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► **To cite this version:**

Patrick Ciarlet, David Lassounon, Mahran Rihani. An optimal control-based numerical method for scalar transmission problems with sign-changing coefficients. *SIAM Journal on Numerical Analysis*, 2023, 61 (3), pp.1316-1339. 10.1137/22M1495998 . hal-03666913v2

HAL Id: hal-03666913

<https://hal.science/hal-03666913v2>

Submitted on 13 Oct 2022

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AN OPTIMAL CONTROL-BASED NUMERICAL METHOD FOR SCALAR TRANSMISSION PROBLEMS WITH SIGN-CHANGING COEFFICIENTS

PATRICK CIARLET JR.*, DAVID LASSOUNON†, AND MAHRAN RIHANI‡

Abstract. In this work, we present a new numerical method for solving the scalar transmission problem with sign-changing coefficients. In electromagnetism, such a transmission problem can occur if the domain of interest is made of a classical dielectric material and a metal or a metamaterial, with for instance an electric permittivity that is strictly negative in the metal or metamaterial. The method is based on an optimal control reformulation of the problem. Contrary to other existing approaches, the convergence of this method is proved without any restrictive condition. In particular, no condition is imposed on the a priori regularity of the solution to the problem, and no condition is imposed on the meshes, other than that they fit with the interface between the two media. Our results are illustrated by some (2D) numerical experiments.

Key words. transmission problem, sign-changing coefficients, fictitious domain methods, optimal control.

MSC codes. 65N30, 78A48

1 Introduction In the present paper, we study the numerical approximation of the scalar transmission problem with sign-changing coefficients in \mathbb{R}^d , for $d \in \{2, 3\}$. To fix ideas, let Ω be an open, bounded, connected subset of \mathbb{R}^d with a Lipschitz boundary, in other words a *domain* of \mathbb{R}^d . Further, consider that Ω is equal to the union of two disjoint (sub)domains Ω_1, Ω_2 . We denote the interface by $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$ (see Figure 1 for an example), and we assume that $\text{meas}_{\partial\Omega}(\partial\Omega_2 \setminus \Sigma) > 0$. The case of an inclusion corresponds to $\partial\Sigma \cap \partial\Omega = \emptyset$ and, since $\text{meas}_{\partial\Omega}(\partial\Omega_2 \setminus \Sigma) > 0$, Ω_1 is always the inclusion in this configuration. When $\partial\Sigma \cap \partial\Omega$ is non-empty, we assume that it is a Lipschitz submanifold of $\partial\Omega_1$ and of $\partial\Omega_2$.

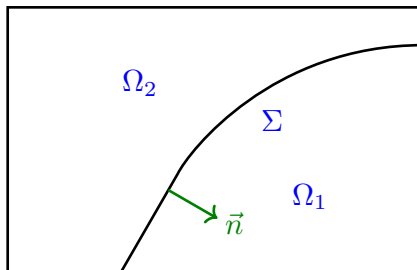


FIG. 1. *Example of geometry.*

We also introduce a coefficient $\varepsilon \in L^\infty(\Omega)$ such that $\varepsilon_1 = \varepsilon|_{\Omega_1} \geq \varepsilon_+ > 0$ a.e. in Ω_1 and $\varepsilon_2 = \varepsilon|_{\Omega_2} \leq \varepsilon_- < 0$ a.e. in Ω_2 . Here ε_+ and ε_- are two real constants. It will be useful to introduce the contrasts $\kappa_\varepsilon^1 := \varepsilon_1^- / \varepsilon_2^+$ and $\kappa_\varepsilon^2 := \varepsilon_2^- / \varepsilon_1^+$ where ε_1^\pm and ε_2^\pm are defined as follows:

$$\varepsilon_1^+ := \sup_{\Omega_1} \varepsilon_1, \quad \varepsilon_1^- := \inf_{\Omega_1} \varepsilon_1, \quad \varepsilon_2^+ := \sup_{\Omega_2} |\varepsilon_2| \quad \text{and} \quad \varepsilon_2^- := \inf_{\Omega_2} |\varepsilon_2|.$$

Note that in the particular case where ε is piecewise constant, we have $\kappa_\varepsilon^1 = 1/\kappa_\varepsilon^2$.

