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Patrick Ciarlet, David Lassounon, Mahran Rihani. An optimal control-based numerical method for scalar transmission problems with sign-changing coefficients. *SIAM Journal on Numerical Analysis*, 2023, 61 (3), pp.1316-1339. <10.1137/22M1495998>. <hal-03666913v2>

**HAL Id: hal-03666913**

**<https://hal.science/hal-03666913v2>**

Submitted on 13 Oct 2022

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# AN OPTIMAL CONTROL-BASED NUMERICAL METHOD FOR SCALAR TRANSMISSION PROBLEMS WITH SIGN-CHANGING COEFFICIENTS

PATRICK CIARLET JR.\*, DAVID LASSOUNON†, AND MAHRAN RIHANI‡

**Abstract.** In this work, we present a new numerical method for solving the scalar transmission problem with sign-changing coefficients. In electromagnetism, such a transmission problem can occur if the domain of interest is made of a classical dielectric material and a metal or a metamaterial, with for instance an electric permittivity that is strictly negative in the metal or metamaterial. The method is based on an optimal control reformulation of the problem. Contrary to other existing approaches, the convergence of this method is proved without any restrictive condition. In particular, no condition is imposed on the a priori regularity of the solution to the problem, and no condition is imposed on the meshes, other than that they fit with the interface between the two media. Our results are illustrated by some (2D) numerical experiments.

**Key words.** transmission problem, sign-changing coefficients, fictitious domain methods, optimal control.

**MSC codes.** 65N30, 78A48

**1 Introduction** In the present paper, we study the numerical approximation of the scalar transmission problem with sign-changing coefficients in  $\mathbb{R}^d$ , for  $d \in \{2, 3\}$ . To fix ideas, let  $\Omega$  be an open, bounded, connected subset of  $\mathbb{R}^d$  with a Lipschitz boundary, in other words a *domain* of  $\mathbb{R}^d$ . Further, consider that  $\Omega$  is equal to the union of two disjoint (sub)domains  $\Omega_1, \Omega_2$ . We denote the interface by  $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$  (see Figure 1 for an example), and we assume that  $\text{meas}_{\partial\Omega}(\partial\Omega_2 \setminus \Sigma) > 0$ . The case of an inclusion corresponds to  $\partial\Sigma \cap \partial\Omega = \emptyset$  and, since  $\text{meas}_{\partial\Omega}(\partial\Omega_2 \setminus \Sigma) > 0$ ,  $\Omega_1$  is always the inclusion in this configuration. When  $\partial\Sigma \cap \partial\Omega$  is non-empty, we assume that it is a Lipschitz submanifold of  $\partial\Omega_1$  and of  $\partial\Omega_2$ .

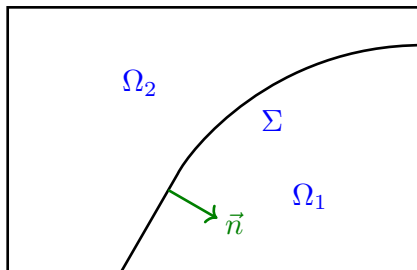


FIG. 1. *Example of geometry.*

We also introduce a coefficient  $\varepsilon \in L^\infty(\Omega)$  such that  $\varepsilon_1 = \varepsilon|_{\Omega_1} \geq \varepsilon_+ > 0$  a.e. in  $\Omega_1$  and  $\varepsilon_2 = \varepsilon|_{\Omega_2} \leq \varepsilon_- < 0$  a.e. in  $\Omega_2$ . Here  $\varepsilon_+$  and  $\varepsilon_-$  are two real constants. It will be useful to introduce the contrasts  $\kappa_\varepsilon^1 := \varepsilon_1^- / \varepsilon_2^+$  and  $\kappa_\varepsilon^2 := \varepsilon_2^- / \varepsilon_1^+$  where  $\varepsilon_1^\pm$  and  $\varepsilon_2^\pm$  are defined as follows:

$$\varepsilon_1^+ := \sup_{\Omega_1} \varepsilon_1, \quad \varepsilon_1^- := \inf_{\Omega_1} \varepsilon_1, \quad \varepsilon_2^+ := \sup_{\Omega_2} |\varepsilon_2| \text{ and } \varepsilon_2^- := \inf_{\Omega_2} |\varepsilon_2|.$$

Note that in the particular case where  $\varepsilon$  is piecewise constant, we have  $\kappa_\varepsilon^1 = 1/\kappa_\varepsilon^2$ .

*Remark 1.1.* In principle,  $\varepsilon$  could be a symmetric tensor-valued coefficient, i.e.,  $\varepsilon = (\varepsilon_{ij})_{1 \leq i, j \leq d}$  with  $\varepsilon_{ij} \in L^\infty(\Omega)$  for all  $1 \leq i, j \leq d$ , and such that

$$\begin{aligned} \exists \varepsilon_+ > 0, \quad \forall \mathbf{z} \in \mathbb{R}^d, \quad \varepsilon_+ |\mathbf{z}|^2 &\leq \varepsilon \mathbf{z} \cdot \mathbf{z} \text{ a.e. in } \Omega_1; \\ \exists \varepsilon_- > 0, \quad \forall \mathbf{z} \in \mathbb{R}^d, \quad \varepsilon_- |\mathbf{z}|^2 &\leq -\varepsilon \mathbf{z} \cdot \mathbf{z} \text{ a.e. in } \Omega_2. \end{aligned}$$

However, for the sake of conciseness, we consider a scalar-valued coefficient.

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29 For a given source term  $f \in L^2(\Omega)$ , we consider the problem

30 (1.1) Find  $u \in H_0^1(\Omega)$  such that  $-\operatorname{div}(\varepsilon \nabla u) = f \in L^2(\Omega)$ .

31 The equivalent variational formulation to (1.1) writes

32 (1.2) Find  $u \in H_0^1(\Omega)$  such that  $\int_{\Omega} \varepsilon \nabla u \cdot \nabla v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}, \quad \forall v \in H_0^1(\Omega)$ .

33 Because of the change of sign of  $\varepsilon$ , the well-posedness of this problem does not fit into the classical theory of  
 34 elliptic PDEs and it can be ill-posed. On the other hand, one can show that when  $\kappa_{\varepsilon}^1$  or  $\kappa_{\varepsilon}^2$  is large enough,  
 35 Problem 1.2 is T-coercive (for instance see [6]), i.e., there exists an operator  $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that  
 36  $(u, v) \mapsto \int_{\Omega} \varepsilon \nabla u \cdot \nabla(T(v))$  is coercive, and then it is well-posed. For the case of polygonal interfaces, the  
 37 construction of such operator T is based on the use of local isometric geometrical transformations (such as  
 38 reflections, rotations, ...) near the interface, see [3].

39 The implementation of a general conforming finite element method to discretize (1.2) leads us to consider the  
 40 problem

41 (1.3) Find  $u_h \in V_h(\Omega)$  such that  $\int_{\Omega} \varepsilon \nabla u_h \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega} f v_h \, d\mathbf{x}, \quad \forall v_h \in V_h(\Omega)$ ,

42 where  $V_h(\Omega)$  is a well-chosen subspace of  $H_0^1(\Omega)$ , and the parameter  $h > 0$  is the so-called meshsize. Even in  
 43 the case where (1.2) is T-coercive, one can not guaranty that Problem (1.3) is also T-coercive. Indeed, it may  
 44 happen that for some  $v_h \in V_h(\Omega)$ , there holds  $T(v_h) \notin V_h(\Omega)$ . To overcome this difficulty, an interesting idea  
 45 is to try to construct meshes such that the approximation spaces  $V_h(\Omega)$  are stable by operators T for which  
 46 Problem (1.2) is T-coercive. This type of meshes are called T-conform meshes. Such an approach has been  
 47 investigated in [26, 12, 10]. It works quite well but presents two main drawbacks:

- 48 • The construction of well-suited meshes for curved interfaces, interfaces with corners or 3D interfaces  
 49 is not a straightforward task [10, 3].
- 50 • Sometimes the operator T for which the problem is T-coercive is constructed by abstract tools and  
 51 therefore is not explicit. In these situations, one cannot find adapted meshes.

52 On general meshes, three alternatives have already been proposed. The first one was introduced in [6] and  
 53 was based on the use of [discrete trace liftings, with quasi-uniform meshes on the interface](#). In addition to this  
 54 constraint on the mesh, one of the limitation of this approach is that, for interfaces with general shapes, the  
 55 convergence can not be assured in all the configurations in which Problem (1.2) is well-posed, because it is  
 56 based on a particular (non-optimal) T-coercivity operator. The [second one](#) is developed in [23] and is based on  
 57 the use of [interpolation](#) techniques. Its essential limitation lies again in the fact that, for interfaces with general  
 58 shapes, the convergence can not be assured for all configurations in which Problem (1.2) is well-posed. The  
 59 [third one](#), presented in [14], consists in adding some dissipation to the problem (considering  $\varepsilon + i\delta$  instead of  $\varepsilon$   
 60 in (1.2) where  $\delta$  depends on the meshsize). Unfortunately, this methods has a sub-optimal order of convergence  
 61 even in the case where the solution and the interface are smooth (see [14]).

62 After that, in 2017, a new technique relying on the use of an optimal control reformulation has been introduced  
 63 by Abdulle et al in [1], [where the auxiliary control function is defined over  \$\Sigma\$](#) . Introducing

64  $\text{PH}^{1+s}(\Omega) := \{u \in H^1(\Omega) \mid u|_{\Omega_1} \in H^{1+s}(\Omega_1) \text{ and } u|_{\Omega_2} \in H^{1+s}(\Omega_2)\}$  for  $s > 0$ ,

65 their method is proved to be convergent for general meshes (that respect the interface) as soon as the exact  
 66 solution to (1.1) belongs to the space  $\text{PH}^{1+s}(\Omega)$  for some  $s > 1/2$ . Unfortunately, this regularity condition is  
 67 not always satisfied, especially when  $\Sigma$  has corners in 2D or conical points in 3D. See the numerical illustration  
 68 in Section 6.3 below.

69  
 70 In this work, we present a new strategy which relies on the use of a different optimal control reformulation and  
 71 which converges without any restriction on the mesh (except the fact of being conforming to the interface),  
 72 and without any restriction on the regularity of the exact solution. [In our approach, the auxiliary control  
 73 function is defined over one subdomain](#). This method is inspired by the smooth extension method that was  
 74 used (without proof of convergence) in [19] to approximate the solution to some classical scalar transmission  
 75 problems. [The key idea is that, given a control, one can construct a pseudo-solution to the problem \(1.1\), and](#)

76 to note that, as soon as one can relate the control to some extension of the solution, then one recovers exactly  
 77 the solution.

78  
 79 The article is organized as follows. In Section 2, we start by giving a detailed description of the problem. Then,  
 80 in Section 3, we explain how to derive an equivalent optimal control reformulation. Section 4 is dedicated to  
 81 the study of some basic properties of the optimization problem and its regularization. The proposed numerical  
 82 method and the proof of its convergence are given in Section 5. Our results are then illustrated by some  
 83 numerical experiments in Section 6. Finally we give concluding remarks, including some possible extensions.

84 **2 Main assumption on  $\varepsilon$  and reformulation of the problem** Introduce the bounded operator  
 85  $A_\varepsilon : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  such that

$$86 \quad (H_0^1(\Omega))^* \langle A_\varepsilon u, v \rangle_{H_0^1(\Omega)} = \int_{\Omega} \varepsilon \nabla u \cdot \nabla v \, d\mathbf{x}, \quad \forall u, v \in H_0^1(\Omega).$$

87 Obviously  $A_\varepsilon$  is an isomorphism if, and only if, Problem (1.1) is well-posed in the Hadamard sense. In this  
 88 article, we shall work under the following

89 *Assumption 2.1.* Assume that the coefficient  $\varepsilon$  is such that  $A_\varepsilon$  is an isomorphism.

90 If  $\varepsilon$  is piecewise constant by subdomain, the previous assumption is satisfied when the contrast  $\kappa_\varepsilon := \varepsilon_2/\varepsilon_1$   
 91 does not belong to the so-called critical interval. The expression of this interval is in general not known  
 92 analytically, except for particular geometries like symmetric domains, simple 2D interface with corners, simple  
 93 3D interfaces with circular conical tips (see [24, Chapter 2]). Under assumption 2.1, one is able to prove the  
 94 accompanying shift theorem. We refer to [18, 7, 13, 12, 5].

