C M_0.$$

By using $h^{\text{nh}} = -\partial_{\varrho} \eta^{\text{nh}}$ and the divergence-free condition

$$w^{\text{nh}} = (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{\text{nh}} - \int_{\varrho}^{\rho_1} \nabla_{\mathbf{x}} \cdot ((\underline{h} + h^{\text{nh}})(\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}})) \, d\varrho'$$

we obtain (augmenting C if necessary)

$$\sup_{t \in [0, T^{\text{h}}]} \left(\|h^{\text{nh}}(t, \cdot)\|_{H^{s+p'-1, k+p'-1}} + \|w^{\text{nh}}(t, \cdot)\|_{H^{s+p'-1, k+p'-1}} \right) \leq C M_0$$

and hence, by Corollary 3.2 (specifically (3.19)), Poincaré inequality (3.10) and choosing p' sufficiently large, that $P_{\text{nh}}(\cdot, \varrho) := P^{\text{nh}}(\cdot, \varrho) - \int_{\rho_0}^{\varrho} \varrho' h^{\text{nh}}(\cdot, \varrho') \, d\varrho'$ satisfies

$$\sup_{t \in [0, T^{\text{h}}]} \|P_{\text{nh}}(t, \cdot)\|_{H^{s+1, k+1}} \leq C \mu M_0.$$

From this estimate we infer (by Lemma 4.1) that $h^{\text{d}} := h^{\text{nh}} - h^{\text{h}}$ and $\mathbf{u}^{\text{d}} := \mathbf{u}^{\text{nh}} - \mathbf{u}^{\text{h}}$ satisfies

$$\begin{aligned} \partial_t \eta^{\text{d}} + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{\text{d}} + \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{\text{d}} \, d\varrho' + \int_{\varrho}^{\rho_1} (\underline{h} + h^{\text{nh}}) \nabla_{\mathbf{x}} \cdot \mathbf{u}^{\text{d}} \, d\varrho' \\ + \int_{\varrho}^{\rho_1} \mathbf{u}^{\text{d}} \cdot \nabla_{\mathbf{x}} h^{\text{h}} + h^{\text{d}} \nabla_{\mathbf{x}} \cdot \mathbf{u}^{\text{h}} \, d\varrho' = \kappa \Delta_{\mathbf{x}} \eta^{\text{d}}, \\ \partial_t \mathbf{u}^{\text{d}} + \left((\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}} - \kappa \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}) \cdot \nabla_{\mathbf{x}} \right) \mathbf{u}^{\text{d}} + \frac{\rho_0}{\varrho} \nabla_{\mathbf{x}} \eta^{\text{d}}|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta^{\text{d}} \, d\varrho' \\ + \left((\mathbf{u}^{\text{d}} - \kappa \left(\frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}} - \frac{\nabla_{\mathbf{x}} h^{\text{h}}}{\underline{h} + h^{\text{h}}} \right)) \cdot \nabla_{\mathbf{x}} \right) \mathbf{u}^{\text{h}} = \mathbf{R}^{\text{nh}}, \end{aligned} \quad (4.12)$$

with $\mathbf{R}^{\text{nh}} := -\frac{\nabla_{\mathbf{x}} P_{\text{nh}}}{\varrho} - \frac{\nabla_{\mathbf{x}} \eta^{\text{nh}}}{\varrho(\underline{h} + h^{\text{nh}})} \partial_{\varrho} P_{\text{nh}}$ satisfying (by Lemma A.3 and Lemma A.6) the bound

$$\sup_{t \in [0, T^{\text{h}}]} \|\mathbf{R}^{\text{nh}}(t, \cdot)\|_{H^{s, k}} \leq C \mu M_0.$$

From this, inspecting the proof of Lemma 4.2, we infer that as long as

$$\mathcal{F}_{s, k} := \|h^{\text{d}}\|_{H^{s-1, k-1}} + \|\eta^{\text{d}}\|_{H^{s, k}} + |\eta^{\text{d}}|_{\varrho=\rho_0}|_{H_{\mathbf{x}}^s} + \|\mathbf{u}^{\text{d}}\|_{H^{s, k}} + \kappa^{1/2} \|h^{\text{d}}\|_{H^{s, k}} \leq \kappa^{1/2} M_0,$$

one has for any $\alpha \in \mathbb{N}^d$ and $j \in \mathbb{N}$ such that $|\alpha| + j \leq s$ that $\eta^{(\alpha,j)} := \partial_{\mathbf{x}}^\alpha \partial_{\varrho}^j \eta^d$, $\mathbf{u}^{(\alpha,j)} := \partial_{\mathbf{x}}^\alpha \partial_{\varrho}^j \mathbf{u}^d$ and $h^{(\alpha,j)} := \partial_{\mathbf{x}}^\alpha \partial_{\varrho}^j h^d$ satisfy

