On the hydrostatic limit of stably stratified fluids

Roberta Bianchini, Vincent Duchêne

On the hydrostatic limit of stably stratified fluids

Roberta Bianchini and Vincent Duchêne

ABSTRACT. This article is concerned with the rigorous justification of the hydrostatic limit for continuously stratified incompressible fluids under the influence of gravity. The main difference with respect to earlier works is that we neglect viscosity effects but rather include diffusivity effects induced by an advection component proposed by Gent and McWilliams in the 90's to model effective eddy correlations for non-eddy-resolving systems. We also crucially make use of the assumption that the fluid is stably stratified, work in functional spaces of high (but finite) regularity and keep track of the various parameters at stake. A key ingredient of our analysis is the reformulation of our problem using isopycnal coordinates, on which careful energy estimates are performed.

1. Introduction

The following system describes the evolution of heterogeneous incompressible flows under the influence of gravity.

$$\begin{aligned}
\partial_t\rho + (\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}}\rho + (w+w_\star)\partial_z\rho = 0,\\
\rho\big(\partial_t\boldsymbol{u} + ((\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u} + (w+w_\star)\partial_z\boldsymbol{u}\big) + \nabla_{\boldsymbol{x}}P = 0,\\
\rho\big(\partial_t w + (\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}}w + (w+w_\star)\partial_z w\big) + \partial_z P + g\,\rho = 0,\\
\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} + \partial_z w = 0,\\
P|_{z=\zeta} - P_{\rm atm} = 0,\\
\partial_t\zeta + (\boldsymbol{u}+\boldsymbol{u}_\star)|_{z=\zeta}\cdot\nabla_{\boldsymbol{x}}\zeta - (w+w_\star)|_{z=\zeta} = 0,\\
w|_{z=-H} = 0.
\end{aligned} \tag{1.1}$$

Here, t and $(\boldsymbol{x},z)$ are the time, and horizontal-vertical space variables, and we denote by $\nabla_{\boldsymbol{x}},\Delta_{\boldsymbol{x}}$ the gradient and Laplacian with respect to $\boldsymbol{x}$. The vector field $(\boldsymbol{u},w)\in\mathbb{R}^d\times\mathbb{R}$ is the (horizontal and vertical) velocity, $\rho>0$ is the density, $P\in\mathbb{R}$ is the incompressible pressure, all being defined in the spatial domain

$$\Omega_t = \{(\boldsymbol{x},z)\ :\ \boldsymbol{x}\in\mathbb{R}^d,\ -H<z<\zeta(t,\boldsymbol{x})\},$$

where $\zeta(t,\boldsymbol{x})$ describes the location of a free surface, and H is the depth of the layer at rest. The gravity field is assumed constant and vertical, and $g>0$ is the gravity acceleration constant. Finally, $(\boldsymbol{u}_\star,w_\star)\in\mathbb{R}^d\times\mathbb{R}$ are correctors of the effective transport velocities which take into account eddy correlations in non-eddy-resolving (large-scale) models, proposed by Gent & McWilliams [12], and read as follows

$$\boldsymbol{u}_\star = \kappa\partial_z\left(\frac{\nabla_{\boldsymbol{x}}\rho}{\partial_z\rho}\right),\quad w_\star = -\kappa\nabla_{\boldsymbol{x}}\cdot\left(\frac{\nabla_{\boldsymbol{x}}\rho}{\partial_z\rho}\right),\qquad \kappa>0\,. \tag{1.2}$$

Disregarding the effective advection terms (i.e. setting $\kappa=0$), one recovers the Euler equations for heterogeneous incompressible fluids under the influence of vertical gravity forces, where the last two lines of (1.1) model the kinematic equation at the free surface and the impermeability condition of the rigid bottom respectively.

In (1.1), the pressure P can be recovered from its (atmospheric) value at the surface, $P_{\rm atm}$, by solving the elliptic boundary-value problem induced by the incompressibility constraint of divergence-free velocity

fields. Yet in the shallow-water regime, where the horizontal scale of the perturbation is large compared with the depth of the layer H, formal computations suggest that vertical accelerations can be neglected and that the pressure P approximately satisfies the hydrostatic balance law, that is

$$\partial_z P + g\,\rho = 0. \tag{1.3}$$

Using identity (1.3) in place of the evolution equation for the vertical velocity, (1.1) becomes the so-called hydrostatic equations:

$$\begin{aligned}
\partial_t \rho + \boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\rho + w\partial_z\rho &= \kappa\left[\nabla_{\boldsymbol{x}}\cdot\left(\frac{\nabla_{\boldsymbol{x}}\rho}{\partial_z\rho}\right)\partial_z\rho - \partial_z\left(\frac{\nabla_{\boldsymbol{x}}\rho}{\partial_z\rho}\right)\cdot\nabla_{\boldsymbol{x}}\rho\right],\\
\rho\big(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u} + w\partial_z\boldsymbol{u}\big) + \nabla_{\boldsymbol{x}}P &= \kappa\rho\left[\nabla_{\boldsymbol{x}}\cdot\left(\frac{\nabla_{\boldsymbol{x}}\rho}{\partial_z\rho}\right)\partial_z\boldsymbol{u} - \partial_z\left(\frac{\nabla_{\boldsymbol{x}}\rho}{\partial_z\rho}\right)\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\right],\\
P &= P_{\rm atm} + g\int_z^\zeta \rho(z',\cdot)\,\mathrm{d}z',\\
w &= -\int_{-H}^z \nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}(z',\cdot)\,\mathrm{d}z,\\
\partial_t\zeta + \boldsymbol{u}|_{z=\zeta}\cdot\nabla_{\boldsymbol{x}}\zeta - w|_{z=\zeta} &= -\kappa\left[\nabla_{\boldsymbol{x}}\cdot\left(\frac{\nabla_{\boldsymbol{x}}\rho}{\partial_z\rho}\right)\Big|_{z=\zeta} + \partial_z\left(\frac{\nabla_{\boldsymbol{x}}\rho}{\partial_z\rho}\right)\Big|_{z=\zeta}\cdot\nabla_{\boldsymbol{x}}\zeta\right].
\end{aligned} \tag{1.4}$$

Our aim in this work is to rigorously justify the hydrostatic equations (1.4) as an asymptotic model for the non-hydrostatic equations (1.1)-(1.2) in the shallow-water regime, for regular and stably stratified flows.

Modeling aspects. Let us now discuss the relevance and our motivation behind the introduction of the additional transport velocities $\boldsymbol{u}_\star$ and $w_\star$ defined in (1.2). While taking into account viscosity effects is standard in mathematical treatments of fluid mechanics, it should be mentioned that the aforementioned shallow-water regime where horizontal scales are larger than vertical scales produces anisotropic viscosity terms which are predominant in the vertical direction. However, it is worth pointing out that in theoretical and laboratory studies on density-stratified geophysical flows, viscosity effects do not model molecular viscosity but rather "turbulent" or "eddy" viscosities, and are widely reported to be anisotropic and only relevant in the horizontal (or more precisely isopycnal) direction; see *e.g.* [16, Section 17.6]. In this work we decide to neglect altogether viscosity effects and rather focus on diffusivity. The deterministic modeling of effective diffusivity caused by eddy correlation that we adopt originated in the 90's and is due to Gent & McWilliams [12], see also [13, 14]. It adds certain correctors, specifically suggesting (1.2), to the velocity field of the system of inhomogeneous incompressible fluids submitted to gravity as in (1.1). Since mesoscale eddies have an averaged dissipative effect on the large-scale flow at the macroscopic level, then it is natural to consider our unknowns $(\rho, \boldsymbol{u}, w)$ as the large-scale components of the density and the velocity field respectively, according to [12].

Let us acknowledge that ruling out the Coriolis force when modeling large-scale flows is certainly arguable. This work is motivated by studying theoretically the interplay between (stable) stratification, shallow water limits, and diffusive effects, and leaves aside other important mechanisms (such as boundaries and bathymetry. We could also consider several tracers —say salinity and temperature— and an equation of state for the density). Returning to the effect of rotation, adding these terms would not substantially modify our local-in-time results, but the rapid rotation limit would deserve a specific treatment.

Let us also mention that there exists a huge mathematical literature dedicated to the investigation of fluid-dynamics equations in the probabilistic setting, where the cumulative effect of mesoscale eddies on the large-scale flow is modeled by means of suitable (additive or multiplicative) noises. For that context, we refer to [8, 9], while our setting will be completely deterministic.

Previous mathematical results and motivation. At a technical level, the main reason for introducing either viscosity or diffusivity components in the equations is that in their absence, the initial-value problem for the hydrostatic equations are not known to be well-posed in finite-regularity functional spaces. In fact,

restricting to homogeneous flows (that is ρ being constant), ill-posedness was established by Renardy [29] at the linear level, and Han-Kwan and Nguyen [17] at the nonlinear level. Yet if we assume additionally that the initial data satisfies the Rayleigh condition of (strict) convexity/concavity in the vertical direction, well-posedness can be restored [3, 15, 24]. Now, assuming stably stratified flows, the celebrated Miles and Howard criterion [18, 25] states that the linearized equations about equilibria $(\underline{\rho}(z), \underline{\boldsymbol{u}}'(z))$ do not exhibit unstable modes provided (in dimension $d = 1$, see [11, Remark 1.3] when $d = 2$) as soon as the local Richardson number is greater than 1/4 everywhere, that is

$$\forall z \in [-H, 0], \qquad |\underline{\boldsymbol{u}}'(z)|^2 \leq \tfrac{g}{4} \frac{-\underline{\rho}'(z)}{\underline{\rho}(z)}.$$

The criterion clearly exhibits the stabilizing (resp. destabilizing) effect of stable stratification (resp. shear velocity). While one would like to obtain the well-posedness at the nonlinear level of the hydrostatic equations for initial data satisfying (strictly) the above inequality, let us repeat that such result is for now open.

This is in sharp constract with known results concerning the non-hydrostatic equations. The result closest to our framework is the one of Desjardins, Lannes and Saut [7], where the authors prove the well-posedness of the (inviscid and non-diffusive) non-hydrostatic equations in Sobolev spaces (using the rigid-lid assumption). The authors do exhibit a stabilizing effect of stable stratification, which is however insufficient to prove that the time of existence of solutions is uniform with respect to the shallowness parameter measuring the ratio of vertical to horizontal lengths, without corresponding smallness assumptions on the data. We discuss in more details their result in comparison with ours later on. At a technical level, the reason for this discrepancy between the non-hydrostatic and hydrostatic equations is that, as the vertical velocity w variable changes from prognostic (namely it belongs to the set of unknowns) to diagnostic (namely it is reconstructed from the unknowns), it loses one order of regularity; see the fourth equation in (1.4).

In order to compensate with this loss of derivatives, and if one does not wish to restrict the analysis to analytic data as in [20, 26], one natural solution consists in adding regularizing viscosity contributions. This is the framework of most theoretical studies concerning the hydrostatic equations and/or hydrostatic limit, starting with the work of Azérad and Guillén [1]. A landmark in the theory is the work of Cao and Titi [6] where the global well-posedness of the initial-value problem for the hydrostatic equations was proved in dimension $d+1 = 3$, a feature which should be contrasted with the current knowledge concerning the Navier-Stokes equations. Many works followed, considering partial viscosities and diffusivities and/or more physical boundary conditions. Instead of mentioning all of them, let us point out [4, 5] which extended the preceding result in the presence of only horizontal velocity and diffusivity, and [10, 22, 23] (in the homogeneous case) and [27, 28] (in the heterogeneous case) for recent results on the hydrostatic limit, including an extended list of references.

A specificity of our analysis compared with previous ones (with the exception of [7]) is that we shall crucially *use* (stable) density stratification, and that we disregard viscosity-induced regularization to consider only diffusivity. What is more, the specific form of the diffusivity contributions we consider, motivated by the work of Gent and McWilliams, had not been examined in theoretical works as far we know. Indeed, the specific form of the effective advection terms in (1.2) introduced therein stems from *isopycnal diffusivity* and, consistently, we will study the systems (1.1) and (1.4) using *isopycnal coordinates* (which inherently relies on stratification).

In the following paragraph, we rewrite the equations using isopycnal coordinates. Our main results are described in the subsequent one, and comment on them and their proof thereafter.

The model in isopycnal coordinates and non-dimensionalization. Let us consider smooth solutions to (1.1) defined on a time interval I_t. Assuming that the flow is *stably stratified*, i.e.

$$\inf(-\partial_z \rho) > 0,$$

the density $\rho : z \to \rho(\cdot, \cdot, z)$ is an invertible function of z. We denote its inverse $\eta : \varrho \to \eta(\cdot, \cdot, \varrho)$, so that

$$\rho(t, \boldsymbol{x}, \eta(t, \boldsymbol{x}, \varrho)) = \varrho, \quad \eta(t, \boldsymbol{x}, \rho(t, \boldsymbol{x}, z)) = z.$$

We also assume that $\rho(t,\boldsymbol{x},-H)=\rho_1,\ \rho(t,\boldsymbol{x},\zeta(t,\boldsymbol{x}))=\rho_0$ for $(t,\boldsymbol{x})\in I_t\times\mathbb{R}^d$, where $\rho_0<\rho_1$ are two fixed and positive constant reference densities. Then we have

$$\eta : I_t\times\mathbb{R}^d\times(\rho_0,\rho_1)\to\mathbb{R},\qquad\text{and}\qquad h:=-\partial_\varrho\eta>0,$$

the latter inequality accounting for the stable stratification assumption. We now introduce

$$\check{\boldsymbol{u}}(t,\boldsymbol{x},\varrho)=\boldsymbol{u}(t,\boldsymbol{x},\eta(t,\boldsymbol{x},\varrho)),\quad \check{w}(t,\boldsymbol{x},\varrho)=w(t,\boldsymbol{x},\eta(t,\boldsymbol{x},\varrho)),\quad \check{P}(t,\boldsymbol{x},\varrho)=P(t,\boldsymbol{x},\eta(t,\boldsymbol{x},\varrho)).$$

From the chain rule, we infer that system (1.1) in isopycnal coordinates reads

$$\begin{aligned}
\partial_t\eta+\check{\boldsymbol{u}}\cdot\nabla_{\boldsymbol{x}}\eta-\check{w}&=\kappa\Delta_{\boldsymbol{x}}\eta,\\
\varrho\Big(\partial_t\check{\boldsymbol{u}}+\big((\check{\boldsymbol{u}}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{h}\big)\cdot\nabla_{\boldsymbol{x}}\big)\check{\boldsymbol{u}}\Big)+\nabla_{\boldsymbol{x}}\check{P}+\frac{\nabla_{\boldsymbol{x}}\eta}{h}\partial_\varrho\check{P}&=0,\\
\varrho\Big(\partial_t\check{w}+\big(\check{\boldsymbol{u}}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{h}\big)\cdot\nabla_{\boldsymbol{x}}\check{w}\Big)-\frac{\partial_\varrho\check{P}}{h}+g\varrho&=0,\\
-h\nabla_{\boldsymbol{x}}\cdot\check{\boldsymbol{u}}-(\nabla_{\boldsymbol{x}}\eta)\cdot(\partial_\varrho\check{\boldsymbol{u}})+\partial_\varrho\check{w}&=0,\\
\check{P}\big|_{\varrho=\rho_0}=P_{\rm atm},\qquad \check{w}\big|_{\varrho=\rho_1}&=0.
\end{aligned}\tag{1.5}$$

Notice that differentiating with respect to ϱ the first equation and using the fourth equation (stemming from the incompressibility constraint), the mass conservation reads

$$\partial_t h+\nabla_{\boldsymbol{x}}\cdot(h\check{\boldsymbol{u}})=\kappa\Delta_{\boldsymbol{x}}h.\tag{1.6}$$

At this point, we are ready to introduce a dimensionless version of the previous system. We are interested in departures from steady solution to the incompressible Euler equations with variable density:

$$(h_{\rm eq},\boldsymbol{u}_{\rm eq},w_{\rm eq},P_{\rm eq})=(\underline{h}(\varrho),\underline{\boldsymbol{u}}(\varrho),0,\underline{P}(\varrho)),$$

which satisfies the equilibrium condition

$$\partial_\varrho\underline{P}(\varrho)=g\varrho\underline{h}(\varrho).$$

Therefore, we consider (non-necessarily small) fluctuations of that steady solution, so that our unknowns admit the following decomposition:

$$\begin{aligned}
h(t,\boldsymbol{x},\varrho)&=\underline{h}(\varrho)+h_{\rm pert}(t,\boldsymbol{x},\varrho),&\qquad \check{\boldsymbol{u}}(t,\boldsymbol{x},\varrho)&=\underline{\boldsymbol{u}}(\varrho)+\boldsymbol{u}_{\rm pert}(t,\boldsymbol{x},\varrho),\\
\check{w}(t,\boldsymbol{x},\varrho)&=0+w_{\rm pert}(t,\boldsymbol{x},\varrho),&\qquad \check{P}(t,\boldsymbol{x},\varrho)&=\underline{P}(\varrho)+P_{\rm pert}(t,\boldsymbol{x},\varrho).
\end{aligned}$$

Furthermore, we non-dimensionalize the equations through the following scaled variables: we set

$$\frac{h(t,\boldsymbol{x},\varrho)}{H}=\underline{\tilde{h}}(\varrho)+\tilde{h}(\tilde{t},\tilde{\boldsymbol{x}},\varrho)\qquad\text{and}\qquad\frac{\check{\boldsymbol{u}}(t,\boldsymbol{x},\varrho)}{\sqrt{gH}}=\underline{\tilde{\boldsymbol{u}}}(\varrho)+\tilde{\boldsymbol{u}}(\tilde{t},\tilde{\boldsymbol{x}},\varrho),$$

and [1]

$$\frac{\lambda}{H}\frac{\check{w}(t,\boldsymbol{x},\varrho)}{\sqrt{gH}}=\tilde{w}(\tilde{t},\tilde{\boldsymbol{x}},\varrho),\qquad\frac{\check{P}(t,\boldsymbol{x},\varrho)}{gH}=\frac{P_{\rm atm}}{gH}+\int_{\rho_0}^{\varrho}\varrho'\underline{\tilde{h}}(\varrho')\,d\varrho'+\rho_0\tilde{P}(\tilde{t},\tilde{\boldsymbol{x}},\varrho),$$

where we use the following scaled coordinates[2]

$$\tilde{\boldsymbol{x}}=\frac{\boldsymbol{x}}{\lambda}\qquad\text{and}\qquad\tilde{t}=\frac{\sqrt{gH}}{\lambda}t.$$

Introducing the dimensionless diffusion parameter, $\tilde{\kappa}$ and the shallowness parameter, μ, through

$$\tilde{\kappa}=\frac{\kappa}{\lambda\sqrt{gH}}\qquad\text{and}\qquad\mu=\frac{H^2}{\lambda^2},$$

[1] Notice the different scaling between the horizontal and vertical velocity fields. There, λ is a reference horizontal length.

[2] Notice we could scale also the ϱ-coordinate. Adjusting accordingly the definition of the pressure, this does not modify the resulting equations.

substituting the scaled coordinates/variables in system (1.5) and the subsequent equation *and dropping the tildes for the sake of readability* yields

$$\begin{aligned}
\partial_t h + \nabla_{\boldsymbol{x}} \cdot \big((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u})\big) &= \kappa \Delta_{\boldsymbol{x}} h,\\
\varrho\Big(\partial_t \boldsymbol{u} + \big((\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\Big) + \rho_0 \nabla_{\boldsymbol{x}} P + \frac{\nabla_{\boldsymbol{x}}\eta(\varrho\underline{h}+\rho_0\partial_\varrho P)}{\underline{h}+h} &= 0,\\
\mu\varrho\Big(\partial_t w + (\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}} w\Big) - \frac{\rho_0\partial_\varrho P}{\underline{h}+h} + \frac{\varrho h}{\underline{h}+h} &= 0,\\
-(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} - \nabla_{\boldsymbol{x}}\eta\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho \boldsymbol{u}) + \partial_\varrho w &= 0, \quad \text{(div.-free cond.)}\\
\eta(\cdot,\varrho) = \int_\varrho^{\rho_1} h(\cdot,\varrho')\,\mathrm{d}\varrho', \qquad P\big|_{\varrho=\rho_0} = 0, \qquad w\big|_{\varrho=\rho_1} &= 0, \quad \text{(bound. cond.)}
\end{aligned} \tag{1.7}$$

The hydrostatic system is obtained by setting $\mu = 0$ in (1.7). Specifically, plugging the hydrostatic balance

$$\frac{\rho_0\partial_\varrho P}{\underline{h}+h} = \frac{\varrho h}{\underline{h}+h} \quad \text{and} \quad P\big|_{\varrho=\rho_0} = 0$$

into the second equation of (1.7) yields

$$\begin{aligned}
\partial_t h + \nabla_{\boldsymbol{x}}\cdot((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u})) &= \kappa\Delta_{\boldsymbol{x}} h,\\
\varrho\Big(\partial_t\boldsymbol{u} + \big((\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\Big) + \nabla_{\boldsymbol{x}}\psi &= 0,
\end{aligned} \tag{1.8a}$$

with

$$\begin{aligned}
\psi(t,\boldsymbol{x},\varrho) &= \int_{\rho_0}^{\varrho} \varrho' h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho' + \varrho\int_\varrho^{\rho_1} h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho'\\
&= \rho_0\int_{\rho_0}^{\rho_1} h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho' + \int_{\rho_0}^{\varrho}\int_{\varrho'}^{\rho_1} h(t,\boldsymbol{x},\varrho'')\,\mathrm{d}\varrho''\,\mathrm{d}\varrho'.
\end{aligned} \tag{1.8b}$$

We shall provide a rigorous proof that regular solutions to (1.7) converge towards solutions to (1.8) as $\mu \searrow 0$, under the stable stratification assumption, $\underline{h}+h>0$.

Our main results. Our main results are given below. Some notations, and in particular the Sobolev spaces $H^{s,k}(\Omega)$, are defined in a subsequent paragraph. First, we prove the existence, uniqueness and control of solutions to the hydrostatic system (1.8) for sufficiently regular initial data. Let us point out that the existence time of our result exhibits the aforementioned stabilizing (resp. destabilizing) effect of stable stratification (resp. shear velocity).

THEOREM 1.1. *Let $s,k\in\mathbb{N}$ be such that $s>2+\frac{d}{2}$, $2\le k\le s$, and $\underline{M},M,h_\star,h^\star>0$ be fixed. Then there exists $C>0$ such that for any $\kappa\in(0,1]$, any $\underline{h},\underline{\boldsymbol{u}}\in W^{k,\infty}((\rho_0,\rho_1))$ satisfying*

$$\big|\underline{h}\big|_{W^{k,\infty}_\varrho} + \big|\underline{\boldsymbol{u}}'\big|_{W^{k-1,\infty}_\varrho} \le \underline{M}$$

and any initial data $(h_0,\boldsymbol{u}_0)\in H^{s,k}(\Omega)$, with $\eta_0(\cdot,\varrho)=\int_\varrho^{\rho_1} h_0(\cdot,\varrho')\,\mathrm{d}\varrho'$, satisfying

- *the following estimate*

$$M_0 := \big\|\eta_0\big\|_{H^{s,k}} + \big\|\boldsymbol{u}_0\big\|_{H^{s,k}} + \big|\eta_0\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}} + \kappa^{1/2}\big\|h_0\big\|_{H^{s,k}} \le M;$$

- *the stable stratification assumption*

$$\forall(\boldsymbol{x},\varrho)\in\Omega, \qquad h_\star \le \underline{h}(\varrho) + h_0(\boldsymbol{x},\varrho) \le h^\star,$$

there exists a unique $(h^{\mathrm{h}},\boldsymbol{u}^{\mathrm{h}})\in L^\infty(0,T;H^{s,k}(\Omega)^{1+d})\cap\mathcal{C}^0([0,T];H^{s',k}(\Omega)^{1+d})$ (for any $0\le s'<s$) strong solution to (1.8) with initial data $(h^{\mathrm{h}},\boldsymbol{u}^{\mathrm{h}})\big|_{t=0}=(h_0,\boldsymbol{u}_0)$, where

$$T^{-1} = C\left(1+\kappa^{-1}\big(\big|\underline{\boldsymbol{u}}'\big|^2_{L^2_\varrho} + M_0^2\big)\right).$$

Moreover, $h^{\rm h} \in L^2(0,T;H^{s+1,k}(\Omega))$ *and one has, for any* $t \in [0,T]$,

$$\forall (\boldsymbol{x},\varrho) \in \Omega, \qquad h_\star/2 \leq \underline{h}(\varrho) + h(t,\boldsymbol{x},\varrho) \leq 2\,h^\star,$$

and, denoting $\eta^{\rm h}(\cdot,\varrho) = \int_\varrho^{\rho_1} h^{\rm h}(\cdot,\varrho')\,{\rm d}\varrho'$,

$$\begin{aligned}\big\|\eta^{\rm h}(t,\cdot)\big\|_{H^{s,k}} + \big\|\boldsymbol{u}^{\rm h}(t,\cdot)\big\|_{H^{s,k}} + \big|\eta^{\rm h}\big|_{\varrho=\rho_0}(t,\cdot)\big|_{H^s_{\boldsymbol{x}}} + \kappa^{1/2}\big\|h^{\rm h}(t,\cdot)\big\|_{H^{s,k}} \\ + \kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta^{\rm h}\big\|_{L^2(0,T;H^{s,k})} + \kappa^{1/2}\big|\nabla_{\boldsymbol{x}}\eta^{\rm h}\big|_{\varrho=\rho_0}\big|_{L^2(0,T;H^s_{\boldsymbol{x}})} + \kappa\big\|\nabla_{\boldsymbol{x}}h^{\rm h}\big\|_{L^2(0,T;H^{s,k})} \leq CM_0.\end{aligned}$$

In our second main result, we prove that on the timescale defined by the previous proposition, solutions to the non-hydrostatic equations (1.7) exist when μ is sufficiently small, and converge towards the corresponding solution of the hydrostatic equations, with the expected $\mathcal{O}(\mu)$ convergence rate.

THEOREM 1.2. *There exists* $p \in \mathbb{N}$ *such that for any* $k = s \in \mathbb{N}$ *and* $\underline{M}, M, h_\star, h^\star > 0$, *there exists* $C > 0$ *such that the following holds. For any* $0 < M_0 \leq M$, $0 < \kappa \leq 1$, *and* $\mu > 0$ *such that*

$$\mu \leq \kappa/(CM_0^2),$$

for any for any $(\underline{h},\underline{\boldsymbol{u}}) \in W^{k+p,\infty}((\rho_0,\rho_1))^2$ *satisfying*

$$\big|\underline{h}\big|_{W^{k+p,\infty}_\varrho} + \big|\underline{\boldsymbol{u}}'\big|_{W^{k+p-1,\infty}_\varrho} \leq \underline{M}\,;$$

for any initial data $(h_0,\boldsymbol{u}_0,w_0) \in H^{s+p,k+p}(\Omega)^{2+d}$ *satisfying the boundary condition* $w_0|_{\varrho=\rho_1} = 0$ *and the incompressibility condition*

$$-(\underline{h}+h_0)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}_0 - (\nabla_{\boldsymbol{x}}\eta_0)\cdot(\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}_0) + \partial_\varrho w_0 = 0,$$

(denoting $\eta_0(\cdot,\varrho) = \int_\varrho^{\rho_1} h_0(\cdot,\varrho')\,{\rm d}\varrho'$*), the bounds*

$$\big\|\eta_0\big\|_{H^{s+p,k+p}} + \big\|\boldsymbol{u}_0\big\|_{H^{s+p,k+p}} + \big|\eta_0\big|_{\varrho=\rho_0}\big|_{H^{s+p}_{\boldsymbol{x}}} + \kappa^{1/2}\big\|h_0\big\|_{H^{s+p,k+p}} = M_0 \leq M$$

and the stable stratification assumption

$$\forall (\boldsymbol{x},\varrho) \in \Omega, \qquad h_\star \leq \underline{h}(\varrho) + h_0(\boldsymbol{x},\varrho) \leq h^\star,$$

the following holds. Denoting $(h^{\rm h},\boldsymbol{u}^{\rm h})$ *the corresponding solution to the hydrostatic equations* (1.8) *as provided by Theorem 1.1 with* $(s,k) = (s+p,k+p)$, $C^{\rm h} > 0$ *the corresponding constant and*

$$(T^{\rm h})^{-1} = C^{\rm h}\left(1 + \kappa^{-1}\big(\big|\underline{\boldsymbol{u}}'\big|^2_{L^2_\varrho} + M_0^2\big)\right),$$

there exists a unique strong solution $(h^{\rm nh},\boldsymbol{u}^{\rm nh},w^{\rm nh}) \in \mathcal{C}([0,T^{\rm h}];H^{s,k}(\Omega)^{1+d})$ *to the non-hydrostatic equations* (1.7) *with initial data* $(h^{\rm nh},\boldsymbol{u}^{\rm nh},w^{\rm nh})\big|_{t=0} = (h_0,\boldsymbol{u}_0,w_0)$. *Moreover, one has*

$$\big\|h^{\rm nh} - h^{\rm h}\big\|_{L^\infty(0,T^{\rm h};H^{s,k})} + \big\|\boldsymbol{u}^{\rm nh} - \boldsymbol{u}^{\rm h}\big\|_{L^\infty(0,T^{\rm h};H^{s,k})} \leq C\,\mu.$$

Strategy of the proofs. In order to obtain our results, we mainly use energy estimates, and product, commutator and composition estimates in the L^2-based Sobolev spaces $H^{s,k}(\Omega)$.

The natural energy functional associated with the hydrostatic equations, (1.8), involves η ,$\boldsymbol{u}$ as well as $\eta|_{\varrho=\rho_0}$ (the latter representing the free surface), and their derivatives. A key comment is that we do *not* control h with the same level of regularity, except when multiplied with the prefactor $\kappa^{1/2}$. We crucially use the diffusivity-induced regularization in order to control terms stemming from the commutator between advection and density integration. This explains why the time of existence vanishes as $\kappa \searrow 0$, yet with a prefactor involving the shear velocity, $\underline{\boldsymbol{u}}'$ (since in that case the advection with velocity $\underline{\boldsymbol{u}}$ commutes with density integration). It is interesting to notice that the index of regularity with respect to the space variable, s, and the one with respect to the density variable, k, are decoupled (yet only for in the hydrostatic framework). This is due to the fact that the isopycnal change of coordinate is semi-Lagrangian: the advection in isopycnal coordinates occurs only with in the horizontal space directions. It would be of utmost interest (but outside of the scope of the present work) to decrease the regularity assumption with respect to the density variable, so as to admit discontinuities, representing density interfaces.

Concerning the non-hydrostatic system, (1.7), the natural energy space involves additionally $\sqrt{\mu}w$ and its derivatives (hence the control vanishes as $\mu \searrow 0$). In order to obtain suitable energy estimates, we decompose the pressure as the sum of the hydrostatic contribution and the non-hydrostatic contribution, the latter being of lower order in terms of regularity and/or smallness with respect to $\mu \ll 1$. Then we use the structure of the hydrostatic equations, which we complement with an additional symmetric structure for the non-hydrostatic contributions. There, the difficulty consists in getting controls in the energy norms and uniform with respect to $\mu \ll 1$ at the same time. Our estimates concerning the non-hydrostatic pressure contribution stem from elliptic estimates on a boundary-value problem. This strategy is heavily inspired by the work of Desjardins, Lannes and Saut, and it is interesting to compare our results with the analogous "large-time" well-posedness result in [7, Theorem 2]. Firstly, due to the choice of isopycnal coordinates, our boundary-value problem is no longer an anisotropic Poisson equation but involves a fully nonlinear elliptic operator. Because this operator involves h which is not controlled in the energy space, we use again use the diffusivity-induced regularization at this stage. However, using isopycnal coordinates rather than Eulerian coordinates allow use to consider the free-surface framework (since isopycnal coordinates readily set the domain as a flat strip, thanks to our assumption that the density is constant at the surface and at the bottom) rather than the rigid-lid situation. We believe that our study can be extended to the rigid-lid framework with little adjustments. Incidentally, we do not employ the often-used Boussinesq approximation, since it is not useful in our context. Additionally, we do not rely on strong boundary conditions concerning initial density and velocities and their derivatives at the surface and the bottom, as used in [7] (and most other works, often put in a periodic framework) and rather use only the natural no-slip boundary condition at the bottom; the former are used to cancel the trace contributions in vertical integration by parts. We also consider the general situation where the velocity field is a deviation of a non-zero background current, $\underline{\boldsymbol{u}}$. The price to pay for this general framework manifests in terms of restrictions on the length of the time of existence of solutions, which is inversely proportional with respect to the size of the fluctuations in [7, Theorem 2].

Our strategy for the proof of the convergence is as follows. The above strategy allows to obtain the existence of solutions of the non-hydrostatic equations in a timescale uniform with respect to μ but not necessarily the same as the existence time of the corresponding solution to the hydrostatic equations, and (for technical reasons) restricted to sufficiently small data. In order to deal with these issues, we consider the hydrostatic solution as an approximate solution to the non-hydrostatic system (in the sense of consistency, that is solving approximately the equations), and control the difference between the two solutions using the aforementioned structure of the non-hydrostatic equations. This allows to bootstrap the control of the difference (and hence the control of the solution to the non-hydrostatic equations) as long as the hydrostatic solution is controlled (in a higher-regularity space) and provided the parameter μ is sufficiently small. Yet the rate of convergence obtained by this method is not optimal. As a second step, we consider the constructed solution to the non-hydrostatic equations as an approximate solution to the hydrostatic equations (again in the sense of consistency) and use the structure of the hydrostatic equations to infer the $\mathcal{O}(\mu)$ convergence rate. Both steps involve loss of derivatives, described by the parameter p in Theorem 1.2.

Plan of the paper. Section 2 is dedicated to the proof of Theorem 1.1 concerning the initial-value problem for the hydrostatic equations, (1.8).

In Section 3, we analyze the non-hydrostatic equations, (1.7). We first provide elliptic estimates for the boundary-value problem of the pressure reconstruction (Lemma 3.1 and Corollary 3.2), and use them to infer two results partial results concerning the initial-value problem : Proposition 3.3 (restricted to small time) and Proposition 3.8 (restricted to small data).

In Section 4, we show the convergence of solutions to the non-hydrostatic equations towards corresponding solutions to the hydrostatic equations as $\mu \searrow 0$, concluding the proof of Theorem 1.2.

Finally, in Appendix A we provide product, commutator and composition estimates in the Sobolev spaces $H^{s,k}(\Omega)$.

Notation and conventions. We highlight the following conventions used throughout the paper.

- For $k, s \in \mathbb{N}$ and $k \leq s$, and $\Omega = \mathbb{R}^d \times (\rho_0, \rho_1)$, we define the functional space

$$H^{s,k}(\Omega) = \left\{ f \ : \ \forall (\alpha, j) \in \mathbb{N}^{d+1}, \ |\alpha| + j \leq s, \ j \leq k, \ \partial_{\boldsymbol{x}}^{\alpha} \partial_{\varrho}^{j} f \in L^2(\Omega) \right\}, \tag{1.9}$$

endowed with the topology of the norm

$$\|f\|_{H^{s,k}}^2 := \sum_{j=0}^{k} \sum_{|\alpha|=0}^{s-j} \|\partial_{\boldsymbol{x}}^{\alpha} \partial_{\varrho}^{j} f\|_{L^2(\Omega)}^2. \tag{1.10}$$

When $s' \in \mathbb{R}$ (and $k \in \mathbb{N}$) we define $H^{s',k}(\Omega) = \left\{ f \ : \ \forall j \in \mathbb{N}, \ j \leq k, \ \Lambda^s \partial_{\varrho}^{j} f \in L^2(\Omega) \right\}$ endowed with

$$\|f\|_{H^{s',k}}^2 := \sum_{j=0}^{k} \|\Lambda^{s'-j} \partial_{\varrho}^{j} f\|_{L^2(\Omega)}^2.$$

where $\Lambda = (\mathrm{Id} - \Delta_{\boldsymbol{x}})^{1/2}$. Of course the two notations are consistent when $s' = s \in \mathbb{N}$, up to harmless factors in the definition of the norm.
- We use both the equivalent notations $H^s(\mathbb{R}^d) = H^s_{\boldsymbol{x}}$ (the usual L^2-based Sobolev space on $\mathbb{R}^d$) and $W^{k,\infty}(\mathbb{R}^d) = W^{k,\infty}_{\boldsymbol{x}}$ (the L^∞-based Sobolev space on $\mathbb{R}^d$), and similarly $L^2((\rho_0, \rho_1)) = L^2_{\varrho}$ and $W^{k,\infty}((\rho_0, \rho_1)) = W^{k,\infty}_{\varrho}$. For functions with variables in Ω we denote for instance

$$L^2_{\varrho} L^\infty_{\boldsymbol{x}} = L^2(\rho_0, \rho_1; L^\infty(\mathbb{R}^d)) = \{ f \ : \ \operatorname{ess\,sup}_{\boldsymbol{x} \in \mathbb{R}^d} |f(\cdot, \boldsymbol{x})| \in L^2((\rho_0, \rho_1)) \}.$$

Notice $L^2_{\varrho} L^2_{\boldsymbol{x}} = L^2_{\boldsymbol{x}} L^2_{\varrho} = L^2(\Omega)$ and $L^\infty_{\varrho} L^\infty_{\boldsymbol{x}} = L^\infty_{\boldsymbol{x}} L^\infty_{\varrho} = L^\infty(\Omega)$. We use similar notations for functions also depending on time. For instance, for $k \in \mathbb{N}$, and X a Banach space as above, $\mathcal{C}^k([0,T]; X)$ is the space of functions with values in X which are continuously differentiable up to order k, and $L^p(0,T;X)$ the p-integrable X-valued functions. All these spaces are endowed with their natural norms.
- For any operator $A : f \to Af$, we denote by $[A, f]g = A(fg) - f(Ag)$ the usual commutator, while $[A; f, g] = A(fg) - f(Ag) - g(Af)$ is the symmetric commutator,
- Sometimes we tacitly use the following formula $\eta(\cdot, \varrho) := \int_{\varrho}^{\rho_1} h(\cdot, \varrho') \, \mathrm{d}\varrho'$.
- $C(\lambda_1, \lambda_2, \dots)$ denotes a constant which depends continuously on its parameters.
- For any $a, b \in \mathbb{R}$, we use the notation $a \lesssim b$ (resp. $a \gtrsim b$) if there exists $C > 0$, independent of relevant parameters, such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \approx b$ if $a \lesssim b$ and $a \gtrsim b$.
- We put $a \vee b := \max(a, b)$. Finally,

$$\big\langle B_a \big\rangle_{a>b} = \begin{cases} 0 & \text{if } a \leq b \,, \\ B_a & \text{otherwise,} \end{cases} \quad \text{and} \quad \big\langle B_a \big\rangle_{a=b} = \begin{cases} 0 & \text{if } a \neq b \,, \\ B_a & \text{otherwise.} \end{cases}$$

2. The hydrostatic system

In this section we study the hydrostatic system in isopycnal coordinates. Specifically, we provide in this section a well-posedness result on the initial-value problem, namely Theorem 1.1. The result follows from careful *a priori* energy estimates, and the standard method of parabolic regularization. Therefore we will first study the system

$$\begin{aligned} \partial_t h + \nabla_{\boldsymbol{x}} \cdot ((\underline{h} + h)(\underline{\boldsymbol{u}} + \boldsymbol{u})) &= \kappa \Delta_{\boldsymbol{x}} h, \\ \partial_t \boldsymbol{u} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h} + h}) \cdot \nabla_{\boldsymbol{x}} \big) \boldsymbol{u} + \tfrac{1}{\varrho} \nabla_{\boldsymbol{x}} \psi &= \nu \Delta_{\boldsymbol{x}} \boldsymbol{u}, \end{aligned} \tag{2.1}$$

with

$$\nabla_{\boldsymbol{x}} \psi(t, \boldsymbol{x}, \varrho) := \rho_0 \int_{\rho_0}^{\rho_1} \nabla_{\boldsymbol{x}} h(t, \boldsymbol{x}, \varrho') \, \mathrm{d}\varrho' + \int_{\rho_0}^{\varrho} \int_{\rho'}^{\rho_1} \nabla_{\boldsymbol{x}} h(t, \boldsymbol{x}, \varrho'') \, \mathrm{d}\varrho'' \, \mathrm{d}\varrho',$$

and $\nu > 0$, and will rigorously establish the limit $\nu \to 0$.

2.1. Well-posedness of the regularized hydrostatic system. We start with proving the well-posedness of the initial value problem.