95 **THEOREM 2.2.** *Assume that  $\Sigma$  is smooth (of class  $\mathcal{C}^2$ ), polygonal (in 2D) or polyhedral (in 3D) and that*  
 96 *Problem (1.1) is well-posed in the Hadamard sense. Then, there exists  $\sigma_D(\varepsilon) \in (0, 1]$  such that*

$$97 \quad \forall f \in L^2(\Omega), \text{ the solution } u \text{ to Problem (1.1) is such that } \begin{cases} u \in \bigcap_{s \in [0, \sigma_D(\varepsilon))} PH^{1+s}(\Omega) & \text{if } \sigma_D(\varepsilon) < 1 \\ u \in PH^2(\Omega) & \text{if } \sigma_D(\varepsilon) = 1 \end{cases},$$

98 *with continuous dependence.*

99 The number  $\sigma_D(\varepsilon)$  in the shift theorem is called the *(limit) regularity exponent*. For instance, when the  
 100 interface is smooth and when it does not intersect with the boundary, then  $\sigma_D(\varepsilon) = 1$  (cf. [18]).

101 *Remark 2.3.* In Problem (1.1), we consider homogeneous Dirichlet boundary conditions. Let us mention that  
 102 the results below extend quite straightforwardly to other situations, for example with Neumann or Robin-  
 103 Fourier boundary conditions which can be homogeneous or not, as long as the associated operator is an  
 104 isomorphism.

105 To introduce the method, we start by writing an equivalent version of (1.1) in which the unknown  $u \in H_0^1(\Omega)$   
 106 is split into two unknowns defined in  $\Omega_1$  and  $\Omega_2$  :  $(u_1, u_2) := (u|_{\Omega_1}, u|_{\Omega_2})$ . To do so, we observe that since  
 107  $f \in L^2(\Omega)$ , the solution  $u$  to (1.1) is such that the vector field  $\varepsilon \nabla u$  belongs to the space  $H(\text{div}, \Omega) = \{ \mathbf{v} \in$   
 108  $(L^2(\Omega))^d \text{ such that } \text{div } \mathbf{v} \in L^2(\Omega) \}$ . Consequently, the pair of functions  $(u_1, u_2)$  satisfies the problem

$$109 \quad (2.1) \quad \text{Find } (u_1, u_2) \in V_1(\Omega_1) \times V_2(\Omega_2) \text{ such that } \begin{cases} -\text{div}(\varepsilon_1 \nabla u_1) = f_1 =: f|_{\Omega_1} \\ -\text{div}(\varepsilon_2 \nabla u_2) = f_2 =: f|_{\Omega_2} \\ \varepsilon_1 \partial_n u_1 = \varepsilon_2 \partial_n u_2 \text{ in } (H_{00}^{1/2}(\Sigma))^* \\ u_1 = u_2 \text{ in } H_{00}^{1/2}(\Sigma) \end{cases}$$

in which  $\mathbf{n}$  stands for the unit normal vector to  $\Sigma$  oriented to the exterior of  $\Omega_2$  (see Figure 1), the spaces  
 $V_1(\Omega_1), V_2(\Omega_2)$  are given by

$$V_1(\Omega_1) := \{v \in H^1(\Omega_1), v = 0 \text{ on } \partial\Omega_1 \setminus \Sigma\}, \quad V_2(\Omega_2) := \{v_2 \in H^1(\Omega_2), v_2 = 0 \text{ on } \partial\Omega_2 \setminus \Sigma\},$$

and the space  $H_{00}^{1/2}(\Sigma)$  is defined as follows

$$H_{00}^{1/2}(\Sigma) = \begin{cases} H^{1/2}(\Sigma) & \text{if } \partial\Sigma \cap \partial\Omega = \emptyset \text{ (inclusion),} \\ \{\lambda \in H^{1/2}(\Sigma), \tilde{\lambda} \in H^{1/2}(\partial\Omega_2)\} & \text{else.} \end{cases}$$

110 Above, in the definition of the space  $H_{00}^{1/2}(\Sigma)$ ,  $\tilde{\lambda}$  denotes the continuation of  $\lambda$  by 0 to  $\partial\Omega_2$  (one can also  
 111 consider the continuation by 0 to  $\partial\Omega_1$ ).

112 Since  $\text{meas}_{\partial\Omega}(\partial\Omega_2 \setminus \Sigma) > 0$ , all elements of  $V_2(\Omega_2)$  fulfill a homogeneous boundary condition on a part of  
 113 the boundary  $\partial\Omega_2$ . On the other hand, one can check that if  $(u_1, u_2)$  is a solution to (2.1), then the function  
 114  $u$  defined by  $u|_{\Omega_j} = u_j$  for  $j = 1, 2$  solves (1.1). The equations satisfied by  $u_1$  and  $u_2$  are elliptic but they  
 115 are coupled by the transmission conditions on  $\Sigma$ . As a consequence, we cannot solve them independently.  
 116 The purpose of the next paragraph is to explain how to proceed to write an alternative formulation (an  
 117 optimization-based one), which can be solved via an iterative procedure such that at each step one has to solve  
 118 a set of elliptic problems.

119 **3 The smooth extension method and optimal control reformulation of the problem** The  
 120 smooth extension method was proposed in [21] and can be considered as a special case of **fictitious domain**  
 121 **methods** (see [2]). It has been adapted to study the classical scalar transmission problem, i.e., with constant  
 122 sign coefficients, in [19]. In this section, we explain how to apply it to our problem.

123 **3.1 Presentation of the smooth extension method** The idea behind the smooth extension method  
 124 is the following: instead of looking for  $(u_1, u_2) \in V_1(\Omega) \times V_2(\Omega_2)$  solution to (2.1), we search for a pair of  
 125 functions  $(\tilde{u}, u_2) \in H_0^1(\Omega) \times V_2(\Omega_2)$  such that  $(\tilde{u}|_{\Omega_1}, u_2)$  is a solution to (2.1).<sup>1</sup> The function  $\tilde{u}$  is then a  
 126 particular continuous extension of  $u_1$  to the whole domain  $\Omega$ . The difficulty is to find a "good" way to define  
 127 the function  $\tilde{u}$  so that it can be approximated by the classical FEM. The function  $u_2$  can then be approximated  
 128 by solving the elliptic problem satisfied by  $u_2$  in  $\Omega_2$  completed by  $\tilde{u}|_{\Sigma}$  (resp.  $\varepsilon_1 \partial_n \tilde{u}|_{\Sigma}$ ) as a Dirichlet (resp.  
 129 Neumann) boundary condition on  $\Sigma$ . Note that at first sight the construction of such  $\tilde{u}$  is not straightforward.  
 130 This will be achieved thanks to an optimal control reformulation of (2.1). This is the main goal of the next  
 131 paragraph in which we also reformulate the idea presented above in a more rigorous way.

132 **3.2 An optimal control reformulation of the problem** Before getting into details, let us first  
 133 introduce  $\tilde{\varepsilon}_1 \in L^\infty(\Omega)$  such that  $\tilde{\varepsilon}_1 \geq \tilde{\varepsilon}^+ > 0$  a. e. in  $\Omega$  and  $\tilde{\varepsilon}_1 = \varepsilon_1$  in  $\Omega_1$ . Then, let  $E : V_1(\Omega_1) \rightarrow H_0^1(\Omega)$  be  
 134 an arbitrary continuous extension operator. By making use of (2.1), one can show easily that

$$135 \left\{ \begin{array}{l} \int_{\Omega} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_1} f_1 v \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v \rangle_{\Sigma} \quad \forall v \in H_0^1(\Omega), \\ \int_{\Omega_2} \varepsilon_2 \nabla u_2 \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Omega_2} f_2 v_2 \, d\mathbf{x} + \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma} \quad \forall v_2 \in V_2(\Omega_2). \end{array} \right.$$

136 Here and elsewhere,  $\langle \cdot, \cdot \rangle_{\Sigma}$  denotes the duality product between  $(H_{00}^{1/2}(\Sigma))^*$  and  $H_{00}^{1/2}(\Sigma)$ .

137 Now, given that the linear form  $v_2 \mapsto \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma}$  is continuous on  $V_2(\Omega_2)$  one can  
 138 define, thanks to the Riesz representation theorem, for each  $E(u_1)$  a unique  $w_{E(u_1)} \in V_2(\Omega_2)$  such that

$$139 (3.1) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_{E(u_1)} \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2).$$

140 Above we have used the fact that  $(u, v) \mapsto (\tilde{\varepsilon}_1 \nabla u, \nabla v)_{L^2(\Omega)^d}$  is an inner product on  $V_2(\Omega_2)$ . As a consequence,  
 141 we have

$$142 \left\{ \begin{array}{l} \int_{\Omega} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_1} f_1 v \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_{E(u_1)} \cdot \nabla v \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega), \\ \int_{\Omega_2} \varepsilon_2 \nabla u_2 \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Omega_2} f_2 v_2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla (E(u_1) - w_{E(u_1)}) \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2). \end{array} \right.$$

Since the coefficients  $\tilde{\varepsilon}_1$  and  $\varepsilon_2$  have fixed signs, the forms

$$(u, v) \mapsto \int_{\Omega} \tilde{\varepsilon}_1 \nabla u \cdot \nabla v \, d\mathbf{x} \quad \text{and} \quad (u_2, v_2) \mapsto - \int_{\Omega_2} \varepsilon_2 \nabla u_2 \cdot \nabla v_2 \, d\mathbf{x},$$

<sup>1</sup>In the text below, we choose an extension from  $\Omega_1$  to  $\Omega^* = \Omega_2$ . Obviously, one could choose an extension from  $\Omega_2$  to  $\Omega^* = \Omega_1$  so that  $(u_1, \tilde{u}|_{\Omega_2})$  is a solution to (2.1). In this case, the condition  $\text{meas}_{\partial\Omega}(\partial\Omega_1 \setminus \Sigma) > 0$  must hold.

143 are coercive, respectively on  $H_0^1(\Omega)$  and on  $V_2(\Omega_2)$ . With this in mind, we define for all  $w \in V_2(\Omega_2)$ , the  
 144 couple of functions  $(u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$  that are solution to the well-posed system of equations:

$$145 \quad (3.2) \quad \begin{cases} \int_{\Omega} \tilde{\varepsilon}_1 \nabla u^w \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_1} f_1 v \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla v \, d\mathbf{x} & \forall v \in H_0^1(\Omega), \\ \int_{\Omega_2} \varepsilon_2 \nabla u_2^w \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Omega_2} f_2 v_2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla(u^w - w) \cdot \nabla v_2 \, d\mathbf{x} & \forall v_2 \in V_2(\Omega_2). \end{cases}$$

146 Well-posedness is achieved by solving the elliptic problem in  $u^w$  first, and then the elliptic problem in  $u_2^w$ .

147 *Remark 3.1.* Observe that if we choose  $w = w_{E(u_1)}$ , then it follows that  $(u^w|_{\Omega_1}, u_2^w)$  is the solution to (2.1).  
 148 Indeed, one finds first that  $u^w = E(u_1)$ , and then that  $u_2^w = u_2$ .

149 Given any auxiliary "control" function  $w$ , the solutions to (3.2) enjoy the properties listed below.

150 PROPOSITION 3.2. For all  $w \in V_2(\Omega_2)$ , the functions  $u_1^w := u^w|_{\Omega_1}$  and  $u_2^w$  are such that

$$151 \quad \begin{cases} -\operatorname{div}(\varepsilon_1 \nabla u_1^w) = f_1 & \text{in } \Omega_1, \\ -\operatorname{div}(\varepsilon_2 \nabla u_2^w) = f_2 & \text{in } \Omega_2 \\ \varepsilon_1 \partial_n u_1^w = \varepsilon_2 \partial_n u_2^w & \text{on } \Sigma. \end{cases}$$

152 *Remark 3.3.* In other words, the introduction of an auxiliary "control" function  $w$  allows us to construct  
 153 pseudo-solutions to the equation (2.1) for which the condition on the normal derivatives is automatically  
 154 satisfied. However we do not have in general continuity across the interface.