$$\begin{aligned} \partial_t \eta^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} + \left\langle \int_{\varrho}^{\rho_1} (\underline{\mathbf{u}}' + \partial_{\varrho} \mathbf{u}^{\text{nh}}) \cdot \nabla_{\mathbf{x}} \eta^{(\alpha,j)} \, d\varrho' \right. \\ \left. + \int_{\varrho}^{\rho_1} (\underline{h} + h^{\text{nh}}) \nabla_{\mathbf{x}} \cdot \mathbf{u}^{(\alpha,j)} \, d\varrho' \right\rangle_{j=0} - \kappa \Delta_{\mathbf{x}} \eta^{(\alpha,j)} = R_{\alpha,j}, \\ \partial_t \mathbf{u}^{(\alpha,j)} + ((\underline{\mathbf{u}} + \mathbf{u}^{\text{nh}} - \kappa \frac{\nabla_{\mathbf{x}} h^{\text{nh}}}{\underline{h} + h^{\text{nh}}}) \cdot \nabla_{\mathbf{x}}) \mathbf{u}^{(\alpha,j)} + \left\langle \frac{\rho_0}{\varrho} \nabla_{\mathbf{x}} \eta^{(\alpha,j)} \Big|_{\varrho=\rho_0} \right. \\ \left. + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\mathbf{x}} \eta^{(\alpha,j)} \, d\varrho' \right\rangle_{j=0} = \mathbf{R}_{\alpha,j}^{\text{nh}}, \end{aligned}$$

and

$$\partial_t h^{(\alpha,j)} + (\underline{\mathbf{u}} + \mathbf{u}) \cdot \nabla_{\mathbf{x}} h^{(\alpha,j)} = \kappa \Delta_{\mathbf{x}} \partial_{\varrho}^j h^{(\alpha,j)} + r_{\alpha,j} + \nabla_{\mathbf{x}} \cdot \mathbf{r}_{\alpha,j},$$

with

$$\|R_{\alpha,j}\|_{L^2(\Omega)} + \|\mathbf{R}_{\alpha,j}^{\text{nh}}\|_{L^2(\Omega)} \leq C (\mathcal{F}_{s,k} + M_0 \kappa \|\nabla_{\mathbf{x}} h^d\|_{H^{s,k}}) + C \mu M_0$$

and

$$\kappa^{1/2} \|r_{\alpha,j}\|_{L^2(\Omega)} + \|\mathbf{r}_{\alpha,j}\|_{L^2(\Omega)} \leq C \mathcal{F}_{s,k}.$$

We may then proceed as in the proof of Proposition 2.6, and bootstrap the control

$$\mathcal{F}_{s,k}(t) + \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^d\|_{L^2(0,t;H^{s,k})} + \kappa^{1/2} \|\nabla_{\mathbf{x}} \eta^d\|_{\varrho=\rho_0} \Big|_{L^2(0,t;H_{\mathbf{x}}^s)} + \kappa \|\nabla_{\mathbf{x}} h^d\|_{L^2(0,t;H^{s,k})} \leq C \mu M_0$$

(choosing C large enough) on the time interval $[0, T^{\text{h}}]$. This concludes the proof. \square

Appendix A. Product, composition and commutator estimates

In this section we collect useful estimates in the spaces $H^{s,k}(\Omega)$ introduced in (1.12). Our results will follow from standard estimates in Sobolev spaces $H^s(\mathbb{R}^d)$ (see *e.g.* [22, Appendix B] and references therein), and the following continuous embedding. Henceforth we denote $\Omega = \mathbb{R}^d \times (\rho_0, \rho_1)$.

LEMMA A.1. For any $s \in \mathbb{R}$ and $\rho_0 < \rho_1$, $H^{s+1/2,1}(\Omega) \subset C^0([\rho_0, \rho_1]; H^s(\mathbb{R}^d))$ and there exists $C > 0$ such that for any $F \in H^{s+1/2,1}(\Omega)$,

$$\max_{\varrho \in [\rho_0, \rho_1]} |F(\cdot, \varrho)|_{H_{\mathbf{x}}^s} \leq C \|F\|_{H^{s+1/2,1}}.$$

More generally, for any $k \geq 1$, $H^{s+1/2,1}(\Omega) \subset \bigcap_{j=0}^{k-1} C^j([\rho_0, \rho_1]; H^{s-j}(\mathbb{R}^d))$, and in particular, for any $s_0 > d/2$ and $j \in \mathbb{N}$, $H^{j+s_0+\frac{1}{2},j+1}(\Omega) \subset (C^j(\Omega) \cap W^{j,\infty}(\Omega))$.

PROOF. By a density argument, we only need to prove the inequality for smooth functions F . Set $\phi : [\rho_0, \rho_1] \rightarrow \mathbb{R}^+$ a smooth function such that $\phi(\rho_0) = 0$ and $\phi(\varrho) = 1$ if $\varrho \geq \frac{\rho_0 + \rho_1}{2}$, and deduce that for any $\varrho \geq \frac{\rho_0 + \rho_1}{2}$, recalling the notation $\Lambda^s := (\text{Id} - \Delta_{\mathbf{x}})^{s/2}$,