PROPOSITION 2.1. *Let $s > \frac{3}{2} + \frac{d}{2}$, $k \in \mathbb{N}$ with $1 \le k \le s$, and $\underline{M}$, M_0, $h_\star$, ν, $\kappa > 0$ and $C_0 > 1$. Then there exists $T = T(s, k, \underline{M}, M_0, h_\star, \nu, \kappa, C_0)$ such that for any $(\underline{h}, \underline{\boldsymbol{u}}) = (\underline{h}(\varrho), \underline{\boldsymbol{u}}(\varrho)) \in W^{k,\infty}((\rho_0, \rho_1))$ and for any $(h_0, \boldsymbol{u}_0) = (h_0(\boldsymbol{x}, \varrho), \boldsymbol{u}_0(\boldsymbol{x}, \varrho)) \in H^{s,k}(\Omega)$ such that*

$$\inf_{(\boldsymbol{x},\varrho)\in\Omega} (\underline{h}(\varrho) + h_0(\boldsymbol{x}, \varrho)) \ge h_\star, \quad \big|(\underline{h}, \underline{\boldsymbol{u}})\big|_{W^{k,\infty}_\varrho} \le \underline{M}, \quad \big\|(h_0, \boldsymbol{u}_0)\big\|_{H^{s,k}} \le M_0,$$

there exists a unique solution $(h, \boldsymbol{u}) \in \mathcal{C}^0([0, T]; H^{s,k}(\Omega))$ to system (2.1) with $(h, \boldsymbol{u})\big|_{t=0} = (h_0, \boldsymbol{u}_0)$. Moreover, $(h, \boldsymbol{u}) \in L^2(0, T; H^{s+1,k}(\Omega))$ and, for a universal constant $c_0 > 0$, the following estimates hold

$$\begin{cases} \big\|h\big\|_{L^\infty(0,T;H^{s,k})} + c_0\kappa^{1/2}\big\|\nabla_{\boldsymbol{x}} h\big\|_{L^2(0,T;H^{s,k})} < C_0\big\|h_0\big\|_{H^{s,k}}; \\ \big\|\boldsymbol{u}\big\|_{L^\infty(0,T;H^{s,k})} + c_0\nu^{1/2}\big\|\nabla_{\boldsymbol{x}} \boldsymbol{u}\big\|_{L^2(0,T;H^{s,k})} < C_0\big\|\boldsymbol{u}_0\big\|_{H^{s,k}}; \\ \inf_{(t,\boldsymbol{x},\varrho)\in(0,T)\times\Omega}(\underline{h}(\varrho) + h(t, \boldsymbol{x}, \varrho)) > h_\star/C_0. \end{cases} \tag{2.2}$$

PROOF. We will construct the solution as the fixed point of the Duhamel formula

$$\begin{aligned} h(t,\cdot) &= e^{\kappa t\Delta_{\boldsymbol{x}}} h_0 + \int_0^t e^{\kappa(t-\tau)\Delta_{\boldsymbol{x}}} f(h(\tau,\cdot), \boldsymbol{u}(\tau,\cdot))\,\mathrm{d}\tau, \\ \boldsymbol{u}(t,\cdot) &= e^{\nu t\Delta_{\boldsymbol{x}}} \boldsymbol{u}_0 + \int_0^t e^{\nu(t-\tau)\Delta_{\boldsymbol{x}}} (\boldsymbol{f}_1 + \boldsymbol{f}_2)(h(\tau,\cdot), \boldsymbol{u}(\tau,\cdot))\,\mathrm{d}\tau \end{aligned}$$

where $e^{\alpha t\Delta_{\boldsymbol{x}}}$ with $\alpha > 0$ is the heat semigroup defined by $\mathcal{F}[e^{\alpha t\Delta_{\boldsymbol{x}}} f](\boldsymbol{\xi}) = e^{-\alpha t|\boldsymbol{\xi}|^2}\mathcal{F}[f](\boldsymbol{\xi})$ where $\mathcal{F}$ is the Fourier transform with respect to the variable $\boldsymbol{x}$, and

$$\begin{aligned} f(h, \boldsymbol{u}) &= -\nabla_{\boldsymbol{x}} \cdot ((\underline{h} + h)(\underline{\boldsymbol{u}} + \boldsymbol{u})), \\ \boldsymbol{f}_1(h, \boldsymbol{u}) &= -\big((\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}) \cdot \nabla_{\boldsymbol{x}}\big)\boldsymbol{u} \\ \boldsymbol{f}_2(h, \boldsymbol{u}) &= -\frac{1}{\varrho}\left(\rho_0 \int_{\rho_0}^{\rho_1} \nabla_{\boldsymbol{x}} h(\cdot, \varrho')\,\mathrm{d}\varrho' + \int_{\rho_0}^{\varrho}\int_{\rho'}^{\rho_1} \nabla_{\boldsymbol{x}} h(\cdot, \varrho'')\,\mathrm{d}\varrho''\,\mathrm{d}\varrho'\right). \end{aligned}$$

Let us first recall the standard regularization properties of the heat flow. For any $\nu > 0$, $T > 0$ and for any $u_0 \in H^{s,k}(\Omega)$ and $g \in L^1(0, T; H^{s,k}(\Omega))$, there exists a unique $u \in \mathcal{C}^0([0, T]; H^{s,k}(\Omega)) \cap L^2(0, T; H^{s+1,k}(\Omega))$ solution to $\partial_t u - \nu\Delta_{\boldsymbol{x}} u = g$ with $u(0, \cdot) = u_0$ which reads by definition

$$u = e^{\nu t\Delta_{\boldsymbol{x}}} u_0 + \int_0^t e^{\nu(t-\tau)\Delta_{\boldsymbol{x}}} g(\tau, \cdot)\,\mathrm{d}\tau,$$

and we have

$$\big\|u\big\|_{L^\infty(0,T;H^{s,k})} + c_0\nu^{1/2}\big\|\nabla_{\boldsymbol{x}} u\big\|_{L^2(0,T;H^{s,k})} \le \big\|u_0\big\|_{H^{s,k}} + \big\|g\big\|_{L^1(0,T;H^{s,k})},$$

where $c_0 > 0$ is a universal constant. The existence and uniqueness of the solution as well as the above estimate easily follow from solving the equation (for almost every $\varrho \in (\rho_0, \rho_1)$) in Fourier space and using Plancherel's formula, then using that ∂_ϱ commutes with ∂_t and $\Delta_{\boldsymbol{x}}$, and invoking Minkowski's integral inequality (resp. Fubini's theorem) to exchange the order of integration in the variables (t, ϱ) (resp. $(\boldsymbol{x}, \varrho)$). We also remark that, by the positivity of the heat kernel and the continuous embedding $H^{s-1,1}(\Omega) \subset L^\infty(\Omega)$ for $s > \frac{3}{2} + \frac{d}{2}$ (see Lemma A.1),

$$\inf_\Omega u \ge \inf_\Omega u_0 - \big\|g\big\|_{L^1(0,T;H^{s-1,1})}.$$

Now we consider $(f, \boldsymbol{f}_1 + \boldsymbol{f}_2)$ as a bounded operator from $H^{s+1,k}(\Omega)^{1+d}$ to $H^{s,k}(\Omega)^{1+d}$. Indeed, there exists $C_{s,k} > 0$ such that for any $(h, \boldsymbol{u}) \in H^{s+1,k}(\Omega)^{1+d}$,

$$\begin{aligned}\big\|f(h,\boldsymbol{u})\big\|_{H^{s,k}} &\leq \big\|\nabla_{\boldsymbol{x}} \cdot (\underline{h}\boldsymbol{u} + h\underline{\boldsymbol{u}} + h\boldsymbol{u})\big\|_{H^{s,k}} \\ &\leq C_{s,k} \times \Big(\big|\underline{h}\big|_{W^{k,\infty}_\varrho} \big\|\boldsymbol{u}\big\|_{H^{s+1,k}} + \big|\underline{\boldsymbol{u}}\big|_{W^{k,\infty}_\varrho} \big\|h\big\|_{H^{s+1,k}} + \big\|h\big\|_{H^{s,k}} \big\|\boldsymbol{u}\big\|_{H^{s+1,k}} + \big\|h\big\|_{H^{s+1,k}} \big\|\boldsymbol{u}\big\|_{H^{s,k}} \Big),\end{aligned}$$

where we used straightforward product estimates for the first two terms, and Lemma A.3 for the last ones. Similarly, we have

$$\begin{aligned}\big\|\boldsymbol{f}_1(h,\boldsymbol{u})\big\|_{H^{s,k}} &\leq \big\|\big((\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}) \cdot \nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\big\|_{H^{s,k}} \\ &\leq C_{s,k} \Big(\big|\underline{\boldsymbol{u}}\big|_{W^{k,\infty}_\varrho} + \big\|\boldsymbol{u}\big\|_{H^{s,k}} \Big) \big\|\boldsymbol{u}\big\|_{H^{s+1,k}} \\ &\quad + \kappa C_{s,k} \big(\big|\underline{h}^{-1}\big|_{W^{k,\infty}_\varrho} + \big\|\tfrac{h}{\underline{h}(\underline{h}+h)}\big\|_{H^{s,k}} \big) \big\|(\nabla_{\boldsymbol{x}} h \cdot \nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{H^{s,k}}.\end{aligned}$$

Using the constraint $\inf_{(\rho_0,\rho_1)} \underline{h} \geq \inf_\Omega(\underline{h} + h_0) \geq h_\star > 0$ and Lemma A.6, we find that for any $h_\star > 0$ and $M_0, \underline{M} \geq 0$ there exists $C_{s,k}(h_\star, \underline{M}, M_0, C_0)$ such that for any $h \in H^{s,k}(\Omega)$ bounded by $\big\|h\big\|_{H^{s,k}} \leq C_0 M_0$ and satisfying $\inf_{(\boldsymbol{x},\varrho)\in\Omega}(\underline{h}(\varrho) + h(\boldsymbol{x},\varrho)) \geq h_\star / C_0$, one has

$$\big|\underline{h}^{-1}\big|_{W^{k,\infty}_\varrho} + \big\|\tfrac{h}{\underline{h}(\underline{h}+h)}\big\|_{H^{s,k}} \leq C_{s,k}(h_\star, \underline{M}, M_0, C_0).$$

Using the last estimates in Lemma A.3, since $s > \frac{3}{2} + \frac{d}{2}$, we have

$$\big\|(\nabla_{\boldsymbol{x}} h \cdot \nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{H^{s,k}} \leq \big\|h\big\|_{H^{s,k}} \big\|\boldsymbol{u}\big\|_{H^{s+1,k}} + \big\|h\big\|_{H^{s+1,k}} \big\|\boldsymbol{u}\big\|_{H^{s,k}}.$$

Finally, from the continuous embedding $L^\infty((\rho_0,\rho_1)) \subset L^2((\rho_0,\rho_1)) \subset L^1((\rho_0,\rho_1))$ we immediately infer

$$\big\|\boldsymbol{f}_2(h,\boldsymbol{u})\big\|_{H^{s,k}} \leq C_{s,k} \big\|h\big\|_{H^{s+1,k}}.$$

Altogether, we find that for any $h_\star, C_0 > 0$ and $\underline{M}, M_0 \geq 0$ there exists $C_{s,k}(h_\star, \underline{M}, M_0, C_0)$ such that for any $(h, \boldsymbol{u}) \in H^{s+1,k}(\Omega)^{1+d}$ satisfying $\big\|(h,\boldsymbol{u})\big\|_{H^{s,k}} \leq C_0 M_0$ and $\inf_{(\boldsymbol{x},\varrho)\in\Omega}(\underline{h}(\varrho) + h(\boldsymbol{x},\varrho)) \geq h_\star / C_0$, we have

$$\big\|\big(f(h,\boldsymbol{u}), \boldsymbol{f}_1(h,\boldsymbol{u}), \boldsymbol{f}_2(h,\boldsymbol{u})\big)\big\|_{H^{s,k}} \leq C_{s,k}(h_\star, \underline{M}, M_0, C_0)\,(1+\kappa)\,\big\|(h,\boldsymbol{u})\big\|_{H^{s+1,k}}.$$

By similar considerations, we find that for any $h_\star, C_0 > 0$ and $\underline{M}, M_0 \geq 0$ there exists $C_{s,k}(h_\star, \underline{M}, M_0, C_0)$ such that for any $(h_1, \boldsymbol{u}_1, h_2, \boldsymbol{u}_2) \in H^{s+1,k}(\Omega)^{2(1+d)}$ satisfying the bound $\big\|(h_i,\boldsymbol{u}_i)\big\|_{H^{s,k}} \leq C_0 M_0$ as well as $\inf_{(\boldsymbol{x},\varrho)\in\Omega}(\underline{h}(\varrho) + h_i(\boldsymbol{x},\varrho)) \geq h_\star / C_0$ (with $i \in \{1,2\}$), one has

$$\begin{aligned}&\big\|\big(f(h_2,\boldsymbol{u}_2) - f(h_1,\boldsymbol{u}_1), \boldsymbol{f}_1(h_2,\boldsymbol{u}_2) - \boldsymbol{f}_1(h_1,\boldsymbol{u}_1), \boldsymbol{f}_2(h_2,\boldsymbol{u}_2) - \boldsymbol{f}_2(h_1,\boldsymbol{u}_1)\big)\big\|_{H^{s,k}} \\ &\qquad \leq C_{s,k}(h_\star, \underline{M}, M_0, C_0)(1+\kappa)\Big(\big\|(h_2 - h_1, \boldsymbol{u}_2 - \boldsymbol{u}_1)\big\|_{H^{s+1,k}} \\ &\qquad\qquad + \big\|(h_1, h_2, \boldsymbol{u}_1, \boldsymbol{u}_2)\big\|_{H^{s+1,k}} \big\|(h_2 - h_1, \boldsymbol{u}_2 - \boldsymbol{u}_1)\big\|_{H^{s,k}} \Big).\end{aligned}$$

From the above estimates, we easily infer that for $T > 0$ sufficiently small (uniquely depending on $s, k, M_0, \underline{M}, h_\star, \nu, \kappa, C_0$),

$$\mathcal{T} : \begin{pmatrix} h \\ \boldsymbol{u} \end{pmatrix} \mapsto \begin{pmatrix} e^{\kappa t \Delta_{\boldsymbol{x}}} h_0 + \int_0^t e^{\kappa(t-\tau)\Delta_{\boldsymbol{x}}} f(h(\tau,\cdot), \boldsymbol{u}(\tau,\cdot))\,\mathrm{d}\tau \\ e^{\nu t \Delta_{\boldsymbol{x}}} \boldsymbol{u}_0 + \int_0^t e^{\nu(t-\tau)\Delta_{\boldsymbol{x}}} (\boldsymbol{f}_1 + \boldsymbol{f}_2)(h(\tau,\cdot), \boldsymbol{u}(\tau,\cdot))\,\mathrm{d}\tau \end{pmatrix}$$

is a contraction mapping on

$$X = \Big\{ (h,\boldsymbol{u}) \in \mathcal{C}^0([0,T]; H^{s,k}(\Omega)) \cap L^2(0,T; H^{s+1,k}(\Omega)) \ : \ (2.2) \text{ holds} \Big\}.$$

The Banach fixed point theorem provides the existence and uniqueness of a fixed point (and hence solution to (2.1)) in X, and uniqueness in $\mathcal{C}^0([0,T]; H^{s,k}(\Omega))$ is easily checked (for instance by the energy method). □

REMARK 2.2. It should be emphasized that the time of existence provided by Proposition 2.1 is not uniform with respect to the parameters $\kappa, \nu > 0$. More precisely, the proof provides a lower bound as

$$T \gtrsim \min(\{1, \kappa, \nu\}), \qquad \textit{i.e.} \quad T^{-1} \lesssim 1 + \kappa^{-1} + \nu^{-1}.$$

2.2. Quasilinearization. In the result below, we apply spatial derivatives to system (2.1) and rewrite it in such a way that the linearized equations satisfied by the highest-order terms exhibit a skew-symmetric structure, which will allow us to obtain improved energy estimates in the subsequent section.

LEMMA 2.3. *Let $s, k \in \mathbb{N}$ such that $s > 2 + \frac{d}{2}$ and $2 \le k \le s$, and $\underline{M}, M, h_\star > 0$. Then there exists $C = C(s, k, \underline{M}, M, h_\star) > 0$ such that for any $\kappa \in [0,1]$, $\nu \ge 0$, for any $(\underline{h}, \underline{\boldsymbol{u}}) \in W^{k,\infty}((\rho_0, \rho_1))$ such that*

$$\big|\underline{h}\big|_{W^{k,\infty}_\varrho} + \big|\underline{\boldsymbol{u}}'\big|_{W^{k-1,\infty}_\varrho} \le \underline{M}\,;$$

and any $(h, \boldsymbol{u}) \in \mathcal{C}^0([0,T]; H^{s,k}(\Omega))$ solution to (2.1) with some $T > 0$ and satisfying

$$\big\|h\big\|_{H^{s-1,k-1}} + \big\|\eta\big\|_{H^{s,k}} + \big\|\boldsymbol{u}\big\|_{H^{s,k}} + \big|\eta\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}} + \kappa^{1/2}\big\|h\big\|_{H^{s,k}} \le M \quad \textit{for } t \in [0,T],$$

(where $\eta(t, \boldsymbol{x}, \varrho) := \int_\varrho^{\rho_1} h(t, \boldsymbol{x}, \varrho')\,\mathrm{d}\varrho'$) and

$$\inf_{(t,\boldsymbol{x},\varrho)\in(0,T)\times\Omega} \underline{h}(\varrho) + h(t, \boldsymbol{x}, \varrho) \ge h_\star,$$

the following hold. Denote, for any multi-index $\alpha \in \mathbb{N}^d$, $\eta^{(\alpha)} = \partial^\alpha_{\boldsymbol{x}}\eta$, $\boldsymbol{u}^{(\alpha)} = \partial^\alpha_{\boldsymbol{x}}\boldsymbol{u}$.

- *For any $\alpha \in \mathbb{N}^d$ with $0 \le |\alpha| \le s$, we have that*

$$\begin{aligned}
&\partial_t\eta^{(\alpha)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}}\eta^{(\alpha)} + \int_\varrho^{\rho_1} (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}}\eta^{(\alpha)}\, d\varrho' \\
&\qquad\qquad + \int_\varrho^{\rho_1} (\underline{h} + h)\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(\alpha)}\,\mathrm{d}\varrho' = \kappa\Delta_{\boldsymbol{x}}\eta^{(\alpha)} + R_{\alpha,0}, \\
&\partial_t\boldsymbol{u}^{(\alpha)} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}) \cdot \nabla_{\boldsymbol{x}}\big)\boldsymbol{u}^{(\alpha)} \\
&\qquad\qquad + \frac{\rho_0}{\varrho}\nabla_{\boldsymbol{x}}\eta^{(\alpha)}\big|_{\varrho=\rho_0} + \frac{1}{\varrho}\int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}}\eta^{(\alpha)}\,\mathrm{d}\varrho' = \nu\Delta_{\boldsymbol{x}}\boldsymbol{u}^{(\alpha)} + \boldsymbol{R}_{\alpha,0},
\end{aligned} \tag{2.3a}$$

  where for every $t \in [0,T]$, $(R_{\alpha,0}(t,\cdot), \boldsymbol{R}_{\alpha,0}(t,\cdot)) \in \mathcal{C}^0([\rho_0,\rho_1]; L^2(\mathbb{R}^d)) \times L^2(\Omega)^d$ and

$$\big\|R_{\alpha,0}\big\|_{L^2(\Omega)} + \big\|\boldsymbol{R}_{\alpha,0}\big\|_{L^2(\Omega)} + \big|R_{\alpha,0}\big|_{\varrho=\rho_0}\big|_{L^2_{\boldsymbol{x}}} \le C\,M\,\big(1 + \kappa\big\|\nabla_{\boldsymbol{x}} h\big\|_{H^{s,k}}\big). \tag{2.3b}$$

- *For any $j \in \mathbb{N}$, $1 \le j \le k$ and any $\alpha \in \mathbb{N}^d$, $0 \le |\alpha| \le s - j$, it holds*

$$\begin{aligned}
\partial_t\partial^j_\varrho\eta^{(\alpha)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}}\partial^j_\varrho\eta^{(\alpha)} &= \kappa\Delta_{\boldsymbol{x}}\partial^j_\varrho\eta^{(\alpha)} + R_{\alpha,j}, \\
\partial_t\partial^j_\varrho\boldsymbol{u}^{(\alpha)} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}) \cdot \nabla_{\boldsymbol{x}}\big)\partial^j_\varrho\boldsymbol{u}^{(\alpha)} &= \nu\Delta_{\boldsymbol{x}}\partial^j_\varrho\boldsymbol{u}^{(\alpha)} + \boldsymbol{R}_{\alpha,j},
\end{aligned} \tag{2.4a}$$

  where for every $t \in [0,T]$, $(R_{\alpha,j}(t,\cdot), \boldsymbol{R}_{\alpha,j}(t,\cdot)) \in L^2(\Omega) \times L^2(\Omega)^d$ and

$$\big\|R_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|\boldsymbol{R}_{\alpha,j}\big\|_{L^2(\Omega)} \le C\,M\,\big(1 + \kappa\big\|\nabla_{\boldsymbol{x}} h\big\|_{H^{s,k}}\big). \tag{2.4b}$$

- *For any $j \in \mathbb{N}$, $0 \le j \le k$ and any multi-index $\alpha \in \mathbb{N}^d$, $0 \le |\alpha| \le s - j$, it holds*

$$\partial_t\partial^j_\varrho h^{(\alpha)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}}\partial^j_\varrho h^{(\alpha)} = \kappa\Delta_{\boldsymbol{x}}\partial^j_\varrho h^{(\alpha)} + r_{\alpha,j} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{r}_{\alpha,j}, \tag{2.5a}$$

  where for every $t \in [0,T]$, $(r_{\alpha,j}(t,\cdot), \boldsymbol{r}_{\alpha,j}(t,\cdot)) \in L^2(\Omega)^{1+d}$ and

$$\kappa^{1/2}\big\|r_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|\boldsymbol{r}_{\alpha,j}\big\|_{L^2(\Omega)} \le C\,M. \tag{2.5b}$$

PROOF. In this proof, we denote $s_0 = s - 2 > \frac{d}{2}$.

Estimate of $R_{\alpha,0}$. First we notice the identity by integration by parts in ϱ,

$$(\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha)} + \int_{\varrho}^{\rho_1} (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha)} \, d\varrho' = \int_{\varrho}^{\rho_1} (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} h^{(\alpha)} \, \mathrm{d}\varrho'.$$

Hence, recalling the notation $[P; u, v] = P(uv) - u(Pv) - v(Pu)$ and integrating by parts in ϱ, we get

$$\begin{aligned}
R_{\alpha,0} &:= -\int_{\varrho}^{\rho_1} [\partial_{\boldsymbol{x}}^\alpha, \boldsymbol{u}] \cdot \nabla_{\boldsymbol{x}} h + [\partial_{\boldsymbol{x}}^\alpha, h] \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \, \mathrm{d}\varrho' \\
&= -\int_{\varrho}^{\rho_1} [\partial_{\boldsymbol{x}}^\alpha; \boldsymbol{u}, \nabla_{\boldsymbol{x}} h] + \boldsymbol{u}^{(\alpha)} \cdot \nabla_{\boldsymbol{x}} h + [\partial_{\boldsymbol{x}}^\alpha; h, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}] + h^{(\alpha)} (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}) \, \mathrm{d}\varrho' \\
&= -[\partial_{\boldsymbol{x}}^\alpha; \boldsymbol{u}, \nabla_{\boldsymbol{x}} \eta] - \eta^{(\alpha)} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \\
&\quad - \int_{\varrho}^{\rho_1} [\partial_{\boldsymbol{x}}^\alpha; \partial_\varrho \boldsymbol{u}, \nabla_{\boldsymbol{x}} \eta] + \boldsymbol{u}^{(\alpha)} \cdot \nabla_{\boldsymbol{x}} h + [\partial_{\boldsymbol{x}}^\alpha; h, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}] + \eta^{(\alpha)} \nabla_{\boldsymbol{x}} \cdot \partial_\varrho \boldsymbol{u} \, \mathrm{d}\varrho'.
\end{aligned}$$

By the standard Sobolev embedding $H^{s_0}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and Lemma A.1, one gets

$$\big\| \eta^{(\alpha)} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \big\|_{L^2(\Omega)} \le \big\| \eta^{(\alpha)} \big\|_{L^2(\Omega)} \big\| \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \big\|_{L^\infty(\Omega)} \lesssim \big\| \eta \big\|_{H^{s,0}} \big\| \boldsymbol{u} \big\|_{H^{s_0 + \frac{3}{2}, 1}}.$$

and

$$\big| \big(\eta^{(\alpha)} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \big) \big|_{\varrho = \rho_0} \big|_{L^2_{\boldsymbol{x}}} \lesssim \big| \eta \big|_{\varrho = \rho_0} \big|_{H^s_{\boldsymbol{x}}} \big\| \boldsymbol{u} \big\|_{H^{s_0 + \frac{3}{2}, 1}}.$$

By Lemma A.7(3), and Lemma A.1, we have

$$\begin{aligned}
\big\| [\partial_{\boldsymbol{x}}^\alpha; \boldsymbol{u}, \nabla_{\boldsymbol{x}} \eta] \big\|_{L^2(\Omega)} &\lesssim \big\| \boldsymbol{u} \big\|_{L^\infty_\varrho H^{s-1}_{\boldsymbol{x}}} \big\| \nabla_{\boldsymbol{x}} \eta \big\|_{L^2_\varrho H^{s_0+1}_{\boldsymbol{x}}} + \big\| \boldsymbol{u} \big\|_{L^\infty_\varrho H^{s_0+1}_{\boldsymbol{x}}} \big\| \nabla_{\boldsymbol{x}} \eta \big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}} \\
&\lesssim \big\| \boldsymbol{u} \big\|_{H^{s-\frac{1}{2}, 1}} \big\| \eta \big\|_{H^{s_0+2,0}} + \big\| \boldsymbol{u} \big\|_{H^{s_0+\frac{3}{2},1}} \big\| \eta \big\|_{H^{s,0}},
\end{aligned}$$

$$\begin{aligned}
\big| [\partial_{\boldsymbol{x}}^\alpha; \boldsymbol{u}|_{\varrho=\rho_0}, \nabla_{\boldsymbol{x}} \eta|_{\varrho=\rho_0}] \big|_{L^2_{\boldsymbol{x}}} &\lesssim \big| \boldsymbol{u} \big|_{\varrho=\rho_0} \big|_{H^{s-1}_{\boldsymbol{x}}} \big| \nabla_{\boldsymbol{x}} \eta \big|_{\varrho=\rho_0} \big|_{H^{s_0+1}_{\boldsymbol{x}}} + \big| \boldsymbol{u} \big|_{\varrho=\rho_0} \big|_{H^{s_0+1}_{\boldsymbol{x}}} \big| \nabla_{\boldsymbol{x}} \eta \big|_{\varrho=\rho_0} \big|_{H^{s-1}_{\boldsymbol{x}}} \\
&\lesssim \big\| \boldsymbol{u} \big\|_{H^{s-\frac{1}{2},1}} \big| \eta \big|_{\varrho=\rho_0} \big|_{H^{s_0+2}_{\boldsymbol{x}}} + \big\| \boldsymbol{u} \big\|_{H^{s_0+\frac{3}{2},1}} \big| \eta \big|_{\varrho=\rho_0} \big|_{H^s_{\boldsymbol{x}}},
\end{aligned}$$

and using additionally the Cauchy-Schwarz inequality,

$$\begin{aligned}
\big\| [\partial_{\boldsymbol{x}}^\alpha; \partial_\varrho \boldsymbol{u}, \nabla_{\boldsymbol{x}} \eta] \big\|_{L^1_\varrho L^2_{\boldsymbol{x}}} &\lesssim \big\| \partial_\varrho \boldsymbol{u} \big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}} \big\| \nabla_{\boldsymbol{x}} \eta \big\|_{L^2_\varrho H^{s_0+1}_{\boldsymbol{x}}} + \big\| \partial_\varrho \boldsymbol{u} \big\|_{L^2_\varrho H^{s_0+1}_{\boldsymbol{x}}} \big\| \nabla_{\boldsymbol{x}} \eta \big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}} \\
&\lesssim \big\| \boldsymbol{u} \big\|_{H^{s,1}} \big\| \eta \big\|_{H^{s_0+2,0}} + \big\| \boldsymbol{u} \big\|_{H^{s_0+2,1}} \big\| \eta \big\|_{H^{s,0}},
\end{aligned}$$

$$\begin{aligned}
\big\| [\partial_{\boldsymbol{x}}^\alpha; h, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}] \big\|_{L^1_\varrho L^2_{\boldsymbol{x}}} &\lesssim \big\| h \big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}} \big\| \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \big\|_{L^2_\varrho H^{s_0+1}_{\boldsymbol{x}}} + \big\| h \big\|_{L^2_\varrho H^{s_0+1}_{\boldsymbol{x}}} \big\| \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}} \\
&\lesssim \big\| h \big\|_{H^{s-1,0}} \big\| \boldsymbol{u} \big\|_{H^{s_0+2,0}} + \big\| h \big\|_{H^{s_0+1,0}} \big\| \boldsymbol{u} \big\|_{H^{s,0}},
\end{aligned}$$

and

$$\begin{aligned}
\big\| \boldsymbol{u}^{(\alpha)} \cdot \nabla_{\boldsymbol{x}} h \big\|_{L^1_\varrho L^2_{\boldsymbol{x}}} &\lesssim \big\| \boldsymbol{u} \big\|_{H^{s,0}} \big\| h \big\|_{H^{s_0+1,0}}, \\
\big\| \eta^{(\alpha)} (\nabla_{\boldsymbol{x}} \cdot \partial_\varrho \boldsymbol{u}) \big\|_{L^1_\varrho L^2_{\boldsymbol{x}}} &\lesssim \big\| \eta \big\|_{H^{s,0}} \big\| \boldsymbol{u} \big\|_{H^{s_0+2,1}}.
\end{aligned}$$

Altogether, using the continuous embedding $L^\infty((\rho_0, \rho_1)) \subset L^2((\rho_0, \rho_1)) \subset L^1((\rho_0, \rho_1))$, the Minkowski and triangle inequalities and $s \ge s_0 + 2$, we get

$$\big| R_{\alpha,0} \big|_{\varrho=\rho_0} \big|_{L^2_{\boldsymbol{x}}} + \big\| R_{\alpha,0} \big\|_{L^2(\Omega)} \lesssim \big(\big\| \eta \big\|_{H^{s,0}} + \big\| h \big\|_{H^{s-1,0}} + \big| \eta \big|_{\varrho=\rho_0} \big|_{H^s_{\boldsymbol{x}}} \big) \big\| \boldsymbol{u} \big\|_{H^{s,1}}. \tag{2.6}$$

Estimate of $R_{\alpha,j}$ for $1 \le j \le k$. We have

$$\begin{aligned} R_{\alpha,j} &:= -[\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j-1}\nabla_{\boldsymbol{x}}\cdot, \underline{\boldsymbol{u}}+\boldsymbol{u}]h - \partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j-1}\nabla_{\boldsymbol{x}}\cdot(\underline{h}(\underline{\boldsymbol{u}}+\boldsymbol{u})) \\ &= -\sum_{i=1}^{d}[\partial_{\boldsymbol{x}}^{\alpha}\partial_{x_i}\partial_{\varrho}^{j-1}, \boldsymbol{u}_i]h - [\partial_{\varrho}^{j-1}, \underline{\boldsymbol{u}}]\cdot\partial_{\boldsymbol{x}}^{\alpha}\nabla_{\boldsymbol{x}}h - \partial_{\varrho}^{j-1}\cdot(\underline{h}\partial_{\boldsymbol{x}}^{\alpha}\nabla_{\boldsymbol{x}}\boldsymbol{u}), \end{aligned}$$

where $\boldsymbol{u}_i$ is the i^{th} component of $\boldsymbol{u}$. By Lemma A.8 and since $(|\alpha|+1)+(j-1) \le s$ and $j-1 \le k-1$, and $s \ge s_0+\frac{3}{2}$, we find for $2 \le k-1 \le s$

$$\big\|[\partial_{\boldsymbol{x}}^{\alpha}\partial_{x_i}\partial_{\varrho}^{j-1}, \boldsymbol{u}_i]h\big\|_{L^2(\Omega)} \lesssim \big\|h\big\|_{H^{s-1,k-1}}\big\|\boldsymbol{u}\big\|_{H^{s,k}}.$$

There remains to consider $1 \le j \le k \le 2$. If $j=1$ we have by Lemma A.7(2) and since $|\alpha| \le s-1$ and $s \ge s_0+\frac{3}{2}$

$$\big\|[\partial_{\boldsymbol{x}}^{\alpha}\partial_{x_i}, \boldsymbol{u}_i]h\big\|_{L^2(\Omega)} \lesssim \big\|h\big\|_{L^{\infty}_{\varrho}H^{s_0}_{\boldsymbol{x}}}\big\|\boldsymbol{u}\big\|_{L^2_{\varrho}H^{s}_{\boldsymbol{x}}} + \big\|h\big\|_{L^2_{\varrho}H^{s-1}_{\boldsymbol{x}}}\big\|\boldsymbol{u}\big\|_{L^{\infty}_{\varrho}H^{s_0+1}_{\boldsymbol{x}}} \lesssim \big\|h\big\|_{H^{s-1,1}}\big\|\boldsymbol{u}\big\|_{H^{s,1}}.$$

If $j=k=2$, and since $|\alpha| \le s-2$ and $s \ge s_0+\frac{3}{2}$,

$$\begin{aligned} \big\|[\partial_{\boldsymbol{x}}^{\alpha}\partial_{x_i}\partial_{\varrho}^{j-1}, \boldsymbol{u}_i]h\big\|_{L^2(\Omega)} &\le \big\|[\partial_{\boldsymbol{x}}^{\alpha}\partial_{x_i}, \boldsymbol{u}_i]\partial_{\varrho}h\big\|_{L^2(\Omega)} + \big\|\partial_{\boldsymbol{x}}^{\alpha}\partial_{x_i}(h\partial_{\varrho}\boldsymbol{u}_i)\big\|_{L^2(\Omega)} \\ &\lesssim \big\|\partial_{\varrho}h\big\|_{L^2_{\varrho}H^{s_0}_{\boldsymbol{x}}}\big\|\boldsymbol{u}\big\|_{L^{\infty}_{\varrho}H^{s-1}_{\boldsymbol{x}}} + \big\|\partial_{\varrho}h\big\|_{L^2_{\varrho}H^{s-2}_{\boldsymbol{x}}}\big\|\boldsymbol{u}\big\|_{L^{\infty}_{\varrho}H^{s_0+1}_{\boldsymbol{x}}} \\ &\quad + \big\|h\big\|_{L^{\infty}_{\varrho}H^{s_0}_{\boldsymbol{x}}}\big\|\partial_{\varrho}\boldsymbol{u}\big\|_{L^2_{\varrho}H^{s-1}_{\boldsymbol{x}}} + \big\|h\big\|_{L^2_{\varrho}H^{s-1}_{\boldsymbol{x}}}\big\|\partial_{\varrho}\boldsymbol{u}\big\|_{L^{\infty}_{\varrho}H^{s_0}_{\boldsymbol{x}}} \\ &\lesssim \big\|h\big\|_{H^{s-1,1}}\big\|\boldsymbol{u}\big\|_{H^{s,2}}. \end{aligned}$$

Finally, we have immediately

$$\begin{aligned} \big\|[\partial_{\varrho}^{j-1}, \underline{\boldsymbol{u}}]\cdot\partial_{\boldsymbol{x}}^{\alpha}\nabla_{\boldsymbol{x}}h\big\|_{L^2(\Omega)} &\lesssim |\underline{\boldsymbol{u}}'|_{W^{j-2,\infty}_{\varrho}}\big\|h\big\|_{H^{s-1,j-2}}, \\ \big\|\partial_{\varrho}^{j-1}(\underline{h}\partial_{\boldsymbol{x}}^{\alpha}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u})\big\|_{L^2(\Omega)} &\lesssim |\underline{h}|_{W^{j-1,\infty}_{\varrho}}\big\|\boldsymbol{u}\big\|_{H^{s,j}}. \end{aligned}$$

Altogether, we find that for any $1 \le j \le k$

$$\big\|R_{\alpha,j}\big\|_{L^2(\Omega)} \lesssim (|\underline{h}|_{W^{k-1,\infty}_{\varrho}} + |\underline{\boldsymbol{u}}'|_{W^{k-2,\infty}_{\varrho}} + \big\|h\big\|_{H^{s-1,k-1}})\big(\big\|\boldsymbol{u}\big\|_{H^{s,k}} + \big\|h\big\|_{H^{s-1,k-2}}\big). \tag{2.7}$$

Estimate of $\boldsymbol{r}_{\alpha,j}$ and $r_{\alpha,j}$ for $0 \le j \le k$. We have (2.5a) with

$$\begin{aligned} \boldsymbol{r}_{\alpha,j} &:= -[\partial_{\varrho}^{j}, \underline{\boldsymbol{u}}]\partial_{\boldsymbol{x}}^{\alpha}h - \partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}(\underline{h}\boldsymbol{u}) - (\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}\boldsymbol{u})h, \\ r_{\alpha,j} &:= -[\partial^{\alpha}\partial_{\varrho}^{j}\nabla_{\boldsymbol{x}}\cdot; \boldsymbol{u}, h] + (\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}h. \end{aligned}$$

We have immediately (since $|\alpha|+j \le s$, $j \le k$, and using Lemma A.1)

$$\begin{aligned} \big\|[\partial_{\varrho}^{j}, \underline{\boldsymbol{u}}]\partial_{\boldsymbol{x}}^{\alpha}h\big\|_{L^2(\Omega)} &\lesssim |\underline{\boldsymbol{u}}'|_{W^{k-1,\infty}_{\varrho}}\big\|h\big\|_{H^{s-1,k-1}}, \\ \big\|\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}(\underline{h}\boldsymbol{u})\big\|_{L^2(\Omega)} &\lesssim |\underline{h}|_{W^{k,\infty}_{\varrho}}\big\|\boldsymbol{u}\big\|_{H^{s,k}}, \\ \big\|(\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}\boldsymbol{u})h\big\|_{L^2(\Omega)} &\lesssim \big\|\boldsymbol{u}\big\|_{H^{s,k}}\big\|h\big\|_{H^{s_0+\frac{1}{2},1}}. \end{aligned}$$

By Lemma A.9 and since $|\alpha|+j+1 \le s+1$, $j \le k \le s$, $s+1 \ge s_0+\frac{5}{2}$, we find for $2 \le k \le s$

$$\big\|[\partial^{\alpha}\partial_{\varrho}^{j}\nabla_{\boldsymbol{x}}\cdot; \boldsymbol{u}, h]\big\|_{L^2(\Omega)} \lesssim \big\|h\big\|_{H^{s,k}}\big\|\boldsymbol{u}\big\|_{H^{s,k}},$$

and we have by Lemma A.1

$$\big\|(\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}h\big\|_{L^2(\Omega)} \lesssim \big\|\boldsymbol{u}\big\|_{H^{s,k}}\big\|h\big\|_{H^{s_0+\frac{3}{2},1}}.$$

Altogether, we find that for any $0 \le j \le k$

$$\big\|\boldsymbol{r}_{\alpha,j}\big\|_{L^2(\Omega)} \lesssim (|\underline{h}|_{W^{k,\infty}_{\varrho}} + |\underline{\boldsymbol{u}}'|_{W^{k-1,\infty}_{\varrho}} + \big\|h\big\|_{H^{s-1,k-1}})\big(\big\|\boldsymbol{u}\big\|_{H^{s,k}} + \big\|h\big\|_{H^{s-1,k-1}}\big), \tag{2.8}$$

$$\big\|r_{\alpha,j}\big\|_{L^2(\Omega)} \lesssim \big\|\boldsymbol{u}\big\|_{H^{s,k}}\big\|h\big\|_{H^{s,k}}. \tag{2.9}$$

Estimate of $\boldsymbol{R}_{\alpha,0}$. The precise expression of the second remainder in (2.3a) is the following:

$$\boldsymbol{R}_{\alpha,0} := -\big([\partial_{\boldsymbol{x}}^{\alpha},\boldsymbol{u}]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u} + \kappa[\partial_{\boldsymbol{x}}^{\alpha},\tfrac{1}{\underline{h}+h}](\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u} + \tfrac{\kappa}{\underline{h}+h}\big([\partial_{\boldsymbol{x}}^{\alpha},\nabla_{\boldsymbol{x}}h]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}.$$

By Lemma A.7(2) and Lemma A.1 we have

$$\big\|\big([\partial_{\boldsymbol{x}}^{\alpha},\boldsymbol{u}]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\big\|_{L^2(\Omega)} \lesssim \|\boldsymbol{u}\|_{L^\infty_\varrho H^{s_0+1}_{\boldsymbol{x}}}\|\boldsymbol{u}\|_{L^2_\varrho H^s_{\boldsymbol{x}}} \lesssim \|\boldsymbol{u}\|_{H^{s_0+\frac32,1}}\|\boldsymbol{u}\|_{H^{s,0}}.$$

Next, appealing again to A.7(2), we have

$$\begin{aligned}\kappa\big\|[\partial_{\boldsymbol{x}}^{\alpha},\tfrac{1}{\underline{h}+h}](\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{L^2(\Omega)} &\lesssim \kappa\big\|\nabla_{\boldsymbol{x}}\tfrac{1}{\underline{h}+h}\big\|_{L^\infty_\varrho H^{s_0}_{\boldsymbol{x}}}\big\|(\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}}\\ &\quad+\kappa\big\|\nabla_{\boldsymbol{x}}\tfrac{1}{\underline{h}+h}\big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}}\big\|(\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{L^\infty_\varrho H^{s_0}_{\boldsymbol{x}}}.\end{aligned}$$

Now, by Lemma A.2(2) and Lemma A.4, one has for any $t\geq 0$ that

$$\begin{aligned}\big|\nabla_{\boldsymbol{x}}\tfrac{1}{\underline{h}+h}\big|_{H^t_{\boldsymbol{x}}} = \big|\tfrac{\nabla_{\boldsymbol{x}}h}{(\underline{h}+h)^2}\big|_{H^t_{\boldsymbol{x}}} &\leq \big|\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}^2}\big|_{H^t_{\boldsymbol{x}}} + \big|(\tfrac{1}{\underline{h}^2}-\tfrac{1}{(\underline{h}+h)^2})\nabla_{\boldsymbol{x}}h\big|_{H^t_{\boldsymbol{x}}}\\ &\lesssim (h_\star)^{-2}\big|\nabla_{\boldsymbol{x}}h\big|_{H^t_{\boldsymbol{x}}} + \big|\tfrac{1}{\underline{h}^2}-\tfrac{1}{(\underline{h}+h)^2}\big|_{H^{s_0}_{\boldsymbol{x}}}\big|\nabla_{\boldsymbol{x}}h\big|_{H^t_{\boldsymbol{x}}} + \Big\langle\big|\tfrac{1}{\underline{h}^2}-\tfrac{1}{(\underline{h}+h)^2}\big|_{H^t_{\boldsymbol{x}}}\big|\nabla_{\boldsymbol{x}}h\big|_{H^{s_0}_{\boldsymbol{x}}}\Big\rangle_{t>s_0}\\ &\leq C(h_\star,\big|h\big|_{H^{s_0}_{\boldsymbol{x}}})\big|\nabla_{\boldsymbol{x}}h\big|_{H^t_{\boldsymbol{x}}},\end{aligned}\tag{2.10}$$

where in the last step we used that, by Lemma A.4,

$$\big|\tfrac{1}{\underline{h}^2}-\tfrac{1}{(\underline{h}+h)^2}\big|_{H^{s_0}_{\boldsymbol{x}}} \leq C(h_\star,\big|h\big|_{H^{s_0}_{\boldsymbol{x}}})$$

and, provided that $t>s_0$,

$$\big|\tfrac{1}{\underline{h}^2}-\tfrac{1}{(\underline{h}+h)^2}\big|_{H^t_{\boldsymbol{x}}} \leq \big|\tfrac{1}{\underline{h}^2}-\tfrac{1}{(\underline{h}+h)^2}\big|_{H^{s_0}_{\boldsymbol{x}}} + \big|\nabla_{\boldsymbol{x}}\tfrac{1}{(\underline{h}+h)^2}\big|_{H^{t-1}_{\boldsymbol{x}}},$$

and a finite induction on t, until $\big|\nabla_{\boldsymbol{x}}\frac{1}{(\underline{h}+h)^2}\big|_{L^2_{\boldsymbol{x}}} = \big|\frac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h}\big|_{L^2_{\boldsymbol{x}}} \leq h_\star^{-2}\big|\nabla_{\boldsymbol{x}}h\big|_{L^2_{\boldsymbol{x}}}$. Then, by Lemma A.2(2) and Lemma A.1, we have

$$\begin{aligned}\big\|(\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}} &\lesssim \big\|\nabla_{\boldsymbol{x}}h\big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}}\|\boldsymbol{u}\|_{L^\infty_\varrho H^{s_0+1}_{\boldsymbol{x}}} + \big\|\nabla_{\boldsymbol{x}}h\big\|_{L^\infty_\varrho H^{s_0}_{\boldsymbol{x}}}\|\boldsymbol{u}\|_{L^2_\varrho H^s_{\boldsymbol{x}}}\\ &\lesssim \|h\|_{H^{s,0}}\|\boldsymbol{u}\|_{H^{s_0+\frac32,1}} + \|h\|_{H^{s_0+\frac32,1}}\|\boldsymbol{u}\|_{H^{s,0}}\end{aligned}$$

and

$$\big\|(\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{L^\infty_\varrho H^{s_0}_{\boldsymbol{x}}} \lesssim \big\|\nabla_{\boldsymbol{x}}h\big\|_{L^\infty_\varrho H^{s_0}_{\boldsymbol{x}}}\|\boldsymbol{u}\|_{L^\infty_\varrho H^{s_0+1}_{\boldsymbol{x}}} \lesssim \|h\|_{H^{s_0+\frac32,1}}\|\boldsymbol{u}\|_{H^{s_0+\frac32,1}}.$$

Finally, we have by Lemma A.7(2) and Lemma A.1

$$\begin{aligned}\big\|\big([\partial_{\boldsymbol{x}}^{\alpha},\nabla_{\boldsymbol{x}}h]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\big\|_{L^2(\Omega)} &\lesssim \big\|\nabla_{\boldsymbol{x}}h\big\|_{L^\infty_\varrho H^{s_0+1}_{\boldsymbol{x}}}\big\|\nabla_{\boldsymbol{x}}\boldsymbol{u}\big\|_{L^2_\varrho H^{s-1}_{\boldsymbol{x}}} + \big\|\nabla_{\boldsymbol{x}}h\big\|_{L^2_\varrho H^s_{\boldsymbol{x}}}\big\|\nabla_{\boldsymbol{x}}\boldsymbol{u}\big\|_{L^\infty_\varrho H^{s_0}_{\boldsymbol{x}}}\\ &\lesssim \big\|\nabla_{\boldsymbol{x}}h\big\|_{H^{s_0+\frac32,1}}\|\boldsymbol{u}\|_{H^{s,0}} + \big\|\nabla_{\boldsymbol{x}}h\big\|_{H^{s,0}}\|\boldsymbol{u}\|_{H^{s_0+\frac32,1}}.\end{aligned}$$