155 *Proof.* Take  $\varphi_1 \in \mathcal{C}_0^\infty(\Omega_1)$  and extend it by 0 to the whole  $\Omega$  to obtain the function  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ . Take  $v = \varphi$   
 156 in the problem satisfied by  $u^w$ . One finds that  $-\operatorname{div}(\varepsilon_1 \nabla u_1^w) = f_1$  in  $\Omega_1$ . Next, take some  $\varphi_2 \in \mathcal{C}_0^\infty(\Omega_2)$ ,  
 157 extend it by 0 in  $\Omega_1$  and denote by  $\varphi$  the new function. By taking  $v = \varphi$  in the problem satisfied by  $u^w$  and  
 158  $v_2 = \varphi_2$  in the problem satisfied by  $u_2^w$  one finds that

$$159 \quad \begin{aligned} \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla u^w \cdot \nabla \varphi_2 \, d\mathbf{x} &= \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla \varphi_2 \, d\mathbf{x}, \\ \int_{\Omega_2} \varepsilon_2 \nabla u_2^w \cdot \nabla \varphi_2 \, d\mathbf{x} &= \int_{\Omega_2} f_2 \varphi_2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla(u^w - w) \cdot \nabla \varphi_2 \, d\mathbf{x}. \end{aligned}$$

160 By considering the sum of the two formulations, we conclude that  $-\operatorname{div}(\varepsilon_2 \nabla u_2^w) = f_2$  in  $\Omega_2$ . To end the proof,  
 161 it remains to show that  $\varepsilon_1 \partial_n u_1^w = \varepsilon_2 \partial_n u_2^w$ . For this, let  $v \in H_0^1(\Omega)$  and define  $v_2 = v|_{\Omega_2} \in V_2(\Omega_2)$ . By taking  
 162  $v$  and  $v_2$  as test functions in (3.2), considering the sum of the two equations, integrating by parts in both  
 163 formulations and then, using the equations satisfied by  $u_1^w$  and  $u_2^w$ , we infer that

$$164 \quad -\langle \varepsilon_1 \partial_n u_1^w, v \rangle_{\Sigma} = -\langle \varepsilon_2 \partial_n u_2^w, v \rangle_{\Sigma}, \quad v \in H_0^1(\Omega).$$

165 According to the surjectivity of the trace mapping on  $\Sigma$ , this gives  $\varepsilon_1 \partial_n u_1^w = \varepsilon_2 \partial_n u_2^w$  on  $\Sigma$ .  $\square$

166 It follows that

167 LEMMA 3.4. If there exists  $w^* \in V_2(\Omega_2)$  such that the solution to (3.2) satisfies  $u^{w^*} = u_2^{w^*}$  on  $\Sigma$ , then  
 168  $(u^{w^*}|_{\Omega_1}, u_2^{w^*})$  solves (2.1).

169 Thanks to what we have explained in Remark 3.1, we know that to every continuous extension of  $u_1$  to  $\Omega$ , one  
 170 can define  $w^* \in V_2(\Omega_2)$  for which  $u^{w^*} = u_2^{w^*}$  on  $\Sigma$ . This leads us to the following result.

171 LEMMA 3.5. Let  $u_1$  be the first part of the solution to (2.1). Then, the set of  $w^* \in V_2(\Omega_2)$  such that the  
 172 solution to (3.2) satisfies  $u^{w^*} = u_2^{w^*}$  on  $\Sigma$  is isomorphic to the set of all possible continuous extensions of  $u_1$   
 173 to  $\Omega$ . Furthermore,  $w^*$  and  $u^{w^*}$  are linked by relation

$$174 \quad (3.3) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla u^{w^*} \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w^* \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2).$$

175 Now, we have all the tools to introduce the optimal control reformulation of the problem (2.1). As a matter  
 176 of fact, in order to find a function  $w^* \in V_2(\Omega_2)$  for which  $u^{w^*} = u_2^{w^*}$  on  $\Sigma$ , it is enough to solve the following  
 177 optimal control problem:

$$178 \quad (3.4) \quad \text{Find } w^* = \operatorname{argmin}_{w \in V_2(\Omega_2)} J(w) \quad \text{with} \quad J(w) = \frac{1}{2} \int_{\Sigma} |u^w - u_2^w|^2 \, d\sigma,$$

179 where  $(u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$  is the solution to (3.2). Note that in (3.4), the functional  $J$  plays the role  
 180 of the cost functional, while (3.2) plays the role of the state equation. Obviously, thanks to Lemma 3.5, the  
 181 problem (3.4) has an infinite number of solutions. As a result, one may need to use a regularization technique  
 182 in order to be able to construct a convergent discretization of the problem: this will be the subject of §4.3  
 183 where we will study the classical Tikhonov regularization method applied to Problem (3.4).

184 **4 Basic properties of the optimization problem and its regularization** In this section, we  
 185 present in §4.1 some useful properties of the cost functional  $J$  and of the set of its minimizers in §4.2. After  
 186 that in §4.3, we study the Tikhonov regularization of the problem. Furthermore, we explain, in §4.4, how to  
 187 use the the adjoint approach in order to find an explicit expression of the gradient of  $J$ .

188 **4.1 Properties of the cost functional** Since we have used the  $L^2(\Sigma)$  norm instead of the  $H^{1/2}(\Sigma)$   
 189 norm in the definition of  $J$ , one has the following results.

190 PROPOSITION 4.1. *The cost functional  $J$  satisfies the following properties:*

- 191 1. *Let  $(w_n)_n$  be a sequence of elements of  $V_2(\Omega_2)$  that converges weakly to  $w_0 \in V_2(\Omega_2)$ . Then,  $(J(w_n))_n$   
 192 converges to  $J(w_0)$ .*
- 193 2. *The functional  $J$  is continuous and convex on  $V_2(\Omega_2)$ .*

194 *Proof.* 1. For all  $n \in \mathbb{N}$ , denote by  $(u^n, u_2^n) \in H_0^1(\Omega) \times V_2(\Omega)$  the solution to (3.2) with  $w = w_n$ . From the  
 195 ellipticity of the problems involved in (3.2), it follows that  $(u^n)_n$  (resp.  $(u_2^n)_n$ ) converges weakly in  $H_0^1(\Omega)$   
 196 (resp.  $V_2(\Omega_2)$ ) to some  $u \in H^1(\Omega)$  (resp.  $u_2 \in V_2(\Omega_2)$ ) such that  $(u, u_2)$  is the solution to (3.2) with  $w = w_0$ .  
 197 The continuity of the trace operator from  $H^1(\Omega)$  to  $H^{1/2}(\Sigma)$  ensures that  $(u_{|\Sigma}^n - u_{2|\Sigma}^n)_n$  converges weakly to  
 198  $u_{|\Sigma} - u_{2|\Sigma}$  in  $H^{1/2}(\Sigma)$ . Given that the embedding of  $H^{1/2}(\Sigma)$  into  $L^2(\Sigma)$  is compact, it actually converges  
 199 strongly to  $u_{|\Sigma} - u_{2|\Sigma}$  in  $L^2(\Sigma)$ . Thus  $(J(w_n))$  converges to  $J(w_0)$ . The result is proved.

200 2. While the continuity is a direct consequence of the first item, the convexity follows from the fact that  $J :$   
 201  $V_2(\Omega_2) \rightarrow \mathbb{R}$  is the composition of the affine map  $j_1 : V_2(\Omega_2) \rightarrow L^2(\Sigma)$  and of the convex map  $j_2 : L^2(\Sigma) \rightarrow \mathbb{R}$   
 202 such that for all  $w \in V_2(\Omega_2)$ ,  $g \in L^2(\Sigma)$  we have

$$203 \quad (4.1) \quad \begin{cases} j_1(w) = (u^w - u_2^w)_{|\Sigma} \text{ where } (u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2) \text{ is the solution to (3.2),} \\ j_2(g) = \frac{1}{2} \int_{\Sigma} |g|^2 \, d\sigma. \end{cases} \quad \square$$

204 **4.2 The set of minimizers of the functional  $J$**  Thanks to Lemma 3.5, we know that  $J$  has an  
 205 infinite number of minimizers. This (non-empty) set will be denoted by  $M_J$ . Without any difficulty, one can  
 206 see that  $M_J$  coincides with the set of zeros of the functional  $J$ . As a result, since  $J$  is continuous, convex and  
 207 positive, the set  $M_J$  is closed and convex in  $V_2(\Omega_2)$ . This allows us to say that the following minimization  
 208 problem:

$$209 \quad \min_{w \in M_J} \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w|^2 \, d\mathbf{x}$$

210 has a unique solution, as a consequence of the strict convexity of  $v_2 \mapsto \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla v_2|^2 \, d\mathbf{x}$  in  $V_2(\Omega_2)$ , and of  
 211 the fact that  $M_J$  is a closed, convex subset of  $V_2(\Omega_2)$ . In the following, we shall denote by  $w_J^*$  the smallest  
 212 minimizer of the functional  $J$ :

$$213 \quad (4.2) \quad w_J^* = \operatorname{argmin}_{w \in M_J} \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w|^2 \, d\mathbf{x}.$$

214 The goal of the rest of this paragraph is to find a characterization of  $E_{w_J^*}(u_1)$ , the continuous extension of  $u_1$   
 215 that is associated with  $w_J^*$ . Note that the link between  $E_{w_J^*}(u_1)$  and  $w_J^*$  is given by the following (see relation  
 216 (3.3)):

$$217 \quad (4.3) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E_{w_J^*}(u_1) \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_J^* \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2).$$

218 To proceed, we define  $E_H(u_1) \in H_0^1(\Omega)$  the continuous extension of  $u_1$  that satisfies

$$219 \quad (4.4) \quad \operatorname{div}(\tilde{\varepsilon}_1 \nabla E_H(u_1)) = 0 \text{ in } \Omega_2.$$

220 In particular, we have

$$221 \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E_H(u_1) \cdot \nabla v_2 \, d\mathbf{x} = 0 \quad \forall v_2 \in H_0^1(\Omega_2).$$

222 Denote by  $w_H \in M_J$  the minimizer associated with  $E_H(u_1)$ . Thanks to (3.3), we know that

$$223 \quad (4.5) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E_H(u_1) \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_\Sigma = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_H \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2).$$

224 We infer that

$$225 \quad (4.6) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_H \cdot \nabla v_2 \, d\mathbf{x} = 0 \quad \forall v_2 \in H_0^1(\Omega_2).$$

226 By taking the difference between (4.3) and (4.5), taking  $v_2 = w_H$ , using the fact that  $E_H(u_1) - E_{w_j^*}(u_1) \in$   
227  $H_0^1(\Omega_2)$  and owing to (4.6), we infer that

$$228 \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla (w_H - w_j^*) \cdot \nabla w_H \, d\mathbf{x} = 0,$$

229 so

$$230 \quad \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w_j^*|^2 \, d\mathbf{x} = \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w_j^* - \nabla w_H|^2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w_H|^2 \, d\mathbf{x} \geq \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w_H|^2 \, d\mathbf{x}.$$

231 Hence, from the definition of  $w_j^*$ , we then obtain the following

232 **PROPOSITION 4.2.** *The functions  $w_H$  and  $w_j^*$  coincide.*

233 *Remark 4.3.* It is worth noting that, thanks to (4.5) and using the definition of  $E_H(u_1)$ , the function  $w_H$   
234 satisfies the problem:

$$235 \quad (4.7) \quad \operatorname{div}(\tilde{\varepsilon} \nabla w_H) = 0 \text{ in } \Omega_2 \text{ and } \tilde{\varepsilon}_1 \partial_n w_H|_\Sigma = \tilde{\varepsilon}_1 \partial_n E_H(u_1)|_\Sigma - \varepsilon_1 \partial_n u_1|_\Sigma.$$

236 Recall that  $\mathbf{n}$  is the unit normal vector to  $\Sigma$  oriented to the exterior of  $\Omega_2$ .

237 **4.3 Tikhonov regularization of the problem** Tikhonov regularization, which was originally intro-  
238 duced in [25], is a classical method to regularize a convex optimization problem. Classically, this method is  
239 used in the context of regularization of ill-posed inverse problems (see [20] and the references therein). In this  
240 paragraph, we study the convergence of such regularization when it is applied to our problem. For  $\delta > 0$ , we  
241 introduce the functional  $J^\delta : V_2(\Omega_2) \rightarrow \mathbb{R}$  defined by

$$242 \quad J^\delta(w) = J(w) + \delta \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w|^2 \, d\mathbf{x} \quad \forall w \in V_2(\Omega_2).$$

243 To simplify notation, we will denote by  $\|\cdot\|_{\tilde{\varepsilon}_1} : V_2(\Omega_2) \rightarrow \mathbb{R}_+$  the norm that is defined as follows:

$$244 \quad \|w\|_{\tilde{\varepsilon}_1} := \left( \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w|^2 \, d\mathbf{x} \right)^{1/2}, \quad \forall w \in V_2(\Omega_2).$$

245 Endowed with the associated scalar product  $(\cdot, \cdot)_{\tilde{\varepsilon}_1}$ , the space  $V_2(\Omega_2)$  is a Hilbert space. Since  $J$  is convex  
246 and  $\delta > 0$ , the functional  $J^\delta$  is strictly convex and coercive. Therefore the minimization problem

$$247 \quad \min_{w \in V_2(\Omega_2)} J^\delta(w)$$

248 has a unique solution that we denote by  $w_\delta^*$ . Our goal is to study the behaviour of  $w_\delta^*$  as  $\delta$  tends to zero. One  
249 expects  $(w_\delta^*)_\delta$  to converge to one of the solutions (3.4). If this is the case and because the problem (3.4) has an  
250 infinite number of solutions, it will be interesting to characterize the particular solution towards which  $(w_\delta^*)_\delta$   
251 converges. Our findings are summarized in the following

252 **PROPOSITION 4.4.** *When  $\delta \rightarrow 0$ , the sequence  $(w_\delta^*)_\delta$  converges towards  $w_j^*$ , the smallest minimizer of  $J$ .*

253 The proof of the previous result is quite classical. However, for the convenience of the reader, we will detail it  
254 in Appendix A.