$$\begin{aligned} \int_{\mathbb{R}^d} (\Lambda^s F)^2(\mathbf{x}, \varrho) \, d\mathbf{x} &= \int_{\mathbb{R}^d} \int_{\rho_0}^{\varrho} \partial_{\varrho}(\phi(\varrho')) (\Lambda^s F)^2(\mathbf{x}, \varrho') \, d\varrho' \, d\mathbf{x} \\ &\leq 2 |\phi|_{L_{\varrho}^{\infty}} \int_{\rho_0}^{\rho_1} |\Lambda^s F(\cdot, \varrho)|_{H_{\mathbf{x}}^{1/2}} |\Lambda^s \partial_{\varrho} F(\cdot, \varrho)|_{H_{\mathbf{x}}^{-1/2}} \, d\varrho \\ &\quad + |\phi'|_{L^{\infty}} \int_{\rho_0}^{\rho_1} |\Lambda^s F(\cdot, \varrho)|_{L^2} |\Lambda^s F(\cdot, \varrho)|_{L^2} \, d\varrho \\ &\lesssim \|F\|_{H^{s+1/2,0}}^2 + \|\partial_{\varrho} F\|_{H^{s-1/2,0}}^2 \end{aligned}$$

Using symmetrical considerations when $\varrho < \frac{\rho_0 + \rho_1}{2}$, we prove the claimed inequality, which yields the first continuous embedding. Higher-order embeddings follow immediately. \square

Recall the notation

$$A_s + \langle B_s \rangle_{s > s_*} = \begin{cases} A_s & \text{if } s \leq s_*, \\ A_s + B_s & \text{otherwise.} \end{cases}$$

Product estimates. Recall the standard product estimates in Sobolev spaces $H^s(\mathbb{R}^d)$.

LEMMA A.2. Let $d \in \mathbb{N}^*$, $s_0 > d/2$.

- (1) For any $s, s_1, s_2 \in \mathbb{R}$ such that $s_1 \geq s, s_2 \geq s$ and $s_1 + s_2 \geq s + s_0$, there exists $C > 0$ such that for any $f \in H^{s_1}(\mathbb{R}^d)$ and $g \in H^{s_2}(\mathbb{R}^d)$, $fg \in H^s(\mathbb{R}^d)$ and

$$|fg|_{H^s} \leq C |f|_{H^{s_1}} |g|_{H^{s_2}}.$$

- (2) For any $s \geq -s_0$, there exists $C > 0$ such that for any $f \in H^s(\mathbb{R}^d)$ and $g \in H^s(\mathbb{R}^d) \cap H^{s_0}(\mathbb{R}^d)$, $fg \in H^s(\mathbb{R}^d)$ and

$$|fg|_{H^s} \leq C |f|_{H^{s_0}} |g|_{H^s} + C \langle |f|_{H^s} |g|_{H^{s_0}} \rangle_{s > s_0}.$$

- (3) For any $s_1, \dots, s_n \in \mathbb{R}$ such that $s_i \geq 0$ and $s_1 + \dots + s_n \geq (n-1)s_0$, there exists $C > 0$ such that for any $(f_1, \dots, f_n) \in H^{s_1}(\mathbb{R}^d) \times \dots \times H^{s_n}(\mathbb{R}^d)$, $\prod_{i=1}^n f_i \in L^2(\mathbb{R}^d)$ and

$$\left| \prod_{i=1}^n f_i \right|_{L^2} \leq C \prod_{i=1}^n |f_i|_{H^{s_i}}.$$

Let us turn to product estimates in $H^{s,k}(\Omega)$ spaces.

LEMMA A.3. Let $d \in \mathbb{N}^*$, $s_0 > d/2$. Let $s, k \in \mathbb{N}$ such that $s \geq s_0 + \frac{1}{2}$ and $1 \leq k \leq s$. Then $H^{s,k}(\Omega)$ is a Banach algebra and there exists $C > 0$ such that for any $F, G \in H^{s,k}(\Omega)$,

$$\|FG\|_{H^{s,k}} \leq C \|F\|_{H^{s,k}} \|G\|_{H^{s,k}}.$$

Moreover, if $s \geq s_0 + \frac{3}{2}$ and $2 \leq k \leq s$, then there exists $C' > 0$ such that for any $F, G \in H^{s,k}(\Omega)$,

$$\|FG\|_{H^{s,k}} \leq C' \|F\|_{H^{s,k}} \|G\|_{H^{s-1,k-1}} + C' \|F\|_{H^{s-1,k-1}} \|G\|_{H^{s,k}},$$

and if $s \geq s_0 + \frac{3}{2}$ and $k = 1$, then there exists $C'' > 0$ such that for any $F, G \in H^{s,k}(\Omega)$,

$$\|FG\|_{H^{s,1}} \leq C'' \|F\|_{H^{s,1}} \|G\|_{H^{s-1,1}} + C'' \|F\|_{H^{s-1,1}} \|G\|_{H^{s,1}}.$$