Collecting the estimates above and using that $s\geq s_0+\frac32$, we obtain

$$\big\|\boldsymbol{R}_{\alpha,0}\big\|_{L^2(\Omega)} \lesssim \|\boldsymbol{u}\|_{H^{s,0}}\|\boldsymbol{u}\|_{H^{s,1}} + \kappa C(h_\star,\|h\|_{H^{s_0+\frac12,1}})\big(\|h\|^2_{H^{s,1}} + \big\|\nabla_{\boldsymbol{x}}h\big\|_{H^{s,1}}\big)\|\boldsymbol{u}\|_{H^{s,1}}.\tag{2.11}$$

Estimate of $\boldsymbol{R}_{\alpha,j}$ *for* $1\leq j\leq k$. The explicit expression of the second remainder in (2.4a) is the following

$$\begin{aligned}\boldsymbol{R}_{\alpha,j} := -\big([\partial_{\boldsymbol{x}}^{\alpha}\partial_\varrho^j,\underline{\boldsymbol{u}}+\boldsymbol{u}]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u} + \kappa[\partial_{\boldsymbol{x}}^{\alpha}\partial_\varrho^j,\tfrac{1}{\underline{h}+h}]\big((\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big) + \tfrac{\kappa}{\underline{h}+h}\big([\partial_{\boldsymbol{x}}^{\alpha}\partial_\varrho^j,\nabla_{\boldsymbol{x}}h]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\\ + \partial_\varrho^j\partial_{\boldsymbol{x}}^{\alpha}\left(\frac{\rho_0}{\varrho}\int_{\rho_0}^{\rho_1}\nabla_{\boldsymbol{x}}h\,\mathrm{d}\varrho' + \frac{1}{\varrho}\int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}\eta\,\mathrm{d}\varrho'\right).\end{aligned}$$

By Lemma A.8 we have for $s\geq s_0+\frac32$ and since $0\leq|\alpha|\leq s-j$ and $j\leq k$ with $k\geq 2$, that

$$\big\|\big([\partial_{\boldsymbol{x}}^{\alpha}\partial_\varrho^j,\boldsymbol{u}]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\big\|_{L^2(\Omega)} \lesssim \|\boldsymbol{u}\|_{H^{s,k}}\big\|\nabla_{\boldsymbol{x}}\boldsymbol{u}\big\|_{H^{s-1,k}}.$$

Then,

$$\big\|\big([\partial_{\boldsymbol{x}}^{\alpha}\partial_\varrho^j,\underline{\boldsymbol{u}}]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\big\|_{L^2(\Omega)} = \big\|[\partial_\varrho^j,\underline{\boldsymbol{u}}]\cdot\nabla_{\boldsymbol{x}}\partial_{\boldsymbol{x}}^{\alpha}\boldsymbol{u}\big\|_{L^2(\Omega)} \lesssim |\underline{\boldsymbol{u}}'|_{W^{j-1,\infty}_\varrho}\|\boldsymbol{u}\|_{H^{s,j-1}}.$$

Next, using Lemma A.3,

$$\begin{aligned}\big\|[\partial_\varrho^j, \tfrac{1}{\underline{h}}]\partial_{\boldsymbol{x}}^\alpha\big((\nabla_{\boldsymbol{x}} h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big)\big\|_{L^2(\Omega)} &\lesssim C(h_\star)\big|\underline{h}'\big|_{W_\varrho^{j-1,\infty}}\big\|(\nabla_{\boldsymbol{x}} h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{H^{s-1,j-1}}\\ &\lesssim C(h_\star)\big|\underline{h}'\big|_{W_\varrho^{j-1,\infty}}\big\|h\big\|_{H^{s,j}}\big\|\boldsymbol{u}\big\|_{H^{s,j}},\end{aligned}$$

and by Lemma A.8, since $s\geq s_0+\frac{3}{2}$ and $2\leq j\leq s$, $|\alpha|+j\leq s$, Lemma A.6 and Lemma A.3,

$$\begin{aligned}\big\|[\partial_{\boldsymbol{x}}^\alpha\partial_\varrho^j, \tfrac{1}{\underline{h}+h}-\tfrac{1}{\underline{h}}]\big((\nabla_{\boldsymbol{x}} h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big)\big\|_{L^2(\Omega)} &\lesssim \big\|\tfrac{1}{\underline{h}+h}-\tfrac{1}{\underline{h}}\big\|_{H^{s,k}}\big\|(\nabla_{\boldsymbol{x}} h\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u}\big\|_{H^{s-1,\min(\{k,s-1\})}}\\ &\lesssim C(h_\star, \big|\underline{h}'\big|_{W_\varrho^{k-1,\infty}}, \big\|h\big\|_{H^{s-1,k-1}})\big\|h\big\|_{H^{s,k}}^2\big\|\boldsymbol{u}\big\|_{H^{s,k}}.\end{aligned}$$

By Lemma A.8 we have for $s\geq s_0+\frac{3}{2}$ and since $|\alpha|+j\leq s$ and $2\leq j\leq s$

$$\big\|\big([\partial_{\boldsymbol{x}}^\alpha\partial_\varrho^j, \nabla_{\boldsymbol{x}} h]\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\big\|_{L^2(\Omega)}\lesssim \big\|\nabla_{\boldsymbol{x}} h\big\|_{H^{s,k}}\big\|\boldsymbol{u}\big\|_{H^{s,k}}.$$

We have immediately since $|\alpha|\leq s-j\leq s-1$,

$$\big\|\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha\Big(\frac{\rho_0}{\varrho}\int_{\rho_0}^{\rho_1}\nabla_{\boldsymbol{x}} h\,\mathrm{d}\varrho'\Big)\big\|_{L^2(\Omega)}\lesssim \big|\partial_{\boldsymbol{x}}^\alpha\nabla_{\boldsymbol{x}}\eta\big|_{\varrho=\rho_0}\big|_{L^2_{\boldsymbol{x}}}\lesssim \big|\eta\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}}$$

and since $(|\alpha|+1)+(j-1)\leq s$,

$$\big\|\partial_\varrho^j\Big(\frac{1}{\varrho}\int_{\rho_0}^{\varrho}\partial_{\boldsymbol{x}}^\alpha\nabla_{\boldsymbol{x}}\eta\,\mathrm{d}\varrho'\Big)\big\|_{L^2(\Omega)}\lesssim\sum_{i=0}^{j-1}\big\|\partial_\varrho^i\partial_{\boldsymbol{x}}^\alpha\nabla_{\boldsymbol{x}}\eta\big\|_{L^2(\Omega)}\lesssim\big\|\eta\big\|_{H^{s,j-1}}.$$

Collecting the estimates above we obtain for $1\leq j\leq k$

$$\begin{aligned}\big\|\boldsymbol{R}_{\alpha,j}\big\|_{L^2(\Omega)}\lesssim\big|\eta\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}}+\big\|\eta\big\|_{H^{s,k-1}}+\big(\big|\underline{\boldsymbol{u}}'\big|_{W_\varrho^{k-1,\infty}}+\big\|\boldsymbol{u}\big\|_{H^{s,k}}\big)\big\|\boldsymbol{u}\big\|_{H^{s,k}}\\ +\kappa C(h_\star, \big|\underline{h}'\big|_{W_\varrho^{k-1,\infty}}, \big\|h\big\|_{H^{s-1,k-1}})\big(\big\|h\big\|_{H^{s,k}}^2+\big\|\nabla_{\boldsymbol{x}} h\big\|_{H^{s,k}}\big)\big\|\boldsymbol{u}\big\|_{H^{s,k}}.\end{aligned}\tag{2.12}$$

We infer the bound (2.3b) from (2.6) and (2.11), the bound (2.4b) from (2.7) and (2.12), and the bound (2.5b) from (2.8) and (2.9), and the proof is complete. □

2.3. A priori energy estimates. In this section we provide *a priori* energy estimates associated with the equations featured in Lemma 2.3. We start with the transport-diffusion equations in (2.4a) and (2.5a), which we rewrite as

$$\partial_t\dot h+\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\dot h=\kappa\Delta_{\boldsymbol{x}}\dot h+r+\nabla_{\boldsymbol{x}}\cdot\boldsymbol{r}.\tag{2.13}$$

LEMMA 2.4. *There exists a universal constant $C_0>0$ such that for any $\kappa>0$ and $T>0$, for any $\boldsymbol{u}\in L^\infty(0,T;L^\infty(\Omega))$ with $\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}\in L^1(0,T;L^\infty(\Omega))$, for any $(r,\boldsymbol{r})\in L^2(0,T;L^2(\Omega))$ and for any $\dot h\in\mathcal{C}^0([0,T);L^2(\Omega))$ with $\nabla_{\boldsymbol{x}}\dot h\in L^2(0,T;L^2(\Omega))$, such that (2.13) holds in $L^2(0,T;H^{1,0}(\Omega)')$, we have*

$$\begin{aligned}&\big\|\dot h\big\|_{L^\infty(0,T;L^2(\Omega))}+\kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\dot h\big\|_{L^2(0,T;L^2(\Omega))}\\ &\qquad\leq C_0\big(\big\|\dot h\big|_{t=0}\big\|_{L^2(\Omega)}+\big\|r\big\|_{L^1(0,T;L^2(\Omega))}+\kappa^{-1/2}\big\|\boldsymbol{r}\big\|_{L^2(0,T;L^2(\Omega))}\big)\\ &\qquad\qquad\times\exp\Big(C_0\int_0^T\big\|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}(t,\cdot)\big\|_{L^\infty(\Omega)}\,\mathrm{d}t\Big).\end{aligned}\tag{2.14}$$

PROOF. Testing the equation against $\dot h$ and integrating by parts (with respect to the variable $\boldsymbol{x}$) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\dot h\big\|_{L^2(\Omega)}^2+\kappa\big\|\nabla_{\boldsymbol{x}}\dot h\big\|_{L^2(\Omega)}^2=\frac{1}{2}\iint_\Omega(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u})\dot h^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho+\iint_\Omega r\dot h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho-\iint_\Omega\boldsymbol{r}\cdot\nabla_{\boldsymbol{x}}\dot h\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho.$$

The estimate follows from the Cauchy-Schwarz inequality and Gronwall's Lemma. □

Next, we consider system (2.3a), which we rewrite as

$$\begin{aligned}
&\partial_t\dot\eta + (\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta + \int_{\varrho}^{\rho_1}(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\,\mathrm{d}\varrho' + \int_{\varrho}^{\rho_1}(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}\varrho' = \kappa\Delta_{\boldsymbol{x}}\dot\eta + R,\\
&\varrho\Big(\partial_t\dot{\boldsymbol{u}} + \big((\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}}\big)\dot{\boldsymbol{u}}\Big) + \rho_0\nabla_{\boldsymbol{x}}\dot\eta\big|_{\varrho=\rho_0} + \int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}\dot\eta\,\mathrm{d}\varrho' = \varrho\nu\Delta_{\boldsymbol{x}}\dot{\boldsymbol{u}} + \boldsymbol{R}.
\end{aligned}\tag{2.15}$$

For the sake of readability, we introduce the following notations

$$X^0 := \mathcal{C}^0([\rho_0,\rho_1];L^2(\mathbb{R}^d))\times L^2(\Omega)^d;\quad X^1 := \mathcal{C}^0([\rho_0,\rho_1];H^1(\mathbb{R}^d))\times H^{1,0}(\Omega)^d.\tag{2.16}$$

LEMMA 2.5. *Let $h_\star, h^\star, \underline{M} > 0$ be fixed. There exists $C(h_\star,h^\star,M)>0$ such that for any $\kappa>0$ and $\nu\in[0,1]$, for any $(\underline{h},\underline{\boldsymbol{u}})\in W^{1,\infty}((\rho_0,\rho_1))$, for any $T>0$ and $(h,\boldsymbol{u})\in L^\infty(0,T;W^{1,\infty}(\Omega))$ with $\Delta_{\boldsymbol{x}}h\in L^1(0,T;L^\infty(\Omega))$ satisfying (2.1) and, for almost any $t\in[0,T]$, the upper bound*

$$\big\|h(t,\cdot)\big\|_{L^\infty(\Omega)} + \big\|\nabla_{\boldsymbol{x}}h(t,\cdot)\big\|_{L^\infty_{\boldsymbol{x}}L^2_\varrho} + \nu^{1/2}\big\|\nabla_{\boldsymbol{x}}h(t,\cdot)\big\|_{L^\infty(\Omega)} + \big\|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}(t,\cdot)\big\|_{L^\infty(\Omega)} \le M$$

and the lower and upper bounds

$$\forall(\boldsymbol{x},\varrho)\in\Omega,\qquad h_\star \le \underline{h}(\varrho)+h(t,\boldsymbol{x},\varrho)\le h^\star;$$

and for any $(\dot\eta,\dot{\boldsymbol{u}})\in\mathcal{C}^0([0,T];X^0)\cap L^2(0,T;X^1)$, with X^0,X^1 in (2.16), and $(R,\boldsymbol{R})\in L^2(0,T;X^0)$ satisfying system (2.15) in $L^2(0,T;X^1)'$, the following estimate holds:

$$\begin{aligned}
&\mathcal{E}(\dot\eta(t,\cdot),\dot{\boldsymbol{u}}(t,\cdot))^{1/2} + \kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\dot\eta\big\|_{L^2(0,t;L^2(\Omega))} + \kappa^{1/2}\big|\nabla_{\boldsymbol{x}}\dot\eta\big|_{\varrho=\rho_0}\big|_{L^2(0,t;L^2_{\boldsymbol{x}})} + \nu\big\|\nabla_{\boldsymbol{x}}\dot{\boldsymbol{u}}\big\|^2_{L^2(\Omega)}\\
&\qquad\le \Big(\mathcal{E}(\dot\eta(0,\cdot),\dot{\boldsymbol{u}}(0,\cdot))^{1/2} + C\int_0^t \mathcal{E}(R(\tau,\cdot),\boldsymbol{R}(\tau,\cdot))^{1/2}\,\mathrm{d}\tau\Big)\\
&\qquad\qquad\times\exp\Big(C\int_0^t\big(1+\kappa^{-1}\big\|\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}(\tau,\cdot)\big\|^2_{L^\infty_{\boldsymbol{x}}L^2_\varrho}\big)\,\mathrm{d}\tau\Big),
\end{aligned}$$

where we denote

$$\mathcal{E}(\dot\eta,\dot{\boldsymbol{u}}) := \frac12\int_{\rho_0}^{\rho_1}\int_{\mathbb{R}^d}\dot\eta^2 + \varrho(\underline{h}+h)\big|\dot{\boldsymbol{u}}\big|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho \;+\; \frac{\rho_0}{2}\int_{\mathbb{R}^d}\dot\eta^2\big|_{\varrho=\rho_0}\,\mathrm{d}\boldsymbol{x}.$$

PROOF. We test the first equation against $\dot\eta\in L^2(0,T;H^{1,0}(\Omega))$, its trace on $\{(\boldsymbol{x},\rho_0),\boldsymbol{x}\in\mathbb{R}^d\}$ against $\rho_0\dot\eta\big|_{\varrho=\rho_0}\in L^2(0,T;H^1(\mathbb{R}^d))$, and the second equation against $(\underline{h}+h)\dot{\boldsymbol{u}}\in L^2(0,T;H^{1,0}(\Omega))$. This yields,

after integration by parts

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\dot\eta,\dot{\boldsymbol{u}})+\kappa\big\|\nabla_{\boldsymbol{x}}\dot\eta\big\|^2_{L^2(\Omega)}+\rho_0\kappa\big|\nabla_{\boldsymbol{x}}\dot\eta\big|_{\varrho=\rho_0}\big|^2_{L^2_{\boldsymbol{x}}}+\nu\sum_{i=1}^d\int_\Omega\varrho(\underline{h}+h)|\partial_{x_i}\dot{\boldsymbol{u}}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho\\
&=-\big((\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta,\dot\eta\big)_{L^2(\Omega)}-\left(\int_\varrho^{\rho_1}(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\,d\varrho',\dot\eta\right)_{L^2(\Omega)} &\text{(i)}\\
&\quad-\left(\int_\varrho^{\rho_1}(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\dot{\boldsymbol{u}}\,d\varrho',\dot\eta\right)_{L^2(\Omega)}+\big(R,\dot\eta\big)_{L^2(\Omega)} &\text{(ii)}\\
&\quad-\big(\varrho(\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot{\boldsymbol{u}},(\underline{h}+h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)}+\kappa\,\big(\varrho(\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\dot{\boldsymbol{u}},\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} &\text{(iii)}\\
&\quad-\big(\rho_0\nabla_{\boldsymbol{x}}\dot\eta\big|_{\varrho=\rho_0},(\underline{h}+h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)}-\big(\int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}\dot\eta\,d\varrho',(\underline{h}+h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} &\text{(iv)}\\
&\quad-\nu\big(\varrho(\nabla_{\boldsymbol{x}}h\cdot\nabla)\dot{\boldsymbol{u}},\dot{\boldsymbol{u}}\big)_{L^2(\Omega)}+\big(\varrho\boldsymbol{R},(\underline{h}+h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} &\text{(v)}\\
&\quad-\rho_0\left(\big((\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\big)\big|_{\varrho=\rho_0},\dot\eta\big|_{\varrho=\rho_0}\right)_{L^2_{\boldsymbol{x}}}-\rho_0\left(\int_{\rho_0}^{\rho_1}(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\,\mathrm{d}\varrho',\dot\eta\big|_{\varrho=\rho_0}\right)_{L^2_{\boldsymbol{x}}} &\text{(vi)}\\
&\quad-\rho_0\left(\int_{\rho_0}^{\rho_1}(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}\varrho',\dot\eta\big|_{\varrho=\rho_0}\right)_{L^2_{\boldsymbol{x}}}+\rho_0\left(R\big|_{\varrho=\rho_0},\dot\eta\big|_{\varrho=\rho_0}\right)_{L^2_{\boldsymbol{x}}} &\text{(vii)}\\
&\quad+\tfrac12(\varrho(\partial_t h)\dot{\boldsymbol{u}},\dot{\boldsymbol{u}})_{L^2(\Omega)}. &\text{(viii)}
\end{aligned}
$$

We consider first the second terms in (i) and (vi). We have by an immediate application of Cauchy-Schwarz inequality and the continuous embedding $L^\infty((\rho_0,\rho_1))\subset L^2((\rho_0,\rho_1))$

$$
\begin{aligned}
&\left|\left(\int_\varrho^{\rho_1}(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\,d\varrho',\dot\eta\right)_{L^2(\Omega)}\right|+\rho_0\left|\left(\int_{\rho_0}^{\rho_1}(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\,d\varrho',\dot\eta\big|_{\varrho=\rho_0}\right)_{L^2_{\boldsymbol{x}}}\right|\\
&\qquad\lesssim\big\|\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}\big\|_{L^\infty_{\boldsymbol{x}}L^2_\varrho}\big\|\nabla_{\boldsymbol{x}}\dot\eta\big\|_{L^2(\Omega)}\Big(\big\|\dot\eta\big\|_{L^2(\Omega)}+\big|\dot\eta\big|_{\varrho=\rho_0}\big|_{L^2_{\boldsymbol{x}}}\Big).
\end{aligned}\tag{2.17}
$$

Notice that the right-hand side (2.17) cannot be bounded by the energy functional $\mathcal{E}(\dot\eta,\dot{\boldsymbol{u}})$, and this is exactly the point where we use the assumption $\kappa>0$. Let us now estimate all other terms.

Using integration by parts in the variable $\boldsymbol{x}$, we estimate the first addend of (i) and (vi) as follows:

$$
\begin{aligned}
&\Big|\big((\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta,\dot\eta\big)_{L^2(\Omega)}\Big|+\rho_0\left|\left(\big((\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\big)\big|_{\varrho=\rho_0},\dot\eta\big|_{\varrho=\rho_0}\right)_{L^2_{\boldsymbol{x}}}\right|\\
&\qquad\lesssim\big\|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}\big\|_{L^\infty(\Omega)}\big\|\dot\eta\big\|^2_{L^2(\Omega)}+\big|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}\big|_{\varrho=\rho_0}\big|_{L^\infty_{\boldsymbol{x}}}\big|\dot\eta\big|_{\varrho=\rho_0}\big|^2_{L^2_{\boldsymbol{x}}}.
\end{aligned}
$$

The contributions in (iii) and (viii) compensate after integration by parts in $\boldsymbol{x}$, using the first equation in (2.1). Now consider the first addend of (ii) together with the second addend of (iv). By application of Fubini's theorem we have

$$
\int_{\mathbb{R}^d}\int_{\rho_0}^{\rho_1}\left(\int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}\dot\eta(\varrho')\,\mathrm{d}\varrho'\right)\cdot(\underline{h}+h)(\varrho)\dot{\boldsymbol{u}}(\varrho)\,\mathrm{d}\varrho\,\mathrm{d}\boldsymbol{x}=\int_{\mathbb{R}^d}\int_{\rho_0}^{\rho_1}\left(\int_{\varrho'}^{\rho_1}(\underline{h}+h)(\varrho)\dot{\boldsymbol{u}}(\varrho)\,\mathrm{d}\varrho\right)\cdot\nabla_{\boldsymbol{x}}\dot\eta(\varrho')\,\mathrm{d}\varrho'\,\mathrm{d}\boldsymbol{x}
$$

and hence, integrating by parts in $\boldsymbol{x}$, we infer

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \int_{\varrho}^{\rho_1} (\underline{h}+h)(\varrho')\nabla_{\boldsymbol{x}} \cdot \dot{\boldsymbol{u}}(\varrho') \,\mathrm{d}\varrho' \,\dot\eta(\varrho) \,\mathrm{d}\varrho \,\mathrm{d}\boldsymbol{x} \right. \\
&\qquad \left. + \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \left(\int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}} \dot\eta(\varrho') \,\mathrm{d}\varrho' \right) \cdot (\underline{h}+h)(\varrho) \dot{\boldsymbol{u}}(\varrho) \,\mathrm{d}\varrho \,\mathrm{d}\boldsymbol{x} \right| \\
&= \left| \int_{\mathbb{R}^d} \int_{\rho_0}^{\rho_1} \int_{\varrho}^{\rho_1} (\nabla_{\boldsymbol{x}} h)(\varrho') \cdot \dot{\boldsymbol{u}}(\varrho') \dot\eta(\varrho) \,\mathrm{d}\varrho' \,\mathrm{d}\varrho \,\mathrm{d}\boldsymbol{x} \right| \lesssim \big\|\nabla_{\boldsymbol{x}} h\big\|_{L^\infty_{\boldsymbol{x}} L^2_\varrho} \big\|\dot{\boldsymbol{u}}\big\|_{L^2(\Omega)} \big\|\dot\eta\big\|_{L^2(\Omega)}.
\end{aligned}$$

Concerning first addend of (iv) and the first addend of (vii), we have after integrating by parts with respect to the $\boldsymbol{x}$ variable and using Cauchy-Schwarz inequality

$$\begin{aligned}
&\left| - \Big(\rho_0 \nabla_{\boldsymbol{x}} \dot\eta\big|_{\varrho=\rho_0}, (\underline{h}+h)\dot{\boldsymbol{u}} \Big)_{L^2(\Omega)} - \rho_0 \left(\int_{\rho_0}^{\rho_1} (\underline{h}+h) \nabla_{\boldsymbol{x}} \cdot \dot{\boldsymbol{u}} \,\mathrm{d}\varrho, \dot\eta\big|_{\varrho=\rho_0} \right)_{L^2_{\boldsymbol{x}}} \right| \\
&\quad = \rho_0 \left| \left(\int_{\rho_0}^{\rho_1} (\nabla_{\boldsymbol{x}} h) \cdot \dot{\boldsymbol{u}} \,\mathrm{d}\varrho, \dot\eta\big|_{\varrho=\rho_0} \right)_{L^2_{\boldsymbol{x}}} \right| \lesssim \big\|\nabla_{\boldsymbol{x}} h\big\|_{L^\infty_{\boldsymbol{x}} L^2_\varrho} \big\|\dot{\boldsymbol{u}}\big\|_{L^2(\Omega)} \big|\dot\eta\big|_{\varrho=\rho_0}\big|_{L^2_{\boldsymbol{x}}}.
\end{aligned}$$

Concerning the first addend of (v), we have for an arbitrarily large constant $K > 0$,

$$\nu \left| \big(\varrho(\nabla_{\boldsymbol{x}} h \cdot \nabla)\dot{\boldsymbol{u}}, \dot{\boldsymbol{u}} \big)_{L^2(\Omega)} \right| \leq \frac{1}{2K} \nu \big\|\nabla \dot{\boldsymbol{u}}\big\|^2_{L^2(\Omega)} + \frac{K\rho_1^2}{2} \nu \big\|\nabla_{\boldsymbol{x}} h\big\|^2_{L^\infty(\Omega)} \big\|\dot{\boldsymbol{u}}\big\|^2_{L^2(\Omega)}.$$

The last contributions, namely

$$\Big| \big(R, \eta\big)_{L^2(\Omega)} + \big(\boldsymbol{R}, \varrho(\underline{h}+h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} + \rho_0 \big(R\big|_{\varrho=\rho_0}, \dot\eta\big|_{\varrho=\rho_0}\big)_{L^2_{\boldsymbol{x}}} \Big|,$$

are easily controlled by means of Cauchy-Schwarz inequality. Collecting all of the above, and using that

$$\mathcal{E}(\dot\eta, \dot{\boldsymbol{u}}) \approx \big\|\dot\eta\big\|^2_{L^2(\Omega)} + \big\|\dot{\boldsymbol{u}}\big\|^2_{L^2(\Omega)} + \big|\dot\eta\big|_{\varrho=\rho_0}\big|^2_{L^2_{\boldsymbol{x}}}$$

and

$$\nu \sum_{i=1}^d \int_\Omega \varrho(\underline{h}+h) |\partial_{x_i} \dot{\boldsymbol{u}}|^2 \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\varrho \gtrsim \nu \big\|\nabla_{\boldsymbol{x}} \dot{\boldsymbol{u}}\big\|^2_{L^2(\Omega)}$$

since $\rho_0 h_\star \leq \varrho(\underline{h}+h) \leq \rho_1 h^\star$, we obtain (choosing K sufficiently large)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\dot\eta, \dot{\boldsymbol{u}}) + \kappa \big\|\nabla_{\boldsymbol{x}} \dot\eta\big\|^2_{L^2(\Omega)} + \rho_0 \kappa \big|\nabla_{\boldsymbol{x}} \dot\eta\big|_{\varrho=\rho_0}\big|^2_{L^2_{\boldsymbol{x}}} & \\
\leq C\, \mathcal{E}(\dot\eta, \dot{\boldsymbol{u}}) + C \big\|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}\big\|_{L^\infty_{\boldsymbol{x}} L^2_\varrho} \mathcal{E}(\dot\eta, \dot{\boldsymbol{u}})^{1/2} \big\|\nabla_{\boldsymbol{x}} \dot\eta\big\|_{L^2(\Omega)} & \\
+ C \mathcal{E}(\dot\eta, \dot{\boldsymbol{u}})^{1/2} \mathcal{E}(R, \boldsymbol{R})^{1/2}, &
\end{aligned}$$

with $C = C(h_\star, h^\star, M)$. We deduce (augmenting C if necessary)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\dot\eta, \dot{\boldsymbol{u}}) + \frac{\kappa}{2} \big\|\nabla_{\boldsymbol{x}} \eta\big\|^2_{L^2(\Omega)} + \rho_0 \kappa \big|\nabla_{\boldsymbol{x}} \dot\eta\big|_{\varrho=\rho_0}\big|^2_{L^2_{\boldsymbol{x}}} & \\
\leq C \left(1 + \kappa^{-1} \big\|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}\big\|^2_{L^\infty_{\boldsymbol{x}} L^2_\varrho}\right) \mathcal{E}(\dot\eta, \dot{\boldsymbol{u}}) + C \mathcal{E}(\dot\eta, \dot{\boldsymbol{u}})^{1/2} \mathcal{E}(R, \boldsymbol{R})^{1/2}, &
\end{aligned}$$

and the desired estimate follows by Gronwall's inequality. □

2.4. Large-time existence; proof of Theorem 1.1. We prove the large-time existence and energy estimates on solutions to the regularized system (2.1) in the following result. Compared with Proposition 2.1, we provide an existence time which is uniformly bounded (from below) with respect to the artificial regularization parameter $\nu > 0$, and specify the dependency with respect to the diffusivity parameter κ, in relation with the size of the data. It is in this sense that the existence of strong solutions to the hydrostatic system holds for *large* times. We then complete the proof of Theorem 1.1 at the end of this section.

PROPOSITION 2.6. *Let $s, k \in \mathbb{N}$ be such that $s > 2 + \frac{d}{2}$, $2 \leq k \leq s$, and $\underline{M}, M^\star, h_\star, h^\star > 0$. Then, there exists $C > 0$ such that, for any $0 < \nu \leq \kappa \leq 1$, and*

- *for any $(\underline{h}, \underline{\boldsymbol{u}}) \in W^{k,\infty}((\rho_0, \rho_1))$ such that*
$$\big|\underline{h}\big|_{W^{k,\infty}_\varrho} + \big|\underline{\boldsymbol{u}}'\big|_{W^{k-1,\infty}_\varrho} \leq \underline{M}\,;$$
- *for any initial data $(h_0, \boldsymbol{u}_0) = (h_0(\boldsymbol{x}, \varrho), \boldsymbol{u}_0(\boldsymbol{x}, \varrho)) \in H^{s,k}(\Omega)$ with*
$$M_0 := \big\|\eta_0\big\|_{H^{s,k}} + \big\|\boldsymbol{u}_0\big\|_{H^{s,k}} + \big|\eta_0\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}} + \kappa^{1/2}\big\|h_0\big\|_{H^{s,k}} \leq M^\star,$$
and
$$\forall (\boldsymbol{x}, \varrho) \in \Omega, \qquad h_\star \leq \underline{h}(\varrho) + h_0(\boldsymbol{x}, \varrho) \leq h^\star,$$

the following holds. Denoting
$$T^{-1} = C\left(1 + \kappa^{-1}\big(\big|\underline{\boldsymbol{u}}'\big|^2_{L^2_\varrho} + M_0^2\big)\right),$$
there exists a unique strong solution $(h, \boldsymbol{u}) \in \mathcal{C}([0,T]; H^{s,k}(\Omega)^{1+d})$ to the Cauchy problem associated with (2.1) and initial data $(h, \boldsymbol{u})\big|_{t=0} = (h_0, \boldsymbol{u}_0)$. Moreover, $h \in L^2(0,T; H^{s+1,k}(\Omega))$ and one has, for any $t \in [0,T]$, the lower and the upper bounds
$$\forall (\boldsymbol{x}, \varrho) \in \Omega, \qquad h_\star/2 \leq \underline{h}(\varrho) + h(t, \boldsymbol{x}, \varrho) \leq 2\, h^\star,$$
and the estimate
$$\begin{aligned}\mathcal{F}(t) := \big\|\eta(t,\cdot)\big\|_{H^{s,k}} + \big\|\boldsymbol{u}(t,\cdot)\big\|_{H^{s,k}} + \big|\eta\big|_{\varrho=\rho_0}(t,\cdot)\big|_{H^s_{\boldsymbol{x}}} + \kappa^{1/2}\big\|h(t,\cdot)\big\|_{H^{s,k}} \\ + \kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{L^2(0,t;H^{s,k})} + \kappa^{1/2}\big|\nabla_{\boldsymbol{x}}\eta\big|_{\varrho=\rho_0}\big|_{L^2(0,t;H^s_{\boldsymbol{x}})} + \kappa\big\|\nabla_{\boldsymbol{x}} h\big\|_{L^2(0,t;H^{s,k})} \leq C M_0.\end{aligned}$$

PROOF. Let us denote by $T^\star \in (0, +\infty]$ the maximal time of existence of $(h, \boldsymbol{u}) \in \mathcal{C}^0([0,T^\star); H^{s,k}(\Omega))$ as provided by Proposition 2.1, and
$$T_\star = \sup\Big\{0 < T < T^\star \,:\, \forall t \in (0,T), \qquad h_\star/2 \leq \underline{h}(\varrho) + h(t,\boldsymbol{x},\varrho) \leq 2\,h^\star \quad \text{and} \quad \mathcal{F}(t) \leq C_0 M_0\Big\},$$
where $C_0 > 1$ will be determined later on. By the continuity in time of the solution, and using that the linear operator $h \mapsto \eta := \int_\varrho^{\rho_1} h(\cdot, \varrho')\,\mathrm{d}\varrho'$ (resp. $h \mapsto \eta\big|_{\varrho=\rho_0}$) is well-defined and bounded from $H^{s,k}(\Omega)$ to itself (resp. $H^s_{\boldsymbol{x}}(\mathbb{R}^d)$) we have $T_\star > 0$. Using Lemma 2.3, 2.4 and 2.5 and, therein, the inequalities $\big\|h\big\|_{H^{s-1,k-1}} = \big\|\partial_\varrho \eta\big\|_{H^{s-1,k-1}} \leq \big\|\eta\big\|_{H^{s,k}}$, and (since $\nu \leq \kappa$) $\nu^{1/2}\big\|\nabla_{\boldsymbol{x}} h\big\|_{L^\infty(\Omega)} \leq \kappa^{1/2}\big\|h\big\|_{H^{s,k}}$, we find that there exists $c_0 > 1$ depending only on $\rho_0 h_\star, \rho_1 h^\star$; and $C > 0$ depending on $\underline{M}, h_\star, h^\star, C_0 M_0$ such that for any $0 < t < T_\star$,
$$\begin{aligned}\big\|\eta(t,\cdot)\big\|_{H^{s,0}} + \big\|\boldsymbol{u}(t,\cdot)\big\|_{H^{s,0}} + \big|\eta\big|_{\varrho=\rho_0}(t,\cdot)\big|_{H^s_{\boldsymbol{x}}} + \kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{L^2(0,t;H^{s,0})} + \kappa^{1/2}\big|\nabla_{\boldsymbol{x}}\eta\big|_{\varrho=\rho_0}\big|_{L^2(0,t;H^s_{\boldsymbol{x}})} \\ \leq c_0\left(\big\|\eta_0\big\|_{H^{s,0}} + \big\|\boldsymbol{u}_0\big\|_{H^{s,0}} + \big|\eta_0\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}} + C\,C_0 M_0\left(t + \sqrt{t}\right)\right) \\ \times \exp\left(C\int_0^t \left(1 + \kappa^{-1}\big\|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}\big\|^2_{L^\infty_{\boldsymbol{x}} L^2_\varrho}\right)\mathrm{d}\tau\right); \quad (2.18)\end{aligned}$$

and (using a slightly adapted version of Lemma 2.5 which does not involve the trace of $\partial_\varrho^j \eta$ at the surface) for any $1 \leq j \leq k$

$$\begin{aligned}\big\|\partial_\varrho^j \eta(t,\cdot)\big\|_{H^{s-j,0}} + \kappa^{1/2}\big\|\nabla_{\boldsymbol{x}} \partial_\varrho^j \eta\big\|_{L^2(0,t;H^{s-j,0})} \\ \leq \Big(\big\|\partial_\varrho^j \eta(0,\cdot)\big\|_{H^{s,0}} + C\, C_0 M_0 \left(t + \sqrt{t}\right)\Big) \\ \times \exp\Big(C \int_0^t \big\|\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}(\tau,\cdot)\big\|_{L^\infty(\Omega)}\, \mathrm{d}\tau\Big), \quad (2.19)\end{aligned}$$

and

$$\begin{aligned}\big\|\partial_\varrho^j \boldsymbol{u}(t,\cdot)\big\|_{H^{s-j,0}} + \nu^{1/2}\big\|\nabla_{\boldsymbol{x}} \partial_\varrho^j \boldsymbol{u}\big\|_{L^2(0,t;H^{s-j,0})} \\ \leq \Big(\big\|\partial_\varrho^j \boldsymbol{u}(0,\cdot)\big\|_{H^{s,0}} + C\, C_0 M_0 \left(t + \sqrt{t}\right)\Big) \\ \times \exp\Big(C \int_0^t \big\|\nabla_{\boldsymbol{x}} \cdot \big(\boldsymbol{u} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}\big)(\tau,\cdot)\big\|_{L^\infty(\Omega)}\, \mathrm{d}\tau\Big); \quad (2.20)\end{aligned}$$

and finally for any $0 \leq j \leq k$

$$\begin{aligned}\kappa^{1/2}\big\|\partial_\varrho^j h(t,\cdot)\big\|_{H^{s-j,0}} + \kappa\big\|\nabla_{\boldsymbol{x}} \partial_\varrho^j h\big\|_{L^2(0,t;H^{s-j,0})} \\ \leq \Big(\kappa^{1/2}\big\|\partial_\varrho^j h(0,\cdot)\big\|_{H^{s,0}} + C\, C_0 M_0 \left(t + \sqrt{t}\right)\Big) \\ \times \exp\Big(C \int_0^t \big\|\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}(\tau,\cdot)\big\|_{L^\infty(\Omega)}\, \mathrm{d}\tau\Big). \quad (2.21)\end{aligned}$$

By the continuous embeddings $H^{s_0+\frac{1}{2},1} \subset L^\infty_\varrho H^{s_0} \subset L^\infty(\Omega)$ for any $s_0 > d/2$ (see Lemma A.1) and since $k \geq 1$ and $s > \frac{3}{2} + \frac{d}{2}$, we have

$$\big\|\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}\big\|_{L^\infty(\Omega)} + \big\|\nabla_{\boldsymbol{x}} \cdot \big(\boldsymbol{u} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}\big)\big\|_{L^\infty(\Omega)} \leq C(h_\star)\big(\big\|\boldsymbol{u}\big\|_{H^{s,k}} + \kappa\big\|h\big\|^2_{H^{s,k}} + \kappa\big\|\nabla_{\boldsymbol{x}} h\big\|_{H^{s,k}}\big).$$

We deduce that

$$\mathcal{F}(t) \leq c\Big(M_0 + C\, C_0 M_0 \left(t + \sqrt{t}\right)\Big) \times \exp\Big(C\big(t + \sqrt{t} + \kappa^{-1} \int_0^t \big\|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}(\tau,\cdot)\big\|^2_{L^\infty_{\boldsymbol{x}} L^2_\varrho}\, \mathrm{d}\tau\big)\Big),$$

where we recall that $c_0 > 1$ depends only on $h_\star$ and $h^\star$; and $C > 0$ depends on $\underline{M}, C_0 M_0, h_\star, h^\star$. Hence choosing $C_0 = 2c_0$ and using that (by Lemma A.1 and since $k \geq 2$ and $s > \frac{3}{2} + \frac{d}{2}$)

$$\big\|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}\big\|_{L^\infty_{\boldsymbol{x}} L^2_\varrho} \leq \big\|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}\big\|^2_{L^2_\varrho L^\infty_{\boldsymbol{x}}} \lesssim \big|\underline{\boldsymbol{u}}'\big|^2_{L^2_\varrho} + \big\|\boldsymbol{u}\big\|^2_{H^{s,k}} \leq \big|\underline{\boldsymbol{u}}'\big|^2_{L^2_\varrho} + (C_0 M_0)^2,$$

we find that there exists $C_0 \geq 1$ depending only on $\underline{M}, M^\star, h_\star, h^\star$ such that

$$t\big(1 + \kappa^{-1}\big(\big|\underline{\boldsymbol{u}}'\big|^2_{L^2_\varrho} + M_0^2\big)\big) \leq C_0^{-1} \quad \Longrightarrow \quad \mathcal{F}(t) \leq \frac{3}{4} C_0 M_0.$$

Now we remark that since

$$\partial_t h + \underline{\boldsymbol{u}} \cdot \nabla_{\boldsymbol{x}} h = \kappa \Delta_{\boldsymbol{x}} h + g \quad \text{ with } \quad g = -\nabla_{\boldsymbol{x}} \cdot (\underline{h}\boldsymbol{u} + h\boldsymbol{u})$$

and by the positivity of the heat kernel we have

$$\inf_\Omega h(t,\cdot) \geq \inf_\Omega h_0 - \big\|g\big\|_{L^1(0,t;L^\infty(\Omega))}, \qquad \sup_\Omega h(t,\cdot) \leq \sup_\Omega h_0 + \big\|g\big\|_{L^1(0,t;L^\infty(\Omega))}.$$

Now, by the continuous embedding $H^{s-1,1}(\Omega) \subset L^\infty(\Omega)$ (since $s > \frac{3}{2} + \frac{d}{2}$), we have that

$$\big\|g\big\|_{L^\infty(\Omega)} \lesssim \big|\underline{h}\big|_{W^{1,\infty}_\varrho} \big\|\boldsymbol{u}\big\|_{H^{s,1}} + \big\|h\big\|_{H^{s,1}} \big\|\boldsymbol{u}\big\|_{H^{s,1}} \leq C(\underline{M})(1 + \kappa^{-1} M_0^2).$$

Hence augmenting C_0 if necessary we find that

$$t(1+\kappa^{-1}M_0^2)\leq C_0^{-1} \quad\Longrightarrow\quad \forall(\boldsymbol{x},\varrho)\in\Omega,\quad \frac{2}{3}h_\star\leq \underline{h}(\varrho)+h(t,\boldsymbol{x},\varrho)\leq\frac{3}{2}h^\star.$$

By a continuity argument we infer $T_\star\geq\Big(C\big(1+\kappa^{-1}\big(\big|\underline{\boldsymbol{u}}'\big|^2_{L^2_\varrho}+M_0^2\big)\big)\Big)^{-1}$, and the proof is complete. □

In order to complete the proof of Theorem 1.1, there remains to consider vanishing viscosity limit, $\nu\searrow 0$, in Proposition 2.6. Let us briefly sketch the standard argument. By Proposition 2.6, we construct a family $(h_\nu,\boldsymbol{u}_\nu)\in\mathcal{C}^0([0,T];H^{s,k}(\Omega))$ of solutions to (2.1) with $(h_\nu,\boldsymbol{u}_\nu)\big|_{t=0}=(h_0,\boldsymbol{u}_0)$ indexed by the parameter $\nu>0$. Notice that the time of existence and associated bounds provided by Proposition 2.6 are uniform with respect to the parameter $\nu>0$. Hence by the Banach-Alaoglu theorem there exists a subsequence which converges weakly towards $(h,\boldsymbol{u})\in L^\infty(0,T;H^{s,k}(\Omega)^{1+d})$, satisfying the estimates of Proposition 2.6. Using the equations, we find that $(\partial_t\zeta_\nu,\partial_t\boldsymbol{u}_\nu)$ are uniformly bounded in $L^\infty(0,T;H^{s-2,k})$. The Aubin-Lions lemma (see [30]) implies that, up to extracting a subsequence, the convergence holds strongly in $(h,\boldsymbol{u})\in\mathcal{C}^0([0,T];H^{s',k}(B)^{1+d})$ for any $0\leq s'<s$ for any bounded $B\subset\mathbb{R}^d\times(\rho_0,\rho_1)$. Choosing $s'>3/2+d/2$ and using Lemma A.1 and Sobolev embedding, we can pass to the limit in the nonlinear terms of the equation and infer that that $(h,\boldsymbol{u})$ is a strong solution to (2.1) with $\nu=0$. Moreover, since $(h,\boldsymbol{u})\in\mathcal{C}^0([0,T];H^{s-2,k}(\Omega)^{1+d})$, we have $(h,\boldsymbol{u})\in\mathcal{C}^0([0,T];H^{s',k}(\Omega)^{1+d})$ for any $0\leq s'<s$.

Uniqueness of the solution $(h,\boldsymbol{u})\in L^\infty(0,T;H^{s,k}(\Omega)^{1+d})$ follows by using Lemma 2.5 on the difference between two solutions, and Gronwall's Lemma.

REMARK 2.7. By the standard result of Strauss [31], we have that $(h,\boldsymbol{u})\in\mathcal{C}^0_{\rm w}([0,T];H^{s,k}(\Omega)^{1+d})$ the subspace of $L^\infty(0,T;H^{s,k}(\Omega)^{1+d})$ of functions which are continuous from $[0,T]$ in $H^{s,k}(\Omega)^{1+d}$, equipped with the weak topology. In order to guarantee the stronger results $(h,\boldsymbol{u})\in\mathcal{C}^0([0,T];H^{s,k}(\Omega)^{1+d})$ (and subsequently the well-posedness in the sense of Hadamard), one typically uses the Bona–Smith technique [2], relying on the propagation of regularity and "tame" estimates which we have decided to omit for the sake of conciseness.