255 In conclusion, we can say that the Tikhonov regularization allows us to obtain a stabilized version of the  
256 optimization problem (3.4). This will be used in order to introduce a stabilization of the discretization of the  
257 problem (3.4), but in that case the stabilization parameter  $\delta$  will be chosen as a function of the discretization  
258 parameter. This will be detailed in §5.3. Note that the same idea was employed in [1].

259 **4.4 Gradient of the functional  $J$**  As indicated in the introduction, the main objective of this work  
 260 is to propose a new numerical method for approximating the solution to (1.1). This method will be based  
 261 on the numerical approximation of the solution to the optimization problem (3.4). In this section, we will  
 262 explain how to obtain an explicit expression of  $J'(w)$  the gradient of  $J$  at some  $w \in V_2(\Omega)$ . We recall that the  
 263 functional  $J$  is differentiable, because it can be written as a composition of the two differentiable maps  $j_1$  and  
 264  $j_2$ , cf. (4.1). Since the functional  $J$  is scalar valued, its differential at  $w \in V_2(\Omega_2)$  can be represented by its  
 265 gradient  $J'(w) \in V_2(\Omega_2)$ :

$$266 \quad \text{For all } h \in V_2(\Omega_2), \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla J'(w) \cdot \nabla h \, d\mathbf{x} = \lim_{t \rightarrow 0} \frac{J(w + th) - J(w)}{t}.$$

267 To find an explicit expression of  $J'(w)$ , we use the adjoint approach [11]. Details about the application of this  
 268 approach to our problem are given in Appendix B (see also [19]). Here, we present final result. To do so, we start  
 269 by introducing the so-called adjoint equations. For all  $w \in V_2(\Omega_2)$ , recalling that  $(u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$   
 270 is the solution to (3.2), we introduce  $(g^w, g_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$  such that

$$271 \quad (4.8) \quad \begin{cases} \int_{\Omega} \tilde{\varepsilon}_1 \nabla g^w \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla g_2^w \cdot \nabla v \, d\mathbf{x} - \int_{\Sigma} (u^w - u_2^w) v \, d\sigma & \forall v \in H_0^1(\Omega) \\ \int_{\Omega_2} \varepsilon_2 \nabla g_2^w \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Sigma} (u^w - u_2^w) v_2 \, d\sigma & \forall v_2 \in V_2(\Omega_2). \end{cases}$$

272 As observed before, the functions  $g^w, g_2^w$  are well-defined. In Appendix B, we prove the

273 LEMMA 4.5. *For all  $w \in V_2(\Omega_2)$ , there holds  $J'(w) = g_2^w - g^w|_{\Omega_2}$ , where  $(g^w, g_2^w)$  solve (4.8).*

274 We have the following optimality result

275 COROLLARY 4.6. *We have the equivalence*

$$276 \quad [w^* \in V_2(\Omega_2) \text{ is such that } J'(w^*) = 0] \iff w^* \in M_J.$$

277 *Proof.* Let us start with the proof of the direct implication. Suppose that there exists some  $w^* \in V_2(\Omega_2)$  such  
 278 that  $g^{w^*}|_{\Omega_2} = g_2^{w^*}$ . By taking the sum of the variational formulations of (4.8), we deduce that

$$279 \quad \int_{\Omega} \varepsilon \nabla g^{w^*} \cdot \nabla v \, d\mathbf{x} = 0 \quad \forall v \in H_0^1(\Omega).$$

280 This means  $A_\varepsilon(g^{w^*}) = 0$  and then, thanks to Assumption 2.1,  $g^{w^*} = 0$ . This implies that  $g_2^{w^*} = 0$  and then by  
 281 using the second equation of (4.8), that  $u^{w^*} = u_2^{w^*}$  on  $\Sigma$ . This shows that  $w^*$  is a minimizer of  $J$ . The reverse  
 282 implication is a consequence of the fact that if  $w^* \in M_J$  we have  $J(w^*) = 0$  and then  $u^{w^*} = u_2^{w^*}$  on  $\Sigma$ . This  
 283 implies that  $g_2^{w^*} = 0$  and that  $g^{w^*} = 0$ .  $\square$

284 We end this paragraph with the following result that can be useful to prove the convergence of the classical  
 285 gradient descent algorithm.

286 COROLLARY 4.7. *The functional  $J' : V_2(\Omega_2) \rightarrow V_2(\Omega_2)$  is Lipschitz continuous.*

287 *Proof.* Starting from (3.2), we deduce that  $w \mapsto u^w, w \mapsto u_2^w$  are Lipschitz continuous. Inserting this into  
 288 (4.8), we obtain the result.  $\square$

289 **5 Numerical discretization of the problem** In this part, we are concerned with the numerical  
 290 approximation of (3.4) by means of the Finite Element Method. To do so, we start by presenting some details  
 291 and notations about the family of meshes that will be used. To simplify the presentation, the domain  $\Omega$  and  
 292 the subdomains  $(\Omega_i)_{i=1,2}$  are supposed to have polygonal (when  $d = 2$ ) or polyhedral (resp.  $d = 3$ ) boundaries.

293 **5.1 Meshes and discrete spaces** Let  $(\mathcal{T}_h)_h$  be a regular family of meshes of  $\overline{\Omega}$  (see [15]), composed  
 294 of (closed) simplices. The subscript  $_h$  stands for the meshsize.

295 *Assumption 5.1.* We suppose that for all  $h$ , every simplex of  $\mathcal{T}_h$  belongs either to  $\overline{\Omega}_1$  or to  $\overline{\Omega}_2$ .

296 According to Assumption 5.1, for  $i = 1, 2$ , one can consider the family of meshes  $(\mathcal{T}_h^i)_h$  made of those simplices  
 297 that belong to  $\overline{\Omega}_i$ .

298 For all  $k \in \mathbb{N}^*$ , we set

$$299 \quad V_h^k(\Omega) := \{v_h \in H_0^1(\Omega) \mid v_h|_T \in P^k(T), \forall T \in \mathcal{T}_h\}.$$

300 Here  $P^k(T)$  stands for the space of polynomials (of  $d$  variables) defined on  $T$  of degree at most equal to  $k$ . In  
 301 the same way, we define for  $i = 1, 2$ ,

$$302 \quad V_h^k(\Omega_i) := \{v_{i,h} \in V_i(\Omega_i) \mid v_{i,h}|_T \in P^k(T), \forall T \in \mathcal{T}_h^i\}.$$

303

304 *Remark 5.2.* Note that for all  $h > 0$ , for  $i = 1, 2$  the space  $V_h^k(\Omega_i)$  coincides with  $\{u|_{\Omega_i} \mid u \in V_h^k(\Omega)\}$ .

305 Finally, we recall the basic approximability properties

$$306 \quad (5.1) \quad \left\{ \begin{array}{l} \forall v \in H_0^1(\Omega), \quad \lim_{h \rightarrow 0} \left( \inf_{v_h \in V_h^k(\Omega)} \|v - v_h\|_{H_0^1(\Omega)} \right) = 0, \\ \forall v_2 \in V_2(\Omega_2), \quad \lim_{h \rightarrow 0} \left( \inf_{v_{2,h} \in V_h^k(\Omega_2)} \|v_2 - v_{2,h}\|_{\tilde{\varepsilon}_1} \right) = 0. \end{array} \right.$$

307

308 **5.2 Discretization strategy** For  $h > 0$  and  $w \in V_2(\Omega)$ , define the functions  $u_h^w \in V_h^k(\Omega)$  and  $u_{2,h}^w \in$   
 309  $V_h^k(\Omega_2)$  as the solutions to the following well-posed discrete problems:

$$310 \quad (5.2) \quad \left\{ \begin{array}{l} \int_{\Omega} \tilde{\varepsilon}_1 \nabla u_h^w \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega_1} f v_h \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla v_h \, d\mathbf{x}, \quad \forall v_h \in V_h^k(\Omega) \\ \int_{\Omega_2} \varepsilon_2 \nabla u_{2,h}^w \cdot \nabla v_{2,h} \, d\mathbf{x} = \int_{\Omega_2} f_2 v_{2,h} \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla (u_h^w - w) \cdot \nabla v_{2,h} \, d\mathbf{x}, \quad \forall v_{2,h} \in V_h^k(\Omega_2). \end{array} \right.$$

311 Then introduce the projection operator  $\pi_h^k : V_2(\Omega_2) \rightarrow V_h^k(\Omega_2)$  such that for all  $w \in V_2(\Omega_2)$ ,  $\pi_h^k w$  is defined  
 312 as the unique element of  $V_h^k(\Omega_2)$  that satisfies the problem

$$313 \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla (\pi_h^k w) \cdot \nabla v_{2,h} \, d\mathbf{x} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla v_{2,h} \, d\mathbf{x} \quad \forall v_{2,h} \in V_h^k(\Omega_2).$$

314 Obviously, one has the estimate

$$315 \quad (5.3) \quad \|\pi_h^k w\|_{\tilde{\varepsilon}_1} \leq \|w\|_{\tilde{\varepsilon}_1}.$$

316 From the definition of  $\pi_h^k w$ , one can easily see that for all  $w \in V_2(\Omega_2)$  we have the identities

$$317 \quad (5.4) \quad u_h^{\pi_h^k w} = u_h^w \quad \text{and} \quad u_{2,h}^{\pi_h^k w} = u_{2,h}^w.$$

318 Now, let us turn our attention to the discretization of the optimization problem (3.4). The natural way to do  
 319 that is to replace it by the problem

$$320 \quad (5.5) \quad \inf_{w_h \in V_h^k(\Omega_2)} J_{0,h}(w_h) := \frac{1}{2} \int_{\Sigma} |u_h^{w_h} - u_{2,h}^{w_h}|^2 \, d\sigma.$$

321 One can proceed as in the proof of proposition 4.1 to show that the cost functional  $J_{0,h} : V_h^k \rightarrow \mathbb{R}$  (defined in  
 322 (5.5)) is convex and continuous. Unfortunately this result is not sufficient to justify that the problem (5.5) is  
 323 well-posed. The difficulty comes from the fact that, even under Assumption 2.1, one can not guarantee that  
 324 the problem

$$325 \quad \text{Find } u_h \in V_h^k(\Omega) \text{ such that } \int_{\Omega} \varepsilon \nabla u_h \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega} f v_h \, d\mathbf{x} \quad \forall v_h \in V_h^k(\Omega)$$

326 is well-posed even for  $h$  small enough. To cope with this difficulty, an idea is to use again the Tikhonov  
 327 regularization (see §4.3), with a regularization parameter that depends now on  $h$ . This idea was originally  
 328 proposed in [22] for the case of elliptic equations and then, was used by Assyr Abdulle et al. in [1] for the case  
 329 of problems with sign-changing coefficients. Here, we explain how to adapt it to our approach. The idea is to  
 330 replace the cost functional  $J_{0,h}$  in (5.5) by the functional  $J_h : V_h^k(\Omega_2) \rightarrow \mathbb{R}_+$  such that for all  $w_h \in V_h^k(\Omega_2)$ ,  
 331 we have

$$332 \quad J_h(w_h) := \frac{1}{2} \int_{\Sigma} |u_h^{w_h} - u_{2,h}^{w_h}|^2 \, d\sigma + \lambda_h \|w_h\|_{\tilde{\varepsilon}_1}^2,$$

333 where  $\lambda_h > 0$  tends to zero as  $h$  goes to 0. Since  $\lambda_h > 0$  for all  $h > 0$ , the functional  $J_h$  is strictly convex and  
 334 coercive. This guarantees that the optimization problem

$$335 \quad (5.6) \quad \min_{w_h \in V_h^k(\Omega_2)} J_h(w_h)$$

336 has a unique solution that we denote by  $w_{k,h}^*$ . All the difficulty now is to choose the parameter  $\lambda_h$  in order to  
 337 be able to ensure the convergence of  $(w_{k,h}^*)_h$  towards a solution to (3.4) as  $h$  tends to zero. This is the main  
 338 goal of the next paragraph.