PROOF. We set two multi-indices $\beta = (\beta_x, \beta_\varrho) \in \mathbb{N}^{d+1}$ and $\gamma = (\gamma_x, \gamma_\varrho) \in \mathbb{N}^{d+1}$ being such that $|\beta| + |\gamma| \leq s$ and $\beta_\varrho + \gamma_\varrho \leq k$. Let us first assume furthermore that $\gamma_\varrho \leq k-1$ and $|\gamma| \leq s-1$. Then

$$\begin{aligned} \|(\partial^\beta F)(\partial^\gamma G)\|_{L^2(\Omega)}^2 &\lesssim \int_{\rho_0}^{\rho_1} |\partial^\beta F(\cdot, \varrho)|_{H_x^{s-|\beta|}}^2 |\partial^\gamma G(\cdot, \varrho)|_{H_x^{s-|\gamma|-\frac{1}{2}}}^2 d\varrho \\ &\lesssim \|\partial^\beta F\|_{H^{s-|\beta|,0}}^2 \|\partial^\gamma G\|_{H^{s-|\gamma|,1}}^2 \leq \|F\|_{H^{s,k}}^2 \|G\|_{H^{s,k}}^2. \end{aligned}$$

where we used Lemma A.2(1) with $(s, s_1, s_2) = (0, s-|\beta|, s-|\gamma|-\frac{1}{2})$, and Lemma A.1. If $\gamma_\varrho = k$ or $|\gamma| = s$, and since $1 \leq k \leq s$, we have $\beta_\varrho \leq k-1$ and $|\beta| \leq s-1$ and we may make use of the symmetric estimate. Hence the proof of the first statement follows from Leibniz rule.

For the second statement, we assume first that $\max(\{\beta_\varrho, \gamma_\varrho\}) \leq k-1$ and $\max(\{|\beta|, |\gamma|\}) \leq s-1$. Then, using Lemma A.2 with $(s, s_1, s_2) = (0, s-|\beta|-\frac{1}{2}, s-|\gamma|-1)$ (recall $s \geq s_0 + \frac{3}{2}$), and Lemma A.1,

$$\begin{aligned} \|(\partial^\beta F)(\partial^\gamma G)\|_{L^2(\Omega)} &\lesssim \|\partial^\beta F(\cdot, \varrho)\|_{H_x^{s-|\beta|-\frac{1}{2}}} \|\partial^\gamma G(\cdot, \varrho)\|_{H_x^{s-|\gamma|-1}}|_{L_\varrho^2} \\ &\lesssim \|F\|_{H^{s,\beta_\varrho+1}} \|G\|_{H^{s-1,\gamma_\varrho}} \leq \|F\|_{H^{s,k}} \|G\|_{H^{s-1,k-1}}, \end{aligned}$$

Then if $\beta_\rho = k$ or $|\beta| = s$, we have (since $s \geq k \geq 2$) $\gamma_\rho \leq k - 2$ and $|\gamma| \leq s - 2$, and we infer

$$\begin{aligned} \|(\partial^\beta F)(\partial^\gamma G)\|_{L^2(\Omega)} &\lesssim \|\partial^\beta F(\cdot, \rho)\|_{H_x^{s-|\beta|}} \|\partial^\gamma G(\cdot, \rho)\|_{H_x^{s-|\gamma|-\frac{3}{2}}} \Big|_{L_\rho^2} \\ &\lesssim \|F\|_{H^{s,\beta_\rho}} \|G\|_{H^{s-1,\gamma_\rho+1}} \leq \|F\|_{H^{s,k}} \|G\|_{H^{s-1,k-1}}. \end{aligned}$$

Of course we have the symmetrical result when $\gamma_\rho = k$ or $|\gamma| = s$, which complete the proof.

Finally, for the last statement, we consider first the case $\beta_\rho = 0$ and $\max(\{|\beta|, |\gamma|\}) \leq s - 1$, and infer as above

$$\|(\partial^\beta F)(\partial^\gamma G)\|_{L^2(\Omega)} \lesssim \|\partial^\beta F(\cdot, \rho)\|_{H_x^{s-|\beta|-\frac{1}{2}}} \|\partial^\gamma G(\cdot, \rho)\|_{H_x^{s-|\gamma|-1}} \Big|_{L_\rho^2} \lesssim \|F\|_{H^{s,1}} \|G\|_{H^{s-1,1}}.$$

The case $\beta_\rho = 1$ (and hence $\gamma_\rho = 0$) and $\max(\{|\beta|, |\gamma|\}) \leq s - 1$ is treated symmetrically. Then if $|\beta| = s$ we have $\gamma_\rho = |\gamma| = 0$, and we infer

$$\|(\partial^\beta F)(\partial^\gamma G)\|_{L^2(\Omega)} \lesssim \|\partial^\beta F(\cdot, \rho)\|_{H_x^{s-|\beta|}} \|\partial^\gamma G(\cdot, \rho)\|_{H_x^{s-|\gamma|-\frac{3}{2}}} \Big|_{L_\rho^2} \lesssim \|F\|_{H^{s,1}} \|G\|_{H^{s-1,1}}.$$

The case $|\gamma| = s$ is treated symmetrically, and the proof is complete. \square

Composition estimates. Let us recall the standard composition estimate in Sobolev spaces $H^s(\mathbb{R}^d)$.