3. The non-hydrostatic system

In this section we study the local well-posedness theory for the non-hydrostatic system in isopycnal coordinates. We set (without loss of generality, see footnote 2) $\rho_0=1$ in (1.7), so that the system reads

$$\begin{aligned}
&\partial_t h+\nabla_{\boldsymbol{x}}\cdot\big((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u})\big)=\kappa\Delta_{\boldsymbol{x}}h,\\
&\varrho\Big(\partial_t\boldsymbol{u}+\big((\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}\Big)+\nabla_{\boldsymbol{x}}P+\frac{\nabla_{\boldsymbol{x}}\eta}{\underline{h}+h}(\partial_\varrho P+\varrho\underline{h})=0,\\
&\mu\varrho\Big(\partial_t w+\big(\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h}\big)\cdot\nabla_{\boldsymbol{x}}w\Big)-\frac{\partial_\varrho P}{\underline{h}+h}+\frac{\varrho h}{\underline{h}+h}=0,\\
&-(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}-\nabla_{\boldsymbol{x}}\eta\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u})+\partial_\varrho w=0,\quad\text{(div.-free cond.)}\\
&\eta(\cdot,\varrho)=\int_\varrho^{\rho_1}h(\cdot,\varrho')\,\mathrm{d}\varrho',\qquad P\big|_{\varrho=\rho_0}=0,\qquad w\big|_{\varrho=\rho_1}=0.\quad\text{(bound. cond.)}
\end{aligned}\tag{3.1}$$

3.1. The pressure reconstruction. The first step of our analysis consists in showing how the pressure variable, P, can be uniquely reconstructed (thanks to the "divergence-free" incompressibility constraint) from prognostic variables $\boldsymbol{u}$, w and h (or, equivalently, η), through an elliptic boundary-value problem. Differentiating the "divergence-free" incompressibility constraint in (3.1) with respect to time yields

$$-(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\partial_t\boldsymbol{u}-(\nabla_{\boldsymbol{x}}\eta)\cdot(\partial_\varrho\partial_t\boldsymbol{u})+\partial_\varrho\partial_t w=(\partial_t h)(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u})+(\nabla_{\boldsymbol{x}}\partial_t\eta)\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}).$$

We plug the expressions for $\partial_t \boldsymbol{u}, \partial_t w, \partial_t h, \partial_t \eta$ provided by (3.1) inside the above identity. Reorganizing terms, this yields the following

$$\begin{aligned}
&(\underline{h}+h)\nabla_{\boldsymbol{x}} \cdot \left(\tfrac{1}{\varrho}\nabla_{\boldsymbol{x}} P + \tfrac{\nabla_{\boldsymbol{x}}\eta(\partial_\varrho P + \varrho \underline{h})}{\varrho(\underline{h}+h)}\right) + (\nabla_{\boldsymbol{x}}\eta)\cdot\left(\partial_\varrho\left(\tfrac{1}{\varrho}\nabla_{\boldsymbol{x}} P + \tfrac{\nabla_{\boldsymbol{x}}\eta(\partial_\varrho P + \varrho \underline{h})}{\varrho(\underline{h}+h)}\right)\right) + \partial_\varrho\left(\tfrac{\partial_\varrho P + \varrho \underline{h}}{\mu\varrho(\underline{h}+h)}\right)\\
&\quad = -(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\left(\left((\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}}\right)\boldsymbol{u}\right) - (\nabla_{\boldsymbol{x}}\eta)\cdot\left(\partial_\varrho\left(\left((\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}}\right)\boldsymbol{u}\right)\right)\\
&\qquad + \partial_\varrho\left(\left(\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}\right)\cdot\nabla_{\boldsymbol{x}} w\right) + \left(\kappa\Delta_{\boldsymbol{x}} h - \nabla_{\boldsymbol{x}}\cdot\left((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u})\right)\right)(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u})\\
&\qquad\qquad + \left(\kappa\nabla_{\boldsymbol{x}}\Delta_{\boldsymbol{x}}\eta - \nabla_{\boldsymbol{x}}\int_\varrho^{\rho_1}\nabla_{\boldsymbol{x}}\cdot\left((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u})\right)\mathrm{d}\varrho'\right)\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}).
\end{aligned}$$

Using that $\partial_\varrho\nabla_{\boldsymbol{x}}\eta = -\nabla_{\boldsymbol{x}} h$ we can rewrite the left-hand side in a compact formulation as

$$(\mathrm{LHS}) = \frac{1}{\mu}\begin{pmatrix}\sqrt{\mu}\nabla_{\boldsymbol{x}}\\ \partial_\varrho\end{pmatrix}\cdot\left(\begin{pmatrix}\frac{\underline{h}+h}{\varrho}\,\mathrm{Id} & \frac{\sqrt{\mu}\nabla_{\boldsymbol{x}}\eta}{\varrho}\\ \frac{\sqrt{\mu}\nabla_{\boldsymbol{x}}^\top\eta}{\varrho} & \frac{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}{\varrho(\underline{h}+h)}\end{pmatrix}\begin{pmatrix}\sqrt{\mu}\nabla_{\boldsymbol{x}}\\ \partial_\varrho\end{pmatrix}(P+\underline{P}_{\mathrm{eq}})\right),$$

with $\underline{P}_{\mathrm{eq}} := \int_{\rho_0}^{\varrho}\varrho'\underline{h}(\varrho')\,\mathrm{d}\varrho'$. As for the right-hand side, we denote

$$\boldsymbol{u}_\star := -\kappa\frac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}, \tag{3.2}$$

and we infer

$$\begin{aligned}
(\mathrm{RHS}) = &-(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\left(\left((\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}}\right)\boldsymbol{u}\right) - (\nabla_{\boldsymbol{x}}\eta)\cdot\left(\partial_\varrho\left(\left(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star\right)\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\right)\right)\\
&+\partial_\varrho\left(\left(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star\right)\cdot\nabla_{\boldsymbol{x}} w\right) - \nabla_{\boldsymbol{x}}\cdot\left((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\right)(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u})\\
&-\left(\nabla_{\boldsymbol{x}}\int_\varrho^{\rho_1}\nabla_{\boldsymbol{x}}\cdot\left((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\right)\mathrm{d}\varrho'\right)\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}).
\end{aligned}$$

Notice the identity (reminiscent of (1.6))

$$\int_\varrho^{\rho_1}\nabla_{\boldsymbol{x}}\cdot\left((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\right)\mathrm{d}\varrho' = (\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}}\eta - w - w_\star, \tag{3.3}$$

where

$$w_\star := \kappa\Delta_{\boldsymbol{x}}\eta - \kappa\frac{\nabla_{\boldsymbol{x}} h\cdot\nabla_{\boldsymbol{x}}\eta}{\underline{h}+h}, \tag{3.4}$$

which is obtained by integrating with respect to ϱ the divergence-free identities

$$\begin{aligned}
-(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} - (\nabla_{\boldsymbol{x}}\eta)\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}) + \partial_\varrho w = 0,\\
-(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}_\star - (\nabla_{\boldsymbol{x}}\eta)\cdot(\partial_\varrho\boldsymbol{u}_\star) + \partial_\varrho w_\star = 0,
\end{aligned}$$

integrating by parts with respect to ϱ, using the boundary condition $w|_{\varrho=\rho_1} = 0 = w_\star|_{\varrho=\rho_1}$ and $h = -\partial_\varrho\eta$. Hence the above can be equivalently written as

$$\begin{aligned}
(\mathrm{RHS}) = &-(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\left(\left(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star\right)\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\right) - (\nabla_{\boldsymbol{x}}\eta)\cdot\left(\partial_\varrho\left(\left(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star\right)\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}\right)\right)\\
&+\partial_\varrho\left(\left(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star\right)\cdot\nabla_{\boldsymbol{x}} w\right) - \nabla_{\boldsymbol{x}}\cdot\left((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\right)(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u})\\
&-\left(\nabla_{\boldsymbol{x}}\left((\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}}\eta - w - w_\star\right)\right)\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}).
\end{aligned} \tag{3.5}$$

Taking into account the boundary conditions in (3.1), we find that the pressure satisfies the following problem (recalling $\underline{P}_{\mathrm{eq}} := \int_{\rho_0}^{\varrho}\varrho'\underline{h}(\varrho')\,\mathrm{d}\varrho'$):

$$\begin{cases}
\dfrac{1}{\mu}\begin{pmatrix}\sqrt{\mu}\nabla_{\boldsymbol{x}}\\ \partial_\varrho\end{pmatrix}\cdot\left(\begin{pmatrix}\frac{\underline{h}+h}{\varrho}\,\mathrm{Id} & \frac{\sqrt{\mu}\nabla_{\boldsymbol{x}}\eta}{\varrho}\\ \frac{\sqrt{\mu}\nabla_{\boldsymbol{x}}^\top\eta}{\varrho} & \frac{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}{\varrho(\underline{h}+h)}\end{pmatrix}\begin{pmatrix}\sqrt{\mu}\nabla_{\boldsymbol{x}}\\ \partial_\varrho\end{pmatrix}(P+\underline{P}_{\mathrm{eq}})\right) = (\mathrm{RHS}),\\
P\big|_{\varrho=\rho_0} = 0, \qquad (\partial_\varrho P)\big|_{\varrho=\rho_1} = \rho_1 h\big|_{\varrho=\rho_1}.
\end{cases} \tag{3.6}$$

This boundary value problem corresponds to [7, (7)] written in isopycnal coordinates, adapting the boundary conditions to the free-surface framework, and taking into account the effective transport velocities from eddy correlation. Following [7], we shall infer the existence and uniqueness as well as estimates on the pressure P from the elliptic theory applied to the above boundary value problem, as stated below.

LEMMA 3.1. *Let $s_0 > d/2$, $s, k \in \mathbb{N}$ such that $s \geq s_0 + \frac{5}{2}$ and $1 \leq k \leq s$. Let $M, h_\star > 0$. There exists $C > 0$ such that for any $\mu \in (0,1]$, and for any $\underline{h}, h, \eta$ satisfying the following bound*

$$\big|\underline{h}\big|_{W^{\max(1,k-1),\infty}_\varrho} + \big\|h\big\|_{H^{s-1,\max(1,k-1)}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s-1,\max(1,k-1)}} \leq M;$$

and the stable stratification assumption

$$\inf_{(\boldsymbol{x},\varrho)\in\Omega} \underline{h}(\varrho) + h(\boldsymbol{x},\varrho) \geq h_\star;$$

and for any $(Q_0, Q_1, \boldsymbol{R}) \in H^{s,k-1}(\Omega)^2 \times H^{s,k}(\Omega)^{d+1}$, there exists a unique $P \in H^{s+1,k+1}(\Omega)$ solution to

$$\left\{ \begin{array}{l} \nabla^\mu_{\boldsymbol{x},\varrho} \cdot \big(A^\mu \nabla^\mu_{\boldsymbol{x},\varrho} P\big) = Q_0 + \sqrt{\mu}\Lambda Q_1 + \nabla^\mu_{\boldsymbol{x},\varrho} \cdot \boldsymbol{R} \\ P\big|_{\varrho=\rho_0} = 0, \qquad \boldsymbol{e}_{d+1} \cdot (A \nabla^\mu_{\boldsymbol{x},\varrho} P)\big|_{\varrho=\rho_1} = \boldsymbol{e}_{d+1} \cdot \boldsymbol{R}\big|_{\varrho=\rho_1} \end{array} \right. \tag{3.7}$$

where we denote $\Lambda := (\mathrm{Id} - \Delta_{\boldsymbol{x}})^{1/2}$,

$$\nabla^\mu_{\boldsymbol{x},\varrho} := \begin{pmatrix} \sqrt{\mu}\nabla_{\boldsymbol{x}} \\ \partial_\varrho \end{pmatrix} \qquad ; \qquad A^\mu := \begin{pmatrix} \frac{\underline{h}+h}{\varrho}\,\mathrm{Id} & \frac{\sqrt{\mu}\nabla_{\boldsymbol{x}}\eta}{\varrho} \\ \frac{\sqrt{\mu}\nabla_{\boldsymbol{x}}^\top \eta}{\varrho} & \frac{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}{\varrho(\underline{h}+h)} \end{pmatrix},$$

and one has, denoting $\big\|(Q_0,Q_1,\boldsymbol{R})\big\|_{r,j} := \big\|Q_0\big\|_{H^{r,j-1}} + \big\|Q_1\big\|_{H^{r,j-1}} + \big\|\boldsymbol{R}\big\|_{H^{r,j}}$ and $a \vee b = \max(a,b)$

$$\begin{aligned} \big\|P\big\|_{L^2(\Omega)} + \big\|\nabla^\mu_{\boldsymbol{x},\varrho} P\big\|_{H^{s,k}} \leq C \times \Big(& \big\|(Q_0,Q_1,\boldsymbol{R})\big\|_{s,k} \\ & + \big(\big\|h\big\|_{H^{s,1}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s,k}}\big)\,\big\|(Q_0,Q_1,\boldsymbol{R})\big\|_{s-1,1\vee k-1}\Big) \end{aligned} \tag{3.8}$$

and, when $k \geq 2$,

$$\big\|P\big\|_{L^2(\Omega)} + \big\|\nabla^\mu_{\boldsymbol{x},\varrho} P\big\|_{H^{s-1,k-1}} \leq C\big\|(Q_0,Q_1,\boldsymbol{R})\big\|_{s-1,k-1}. \tag{3.9}$$

PROOF. Testing (3.7) with P, using integration by parts and the boundary conditions, we find

$$-\int_{\mathbb{R}^d}\int_{\rho_0}^{\rho_1} A^\mu \nabla^\mu_{\boldsymbol{x},\varrho} P \cdot \nabla^\mu_{\boldsymbol{x},\varrho} P \,\mathrm{d}\varrho\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d}\int_{\rho_0}^{\rho_1} Q_0 P + Q_1(\sqrt{\mu}\Lambda P) - \tilde{\boldsymbol{R}} \cdot \nabla^\mu_{\boldsymbol{x},\varrho} P \,\mathrm{d}\varrho\,\mathrm{d}\boldsymbol{x}.$$

For $(Q_0, Q_1, \boldsymbol{R}) \in L^2(\Omega)^{2+d+1}$, the existence and uniqueness of a (variational) solution to (3.7) in the functional space

$$H^1_0(\Omega) := \{P \in L^2(\Omega) \,:\, \nabla_{\boldsymbol{x},\varrho} P \in L^2(\Omega),\ P\big|_{\varrho=\rho_0} = 0\}$$

classically follows from the Lax-Milgram Lemma thanks to the boundedness and the coercivity of the matrix A (recall that $\underline{h} + h \geq h_\star > 0$ and the embedding of Lemma A.1), and the Poincaré inequality

$$\forall P \in H^1_0(\Omega), \qquad \big\|P\big\|^2_{L^2(\Omega)} = \int_{\mathbb{R}^d}\int_{\rho_0}^{\rho_1}\left|\int_{\rho_0}^{\varrho} \partial_{\varrho'} P \,\mathrm{d}\varrho'\right|^2 \mathrm{d}\varrho\,\mathrm{d}\boldsymbol{x} \leq (\rho_1-\rho_0)^2\big\|\partial_\varrho P\big\|^2_{L^2(\Omega)}, \tag{3.10}$$

and we have

$$\big\|\nabla^\mu_{\boldsymbol{x},\varrho} P\big\|_{L^2(\Omega)} \lesssim \big\|Q_0\big\|_{L^2(\Omega)} + \big\|Q_1\big\|_{L^2(\Omega)} + \big\|\boldsymbol{R}\big\|_{L^2(\Omega)}. \tag{3.11}$$

The desired regularity for $(Q_0, Q_1, \boldsymbol{R}) \in H^{s,k-1}(\Omega)^2 \times H^{s,k}(\Omega)^{d+1}$ is then deduced following the standard approach for elliptic equations (notice the domain is flat) from the estimates which we obtain below. For more details, we refer for instance to [21, Chapter 2] where a very similar elliptic problem is thoroughly studied. We now focus on the estimates, assuming *a priori* the needed regularity to justify the following computations.

First, we provide an estimate for $\big\|\nabla_{\boldsymbol{x},\varrho} P\big\|_{H^{r,0}(\Omega)}$ for $1 \leq r \leq s$. One readily checks that $P_r := \Lambda^r P$ with $\Lambda^r := (\mathrm{Id} - \Delta_{\boldsymbol{x}})^{r/2}$ satisfies (3.7) with $Q_0 \leftarrow \Lambda^r Q_0$, $Q_1 \leftarrow \Lambda^r Q_1$ and $\boldsymbol{R} \leftarrow \Lambda^r \boldsymbol{R} - [\Lambda^r, A^\mu]\nabla_{\boldsymbol{x},\varrho} P$.

We focus now on the contribution of $\boldsymbol{P}_r := [\Lambda^r, A^\mu]\nabla^\mu_{\boldsymbol{x},\varrho}P$. By continuous embedding (Lemma A.1) and commutator estimates (Lemma A.7), we have

$$\big\|\boldsymbol{P}_r\big\|_{L^2(\Omega)} \lesssim \big\|\nabla_{\boldsymbol{x}}A^\mu\big\|_{H^{s_0+\frac12,1}}\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{r-1,0}} + \Big\langle \big\|\nabla_{\boldsymbol{x}}A^\mu\big\|_{H^{r-1,0}}\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{s_0+\frac12,1}}\Big\rangle_{r>s_0+1}.$$

Hence, using product (Lemma A.2,A.3) and composition (Lemma A.4,A.6) estimates, we deduce

$$\big\|\boldsymbol{P}_r\big\|_{L^2(\Omega)} \le C\left(\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{r-1,0}} + \Big\langle \big(\|h\|_{H^{r,0}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,0}}\big)\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{s_0+\frac12,1}}\Big\rangle_{r>s_0+1}\right) \tag{3.12}$$

with $C = C(h_\star, |\underline{h}|_{W^{1,\infty}_\varrho}, \big\|(h, \sqrt{\mu}\nabla_{\boldsymbol{x}}\eta)\big\|_{H^{s_0+\frac32,1}})$.

Plugging (3.12) in (3.11) and using continuous embedding (Lemma A.1) and $s_0 + \frac32 \le s-1$ yields

$$\begin{aligned}\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{r,0}} \lesssim \big\|Q_0\big\|_{H^{r,0}} + \big\|Q_1\big\|_{H^{r,0}} + \big\|\boldsymbol{R}\big\|_{H^{r,0}} + \big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{r-1,0}}\\ + C\left(\|h\|_{H^{r,0}} + \big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,0}}\right)\Big\langle \big\|\nabla_{\boldsymbol{x},\varrho}P\big\|_{H^{s_0+\frac12,1}}\Big\rangle_{r>s_0+1},\end{aligned} \tag{3.13}$$

where we denote, here and thereafter, $a \lesssim b$ for $a \le Cb$ with

$$C = C(h_\star, |\underline{h}|_{W^{1,\infty}_\varrho}, \|h\|_{H^{s-1,1}}, \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s-1,1}}) = C(h_\star, M).$$

Next we provide an estimate for $\big\|\nabla_{\boldsymbol{x},\varrho}P\big\|_{H^{r,1}(\Omega)}$ appearing in the above right-hand side. This term involves $\partial^2_\varrho P$, which we control by rewriting (3.7) as

$$\tfrac{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}{\varrho(\underline{h}+h)}\partial^2_\varrho P = -\partial_\varrho\big(\tfrac{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}{\varrho(\underline{h}+h)}\big)(\partial_\varrho P) - \nabla^\mu_{\boldsymbol{x},\varrho}\cdot A^\mu_0\nabla^\mu_{\boldsymbol{x},\varrho}P + Q_0 + \sqrt{\mu}\Lambda Q_1 + \nabla^\mu_{\boldsymbol{x},\varrho}\cdot\boldsymbol{R} =: \widetilde{\boldsymbol{R}} \tag{3.14}$$

where we denote

$$\begin{aligned}\nabla^\mu_{\boldsymbol{x},\varrho}\cdot A^\mu_0\nabla^\mu_{\boldsymbol{x},\varrho}P &:= \nabla^\mu_{\boldsymbol{x},\varrho}\cdot\left(\begin{pmatrix}\frac{\underline{h}+h}{\varrho}\,\mathrm{Id} & \frac{\sqrt{\mu}\nabla_{\boldsymbol{x}}\eta}{\varrho}\\ \frac{\sqrt{\mu}\nabla^\top_{\boldsymbol{x}}\eta}{\varrho} & 0\end{pmatrix}\nabla^\mu_{\boldsymbol{x},\varrho}P\right)\\ &= \frac{\mu}{\varrho}\nabla_{\boldsymbol{x}}\cdot\big(h\nabla_{\boldsymbol{x}}P + (\nabla_{\boldsymbol{x}}\eta)(\partial_\varrho P)\big) + \partial_\varrho\big(\frac{\mu}{\varrho}(\nabla_{\boldsymbol{x}}\eta)\cdot(\nabla_{\boldsymbol{x}}P)\big).\end{aligned}$$

When estimating the above, we use product estimates (Lemma A.2) and then continuous embedding (Lemma A.1), treating differently terms involving $\Delta_{\boldsymbol{x}}P$ or $\nabla_{\boldsymbol{x}}\partial_\varrho P$: for instance

$$\big\|\Lambda^{r-1}(h\Delta_{\boldsymbol{x}}P)\big\|_{L^2(\Omega)} \lesssim \|h\|_{H^{s_0+\frac12,1}}\big\|\Delta_{\boldsymbol{x}}P\big\|_{H^{r-1,0}} + \big\langle\|h\|_{H^{r-\frac12,1}}\big\|\Delta_{\boldsymbol{x}}P\big\|_{H^{s_0,0}}\big\rangle_{r-1>s_0};$$

and terms involving only $\nabla_{\boldsymbol{x}}P$ or $\partial_\varrho P$: for instance

$$\big\|\Lambda^{r-1}((\Delta_{\boldsymbol{x}}\eta)(\partial_\varrho P))\big\|_{L^2(\Omega)} \lesssim \big\|\Delta_{\boldsymbol{x}}\eta\big\|_{H^{s_0+\frac12,1}}\big\|\partial_\varrho P\big\|_{H^{r-1,0}} + \big\langle\big\|\Delta_{\boldsymbol{x}}\eta\big\|_{H^{r-1,0}}\big\|\partial_\varrho P\big\|_{H^{s_0+\frac12,1}}\big\rangle_{r-1>s_0}.$$

We infer, using Lemma A.1, $\mu \in (0,1]$ and $s_0 + \frac32 \le s-1$, that for any $1 \le r \le s$,

$$\begin{aligned}\big\|\partial^2_\varrho P\big\|_{H^{r-1,0}} \lesssim \big\|Q_0\big\|_{H^{r-1,0}} + \big\|Q_1\big\|_{H^{r,0}} + \big\|\boldsymbol{R}\big\|_{H^{r,1}} + \big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{r,0}}\\ + \left(\|h\|_{H^{r,1}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,1}}\right)\big\langle\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{s_0+1,1}}\big\rangle_{r>s_0+1}.\end{aligned} \tag{3.15}$$

By combining (3.13) and (3.15) we obtain

$$\begin{aligned}\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{r,1}} \lesssim \big\|Q_0\big\|_{H^{r,0}} + \big\|Q_1\big\|_{H^{r,0}} + \big\|\boldsymbol{R}\big\|_{H^{r,1}} + \big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{r-1,0}}\\ + C\left\langle\left(\|h\|_{H^{r,1}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,1}}\right)\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{s_0+1,1}}\right\rangle_{r>s_0+1}\end{aligned}$$

which, after finite induction on $1 \le r \le s$ and using (3.11) for the initialization, yields

$$\begin{aligned}\big\|\nabla^\mu_{\boldsymbol{x},\varrho}P\big\|_{H^{r,1}} \lesssim \big\|Q_0\big\|_{H^{r,0}} + \big\|Q_1\big\|_{H^{r,0}} + \big\|\boldsymbol{R}\big\|_{H^{r,1}}\\ + \left(\|h\|_{H^{r,1}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,1}}\right)\times\big\langle\big\|Q_0\big\|_{H^{r-1,0}} + \big\|Q_1\big\|_{H^{r-1,0}} + \big\|\boldsymbol{R}\big\|_{H^{r-1,1}}\big\rangle_{r>s_0+1}.\end{aligned} \tag{3.16}$$

This, together with (3.10), proves (3.8) when $k = 1$.

We now proceed to estimate higher ϱ-derivatives. In what follows, we denote

$$C = C(h_\star, |\underline{h}|_{W^{k-1,\infty}_\varrho}, \|h\|_{H^{s-1,k-1}}, \sqrt{\mu}\|\nabla_{\boldsymbol{x}}\eta\|_{H^{s-1,k-1}}) = C(h_\star, M).$$

Let $2 \leq j \leq k$. By definition, and using $\mu \in (0,1]$, we have

$$\|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s,j}} \leq \|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s,j-1}} + \|\partial_\varrho \nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s-1,j-1}} \lesssim \|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s,j-1}} + \|\partial^2_\varrho P\|_{H^{s-1,j-1}}.$$

We shall also use, when $j \leq k-1$, the corresponding estimate

$$\|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s-1,j}} \lesssim \|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s-1,j-1}} + \|\partial^2_\varrho P\|_{H^{s-2,j-1}}.$$

By using (3.14) (according to which $\partial^2_\varrho P = \frac{\varrho(\underline{h}+h)}{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}\tilde{\boldsymbol{R}}$), and since $1 \leq j-1 \leq s-1$, using Lemma A.3 and Lemma A.5 yields

$$\|\partial^2_\varrho P\|_{H^{s-1,j-1}} \lesssim \|\tfrac{\varrho(\underline{h}+h)}{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}\|_{H^{s-1,j-1}} \|\widetilde{\boldsymbol{R}}\|_{H^{s-1,j-1}} \leq C\|\widetilde{\boldsymbol{R}}\|_{H^{s-1,j-1}}.$$

If moreover $j \leq k-1 \leq s-1$, then

$$\|\partial^2_\varrho P\|_{H^{s-2,j-1}} \lesssim \|\tfrac{\varrho(\underline{h}+h)}{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}\|_{H^{s-2,j-1}} \|\widetilde{\boldsymbol{R}}\|_{H^{s-2,j-1}} \leq C\|\widetilde{\boldsymbol{R}}\|_{H^{s-2,j-1}}.$$

Applying Lemma A.3 and Lemma A.5 to $\widetilde{\boldsymbol{R}}$ defined in (3.14), we obtain

$$\begin{aligned}\|\widetilde{\boldsymbol{R}}\|_{H^{s-1,j-1}} \leq \|Q_0\|_{H^{s-1,j-1}} + \|Q_1\|_{H^{s,j-1}} + \|\boldsymbol{R}\|_{H^{s,j}} \\ + C\|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s,j-1}} + C \times \left(\|h\|_{H^{s,j}} + \sqrt{\mu}\|\nabla_{\boldsymbol{x}}\eta\|_{H^{s,j}}\right) \|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s-1,j-1}}\end{aligned}$$

and, if moreover $j \leq k-1 \leq s-1$,

$$\|\widetilde{\boldsymbol{R}}\|_{H^{s-2,j-1}} \leq \|Q_0\|_{H^{s-2,j-1}} + \|Q_1\|_{H^{s-1,j-1}} + \|\boldsymbol{R}\|_{H^{s-1,j}} + C\|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s-1,j-1}}.$$

From the second set of inequalities, (3.16) with $r = s-1$ and finite induction on $2 \leq j \leq k-1$ we infer

$$\|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s-1,j}} \leq C\left(\|Q_0\|_{H^{s-1,j-1}} + \|Q_1\|_{H^{s-1,j-1}} + \|\boldsymbol{R}\|_{H^{s-1,j}}\right).$$

Then, from the first set of inequalities, (3.16) with $r = s$ and the previous result, we infer by finite induction on $2 \leq j \leq k$

$$\begin{aligned}\|\nabla^\mu_{\boldsymbol{x},\varrho} P\|_{H^{s,j}} \leq C\left(\|Q_0\|_{H^{s,j-1}} + \|Q_1\|_{H^{s,j-1}} + \|\boldsymbol{R}\|_{H^{s,j}}\right) \\ + C\left(\|h\|_{H^{s,j}} + \sqrt{\mu}\|\nabla_{\boldsymbol{x}}\eta\|_{H^{s,j}}\right)\left(\|Q_0\|_{H^{s-1,j-2}} + \|Q_1\|_{H^{s-1,j-2}} + \|\boldsymbol{R}\|_{H^{s-1,j-1}}\right).\end{aligned}$$

The result is proved. □

We now apply Lemma 3.1 to obtain several estimates on the solution to (3.5)-(3.6).

COROLLARY 3.2. *Let $s_0 > d/2$, $s, k \in \mathbb{N}$ such that $s \geq s_0 + \frac{5}{2}$ and $2 \leq k \leq s$. Let $h_\star, M > 0$. There exists $C > 0$ such that for any $\mu \in (0,1]$ and $\kappa \in \mathbb{R}$, for any $(\underline{h}, \underline{\boldsymbol{u}}) \in W^{k,\infty}((\rho_0,\rho_1))^2$, and for any $(h, \boldsymbol{u}, w) \in H^{s+2,k}(\Omega) \times H^{s,k}(\Omega)^d \times H^{s,k-1}(\Omega)$ satisfying*

- *the following bound*

$$|\underline{h}|_{W^{k,\infty}_\varrho} + |\underline{\boldsymbol{u}}'|_{W^{k-1,\infty}_\varrho} + \|h\|_{H^{s-1,k-1}} + \sqrt{\mu}\|\nabla_{\boldsymbol{x}}\eta\|_{H^{s-1,k-1}} + \|\boldsymbol{u}\|_{H^{s,k}} + \sqrt{\mu}\|w\|_{H^{s,k-1}} \leq M;$$

- *the stable stratification assumption*

$$\inf_{(\boldsymbol{x},\varrho)\in\Omega} \underline{h}(\varrho) + h(\boldsymbol{x},\varrho) \geq h_\star;$$

- *the incompressibility condition*

$$-(\underline{h}+h)\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} - (\nabla_{\boldsymbol{x}}\eta) \cdot (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}) + \partial_\varrho w = 0,$$

there exists a unique solution $P \in H^{s+1,k+1}(\Omega)$ *to* (3.5)-(3.6) *and one has*

$$\begin{aligned}\big\|P\big\|_{L^2(\Omega)} + \big\|\nabla^\mu_{\boldsymbol{x},\varrho} P\big\|_{H^{s,k}} &\le C\,(1+\big\|h\big\|_{H^{s,k}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s,k}}) \\ &\times \big(\big\|h\big\|_{H^{s,k}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s,k}} + \big\|(\boldsymbol{u},\boldsymbol{u}_\star)\big\|_{H^{s,k}} + \sqrt{\mu}\big\|(w,w_\star)\big\|_{H^{s,k-1}}\big) \quad (3.17)\end{aligned}$$

where we recall the notations $\boldsymbol{u}_\star := -\kappa\frac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}$ *and* $w_\star := \kappa\Delta_{\boldsymbol{x}}\eta - \kappa\frac{\nabla_{\boldsymbol{x}} h\cdot\nabla_{\boldsymbol{x}}\eta}{\underline{h}+h}$.

Moreover, decomposing

$$P = P_{\rm h} + P_{\rm nh}, \qquad P_{\rm h} := \int_{\rho_0}^{\varrho} \varrho' h(\cdot,\varrho')\,\mathrm{d}\varrho',$$

and assuming (in addition to the above) that

$$\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s-1,k-1}} + \big|\eta\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}} \le M$$

we have

$$\begin{aligned}&\big\|P_{\rm nh}\big\|_{L^2(\Omega)} + \big\|\nabla^\mu_{\boldsymbol{x},\varrho} P_{\rm nh}\big\|_{H^{s,k}} \le C\,(1+\big\|h\big\|_{H^{s,k}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s,k}}) \\ &\times\Big(\sqrt{\mu}\big\|h\big\|_{H^{s,k}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s,k}} + \sqrt{\mu}\big|\eta\big|_{\varrho=\rho_0}\big|_{H^{s+1}_{\boldsymbol{x}}} + \big\|(\boldsymbol{u},\boldsymbol{u}_\star)\big\|_{H^{s,k}} + \sqrt{\mu}\big\|(w,w_\star)\big\|_{H^{s,k-1}}\Big), \quad (3.18)\end{aligned}$$

and

$$\begin{aligned}&\big\|P_{\rm nh}\big\|_{L^2(\Omega)} + \big\|\nabla^\mu_{\boldsymbol{x},\varrho} P_{\rm nh}\big\|_{H^{s-1,k-1}} \le C\sqrt{\mu} \\ &\times\Big(\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s-1,k-1}} + \big|\eta\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}} + \big\|(\boldsymbol{u},\boldsymbol{u}_\star)\big\|_{H^{s,k}} + \sqrt{\mu}\big\|(w,w_\star)\big\|_{H^{s,k-1}}\Big), \quad (3.19)\end{aligned}$$

and, setting $\Lambda^\mu = 1+\sqrt{\mu}|D|$,

$$\begin{aligned}\big\|\nabla^\mu_{\boldsymbol{x},\varrho} P_{\rm nh}\big\|_{H^{s-1,k-1}} &\le C\,\mu\Big(\big\|(\Lambda^\mu)^{-1}h\big\|_{H^{s,k-1}} + \big\|(\Lambda^\mu)^{-1}\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s,k}} + \big|(\Lambda^\mu)^{-1}\eta\big|_{\varrho=\rho_0}\big|_{H^{s+1}_{\boldsymbol{x}}} \\ &+ \big(\big|\underline{\boldsymbol{u}}'\big|_{W^{s-1,\infty}_\varrho} + \big\|\boldsymbol{u}\big\|_{H^{s,k}}\big)\big(\big\|(\boldsymbol{u},\boldsymbol{u}_\star)\big\|_{H^{s,k}} + \big\|(w,w_\star)\big\|_{H^{s,k-1}}\big) + \big\|\boldsymbol{u}_\star\big\|_{H^{s,k}}\big\|w\big\|_{H^{s,k-1}}\Big). \quad (3.20)\end{aligned}$$

PROOF. In view of Lemma 3.1, we shall first estimate (RHS), defined in (3.5). We decompose

$$(\mathrm{RHS}) = R_1 + R_2$$

where R_1 is constituted by terms involving maximum one derivative on $h,\eta,\boldsymbol{u},\boldsymbol{u}_\star,w,w_\star$, while

$$\begin{aligned}R_2 := -(\underline{h}+h)\big((\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}}(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u})\big) - (\nabla_{\boldsymbol{x}}\eta)\cdot\big((\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}}\partial_\varrho\boldsymbol{u}\big) \\ + \big(\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star\big)\cdot\nabla_{\boldsymbol{x}}\partial_\varrho w - \big((\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\cdot\nabla_{\boldsymbol{x}}(\nabla_{\boldsymbol{x}}\eta)\big)\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}).\end{aligned}$$

Appealing to the incompressibility condition

$$-(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} - (\nabla_{\boldsymbol{x}}\eta)\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}) + \partial_\varrho w = 0,$$

we have simply

$$R_2 = \big((\underline{\boldsymbol{u}}+\boldsymbol{u}+\boldsymbol{u}_\star)\cdot(\nabla_{\boldsymbol{x}}h)\big)(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}).$$

As a matter of fact, this term compensates with the second addend of R_1, so that contributions from h are not differentiated. By inspecting the remaining terms and using Lemma A.3, we infer that for any $r \ge s_0+1/2$ and $1\le j\le r\le s-1$,

$$\begin{aligned}&\big\|(\mathrm{RHS})\big\|_{H^{r,j}} \lesssim (1+\big|\underline{h}\big|_{W^{j,\infty}_\varrho} + \big\|h\big\|_{H^{r,j}} + \big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,j}}) \\ &\times\Big(\big(\big|\underline{\boldsymbol{u}}'\big|_{W^{j,\infty}_\varrho} + \big\|\boldsymbol{u}\big\|_{H^{r+1,j+1}}\big)\big(\big\|(\boldsymbol{u},\boldsymbol{u}_\star)\big\|_{H^{r+1,j+1}} + \big\|(w,w_\star)\big\|_{H^{r+1,j}}\big) + \big\|w\big\|_{H^{r+1,j}}\big\|\boldsymbol{u}_\star\big\|_{H^{r+1,j+1}}\Big).\end{aligned} \quad (3.21)$$

Owing to the fact that contributions from $(h,\eta,w,w^\star)$ to (RHS) are affine, and $\mu\in(0,1]$, we have for any $r\geq s_0+3/2$ and $2\leq j\leq r\leq s$

$$\begin{aligned}\sqrt{\mu}\big\|(\mathrm{RHS})\big\|_{H^{r-1,j-1}}&\lesssim (1+|\underline{h}|_{W^{j-1,\infty}_\varrho}+\sqrt{\mu}\big\|h\big\|_{H^{r-1,j-1}}+\sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r-1,j-1}})\\&\times\Big((|\underline{\boldsymbol{u}}'|_{W^{j-1,\infty}_\varrho}+\big\|\boldsymbol{u}\big\|_{H^{r,j}})\big(\big\|(\boldsymbol{u},\boldsymbol{u}_\star)\big\|_{H^{r,j}}+\sqrt{\mu}\big\|(w,w_\star)\big\|_{H^{r,j-1}}\big)+\sqrt{\mu}\big\|w\big\|_{H^{r,j-1}}\big\|\boldsymbol{u}_\star\big\|_{H^{r,j}}\Big).\end{aligned}\tag{3.22}$$

For the first estimate, we write (3.6) as (3.7) with $Q_0=0$, $Q_1=\sqrt{\mu}\Lambda^{-1}(\mathrm{RHS})$ and

$$\boldsymbol{R}=\begin{pmatrix}-\sqrt{\mu}\underline{h}\nabla_{\boldsymbol{x}}\eta\\ \frac{h}{\underline{h}+h}(1+\mu|\nabla_{\boldsymbol{x}}\eta|^2)-\mu|\nabla_{\boldsymbol{x}}\eta|^2\end{pmatrix}$$

where we used the identities

$$\frac{1+\mu|\nabla_{\boldsymbol{x}}\eta|^2}{\varrho(\underline{h}+h)}\partial_\varrho P_{\mathrm{eq}}=\frac{\underline{h}(1+\mu|\nabla_{\boldsymbol{x}}\eta|^2)}{\underline{h}+h}=1+\mu|\nabla_{\boldsymbol{x}}\eta|^2-\frac{h}{\underline{h}+h}(1+\mu|\nabla_{\boldsymbol{x}}\eta|^2).$$

Product (Lemma A.3) and composition (Lemma A.6) estimates yield if $r\geq s_0+1/2$ and $1\leq j\leq r\leq s-1$

$$\big\|\boldsymbol{R}\big\|_{H^{r,j}}\leq C(h_\star,|\underline{h}|_{W^{j,\infty}_\varrho},\big\|h\big\|_{H^{r,j}},\sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,j}})\times\big(\big\|h\big\|_{H^{r,j}}+\sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,j}}\big);\tag{3.23}$$

and, if $r\geq s_0+3/2$ and $2\leq j\leq r\leq s$, using the tame estimates, we obtain

$$\big\|\boldsymbol{R}\big\|_{H^{r,j}}\leq C(h_\star,|\underline{h}|_{W^{j,\infty}_\varrho},\big\|h\big\|_{H^{r-1,j-1}},\sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r-1,j-1}})\times\big(\big\|h\big\|_{H^{r,j}}+\sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,j}}\big).\tag{3.24}$$

Plugging in (3.8) the estimates (3.22) and (3.24) with $(r,j)=(s,k)$, and (3.23) with $(r,j)=(s-1,k-1)$, yields (3.17).