339 **5.3 Convergence of the method** The starting point of our discussion is the following

340 LEMMA 5.3. *We have the estimate*

$$341 \quad (5.7) \quad J_h(w_{k,h}^*) \leq \frac{1}{2} \int_{\Sigma} |u_h^{w_J^*} - u_{2,h}^{w_J^*}|^2 \, d\sigma + \lambda_h \|w_J^*\|_{\tilde{\varepsilon}_1}^2$$

342 where  $w_J^*$  is defined by (4.2).

343 *Proof.* Starting from the fact that  $\pi_h^k w_J^* \in V_h^k(\Omega_2)$  and using that  $w_{k,h}^*$  is the unique solution to the optimiza-  
 344 tion problem (5.6), we conclude that  $J_h(w_{k,h}^*) \leq J_h(\pi_h^k w_J^*)$ . On the other hand, the identity (5.4) allows us to  
 345 write

$$346 \quad J_h(\pi_h^k w_J^*) = \frac{1}{2} \int_{\Sigma} |u_h^{w_J^*} - u_{2,h}^{w_J^*}|^2 \, d\sigma + \lambda_h \|\pi_h^k w_J^*\|_{\tilde{\varepsilon}_1}^2.$$

347 The Lemma is then proved by recalling the estimate (5.3). □

348 In order to simplify notations, for  $h > 0$  and  $w \in V_2(\Omega_2)$ , we denote by  $A_h(w)$  the real number

$$349 \quad A_h(w) = \frac{1}{2} \int_{\Sigma} |u_h^w - u_{2,h}^w|^2 \, d\sigma.$$

350 From (5.4), we know that for all  $w \in V_2(\Omega_2)$ , we have  $A_h(w) = J_0^h(\pi_h^k w)$ . The main result of this paragraph  
 351 is the following theorem.

352 THEOREM 5.4. *Assume that the parameter  $\lambda_h$  can be chosen such that the sequences  $(\lambda_h)_h$  and  $(A_h(w_J^*)/\lambda_h)_h$   
 353 tend to zero as  $h$  tends to zero. Then, as  $h$  goes to 0:*

- 354 • *the sequence  $(w_{k,h}^*)_h$  converges to  $w_J^*$  in  $V_2(\Omega_2)$  ;*
- 355 • *the sequence  $(u_h^{w_{k,h}^*})_h$  converges to  $E_H(u_1)$  in  $H_0^1(\Omega)$ , resp. the sequence  $(u_{2,h}^{w_{k,h}^*})_h$  converges to  $u_2$  in  
 356  $V_2(\Omega_2)$ , where  $(u_1, u_2)$  is the solution to (2.1) and  $E_H(u_1)$  is the extension of  $u_1$  defined in (4.4).*

357 *Proof.* The strategy of proof is similar to the one of proposition 4.4. To simplify notations, we denote by  
 358  $u^{k,h} \in V_h^k(\Omega)$  and  $u_2^{k,h} \in V_h^k(\Omega_2)$  the functions

$$359 \quad u^{k,h} = u_h^{w_{k,h}^*} \quad \text{and} \quad u_2^{k,h} = u_{2,h}^{w_{k,h}^*}.$$

360 In order to make the proof as clear as possible, we divide it into four steps.

361 **Step 1: weak convergence of  $(w_{k,h}^*)_h$ ,  $(u^{k,h})_h$  and  $(u_2^{k,h})_h$ .** Starting from the estimate

$$362 \quad (5.8) \quad \|w_{k,h}^*\|_{\tilde{\varepsilon}_1}^2 \leq J_h(w_{k,h}^*)/\lambda_h \leq A_h(w_J^*)/\lambda_h + \|w_J^*\|_{\tilde{\varepsilon}_1}^2$$

363 and using the fact that  $(A_h(w_J^*)/\lambda_h)_h$  tends to 0 as  $h$  goes to 0, we infer that  $(w_{k,h}^*)_h$  is bounded in  $V_2(\Omega_2)$ .  
 364 This implies that, up to a sub-sequence,  $(w_{k,h}^*)_h$  converges weakly to some  $w_0 \in V_2(\Omega)$ . For the reader's  
 365 convenience, this sub-sequence is still denoted by  $(w_{k,h}^*)_h$ .

366 Since the problems in (5.2) are uniformly elliptic with respect to  $h$ , we know that the sequence  $(u^{k,h})_h$  (resp.  
 367  $(u_2^{k,h})_h$ ) converges weakly in  $H_0^1(\Omega)$  (resp. in  $V_2(\Omega_2)$ ) to some  $u \in H_0^1(\Omega)$  (resp.  $u_2 \in V_2(\Omega_2)$ ). Using the basic  
 368 approximability property (5.1), we infer that  $u = u^{w_0}$  and  $u_2 = u_2^{w_0}$ .

369 **Step 2:  $w_0$  is a minimizer of  $J$ .** The continuity of the trace operator and the compactness of the embedding  
 370  $H^{1/2}(\Sigma) \subset L^2(\Sigma)$  ensure that

$$371 \quad u^{k,h}|_{\Sigma} - u_2^{k,h}|_{\Sigma} \rightarrow u^{w_0}|_{\Sigma} - u_2^{w_0}|_{\Sigma}$$

372 in  $L^2(\Sigma)$  as  $h \rightarrow 0$ . By noticing that

$$373 \quad \frac{1}{2} \int_{\Sigma} |u^{k,h} - u_2^{k,h}|^2 d\sigma = J_0^h(w_{k,h}^*) \leq J_h(w_{k,h}^*) \leq \lambda_h(A_h(w_j^*)/\lambda_h + \|w_j^*\|_{\tilde{\varepsilon}_1}^2)$$

374 and using that  $\lambda_h, A_h(w_j^*)/\lambda_h \rightarrow 0$  as  $h$  goes to zero, we deduce that  $u^{w_0}|_{\Sigma} - u_2^{w_0}|_{\Sigma} = 0$ . This shows that  $w_0$   
375 is a minimizer of  $J$ .

376 **Step 3: strong convergence of  $(w_{k,h}^*)_h$  to  $w_j^*$ .** Thanks to the fact that  $A_h(w_j^*)/\lambda_h \rightarrow 0$  as  $h \rightarrow 0$  and by  
377 means of the estimate (5.8), we can write

$$378 \quad \limsup_{h \rightarrow 0} \|w_{k,h}^*\|_{\tilde{\varepsilon}_1} \leq \|w_j^*\|_{\tilde{\varepsilon}_1}.$$

379 On the other hand, since  $(w_{k,h}^*)_h$  converges weakly to  $w_0$  as  $h \rightarrow 0$ , we infer that

$$380 \quad \|w_0\|_{\tilde{\varepsilon}_1} \leq \liminf_{h \rightarrow 0} \|w_{k,h}^*\|_{\tilde{\varepsilon}_1},$$

381 which is a consequence of the fact that the norm of a Banach space is weakly lower semicontinuous, see [9,  
382 Proposition III.5 (iii)]. This implies that  $\|w_0\|_{\tilde{\varepsilon}_1} \leq \|w_j^*\|_{\tilde{\varepsilon}_1}$ . Since  $w_0$  is a minimizer of  $J$ , we conclude that  
383  $w_0 = w_j^*$ . Furthermore, we also deduce that

$$384 \quad \lim_{h \rightarrow 0} \|w_{k,h}^*\|_{\tilde{\varepsilon}_1} = \|w_0\|_{\tilde{\varepsilon}_1}.$$

385 As a result, by applying [9, Proposition III.32], we infer that  $(w_{k,h}^*)_h$  converges, strongly, in  $V_2(\Omega_2)$  to  $w_0 = w_j^*$ .

386 **Step 4: strong convergence of  $(u^{k,h})_h$  and  $(u_2^{k,h})_h$ .** The ellipticity of the problems in (3.2), combined  
387 with the strong convergence of  $(w_{k,h}^*)_h$  to  $w_j^*$ , imply the convergence of  $(u^{k,h})_h$  in  $H_0^1(\Omega)$  to  $u^{w_j^*}$  and of  $(u_2^{k,h})_h$   
388 in  $V_2(\Omega_2)$  to  $u_2^{w_j^*}$ .

389 The result is then proved by using that  $u^{w_j^*} = E_H(u_1)$  and by observing that these limits are independent of  
390 the chosen sub-sequences.  $\square$

391 The rest of this paragraph is devoted to explain why it is possible to choose the parameter  $\lambda_h$  in such a way  
392 that  $(\lambda_h)_h$  and  $(A_h(w_j^*)/\lambda_h)_h$  both converge to 0 as  $h$  tends to 0. To do so, one needs to study the behaviour  
393 of  $A_h(w_j^*)$  as  $h$  tends to 0. We recall that, according to theorem 2.2, we know that  $u \in \cap_{s \in [0, \sigma_D(\varepsilon)]} PH^{1+s}(\Omega)$ ,  
394 where  $\sigma_D(\varepsilon) \in (0, 1]$  is the so-called regularity exponent. Let us start with the following

395 **PROPOSITION 5.5.** *Suppose that the coefficients  $\tilde{\varepsilon}_1$  and  $\varepsilon_2$  are smooth, or piecewise smooth. Assume that the*  
396 *solution  $u$  to (1.1) belongs to  $PH^{1+s}(\Omega)$  for some  $s > 0$ . Then there exists  $s' \in (0, s]$  that depends only on the*  
397 *geometry of  $\Omega_2$  and on the coefficient  $\varepsilon_2$ , and there exists  $\sigma \in (0, 1]$  that depends only on the geometry of  $\Omega$*   
398 *and of  $\Omega_2$  such that*

$$399 \quad \begin{aligned} \|u^{w_j^*} - u_h^{w_j^*}\|_{H_0^1(\Omega)} &\leq Ch^{p'} \|u\|_{PH^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{\tilde{\varepsilon}_1} \leq Ch^{p'} \|u_2\|_{H^{1+p'}(\Omega_2)}, \\ \|u^{w_j^*} - u_h^{w_j^*}\|_{L^2(\Omega)} &\leq Ch^{p'+\sigma} \|u\|_{PH^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{L^2(\Omega_2)} \leq Ch^{p'+\sigma} \|u_2\|_{H^{1+p'}(\Omega_2)} \end{aligned}$$

400 with  $C$  independent of  $h$  and  $p' = \min(s', k)$ .

401 *Proof.* Along this proof,  $C$  denotes a positive constant whose value can change from one line to the next but  
402 does not depend on  $h$ .

403 Given that  $u^{w_j^*} = E_H(u_1)$  solves (4.4), and since  $u_1 \in H^{1+s}(\Omega_1)$ , it follows that  $E_H(u_1)|_{\Omega_2}$  exhibits some  
404 extra-regularity because  $\varepsilon_2$  is (piecewise) smooth (via a classical shift theorem). In other words, there exists  
405  $s' \in (0, s]$  such that  $u^{w_j^*} \in PH^{1+s'}(\Omega)$ .