Remark 1.1. In principle, ε could be a symmetric tensor-valued coefficient, i.e., $\varepsilon = (\varepsilon_{ij})_{1 \leq i, j \leq d}$ with $\varepsilon_{ij} \in L^\infty(\Omega)$ for all $1 \leq i, j \leq d$, and such that

$$\begin{aligned} \exists \varepsilon_+ > 0, \quad \forall \mathbf{z} \in \mathbb{R}^d, \quad \varepsilon_+ |\mathbf{z}|^2 &\leq \varepsilon \mathbf{z} \cdot \mathbf{z} \quad \text{a.e. in } \Omega_1; \\ \exists \varepsilon_- > 0, \quad \forall \mathbf{z} \in \mathbb{R}^d, \quad \varepsilon_- |\mathbf{z}|^2 &\leq -\varepsilon \mathbf{z} \cdot \mathbf{z} \quad \text{a.e. in } \Omega_2. \end{aligned}$$

However, for the sake of conciseness, we consider a scalar-valued coefficient.

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29 For a given source term $f \in L^2(\Omega)$, we consider the problem

30 (1.1) Find $u \in H_0^1(\Omega)$ such that $-\operatorname{div}(\varepsilon \nabla u) = f \in L^2(\Omega)$.

31 The equivalent variational formulation to (1.1) writes

32 (1.2) Find $u \in H_0^1(\Omega)$ such that $\int_{\Omega} \varepsilon \nabla u \cdot \nabla v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}, \quad \forall v \in H_0^1(\Omega)$.

33 Because of the change of sign of ε , the well-posedness of this problem does not fit into the classical theory of
 34 elliptic PDEs and it can be ill-posed. On the other hand, one can show that when κ_{ε}^1 or κ_{ε}^2 is large enough,
 35 Problem 1.2 is T-coercive (for instance see [6]), i.e., there exists an operator $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that
 36 $(u, v) \mapsto \int_{\Omega} \varepsilon \nabla u \cdot \nabla(T(v))$ is coercive, and then it is well-posed. For the case of polygonal interfaces, the
 37 construction of such operator T is based on the use of local isometric geometrical transformations (such as
 38 reflections, rotations, ...) near the interface, see [3].

39 The implementation of a general conforming finite element method to discretize (1.2) leads us to consider the
 40 problem

41 (1.3) Find $u_h \in V_h(\Omega)$ such that $\int_{\Omega} \varepsilon \nabla u_h \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega} f v_h \, d\mathbf{x}, \quad \forall v_h \in V_h(\Omega),$

42 where $V_h(\Omega)$ is a well-chosen subspace of $H_0^1(\Omega)$, and the parameter $h > 0$ is the so-called meshsize. Even in
 43 the case where (1.2) is T-coercive, one can not guaranty that Problem (1.3) is also T-coercive. Indeed, it may
 44 happen that for some $v_h \in V_h(\Omega)$, there holds $T(v_h) \notin V_h(\Omega)$. To overcome this difficulty, an interesting idea
 45 is to try to construct meshes such that the approximation spaces $V_h(\Omega)$ are stable by operators T for which
 46 Problem (1.2) is T-coercive. This type of meshes are called T-conform meshes. Such an approach has been
 47 investigated in [26, 12, 10]. It works quite well but presents two main drawbacks:

- 48 • The construction of well-suited meshes for curved interfaces, interfaces with corners or 3D interfaces
 49 is not a straightforward task [10, 3].
- 50 • Sometimes the operator T for which the problem is T-coercive is constructed by abstract tools and
 51 therefore is not explicit. In these situations, one cannot find adapted meshes.