For the next set of estimates, we notice that, by (3.6), $P-P_{\mathrm{h}}$ satisfies (3.7) with $\boldsymbol{e}_{d+1}\cdot\boldsymbol{R}\big|_{\varrho=\rho_1}=0$ and $Q_0+\sqrt{\mu}\Lambda Q_1+\nabla^\mu_{\boldsymbol{x},\varrho}\cdot\boldsymbol{R}=\mu(\mathrm{RHS})+\nabla^\mu_{\boldsymbol{x},\varrho}\cdot\widetilde{\boldsymbol{R}}$ where

$$\widetilde{\boldsymbol{R}}:=-\begin{pmatrix}\sqrt{\mu}\varrho^{-1}(\underline{h}+h)\nabla_{\boldsymbol{x}}\psi\\ \mu\varrho^{-1}(\nabla_{\boldsymbol{x}}\eta)\cdot(\nabla_{\boldsymbol{x}}\psi)\end{pmatrix},\quad \psi:=\int_{\rho_0}^{\varrho}\eta(\cdot,\varrho')\,\mathrm{d}\varrho'+\rho_0\eta\big|_{\varrho=\rho_0}.$$

Indeed, we have immediately $\boldsymbol{e}_{d+1}\cdot\widetilde{\boldsymbol{R}}\big|_{\varrho=\rho_1}=0=\partial_\varrho\big(P-P_{\mathrm{h}}\big)\big|_{\varrho=\rho_1}$ and we infer

$$-\nabla^\mu_{\boldsymbol{x},\varrho}\cdot\widetilde{\boldsymbol{R}}=\nabla^\mu_{\boldsymbol{x},\varrho}\cdot\big(A^\mu\nabla^\mu_{\boldsymbol{x},\varrho}(\underline{P}_{\mathrm{eq}}+P_{\mathrm{h}})\big)$$

from (integrating by parts as in (1.8b))

$$P_{\mathrm{h}}:=\int_{\rho_0}^{\varrho}\varrho' h(\cdot,\varrho')\,\mathrm{d}\varrho'=-\varrho\eta+\int_{\rho_0}^{\varrho}\eta(\cdot,\varrho')\,\mathrm{d}\varrho'+\rho_0\eta\big|_{\varrho=\rho_0}.\tag{3.25}$$

Using that by product estimates (Lemma A.3), we infer immediately for any $r\geq s_0+1/2$ and $1\leq j\leq r$

$$\big\|\widetilde{\boldsymbol{R}}\big\|_{H^{r,j}}\lesssim\Big(|\underline{h}|_{W^{j,\infty}_\varrho}+\big\|h\big\|_{H^{r,j}}+\sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,j}}\Big)\Big(\sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,j}}+\sqrt{\mu}|\eta|_{\varrho=\rho_0}|_{H^{r+1}_{\boldsymbol{x}}}\Big).\tag{3.26}$$

and using tame product estimates if $r\geq s_0+3/2$ and $2\leq j\leq r$;

$$\begin{aligned}\big\|\widetilde{\boldsymbol{R}}\big\|_{H^{r,j}}\lesssim\Big(|\underline{h}|_{W^{j,\infty}_\varrho}+\big\|h\big\|_{H^{r-1,j-1}}+\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r-1,j-1}}+|\eta|_{\varrho=\rho_0}|_{H^{r}_{\boldsymbol{x}}}\Big)\\\times\Big(\sqrt{\mu}\big\|h\big\|_{H^{r,j}}+\sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,j}}+\sqrt{\mu}|\eta|_{\varrho=\rho_0}|_{H^{r+1}_{\boldsymbol{x}}}\Big).\end{aligned}\tag{3.27}$$

Moreover, using again the tame estimates, we have for any $r\geq s_0+1/2$ and $1\leq j\leq r$,

$$\begin{aligned}\big\|\nabla^\mu_{\boldsymbol{x},\varrho}\cdot\widetilde{\boldsymbol{R}}\big\|_{H^{r,j}}\lesssim\mu\Big(|\underline{h}|_{W^{j,\infty}_\varrho}+\big\|h\big\|_{H^{r,j}}+\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r,j}}+|\eta|_{\varrho=\rho_0}|_{H^{r+1}_{\boldsymbol{x}}}\Big)\\\times\Big(\big\|h\big\|_{H^{r+1,j}}+\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{r+1,j+1}}+|\eta|_{\varrho=\rho_0}|_{H^{r+2}_{\boldsymbol{x}}}\Big).\end{aligned}\tag{3.28}$$

Finally, recalling $\Lambda^\mu = 1 + \sqrt{\mu}|D|$ and introduce

$$\widetilde{\boldsymbol{R}}_0 := -\begin{pmatrix} \sqrt{\mu}\varrho^{-1}(\underline{h} + (\Lambda^\mu)^{-1}h)((\Lambda^\mu)^{-1}\nabla_{\boldsymbol{x}}\psi) \\ \mu\varrho^{-1}((\Lambda^\mu)^{-1}\nabla_{\boldsymbol{x}}\eta)\cdot((\Lambda^\mu)^{-1}\nabla_{\boldsymbol{x}}\psi)\end{pmatrix}$$

and obtain, as for (3.26) and (3.28) and using that $\mathrm{Id} - (\Lambda^\mu)^{-1} = \sqrt{\mu}|D|(\Lambda^\mu)^{-1}$ and $\|(\Lambda^\mu)^{-1}\|_{L^2_{\boldsymbol{x}}\to L^2_{\boldsymbol{x}}} = 1$, that for any $r \geq s_0 + 1/2$ and $1 \leq j \leq r$, one has

$$\begin{aligned}\|\widetilde{\boldsymbol{R}} - \widetilde{\boldsymbol{R}}_0\|_{H^{r,j}} + \|\nabla^\mu_{\boldsymbol{x},\varrho}\cdot\widetilde{\boldsymbol{R}}_0\|_{H^{r,j}} \lesssim \mu\Big(|\underline{h}|_{W^{j,\infty}_\varrho} + \|h\|_{H^{r,j}} + \|\nabla_{\boldsymbol{x}}\eta\|_{H^{r,j}} + |\eta|_{\varrho=\rho_0}|_{H^{r+1}_{\boldsymbol{x}}}\Big) \\ \times\Big(\|(\Lambda^\mu)^{-1}h\|_{H^{r+1,j}} + \|(\Lambda^\mu)^{-1}\nabla_{\boldsymbol{x}}\eta\|_{H^{r+1,j}} + |(\Lambda^\mu)^{-1}\eta|_{\varrho=\rho_0}|_{H^{r+2}_{\boldsymbol{x}}}\Big). \quad (3.29)\end{aligned}$$

We obtain (3.18) by setting $Q_0 = 0$, $Q_1 = \sqrt{\mu}\Lambda^{-1}(\mathrm{RHS})$ and $\boldsymbol{R} = \widetilde{\boldsymbol{R}}$, and plugging in (3.8) the estimates (3.22) and (3.27) with $(r,j) = (s,k)$, and (3.26) with $(r,j) = (s-1,k-1)$. For (3.19), we set instead $Q_0 = \mu(\mathrm{RHS})$ and $Q_1 = 0$, and plug in (3.9) the estimates (3.22) with $(r,j) = (s,k)$, and (3.26) with $(r,j) = (s-1,k-1)$. Finally, for (3.20), we set $Q_0 = \mu(\mathrm{RHS}) + \nabla^\mu_{\boldsymbol{x},\varrho}\cdot\widetilde{\boldsymbol{R}}_0$ and $Q_1 = 0$, $\boldsymbol{R} = \widetilde{\boldsymbol{R}} - \widetilde{\boldsymbol{R}}_0$, and plug in (3.9) the estimates (3.21) with $(r,j) = (s,k)$, and (3.29) with $(r,j) = (s-1,k-1)$. □

3.2. Small-time well-posedness. We infer small-time existence and uniqueness of regular solutions to the Cauchy problem associated with the non-hydrostatic problem, (3.1), proceeding as for the hydrostatic system in Section 2, that is considering the system as the combination of a transport-diffusion equation and transport equations, coupled through order-zero source terms (by the first estimate in Corollary 3.2). Specifically, we rewrite (3.1) as

$$\begin{aligned}\partial_t h + \nabla_{\boldsymbol{x}}\cdot\big((\underline{h}+h)(\underline{\boldsymbol{u}}+\boldsymbol{u})\big) &= \kappa\Delta_{\boldsymbol{x}}h, \\ \partial_t\boldsymbol{u} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u} + \frac{1}{\varrho}\nabla_{\boldsymbol{x}}P + \big(1 + \frac{\partial_\varrho P - \varrho h}{\varrho(\underline{h}+h)}\big)\nabla_{\boldsymbol{x}}\eta &= 0, \\ \partial_t w + \big(\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h}\big)\cdot\nabla_{\boldsymbol{x}}w - \frac{1}{\mu}\frac{\partial_\varrho P - \varrho h}{\varrho(\underline{h}+h)} &= 0,\end{aligned} \quad (3.30)$$

where $\eta(\cdot,\varrho) = \int_\varrho^{\rho_1} h(\cdot,\varrho')\,\mathrm{d}\varrho'$ and P is defined by Corollary 3.2. Systems (3.1) and (3.30) are equivalent (for sufficiently regular data) by the computations of Section 3.1, and in particular regular solutions to (3.30) satisfy the boundary condition $w|_{\varrho=\rho_1} = 0$ and the incompressibility constraint

$$(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} + \nabla_{\boldsymbol{x}}\eta\cdot(\underline{\boldsymbol{u}}' + \partial_\varrho\boldsymbol{u}) - \partial_\varrho w = 0 \quad (3.31)$$

provided these identities hold initially.

PROPOSITION 3.3. *Let $s_0 > d/2$, $s,k \in \mathbb{N}$ such that $s \geq s_0 + \frac{5}{2}$ and $2 \leq k \leq s$. Let $h_\star, M, \mu, \kappa > 0$ and $C_0 > 1$. There exists $T > 0$ such that for any $(\underline{h},\underline{\boldsymbol{u}}) \in W^{k,\infty}((\rho_0,\rho_1))^2$, and for any initial data $(h_0,\boldsymbol{u}_0,w_0) \in H^{s+1,k}(\Omega)\times H^{s,k}(\Omega)^d \times H^{s,k}(\Omega)$ satisfying*

- *the following bounds (where $\eta_0(\cdot,\varrho) := \int_\varrho^{\rho_1} h_0(\cdot,\varrho')\,\mathrm{d}\varrho'$)*

$$|\underline{h}|_{W^{k,\infty}_\varrho} + |\underline{\boldsymbol{u}}'|_{W^{k-1,\infty}_\varrho} \leq M, \qquad M_0 := \|h_0\|_{H^{s,k}} + \|\nabla_{\boldsymbol{x}}\eta_0\|_{H^{s,k}} + \|\boldsymbol{u}_0\|_{H^{s,k}} + \|w_0\|_{H^{s,k}} \leq M,$$

- *the stable stratification assumption*

$$\inf_{(\boldsymbol{x},\varrho)\in\Omega} \underline{h}(\varrho) + h_0(\boldsymbol{x},\varrho) \geq h_\star,$$

- *the boundary condition $w_0|_{\varrho=\rho_1} = 0$ and the incompressibility condition*

$$-(\underline{h}+h_0)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}_0 - (\nabla_{\boldsymbol{x}}\eta_0)\cdot(\underline{\boldsymbol{u}}' + \partial_\varrho\boldsymbol{u}_0) + \partial_\varrho w_0 = 0,$$

there exists a unique $(h,\boldsymbol{u},w)\in L^\infty(0,T;H^{s,k}(\Omega)^{2+d})\cap\mathcal{C}^0([0,T];H^{s',k}(\Omega)^{2+d})$ *(for any* $0\leq s'<s$*) and* $P\in L^2(0,T;H^{s+1,k+1}(\Omega))$ *solution to* (3.30) *and* $(h,\boldsymbol{u},w)\big|_{t=0}=(h_0,\boldsymbol{u}_0,w_0)$*. Moreover, one has* $\eta\in L^\infty(0,T;H^{s+1,k}(\Omega))$ *and* $(h,\nabla_{\boldsymbol{x}}\eta)\in L^2(0,T;H^{s+1,k}(\Omega))$ *and*

$$\mathcal{F}_{s,k,T}:=\big\|h\big\|_{L^\infty(0,T;H^{s,k})}+\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{L^\infty(0,T;H^{s,k})}+\big\|\boldsymbol{u}\big\|_{L^\infty(0,T;H^{s,k})}+\big\|w\big\|_{L^\infty(0,T;H^{s,k})}$$
$$+c_0\kappa^{1/2}\big\|h\big\|_{L^2(0,T;H^{s,k})}+c_0\kappa^{1/2}\big\|\nabla^2_{\boldsymbol{x}}\eta\big\|_{L^2(0,T;H^{s,k})}\leq C_0M_0$$

with c_0 *a universal constant.*

PROOF. Since a very similar proof has been detailed in the hydrostatic framework in Section 2, we will only briefly sketch the main arguments. As aforementioned, thanks to Corollary 3.2, we may consider the contributions of the pressure as zero-order source terms in the energy space displayed in the statement, and (3.30) is then interpreted as a standard set of evolution equations. We now explain how to infer the necessary bounds on all contributions to $\mathcal{F}_{s,k,T}$, assuming enough regularity.

The desired control of $\big\|h\big\|_{L^\infty(0,T;H^{s,k})}+c_0\kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}h\big\|_{L^2(0,T;H^{s,k})}$ is a direct consequence of the first equation of (3.30), and the regularization properties of the heat semigroup already summoned in Proposition 2.1. The corresponding control of $\big\|\nabla_x\eta\big\|_{L^\infty(0,T;H^{s,k})}+c_0\kappa^{1/2}\big\|\nabla^2_{\boldsymbol{x}}\eta\big\|_{L^2(0,T;H^{s,k})}$ demands an additional structure. We recall (see (1.6) or (3.3)) that by the identity (3.31) and integrating the first equation of (3.30), one has

$$\partial_t\eta+(\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\eta-w=\kappa\Delta_{\boldsymbol{x}}\eta. \tag{3.32}$$

By the regularization properties of the heat semigroup, we infer (with c_0 a universal constant)

$$\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{L^\infty(0,T;H^{s,k})}+c_0\kappa^{1/2}\big\|\nabla^2_{\boldsymbol{x}}\eta\big\|_{L^2(0,T;H^{s,k})}\leq\big\|\nabla_{\boldsymbol{x}}\eta_0\big\|_{H^{s,k}}+\frac{1}{c_0\kappa^{1/2}}\big\|(\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\eta-w\big\|_{L^2(0,T;H^{s,k})},$$

and the right-hand side is estimated by product estimates (Lemma A.3). Finally, the desired *a priori* estimates on $\big\|\boldsymbol{u}\big\|_{L^\infty(0,T;H^{s,k})}$ and $\big\|w\big\|_{L^\infty(0,T;H^{s,k})}$ for sufficiently regular solutions follow by the energy method (that is integrating by parts in the variable $\boldsymbol{x}$) on the second and third equations of (3.30), which can be seen as transport equations with source terms. More precisely, by Corollary 3.2, we have the existence and uniqueness of $P\in L^2(0,T;H^{s+1,k+1}(\Omega))$, satisfying the bound

$$\big\|P\big\|_{L^2(0,T;H^{s+1,k+1})}\leq C(h_\star,\mu,\kappa,M,\mathcal{F}_{s,k,T})\mathcal{F}_{s,k,T}.$$

Moreover, the advection velocity is controlled (using Lemma A.1, $s-2\geq s_0+\frac12$, $k\geq1$) by

$$\big\|\nabla\cdot\big(\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h}\big)\big\|_{L^\infty(0,T;L^\infty(\Omega))}\leq C(h_\star,\kappa,\mathcal{F}_{s,k,T}),$$

and using commutator (Lemma A.8) and composition (Lemma A.6) estimates, one has for any $f\in H^{s,k}(\Omega)$, and any $\alpha\in\mathbb{N}^d$, $j\in\mathbb{N}$ with $0\leq j\leq k$ and $|\alpha|+j\leq s$,

$$\big\|[\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho,\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h}]\nabla_{\boldsymbol{x}}f\big\|_{L^2(0,T;L^2(\Omega))}\leq C(h_\star,\kappa,M,\mathcal{F}_{s,k,T})\big\|f\big\|_{H^{s,k}}.$$

It follows

$$\big\|\boldsymbol{u}\big\|_{H^{s,k}}+\big\|w\big\|_{H^{s,k}}\leq\Big(\big\|\boldsymbol{u}_0\big\|_{H^{s,k}}+\big\|w_0\big\|_{H^{s,k}}+C\sqrt{T}\Big)\exp(C\sqrt{T}),$$

with $C=C(h_\star,\mu,\kappa,M,\mathcal{F}_{s,k,T})$.

Altogether, and using standard continuity arguments, we find that for any $C_0>1$ we can restrict the time $T=T(h_\star,\mu,\kappa,M,C_0)>0$ so that all sufficiently regular solutions to (3.30) satisfy the bound $\mathcal{F}_{s,k,T}\leq C_0M_0$. We may infer the existence of solutions using for instance the parabolic regularization approach (see the closing paragraph of Section 2), and uniqueness is straightforward. This concludes the proof. □

REMARK 3.4. Proposition 3.3 does not provide any lower bound on the time of existence (and control) of solutions with respect to either $\mu\ll1$ or $\kappa\ll1$, hence the "small-time" terminology.

3.3. Quasi-linearization of the non-hydrostatic system. In this section we extract the leading-order terms of the equations satisfied by spatial derivatives of solutions to the system (3.30). This will allow us to obtain improved energy estimates in the subsequent section. Notice that starting from here, our study is restricted to the situation $k = s$.

LEMMA 3.5. *Let $s, k \in \mathbb{N}$ such that $k = s > \frac{5}{2} + \frac{d}{2}$ and $\underline{M}, M, h_\star > 0$. Then there exists $C > 0$ such that for any $\mu, \kappa \in (0,1]$, and for any $(\underline{h}, \underline{\boldsymbol{u}}) \in W^{k,\infty}((\rho_0,\rho_1)) \times W^{k+1,\infty}((\rho_0,\rho_1))$ satisfying*

$$\big|\underline{h}\big|_{W^{k,\infty}_\varrho} + \big|\underline{\boldsymbol{u}}'\big|_{W^{k,\infty}_\varrho} \leq \underline{M}\,;$$

and any $(h, \boldsymbol{u}, w) \in L^\infty(0,T; H^{s,k}(\Omega)^{d+2})$ solution to (3.30) (with P defined by Corollary 3.2) with any $T > 0$ and satisfying for almost any $t \in [0,T]$,

$$\big\|h(t,\cdot)\big\|_{H^{s-1,k-1}} + \big\|\eta(t,\cdot)\big\|_{H^{s,k}} + \big|\eta\big|_{\varrho=\rho_0}(t,\cdot)\big|_{H^s_{\boldsymbol{x}}} + \big\|\boldsymbol{u}(t,\cdot)\big\|_{H^{s,k}} + \sqrt{\mu}\big\|w(t,\cdot)\big\|_{H^{s,k}} + \kappa^{1/2}\big\|h(t,\cdot)\big\|_{H^{s,k}} \leq M$$

(where $\eta(t,\boldsymbol{x},\varrho) := \int_\varrho^{\rho_1} h(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho'$) and

$$\inf_{(t,\boldsymbol{x},\varrho)\in(0,T)\times\Omega} \underline{h}(\varrho) + h(t,\boldsymbol{x},\varrho) \geq h_\star,$$

the following results hold.

Denote, for any multi-index $\alpha \in \mathbb{N}^d$ and any $j \in \mathbb{N}$ such that $|\alpha| + j \leq s$, $h^{(\alpha,j)} = \partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha h$, $\eta^{(\alpha,j)} = \partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha \eta$, $\boldsymbol{u}^{(\alpha,j)} = \partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha \boldsymbol{u}$, $w^{(\alpha,j)} = \partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha w$, and $P^{(\alpha)}_{\mathrm{nh}} = \partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha P_{\mathrm{nh}}$, with

$$P_{\mathrm{nh}} := P - P_{\mathrm{h}}, \qquad P_{\mathrm{h}} := \int_{\rho_0}^{\varrho} \varrho' h(\cdot,\varrho')\,\mathrm{d}\varrho'.$$

We have

$$\begin{aligned}
\partial_t \eta^{(\alpha,j)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)} - w^{(\alpha,j)} &= \kappa \Delta_{\boldsymbol{x}} \eta^{(\alpha,j)} + \widetilde{R}_{\alpha,j},\\
\partial_t \eta^{(\alpha,j)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)} + \Big\langle \int_\varrho^{\rho_1} (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)}\,\mathrm{d}\varrho' &\\
+ \int_\varrho^{\rho_1} (\underline{h} + h) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(\alpha,j)}\,\mathrm{d}\varrho' \Big\rangle_{j=0} &= \kappa \Delta_{\boldsymbol{x}} \eta^{(\alpha,j)} + R_{\alpha,j},\\
\partial_t \boldsymbol{u}^{(\alpha,j)} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}) \cdot \nabla_{\boldsymbol{x}}\big) \boldsymbol{u}^{(\alpha,j)} + \Big\langle \frac{\rho_0}{\varrho} \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)}\big|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)}\,\mathrm{d}\varrho' \Big\rangle_{j=0} &\\
+ \frac{1}{\varrho} \nabla_{\boldsymbol{x}} P^{(\alpha,j)}_{\mathrm{nh}} + \frac{\nabla_{\boldsymbol{x}} \eta}{\varrho(\underline{h}+h)} \partial_\varrho P^{(\alpha,j)}_{\mathrm{nh}} &= \boldsymbol{R}^{\mathrm{nh}}_{\alpha,j},\\
\sqrt{\mu}\Big(\partial_t w^{(\alpha,j)} + (\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}) \cdot \nabla_{\boldsymbol{x}} w^{(\alpha,j)}\Big) - \frac{1}{\sqrt{\mu}} \frac{\partial_\varrho P^{(\alpha,j)}_{\mathrm{nh}}}{\varrho(\underline{h}+h)} &= R^{\mathrm{nh}}_{\alpha,j},\\
-\partial_\varrho w^{(\alpha)} + (\underline{h} + h) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(\alpha,j)} + (\nabla_{\boldsymbol{x}} \eta) \cdot (\partial_\varrho \boldsymbol{u}^{(\alpha,j)}) &\\
+ (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}) h^{(\alpha,j)} + (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)} &= R^{\mathrm{div}}_{\alpha,j},
\end{aligned} \tag{3.33a}$$

where for every $t \in [0,T]$, one has $(R_{\alpha,j}(t,\cdot), \boldsymbol{R}^{\mathrm{nh}}_{\alpha,j}(t,\cdot), R^{\mathrm{nh}}_{\alpha,j}(t,\cdot), R^{\mathrm{div}}_{\alpha,j}(t,\cdot)) \in L^2(\Omega)^{d+3}$, additionally $R_{\alpha,0}(t,\cdot) \in \mathcal{C}((\rho_0,\rho_1); L^2(\mathbb{R}^d))$ and

$$\begin{aligned}
&\big\|R_{\alpha,j}\big\|_{L^2(\Omega)} + \big|R_{\alpha,0}|_{\varrho=\rho_0}\big|_{L^2_{\boldsymbol{x}}} + \big\|\widetilde{R}_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|R^{\mathrm{div}}_{\alpha,j}\big\|_{L^2(\Omega)} \leq C\,M\,,\\
&\big\|\boldsymbol{R}^{\mathrm{nh}}_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|R^{\mathrm{nh}}_{\alpha,j}\big\|_{L^2(\Omega)} \leq C\,M\,\big(1 + \kappa\big\|\nabla_{\boldsymbol{x}} h\big\|_{H^{s,k}}\big)\\
&\qquad + C\,\big(\big\|h\big\|_{H^{s,k}} + \sqrt{\mu}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s,k}}\big)\big(M + \big\|\boldsymbol{u}_\star\big\|_{H^{s,k}} + \sqrt{\mu}\big\|w_\star\big\|_{H^{s,k-1}}\big)\,,
\end{aligned} \tag{3.33b}$$

and

$$\partial_t h^{(\alpha,j)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} h^{(\alpha,j)} = \kappa \Delta_{\boldsymbol{x}} h^{(\alpha,j)} + r_{\alpha,j} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{r}_{\alpha,j}, \tag{3.34a}$$

where for every $t \in [0,T]$, *one has* $(r_{\alpha,j}(t,\cdot), \boldsymbol{r}_{\alpha,j}(t,\cdot)) \in L^2(\Omega)^{1+d}$ *and*

$$\kappa^{1/2} \|r_{\alpha,j}\|_{L^2(\Omega)} + \|\boldsymbol{r}_{\alpha,j}\|_{L^2(\Omega)} \leq C\, M. \tag{3.34b}$$

PROOF. Let us first point out that the estimates for $|R_{\alpha,0}|_{\varrho=\rho_0}|_{L^2_{\boldsymbol{x}}}$, $\|R_{\alpha,j}\|_{L^2(\Omega)}$, $\|r_{\alpha,j}\|_{L^2(\Omega)}$ and $\|\boldsymbol{r}_{\alpha,j}\|_{L^2(\Omega)}$ have been stated and proved in Lemma 2.3. Thus we only need to focus on the other terms. In the following, we denote $s_0 = s - \frac{5}{2} > \frac{d}{2}$.

Using the identity (already pointed out in (3.32))

$$\partial_t \eta + (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} \eta - w = \kappa \Delta_{\boldsymbol{x}} \eta,$$

and the commutator estimate in Lemma A.8, we find immediately

$$\widetilde{R}_{\alpha,j} = [\partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha, \underline{\boldsymbol{u}} + \boldsymbol{u}] \cdot \nabla_{\boldsymbol{x}} \eta, \qquad \|\widetilde{R}_{\alpha,j}\|_{L^2(\Omega)} \lesssim \big(|\underline{\boldsymbol{u}}'|_{W^{k-1,\infty}_\varrho} + \|\boldsymbol{u}\|_{H^{s,k}}\big) \|\eta\|_{H^{s,k}}.$$

Using the identity (3.25) (stemming from integrating by parts) we have

$$\nabla_{\boldsymbol{x}} P_{\rm h} := \int_{\rho_0}^{\varrho} \varrho' \nabla_{\boldsymbol{x}} h(\cdot,\varrho')\, \mathrm{d}\varrho' = -\varrho \nabla_{\boldsymbol{x}} \eta + \rho_0 \nabla_{\boldsymbol{x}} \eta|_{\varrho=\rho_0} + \int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}} \eta(\cdot,\varrho')\, \mathrm{d}\varrho',$$

and hence the evolution equation for $\boldsymbol{u}$ reads

$$\partial_t \boldsymbol{u} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}) \cdot \nabla_{\boldsymbol{x}}\big)\boldsymbol{u} + \frac{\rho_0}{\varrho} \nabla_{\boldsymbol{x}} \eta|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}} \eta(\cdot,\varrho')\, \mathrm{d}\varrho' + \frac{1}{\varrho} \nabla_{\boldsymbol{x}} P_{\rm nh} + \frac{\nabla_{\boldsymbol{x}} \eta}{\varrho(\underline{h}+h)} \partial_\varrho P_{\rm nh} = 0.$$

Differentiating α times with respect to $\boldsymbol{x}$ and j times with respect to ϱ yields the corresponding equations in (3.33a), with remainder terms

$$\boldsymbol{R}^{\rm nh}_{\alpha,j} := \boldsymbol{R}_{\alpha,j} - \big[\partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha, \tfrac{\nabla_{\boldsymbol{x}} \eta}{\varrho(\underline{h}+h)}\big] \partial_\varrho P_{\rm nh},$$

using the notation $\boldsymbol{R}_{\alpha,j}$ for the hydrostatic contributions introduced in Lemma 2.3. The first addends have been estimated in Lemma 2.3, (2.3b) (when $j=0$) and (2.4b) (when $j \geq 1$). We now estimate the second addend as follows. By the commutator estimate in Lemma A.8 with $k = s \geq s_0 + 3/2$, we have

$$\big\|\big[\partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha, \tfrac{\nabla_{\boldsymbol{x}} \eta}{\varrho(\underline{h}+h)}\big] \partial_\varrho P_{\rm nh}\big\|_{L^2(\Omega)} \lesssim \big\|\tfrac{\nabla_{\boldsymbol{x}} \eta}{\varrho(\underline{h}+h)}\big\|_{H^{s,k}} \big\|\partial_\varrho P_{\rm nh}\big\|_{H^{s-1,\min(\{k,s-1\})}}.$$

Then by tame product estimate Lemma A.3 and composition estimates in Lemma A.6, we have

$$\big\|\tfrac{\nabla_{\boldsymbol{x}} \eta}{\varrho(\underline{h}+h)}\big\|_{H^{s,k}} \leq C(h_\star, |\underline{h}|_{W^{k,\infty}}, \|h\|_{H^{s-1,k-1}}) \big(\|h\|_{H^{s,k}} \|\nabla_{\boldsymbol{x}} \eta\|_{H^{s-1,k-1}} + \|\nabla_{\boldsymbol{x}} \eta\|_{H^{s,k}}\big)$$

and there remains to use estimate (3.19) in Corollary 3.2 to infer

$$\begin{aligned} \|\boldsymbol{R}^{\rm nh}_{\alpha,j}\|_{L^2(\Omega)} &\leq C(h_\star, \underline{M}, M)\, M \big(1 + \kappa \|\nabla_{\boldsymbol{x}} h\|_{H^{s,k}}\big) \\ &\quad + C(h_\star, \underline{M}, M)\, \sqrt{\mu} \big(\|h\|_{H^{s,k}} + \|\nabla_{\boldsymbol{x}} \eta\|_{H^{s,k}}\big)\big(M + \|\boldsymbol{u}_\star\|_{H^{s,k}} + \sqrt{\mu} \|w_\star\|_{H^{s,k-1}}\big). \end{aligned}$$

Now consider

$$R^{\rm nh}_{\alpha,j} := -\sqrt{\mu} [\partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha, \underline{\boldsymbol{u}} + \boldsymbol{u}] \cdot \nabla_{\boldsymbol{x}} w + \kappa \sqrt{\mu} \Big[\partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha, \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}\Big] \cdot \nabla_{\boldsymbol{x}} w + \tfrac{1}{\sqrt{\mu}} [\partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha, \tfrac{1}{\varrho(\underline{h}+h)}] \partial_\varrho P.$$

We have, by Lemma A.8 with $k = s \geq s_0 + 3/2$,

$$\sqrt{\mu} \|[\partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha, \underline{\boldsymbol{u}} + \boldsymbol{u}] \cdot \nabla_{\boldsymbol{x}} w\|_{L^2(\Omega)} \lesssim \sqrt{\mu} \big(|\underline{\boldsymbol{u}}'|_{W^{k-1,\infty}_\varrho} + \|\boldsymbol{u}\|_{H^{s,k}}\big) \|\nabla_{\boldsymbol{x}} w\|_{H^{s-1,k-1}},$$

and similarly, using tame product estimate Lemma A.3 and composition estimates in Lemma A.6 as above,

$$\kappa \sqrt{\mu} \big\|\big[\partial_\varrho^j \partial_{\boldsymbol{x}}^\alpha, \tfrac{\nabla_{\boldsymbol{x}} h}{\underline{h}+h}\big] \cdot \nabla_{\boldsymbol{x}} w\big\|_{L^2(\Omega)} \leq \kappa \sqrt{\mu}\, C(h_\star, \underline{M}, M) \big(\|h\|^2_{H^{s,k}} + \|\nabla_{\boldsymbol{x}} h\|_{H^{s,k}}\big) \|w\|_{H^{s,k-1}}.$$

and

$$\tfrac{1}{\sqrt{\mu}}\big\|[\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha, \tfrac{1}{\varrho(\underline{h}+h)}]\partial_\varrho P_{\rm nh}\big\|_{L^2(\Omega)} \le \tfrac{1}{\sqrt{\mu}} C(h_\star, \underline{M}, M)\big\|h\big\|_{H^{s,k}}\big\|\partial_\varrho P_{\rm nh}\big\|_{H^{s-1,k-1}}.$$

Collecting the above and using estimate (3.19) in Corollary 3.2 yields

$$\begin{aligned}\big\|R^{\rm nh}_{\alpha,j}\big\|_{L^2(\Omega)} &\le C(h_\star, \underline{M}, M)\, M\,\big(1+\kappa\big\|\nabla_{\boldsymbol{x}} h\big\|_{H^{s,k}}\big)\\ &\quad + C(h_\star, \underline{M}, M)\,\big\|h\big\|_{H^{s,k}}\big(M + \big\|\boldsymbol{u}_\star\big\|_{H^{s,k}} + \sqrt{\mu}\big\|w_\star\big\|_{H^{s,k-1}}\big).\end{aligned}$$

Finally, we consider the remainder (stemming from differentiating the incompressibility condition (3.31))

$$\begin{aligned}R^{\rm div}_{\alpha,j} &= \big(\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha(h\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}) - (h^{(\alpha,j)})\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} - h\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{(\alpha,j)}\big)\\ &\quad + \big(\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha((\partial_\varrho\boldsymbol{u})\cdot(\nabla_{\boldsymbol{x}}\eta)) - (\partial_\varrho\boldsymbol{u}^{(\alpha,j)})\cdot(\nabla_{\boldsymbol{x}}\eta) - (\partial_\varrho\boldsymbol{u})\cdot(\nabla_{\boldsymbol{x}}\eta^{(\alpha,j)})\big)\\ &\quad + \big(\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha(\underline{h}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}) - \underline{h}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{(\alpha,j)}\big) + \big(\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha(\underline{\boldsymbol{u}}'\cdot\nabla_{\boldsymbol{x}}\eta) - \underline{\boldsymbol{u}}'\cdot\nabla_{\boldsymbol{x}}\eta^{(\alpha,j)}\big).\end{aligned}$$

Using Lemma A.9 for the first two terms and direct estimates for the last to terms, and $k = s \ge s_0 + 5/2$,

$$\begin{aligned}\big\|R^{\rm div}_{\alpha,j}\big\| &\lesssim \big\|h\big\|_{H^{s-1,k-1}}\big\|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}\big\|_{H^{s-1,k-1}} + \big\|\partial_\varrho\boldsymbol{u}\big\|_{H^{s-1,k-1}}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s-1,k-1}}\\ &\quad + \big\|\underline{h}\big\|_{W^{k,\infty}_\varrho}\big\|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}\big\|_{H^{s-1,k-1}} + \big\|\underline{\boldsymbol{u}}'\big\|_{W^{k,\infty}_\varrho}\big\|\nabla_{\boldsymbol{x}}\eta\big\|_{H^{s-1,j-1}}\\ &\lesssim (M+\underline{M})M.\end{aligned}$$

Identity (3.34a) with the estimate (3.34b) have been already obtained in the proof of Lemma 2.3. This concludes the proof. □

3.4. A priori energy estimates. In this section we provide *a priori* energy estimates associated with the equations featured in Lemma 3.5. Recalling the notation $\dot\eta = \eta^{(\alpha,j)}, \dot h = h^{(\alpha,j)}, \dot{\boldsymbol{u}} = \boldsymbol{u}^{(\alpha,j)}, \dot w = w^{(\alpha,j)}, \dot P_{\rm nh} = P_{\rm nh}^{(\alpha,j)}$, we point out that such estimates concerning $h^{(\alpha,j)}$ solving the transport-diffusion equation (2.13) have been provided in Lemma 2.4. Now, we consider the following linearized system:

$$\begin{aligned}&\partial_t\dot\eta + (\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta + \int_\varrho^{\rho_1}\big((\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta + (\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\dot{\boldsymbol{u}}\big)\,\mathrm{d}\varrho' - \kappa\Delta_{\boldsymbol{x}}\dot\eta = R,\\ &\varrho\big(\partial_t\dot{\boldsymbol{u}} + \big((\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h})\cdot\nabla_{\boldsymbol{x}}\big)\dot{\boldsymbol{u}}\big) + \rho_0\nabla_{\boldsymbol{x}}\dot\eta|_{\varrho=\rho_0} + \int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}\dot\eta\,\mathrm{d}\varrho' + \nabla_{\boldsymbol{x}}\dot P_{\rm nh} + \frac{\nabla_{\boldsymbol{x}}\eta}{\underline{h}+h}\partial_\varrho\dot P_{\rm nh} = \boldsymbol{R}^{\rm nh},\\ &\sqrt{\mu}\varrho\big(\partial_t\dot w + \mu\big(\underline{\boldsymbol{u}}+\boldsymbol{u}-\kappa\tfrac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h}\big)\cdot\nabla_{\boldsymbol{x}}\dot w\big) - \frac{1}{\sqrt{\mu}}\frac{\partial_\varrho\dot P_{\rm nh}}{\underline{h}+h} = R^{\rm nh},\\ &-\partial_\varrho\dot w + (\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\dot{\boldsymbol{u}} + \nabla_{\boldsymbol{x}}\eta\cdot\partial_\varrho\dot{\boldsymbol{u}} - (\partial_\varrho\dot\eta)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} + \nabla_{\boldsymbol{x}}\dot\eta\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}) = R^{\rm div},\end{aligned}\tag{3.35}$$

where we denote as always $\eta(\cdot,\varrho) = \int_\varrho^{\rho_1} h(\cdot,\varrho)\,\mathrm{d}\varrho$.

We shall use the following definitions of the spaces Y^0 and Y^1

$$\begin{aligned}&Y^0 := \mathcal{C}^0([\rho_0,\rho_1];L^2(\mathbb{R}^d))\times L^2(\Omega)^d\times L^2(\Omega)\times L^2(\Omega)\,, \qquad \text{and}\\ &Y^1 := \Big\{(\eta,\boldsymbol{u},w,P)\in H^{1,1}(\Omega)^{d+3}\ :\ \eta\big|_{\varrho=\rho_0}\in H^1(\mathbb{R}^d),\ w\big|_{\varrho=\rho_1}=0,\ P\big|_{\varrho=\rho_0}=0\Big\}.\end{aligned}\tag{3.36}$$

LEMMA 3.6. *Let $M, h_\star, h^\star > 0$ be fixed. There exists $C(h_\star, M, h^\star) > 0$ such that for any $\kappa > 0$ and $\mu > 0$, and for any $(\underline{h},\underline{\boldsymbol{u}}) \in W^{1,\infty}((\rho_0,\rho_1))$ and any $T > 0$ and $(h,\boldsymbol{u},w) \in L^\infty(0,T;W^{1,\infty}(\Omega))$ with $\Delta_{\boldsymbol{x}}h \in L^1(0,T;L^\infty(\Omega))$ satisfying (3.30) and, for almost any $t\in[0,T]$, the estimate*

$$\big\|h(t,\cdot)\big\|_{L^\infty(\Omega)} + \big\|\nabla_{\boldsymbol{x}}h(t,\cdot)\big\|_{L^\infty_{\boldsymbol{x}}L^2_\varrho} + \big\|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}(t,\cdot)\big\|_{L^\infty(\Omega)} \le M$$

and the upper and lower bounds

$$\forall(\boldsymbol{x},\varrho)\in\Omega, \qquad h_\star \le \underline{h}(\varrho) + h(t,\boldsymbol{x},\varrho) \le h^\star;$$

and for any $(\dot\eta, \dot{\boldsymbol{u}}, \dot w, \dot{P}_{\rm nh}) \in \mathcal{C}^0([0,T];Y^0) \cap L^2(0,T;Y^1)$ *and* $(R, \boldsymbol{R}^{\rm nh}, R^{\rm nh}, R^{\rm div}) \in L^2(0,T;Y^0)$ *satisfying system* (3.35) *in* $L^2(0,T;Y^1)'$*, the following inequality holds:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\dot\eta,\dot{\boldsymbol{u}},\dot w) + \tfrac{\kappa}{2}\big\|\nabla_{\boldsymbol{x}}\dot\eta\big\|^2_{L^2(\Omega)} + \rho_0\kappa\big|\nabla_{\boldsymbol{x}}\dot\eta\big|_{\varrho=\rho_0}\big|^2_{L^2_{\boldsymbol{x}}}&\\ \le C\,(1+\kappa^{-1}\big\|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}\big\|^2_{L^\infty_{\boldsymbol{x}}L^2_\varrho})\mathcal{E}(\dot\eta,\dot{\boldsymbol{u}},\dot w)&\\ + \big(M + \big\|\underline{\boldsymbol{u}}' + \partial_\varrho\boldsymbol{u}\big\|_{L^\infty_{\boldsymbol{x}}L^\infty_\varrho}\big)\big\|\dot{P}_{\rm nh}\big\|_{L^2(\Omega)}\big(\big\|\partial_\varrho\dot\eta\big\|_{L^2(\Omega)} + \big\|\nabla_{\boldsymbol{x}}\dot\eta\big\|_{L^2(\Omega)}\big)&\\ + \big\|\dot{P}_{\rm nh}\big\|_{L^2(\Omega)}\big\|R^{\rm div}\big\|_{L^2(\Omega)} + C\,\mathcal{E}(\dot\eta,\dot{\boldsymbol{u}},\dot w)^{1/2}\mathcal{E}(R,\boldsymbol{R}^{\rm nh},R^{\rm nh})^{1/2},&\end{aligned}$$

where we denote

$$\mathcal{E}(\dot\eta,\dot{\boldsymbol{u}},\dot w) = \frac12\int_{\rho_0}^{\rho_1}\int_{\mathbb{R}^d}\dot\eta^2 + \varrho(\underline h + h)|\dot{\boldsymbol{u}}|^2 + \mu\varrho(\underline h + h)\dot w^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho + \frac12\int_{\mathbb{R}^d}\dot\eta^2|_{\varrho=\rho_0}\,\mathrm{d}\boldsymbol{x}. \tag{3.37}$$

PROOF. We test the first equation against $\dot\eta \in L^2(0,T;H^{1,1}(\Omega))$ and its trace on $\{(\boldsymbol{x},\rho_0),\ \boldsymbol{x}\in\mathbb{R}^d\}$ against $\rho_0\dot\eta|_{\varrho=\rho_0} \in L^2(0,T;H^1_{\boldsymbol{x}}(\mathbb{R}^d))$, the second equation against $(\underline h + h)\dot{\boldsymbol{u}} \in L^2(0,T;H^{1,1}(\Omega)^d)$ and the third equation against $\sqrt{\mu}(\underline h + h)\dot w \in L^2(0,T;H^{1,1}(\Omega))$. This yields:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\dot\eta,\dot{\boldsymbol{u}},\dot w) + \kappa\big\|\nabla_{\boldsymbol{x}}\dot\eta\big\|^2_{L^2(\Omega)} + \kappa\big|\nabla_{\boldsymbol{x}}\dot\eta|_{\varrho=\rho_0}\big|^2_{L^2_{\boldsymbol{x}}}\\
&= -\big((\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta,\dot\eta\big)_{L^2(\Omega)} - \left(\int_\varrho^{\rho_1}(\underline{\boldsymbol{u}}' + \partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\,\mathrm{d}\varrho', \dot\eta\right)_{L^2(\Omega)} && \text{(i)}\\
&\quad - \left(\int_\varrho^{\rho_1}(\underline h + h)\nabla_{\boldsymbol{x}}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}\varrho', \dot\eta\right)_{L^2(\Omega)} + \big(R,\dot\eta\big)_{L^2(\Omega)} && \text{(ii)}\\
&\quad - \big(\varrho(\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot{\boldsymbol{u}}, (\underline h + h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} + \kappa\big(\varrho(\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\dot{\boldsymbol{u}},\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} && \text{(iii)}\\
&\quad - \big(\rho_0\nabla_{\boldsymbol{x}}\dot\eta|_{\varrho=\rho_0}, (\underline h + h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} - \left(\int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}\dot\eta\,\mathrm{d}\varrho', (\underline h + h)\dot{\boldsymbol{u}}\right)_{L^2(\Omega)} && \text{(iv)}\\
&\quad - \big(\nabla_{\boldsymbol{x}}\dot{P}_{\rm nh}, (\underline h + h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} - \big((\partial_\varrho\dot{P}_{\rm nh})\,\nabla_{\boldsymbol{x}}\eta, \dot{\boldsymbol{u}}\big)_{L^2(\Omega)} + \big(\boldsymbol{R}^{\rm nh}, (\underline h + h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} && \text{(v)}\\
&\quad - \mu\big(\varrho(\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot w, (\underline h + h)\dot w\big)_{L^2(\Omega)} + \mu\kappa\big(\varrho(\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}})\dot w,\dot w\big)_{L^2(\Omega)} && \text{(vi)}\\
&\quad + \big(\partial_\varrho\dot{P}_{\rm nh},\dot w\big)_{L^2(\Omega)} + \sqrt{\mu}\big(R^{\rm nh}, (\underline h + h)\dot w\big)_{L^2(\Omega)} && \text{(vii)}\\
&\quad - \rho_0\,\big((\underline{\boldsymbol{u}}+\boldsymbol{u})(\nabla_{\boldsymbol{x}}\dot\eta|_{\varrho=\rho_0}), \dot\eta|_{\varrho=\rho_0}\big)_{L^2_{\boldsymbol{x}}} - \rho_0\left(\int_{\rho_0}^{\rho_1}(\underline{\boldsymbol{u}}' + \partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot\eta\,\mathrm{d}\varrho', \dot\eta|_{\varrho=\rho_0}\right)_{L^2_{\boldsymbol{x}}} && \text{(viii)}\\
&\quad - \rho_0\left(\int_{\rho_0}^{\rho_1}(h+\underline h)(\nabla_{\boldsymbol{x}}\cdot\dot{\boldsymbol{u}})\,\mathrm{d}\varrho', \dot\eta|_{\varrho=\rho_0}\right)_{L^2_{\boldsymbol{x}}} + \big(R|_{\varrho=\rho_0},\dot\eta|_{\varrho=\rho_0}\big)_{L^2_{\boldsymbol{x}}} && \text{(ix)}\\
&\quad + \frac12\big(\varrho(\partial_t h)\dot{\boldsymbol{u}},\dot{\boldsymbol{u}}\big)_{L^2(\Omega)} + \frac{\mu}{2}\big(\varrho(\partial_t h)\dot w,\dot w\big)_{L^2(\Omega)}. && \text{(x)}\end{aligned}$$

Some terms have already been treated in the course of the proof of Lemma 2.5: the second term in (i) and the second term in (viii) require $\kappa > 0$; the first terms in (i), (viii) are advection terms; the first addend of (ii) together with the second term in (iv) after integration by parts; the first addend of (iv) with the first term in (ix).The contributions in (iii) compensate with the first addend of (x), using the first equation of (3.30) and, in the same way, the contributions in (vi) compensate with the second addend of (x). It remains only to deal with the contribution frm the non-hydrostatic pressure terms in (v) and (vii), and remainder terms.

Consider the sum of the first two terms in (v) and the first term in (vii). We integrate by parts in $\boldsymbol{x}$ the first term and in ϱ in the last two terms. Thus we have

$$-\big(\nabla_{\boldsymbol{x}}\dot{P_{\rm nh}},(\underline{h}+h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)}-\big((\partial_\varrho\dot{P_{\rm nh}})\nabla_{\boldsymbol{x}}\eta,\dot{\boldsymbol{u}}\big)_{L^2(\Omega)}+\big(\partial_\varrho\dot{P_{\rm nh}},\dot{w}\big)_{L^2(\Omega)}$$
$$=\big(\dot{P_{\rm nh}},(\underline{h}+h)\nabla_{\boldsymbol{x}}\cdot\dot{\boldsymbol{u}}\big)_{L^2(\Omega)}+\big(\dot{P_{\rm nh}}\nabla_{\boldsymbol{x}}\eta,\partial_\varrho\dot{\boldsymbol{u}}\big)_{L^2(\Omega)}-\big(\dot{P_{\rm nh}},\partial_\varrho\dot{w}\big)_{L^2(\Omega)},$$

where we used the identity $h=-\partial_\varrho\eta$ and the boundary conditions $\dot{P_{\rm nh}}|_{\varrho=\rho_0}=\dot{\eta}|_{\varrho=\rho_1}=\dot{w}|_{\varrho=\rho_1}=0$ when integrating by parts with respect to ϱ. Using the last equation in (3.35) (stemming from the incompressibility condition), the above term reads

$$\big(\dot{P_{\rm nh}},(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u})(\partial_\varrho\dot{\eta})-(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\dot{\eta}+R^{\rm div}\big)_{L^2(\Omega)}.$$

These terms, alike remainder terms

$$\big|\big(R,\dot{\eta}\big)_{L^2(\Omega)}\big|+\big|\big(R|_{\varrho=\rho_0},\dot{\eta}|_{\varrho=\rho_0}\big)_{L^2_{\boldsymbol{x}}}\big|+\big|\big(\boldsymbol{R}^{\rm nh},(\underline{h}+h)\dot{\boldsymbol{u}}\big)_{L^2(\Omega)}\big|+\sqrt{\mu}\big|\big(R^{\rm nh},(\underline{h}+h)\dot{w}\big)_{L^2(\Omega)}\big|\,,$$

are bounded by Cauchy-Schwartz inequality and using $\rho_0 h_\star\le\varrho(\underline{h}+h)\le\rho_1 h^\star$.