406 Given that  $u_2^{w_j^*} = u_2 \in PH^{1+s}(\Omega_2) \subset PH^{1+s'}(\Omega_2)$  and since the problems in (3.2) are elliptic with (piecewise)  
407 smooth coefficients  $\tilde{\varepsilon}_1$  and  $\varepsilon_2$ , we obtain the estimates (see [15])

$$408 \quad \|u^{w_j^*} - u_h^{w_j^*}\|_{H_0^1(\Omega)} \leq Ch^{p'} \|u\|_{PH^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{\tilde{\varepsilon}_1} \leq Ch^{p'} \|u_2\|_{H^{1+p'}(\Omega_2)},$$

409 where  $p' = \min(s', k)$ . By applying the classical Aubin-Nitsche's lemma (see [15, Theorem 3.2.4]), we infer  
410 that there exists  $\sigma \in (0, 1]$  such that

$$411 \quad \|u^{w_j^*} - u_h^{w_j^*}\|_{L^2(\Omega)} \leq Ch^{p'+\sigma} \|u\|_{PH^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{L^2(\Omega_2)} \leq Ch^{p'+\sigma} \|u_2\|_{H^{1+p'}(\Omega_2)}. \quad \square$$

412 *Remark 5.6.* It is worth to note that the value of  $s'$  is prescribed by the regularity of  $E_H(u_1)$ , the harmonic-like  
413 extension  $u_1$  that satisfies (4.4). More precisely,  $s'$  depends both on the regularity exponent  $\sigma_D(\tilde{\varepsilon}_1) \in (0, 1]$ ,  
414 and on the regularity of the boundary data on  $\partial\Omega_2$ , because  $u_{2H} := E_H(u_1)|_{\Omega_2} \in V_2(\Omega_2)$  solves the Dirichlet  
415 problem (cf. (4.4)):  $-\operatorname{div}(\tilde{\varepsilon}_1 \nabla u_{2H}) = 0$  in  $\Omega_2$ , with  $u_{2H} = u_1$  in  $H_{00}^{1/2}(\Sigma)$ . Assume for instance that  $\Omega$  and  
416  $\Omega_2$  are convex domains, and that the coefficients  $\varepsilon_1$  and  $\varepsilon_2$  are constant. In this case, one can choose  $\tilde{\varepsilon}_1$   
417 to be constant over  $\Omega$ . We recall that the solution  $u$  to (1.1) belongs to  $\text{PH}^{1+s}(\Omega)$  for all  $s \in (0, \sigma_D(\varepsilon))$ .  
418 Then, because  $u_{2H} \in H^1(\Omega_2)$  is governed by:  $-\Delta u_{2H} = 0$  in the convex domain  $\Omega_2$  with Dirichlet data in  
419  $H^{1/2+s}(\partial\Omega_2)$ , one has  $u_{2H} \in H^{1+s}(\Omega_2)$ . In other words,  $s' = s$ , and  $p' = \min(s', k) = s' = s$ . Finally, because  
420  $\Omega$  and  $\Omega_2$  are convex, one finds that  $\sigma = 1$ .

421 Now we have all the tools to study the behavior  $A_h(w_J^*)$  as  $h$  goes to 0.

422 **COROLLARY 5.7.** *Under the same assumptions as in proposition 5.5, one has*

$$423 \quad A_h(w_J^*) \leq Ch^{2p'+\sigma}$$

424 *with  $C$  independent of  $h$  and  $p' = \min(s', k)$ .*

425 *Proof.* Applying the multiplicative trace inequality (recalled in proposition A.1) and using the estimates of  
426 proposition 5.5 yield the estimates

$$427 \quad \|u^{w_J^*} - u_h^{w_J^*}\|_{L^2(\Sigma)}^2 \leq Ch^{2p'+\sigma} \|u\|_{\text{PH}^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_J^*} - u_{2,h}^{w_J^*}\|_{L^2(\Sigma)}^2 \leq Ch^{2p'+\sigma} \|u_2\|_{H^{1+p'}(\Omega_2)}.$$

428 By design, one has  $u^{w_J^*}|_{\Sigma} = u_2^{w_J^*}|_{\Sigma}$ . So, observing that

$$429 \quad \|u_h^{w_J^*} - u_{2,h}^{w_J^*}\|_{L^2(\Sigma)}^2 \leq 2(\|u^{w_J^*} - u_h^{w_J^*}\|_{L^2(\Sigma)}^2 + \|u_2^{w_J^*} - u_{2,h}^{w_J^*}\|_{L^2(\Sigma)}^2),$$

430 we conclude that  $A_h(w_J^*) \leq Ch^{2p'+\sigma}$ . □

431 The previous result gives us a simple way to choose the parameter  $\lambda_h$  in order to ensure that both  $(\lambda_h)_h$  and  
432  $(A_h(w_J^*)/\lambda_h)_h$  tend to 0 as  $h$  tends to 0.

433 **COROLLARY 5.8.** *Under the same assumptions as in proposition 5.5, any parameter  $\lambda_h$  of the form  $\lambda_h = Ch^q$   
434 with  $C > 0$  independent of  $h$  and  $q \in (0, 2p' + \sigma)$  satisfies the conditions of theorem 5.4.*

435 *Remark 5.9.* Within the framework of remark 5.6, one may choose  $q \in (0, 2\sigma_D(\varepsilon) + 1)$  in the statement of  
436 corollary 5.8.

437 Thanks to theorem 5.4, using the conditions of corollary 5.8, one obtains the convergence of the discrete  
438 solutions to the exact solution.

439 On the one hand, convergence is guaranteed even on meshes that are not T-conforming. Compared to [1],  
440 convergence holds in very general situations, namely as soon as there is a shift theorem for problem (1.1), cf.  
441 theorem 2.2, even with a regularity exponent  $\sigma_D(\varepsilon) < 1/2$ .

442 On the other hand, there is no associated convergence rate. Assuming a Céa lemma-like result, and using the  
443 same notations as above, the *expected* convergence rate is  $h^{p'}$  in  $H_0^1$ -norm, and  $h^{p'+\sigma}$  in  $L^2$ -norm. Whereas,  
444 classically, the *optimal* convergence rate is  $h^k$  in  $H_0^1$ -norm, and  $h^{k+1}$  in  $L^2$ -norm.

445 **6 Numerical experiments** In this section we turn our attention to the validation of the numerical  
446 method that we have proposed. We limit ourselves to the case of 2D domains and use  $P^1$  Lagrange finite  
447 elements. The numerical results that we present below have been obtained with the help of the library  
448 **FreeFem++**<sup>2</sup>. Since the well-posedness of (1.1) depends on the shape of the interface  $\Sigma$ , we test the performance  
449 of our method in three different configurations. In the first one,  $\Sigma$  is flat, in the second one,  $\Sigma$  is a circular  
450 interface and in the last one,  $\Sigma$  has a "corner", in the sense that the angle at the intersection with the boundary  
451 is not a right angle. In all these experiments, we suppose that the coefficients  $\varepsilon_1$  and  $\varepsilon_2$  are constant with  
452  $\varepsilon_1 = 1$ . We denote by  $\kappa_\varepsilon$  the contrast  $\kappa_\varepsilon = \varepsilon_2/\varepsilon_1$ .

453 The shape, smoothness and (respective) volumes of  $\Omega_1$  and  $\Omega_2$  are taken into account to choose the domain  
454  $\Omega^* \in \{\Omega_1, \Omega_2\}$  to which the extension is performed (we recall that one must have  $\text{meas}_{\partial\Omega}(\partial\Omega^* \setminus \Sigma) > 0$ , see  
455 footnote<sup>1</sup> on page 4). Indeed, to have a better convergence rate, one should choose  $\Omega^*$  convex, or with as

<sup>2</sup>See <https://freefem.org/>.

456 smooth a boundary as possible. Also, in order to speed up the convergence of the optimization algorithm,  
457 we must choose  $\Omega^*$  as small as possible. Once  $\Omega^*$  is fixed, one has to extend the function  $\varepsilon_1$  or  $\varepsilon_2$  to all  
458 the domain  $\Omega$ . Because the coefficients are constant, we extend  $\varepsilon_1$  (resp.  $\varepsilon_2$ ) by  $\varepsilon_1$  in  $\Omega_2$  (resp. in  $\Omega_1$ ).  
459 In the case where  $\Sigma$  is flat or circular, we take  $\Omega^* = \Omega_2$ . In the third configuration, we take  $\Omega^* = \Omega_1$ . To  
460 solve the optimization problem, we will use two different algorithms that are available in `FreeFem++`. The  
461 first one is the algorithm `BFGS` (Broyden-Fletcher-Goldfarb-Shanno) and the second one is the algorithm `NLCG`  
462 (Nonlinear Conjugate Gradient). Compared to the `NLCG` algorithm which uses only the gradient of the cost  
463 function to solve the optimization problem, the `BFGS` algorithm, which belongs to the class of quasi-Newton  
464 methods, uses a particular approximation of the hessian of the objective function. As already mentioned in  
465 the documentation of `FreeFem++` (see page 65 of the third version), when the unknown of the optimization  
466 problem is a finite element function with a large size, it is preferable to work with the `NLCG` algorithm because  
467 the `BFGS` algorithm can be very memory consuming and its convergence rate can be low. In our numerical  
468 experiments, we observed that in general both algorithms work similarly for the case where the interface is flat  
469 or circular, however for the case of the interface with corner, the algorithm `NLCG` performs better (the observed  
470 convergence rate is better). Below we present the numerical results we obtained using the `BFGS` algorithm  
471 for the case where the interface is flat or circular and the results we obtained using the `NLCG` algorithm for  
472 the case where the interface has a corner. In our numerical experiments, the `BFGS` function was used with  
473 the following parameters: `eps=1.e-11,nbiter=10,nbiterline=1` and the `NLCG` function with the following:  
474 `nbiter=10,eps=1.e-11`.

475 **6.1 Flat interface** In this paragraph, we take

$$476 \quad \Omega_1 = \{(x, y) \in (0; 1/2) \times (0; 1)\} \quad \text{and} \quad \Omega_2 = \{(x, y) \in (1/2; 1) \times (0; 1)\}$$

477 (a flat interface and a domain which is symmetric with respect to  $\Sigma$ ). We consider a family of meshes of  $\bar{\Omega}$   
478 satisfying Assumption 5.1 (see Figure 2). In the rest of this paragraph we suppose that  $\kappa_\varepsilon \neq -1$ . To test the  
479 performance of our method, we work with the same example considered in [1, 14]. Define the function  $u_{\kappa_\varepsilon}$   
480 such that

$$481 \quad u_{\kappa_\varepsilon}(x, y) = \begin{cases} (x^2 + bx) \sin(\pi y) & \text{if } x < 1/2 \\ a(x - 1) \sin(\pi y) & \text{if } 1/2 < x \end{cases}, \quad \text{where } a = \frac{1}{2(\kappa_\varepsilon + 1)} \quad \text{and} \quad b = -\frac{\kappa_\varepsilon + 2}{2(\kappa_\varepsilon + 1)}.$$

482 and consider it as an exact solution to (1.1). This is possible because  $\text{div}(\varepsilon \nabla u_{\kappa_\varepsilon}) \in L^2(\Omega)$ . The source term  
483  $f$  is computed accordingly. As observed in remark 5.6, by choosing  $\lambda_h = Ch^q$  with  $q \in (0, 3)$ , the method  
484 is convergent. In our experiment, we take  $\lambda_h = 0.002h^2$ . We work with  $\kappa_\varepsilon = -1.001$ . The behaviors of the  
485 relative  $L^2$ -norm error ( $\|e_h^r\|_0$ ) and the relative  $H_0^1$ -norm error ( $\|e_h^r\|_1$ ) between the exact solution and the  
numerical one are reported in Figure 2. We observe that both rates of convergence are equal to 2.

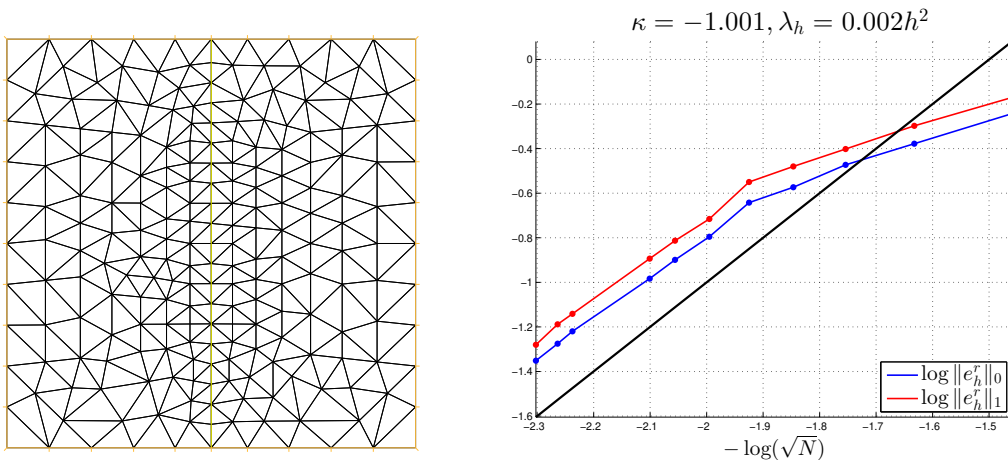


FIG. 2. A given mesh (left). Behavior of the relative  $L^2$  and  $H_0^1$  errors with respect to the meshsize  $h \sim \sqrt{N}$ , where  $N$  is the total number of nodes of the mesh (right).

486

487 *Remark 6.1.* The constant  $C$  in  $\lambda_h = Ch^q$  must be adjusted by the user according to the contrast  $\kappa_\varepsilon$  in order  
 488 to obtain a fast convergence of the method. Clearly this depends on  $\|w_J^*\|_{\varepsilon_1}$ . Using the fact that  $w_J^* = w_H$   
 489 and owing to (4.7) we see that this depends on the jump of the normal derivative (across  $\Sigma$ ) between  $u_1$  and  
 490 its harmonic extension. It is also important to note that, once  $q$  is fixed and when  $h$  is small enough, the  
 491 choice of  $C$  has little influence on the convergence of the method.