LEMMA A.4. Let $d \in \mathbb{N}^*$, $s_0 > d/2$. For any $\varphi \in C^\infty(\mathbb{R}; \mathbb{R})$ such that $\varphi(0) = 0$, and any $M > 0$, there exists $C > 0$ such that for any $f \in H^{s_0}(\mathbb{R}^d) \cap H^s(\mathbb{R}^d)$ with $|f|_{H^{s_0}} \leq M$, one has $\varphi(f) \in H^s(\mathbb{R}^d)$ and

$$|\varphi(f)|_{H^s} \leq C|f|_{H^s}.$$

We now consider composition estimates in $H^{s,k}(\Omega)$.

LEMMA A.5. Let $d \in \mathbb{N}^*$, $s_0 > d/2$. Let $s, k \in \mathbb{N}$ with $s \geq s_0 + \frac{1}{2}$ and $1 \leq k \leq s$, and $M > 0$. There exists $C > 0$ such that for any $\varphi \in W^{s,\infty}(\mathbb{R}; W^{k,\infty}((\rho_0, \rho_1)))$ with $\varphi(0; \cdot) \equiv 0$, and any $F \in H^{s,k}(\Omega)$ such that $\|F\|_{H^{s,k}} \leq M$, then $\varphi \circ F : (\mathbf{x}, \rho) \mapsto \varphi(F(\mathbf{x}, \rho); \rho) \in H^{s,k}(\Omega)$ and

$$\|\varphi \circ F\|_{H^{s,k}} \leq C|\varphi|_{W^{s,\infty}(\mathbb{R}; W^{k,\infty}((\rho_0, \rho_1)))} \|F\|_{H^{s,k}}.$$

If moreover $s \geq s_0 + \frac{3}{2}$ and $2 \leq k \leq s$, then there exists $C' > 0$ such that for any $F \in H^{s,k}(\Omega)$ such that $\|F\|_{H^{s-1,k-1}} \leq M$,

$$\|\varphi \circ F\|_{H^{s,k}} \leq C'|\varphi|_{W^{s,\infty}(\mathbb{R}; W^{k,\infty}((\rho_0, \rho_1)))} \|F\|_{H^{s,k}}.$$

PROOF. Let $\alpha = (\alpha_x, \alpha_\rho) \in \mathbb{N}^{d+1} \setminus \{\mathbf{0}\}$ with $0 \leq |\alpha| \leq s$ and $0 \leq \alpha_\rho \leq k$. We have by Faà di Bruno's formula

$$\|\partial^\alpha(\varphi \circ F)\|_{L^2(\Omega)} \lesssim \sum \|((\partial_1^i \partial_2^j \varphi) \circ F) (\partial^{\alpha_1^{i,j}} F) \dots (\partial^{\alpha_i^{i,j}} F)\|_{L^2(\Omega)},$$

where $i, j \in \mathbb{N}$ with $i + j \leq |\alpha| \leq s$, and the multi-indices $\alpha_\ell^{i,j} = (\alpha_{\ell,x}^{i,j}, \alpha_{\ell,\rho}^{i,j}) \in \mathbb{N}^{d+1} \setminus \{\mathbf{0}\}$ satisfy $\sum_{\ell=1}^i \alpha_{\ell,x}^{i,j} = \alpha_x$ and $j + \sum_{\ell=1}^i \alpha_{\ell,\rho}^{i,j} = \alpha_\rho$. If $i = 0$ then we have from the mean value theorem that for any $0 \leq j \leq k$

$$\|(\partial_2^j \varphi) \circ F\|_{L^2(\Omega)} = \|(\partial_2^j \varphi) \circ F - (\partial_2^j \varphi) \circ 0\|_{L^2(\Omega)} \leq |\partial_1 \partial_2^j \varphi|_{L^\infty(\mathbb{R} \times (\rho_0, \rho_1))} \|F\|_{L^2(\Omega)}.$$

The case $i = 1$ is straightforward, and we now focus on the case $i \geq 2$. We assume without loss of generality that $|\alpha_{1,\rho}^{i,j}| \geq |\alpha_{2,\rho}^{i,j}| \geq \dots \geq |\alpha_{i,\rho}^{i,j}|$ and remark that for $\ell \neq 1$, $|\alpha_{\ell,\rho}^{i,j}| \leq k - 1$ (otherwise $|\alpha_{1,\rho}^{i,j}| + |\alpha_{\ell,\rho}^{i,j}| = 2k > k \geq |\alpha_\rho|$) and $|\alpha_\ell^{i,j}| \leq s - |\alpha_1^{i,j}| \leq s - 1$. Hence we have

$$\begin{aligned} \|\prod_{\ell=1}^i (\partial^{\alpha_\ell^{i,j}} F)\|_{L^2(\Omega)} &\lesssim \|\partial^{\alpha_1^{i,j}} F\|_{H_x^{s-|\alpha_1^{i,j}|}} \left(\prod_{\ell=2}^i \|\partial^{\alpha_\ell^{i,j}} F\|_{H_x^{s-|\alpha_\ell^{i,j}|-\frac{1}{2}}} \right) \Big|_{L_\rho^2} \\ &\lesssim \|F\|_{H^{s,\alpha_{1,\rho}^{i,j}}} \left(\prod_{\ell=2}^i \|F\|_{H^{s,\alpha_{\ell,\rho}^{i,j}+1}} \right) \leq \|F\|_{H^{s,k}}^i \end{aligned}$$

where we used Lemma A.2(3) and $(i-1)(s-\frac{1}{2}) \geq (i-1)s_0$ and Lemma A.1. The first claim follows.