Altogether, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\dot{\eta},\dot{\boldsymbol{u}},\dot{w})+\kappa\big\|\nabla_{\boldsymbol{x}}\eta\big\|^2_{L^2(\Omega)}+\rho_0\kappa\big|\nabla_{\boldsymbol{x}}\eta\big|_{\varrho=\rho_0}\big|^2_{L^2_{\boldsymbol{x}}}$$
$$\le C\,\mathcal{E}(\dot{\eta},\dot{\boldsymbol{u}},0)+C\big\|\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}\big\|_{L^2_\varrho L^\infty_{\boldsymbol{x}}}\mathcal{E}(\dot{\eta},\dot{\boldsymbol{u}},0)^{1/2}\big\|\nabla_{\boldsymbol{x}}\dot{\eta}\big\|_{L^2(\Omega)}$$
$$+\big(M+\big\|\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}\big\|_{L^\infty_\varrho L^\infty_{\boldsymbol{x}}}\big)\big\|\dot{P_{\rm nh}}\big\|_{L^2(\Omega)}\big(\big\|\partial_\varrho\dot{\eta}\big\|_{L^2(\Omega)}+\big\|\nabla_{\boldsymbol{x}}\dot{\eta}\big\|_{L^2(\Omega)}\big)$$
$$+\big\|\dot{P_{\rm nh}}\big\|_{L^2(\Omega)}\big\|R^{\rm div}\big\|_{L^2(\Omega)}+C\mathcal{E}(\dot{\eta},\dot{\boldsymbol{u}},\dot{w})^{1/2}\mathcal{E}(R,\boldsymbol{R}^{\rm nh},R^{\rm nh})^{1/2}$$

with $C=C(h_\star,h^\star,M)$, and the desired estimate follows straightforwardly. □

REMARK 3.7. Lemma 3.6 will be applied to the system (3.33a)-(3.33b) appearing in Lemma 3.5, *when* $j=0$. A similar result holds for the simplified system when $j\neq0$. The main difference is that the result does not require nor provide the control of the trace $\partial^j_\varrho\eta\big|_{\varrho=\rho_0}$.

3.5. Large-time well-posedness. We prove the large-time existence of strong solutions to system (3.1). As for the hydrostatic system, *large time* underlines the fact that the existence time that is provided by the following result is uniformly bounded (from below) with respect to the vanishing parameter $\mu\in(0,1]$. Besides, the result below keeps track of the dependency of this large time-scale on the diffusivity parameter $\kappa\in[\mu,1]$.

PROPOSITION 3.8. *Let $s,k\in\mathbb{N}$ be such that $k=s>\frac52+\frac{d}{2}$ and $\underline{M},M^\star,h_\star,h^\star>0$. Then, there exists $C>0$ such that, for any $0<\mu\le\kappa\le1$, and any $(\underline{h},\underline{\boldsymbol{u}})\in W^{k,\infty}((\rho_0,\rho_1))\times W^{k+1,\infty}((\rho_0,\rho_1))$ such that*

$$\big|\underline{h}\big|_{W^{k,\infty}_\varrho}+\big|\underline{\boldsymbol{u}}'\big|_{W^{k,\infty}_\varrho}\le\underline{M}\,;$$

for any initial data $(h_0,\boldsymbol{u}_0,w_0)\in H^{s,k}(\Omega)^{d+2}$ with

$$M_0:=\big\|\eta_0\big\|_{H^{s,k}}+\big\|\boldsymbol{u}_0\big\|_{H^{s,k}}+\sqrt{\mu}\big\|w_0\big\|_{H^{s,k}}+\big|\eta_0\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}}+\kappa^{1/2}\big\|h_0\big\|_{H^{s,k}}+\kappa\big\|\nabla_{\boldsymbol{x}}\eta_0\big\|_{H^{s,k}}\le M^\star\,,$$

and satisfying the boundary condition $w_0|_{\varrho=\rho_1}=0$ and the incompressibility condition

$$-(\underline{h}+h_0)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}_0-(\nabla_{\boldsymbol{x}}\eta_0)\cdot(\underline{\boldsymbol{u}}'+\partial_\varrho\boldsymbol{u}_0)+\partial_\varrho w_0=0,$$

the lower and upper bounds

$$\forall(\boldsymbol{x},\varrho)\in\Omega,\qquad h_\star\le\underline{h}(\varrho)+h_0(\boldsymbol{x},\varrho)\le h^\star\,,$$

and the smallness assumption

$$C\kappa^{-1}\left(\left|\underline{\boldsymbol{u}}'\right|^2_{L^\infty_\varrho}+M_0\right)\leq 1\,,$$

the following holds. Denoting by

$$T^{-1}=C\left(1+\kappa^{-1}\left(\left|\underline{\boldsymbol{u}}'\right|^2_{L^2_\varrho}+M_0^2\right)\right),$$

there exists a unique $(h,\boldsymbol{u},w)\in L^\infty(0,T;H^{s,k}(\Omega)^{2+d})$ *and* $P\in L^2(0,T;H^{s+1,k+1}(\Omega))$ *strong solution to* (3.30) *with initial data* $(h,\boldsymbol{u},w)\big|_{t=0}=(h_0,\boldsymbol{u}_0,w_0)$. *Moreover, one has* $\eta\in L^\infty(0,T;H^{s+1,k}(\Omega))$ *and* $(h,\nabla_{\boldsymbol{x}}\eta)\in L^2(0,T;H^{s+1,k}(\Omega))$ *and , for any* $t\in[0,T]$, *the lower and the upper bounds hold*

$$\forall(\boldsymbol{x},\varrho)\in\Omega,\qquad h_\star/2\leq\underline{h}(\varrho)+h(t,\boldsymbol{x},\varrho)\leq 2h^\star\,,$$

and the estimate below holds true

$$\begin{aligned}\mathcal{F}(t):=&\left\|\eta(t,\cdot)\right\|_{H^{s,k}}+\left\|\boldsymbol{u}(t,\cdot)\right\|_{H^{s,k}}+\mu^{1/2}\left\|w(t,\cdot)\right\|_{H^{s,k}}+\left|\eta\big|_{\varrho=\rho_0}(t,\cdot)\right|_{H^s_{\boldsymbol{x}}}\\&+\kappa^{1/2}\left\|h(t,\cdot)\right\|_{H^{s,k}}+\mu^{1/2}\kappa^{1/2}\left\|\nabla_{\boldsymbol{x}}\eta(t,\cdot)\right\|_{H^{s,k}}\\&+\kappa^{1/2}\left\|\nabla_{\boldsymbol{x}}\eta\right\|_{L^2(0,t;H^{s,k})}+\kappa^{1/2}\left|\nabla_{\boldsymbol{x}}\eta\big|_{\varrho=\rho_0}\right|_{L^2(0,t;H^s_{\boldsymbol{x}})}\\&+\kappa\left\|\nabla_{\boldsymbol{x}}h\right\|_{L^2(0,t;H^{s,k})}+\mu^{1/2}\kappa\left\|\nabla^2_{\boldsymbol{x}}\eta\right\|_{L^2(0,t;H^{s,k})}\leq C\,M_0.\end{aligned}\tag{3.38}$$

PROOF. As for the large-time existence for the hydrostatic system (see Proposition 2.6), the proof is based on a bootstrap argument on the functional $\mathcal{F}$. Recalling that the (short-time) existence and uniqueness of the solution has been provided in Proposition 3.3, we denote by $T^\star$ the maximal existence time, and set

$$T_\star=\sup\{0<T<T^\star\,:\,\forall\,t\in(0,T),\ h_\star/2\leq\underline{h}(\varrho)+h(t,\boldsymbol{x},\varrho)\leq 2h^\star\quad\text{and}\quad\mathcal{F}(t)\leq C_0M_0\},\tag{3.39}$$

with $C_0=C(h_\star,h^\star,\underline{M},M_0)$ sufficiently large (to be determined). Henceforth, we restrain to $0<T<T_\star$, and and denote by C any positive constant depending uniquely on $\underline{M},h_\star,h^\star,C_0M_0$ and s,k.

By means of (3.34a)-(3.34b) in Lemma 3.5 and Lemma 2.4, we infer as in the proof of Proposition 2.6 the control

$$\kappa^{1/2}\left\|h\right\|_{L^\infty(0,T;H^{s,k})}+\kappa\left\|\nabla_{\boldsymbol{x}}h\right\|_{L^2(0,T;H^{s,k})}\leq\Big(c_0M_0+C\,C_0M_0\left(T+\sqrt{T}\right)\Big)\times\exp\Big(CC_0M_0\,T\Big).\tag{3.40}$$

with the same notations as above and c_0 a universal constant. In the non-hydrostatic situation, additional controls can be inferred on η. Indeed, from the first equation in in Lemma 3.5, (3.33a)-(3.33b), we find that

$$\partial_t\eta^{(\alpha,j)}+(\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}\eta^{(\alpha,j)}=\kappa\Delta_{\boldsymbol{x}}\eta^{(\alpha)}+\widetilde{R}_{\alpha,j}+w^{(\alpha,j)}$$

with

$$\sqrt{\mu}\left\|\widetilde{R}_{\alpha,j}+w^{(\alpha,j)}\right\|_{L^2(\Omega)}\leq C\,C_0M_0\,.$$

Differentiating once with respect to the space variables and proceeding as in Lemma 2.4, we infer

$$\begin{aligned}&\mu^{1/2}\kappa^{1/2}\left\|\nabla_{\boldsymbol{x}}\eta\right\|_{L^\infty(0,T;H^{s,k})}+\mu^{1/2}\kappa\left\|\nabla^2_{\boldsymbol{x}}\eta\right\|_{L^2(0,T;H^{s,k})}\\&\qquad\leq\big(c_0M_0+CC_0M_0(T+\sqrt{T})\big)\times\exp\Big(CC_0M_0T\Big).\end{aligned}\tag{3.41}$$

Next we use again Lemma 3.5, (3.33a)-(3.33b), together with Lemma 3.6 (see also Remark 3.7) to obtain that the functional

$$\mathcal{E}^{s,k}:=\frac12\sum_{j=0}^k\sum_{|\alpha|=0}^{s-j}\iint_\Omega(\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha\eta)^2+\varrho(\underline{h}+h)|\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha\boldsymbol{u}|^2+\mu\varrho(\underline{h}+h)(\partial_\varrho^j\partial_{\boldsymbol{x}}^\alpha w)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\varrho+\frac12\sum_{|\alpha|=0}^s\int_{\mathbb{R}^d}(\partial_{\boldsymbol{x}}^\alpha\eta|_{\varrho=\rho_0})^2\,\mathrm{d}\boldsymbol{x},$$

satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^{s,k}+\tfrac{\kappa}{2}\left\|\nabla_{\boldsymbol{x}}\eta\right\|^2_{H^{s,k}}+\rho_0\kappa\left|\nabla_{\boldsymbol{x}}\eta\big|_{\varrho=\rho_0}\right|^2_{H^s_{\boldsymbol{x}}}\leq C\left(R_1+R_2+R_3\right);\tag{3.42}$$

with

$$R_1 := (1+\kappa^{-1}\|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}\|^2_{L^\infty_{\boldsymbol{x}} L^2_\varrho})\mathcal{E}^{s,k},$$
$$R_2 := \big(C_0M_0 + \|\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}\|_{L^\infty_\varrho L^\infty_{\boldsymbol{x}}}\big)\|P_{\rm nh}\|_{H^{s,k}}\big(\|h\|_{H^{s,k}} + \|\nabla_{\boldsymbol{x}}\eta\|_{H^{s,k}}\big),$$
$$R_3 := \|P_{\rm nh}\|_{H^{s,k}}\|R^{\rm div}_{s,k}\|_{L^2(\Omega)} + (\mathcal{E}^{s,k})^{1/2}\|\mathcal{R}_{s,k}\|_{L^2(\Omega)},$$

and

$$\|R^{\rm div}_{s,k}\|_{L^2(\Omega)} \le C\, C_0M_0\,, \tag{3.43}$$
$$\|\mathcal{R}_{s,k}\|_{L^2(\Omega)} \le C\, C_0M_0\,\big(1+\kappa\|\nabla_{\boldsymbol{x}}h\|_{H^{s,k}}\big) + C\,\big(\|h\|_{H^{s,k}} + \mu^{1/2}\|\nabla_{\boldsymbol{x}}\eta\|_{H^{s,k}}\big)\big(C_0M_0 + \|\boldsymbol{u}_\star\|_{H^{s,k}} + \mu^{1/2}\|w_\star\|_{H^{s,k}}\big)\,. \tag{3.44}$$

By (3.39), we have obviously for any $0<t<T_\star$,

$$\frac{1}{2\rho_1 h^\star}\mathcal{E}^{s,k}(t) \le \|\eta(t,\cdot)\|^2_{H^{s,k}} + \|\boldsymbol{u}(t,\cdot)\|^2_{H^{s,k}} + \mu\|w(t,\cdot)\|^2_{H^{s,k}} + |\eta|_{\varrho=\rho_0}(t,\cdot)|^2_{H^s_{\boldsymbol{x}}} \le \frac{2}{\rho_0 h_\star}\mathcal{E}^{s,k}(t).$$

Moreover, we have the following control on $\boldsymbol{u}_\star := -\kappa\frac{\nabla_{\boldsymbol{x}}h}{\underline{h}+h}$ and $w_\star := \kappa\Delta_{\boldsymbol{x}}\eta - \kappa\frac{\nabla_{\boldsymbol{x}}h\cdot\nabla_{\boldsymbol{x}}\eta}{\underline{h}+h}$ stemming from (tame) product and composition estimates (Lemma A.3 and A.6), and using that $\mu\le\kappa\le 1$:

$$\|\boldsymbol{u}_\star\|_{L^2(0,T;H^{s,k})} + \mu^{1/2}\|w_\star\|_{L^2(0,T;H^{s,k})} \le C\,C_0M_0(1+\sqrt{T})\,. \tag{3.45}$$

Finally, using estimate (3.19) in Corollary 3.2 yields

$$\|P_{\rm nh}\|_{H^{s,k}} \le \|P_{\rm nh}\|_{L^2} + \|\nabla_{\boldsymbol{x},\varrho}P_{\rm nh}\|_{H^{s-1,k-1}} \le \|P_{\rm nh}\|_{L^2} + \mu^{-1/2}\|\nabla^\mu_{\boldsymbol{x},\varrho}P_{\rm nh}\|_{H^{s-1,k-1}}$$
$$\le C\,\Big(\|\nabla_{\boldsymbol{x}}\eta\|_{H^{s-1,k-1}} + |\eta|_{\varrho=\rho_0}|_{H^s_{\boldsymbol{x}}} + \|(\boldsymbol{u},\boldsymbol{u}_\star)\|_{H^{s,k}} + \mu^{1/2}\|(w,w_\star)\|_{H^{s,k-1}}\Big)\,,$$

from which we infer, using the controls (3.39) and (3.45), that

$$\|P_{\rm nh}\|_{L^2(0,T;H^{s,k})} \le C\,C_0M_0(1+\sqrt{T}). \tag{3.46}$$

From (3.39) and (3.43)-(3.44)-(3.45)-(3.46) we infer

$$\int_0^T R_1(t)\,{\rm d}t \le C\,(C_0M_0)^2(1+\kappa^{-1}|\underline{\boldsymbol{u}}'|^2_{L^2_\varrho} + \kappa^{-1}(C_0M_0)^2)\,T,$$
$$\int_0^T R_2(t)\,{\rm d}t \le C\,\kappa^{-1/2}\,\big(C_0M_0 + |\underline{\boldsymbol{u}}'|_{L^\infty_\varrho}\big)\,(C_0M_0)^2(1+\sqrt{T})^2\,,$$
$$\int_0^T R_3(t)\,{\rm d}t \le C\,(C_0M_0)^2(T+\sqrt{T}) \;+\; C\,(C_0M_0)^2\,\big(T + C_0M_0\sqrt{T} + \kappa^{-1/2}(C_0M_0)(T+\sqrt{T})\big).$$

Hence there exists $C>0$, depending on $\underline{M}, h_\star, h^\star, C_0, M_0$ (and s,k), such that if

$$C\,T\,\big(1+\kappa^{-1}(|\boldsymbol{u}'|^2_{L^2_\varrho} + M_0^2)\big) \le 1,$$

and imposing additionally that

$$C\,\kappa^{-1/2}\,\big(C_0M_0 + |\underline{\boldsymbol{u}}'|_{L^\infty_\varrho}\big) \le \tfrac{1}{16}\rho_0 h_\star \tag{3.47}$$

we have, when integrating the differential inequality (3.42) and combine with (3.40) and (3.41),

$$\mathcal{E}^{s,k}(t) \le \mathcal{E}^{s,k}(0) + \tfrac{1}{8}(\rho_0 h_\star)(C_0M_0)^2\,.$$

We point out that the only term requiring the above smallness condition (3.47) on the initial data is (the time integral of) R_2, and more precisely the product $\|P_{\rm nh}\|_{H^{s,k}}\|\nabla_{\boldsymbol{x}}\eta\|_{H^{s,k}}$, where both terms are controlled in L^2_T (but not in L^∞_T).

Now, setting $C_0 = \max(\{4(\frac{\rho_1 h^\star}{\rho_0 h_\star})^{1/2}, 8c_0\}$, and C accordingly, one has $\mathcal{F}(t) \le C_0M_0/2$ for all $0<t<T$. We obtain as in the proof of Proposition 2.6 the lower and upper bounds $2h_\star/3 \le \underline{h}(\varrho) +$

$h(t,\boldsymbol{x},\varrho) \leq 3h^\star/2$, augmenting C if necessary, and the standard continuity argument allows to conclude the proof.

Let us acknowledge however that the continuity of the functional $\mathcal{F}(t)$ is not provided by Proposition 3.3. In order to deal with this issue, we can, as detailed in Section 2, consider a parabolic regularization method. □

4. Convergence

This section is devoted to the proof of the convergence of regular solutions to the (non-hydrostatic) equations (1.7) towards the corresponding solutions to the limit (hydrostatic) equations (1.8), namely Theorem 1.2. The convergence result provided here holds in the *strong* sense and "with loss of derivatives": we prove that the solutions to the approximating (non-hydrostatic) system converge towards the solutions to the hydrostatic equations in a suitable strong topology that is strictly weaker than the one measuring the size of the initial data.

For a given set of initial data, we use the apex h to refer to the solution to the hydrostatic equations (provided by the analysis of Section 2 culminating with Theorem 1.1), and the apex nh for the corresponding solution to the non-hydrostatic equations (provided by the analysis of Section 3, specifically Proposition 3.3). The apex d denotes the difference between the non-hydrostatic solution and the hydrostatic one, whose size will be controlled in the limit $\mu \searrow 0$.

While we can appeal to Theorem 1.1 to obtain the existence, uniqueness and control of solutions to the hydrostatic equations over a large time interval, Proposition 3.3 provides only a time interval which *a priori* vanishes as $\mu \searrow 0$, and Proposition 3.8 only applies to sufficiently small initial data. The (standard) strategy (used for instance in [19] in the context of weakly compressible flows) that we apply here relies on a bootstrap argument to control the difference between the non-hydrostatic solution and the hydrostatic one in the time-interval established by the hydrostatic solution, within which the existence and control of the non-hydrostatic solution (again, with loss of derivatives) can be inferred. We perform this analysis in Sections 4.1 to 4.3, where we first provide a consistency result (Lemma 4.1), then exhibit the (non-hydrostatic) quasilinear structure of the equations satisfied by the difference (Lemma 4.2), and finally infer the uniform control of the non-hydrostatic solution and the strong convergence towards the corresponding hydrostatic solution (Proposition 4.3). In a last step, in Section 4.4, we use this uniform control to offer an improved convergence rate based on the structure of the hydrostatic equations (Proposition 4.4). Propositions 4.3 and 4.4 immediately yield Theorem 1.2.

4.1. Consistency.

In the following result we prove that solutions to the hydrostatic equations (1.8) emerging from smooth initial data satisfy (suitably defining the horizontal velocity and pressure variables) the non-hydrostatic equations (1.7), up to small remainder terms.

LEMMA 4.1. *There exists $p \in \mathbb{N}$ such that for any $s \geq 0$ and $0 \leq k \leq s$, the following holds. Let $\underline{M}, M, h_\star, h^\star > 0$ be fixed. Then there exists $C_0, C_1 > 0$ such that for any $\kappa \in (0,1]$, any $\underline{h}, \underline{\boldsymbol{u}} \in W^{k+p,\infty}((\rho_0,\rho_1))$ satisfying*

$$\big|\underline{h}\big|_{W^{k+p,\infty}_\varrho} + \big|\underline{\boldsymbol{u}}'\big|_{W^{k+p-1,\infty}_\varrho} \leq \underline{M}$$

and any initial data $(h_0,\boldsymbol{u}_0) \in H^{s+p,k+p}(\Omega)$ satisfying the following estimate

$$M_0 := \big\|\eta_0\big\|_{H^{s+p,k+p}} + \big\|\boldsymbol{u}_0\big\|_{H^{s+p,k+p}} + \big|\eta_0\big|_{\varrho=\rho_0}\big|_{H^{s+p}_{\boldsymbol{x}}} + \kappa^{1/2}\big\|h_0\big\|_{H^{s+p,k+p}} \leq M$$

(where we denote $\eta_0(\cdot,\varrho) := \int_\varrho^{\rho_1} h_0(\cdot,\varrho')\,\mathrm{d}\varrho'$) and the stable stratification assumption

$$\inf_{(\boldsymbol{x},\varrho)\in\Omega} h_\star \leq \underline{h}(\varrho) + h_0(\boldsymbol{x},\varrho) \leq h^\star,$$

there exists a unique $(h^{\mathrm{h}},\boldsymbol{u}^{\mathrm{h}}) \in L^\infty(0,T;H^{s+p,k+p}(\Omega)^{1+d})$ strong solution to (1.8) with initial data $(h^{\mathrm{h}},\boldsymbol{u}^{\mathrm{h}})\big|_{t=0} = (h_0,\boldsymbol{u}_0)$, where

$$T^{-1} = C_0\left(1+\kappa^{-1}\big(\big|\underline{\boldsymbol{u}}'\big|^2_{L^2_\varrho} + M_0^2\big)\right).$$

Moreover, one has for all $t \in [0,T]$,

$$\forall (\boldsymbol{x},\varrho) \in \Omega, \qquad h_\star/2 \leq \underline{h}(\varrho) + h^{\mathrm{h}}(t,\boldsymbol{x},\varrho) \leq 2\,h^\star,$$

and, denoting $\eta^{\mathrm{h}}(\cdot,\varrho) := \int_\varrho^{\rho_1} h^{\mathrm{h}}(\cdot,\varrho')\,\mathrm{d}\varrho'$ *and*

$$w^{\mathrm{h}}(\cdot,\varrho) := -\int_\varrho^{\rho_1} (\underline{h}(\varrho') + h^{\mathrm{h}}(\cdot,\varrho'))\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\mathrm{h}} + \nabla_{\boldsymbol{x}}\eta^{\mathrm{h}}(\cdot,\varrho')\cdot(\underline{\boldsymbol{u}}'(\varrho') + \partial_\varrho \boldsymbol{u}^{\mathrm{h}}(\cdot,\varrho'))\,\mathrm{d}\varrho'\,, \tag{4.1}$$

$$P^{\mathrm{h}}(\cdot,\varrho) := \int_{\rho_0}^{\varrho} \varrho' h^{\mathrm{h}}(\cdot,\varrho')\,\mathrm{d}\varrho'\,, \tag{4.2}$$

one has for any $t \in [0,T]$,

$$\big\|\big(h^{\mathrm{h}}(t,\cdot),\eta^{\mathrm{h}}(t,\cdot),\boldsymbol{u}^{\mathrm{h}}(t,\cdot),w^{\mathrm{h}}(t,\cdot),P^{\mathrm{h}}(t,\cdot)\big)\big\|_{H^{s+1,k+1}} \leq C_1\,M_0. \tag{4.3}$$

and

$$\begin{aligned}
\partial_t h^{\mathrm{h}} + \nabla_{\boldsymbol{x}}\cdot\big((\underline{h}+h^{\mathrm{h}})(\underline{\boldsymbol{u}}+\boldsymbol{u}^{\mathrm{h}})\big) &= \kappa\Delta_{\boldsymbol{x}} h^{\mathrm{h}},\\
\varrho\Big(\partial_t \boldsymbol{u}^{\mathrm{h}} + \big((\underline{\boldsymbol{u}}+\boldsymbol{u}^{\mathrm{h}} - \kappa\tfrac{\nabla_{\boldsymbol{x}}h^{\mathrm{h}}}{\underline{h}+h^{\mathrm{h}}})\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}^{\mathrm{h}}\Big) + \nabla_{\boldsymbol{x}}P^{\mathrm{h}} + \frac{\nabla_{\boldsymbol{x}}\eta^{\mathrm{h}}}{\underline{h}+h^{\mathrm{h}}}(\partial_\varrho P^{\mathrm{h}} + \varrho\underline{h}^{\mathrm{h}}) &= 0,\\
\mu\varrho\Big(\partial_t w^{\mathrm{h}} + \big(\underline{\boldsymbol{u}}+\boldsymbol{u}^{\mathrm{h}} - \kappa\tfrac{\nabla_{\boldsymbol{x}}h^{\mathrm{h}}}{\underline{h}+h^{\mathrm{h}}}\big)\cdot\nabla_{\boldsymbol{x}} w^{\mathrm{h}}\Big) - \frac{\partial_\varrho P^{\mathrm{h}}}{\underline{h}+h^{\mathrm{h}}} + \frac{\varrho h^{\mathrm{h}}}{\underline{h}+h^{\mathrm{h}}} &= \mu\,R^{\mathrm{h}},\\
-(\underline{h}+h^{\mathrm{h}})\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\mathrm{h}} - \nabla_{\boldsymbol{x}}\eta^{\mathrm{h}}\cdot(\underline{\boldsymbol{u}}' + \partial_\varrho\boldsymbol{u}^{\mathrm{h}}) + \partial_\varrho w^{\mathrm{h}} &= 0,
\end{aligned} \tag{4.4a}$$

with $R^{\mathrm{h}}(t,\cdot) \in \mathcal{C}^0([0,T];H^{s,k}(\Omega))$ *and satisfying*

$$\forall t\in[0,T], \quad \big\|R^{\mathrm{h}}(t,\cdot)\big\|_{H^{s,k}} \leq C_1\,M_0. \tag{4.4b}$$

PROOF. From Theorem 1.1 we infer immediately (for $p > 2 + d/2$) the existence, uniqueness and control of the hydrostatic solution $(h^{\mathrm{h}},\boldsymbol{u}^{\mathrm{h}}) \in L^\infty(0,T;H^{s+p,k+p}(\Omega)^{1+d})$, and $C_0 > 0$. From the formula (4.1), (4.2), product (Lemma A.3) and commutator Lemma A.8) estimates in the space $H^{s+p',k+p'}(\Omega)$ (for $1 \leq p' \leq p$ sufficiently large) we infer the estimate (4.3).

We obtain similarly the desired consistency estimate, (4.4a)-(4.4b), using the identity (recall (3.25))

$$P^{\mathrm{h}} + \varrho\eta^{\mathrm{h}} = \int_{\rho_0}^{\varrho} \eta^{\mathrm{h}}(\cdot,\varrho')\,\mathrm{d}\varrho' + \rho_0\eta^{\mathrm{h}}\big|_{\varrho=\rho_0}\,,$$

and denoting

$$R^{\mathrm{h}} := \varrho\Big(\partial_t w^{\mathrm{h}} + \big(\underline{\boldsymbol{u}}+\boldsymbol{u}^{\mathrm{h}} - \kappa\tfrac{\nabla_{\boldsymbol{x}}h^{\mathrm{h}}}{\underline{h}+h^{\mathrm{h}}}\big)\cdot\nabla_{\boldsymbol{x}} w^{\mathrm{h}}\Big),$$

differentiating with respect to time the identity (4.1), and using (1.8) to infer the control of $\partial_t\boldsymbol{u}^{\mathrm{h}}$ and $\partial_t w^{\mathrm{h}}$. □

As a corollary to the above, we can write the equations satisfied by the difference between $(h^{\mathrm{h}},\boldsymbol{u}^{\mathrm{h}},w^{\mathrm{h}})$, i.e. the maximal solution to the hydrostatic equations emerging from given regular, well-prepared initial data, and $(h^{\mathrm{nh}},\boldsymbol{u}^{\mathrm{nh}},w^{\mathrm{nh}})$, i.e. the maximal solution to the non-hydrostatic with the same data (see Proposition 3.3). Specifically, under the assumptions and using the notations of Lemma 4.1, we have that

$$h^{\mathrm{d}} := h^{\mathrm{nh}} - h^{\mathrm{h}}; \quad \boldsymbol{u}^{\mathrm{d}} := \boldsymbol{u}^{\mathrm{nh}} - \boldsymbol{u}^{\mathrm{h}}; \quad w^{\mathrm{d}} := w^{\mathrm{nh}} - w^{\mathrm{h}};$$

satisfies $(\eta^d, \boldsymbol{u}^{\mathrm{d}}, w^{\mathrm{d}})\big|_{t=0} = (0,0,0)$ and

$$\begin{aligned}
\partial_t h^{\mathrm{d}} + \nabla_{\boldsymbol{x}} \cdot \big((\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}})h^{\mathrm{d}} + (\underline{h} + h^{\mathrm{h}})\boldsymbol{u}^{\mathrm{d}}\big) &= \kappa \Delta_{\boldsymbol{x}} h^{\mathrm{d}},\\
\partial_t \eta^{\mathrm{d}} + (\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}}) \cdot \nabla_{\boldsymbol{x}} \eta^{\mathrm{d}} + \int_{\varrho}^{\rho_1} (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}^{\mathrm{nh}}) \cdot \nabla_{\boldsymbol{x}} \eta^{\mathrm{d}} \,\mathrm{d}\varrho' + \int_{\varrho}^{\rho_1} (\underline{h} + h^{\mathrm{nh}}) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{\mathrm{d}} \,\mathrm{d}\varrho'&\\
+ \int_{\varrho}^{\rho_1} \boldsymbol{u}^{\mathrm{d}} \cdot \nabla_{\boldsymbol{x}} h^{\mathrm{h}} + h^{\mathrm{d}} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{\mathrm{h}} \,\mathrm{d}\varrho' &= \kappa \Delta_{\boldsymbol{x}} \eta^{\mathrm{d}},\\
\partial_t \boldsymbol{u}^{\mathrm{d}} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h^{\mathrm{nh}}}{\underline{h} + h^{\mathrm{nh}}}) \cdot \nabla_{\boldsymbol{x}}\big)\boldsymbol{u}^{\mathrm{d}} + \frac{\rho_0}{\varrho} \nabla_{\boldsymbol{x}} \eta^{\mathrm{d}}|_{\varrho=\rho_0} + \frac{1}{\varrho}\int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}} \eta^{\mathrm{d}} \,\mathrm{d}\varrho'&\\
+\big((\boldsymbol{u}^{\mathrm{d}} - \kappa(\tfrac{\nabla_{\boldsymbol{x}} h^{\mathrm{nh}}}{\underline{h} + h^{\mathrm{nh}}} - \tfrac{\nabla_{\boldsymbol{x}} h^{\mathrm{h}}}{\underline{h} + h^{\mathrm{h}}})) \cdot \nabla_{\boldsymbol{x}}\big)\boldsymbol{u}^{\mathrm{h}} + \frac{\nabla_{\boldsymbol{x}} P_{\mathrm{nh}}}{\varrho} + \frac{\nabla_{\boldsymbol{x}} \eta^{\mathrm{nh}}}{\varrho(\underline{h} + h^{\mathrm{nh}})} \partial_\varrho P_{\mathrm{nh}} &= 0,\\
\mu\big(\partial_t w^{\mathrm{d}} + \big(\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h^{\mathrm{nh}}}{\underline{h} + h^{\mathrm{nh}}}\big) \cdot \nabla_{\boldsymbol{x}} w^{\mathrm{d}}\big) + \mu\big((\boldsymbol{u}^{\mathrm{d}} - \kappa(\tfrac{\nabla_{\boldsymbol{x}} h^{\mathrm{nh}}}{\underline{h} + h^{\mathrm{nh}}} - \tfrac{\nabla_{\boldsymbol{x}} h^{\mathrm{h}}}{\underline{h} + h^{\mathrm{h}}})) \cdot \nabla_{\boldsymbol{x}} w^{\mathrm{h}}\big) - \frac{\partial_\varrho P_{\mathrm{nh}}}{\varrho(\underline{h} + h^{\mathrm{nh}})} &= -\mu R^{\mathrm{h}},\\
-(\underline{h} + h^{\mathrm{nh}}) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{\mathrm{d}} - (\underline{h} + h^{\mathrm{d}}) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{\mathrm{h}} - \nabla_{\boldsymbol{x}} \eta^{\mathrm{d}} \cdot (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}^{\mathrm{nh}}) - \nabla_{\boldsymbol{x}} \eta^{\mathrm{h}} \cdot (\partial_\varrho \boldsymbol{u}^{\mathrm{d}}) + \partial_\varrho w^{\mathrm{d}} &= 0,
\end{aligned} \tag{4.5}$$

where we denote as usual $\eta^{\mathrm{h}}(\cdot,\varrho) = \int_{\varrho}^{\rho_1} h^{\mathrm{h}}(\cdot,\varrho')\,\mathrm{d}\varrho'$ (and analogously η^{nh}, η^{d}), and define the non-hydrostatic pressure $P_{\mathrm{nh}}(\cdot,\varrho) := P^{\mathrm{nh}}(\cdot,\varrho) - \int_{\rho_0}^{\varrho} \varrho' h^{\mathrm{nh}}(\cdot,\varrho')\,\mathrm{d}\varrho'$ where P^{nh} is defined by Corollary 3.2.

4.2. Quasi-linearization. In this section we extract the leading order terms of the system (4.5), in the spirit of Lemma 3.5.

LEMMA 4.2. *There exists $p \in \mathbb{N}$ such that for any $s, k \in \mathbb{N}$ such that $k = s > \frac{5}{2} + \frac{d}{2}$ and $\underline{M}, M, h_\star > 0$, there exists $C = C(s,k,\underline{M},M,h_\star) > 0$ such that the following holds. For any $0 < \mu \le \kappa \le 1$, and for any $(\underline{h}, \underline{\boldsymbol{u}}) \in W^{k+p,\infty}((\rho_0,\rho_1))^2$ satisfying*

$$|\underline{h}|_{W^{k+p,\infty}_\varrho} + |\underline{\boldsymbol{u}}'|_{W^{k+p-1,\infty}_\varrho} \le \underline{M};$$

and any $(h^{\mathrm{nh}}, \boldsymbol{u}^{\mathrm{nh}}, w^{\mathrm{nh}}) \in L^\infty(0,T^{\mathrm{nh}}; H^{s,k}(\Omega)^{d+2})$ and $P^{\mathrm{nh}} \in L^2(0,T^{\mathrm{nh}}; H^{s+1,k+1}(\Omega))$ solution to (3.30) with some $T^{\mathrm{nh}} > 0$ and satisfying for almost any $t \in [0,T^{\mathrm{nh}}]$

$$\begin{aligned}
\big\|h^{\mathrm{nh}}\big\|_{H^{s-1,k-1}}(t,\cdot) + \big\|\eta^{\mathrm{nh}}\big\|_{H^{s,k}}(t,\cdot) + \big|\eta^{\mathrm{nh}}\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}}(t,\cdot) + \big\|\boldsymbol{u}^{\mathrm{nh}}\big\|_{H^{s,k}}(t,\cdot) + \mu^{1/2}\big\|w^{\mathrm{nh}}\big\|_{H^{s,k}}(t,\cdot)&\\
+ \kappa^{1/2}\big\|h^{\mathrm{nh}}\big\|_{H^{s,k}}(t,\cdot) + \mu^{1/2}\kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta^{\mathrm{nh}}\big\|_{H^{s,k}}(t,\cdot) &\le M
\end{aligned}$$

(where $\eta^{\mathrm{nh}}(t,\boldsymbol{x},\varrho) := \int_{\varrho}^{\rho_1} h^{\mathrm{nh}}(t,\boldsymbol{x},\varrho')\,\mathrm{d}\varrho'$),

$$M_0 := \big\|\eta^{\mathrm{nh}}\big|_{t=0}\big\|_{H^{s+p,k+p}} + \big\|\boldsymbol{u}^{\mathrm{nh}}\big|_{t=0}\big\|_{H^{s+p,k+p}} + \big|(\eta^{\mathrm{nh}}\big|_{\varrho=\rho_0})\big|_{t=0}\big|_{H^{s+p}_{\boldsymbol{x}}} + \kappa^{1/2}\big\|h^{\mathrm{nh}}\big|_{t=0}\big\|_{H^{s+p,k+p}} \le M$$

and the stable stratification assumption

$$\inf_{(t,\boldsymbol{x},\varrho)\in(0,T)\times\Omega} \underline{h}(\varrho) + h^{\mathrm{nh}}(t,\boldsymbol{x},\varrho) \ge h_\star,$$

we have the following.

Denote $(h^{\mathrm{h}}, \boldsymbol{u}^{\mathrm{h}}, w^{\mathrm{h}}) \in \mathcal{C}^1([0,T^{\mathrm{h}}]; H^{s+1,k+1}(\Omega)^{2+d})$ the corresponding strong solution to the hydrostatic equations (2.1) (see Lemma 4.1) satisfying

$$\big\|h^{\mathrm{h}}(t,\cdot)\big\|_{H^{s+1,k+1}} + \big\|\boldsymbol{u}^{\mathrm{h}}(t,\cdot)\big\|_{H^{s+1,k+1}} + \big\|\eta^{\mathrm{h}}(t,\cdot)\big\|_{H^{s+1,k+1}} + \big\|w^{\mathrm{h}}(t,\cdot)\big\|_{H^{s+1,k+1}} \le CM_0$$

and, for any multi-index $\alpha \in \mathbb{N}^d$ and $j \in \mathbb{N}$ such that $0 \le |\alpha| + j \le s$,

$$\eta^{(\alpha,j)} := \partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho \eta^{\mathrm{nh}} - \partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho \eta^{\mathrm{h}}; \quad \boldsymbol{u}^{(\alpha,j)} := \partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho \boldsymbol{u}^{\mathrm{nh}} - \partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho \boldsymbol{u}^{\mathrm{h}}; \quad w^{(\alpha,j)} := \partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho w^{\mathrm{nh}} - \partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho w^{\mathrm{h}};$$

and $P^{(\alpha,j)}_{\mathrm{nh}}(\cdot,\varrho) = \partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho \big(P^{\mathrm{nh}}(\cdot,\varrho) - \int_{\rho_0}^{\varrho} \varrho' h^{\mathrm{nh}}(\cdot,\varrho')\,\mathrm{d}\varrho'\big)$.

Then restricting to $t \in [0, \min(T^{\mathrm{h}}, T^{\mathrm{nh}})]$ *and such that*

$$\mathcal{F}_{s,k} := \big\|h^{\mathrm{d}}\big\|_{H^{s-1,k-1}} + \big\|\eta^{\mathrm{d}}\big\|_{H^{s,k}} + \big|\eta^{\mathrm{d}}\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}} + \big\|\boldsymbol{u}^{\mathrm{d}}\big\|_{H^{s,k}} + \mu^{1/2}\big\|w^{\mathrm{d}}\big\|_{H^{s,k}} + \kappa^{1/2}\big\|h^{\mathrm{d}}\big\|_{H^{s,k}} + \mu^{1/2}\kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta^{\mathrm{d}}\big\|_{H^{s,k}} \leq \kappa^{1/2} M$$

we have

$$\begin{aligned}
&\partial_t \eta^{(\alpha,j)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)} + w^{(\alpha,j)} = \kappa \Delta_{\boldsymbol{x}} \eta^{(\alpha)} + \widetilde{R}_{\alpha,j} \\
&\partial_t \eta^{(\alpha,j)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)} + \Big\langle \int_{\varrho}^{\rho_1} (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}^{\mathrm{nh}}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)} \, \mathrm{d}\varrho' \\
&\qquad + \int_{\varrho}^{\rho_1} (\underline{h} + h^{\mathrm{nh}}) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(\alpha,j)} \, \mathrm{d}\varrho' \Big\rangle_{j=0} = \kappa \Delta_{\boldsymbol{x}} \eta^{(\alpha,j)} + R_{\alpha,j}, \\
&\partial_t \boldsymbol{u}^{(\alpha,j)} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h^{\mathrm{nh}}}{\underline{h} + h^{\mathrm{nh}}}) \cdot \nabla_{\boldsymbol{x}}\big) \boldsymbol{u}^{(\alpha,j)} + \Big\langle \frac{\rho_0}{\varrho} \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)}\big|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)} \, \mathrm{d}\varrho' \Big\rangle_{j=0} \\
&\qquad + \frac{1}{\varrho} \nabla_{\boldsymbol{x}} P^{(\alpha,j)}_{\mathrm{nh}} + \frac{\nabla_{\boldsymbol{x}} \eta^{\mathrm{nh}}}{\varrho(\underline{h} + h^{\mathrm{nh}})} \partial_\varrho P^{(\alpha,j)}_{\mathrm{nh}} = \boldsymbol{R}^{\mathrm{nh}}_{\alpha,j}, \\
&\mu^{1/2} \Big(\partial_t w^{(\alpha,j)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h^{\mathrm{nh}}}{\underline{h} + h^{\mathrm{nh}}}) \cdot \nabla_{\boldsymbol{x}} w^{(\alpha,j)}\Big) - \frac{1}{\mu^{1/2}} \frac{\partial_\varrho P^{(\alpha,j)}_{\mathrm{nh}}}{\varrho(\underline{h} + h^{\mathrm{nh}})} = R^{\mathrm{nh}}_{\alpha,j}, \\
&-\partial_\varrho w^{(\alpha,j)} + (\underline{h} + h^{\mathrm{nh}}) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(\alpha,j)} + (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}^{\mathrm{nh}}) \cdot \nabla_{\boldsymbol{x}} \eta^{(\alpha,j)} \\
&\qquad + (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{\mathrm{nh}}) h^{(\alpha,j)} + (\nabla_{\boldsymbol{x}} \eta^{\mathrm{nh}}) \cdot (\partial_\varrho \boldsymbol{u}^{(\alpha,j)}) = R^{\mathrm{div}}_{\alpha,j},
\end{aligned} \tag{4.6a}$$

where $(R_{\alpha,j}(t,\cdot), \boldsymbol{R}^{\mathrm{nh}}_{\alpha,j}(t,\cdot), R^{\mathrm{nh}}_{\alpha,j}(t,\cdot), R^{\mathrm{div}}_{\alpha,j}) \in L^2(\Omega)^{d+3}$, $R_{\alpha,0}(t,\cdot) \in \mathcal{C}((\rho_0,\rho_1); L^2(\mathbb{R}^d))$ *and*

$$\begin{aligned}
&\big\|R_{\alpha,j}\big\|_{L^2(\Omega)} + \big|R_{\alpha,0}\big|_{\varrho=\rho_0}\big|_{L^2_{\boldsymbol{x}}} + \big\|\widetilde{R}_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|R^{\mathrm{div}}_{\alpha,j}\big\|_{L^2(\Omega)} \leq C\, \mathcal{F}_{s,k}, \\
&\big\|\boldsymbol{R}^{\mathrm{nh}}_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|R^{\mathrm{nh}}_{\alpha,j}\big\|_{L^2(\Omega)} \leq C\, \big(\mathcal{F}_{s,k} + M\kappa \big\|\nabla_{\boldsymbol{x}} h^{\mathrm{d}}\big\|_{H^{s,k}} + M\mu^{1/2}\kappa \big\|\Delta_{\boldsymbol{x}} \eta^{\mathrm{d}}\big\|_{H^{s,k}}\big) + C\, \mu^{1/2} M,
\end{aligned} \tag{4.6b}$$

and

$$\partial_t h^{(\alpha,j)} + (\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}}) \cdot \nabla_{\boldsymbol{x}} h^{(\alpha,j)} = \kappa \Delta_{\boldsymbol{x}} \partial^j_\varrho h^{(\alpha,j)} + r_{\alpha,j} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{r}_{\alpha,j}, \tag{4.7a}$$

where $(r_{\alpha,j}(t,\cdot), \boldsymbol{r}_{\alpha,j}(t,\cdot)) \in L^2(\Omega)^{1+d}$ *and*

$$\kappa^{1/2} \big\|r_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|\boldsymbol{r}_{\alpha,j}\big\|_{L^2(\Omega)} \leq C\, \mathcal{F}_{s,k}. \tag{4.7b}$$

PROOF. Explicit expressions for the remainder terms follow from (4.5). Specifically, the following equation is obtained by combining the second and last equation (recall (3.32)) combining the second and last equation (recall (3.32))

$$\partial_t \eta^{\mathrm{d}} + (\underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}}) \cdot \nabla_{\boldsymbol{x}} \eta^{\mathrm{d}} + \boldsymbol{u}^{\mathrm{d}} \cdot \nabla_{\boldsymbol{x}} \eta^{\mathrm{h}} - w^{\mathrm{d}} = \kappa \Delta_{\boldsymbol{x}} \eta^{\mathrm{d}}$$

and hence

$$\widetilde{R}_{\alpha,j} := -[\partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho, \underline{\boldsymbol{u}} + \boldsymbol{u}^{\mathrm{nh}}] \cdot \nabla_{\boldsymbol{x}} \eta^{\mathrm{d}} - \partial^\alpha_{\boldsymbol{x}} \partial^j_\varrho (\boldsymbol{u}^{\mathrm{d}} \cdot \nabla_{\boldsymbol{x}} \eta^{\mathrm{h}}),$$

and the estimate (4.6b) for $\big\|\widetilde{R}_{\alpha,j}\big\|_{L^2(\Omega)}$ follows from product (Lemma A.3) and commutator (Lemma A.8) estimates.