492 **6.2 The case of a circular interface** In this paragraph, we consider the case of a circular inclusion,  
 493 precisely the domains  $\Omega_1$  and  $\Omega_2$  are such that  $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\}$  and  $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 < |\mathbf{x}| < 2\}$ .  
 494 In proposition A.2, we prove that  $A_\varepsilon$  is an isomorphism  $\kappa_\varepsilon \notin \{-1\} \cup \mathcal{S}$  with  $\mathcal{S} := \{-1 - (1/2)^{2n}/(1 +$   
 495  $(1/2)^{2n}) \mid n \in \mathbb{N}^*\}$ . For this reason, we consider the case where  $\kappa_\varepsilon = -2 \notin \mathcal{S}$ . Given that both  $\Omega_2$  and  $\Omega$  have  
 496 smooth boundaries, we infer that  $\sigma = 1$  and  $s' = s$ . By taking  $f$  as the source term associated to the function

$$497 \quad u_{\kappa_\varepsilon}(x, y) = \begin{cases} r^2 + b & \text{if } r < 1 \\ a(r-2)^2 & \text{if } 1 < r < 2. \end{cases}, \text{ with } r = \sqrt{x^2 + y^2}, a = -1/\kappa_\varepsilon \text{ and } b = a - 1$$

498 and by taking  $\lambda_h = 0.002h^2$ , we obtain the results displayed in Figure 3. We observe that the method converges  
 499 with optimal rate (i.e., the relative  $L^2$ -norm error ( $\|e_h^r\|_0$ ) is of order 2, while the relative  $H_0^1$ -norm error is of  
 500 order 1), even though the exterior boundary and the interface are curved.

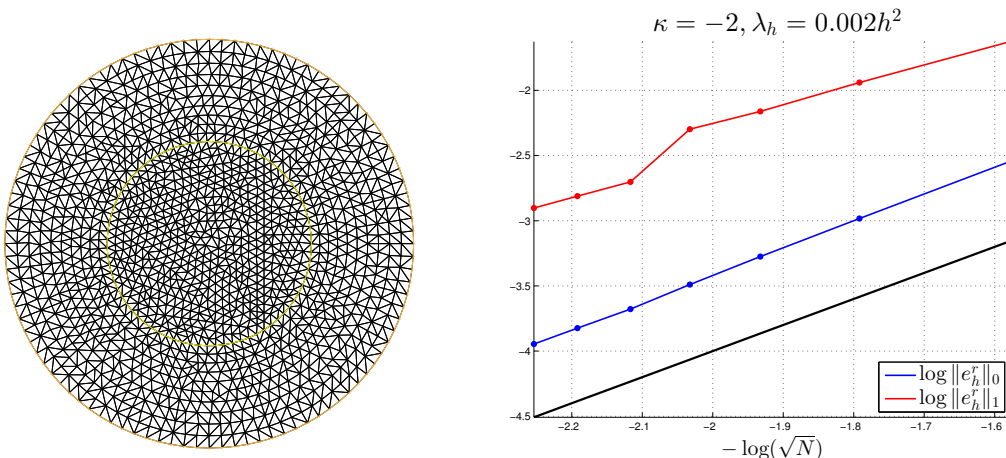


FIG. 3. A given mesh (left). Behavior of the relative  $L^2$  and  $H_0^1$  errors with respect to the meshsize  $h \sim \sqrt{N}$ , where  $N$  is the total number of nodes of the mesh (right)

501 **6.3 The case of an interface with corner** Now, we consider the configuration where the interface  
 502  $\Sigma$  has a corner. More precisely, we assume that  $\Omega := \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1 \text{ and } \arg(\mathbf{x}) \in (0; \pi)\}$  and  $\Omega_1 := \{\mathbf{x} \in$   
 503  $\Omega \mid \arg(\mathbf{x}) \in (0; \pi/4)\}$  (see Figure 4). In such configuration, it can be proved (see [4]) that  $A_\varepsilon$  is an isomorphism  
 504 if and only if  $\kappa_\varepsilon \in \mathbb{R}_-^* \setminus [-3, -1]$ . Furthermore, contrarily to the two previous cases, in this configuration the  
 505 solution to (1.1) can be very singular near the origin. Indeed, it was proved in [12, Chapter 2] that the  
 506 regularity of the solution to (1.1) depends in  $\kappa_\varepsilon$  and can be very low as  $\kappa_\varepsilon$  approaches  $[-3, -1]$ : more precisely,

$$507 \quad \lim_{\kappa_\varepsilon \rightarrow -3^-} \sigma_D(\varepsilon) = \lim_{\kappa_\varepsilon \rightarrow -1^+} \sigma_D(\varepsilon) = 0.$$

508 As a matter of fact, the value of the regularity exponent  $\sigma_D(\varepsilon)$  is  $\Re e(\lambda_0)$ , where  $\lambda_0$  is the solution to

$$509 \quad (6.1) \quad \kappa_\varepsilon = -\tan(3\lambda\pi/4)/\tan(\lambda\pi/4)$$

510 that has the smallest positive real part. Note that one can show (see [12, Chapter 3]) that all the solutions  
 511 to (6.1) are real-valued. In the particular cases where  $\kappa_\varepsilon = -5$  and  $\kappa_\varepsilon = -3.1$ , one finds, respectively that  
 512  $\lambda_0 \approx 0.458$  and  $\lambda_0 \approx 0.139$ . As mentioned previously this regularity result is optimal. Indeed, one can check  
 513 that the function

$$514 \quad u_{\lambda_0}(r, \theta) := (1-r)r^{\lambda_0} \begin{cases} \sin(\lambda_0\theta)/\sin(\lambda_0\pi/4) & \theta \in (0; \pi/4), \\ \sin(\lambda_0(\pi-\theta))/\sin(3\lambda_0\pi/4) & \theta \in (\pi/4; \pi) \end{cases}$$

515 satisfies  $\text{div}(\varepsilon \nabla u_{\lambda_0}) \in L^2(\Omega)$ . Observe that  $u_{\lambda_0} \notin \text{PH}^{\lambda_0}(\Omega)$ . This means that  $u_{\lambda_0} \notin \text{PH}^{3/2}(\Omega)$ . Now, given  
516 that  $\Omega$  and  $\Omega_2$  are both convex, owing to proposition 5.4, we can say that by choosing  $\lambda_h = Ch^q$  with  
517  $q \in (0, 1 + 2\lambda_0)$ , the convergence of the method can be guaranteed. In the case  $\kappa_\varepsilon = -5$  (resp.  $\kappa_\varepsilon = -3.1$ ),  
518 we work with  $\lambda_h = 6h^{1.8}$  (resp.  $\lambda_h = 1.5h^{1.2}$ ).  
519 The behaviors of the relative  $L^2$ -norm error and the relative  $H_0^1$ -norm error (for the cases  $\kappa_\varepsilon = -5$  and  
520  $\kappa_\varepsilon = -3.1$ ) are given in Figure 4. In either case, the expected rate of convergence is equal to  $\lambda_0$  ( $\approx 0.458$  when  
521  $\kappa_\varepsilon = -5$  and  $\approx 0.139$  when  $\kappa_\varepsilon = -3.1$ ) for the case of the  $H_0^1$ -norm error, while it is equal to  $2\lambda_0$  for the case  
522 of the  $L^2$ -norm error. We observe that, in both cases, the method converges with **optimal rate of convergence**  
523 **for the  $H^1$ -norm and the  $L^2$  one.**

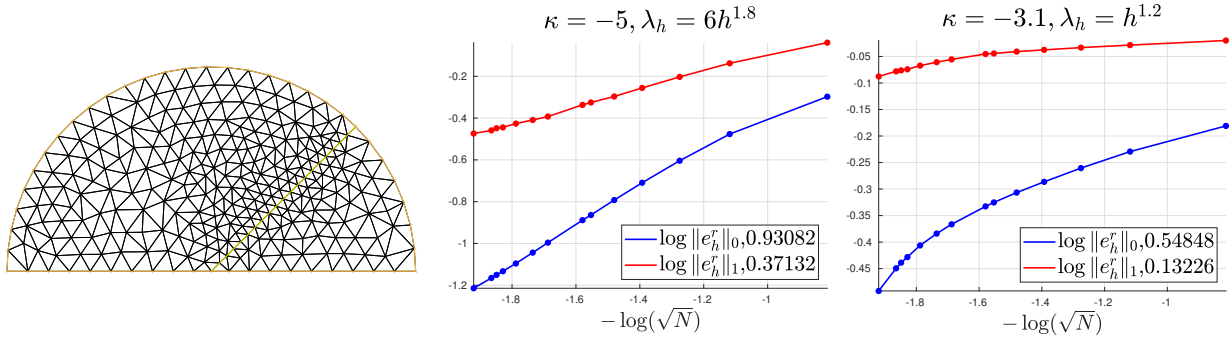


FIG. 4. A given mesh (left). Behavior of the relative  $L^2$  and  $H_0^1$  errors with respect to the meshsize  $h \sim \sqrt{N}$ , where  $N$  is the total number of nodes of the mesh, with the observed convergence rates, when  $\kappa_\varepsilon = -5$  (center) and  $\kappa_\varepsilon = -3.1$  (right).

524 **7 Concluding remarks** In this work, we have presented a new numerical method to approximate  
525 the solution to the scalar transmission problem with sign-changing coefficients. We proved that the method  
526 converges without any restriction on the mesh sequence, nor on the regularity of the solution. This result  
527 has been illustrated by several 2D numerical experiments. The convergence rate of our method seems to be  
528 optimal. In order to improve the performance of the method, several questions can be studied:

- 529 1. Choose the parameter  $\lambda_h$  in order to accelerate convergence. An interesting idea would be to find  
530 an adaptive approach to fit its value. Also, one could use adaptive mesh refinement, together with a  
531 posteriori estimates. We refer to [17] for estimators that deliver guaranteed error bounds, and that  
532 are robust with respect to the sign-changing coefficient  $\varepsilon$ .
- 533 2. Work with other regularization approaches, i.e., other choices for the coefficient  $\tilde{\varepsilon}_1$ , and/or an alter-  
534 native to the Tikhonov regularization method.
- 535 3. In the case where the interface is not regular, it would be interesting to combine our approach with  
536 other existing methods for solving PDE with singular solution such as the Singular Complement  
537 Method [16].

538 Besides that, it will be also interesting to extend this approach to other models involving sign-changing  
539 coefficients.

540 **Acknowledgments** The authors would like to thank Anne-Sophie Bonnet-Ben Dhia and Lucas Chesnel  
541 for their helpful comments on the first draft of this article. We also thank Farah Chaaban, for her constructive  
542 remarks during her internship, regarding the implementation of the method.

## 543 Appendix A. Missing results.

544 *Proof of Proposition 4.4.* From the definition of  $w_\delta^*$ , we can write that

$$545 \quad \delta \|w_\delta^*\|_{\tilde{\varepsilon}_1}^2 \leq J^\delta(w_\delta^*) \leq J^\delta(w_J^*) = J(w_J^*) + \delta \|w_J^*\|_{\tilde{\varepsilon}_1}^2 = \delta \|w_J^*\|_{\tilde{\varepsilon}_1}^2.$$

546 This means that for all  $0 < \delta$ , there holds  $\|w_\delta^*\|_{\tilde{\varepsilon}_1} \leq \|w_J^*\|_{\tilde{\varepsilon}_1}$ . As a result  $(w_\delta^*)$  is bounded in  $V_2(\Omega_2)$ . This  
547 implies that, up to a sub-sequence,  $(w_\delta^*)_\delta$  converges, as  $\delta$  tends to 0, weakly in  $V_2(\Omega_2)$  to some  $w_0 \in V_2(\Omega_2)$ .  
548 For the reader's convenience, this sequence is also denoted by  $(w_\delta^*)_\delta$ . Now, let us prove that  $w_0$  is a minimizer  
549 of  $J$ . To do that, we start by observing that for all  $\delta > 0$ , we have

$$550 \quad 0 \leq J(w_\delta^*) \leq J^\delta(w_\delta^*) \leq J^\delta(w_J^*) = \delta \|w_J^*\|_{\tilde{\varepsilon}_1}^2.$$

551 This shows that  $(J(w_\delta^*))_\delta$  converges to zero as  $\delta$  tends to zero. On the other hand, by using the result of  
 552 proposition 4.1, we know that  $(J(w_\delta^*))_\delta$  converges to  $J(w_0)$ . Consequently,  $J(w_0) = 0$  and then  $w_0$  is a  
 553 minimizer of  $J$ .