Now we assume additionally that $k \geq 2$ and $s \geq s_0 + \frac{3}{2}$. The cases $i \in \{0, 1\}$ can be treated exactly as above and we deal only with the case $i \geq 2$, ordering $|\alpha_{1,\varrho}^{i,j}| \geq |\alpha_{2,\varrho}^{i,j}| \geq \dots \geq |\alpha_{i,\varrho}^{i,j}|$ as above. Assume first that $|\alpha_{1,\varrho}^{i,j}| = k \geq 2$. Then for all $\ell \neq 1$, $|\alpha_{\ell,\varrho}^{i,j}| = 0$ and $|\alpha_{\ell}^{i,j}| \leq s - 2$, and we conclude as before with

$$\|\prod_{\ell=1}^i (\partial^{\alpha_{\ell}^{i,j}} F)\|_{L^2(\Omega)} \lesssim \|\partial^{\alpha_1^{i,j}} F\|_{H_{\mathbf{x}}^{s-|\alpha_1^{i,j}|}} \left(\prod_{\ell=1}^i \|\partial^{\alpha_{\ell}^{i,j}} F\|_{H_{\mathbf{x}}^{s-|\alpha_{\ell}^{i,j}|-\frac{3}{2}}} \right) \Big|_{L_{\varrho}^2} \lesssim \|F\|_{H^{s,k}} \|F\|_{H^{s-1,1}}^{i-1}.$$

Otherwise we have $|\alpha_{2,\varrho}^{i,j}| \leq |\alpha_{1,\varrho}^{i,j}| \leq k - 1$ and $|\alpha_2^{i,j}| \leq s - |\alpha_1^{i,j}| \leq s - 1$ and notice that for $\ell \geq 3$, $|\alpha_{\ell,\varrho}^{i,j}| \leq k - 2$ (since otherwise we have the contradiction $|\alpha_{1,\varrho}^{i,j}| + |\alpha_{2,\varrho}^{i,j}| + |\alpha_{3,\varrho}^{i,j}| \geq 3(k - 1) \geq k + 1 \geq |\alpha_{\varrho}| + 1$) and $|\alpha_{\ell}^{i,j}| \leq s - |\alpha_1^{i,j}| - |\alpha_2^{i,j}| \leq s - 2$. Hence

$$\begin{aligned} \|\prod_{\ell=1}^i (\partial^{\alpha_{\ell}^{i,j}} F)\|_{L^2(\Omega)} &\lesssim \|\partial^{\alpha_1^{i,j}} F\|_{H_{\mathbf{x}}^{s-|\alpha_1^{i,j}|-\frac{1}{2}}} \|\partial^{\alpha_2^{i,j}} F\|_{H_{\mathbf{x}}^{s-|\alpha_2^{i,j}|-1}} \left(\prod_{\ell=3}^i \|\partial^{\alpha_{\ell}^{i,j}} F\|_{H_{\mathbf{x}}^{s-|\alpha_{\ell}^{i,j}|-\frac{3}{2}}} \right) \Big|_{L_{\varrho}^2} \\ &\lesssim \|F\|_{H^{s,\alpha_1^{i,j}+1}} \|F\|_{H^{s-1,\alpha_2^{i,j}}} \left(\prod_{\ell=3}^i \|F\|_{H^{s-1,\alpha_{\ell}^{i,j}+1}} \right) \lesssim \|F\|_{H^{s,k}} \|F\|_{H^{s-1,k-1}}^{i-1}. \end{aligned}$$

This concludes the proof. \square

We shall apply the above to estimate quantities such as (but not restricted to)

$$\Phi : (\mathbf{x}, \varrho) \in \Omega \mapsto \frac{h(\mathbf{x}, \varrho)}{\underline{h}(\varrho) + h(\mathbf{x}, \varrho)},$$

with $\underline{h} \in W^{k,\infty}((\rho_0, \rho_1))$ and $h \in H^{s,k}(\mathbb{R}^d)$ satisfying the condition $\inf_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h(\mathbf{x}, \varrho) \geq h_{\star} > 0$. Let us detail the result and its proof for this specific example.