Then, from the second equation we have

$$R_{\alpha,j} := R^{(i)}_{\alpha,j} + R^{(ii)}_{\alpha,j}$$

with $R^{(i)}_{\alpha,j} := -[\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho, \underline{\boldsymbol{u}} + \boldsymbol{u}^{\rm nh}]\cdot\nabla_{\boldsymbol{x}}\eta^{\rm d}$ and

$$R^{(ii)}_{\alpha,j} := \begin{cases} -\int_\varrho^{\rho_1}[\partial^\alpha_{\boldsymbol{x}}, \partial_\varrho \boldsymbol{u}^{\rm nh}]\cdot\nabla_{\boldsymbol{x}}\eta^{\rm d} + [\partial^\alpha_{\boldsymbol{x}}, h^{\rm nh}]\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm d} + \partial^\alpha_{\boldsymbol{x}}\big(\boldsymbol{u}^{\rm d}\cdot\nabla_{\boldsymbol{x}}h^{\rm h} + h^{\rm d}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm h}\big)\,{\rm d}\varrho' & \text{if } j=0,\\ \partial^{j-1}_\varrho\partial^\alpha_{\boldsymbol{x}}\big((\underline{\boldsymbol{u}}' + \partial_\varrho\boldsymbol{u}^{\rm nh})\cdot\nabla_{\boldsymbol{x}}\eta^{\rm d} + (\underline{h}+h^{\rm nh})\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm d} + \boldsymbol{u}^{\rm d}\cdot\nabla_{\boldsymbol{x}}h^{\rm h} + h^{\rm d}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm h} & \text{if } j\geq 1.\end{cases}$$

Using Lemma A.3, Lemma A.8 and the continuous embedding $L^\infty((\rho_0,\rho_1))\subset L^2((\rho_0,\rho_1))\subset L^1((\rho_0,\rho_1))$ we find

$$\begin{aligned}\big\|R_{\alpha,j}\big\|_{L^2(\Omega)} \lesssim \big(|\underline{\boldsymbol{u}}'|_{W^{k-1,\infty}_\varrho} + \big\|\boldsymbol{u}^{\rm nh}\big\|_{H^{s,k}} + \big\|h^{\rm nh}\big\|_{H^{s-1,k-1}} + \big\|\eta^{\rm nh}\big\|_{H^{s,k-1}} + \big\|h^{\rm h}\big\|_{H^{s+1,k-1}} + \big\|\boldsymbol{u}^{\rm h}\big\|_{H^{s+1,k-1}}\big)\\ \times\big(\big\|\eta^{\rm d}\big\|_{H^{s,k-1}} + \big\|\boldsymbol{u}^{\rm d}\big\|_{H^{s,k-1}} + \big\|h^{\rm d}\big\|_{H^{s-1,k-1}}\big)\end{aligned}$$

where for $j=0$ we used the identities (and Lemma A.1 and Lemma A.7,(2) and (3))

$$\begin{aligned}\int_\varrho^{\rho_1}[\partial^\alpha_{\boldsymbol{x}}, \partial_\varrho\boldsymbol{u}^{\rm nh}]\cdot\nabla_{\boldsymbol{x}}\eta^{\rm d}\,{\rm d}\varrho' + [\partial^\alpha_{\boldsymbol{x}}, h^{\rm nh}]\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm d} &= \int_\varrho^{\rho_1}[\partial^\alpha_{\boldsymbol{x}}; \partial_\varrho\boldsymbol{u}^{\rm nh}, \nabla_{\boldsymbol{x}}\eta^{\rm d}] + [\partial^\alpha_{\boldsymbol{x}}; h^{\rm nh}, \nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm d}]\,{\rm d}\varrho'\\ &+\int_\varrho^{\rho_1}\partial^\alpha_{\boldsymbol{x}}\boldsymbol{u}^{\rm nh}\cdot\nabla_{\boldsymbol{x}}h^{\rm d} + (\partial^\alpha_{\boldsymbol{x}}\eta^{\rm nh})(\nabla_{\boldsymbol{x}}\cdot\partial_\varrho\boldsymbol{u}^{\rm d})\,{\rm d}\varrho' + \partial^\alpha_{\boldsymbol{x}}\boldsymbol{u}^{\rm nh}\cdot\nabla_{\boldsymbol{x}}\eta^{\rm d} + (\partial^\alpha_{\boldsymbol{x}}\eta^{\rm nh})(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm d})\end{aligned}$$

and

$$\int_\varrho^{\rho_1}\partial^\alpha_{\boldsymbol{x}}(h^{\rm d}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm h})\,{\rm d}\varrho' = \int_\varrho^{\rho_1}[\partial^\alpha_{\boldsymbol{x}}, \nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm h}]h^{\rm d} + (\partial^\alpha_{\boldsymbol{x}}\eta^{\rm d})(\nabla_{\boldsymbol{x}}\cdot\partial_\varrho\boldsymbol{u}^{\rm h})\,{\rm d}\varrho' + (\partial^\alpha_{\boldsymbol{x}}\eta^{\rm d})(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm h}).$$

The corresponding estimate for $\big|R_{\alpha,0}\big|_{\varrho=\rho_0}\big|_{L^2_{\boldsymbol{x}}}$ is obtained in the same way.

Now, we have

$$\begin{aligned}R^{\rm div}_{\alpha,j} :=\ & [\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho, \underline{h}]\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm d} + [\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho, \underline{\boldsymbol{u}}']\cdot\nabla_{\boldsymbol{x}}\eta^{\rm d}\\ &+ [\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho, h^{\rm nh}]\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm d} + [\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho, \nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm h}]h^{\rm d} + \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{u}^{\rm h}-\boldsymbol{u}^{\rm nh})\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho h^{\rm d}\\ &+ [\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho, \partial_\varrho\boldsymbol{u}^{\rm nh}]\cdot\nabla_{\boldsymbol{x}}\eta^{\rm d} + [\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho, \nabla_{\boldsymbol{x}}\eta^{\rm h}]\cdot\partial_\varrho\boldsymbol{u}^{\rm d} + \nabla_{\boldsymbol{x}}(\eta^{\rm h}-\eta^{\rm nh})\cdot\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho\partial_\varrho\boldsymbol{u}^{\rm d}\end{aligned}$$

Then, writing $h^{\rm nh} = h^{\rm h} + h^{\rm d}$, $\partial_\varrho\boldsymbol{u}^{\rm nh} = \partial_\varrho\boldsymbol{u}^{\rm h} + \partial_\varrho\boldsymbol{u}^{\rm d}$, some manipulations of the terms and the use of Lemma A.8 and A.9 lead to

$$\begin{aligned}\big\|R^{\rm div}_{\alpha,j}\big\|_{L^2(\Omega)} \lesssim\ & \big(|\underline{h}|_{W^{k,\infty}_\varrho} + |\underline{\boldsymbol{u}}'|_{W^{k,\infty}_\varrho} + \big\|h^{\rm d}\big\|_{H^{s-1,k-1}} + \big\|\partial_\varrho\boldsymbol{u}^{\rm d}\big\|_{H^{s-1,k-1}}\\ &+ \big\|h^{\rm h}\big\|_{H^{s,k}} + \big\|\nabla_{\boldsymbol{x}}\eta^{\rm h}\big\|_{H^{s,k}} + \big\|\boldsymbol{u}^{\rm h}\big\|_{H^{s+1,k+1}}\big)\times\big(\big\|\boldsymbol{u}^{\rm d}\big\|_{H^{s,k}} + \big\|\eta^{\rm d}\big\|_{H^{s,k-1}} + \big\|h^{\rm d}\big\|_{H^{s-1,k-1}}\big).\end{aligned}$$

which provides the desired estimate.

We focus now on $\big\|\boldsymbol{R}^{\rm nh}_{\alpha,j}\big\|_{L^2(\Omega)}$ and $\big\|R^{\rm nh}_{\alpha,j}\big\|_{L^2(\Omega)}$. Let us estimate some relevant contributions. In the following, we shall use repeatedly that $\big\|h^{\rm d}\big\|_{H^{s,k}}\leq M$ and hence, for instance, by Lemma A.6 and triangular inequality,

$$\big\|\tfrac{h^{\rm nh}}{\underline{h}+h^{\rm nh}}\big\|_{H^{s,k}} \leq C(h_\star, |\underline{h}|_{W^{k,\infty}_\varrho}\big\|h^{\rm nh}\big\|_{H^{s-1,k-1}})\big\|h^{\rm nh}\big\|_{H^{s,k}} \leq C(h_\star, \underline{M}, M)M.$$

- By Lemma A.9, Lemma A.3 and Lemma A.6

$$\begin{aligned}\big\|[\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho, \tfrac{\nabla_{\boldsymbol{x}}h^{\rm nh}}{\underline{h}+h^{\rm nh}}]\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u}^{\rm d}\big\|_{L^2(\Omega)} &\lesssim \big\|\tfrac{\nabla_{\boldsymbol{x}}h^{\rm nh}}{\underline{h}+h^{\rm nh}}\big\|_{H^{s,k}}\big\|\nabla_{\boldsymbol{x}}\boldsymbol{u}^{\rm d}\big\|_{H^{s-1,k}}\\ &\lesssim \big\|\tfrac{\nabla_{\boldsymbol{x}}h^{\rm nh}}{\underline{h}}\big\|_{H^{s,k}}\big(1+\big\|\tfrac{h^{\rm nh}}{\underline{h}+h^{\rm nh}}\big\|_{H^{s,k}}\big)\mathcal{F}_{s,k}\\ &\leq C(h_\star, \underline{M}, M)\big(M + \big\|\nabla_{\boldsymbol{x}}h^{\rm d}\big\|_{H^{s,k}}\big)\mathcal{F}_{s,k}.\end{aligned}$$

- By Lemma A.5 (or rather its proof)

$$\begin{aligned}\big\|\partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho\big((\tfrac{\nabla_{\boldsymbol{x}}h^{\rm nh}}{\underline{h}+h^{\rm nh}} - \tfrac{\nabla_{\boldsymbol{x}}h^{\rm h}}{\underline{h}+h^{\rm h}})\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}^{\rm h}\big)\big\|_{L^2(\Omega)} &\lesssim \big\|\tfrac{\nabla_{\boldsymbol{x}}h^{\rm d}}{\underline{h}+h^{\rm h}} + \tfrac{h^{\rm d}\nabla_{\boldsymbol{x}}h^{\rm nh}}{(\underline{h}+h^{\rm nh})(\underline{h}+h^{\rm h})}\big\|_{H^{s,k}}\big\|\nabla_{\boldsymbol{x}}\boldsymbol{u}^{\rm h}\big\|_{H^{s,k}}\\ &\leq C(h_\star, \underline{M}, M)M(\big\|\nabla_{\boldsymbol{x}}h^{\rm d}\big\|_{H^{s,k}} + M\big\|h^{\rm d}\big\|_{H^{s,k}}).\end{aligned}$$

- By direct inspection, and since $|\alpha| + j - 1 \leq s - 1$,
$$\big\|[\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}, \tfrac{1}{\varrho}]\nabla_{\boldsymbol{x}} P_{\rm nh}\big\|_{L^2(\Omega)} \lesssim \big\|\nabla_{\boldsymbol{x}} P_{\rm nh}\big\|_{H^{s-1,k-1}}.$$
- By Lemma A.8 and since $s = k > \frac{5}{2} + \frac{d}{2}$, using the above and Lemma A.1
$$\begin{aligned}\big\|[\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}, \tfrac{\nabla_{\boldsymbol{x}}\eta^{\rm nh}}{\varrho(\underline{h}+h^{\rm nh})}]\partial_{\varrho} P_{\rm nh}\big\|_{L^2(\Omega)} &\lesssim \big\|\tfrac{\nabla_{\boldsymbol{x}}\eta^{\rm nh}}{\varrho(\underline{h}+h^{\rm nh})}\big\|_{H^{s,k}}\big\|\partial_{\varrho} P_{\rm nh}\big\|_{H^{s-1,k-1}}\\ &\leq C(h_\star, \underline{M}, M)\,\big(M + \big\|\nabla_{\boldsymbol{x}}\eta^{\rm d}\big\|_{H^{s,k}}\big)\big\|\partial_{\varrho} P_{\rm nh}\big\|_{H^{s-1,k-1}}\big).\end{aligned}$$
- Similarly,
$$\begin{aligned}\big\|[\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}, \tfrac{1}{\varrho(\underline{h}+h^{\rm nh})} - \tfrac{1}{\varrho\underline{h}}]\partial_{\varrho} P_{\rm nh}\big\|_{L^2(\Omega)} &\lesssim \big\|\tfrac{h^{\rm nh}}{\varrho(\underline{h}+h^{\rm nh})}\big\|_{H^{s,k}}\big\|\partial_{\varrho} P_{\rm nh}\big\|_{H^{s-1,k-1}}\\ &\leq C(h_\star, \underline{M}, M)\, M\,\big\|\partial_{\varrho} P_{\rm nh}\big\|_{H^{s-1,k-1}}.\end{aligned}$$

Altogether, and using $\mathcal{F}_{s,k} \leq \kappa^{1/2} M$, we find

$$\begin{aligned}\big\|\boldsymbol{R}_{\alpha,j}^{\rm nh}\big\|_{L^2(\Omega)} + \big\|R_{\alpha,j}^{\rm nh}\big\|_{L^2(\Omega)} &\leq C_0\,\big(\mathcal{F}_{s,k}(1 + \kappa\big\|\nabla_{\boldsymbol{x}} h^{\rm d}\big\|_{H^{s,k}}) + M\kappa\big\|\nabla_{\boldsymbol{x}} h^{\rm d}\big\|_{H^{s,k}}\big)\\ &\quad + C_0\,\mu^{-1/2} M\big\|\partial_{\varrho} P_{\rm nh}\big\|_{H^{s-1,k-1}} \qquad (4.8)\end{aligned}$$

with $C_0 = C(h_\star, \underline{M}, M, \big\|h^{\rm d}\big\|_{H^{s,k}})$. Now, recalling that $\Lambda^\mu = 1 + \sqrt{\mu}|D|$, we use Corollary 3.2, specifically (3.20):

$$\begin{aligned}\big\|\nabla_{\boldsymbol{x},\varrho}^{\mu} P_{\rm nh}\big\|_{H^{s-1,k-1}} \leq C_0\,\mu\,\Big(&\big\|(\Lambda^\mu)^{-1} h^{\rm nh}\big\|_{H^{s,k-1}} + \big\|(\Lambda^\mu)^{-1}\nabla_{\boldsymbol{x}}\eta^{\rm nh}\big\|_{H^{s,k}} + \big|(\Lambda^\mu)^{-1}\eta^{\rm nh}\big|_{\varrho=\rho_0}\big|_{H^{s+1}_{\boldsymbol{x}}}\\ &+\big\|(\boldsymbol{u}^{\rm nh}, \boldsymbol{u}_\star^{\rm nh})\big\|_{H^{s,k}} + \big\|(w^{\rm nh}, w_\star^{\rm nh})\big\|_{H^{s,k-1}} + \big\|\boldsymbol{u}_\star^{\rm nh}\big\|_{H^{s,k}}\big\|w^{\rm nh}\big\|_{H^{s,k-1}}\Big).\end{aligned}$$

Then we use that

$$\big\|(\Lambda^\mu)^{-1} h^{\rm nh}\big\|_{H^{s,k-1}} \leq \big\|h^{\rm h}\big\|_{H^{s,k-1}} + \mu^{-1/2}\big\|h^{\rm d}\big\|_{H^{s-1,k-1}} \lesssim M + \mu^{-1/2}\mathcal{F}_{s,k},$$

and, similarly,

$$\big\|(\Lambda^\mu)^{-1}\nabla_{\boldsymbol{x}}\eta^{\rm nh}\big\|_{H^{s,k}} + \big|(\Lambda^\mu)^{-1}\eta^{\rm nh}\big|_{\varrho=\rho_0}\big|_{H^{s+1}_{\boldsymbol{x}}} \lesssim M + \mu^{-1/2}\mathcal{F}_{s,k}.$$

Now,

$$\big\|w^{\rm nh}\big\|_{H^{s,k-1}} \leq \big\|w^{\rm h}\big\|_{H^{s,k-1}} + \big\|w^{\rm d}\big\|_{H^{s,k-1}} \lesssim M + \mu^{-1/2}\mathcal{F}_{s,k}$$

where, for the first contribution, we applied the product estimates to the expression in (4.1). Then, we have

$$\begin{aligned}\big\|\boldsymbol{u}_\star\big\|_{H^{s,k}} &\leq \kappa\big\|\tfrac{\nabla_{\boldsymbol{x}} h^{\rm nh}}{\underline{h}+h^{\rm nh}}\big\|_{H^{s,k}} \leq C_0\kappa\big(\big\|\nabla_{\boldsymbol{x}} h^{\rm nh}\big\|_{H^{s,k}} + \big\|h^{\rm nh}\big\|^2_{H^{s,k}}\big)\\ &\leq C_0 M + C_0\kappa\big\|\nabla_{\boldsymbol{x}} h^{\rm d}\big\|_{H^{s,k}},\\ \big\|w_\star^{\rm nh}\big\|_{H^{s,k-1}} &\leq \kappa\big\|\Delta_{\boldsymbol{x}}\eta^{\rm nh}\big\|_{H^{s,k-1}} + C_0\kappa\big\|\nabla_{\boldsymbol{x}} h^{\rm nh}\big\|_{H^{s,k-1}}\big\|\nabla_{\boldsymbol{x}}\eta^{\rm nh}\big\|_{H^{s,k-1}}\\ &\leq C_0(M + \kappa\big\|\Delta_{\boldsymbol{x}}\eta^{\rm d}\big\|_{H^{s,k-1}} + \kappa^{1/2} M\big\|\nabla_{\boldsymbol{x}} h^{\rm d}\big\|_{H^{s,k-1}}).\end{aligned}$$

Altogether, this yields

$$\begin{aligned}\mu^{-1/2}\big\|\nabla_{\boldsymbol{x},\varrho}^{\mu} P_{\rm nh}\big\|_{H^{s-1,k-1}} \leq C_0\,\Big(&\mu^{1/2} M + \mathcal{F}_{s,k} + \mu^{1/2}\kappa^{1/2}\big\|\nabla_{\boldsymbol{x}} h^{\rm d}\big\|_{H^{s,k}} + \mu^{1/2}\kappa\big\|\Delta_{\boldsymbol{x}}\eta^{\rm d}\big\|_{H^{s,k}}\\ &+ \big(M + \kappa\big\|\nabla_{\boldsymbol{x}} h^{\rm d}\big\|_{H^{s,k}}\big)\,\big(\mu^{1/2} M + \mathcal{F}_{s,k}\big)\Big).\end{aligned}$$

Plugging this estimate in (4.8), using $\mathcal{F}_{s,k} \leq \kappa^{1/2} M$ and $\mu \leq \kappa$, and combining them with the consistency estimate of Lemma 4.1, we obtain (4.6b).

Now, we set

$$\boldsymbol{r}_{\alpha,j} := -[\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}, \underline{\boldsymbol{u}} + \boldsymbol{u}^{\rm nh}]h^{\rm d} - \partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j}\big((\underline{h} + h^{\rm h})\boldsymbol{u}^{\rm d}\big), \qquad r_{\alpha,j} := -(\partial_{\boldsymbol{x}}^{\alpha}\partial_{\varrho}^{j} h^{\rm d})\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{\rm nh}.$$

By Lemma A.3 and Lemma A.8 and since $s \geq s_0 + \frac{3}{2}$ and $2 \leq k = s$, we have

$$\|\boldsymbol{r}_{\alpha,j}\|_{L^2(\Omega)} \lesssim \big(|\underline{h}|_{W^{k,\infty}} + \|h^{\mathrm{h}}\|_{H^{s,k}}\big)\|\boldsymbol{u}^{\mathrm{d}}\|_{H^{s,k}} + \big(|\underline{\boldsymbol{u}}'|_{W^{k-1,\infty}} + \|\boldsymbol{u}^{\mathrm{nh}}\|_{H^{s,k}}\big)\|h^{\mathrm{d}}\|_{H^{s-1,k-1}}$$

and (by Lemma A.1)

$$\|r_{\alpha,j}\|_{L^2(\Omega)} \lesssim \|h^{\mathrm{d}}\|_{H^{s,k}}\|\boldsymbol{u}^{\mathrm{nh}}\|_{H^{s,k}}.$$

This yields immediately (4.7b). The proof is complete. □

4.3. Strong convergence. In this section, we prove that for μ sufficiently small and starting from regular and well-prepared initial data, the solution to the non-hydrostatic equations exists at least within the existence time of the solution to the hydrostatic equation. We also prove the strong convergence of the non-hydrostatic system to the hydrostatic one as $\mu \searrow 0$.

PROPOSITION 4.3. *There exists $p \in \mathbb{N}$ such that for any $s, k \in \mathbb{N}$ such that $k = s > \frac{5}{2} + \frac{d}{2}$ and any $\underline{M}, M, h_\star, h^\star > 0$, there exists $C = C(s, k, \underline{M}, M, h_\star, h^\star) > 0$ such that the following holds. For any $0 < M_0 \leq M$, $0 < \kappa \leq 1$, and $\mu > 0$ such that*

$$\mu \leq \kappa/(CM_0^2),$$

for any $(\underline{h}, \underline{\boldsymbol{u}}) \in W^{k+p,\infty}((\rho_0, \rho_1))^2$ satisfying

$$|\underline{h}|_{W^{k+p,\infty}_{\varrho}} + |\underline{\boldsymbol{u}}'|_{W^{k+p-1,\infty}_{\varrho}} \leq \underline{M};$$

for any initial data $(h_0, \boldsymbol{u}_0, w_0) \in H^{s,k}(\Omega)^{2+d}$ satisfying the boundary condition $w_0|_{\varrho=\rho_1} = 0$ and the incompressibility condition

$$-(\underline{h} + h_0)\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}_0 - (\nabla_{\boldsymbol{x}}\eta_0) \cdot (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}_0) + \partial_\varrho w_0 = 0,$$

(denoting $\eta_0(\cdot, \varrho) = \int_\varrho^{\rho_1} h_0(\cdot, \varrho')\,\mathrm{d}\varrho'$), the bounds

$$\|\eta_0\|_{H^{s+p,k+p}} + \|\boldsymbol{u}_0\|_{H^{s+p,k+p}} + |\eta_0|_{\varrho=\rho_0}|_{H^{s+p}_{\boldsymbol{x}}} + \kappa^{1/2}\|h_0\|_{H^{s+p,k+p}} = M_0 \leq M$$

and the stable stratification assumption

$$\inf_{(\boldsymbol{x},\varrho)\in\Omega} h_\star \leq \underline{h}(\varrho) + h_0(\boldsymbol{x}, \varrho) \leq h^\star,$$

the following holds. Denoting

$$(T^{\mathrm{h}})^{-1} = C^{\mathrm{h}}\big(1 + \kappa^{-1}\big(|\underline{\boldsymbol{u}}'|^2_{L^2_\varrho} + M_0^2\big)\big),$$

as in Lemma 4.1 there exists a unique strong solution $(h^{\mathrm{nh}}, \boldsymbol{u}^{\mathrm{nh}}, w^{\mathrm{nh}}) \in L^\infty(0, T^{\mathrm{h}}; H^{s,k}(\Omega)^{1+d})$ to the non-hydrostatic equations (3.30) with initial data $(h^{\mathrm{nh}}, \boldsymbol{u}^{\mathrm{nh}}, w^{\mathrm{nh}})|_{t=0} = (h_0, \boldsymbol{u}_0, w_0)$. Moreover, one has $h^{\mathrm{nh}} \in L^2(0, T^{\mathrm{h}}; H^{s+1,k}(\Omega))$, $\eta^{\mathrm{nh}} \in L^2(0, T^{\mathrm{h}}; H^{s+2,k}(\Omega))$ and, for any $t \in [0, T^{\mathrm{h}}]$, the lower and the upper bounds hold

$$\inf_{(\boldsymbol{x},\varrho)\in\Omega} \underline{h}(\varrho) + h^{\mathrm{nh}}(t, \boldsymbol{x}, \varrho) \geq h_\star/3, \qquad \sup_{(\boldsymbol{x},\varrho)\in\Omega} \underline{h}(\varrho) + h^{\mathrm{nh}}(t, \boldsymbol{x}, \varrho) \leq 3h^\star,$$

and the estimate below holds true

$$\begin{aligned}
&\|\eta^{\mathrm{nh}}(t,\cdot)\|_{H^{s,k}} + \|\boldsymbol{u}^{\mathrm{nh}}(t,\cdot)\|_{H^{s,k}} + \mu^{1/2}\|w^{\mathrm{nh}}(t,\cdot)\|_{H^{s,k}} + |\eta^{\mathrm{nh}}|_{\varrho=\rho_0}(t,\cdot)|_{H^s_{\boldsymbol{x}}} \\
&\quad + \kappa^{1/2}\|h^{\mathrm{nh}}(t,\cdot)\|_{H^{s,k}} + \mu^{1/2}\kappa^{1/2}\|\nabla_{\boldsymbol{x}}\eta^{\mathrm{nh}}(t,\cdot)\|_{H^{s,k}} \\
&\quad + \kappa^{1/2}\|\nabla_{\boldsymbol{x}}\eta^{\mathrm{nh}}\|_{L^2(0,t;H^{s,k})} + \kappa^{1/2}|\nabla_{\boldsymbol{x}}\eta^{\mathrm{nh}}|_{\varrho=\rho_0}|_{L^2(0,t;H^s_{\boldsymbol{x}})} \\
&\quad + \kappa\|\nabla_{\boldsymbol{x}}h^{\mathrm{nh}}\|_{L^2(0,t;H^{s,k})} + \mu^{1/2}\kappa\|\nabla^2_{\boldsymbol{x}}\eta^{\mathrm{nh}}\|_{L^2(0,t;H^{s,k})} \leq C\,M_0,
\end{aligned} \tag{4.9}$$

and $(h^{\mathrm{nh}}, \boldsymbol{u}^{\mathrm{nh}})$ converges strongly in $L^\infty(0, T; H^{s,k}(\Omega)^{1+d})$ towards $(h^{\mathrm{h}}, \boldsymbol{u}^{\mathrm{h}})$ the corresponding solution to the hydrostatic equations (2.1), as $\mu \searrow 0$.

PROOF. We closely follow the proof of Proposition 3.8 and exhibit a bootstrap argument on the functional

$$\begin{aligned}\mathcal{F}(t) :=& \big\|\eta^{\mathrm{d}}(t,\cdot)\big\|_{H^{s,k}} + \big\|\boldsymbol{u}^{\mathrm{d}}(t,\cdot)\big\|_{H^{s,k}} + \mu^{1/2}\big\|w^{\mathrm{d}}(t,\cdot)\big\|_{H^{s,k}} + \big|\eta^{\mathrm{d}}\big|_{\varrho=\rho_0}(t,\cdot)\big|_{H^s_{\boldsymbol{x}}}\\ &+ \kappa^{1/2}\big\|h^{\mathrm{d}}(t,\cdot)\big\|_{H^{s,k}} + \mu^{1/2}\kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta^{\mathrm{d}}(t,\cdot)\big\|_{H^{s,k}}\\ &+ \kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta^{\mathrm{d}}\big\|_{L^2(0,t;H^{s,k})} + \kappa^{1/2}\big|\nabla_{\boldsymbol{x}}\eta^{\mathrm{d}}\big|_{\varrho=\rho_0}\big|_{L^2(0,t;H^s_{\boldsymbol{x}})}\\ &+ \kappa\big\|\nabla_{\boldsymbol{x}}h^{\mathrm{d}}\big\|_{L^2(0,t;H^{s,k})} + \mu^{1/2}\kappa\big\|\nabla^2_{\boldsymbol{x}}\eta^{\mathrm{d}}\big\|_{L^2(0,t;H^{s,k})}\end{aligned}$$

where we denote

$$h^{\mathrm{d}} := h^{\mathrm{nh}} - h^{\mathrm{h}}; \quad \eta^{\mathrm{d}} := \eta^{\mathrm{nh}} - \eta^{\mathrm{h}}; \quad \boldsymbol{u}^{\mathrm{d}} := \boldsymbol{u}^{\mathrm{nh}} - \boldsymbol{u}^{\mathrm{h}}; \quad w^{\mathrm{d}} := w^{\mathrm{nh}} - w^{\mathrm{h}}$$

with the usual notation for η^{nh}, η^{h}, and w^{h} is defined by (4.1). Denoting by $T^\star$ the maximal existence time of the non-hydrostatic solution provided by Proposition 3.3, we set

$$\begin{aligned}T_\star = \sup\Big\{0 < T < \min(T^\star, T^{\mathrm{h}}) \,:\, \forall\, t\in(0,T),\ h_\star/3 \le \underline{h}(\varrho) + h^{\mathrm{nh}}(t,\boldsymbol{x},\varrho) \le 3h^\star\\ \text{and} \quad \mathcal{F}(t) \le \mu^{1/2}M_0\exp(C_0 t), \quad \mathcal{F}(t) \le \kappa^{1/2}M_0\Big\},\end{aligned} \tag{4.10}$$

with C_0 sufficiently large (to be determined later on). We will show by the standard continuity argument that $T_\star = \min(T^\star, T^{\mathrm{h}})$, which in turns yields $T^\star > T^{\mathrm{h}}$ and shows the result. Indeed, the converse inequality $T_\star = T^\star \le T^{\mathrm{h}}$ yields a contradiction by Proposition 3.3 and the desired estimates immediately follow from the control of $\mathcal{F}$, the bound

$$\big\|h^{\mathrm{h}}(t,\cdot)\big\|_{H^{s+1,k+1}} + \big\|\boldsymbol{u}^{\mathrm{h}}(t,\cdot)\big\|_{H^{s+1,k+1}} + \big\|\eta^{\mathrm{h}}(t,\cdot)\big\|_{H^{s+1,k+1}} + \big\|w^{\mathrm{h}}(t,\cdot)\big\|_{H^{s+1,k+1}} \le C^{\mathrm{h}}M_0 \tag{4.11}$$

provided by Lemma 4.1, and triangular inequality (when C is chosen sufficiently large).

Let us now derive from Lemma 4.2 the necessary estimates for the bootstrap argument. In the following we repeatedly use the triangular inequality to infer from (4.10) and (4.11) the corresponding control (4.9) with C depending only C^{h}, T^{h} (and $\kappa \le 1$). We shall denote by C a constant depending uniquely on $s, k, \underline{M}, M, h_\star, h^\star$ and C^{h}, T^{h}, but not on C_0, and which may change from line to line.

By means of (4.7a)-(4.7b) and Lemma 2.4, we infer from (4.10)-(4.11)

$$\kappa^{1/2}\big\|h^{\mathrm{d}}\big\|_{L^\infty(0,T;H^{s,k})} + \kappa\big\|\nabla_{\boldsymbol{x}}h^{\mathrm{d}}\big\|_{L^2(0,T;H^{s,k})} \le C\left(\big|\mathcal{F}\big|_{L^1_T} + \big|\mathcal{F}\big|_{L^2_T}\right).$$

Next, by differentiating with respect to space the first equation of (4.6a) using (4.6b) and Lemma 2.4, we infer

$$\mu^{1/2}\kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta^{\mathrm{d}}\big\|_{L^\infty(0,T;H^{s,k})} + \kappa\big\|\nabla^2_{\boldsymbol{x}}\eta^{\mathrm{d}}\big\|_{L^2(0,T;H^{s,k})} \le C\left(\big|\mathcal{F}\big|_{L^1_T} + \big|\mathcal{F}\big|_{L^2_T}\right).$$

Now, we use (4.6a)-(4.6b) and proceeding as in the proof of Proposition 3.8 (together with the above estimates) we infer that for any $t \in (0,T)$,

$$\mathcal{F}(t) \le C_1\big|\mathcal{F}\big|_{L^1_t} + C_2\big|\mathcal{F}\big|_{L^2_t} + C_3\,\mu^{1/2}M_0\,t$$

with C_i ($i \in \{1,2,3\}$) depending uniquely on $s, k, \underline{M}, M, h_\star, h^\star$ and C^{h}, T^{h}. By using the inequality $\mathcal{F}(t) \le \mu^{1/2}M_0\exp(C_0 t)$ from (4.10) and the inequality $\tau \le \exp(\tau)$ ($\tau \ge 0$), we deduce

$$\mathcal{F}(t) \le C_1\mu^{1/2}M_0C_0^{-1}\exp(C_0t) + C_2\mu^{1/2}M_0(2C_0)^{-1/2}\exp(C_0t) + C_3\mu^{1/2}M_0C_0^{-1}\exp(C_0t).$$

There remains to choose C_0 sufficiently large so that $C_1C_0^{-1} + C_2(2C_0)^{-1/2} + C_3C_0^{-1} < 1$, and the constant C sufficiently large (whence μ sufficiently small) so that $\mu^{1/2}M_0\exp(C_0T^{\mathrm{h}}) \le \mu^{1/2}M_0C^{1/2} \le \kappa^{1/2}/2$. The upper and lower bounds for $\underline{h} + h^{\mathrm{nh}}$ follow immediately from the corresponding ones for $\underline{h} + h^{\mathrm{h}}$ provided by Lemma 4.1 and triangular inequality, augmenting C if necessary. Then the usual continuity argument yields, as desired, $T_\star = \min(T^\star, T^{\mathrm{h}})$.

Let us acknowledge that the necessary continuity of the functional $\mathcal{F}(t)$ is not provided by Proposition 3.3. In order to deal with this issue, we can, as detailed in Section 2, consider a parabolic regularization method. □

4.4. Improved convergence rate. Proposition 4.3 established the strong convergence for regular well-prepared initial data of the solution to the non-hydrostatic equations towards the corresponding solution to the hydrostatic equations as $\mu \searrow 0$. The convergence rate given by the proof is $\mathcal{O}(\mu^{1/2})$. The aim of this section is to provide an improved and *optimal* convergence rate $\mathcal{O}(\mu)$. The strategy is based on the interpretation of the non-hydrostatic solution as an approximate solution to the hydrostatic equations (in the sense of consistency) and the use of the uniform control obtained in Proposition 4.3.

PROPOSITION 4.4. *There exists $p \in \mathbb{N}$ such that for any $s, k \in \mathbb{N}$ with $k = s > \frac{5}{2} + \frac{d}{2}$ and $\underline{M}, M, h_\star, h^\star > 0$, there exists $C = C(s, k, \underline{M}, M, h_\star, h^\star) > 0$ such that under the assumptions of Proposition 4.3 and using the notations therein,*

$$\big\|h^{\rm nh} - h^{\rm h}\big\|_{L^\infty(0,T^{\rm h};H^{s-1,k-1})} + \big\|\eta^{\rm nh} - \eta^{\rm h}\big\|_{L^\infty(0,T^{\rm h};H^{s,k})} + \big\|\boldsymbol{u}^{\rm nh} - \boldsymbol{u}^{\rm h}\big\|_{L^\infty(0,T^{\rm h};H^{s,k})} \leq C\,\mu.$$

COROLLARY 4.5. *Incrementing $p \in \mathbb{N}$, we find that for any $s, k \in \mathbb{N}$ such that $k = s > \frac{3}{2} + \frac{d}{2}$,*

$$\big\|h^{\rm nh} - h^{\rm h}\big\|_{L^\infty(0,T^{\rm h};H^{s,k})} + \big\|\eta^{\rm nh} - \eta^{\rm h}\big\|_{L^\infty(0,T^{\rm h};H^{s+1,k+1})} + \big\|\boldsymbol{u}^{\rm nh} - \boldsymbol{u}^{\rm h}\big\|_{L^\infty(0,T^{\rm h};H^{s+1,k+1})} \leq C\,\mu$$

with $C = C(s+1, k+1, \underline{M}, M, h_\star, h^\star) > 0$.

PROOF. Since all arguments of the proof have been already used in slightly different context, we only quickly sketch the argument.

For any $p' \in \mathbb{N}$, we may use Proposition 4.3 with indices $s + p'$ and $k + p'$ to infer the existence of the non-hydrostatic solution $(h^{\rm nh}, \boldsymbol{u}^{\rm nh}, w^{\rm nh}) \in \mathcal{C}([0, T^{\rm h}]; H^{s+p',k+p'}(\Omega)^{1+d})$ and the control

$$\sup_{t\in[0,T^{\rm h}]} \Big(\big\|\eta^{\rm nh}(t,\cdot)\big\|_{H^{s+p',k+p'}} + \big\|\boldsymbol{u}^{\rm nh}(t,\cdot)\big\|_{H^{s+p',k+p'}} + \big|\eta^{\rm nh}\big|_{\varrho=\rho_0}(t,\cdot)\big|_{H^{s+p'}_{\boldsymbol{x}}} \Big) \leq C\,M_0.$$

By using $h = -\partial_\varrho \eta$ and the divergence-free condition

$$w = (\underline{\boldsymbol{u}} + \boldsymbol{u}) \cdot \nabla_{\boldsymbol{x}} \eta - \int_\varrho^{\rho_1} \nabla_{\boldsymbol{x}} \cdot \big((\underline{h} + h)(\underline{\boldsymbol{u}} + \boldsymbol{u})\big)\,\mathrm{d}\varrho'$$

we obtain (augmenting C if necessary)

$$\sup_{t\in[0,T^{\rm h}]} \Big(\big\|h^{\rm nh}(t,\cdot)\big\|_{H^{s+p'-1,k+p'-1}} + \big\|w^{\rm nh}(t,\cdot)\big\|_{H^{s+p'-1,k+p'-1}} \Big) \leq C\,M_0$$

and hence, by Corollary 3.2 (specifically (3.20)), Poincaré inequality (3.10) and choosing p' sufficiently large, that $P_{\rm nh}(\cdot,\varrho) := P^{\rm nh}(\cdot,\varrho) - \int_{\rho_0}^{\varrho} \varrho' h^{\rm nh}(\cdot,\varrho')\,\mathrm{d}\varrho'$ satisfies

$$\sup_{t\in[0,T^{\rm h}]} \big\|P_{\rm nh}(t,\cdot)\big\|_{H^{s+1,k+1}} \leq C\,\mu\,M_0.$$

From this estimate we infer (by Lemma 4.1) that $h^{\rm d} := h^{\rm nh} - h^{\rm h}$ and $\boldsymbol{u}^{\rm d} := \boldsymbol{u}^{\rm nh} - \boldsymbol{u}^{\rm h}$ satisfies $(\eta^d, \boldsymbol{u}^{\rm d})\big|_{t=0} = (0,0)$ and

$$\begin{aligned}
&\partial_t \eta^{\rm d} + (\underline{\boldsymbol{u}} + \boldsymbol{u}^{\rm nh}) \cdot \nabla_{\boldsymbol{x}} \eta^{\rm d} + \int_\varrho^{\rho_1} (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}^{\rm nh}) \cdot \nabla_{\boldsymbol{x}} \eta^{\rm d}\,\mathrm{d}\varrho' + \int_\varrho^{\rho_1} (\underline{h} + h^{\rm nh}) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{\rm d}\,\mathrm{d}\varrho' \\
&\qquad\qquad + \int_\varrho^{\rho_1} \boldsymbol{u}^{\rm d} \cdot \nabla_{\boldsymbol{x}} h^{\rm h} + h^{\rm d} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^{\rm h}\,\mathrm{d}\varrho' = \kappa \Delta_{\boldsymbol{x}} \eta^{\rm d}, \\
&\partial_t \boldsymbol{u}^{\rm d} + \big((\underline{\boldsymbol{u}} + \boldsymbol{u}^{\rm nh} - \kappa \tfrac{\nabla_{\boldsymbol{x}} h^{\rm nh}}{\underline{h} + h^{\rm nh}}) \cdot \nabla_{\boldsymbol{x}}\big) \boldsymbol{u}^{\rm d} + \frac{\rho_0}{\varrho} \nabla_{\boldsymbol{x}} \eta^{\rm d}|_{\varrho=\rho_0} + \frac{1}{\varrho} \int_{\rho_0}^{\varrho} \nabla_{\boldsymbol{x}} \eta^{\rm d}\,\mathrm{d}\varrho' \\
&\qquad\qquad + \big((\boldsymbol{u}^{\rm d} - \kappa(\tfrac{\nabla_{\boldsymbol{x}} h^{\rm nh}}{\underline{h} + h^{\rm nh}} - \tfrac{\nabla_{\boldsymbol{x}} h^{\rm h}}{\underline{h} + h^{\rm h}})) \cdot \nabla_{\boldsymbol{x}}\big) \boldsymbol{u}^{\rm h} = \boldsymbol{R}^{\rm nh},
\end{aligned} \tag{4.12}$$

with $\boldsymbol{R}^{\rm nh} := -\frac{\nabla_{\boldsymbol{x}} P_{\rm nh}}{\varrho} - \frac{\nabla_{\boldsymbol{x}}\eta^{\rm nh}}{\varrho(\underline{h}+h^{\rm nh})}\partial_\varrho P_{\rm nh}$ satisfying (by Lemma A.3 and Lemma A.6)

$$\sup_{t\in[0,T^{\rm h}]} \big\|\boldsymbol{R}^{\rm nh}(t,\cdot)\big\|_{H^{s,k}} \leq C\,\mu\,M_0.$$

From this, inspecting the proof of Proposition 4.2, we infer that as long as

$$\mathcal{F}_{s,k} := \big\|h^{\rm d}\big\|_{H^{s-1,k-1}} + \big\|\eta^{\rm d}\big\|_{H^{s,k}} + \big|\eta^{\rm d}\big|_{\varrho=\rho_0}\big|_{H^s_{\boldsymbol{x}}} + \big\|\boldsymbol{u}^{\rm d}\big\|_{H^{s,k}} + \kappa^{1/2}\big\|h^{\rm d}\big\|_{H^{s,k}} \leq \kappa^{1/2} M_0,$$

one has for any $\alpha\in\mathbb{N}^d$ and $j\in\mathbb{N}$ such that $|\alpha|+j\leq s$ that $\eta^{(\alpha,j)} := \partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho\eta^{\rm d}$, $\boldsymbol{u}^{(\alpha,j)} := \partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho\boldsymbol{u}^{\rm d}$ and $\mathrm{h}^{(\alpha,j)} := \partial^\alpha_{\boldsymbol{x}}\partial^j_\varrho h^{\rm d}$ satisfy

$$\begin{aligned}\partial_t\eta^{(\alpha,j)} + (\underline{\boldsymbol{u}}+\boldsymbol{u}^{\rm nh})\cdot\nabla_{\boldsymbol{x}}\eta^{(\alpha,j)} + \Big\langle \int_\varrho^{\rho_1} (\underline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}^{\rm nh})\cdot\nabla_{\boldsymbol{x}}\eta^{(\alpha,j)}\,\mathrm{d}\varrho' \\ + \int_\varrho^{\rho_1} (\underline{h}+h^{\rm nh})\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}^{(\alpha,j)}\,\mathrm{d}\varrho'\Big\rangle_{j=0} = \kappa\Delta_{\boldsymbol{x}}\eta^{(\alpha,j)} + R_{\alpha,j},\end{aligned}$$

$$\partial_t\boldsymbol{u}^{(\alpha,j)} + \big((\underline{\boldsymbol{u}}+\boldsymbol{u}^{\rm nh} - \kappa\tfrac{\nabla_{\boldsymbol{x}}h^{\rm nh}}{\underline{h}+h^{\rm nh}})\cdot\nabla_{\boldsymbol{x}}\big)\boldsymbol{u}^{(\alpha,j)} + \Big\langle \frac{\rho_0}{\varrho}\nabla_{\boldsymbol{x}}\eta^{(\alpha,j)}\big|_{\varrho=\rho_0} + \frac{1}{\varrho}\int_{\rho_0}^{\varrho}\nabla_{\boldsymbol{x}}\eta^{(\alpha,j)}\,\mathrm{d}\varrho'\Big\rangle_{j=0} = \boldsymbol{R}^{\rm nh}_{\alpha,j},$$

and

$$\partial_t h^{(\alpha,j)} + (\underline{\boldsymbol{u}}+\boldsymbol{u})\cdot\nabla_{\boldsymbol{x}}h^{(\alpha,j)} = \kappa\Delta_{\boldsymbol{x}}\partial^j_\varrho h^{(\alpha,j)} + r_{\alpha,j} + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{r}_{\alpha,j},$$

with

$$\big\|R_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|\boldsymbol{R}^{\rm nh}_{\alpha,j}\big\|_{L^2(\Omega)} \leq C\left(\mathcal{F}_{s,k} + M_0\kappa\big\|\nabla_{\boldsymbol{x}}h^{\rm d}\big\|_{H^{s,k}}\right) + C\,\mu\,M_0$$

and

$$\kappa^{1/2}\big\|r_{\alpha,j}\big\|_{L^2(\Omega)} + \big\|\boldsymbol{r}_{\alpha,j}\big\|_{L^2(\Omega)} \leq C\,\mathcal{F}_{s,k}.$$

We may then proceed as in the proof of Proposition 2.6, and bootstrap the control

$$\mathcal{F}(t) := \mathcal{F}_{s,k}(t) + \kappa^{1/2}\big\|\nabla_{\boldsymbol{x}}\eta^{\rm d}\big\|_{L^2(0,t;H^{s,k})} + \kappa^{1/2}\big|\nabla_{\boldsymbol{x}}\eta^{\rm d}\big|_{\varrho=\rho_0}\big|_{L^2(0,t;H^s_{\boldsymbol{x}})} + \kappa\big\|\nabla_{\boldsymbol{x}}h^{\rm d}\big\|_{L^2(0,t;H^{s,k})} \leq C\,\mu\,M_0,$$

on the time interval $[0,T^{\rm h}]$. This concludes the proof. □

Appendix A. Product, composition and commutator estimates

In this section we collect useful estimates in the spaces $H^{s,k}(\Omega)$ introduced in (1.10). Our results will follow from standard estimates in Sobolev spaces $H^s(\mathbb{R}^d)$ (see *e.g.* [21, Appendix B] and references therein), and the following continuous embedding. Henceforth we denote $\Omega = \mathbb{R}^d\times(\rho_0,\rho_1)$.