554 The next step is to show that the convergence of  $(w_\delta^*)_\delta$  to  $w_0$  occurs in the strong sense and that  $w_0 = w_J^*$ .  
 555 To do so, we observe that

$$556 \quad \|w_\delta^*\|_{\varepsilon_1} \leq \|w_J^*\|_{\varepsilon_1} \quad \forall \delta \implies \limsup_{\delta \rightarrow 0} \|w_\delta^*\|_{\varepsilon_1} \leq \|w_J^*\|_{\varepsilon_1}, \quad \text{and} \quad w_\delta^* \rightharpoonup w_0 \text{ in } V_2(\Omega_2) \implies \|w_0\|_{\varepsilon_1} \leq \liminf_{\delta \rightarrow 0} \|w_\delta^*\|_{\varepsilon_1},$$

557 where the latter is a consequence of the fact that the norm of a Banach space is weakly lower semicontinuous,  
 558 see [9, Proposition III.5 (iii)]. This implies that  $\|w_0\|_{\varepsilon_1} \leq \|w_J^*\|_{\varepsilon_1}$ . Thanks to the definition of  $w_J^*$ , we deduce  
 559 that  $w_0 = w_J^*$ . With this in mind and with the help of the previous inequality, we conclude that

$$560 \quad \lim_{\delta \rightarrow 0} \|w_\delta^*\|_{\varepsilon_1} = \|w_J^*\|_{\varepsilon_1}.$$

561 Since  $V_2(\Omega_2)$  is a Hilbert space, it follows (see [9, Proposition III.32]) that  $w_\delta \rightarrow w_J^*$  in  $V_2(\Omega_2)$ . By noticing  
 562 that  $w_J^*$  is independent of the considered sub-sequence, the result is then proved.  $\square$

563 PROPOSITION A.1. [8, Theorem 1.6.6] Let  $\Omega$  be an open, bounded, connected subset of  $\mathbb{R}^d$  ( $d = 2, 3$ ) with a  
 564 Lipschitz boundary. Then the estimate

$$565 \quad \|u\|_{L^2(\partial\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2} \quad \forall u \in H^1(\Omega)$$

566 holds with  $0 < C$  independent of  $u$ .

567 PROPOSITION A.2. Let  $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\}$  and  $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 < |\mathbf{x}| < 2\}$ . Assume that  $\kappa_\varepsilon := \varepsilon_2/\varepsilon_1 \notin$   
 568  $\{-1\} \cup \mathcal{S}$  with

$$569 \quad \mathcal{S} := \left\{ -\frac{1 - (1/2)^{2n}}{1 + (1/2)^{2n}} \mid n \in \mathbb{N}^* \right\}.$$

570 Then the operator  $A_\varepsilon : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism.

571 Remark A.3. Note that in accordance with the results concerning the Neumann-Poincaré operator [24, Chapter  
 572 1], we observe that  $-1$  is an accumulation point of  $\mathcal{S}$ .

573 Proof. [12, Theorem 1.3.3] guarantees that  $A_\varepsilon$  is Fredholm of index 0 when  $\kappa_\varepsilon \neq -1$ . Therefore it suffices to  
 574 study its kernel. Let  $u \in H_0^1(\Omega)$  be such that  $A_\varepsilon u = 0$ . Then  $u_1 := u|_{\Omega_1}$  and  $u_2 = u|_{\Omega_2}$  satisfy

$$575 \quad \begin{cases} \Delta u_1 = 0 & \text{in } \Omega_1 \\ \Delta u_2 = 0 & \text{in } \Omega_2 \\ u_1(1, \theta) = u_2(1, \theta) & \text{and} \quad \partial_r u_1(1, \theta) = \kappa_\varepsilon \partial_r u_2(1, \theta) \quad \forall \theta \in [0; 2\pi]. \end{cases}$$

576 Since the problem is invariant with respect to  $\theta$ , by Fourier decomposition for  $u_1, u_2$  we have the representa-  
 577 tions:

$$578 \quad u_1(r, \theta) = \sum_{n \in \mathbb{N}} a_n r^n e^{in\theta} \quad \text{and} \quad u_2(r, \theta) = b_0 \ln(r/2) + \sum_{n \in \mathbb{Z}^*} b_n ((r/2)^n - (r/2)^{-n}) e^{in\theta},$$

579 where  $a_n, b_n \in \mathbb{C}$ . Using the transmission conditions, we get

$$580 \quad \begin{cases} a_0 = b_0 \ln(1/2), & 0 = b_0 \kappa_\varepsilon \\ a_n = b_n ((1/2)^n - (1/2)^{-n}), & a_n = b_n ((1/2)^n + (1/2)^{-n}) \kappa_\varepsilon, \quad n \in \mathbb{N}^* \\ 0 = b_n ((1/2)^n - (1/2)^{-n}), & 0 = b_n ((1/2)^n + (1/2)^{-n}) \kappa_\varepsilon, \quad -n \in \mathbb{N}^*. \end{cases}$$

581 Therefore we deduce that  $A_\varepsilon$  is injective when  $\kappa_\varepsilon \notin \mathcal{S}$ .  $\square$

582 **Appendix B. On the use of the adjoint approach to compute the gradient of the cost functional**  
 583  $J$ . The adjoint approach was introduced in [11] as a method for computing the gradient of cost functions  
 584 that depend in non-explicit way on the main variable of the problem, namely via the solution of PDEs (the  
 585 state equations) in which the main variable plays the role of a parameter. Here, we are going to explain how

586 to apply this method to our case. The idea is to introduce a Lagrangian functional  $\mathcal{L} : V_2(\Omega_2) \times H_0^1(\Omega) \times$   
 587  $V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2) \rightarrow \mathbb{R}$  such that

$$588 \quad \mathcal{L}(w, u, u_2, g, g_2) = \frac{1}{2} \int_{\Sigma} |u - u_2|^2 \, d\sigma + a_1(w, u, g) + a_2(w, u, u_2, g_2)$$

589 in which  $a_1(w, u, g)$  and  $a_2(w, u, u_2, g_2)$  are respectively given by

$$590 \quad \begin{cases} a_1(w, u, g) = \int_{\Omega} \tilde{\varepsilon}_1 \nabla u \cdot \nabla g \, d\mathbf{x} - \int_{\Omega_1} f g \, d\mathbf{x} - \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla g \, d\mathbf{x} \\ a_2(w, u, u_2, g_2) = \int_{\Omega_2} \varepsilon_2 \nabla u_2 \cdot \nabla g_2 \, d\mathbf{x} - \int_{\Omega_2} f_2 g_2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla(w - u) \cdot \nabla g_2 \, d\mathbf{x}. \end{cases}$$

591 The functions  $g \in H_0^1(\Omega)$ ,  $g_2 \in V_2(\Omega_2)$  are the adjoint variables associated to  $u, u_2$  respectively. Let  $(u^w, u_2^w)$   
 592 be the solution to (3.2). By design,  $a_1(w, u^w, g) = 0$  for all  $g \in H_0^1(\Omega)$ , and  $a_2(w, u^w, u_2^w, g_2) = 0$  for all  
 593  $g_2 \in V_2(\Omega_2)$ , so one has

$$594 \quad (\text{B.1}) \quad \mathcal{L}(w, u^w, u_2^w, g, g_2) = J(w) \quad \forall g \in H_0^1(\Omega), \quad \forall g_2 \in V_2(\Omega_2).$$

595 Clearly, the functional  $\mathcal{L}$  is differentiable with respect to all its variables. For all  $\mathbf{v} = (w, u, u_2, g, g_2) \in$   
 596  $V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2)$ , the partial derivatives of  $\mathcal{L}$  at  $\mathbf{v}$  belong respectively to  
 597  $\partial_w \mathcal{L}(\mathbf{v}) \in (V_2(\Omega_2))^*$ ,  $\partial_u \mathcal{L}(\mathbf{v}) \in (H_0^1(\Omega))^*$ ,  $\partial_{u_2} \mathcal{L}(\mathbf{v}) \in (V_2(\Omega_2))^*$ ,  $\partial_g \mathcal{L}(\mathbf{v}) \in (H_0^1(\Omega))^*$ ,  $\partial_{g_2} \mathcal{L}(\mathbf{v}) \in (V_2(\Omega_2))^*$ .  
 598 Let  $g \in H_0^1(\Omega)$  and  $g_2 \in V_2(\Omega_2)$  be given, and  $\mathbf{v}^w = (w, u^w, u_2^w, g, g_2)$ . By taking the derivative of the relation  
 599 (B.1) with respect to  $w$ , we find that, by applying the chain rule formula,

$$600 \quad (J'(w), h)_{\tilde{\varepsilon}_1} = \langle \partial_w \mathcal{L}(\mathbf{v}^w), h \rangle + \langle \partial_u \mathcal{L}(\mathbf{v}^w), \frac{du^w}{dw}(h) \rangle + \langle \partial_{u_2} \mathcal{L}(\mathbf{v}^w), \frac{du_2^w}{dw}(h) \rangle, \quad \forall h \in V_2(\Omega_2).$$

601 Now, if there exists  $(g^w, g_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$  for which the equations

$$602 \quad \partial_u \mathcal{L}(w, u^w, u_2^w, g^w, g_2^w) = 0 \quad \text{and} \quad \partial_{u_2} \mathcal{L}(w, u^w, u_2^w, g^w, g_2^w) = 0$$

603 are satisfied for all  $w \in V_2(\Omega_2)$ , this yields

$$604 \quad (J'(w), h)_{\tilde{\varepsilon}_1} = \langle \partial_w \mathcal{L}(w, u^w, u_2^w, g_1^w, g_2^w), h \rangle \quad \forall w \in V_2(\Omega_2), \quad \forall h \in V_2(\Omega_2).$$

605 To investigate the existence of  $(g^w, g_2^w)$ , we need to write down the expression of  $\partial_u \mathcal{L}(\mathbf{v}^w)$  and  $\partial_{u_2} \mathcal{L}(\mathbf{v}^w)$ : By  
 606 a direct calculus, one checks that

$$607 \quad \begin{aligned} \langle \partial_u \mathcal{L}(\mathbf{v}^w), v \rangle &= \int_{\Omega} \tilde{\varepsilon}_1 \nabla g \cdot \nabla v \, d\mathbf{x} - \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla g_2 \cdot \nabla v \, d\mathbf{x} + \int_{\Sigma} (u^w - u_2^w) v \, d\sigma \quad \forall v \in H_0^1(\Omega) \\ \langle \partial_{u_2} \mathcal{L}(\mathbf{v}^w), v_2 \rangle &= \int_{\Omega_2} \varepsilon_2 \nabla g_2 \cdot \nabla v_2 \, d\mathbf{x} - \int_{\Sigma} (u^w - u_2^w) v_2 \, d\sigma \quad \forall v_2 \in V_2(\Omega_2). \end{aligned}$$

608 Hence, the functions  $(g^w, g_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$  are governed by the following system of equations:

$$609 \quad (\text{B.2}) \quad \begin{cases} \int_{\Omega} \tilde{\varepsilon}_1 \nabla g^w \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla g_2^w \cdot \nabla v \, d\mathbf{x} - \int_{\Sigma} (u^w - u_2^w) v \, d\sigma & \forall v \in H_0^1(\Omega) \\ \int_{\Omega_2} \varepsilon_2 \nabla g_2^w \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Sigma} (u^w - u_2^w) v_2 \, d\sigma & \forall v_2 \in V_2(\Omega_2). \end{cases}$$

610 Clearly the previous system of equations is well-posed. Therefore the functions  $g^w, g_2^w$  are well-defined. We  
 611 then have all the tools to prove the result stated in Lemma 4.5.

612 *Proof of Lemma 4.5.* Take  $w \in V_2(\Omega_2)$ . From the characterization (B.2) of  $g^w$  and  $g_2^w$ , we deduce that for all  
 613  $h \in V_2(\Omega_2)$ , we have

$$614 \quad (J'(w), h)_{\tilde{\varepsilon}_1} = \langle \partial_w \mathcal{L}(w, u^w, u_2^w, g_1^w, g_2^w), h \rangle.$$

615 On the other hand, one can compute explicitly the value of  $\langle \partial_w \mathcal{L}(w, u, u_2, g, g_2), h \rangle$ :

$$616 \quad \langle \partial_w \mathcal{L}(w, u, u_2, g, g_2), h \rangle = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla h \cdot \nabla (g_2 - g|_{\Omega_2}) \, d\mathbf{x}.$$

617 This shows that  $J'(w) = g_2^w - g^w|_{\Omega_2}$  and then the result is proved.  $\square$

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