LEMMA A.6. Let $d \in \mathbb{N}^*$, $s_0 > d/2$. Let $s, k \in \mathbb{N}$ with $s \geq s_0 + \frac{1}{2}$ and $1 \leq k \leq s$, and $M, \underline{M}, h_{\star} > 0$. There exists $C > 0$ such that for any $\underline{h} \in W^{k,\infty}((\rho_0, \rho_1))$ with $|\underline{h}|_{W_{\varrho}^{k,\infty}} \leq \underline{M}$ and any $h \in H^{s,k}(\Omega)$ with $\|h\|_{H^{s,k}} \leq M$ and satisfying the condition $\inf_{(\mathbf{x}, \varrho) \in \Omega} \underline{h}(\varrho) + h(\mathbf{x}, \varrho) \geq h_{\star}$, then

$$\Phi : (\mathbf{x}, \varrho) \mapsto \frac{h(\mathbf{x}, \varrho)}{\underline{h}(\varrho) + h(\mathbf{x}, \varrho)} \in H^{s,k}(\Omega),$$

and

$$\|\Phi\|_{H^{s,k}} \leq C \|h\|_{H^{s,k}}.$$

If moreover $s > \frac{d}{2} + \frac{3}{2}$ and $2 \leq k \leq s$, then the above holds for any $h \in H^{s,k}(\Omega)$ with $\|h\|_{H^{s-1,k-1}} \leq M$.

PROOF. We can write $\Phi = \varphi \circ h$ with $\varphi(\cdot, \varrho) = f(\cdot, \underline{h}(\varrho))$ where $f \in C^{\infty}(\mathbb{R}^2)$ is set such that $f(y, z) = \frac{y}{y+z}$ on $\omega := \{(y, z) : |y| \leq \|h\|_{L^{\infty}(\Omega)}, |z| \leq |\underline{h}|_{L^{\infty}((\rho_0, \rho_1))}, y + z \geq h_{\star}\}$. We can construct f as above such that the control of $|\varphi|_{W^{s,\infty}(\mathbb{R}; W^{k,\infty}((\rho_0, \rho_1)))}$ depends only on $\|h\|_{L^{\infty}(\Omega)}$ (which is bounded appealing to Lemma A.1, if $h \in H^{s,k}$ with $s > \frac{d}{2} + \frac{1}{2}$, $1 \leq k \leq s$), $|\underline{h}|_{W^{k,\infty}((\rho_0, \rho_1))}$ and $h_{\star} > 0$. The result is now a direct application of Lemma A.5. \square

Commutator estimates. We now recall standard commutator estimates in $H^s(\mathbb{R}^d)$.

LEMMA A.7. Let $d \in \mathbb{N}^*$, $s_0 > d/2$ and $s \geq 0$.

- (1) For any $s_1, s_2 \in \mathbb{R}$ such that $s_1 \geq s$, $s_2 \geq s - 1$ and $s_1 + s_2 \geq s + s_0$, there exists $C > 0$ such that for any $f \in H^{s_1}(\mathbb{R}^d)$ and $g \in H^{s_2}(\mathbb{R}^d)$, $[\Lambda^s, f]g := \Lambda^s(fg) - f\Lambda^s g \in L^2(\mathbb{R}^d)$ and

$$|[\Lambda^s, f]g|_{L^2} \leq C |f|_{H^{s_1}} |g|_{H^{s_2}}.$$

- (2) There exists $C > 0$ such that for any $f \in L^{\infty}(\mathbb{R}^d)$ such that $\nabla f \in H^{s-1}(\mathbb{R}^d) \cap H^{s_0}(\mathbb{R}^d)$ and for any $g \in H^{s-1}(\mathbb{R}^d)$, one has $[\Lambda^s, f]g \in L^2(\mathbb{R}^d)$ and

$$|[\Lambda^s, f]g|_{L^2} \leq C |\nabla f|_{H^{s_0}} |g|_{H^{s-1}} + C \langle |\nabla f|_{H^{s-1}} |g|_{H^{s_0}} \rangle_{s > s_0+1}.$$

- (3) There exists $C > 0$ such that for any $f, g \in H^s(\mathbb{R}^d) \cap H^{s_0+1}(\mathbb{R}^d)$, the symmetric commutator $[\Lambda^s; f, g] := \Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f \in L^2(\mathbb{R}^d)$ and

$$|[\Lambda^s; f, g]|_{L^2} \leq C|f|_{H^{s_0+1}}|g|_{H^{s-1}} + C|f|_{H^{s-1}}|g|_{H^{s_0+1}}.$$

The validity of the above estimates persist when replacing the operator Λ^s with the operator ∂^α with $\alpha \in \mathbb{N}^d$ a multi-index such that $|\alpha| \leq s$.

We conclude with commutator estimates in the spaces $H^{s,k}(\Omega)$.

LEMMA A.8. Let $d \in \mathbb{N}^*$, $s_0 > d/2$. Let $s \geq s_0 + \frac{3}{2}$ and $k \in \mathbb{N}$ such that $2 \leq k \leq s$. Then there exists $C > 0$ such that for any $\alpha = (\alpha_x, \alpha_\varrho) \in \mathbb{N}^{d+1}$ with $|\alpha| \leq s$ and $\alpha_\varrho \leq k$, one has

$$\|[\partial^\alpha, F]G\|_{L^2(\Omega)} \leq C\|F\|_{H^{s,k}}\|G\|_{H^{s-1, \min(\{k, s-1\})}}.$$