LEMMA A.1. *For any $s\in\mathbb{R}$ and $\rho_0<\rho_1$, $H^{s+1/2,1}(\Omega)\subset\mathcal{C}^0([\rho_0,\rho_1];H^s(\mathbb{R}^d))$ and there exists $C>0$ such that for any $F\in H^{s+1/2,1}(\Omega)$,*

$$\max_{\varrho\in[\rho_0,\rho_1]}\big|F(\cdot,\varrho)\big|_{H^s_{\boldsymbol{x}}} \leq C\big\|F\big\|_{H^{s+1/2,1}}.$$

More generally, for any $k\geq 1$, $H^{s+1/2,1}(\Omega)\subset\bigcap_{j=0}^{k-1}\mathcal{C}^j([\rho_0,\rho_1];H^{s-j}(\mathbb{R}^d))$, and in particular, for any $s_0>d/2$ and $j\in\mathbb{N}$, $H^{j+s_0+\frac12,j+1}(\Omega)\subset\big(\mathcal{C}^j(\Omega)\cap W^{j,\infty}(\Omega)\big)$.

PROOF. By a density argument, we only need to prove the inequality for smooth functions F. Set $\phi:[\rho_0,\rho_1]\to\mathbb{R}^+$ a smooth function such that $\phi(\rho_0)=0$ and $\phi(\varrho)=1$ if $\varrho\geq\frac{\rho_0+\rho_1}{2}$, and deduce that for

any $\varrho \geq \frac{\rho_0+\rho_1}{2}$, recalling the notation $\Lambda^s := (\mathrm{Id} - \Delta_{\boldsymbol{x}})^{s/2}$,

$$\begin{aligned}\int_{\mathbb{R}^d} (\Lambda^s F)^2(\boldsymbol{x},\varrho)\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^d}\int_{\rho_0}^{\varrho} \partial_\varrho\big(\phi(\varrho')(\Lambda^s F)^2(\boldsymbol{x},\varrho')\big)\,\mathrm{d}\varrho'\,\mathrm{d}\boldsymbol{x}\\ &\leq 2\big|\phi\big|_{L^\infty_\varrho}\int_{\rho_0}^{\rho_1}\big|\Lambda^s F(\cdot,\varrho)\big|_{H^{1/2}_{\boldsymbol{x}}}\big|\Lambda^s\partial_\varrho F(\cdot,\varrho)\big|_{H^{-1/2}_{\boldsymbol{x}}}\,\mathrm{d}\varrho\\ &\quad + \big|\phi'\big|_{L^\infty}\int_{\rho_0}^{\rho_1}\big|\Lambda^s F(\cdot,\varrho)\big|_{L^2}\big|\Lambda^s F(\cdot,\varrho)\big|_{L^2}\,\mathrm{d}\varrho\\ &\lesssim \big\|F\big\|^2_{H^{s+1/2,0}} + \big\|\partial_\varrho F\big\|^2_{H^{s-1/2,0}}\end{aligned}$$

Using symmetrical considerations when $\varrho < \frac{\rho_0+\rho_1}{2}$, we prove the claimed inequality, which yields the first continuous embedding. Higher-order embeddings follow immediately. □

Recall the notation

$$A_s + \big\langle B_s\big\rangle_{s>s_\star} = \begin{cases} A_s & \text{if } s \leq s_\star\,,\\ A_s + B_s & \text{otherwise.}\end{cases}$$

Product estimates. Recall the standard product estimates in Sobolev spaces $H^s(\mathbb{R}^d)$.

LEMMA A.2. *Let $d \in \mathbb{N}^\star$, $s_0 > d/2$.*

(1) *For any $s, s_1, s_2 \in \mathbb{R}$ such that $s_1 \geq s$, $s_2 \geq s$ and $s_1 + s_2 \geq s + s_0$, there exists $C > 0$ such that for any $f \in H^{s_1}(\mathbb{R}^d)$ and $g \in H^{s_2}(\mathbb{R}^d)$, $fg \in H^s(\mathbb{R}^d)$ and*

$$\big|fg\big|_{H^s} \leq C\big|f\big|_{H^{s_1}}\big|g\big|_{H^{s_2}}.$$

(2) *For any $s \geq -s_0$, there exists $C > 0$ such that for any $f \in H^s(\mathbb{R}^d)$ and $g \in H^s(\mathbb{R}^d) \cap H^{s_0}(\mathbb{R}^d)$, $fg \in H^s(\mathbb{R}^d)$ and*

$$\big|fg\big|_{H^s} \leq C\big|f\big|_{H^{s_0}}\big|g\big|_{H^s} + C\,\big\langle\big|f\big|_{H^s}\big|g\big|_{H^{s_0}}\big\rangle_{s>s_0}.$$

(3) *For any $s_1, \dots, s_n \in \mathbb{R}$ such that $s_i \geq 0$ and $s_1 + \cdots + s_n \geq (n-1)s_0$, there exists $C > 0$ such that for any $(f_1, \dots, f_n) \in H^{s_1}(\mathbb{R}^d) \times \cdots \times H^{s_n}(\mathbb{R}^d)$, $\prod_{i=1}^n f_i \in L^2(\mathbb{R}^d)$ and*

$$\big|\prod_{i=1}^n f_i\big|_{L^2} \leq C\prod_{i=1}^n \big|f_i\big|_{H^{s_i}}.$$

Let us turn to product estimates in $H^{s,k}(\Omega)$ spaces.

LEMMA A.3. *Let $d \in \mathbb{N}^\star$, $s_0 > d/2$. Let $s, k \in \mathbb{N}$ such that $s \geq s_0 + \frac{1}{2}$ and $1 \leq k \leq s$. Then $H^{s,k}(\Omega)$ is a Banach algebra and there exists $C > 0$ such that for any $F, G \in H^{s,k}(\Omega)$,*

$$\big\|FG\big\|_{H^{s,k}} \leq C\big\|F\big\|_{H^{s,k}}\big\|G\big\|_{H^{s,k}}.$$

Moreover, if $s \geq s_0 + \frac{3}{2}$ and $2 \leq k \leq s$, then there exists $C' > 0$ such that for any $F, G \in H^{s,k}(\Omega)$,

$$\big\|FG\big\|_{H^{s,k}} \leq C'\big\|F\big\|_{H^{s,k}}\big\|G\big\|_{H^{s-1,k-1}} + C'\big\|F\big\|_{H^{s-1,k-1}}\big\|G\big\|_{H^{s,k}},$$

and if $s \geq s_0 + \frac{3}{2}$ and $k = 1$, then there exists $C'' > 0$ such that for any $F, G \in H^{s,k}(\Omega)$,

$$\big\|FG\big\|_{H^{s,1}} \leq C''\big\|F\big\|_{H^{s,1}}\big\|G\big\|_{H^{s-1,1}} + C''\big\|F\big\|_{H^{s-1,1}}\big\|G\big\|_{H^{s,1}}.$$

PROOF. We set two multi-indices $\beta = (\beta_{\boldsymbol{x}}, \beta_\varrho) \in \mathbb{N}^{d+1}$ and $\gamma = (\gamma_{\boldsymbol{x}}, \gamma_\varrho) \in \mathbb{N}^{d+1}$ being such that $|\beta| + |\gamma| \leq s$ and $\beta_\varrho + \gamma_\varrho \leq k$. Let us first assume furthermore that $\gamma_\varrho \leq k-1$ and $|\gamma| \leq s-1$. Then

$$\begin{aligned}\big\|(\partial^\beta F)(\partial^\gamma G)\big\|^2_{L^2(\Omega)} &\lesssim \int_{\rho_0}^{\rho_1}\big|\partial^\beta F(\cdot,\varrho)\big|^2_{H^{s-|\beta|}_{\boldsymbol{x}}}\big|\partial^\gamma G(\cdot,\varrho)\big|^2_{H^{s-|\gamma|-\frac{1}{2}}_{\boldsymbol{x}}}\,\mathrm{d}\varrho\\ &\lesssim \big\|\partial^\beta F\big\|^2_{H^{s-|\beta|,0}}\big\|\partial^\gamma G\big\|^2_{H^{s-|\gamma|,1}} \leq \big\|F\big\|^2_{H^{s,k}}\big\|G\big\|^2_{H^{s,k}}.\end{aligned}$$

where we used Lemma A.2(1) with $(s, s_1, s_2) = (0, s-|\beta|, s-|\gamma|-\frac{1}{2})$, and Lemma A.1. If $\gamma_\varrho = k$ or $|\gamma| = s$, and since $1 \le k \le s$, we have $\beta_\varrho \le k-1$ and $|\beta| \le s-1$ and we may make use of the symmetric estimate. Hence the proof of the first statement follows from Leibniz rule.

For the second statement, we assume first that $\max(\{\beta_\varrho, \gamma_\varrho\}) \le k-1$ and $\max(\{|\beta|, |\gamma|\}) \le s-1$. Then, using Lemma A.2 with $(s, s_1, s_2) = (0, s-|\beta|-\frac{1}{2}, s-|\gamma|-1)$ (recall $s \ge s_0 + \frac{3}{2}$), and Lemma A.1,

$$\begin{aligned}\big\|(\partial^\beta F)(\partial^\gamma G)\big\|_{L^2(\Omega)} &\lesssim \big|\big|\partial^\beta F(\cdot,\varrho)\big|_{H^{s-|\beta|-\frac{1}{2}}_{\boldsymbol{x}}}\big|\partial^\gamma G(\cdot,\varrho)\big|_{H^{s-|\gamma|-1}_{\boldsymbol{x}}}\big|_{L^2_\varrho}\\ &\lesssim \big\|F\big\|_{H^{s,\beta_\varrho+1}}\big\|G\big\|_{H^{s-1,\gamma_\varrho}} \le \big\|F\big\|_{H^{s,k}}\big\|G\big\|_{H^{s-1,k-1}},\end{aligned}$$

Then if $\beta_\varrho = k$ or $|\beta| = s$, we have (since $s \ge k \ge 2$) $\gamma_\varrho \le k-2$ and $|\gamma| \le s-2$, and we infer

$$\begin{aligned}\big\|(\partial^\beta F)(\partial^\gamma G)\big\|_{L^2(\Omega)} &\lesssim \big|\big|\partial^\beta F(\cdot,\varrho)\big|_{H^{s-|\beta|}_{\boldsymbol{x}}}\big|\partial^\gamma G(\cdot,\varrho)\big|_{H^{s-|\gamma|-\frac{3}{2}}_{\boldsymbol{x}}}\big|_{L^2_\varrho}\\ &\lesssim \big\|F\big\|_{H^{s,\beta_\varrho}}\big\|G\big\|_{H^{s-1,\gamma_\varrho+1}} \le \big\|F\big\|_{H^{s,k}}\big\|G\big\|_{H^{s-1,k-1}}.\end{aligned}$$

Of course we have the symmetrical result when $\gamma_\varrho = k$ or $|\gamma| = s$, which complete the proof.

Finally, for the last statement, we consider first the case $\beta_\varrho = 0$ and $\max(\{|\beta|, |\gamma|\}) \le s-1$, and infer as above

$$\big\|(\partial^\beta F)(\partial^\gamma G)\big\|_{L^2(\Omega)} \lesssim \big|\big|\partial^\beta F(\cdot,\varrho)\big|_{H^{s-|\beta|-\frac{1}{2}}_{\boldsymbol{x}}}\big|\partial^\gamma G(\cdot,\varrho)\big|_{H^{s-|\gamma|-1}_{\boldsymbol{x}}}\big|_{L^2_\varrho} \lesssim \big\|F\big\|_{H^{s,1}}\big\|G\big\|_{H^{s-1,1}}.$$

The case $\beta_\varrho = 1$ (and hence $\gamma_\varrho = 0$) and $\max(\{|\beta|, |\gamma|\}) \le s-1$ is treated symmetrically. Then if $|\beta| = s$ we have $\gamma_\varrho = |\gamma| = 0$, and we infer

$$\big\|(\partial^\beta F)(\partial^\gamma G)\big\|_{L^2(\Omega)} \lesssim \big|\big|\partial^\beta F(\cdot,\varrho)\big|_{H^{s-|\beta|}_{\boldsymbol{x}}}\big|\partial^\gamma G(\cdot,\varrho)\big|_{H^{s-|\gamma|-\frac{3}{2}}_{\boldsymbol{x}}}\big|_{L^2_\varrho} \lesssim \big\|F\big\|_{H^{s,1}}\big\|G\big\|_{H^{s-1,1}}.$$

The case $|\gamma| = s$ is treated symmetrically, and the proof is complete. □

Composition estimates. Let us recall the standard composition estimate in Sobolev spaces $H^s(\mathbb{R}^d)$.

LEMMA A.4. *Let $d \in \mathbb{N}^\star$, $s_0 > d/2$. For any $\varphi \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ such that $\varphi(0) = 0$, and any $M > 0$, there exists $C > 0$ such that for any $f \in H^{s_0}(\mathbb{R}^d) \cap H^s(\mathbb{R}^d)$ with $\big|f\big|_{H^{s_0}} \le M$, one has $\varphi(f) \in H^s(\mathbb{R}^d)$ and*

$$\big|\varphi(f)\big|_{H^s} \le C\big|f\big|_{H^s}.$$

We now consider composition estimates in $H^{s,k}(\Omega)$.

LEMMA A.5. *Let $d \in \mathbb{N}^\star$, $s_0 > d/2$. Let $s, k \in \mathbb{N}$ with $s \ge s_0 + \frac{1}{2}$ and $1 \le k \le s$, and $M > 0$. There exists $C > 0$ such that for any $\varphi \in W^{s,\infty}(\mathbb{R}; W^{k,\infty}((\rho_0,\rho_1)))$ with $\varphi(0;\cdot) \equiv 0$, and any $F \in H^{s,k}(\Omega)$ such that $\big\|F\big\|_{H^{s,k}} \le M$, then $\varphi \circ F : (\boldsymbol{x},\varrho) \mapsto \varphi(F(\boldsymbol{x},\varrho);\varrho) \in H^{s,k}(\Omega)$ and*

$$\big\|\varphi\circ F\big\|_{H^{s,k}} \le C\big|\varphi\big|_{W^{s,\infty}(\mathbb{R};W^{k,\infty}((\rho_0,\rho_1)))}\big\|F\big\|_{H^{s,k}}.$$

If moreover $s \ge s_0 + \frac{3}{2}$ and $2 \le k \le s$, then there exists $C' > 0$ such that for any $F \in H^{s,k}(\Omega)$ such that $\big\|F\big\|_{H^{s-1,k-1}} \le M$,

$$\big\|\varphi\circ F\big\|_{H^{s,k}} \le C'\big|\varphi\big|_{W^{s,\infty}(\mathbb{R};W^{k,\infty}((\rho_0,\rho_1)))}\big\|F\big\|_{H^{s,k}}.$$

PROOF. Let $\alpha = (\alpha_{\boldsymbol{x}}, \alpha_\varrho) \in \mathbb{N}^{d+1} \setminus \{\mathbf{0}\}$ with $0 \le |\alpha| \le s$ and $0 \le \alpha_\varrho \le k$. We have by Faà di Bruno's formula

$$\big\|\partial^\alpha(\varphi\circ F)\big\|_{L^2(\Omega)} \lesssim \sum \big\|\big((\partial_1^i\partial_2^j\varphi)\circ F\big)\,(\partial^{\alpha_1^{i,j}}F)\cdots(\partial^{\alpha_i^{i,j}}F)\big\|_{L^2(\Omega)},$$

where $i, j \in \mathbb{N}$ with $i + j \le |\alpha| \le s$, and the multi-indices $\alpha_\ell^{i,j} = (\alpha_{\ell,\boldsymbol{x}}^{i,j}, \alpha_{\ell,\varrho}^{i,j}) \in \mathbb{N}^{d+1} \setminus \{\mathbf{0}\}$ satisfy $\sum_{\ell=1}^i \alpha_{\ell,\boldsymbol{x}}^{i,j} = \alpha_{\boldsymbol{x}}$ and $j + \sum_{\ell=1}^i \alpha_{\ell,\varrho}^{i,j} = \alpha_\varrho$. If $i = 0$ then we have from the mean value theorem that for any $0 \le j \le k$

$$\big\|(\partial_2^j\varphi)\circ F\big\|_{L^2(\Omega)} = \big\|(\partial_2^j\varphi)\circ F - (\partial_2^j\varphi)\circ 0\big\|_{L^2(\Omega)} \le \big|\partial_1\partial_2^j\varphi\big|_{L^\infty(\mathbb{R}\times(\rho_0,\rho_1))}\big\|F\big\|_{L^2(\Omega)}.$$

The case $i=1$ is straightforward, and we now focus on the case $i\geq 2$. We assume without loss of generality that $\alpha^{i,j}_{1,\varrho}\geq\alpha^{i,j}_{2,\varrho}\geq\cdots\geq\alpha^{i,j}_{i,\varrho}$ and remark that for $\ell\neq 1$, $\alpha^{i,j}_{\ell,\varrho}\leq k-1$ (otherwise $\alpha^{i,j}_{1,\varrho}+\alpha^{i,j}_{\ell,\varrho}=2k>k\geq\alpha_\varrho$) and $|\alpha^{i,j}_\ell|\leq s-|\alpha^{1,j}_1|\leq s-1$. Hence we have

$$\begin{aligned}\big\|\textstyle\prod_{\ell=1}^i(\partial^{\alpha^{i,j}_\ell}F)\big\|_{L^2(\Omega)}&\lesssim\big|\big|\partial^{\alpha^{i,j}_1}F\big|_{H^{s-|\alpha^{i,j}_1|}_{\boldsymbol{x}}}\big(\textstyle\prod_{\ell=2}^i\big|\partial^{\alpha^{i,j}_\ell}F\big|_{H^{s-|\alpha^{i,j}_2|-\frac12}_{\boldsymbol{x}}}\big)\big|_{L^2_\varrho}\\&\lesssim\big\|F\big\|_{H^{s,\alpha^{i,j}_{1,\varrho}}}\big(\textstyle\prod_{\ell=2}^i\big\|F\big\|_{H^{s,\alpha^{i,j}_{\ell,\varrho}+1}}\big)\leq\big\|F\big\|^i_{H^{s,k}}\end{aligned}$$

where we used Lemma A.2(3) and $(i-1)(s-\frac12)\geq(i-1)s_0$ and Lemma A.1. The first claim follows.

Now we assume additionally that $k\geq 2$ and $s\geq s_0+\frac32$. The cases $i\in\{0,1\}$ can be treated exactly as above and we deal only with the case $i\geq 2$, ordering $\alpha^{i,j}_{1,\varrho}\geq\alpha^{i,j}_{2,\varrho}\geq\cdots\geq\alpha^{i,j}_{i,\varrho}$ as above. Assume first that $\alpha^{i,j}_{1,\varrho}=k\geq 2$. Then for all $\ell\neq 1$, $\alpha^{i,j}_{\ell,\varrho}=0$ and $|\alpha^{i,j}_\ell|\leq s-2$, and we conclude as before with

$$\big\|\textstyle\prod_{\ell=1}^i(\partial^{\alpha^{i,j}_\ell}F)\big\|_{L^2(\Omega)}\lesssim\big|\big|\partial^{\alpha^{i,j}_1}F\big|_{H^{s-|\alpha^{i,j}_1|}_{\boldsymbol{x}}}\big(\textstyle\prod_{\ell=1}^i\big|\partial^{\alpha^{i,j}_\ell}F\big|_{H^{s-|\alpha^{i,j}_\ell|-\frac32}_{\boldsymbol{x}}}\big)\big|_{L^2_\varrho}\lesssim\big\|F\big\|_{H^{s,k}}\big\|F\big\|^{i-1}_{H^{s-1,1}}.$$

Otherwise we have $\alpha^{i,j}_{2,\varrho}\leq\alpha^{i,j}_{1,\varrho}\leq k-1$ and $|\alpha^{i,j}_2|\leq s-|\alpha^{i,j}_1|\leq s-1$ and notice that for $\ell\geq 3$, $\alpha^{i,j}_{\ell,\varrho}\leq k-2$ (since otherwise we have the contradiction $\alpha^{i,j}_{1,\varrho}+\alpha^{i,j}_{2,\varrho}+\alpha^{i,j}_{3,\varrho}\geq 3(k-1)\geq k+1\geq\alpha_\varrho+1$) and $|\alpha^{i,j}_\ell|\leq s-|\alpha^{i,j}_1|-|\alpha^{i,j}_2|\leq s-2$. Hence

$$\begin{aligned}\big\|\textstyle\prod_{\ell=1}^i(\partial^{\alpha^{i,j}_\ell}F)\big\|_{L^2(\Omega)}&\lesssim\big|\big|\partial^{\alpha^{i,j}_1}F\big|_{H^{s-|\alpha^{i,j}_1|-\frac12}_{\boldsymbol{x}}}\big|\partial^{\alpha^{i,j}_2}F\big|_{H^{s-|\alpha^{i,j}_2|-1}_{\boldsymbol{x}}}\big(\textstyle\prod_{\ell=3}^i\big|\partial^{\alpha^{i,j}_\ell}F\big|_{H^{s-|\alpha^{i,j}_\ell|-\frac32}_{\boldsymbol{x}}}\big)\big|_{L^2_\varrho}\\&\lesssim\big\|F\big\|_{H^{s,\alpha^{i,j}_{1,\varrho}+1}}\big\|F\big\|_{H^{s-1,\alpha^{i,j}_{2,\varrho}}}\big(\textstyle\prod_{\ell=3}^i\big\|F\big\|_{H^{s-1,\alpha^{i,j}_{\ell,\varrho}+1}}\big)\lesssim\big\|F\big\|_{H^{s,k}}\big\|F\big\|^{i-1}_{H^{s-1,k-1}}.\end{aligned}$$

This concludes the proof. □

We shall apply the above to estimate quantities such as (but not restricted to)

$$\Phi:(\boldsymbol{x},\varrho)\in\Omega\mapsto\frac{h(\boldsymbol{x},\varrho)}{\underline{h}(\varrho)+h(\boldsymbol{x},\varrho)},$$

with $\underline{h}\in W^{k,\infty}((\rho_0,\rho_1))$ and $h\in H^{s,k}(\mathbb{R}^d)$ satisfying the condition $\inf_{(\boldsymbol{x},\varrho)\in\Omega}\underline{h}(\varrho)+h(\boldsymbol{x},\varrho)\geq h_\star>0$. Let us detail the result and its proof for this specific example.

LEMMA A.6. *Let $d\in\mathbb{N}^\star$, $s_0>d/2$. Let $s,k\in\mathbb{N}$ with $s\geq s_0+\frac12$ and $1\leq k\leq s$, and $M,\underline{M},h_\star>0$. There exists $C>0$ such that for any $\underline{h}\in W^{k,\infty}((\rho_0,\rho_1))$ with $\big|\underline{h}\big|_{W^{k,\infty}_\varrho}\leq\underline{M}$ and any $h\in H^{s,k}(\Omega)$ with $\big\|h\big\|_{H^{s,k}}\leq M$ and satisfying the condition $\inf_{(\boldsymbol{x},\varrho)\in\Omega}\underline{h}(\varrho)+h(\boldsymbol{x},\varrho)\geq h_\star$, then*

$$\Phi:(\boldsymbol{x},\varrho)\mapsto\frac{h(\boldsymbol{x},\varrho)}{\underline{h}(\varrho)+h(\boldsymbol{x},\varrho)}\in H^{s,k}(\Omega),$$

and

$$\big\|\Phi\big\|_{H^{s,k}}\leq C\big\|h\big\|_{H^{s,k}}.$$

If moreover $s>\frac{d}{2}+\frac32$ and $2\leq k\leq s$, then the above holds for any $h\in H^{s,k}(\Omega)$ with $\big\|h\big\|_{H^{s-1,k-1}}\leq M$.

PROOF. We can write $\Phi=\varphi\circ h$ with $\varphi(\cdot,\varrho)=f(\cdot,\underline{h}(\varrho))$ where $f\in\mathcal{C}^\infty(\mathbb{R}^2)$ is set such that $f(y,z)=\frac{y}{y+z}$ on $\omega:=\{(y,z)\,:\,|y|\leq\big\|h\big\|_{L^\infty(\Omega)},\,|z|\leq\big|\underline{h}\big|_{L^\infty((\rho_0,\rho_1))},\,y+z\geq h_\star\}$. We can construct f as above such that the control of $\big|\varphi\big|_{W^{s,\infty}(\mathbb{R};W^{k,\infty}((\rho_0,\rho_1)))}$ depends only on $\big\|h\big\|_{L^\infty(\Omega)}$ (which is bounded appealing to Lemma A.1, if $h\in H^{s,k}$ with $s>\frac{d}{2}+\frac12$, $1\leq k\leq s$), $\big|\underline{h}\big|_{W^{k,\infty}((\rho_0,\rho_1))}$ and $h_\star>0$. The result is now a direct application of Lemma A.5. □

Commutator estimates. We now recall standard commutator estimates in $H^s(\mathbb{R}^d)$.

LEMMA A.7. *Let $d \in \mathbb{N}^\star$, $s_0 > d/2$ and $s \geq 0$.*

(1) *For any $s_1, s_2 \in \mathbb{R}$ such that $s_1 \geq s$, $s_2 \geq s-1$ and $s_1 + s_2 \geq s + s_0$, there exists $C > 0$ such that for any $f \in H^{s_1}(\mathbb{R}^d)$ and $g \in H^{s_2}(\mathbb{R}^d)$, $[\Lambda^s, f]g := \Lambda^s(fg) - f\Lambda^s g \in L^2(\mathbb{R}^d)$ and*
$$\big|[\Lambda^s, f]g\big|_{L^2} \leq C|f|_{H^{s_1}}|g|_{H^{s_2}}.$$

(2) *There exists $C > 0$ such that for any $f \in L^\infty(\mathbb{R}^d)$ such that $\nabla f \in H^{s-1}(\mathbb{R}^d) \cap H^{s_0}(\mathbb{R}^d)$ and for any $g \in H^{s-1}(\mathbb{R}^d)$, one has $[\Lambda^s, f]g \in L^2(\mathbb{R}^d)$ and*
$$\big|[\Lambda^s, f]g\big|_{L^2} \leq C|\nabla f|_{H^{s_0}}|g|_{H^{s-1}} + C\left\langle |\nabla f|_{H^{s-1}}|g|_{H^{s_0}}\right\rangle_{s>s_0+1}.$$

(3) *There exists $C > 0$ such that for any $f, g \in H^s(\mathbb{R}^d) \cap H^{s_0+1}(\mathbb{R}^d)$, the symmetric commutator $[\Lambda^s; f, g] := \Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f \in L^2(\mathbb{R}^d)$ and*
$$\big|[\Lambda^s; f, g]\big|_{L^2} \leq C|f|_{H^{s_0+1}}|g|_{H^{s-1}} + C|f|_{H^{s-1}}|g|_{H^{s_0+1}}.$$

The validity of the above estimates persist when replacing the operator Λ^s with the operator ∂^α with $\alpha \in \mathbb{N}^d$ a multi-index such that $|\alpha| \leq s$.

We conclude with commutator estimates in the spaces $H^{s,k}(\Omega)$.

LEMMA A.8. *Let $d \in \mathbb{N}^\star$, $s_0 > d/2$. Let $s \geq s_0 + \frac{3}{2}$ and $k \in \mathbb{N}$ such that $2 \leq k \leq s$. Then there exists $C > 0$ such that for any $\alpha = (\alpha_{\boldsymbol{x}}, \alpha_\varrho) \in \mathbb{N}^{d+1}$ with $|\alpha| \leq s$ and $\alpha_\varrho \leq k$, one has*
$$\big\|[\partial^\alpha, F]G\big\|_{L^2(\Omega)} \leq C\big\|F\big\|_{H^{s,k}}\big\|G\big\|_{H^{s-1,\min(\{k,s-1\})}}.$$

PROOF. We set two multi-indices $\beta = (\beta_{\boldsymbol{x}}, \beta_\varrho) \in \mathbb{N}^{d+1}$ and $\gamma = (\gamma_{\boldsymbol{x}}, \gamma_\varrho) \in \mathbb{N}^{d+1}$ with $\beta + \gamma = \alpha$, and $|\gamma| \leq s-1$. Assume first that $\beta_\varrho \leq k-1$ and $|\beta| \leq s-1$. Then
$$\begin{aligned}\big\|(\partial^\beta F)(\partial^\gamma G)\big\|_{L^2(\Omega)} &\lesssim \big|\big|\partial^\beta F(\cdot,\varrho)\big|_{H^{s-|\beta|-\frac{1}{2}}_{\boldsymbol{x}}}\big|\partial^\gamma G(\cdot,\varrho)\big|_{H^{s-|\gamma|-1}_{\boldsymbol{x}}}\big|_{L^2_\varrho}\\ &\lesssim \big\|F\big\|_{H^{s,\beta_\varrho+1}}\big\|G\big\|_{H^{s-1,\gamma_\varrho}} \leq \big\|F\big\|_{H^{s,k}}\big\|G\big\|_{H^{s-1,\min(\{k,s-1\})}},\end{aligned}$$
where we used Lemma A.2(1) with $(s, s_1, s_2) = (0, s-|\beta|-\frac{1}{2}, s-|\gamma|-1)$, and Lemma A.1. Otherwise $\gamma_\varrho = 0$ and $|\gamma| \leq s - |\beta| \leq s-2$, and we have
$$\begin{aligned}\big\|(\partial^\beta F)(\partial^\gamma G)\big\|_{L^2(\Omega)} &\lesssim \big|\big|\partial^\beta F(\cdot,\varrho)\big|_{H^{s-|\beta|}_{\boldsymbol{x}}}\big|\partial^\gamma G(\cdot,\varrho)\big|_{H^{s-|\gamma|-\frac{3}{2}}_{\boldsymbol{x}}}\big|_{L^2_\varrho}\\ &\lesssim \big\|F\big\|_{H^{s,\beta_\varrho}}\big\|G\big\|_{H^{s-1,1}} \leq \big\|F\big\|_{H^{s,k}}\big\|G\big\|_{H^{s-1,\min(\{k,s-1\})}}.\end{aligned}$$
The claim follows from decomposing $[\partial^\alpha, F]G$ as a sum of products as above. □

LEMMA A.9. *Let $d \in \mathbb{N}^\star$, $s_0 > d/2$. Let $s \geq s_0 + \frac{5}{2}$ and $k \in \mathbb{N}$ such that $2 \leq k \leq s$. Then there exists $C > 0$ such that for any $\alpha = (\alpha_{\boldsymbol{x}}, \alpha_\varrho) \in \mathbb{N}^{d+1}$ with $|\alpha| \leq s$ and $\alpha_\varrho \leq k$, one has*
$$\big\|[\partial^\alpha; F, G]\big\|_{L^2(\Omega)} \leq C\big\|F\big\|_{H^{s-1,\min(\{k,s-1\})}}\big\|G\big\|_{H^{s-1,\min(\{k,s-1\})}}.$$

PROOF. We can decompose
$$[\partial^\alpha; F, G] = \sum_{\beta+\gamma=\alpha}(\partial^\beta F)(\partial^\gamma G)$$
with multi-indices $\beta = (\beta_{\boldsymbol{x}}, \beta_\varrho) \in \mathbb{N}^{d+1}$ and $\gamma = (\gamma_{\boldsymbol{x}}, \gamma_\varrho) \in \mathbb{N}^{d+1}$ such that $|\beta| + |\gamma| \leq s$ and $\beta_\varrho + \gamma_\varrho \leq k$, and $1 \leq |\beta|, |\gamma| \leq s-1$. Assume furthermore that $\beta_\varrho \leq k-1$ and $|\beta| \leq s-2$. Then
$$\begin{aligned}\big\|(\partial^\beta F)(\partial^\gamma G)\big\|_{L^2(\Omega)} &\lesssim \big|\big|\partial^\beta F(\cdot,\varrho)\big|_{H^{s-|\beta|-\frac{3}{2}}_{\boldsymbol{x}}}\big|\partial^\gamma G(\cdot,\varrho)\big|_{H^{s-|\gamma|-1}_{\boldsymbol{x}}}\big|_{L^2_\varrho}\\ &\lesssim \big\|F\big\|_{H^{s-1,\beta_\varrho+1}}\big\|G\big\|_{H^{s-1,\gamma_\varrho}} \leq \big\|F\big\|_{H^{s-1,\min(\{k,s-1\})}}\big\|G\big\|_{H^{s-1,\min(\{k,s-1\})}},\end{aligned}$$

where we used Lemma A.2(1) with $(s, s_1, s_2) = (0, s-|\beta|-\frac{3}{2}, s-|\gamma|-1)$, and Lemma A.1. By symmetry, the result holds if $\gamma_\varrho \leq k-1$ and $|\gamma| \leq s-2$. Hence there remains to consider the situation where ($\beta_\varrho = k$ or $|\beta| = s-1$) and ($\gamma_\varrho = k$ or $|\gamma| = s-1$). Since $s > 2$ and $|\beta|+|\gamma| \leq s$, we cannot have $|\beta| = |\gamma| = s-1$. In the same way, we cannot have $\beta_\varrho = \gamma_\varrho = k$ since $k > 0$. Furthermore , we cannot have $\beta_\varrho = k$ and $|\gamma| = s-1$, since the former implies $|\beta| \geq \beta_\varrho = k \geq 2$ and the latter implies $|\beta| \leq 1$. Symmetrically, we cannot have $\gamma_\varrho = k$ and $|\beta| = s-1$. This concludes the proof. □

Acknowledgments. VD would like to thank Eric Blayo for pointing out the Gent&Mc Williams paper, and Mahieddine Adim for his careful proofreading. RB is partially supported by the GNAMPA group of INdAM.

References

[1] P. Azérad and F. Guillén, *Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics*, SIAM J. Math. Anal. **33** (2001), no. 4, 847–859. MR1884725
[2] J. L. Bona and R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London Ser. A **278** (1975), no. 1287, 555–601.
[3] Y. Brenier, *Homogeneous hydrostatic flows with convex velocity profiles*, Nonlinearity **12** (1999), no. 3, 495–512. MR1690189
[4] C. Cao, J. Li, and E. S. Titi, *Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity*, Arch. Ration. Mech. Anal. **214** (2014), no. 1, 35–76. MR3237881
[5] C. Cao, J. Li, and E. S. Titi, *Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion*, Comm. Pure Appl. Math. **69** (2016), no. 8, 1492–1531. MR3518238
[6] C. Cao and E. S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. of Math. (2) **166** (2007), no. 1, 245–267. MR2342696
[7] B. Desjardins, D. Lannes, and J.-C. Saut, *Normal mode decomposition and dispersive and nonlinear mixing in stratified fluids*, Water Waves (2020), 1–40, available at 1912.05346.
[8] F. Flandoli, L. Galeati, and D. Luo, *Eddy heat exchange at the boundary under white noise turbulence* (2021), available at arXiv:2103.08098.
[9] F. Flandoli, L. Galeati, and D. Luo, *Mixing, dissipation enhancement and convergence rates for scaling limit of spdes with transport noise* (2021), available at arXiv:2104.01740.
[10] K. Furukawa, Y. Giga, M. Hieber, A. Hussein, T. Kashiwabara, and M. Wrona, *Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier-Stokes equations*, Nonlinearity **33** (2020), no. 12, 6502–6516. MR4164683
[11] T. Gallay, *Stability of vortices in ideal fluids: the legacy of kelvin and rayleigh* **10** (2018), 42–59. Hyperbolic Problems: Theory, Numerics, Applications, Proceedings of HYP2018.
[12] P. R. Gent and J. C. McWilliams, *Isopycnal mixing in ocean circulation models*, J. Phys. Oceanogr. **20** (1990), no. 1, 150–155.
[13] P. R. Gent and J. C. McWilliams, *Eliassen–palm fluxes and the momentum equation in non-eddy-resolving ocean circulation models*, J. of Phys. Oceanogr. **6** (1996), no. 11, 2539–2546.
[14] P. R. Gent, J. Willebrand, T. J. McDougall, and J. C. McWilliams, *Parameterizing eddy-induced tracer transports in ocean circulation models*, J. of Phys. Oceanogr. **25** (1995), no. 4, 463–474.
[15] E. Grenier, *On the derivation of homogeneous hydrostatic equations*, M2AN Math. Model. Numer. Anal. **33** (1999), no. 5, 965–970. MR1726718
[16] S. M. Griffies, *Fundamentals of ocean climate models*, Princeton University Press, Princeton, NJ, 2004. With a foreword by Trevor J. McDougall. MR2079980
[17] D. Han-Kwan and T. T. Nguyen, *Ill-posedness of the hydrostatic Euler and singular Vlasov equations*, Arch. Ration. Mech. Anal. **221** (2016), no. 3, 1317–1344. MR3509003
[18] L. N. Howard, *A note on the existence of certain viscous flows*, J. Math. and Phys. **40** (1961), 172–176.
[19] S. Klainerman and A. Majda, *Compressible and incompressible fluids*, Comm. Pure Appl. Math. **35** (1982), no. 5, 629–651. MR668409 (84a:35264)
[20] I. Kukavica, R. Temam, V. C. Vicol, and M. Ziane, *Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain*, J. Differential Equations **250** (2011), no. 3, 1719–1746. MR2737223
[21] D. Lannes, *The water waves problem*, Mathematical Surveys and Monographs, vol. 188, American Mathematical Society, Providence, RI, 2013. Mathematical analysis and asymptotics. MR3060183
[22] J. Li and E. S. Titi, *The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation*, J. Math. Pures Appl. (9) **124** (2019), 30–58. MR3926040
[23] J. Li, E. S. Titi, and G. Yuan, *The primitive equations approximation of the anisotropic horizontally viscous Navier-Stokes equations*, 2021. arXiv preprint:2106.00201.

[24] N. Masmoudi and T. K. Wong, *On the H^s theory of hydrostatic Euler equations*, Arch. Ration. Mech. Anal. **204** (2012), no. 1, 231–271. MR2898740
[25] J. W. Miles, *On the stability of heterogeneous shear flows*, J. Fluid Mech. **10** (1961), 496–508.
[26] M. Paicu, P. Zhang, and Z. Zhang, *On the hydrostatic approximation of the Navier-Stokes equations in a thin strip*, Adv. Math. **372** (2020), 107293, 42. MR4125518
[27] X. Pu and W. Zhou, *Rigorous derivation of the full primitive equations by scaled boussinesq equations*, 2021. arXiv preprint:2105.10621.
[28] X. Pu and W. Zhou, *On the rigorous mathematical derivation for the viscous primitive equations with density stratification*, 2022. arXiv preprint:2203.10529.
[29] M. Renardy, *Ill-posedness of the hydrostatic Euler and Navier-Stokes equations*, Arch. Ration. Mech. Anal. **194** (2009), no. 3, 877–886. MR2563627
[30] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.
[31] W. A. Strauss, *On continuity of functions with values in various Banach spaces*, Pacific J. Math. **19** (1966), 543–551.

IAC, CONSIGLIO NAZIONALE DELLE RICERCHE, 00185 ROME, ITALY
Email address: roberta.bianchini@cnr.it

IRMAR, CNRS, UNIV. RENNES, F-35000 RENNES, FRANCE
Email address: vincent.duchene@univ-rennes1.fr