PROOF. We set two multi-indices $\beta = (\beta_x, \beta_\varrho) \in \mathbb{N}^{d+1}$ and $\gamma = (\gamma_x, \gamma_\varrho) \in \mathbb{N}^{d+1}$ with $\beta + \gamma = \alpha$, and $|\gamma| \leq s - 1$. Assume first that $\beta_\varrho \leq k - 1$ and $|\beta| \leq s - 1$. Then

$$\begin{aligned} \|(\partial^\beta F)(\partial^\gamma G)\|_{L^2(\Omega)} &\lesssim \| \partial^\beta F(\cdot, \varrho) \Big|_{H_x^{s-|\beta|-\frac{1}{2}}} \partial^\gamma G(\cdot, \varrho) \Big|_{H_x^{s-|\gamma|-1}} \|_{L_\varrho^2} \\ &\lesssim \|F\|_{H^{s, \beta_\varrho+1}}\|G\|_{H^{s-1, \gamma_\varrho}} \leq \|F\|_{H^{s,k}}\|G\|_{H^{s-1, \min(\{k, s-1\})}}, \end{aligned}$$

where we used Lemma A.2(1) with $(s, s_1, s_2) = (0, s - |\beta| - \frac{1}{2}, s - |\gamma| - 1)$, and Lemma A.1. Otherwise $\gamma_\varrho = 0$ and $|\gamma| \leq s - |\beta| \leq s - 2$, and we have

$$\begin{aligned} \|(\partial^\beta F)(\partial^\gamma G)\|_{L^2(\Omega)} &\lesssim \| \partial^\beta F(\cdot, \varrho) \Big|_{H_x^{s-|\beta|}} \partial^\gamma G(\cdot, \varrho) \Big|_{H_x^{s-|\gamma|-\frac{3}{2}}} \|_{L_\varrho^2} \\ &\lesssim \|F\|_{H^{s, \beta_\varrho}}\|G\|_{H^{s-1, 1}} \leq \|F\|_{H^{s,k}}\|G\|_{H^{s-1, \min(\{k, s-1\})}}. \end{aligned}$$

The claim follows from decomposing $[\partial^\alpha, F]G$ as a sum of products as above. \square

LEMMA A.9. Let $d \in \mathbb{N}^*$, $s_0 > d/2$. Let $s \geq s_0 + \frac{5}{2}$ and $k \in \mathbb{N}$ such that $2 \leq k \leq s$. Then there exists $C > 0$ such that for any $\alpha = (\alpha_x, \alpha_\varrho) \in \mathbb{N}^{d+1}$ with $|\alpha| \leq s$ and $\alpha_\varrho \leq k$, one has

$$\|[\partial^\alpha; F, G]\|_{L^2(\Omega)} \leq C\|F\|_{H^{s-1, \min(\{k, s-1\})}}\|G\|_{H^{s-1, \min(\{k, s-1\})}}.$$

PROOF. We can decompose

$$[\partial^\alpha; F, G] = \sum_{\beta+\gamma=\alpha} (\partial^\beta F)(\partial^\gamma G)$$

with multi-indices $\beta = (\beta_x, \beta_\varrho) \in \mathbb{N}^{d+1}$ and $\gamma = (\gamma_x, \gamma_\varrho) \in \mathbb{N}^{d+1}$ such that $|\beta|+|\gamma| \leq s$ and $\beta_\varrho+\gamma_\varrho \leq k$, and $1 \leq |\beta|, |\gamma| \leq s - 1$. Assume furthermore that $\beta_\varrho \leq k - 1$ and $|\beta| \leq s - 2$. Then

$$\begin{aligned} \|(\partial^\beta F)(\partial^\gamma G)\|_{L^2(\Omega)} &\lesssim \| \partial^\beta F(\cdot, \varrho) \Big|_{H_x^{s-|\beta|-\frac{3}{2}}} \partial^\gamma G(\cdot, \varrho) \Big|_{H_x^{s-|\gamma|-1}} \|_{L_\varrho^2} \\ &\lesssim \|F\|_{H^{s-1, \beta_\varrho+1}}\|G\|_{H^{s-1, \gamma_\varrho}} \leq \|F\|_{H^{s-1, \min(\{k, s-1\})}}\|G\|_{H^{s-1, \min(\{k, s-1\})}}, \end{aligned}$$

where we used Lemma A.2(1) with $(s, s_1, s_2) = (0, s - |\beta| - \frac{3}{2}, s - |\gamma| - 1)$, and Lemma A.1. By symmetry, the result holds if $\gamma_\varrho \leq k - 1$ and $|\gamma| \leq s - 2$. Hence there remains to consider the situation where $(\beta_\varrho = k$ or $|\beta| = s - 1)$ and $(\gamma_\varrho = k$ or $|\gamma| = s - 1)$. Since $s > 2$ and $|\beta|+|\gamma| \leq s$, we cannot have $|\beta| = |\gamma| = s - 1$. In the same way, we cannot have $\beta_\varrho = \gamma_\varrho = k$ since $k > 0$. Furthermore, we cannot have $\beta_\varrho = k$ and $|\gamma| = s - 1$, since the former implies $|\beta| \geq \beta_\varrho = k \geq 2$ and the latter implies $|\beta| \leq 1$. Symmetrically, we cannot have $\gamma_\varrho = k$ and $|\beta| = s - 1$. This concludes the proof. \square

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