52 On general meshes, three alternatives have already been proposed. The first one was introduced in [6] and
 53 was based on the use of [discrete trace liftings, with quasi-uniform meshes on the interface](#). In addition to this
 54 constraint on the mesh, one of the limitation of this approach is that, for interfaces with general shapes, the
 55 convergence can not be assured in all the configurations in which Problem (1.2) is well-posed, because it is
 56 based on a particular (non-optimal) T-coercivity operator. The [second one](#) is developed in [23] and is based on
 57 the use of [interpolation](#) techniques. Its essential limitation lies again in the fact that, for interfaces with general
 58 shapes, the convergence can not be assured for all configurations in which Problem (1.2) is well-posed. The
 59 [third one](#), presented in [14], consists in adding some dissipation to the problem (considering $\varepsilon + i\delta$ instead of ε
 60 in (1.2) where δ depends on the meshsize). Unfortunately, this methods has a sub-optimal order of convergence
 61 even in the case where the solution and the interface are smooth (see [14]).

62 After that, in 2017, a new technique relying on the use of an optimal control reformulation has been introduced
 63 by Abdulle et al in [1], [where the auxiliary control function is defined over \$\Sigma\$](#) . Introducing

64 $\text{PH}^{1+s}(\Omega) := \{u \in H^1(\Omega) \mid u|_{\Omega_1} \in H^{1+s}(\Omega_1) \text{ and } u|_{\Omega_2} \in H^{1+s}(\Omega_2)\}$ for $s > 0$,

65 their method is proved to be convergent for general meshes (that respect the interface) as soon as the exact
 66 solution to (1.1) belongs to the space $\text{PH}^{1+s}(\Omega)$ for some $s > 1/2$. Unfortunately, this regularity condition is
 67 not always satisfied, especially when Σ has corners in 2D or conical points in 3D. See the numerical illustration
 68 in Section 6.3 below.

69
 70 In this work, we present a new strategy which relies on the use of a different optimal control reformulation and
 71 which converges without any restriction on the mesh (except the fact of being conforming to the interface),
 72 and without any restriction on the regularity of the exact solution. [In our approach, the auxiliary control
 73 function is defined over one subdomain](#). This method is inspired by the smooth extension method that was
 74 used (without proof of convergence) in [19] to approximate the solution to some classical scalar transmission
 75 problems. [The key idea is that, given a control, one can construct a pseudo-solution to the problem \(1.1\), and](#)

76 to note that, as soon as one can relate the control to some extension of the solution, then one recovers exactly
77 the solution.

78
79 The article is organized as follows. In Section 2, we start by giving a detailed description of the problem. Then,
80 in Section 3, we explain how to derive an equivalent optimal control reformulation. Section 4 is dedicated to
81 the study of some basic properties of the optimization problem and its regularization. The proposed numerical
82 method and the proof of its convergence are given in Section 5. Our results are then illustrated by some
83 numerical experiments in Section 6. Finally we give concluding remarks, including some possible extensions.

84 **2 Main assumption on ε and reformulation of the problem** Introduce the bounded operator
85 $A_\varepsilon : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$ such that

$$86 \quad (H_0^1(\Omega))^* \langle A_\varepsilon u, v \rangle_{H_0^1(\Omega)} = \int_{\Omega} \varepsilon \nabla u \cdot \nabla v \, d\mathbf{x}, \quad \forall u, v \in H_0^1(\Omega).$$

87 Obviously A_ε is an isomorphism if, and only if, Problem (1.1) is well-posed in the Hadamard sense. In this
88 article, we shall work under the following

89 *Assumption 2.1.* Assume that the coefficient ε is such that A_ε is an isomorphism.

90 If ε is piecewise constant by subdomain, the previous assumption is satisfied when the contrast $\kappa_\varepsilon := \varepsilon_2/\varepsilon_1$
91 does not belong to the so-called critical interval. The expression of this interval is in general not known
92 analytically, except for particular geometries like symmetric domains, simple 2D interface with corners, simple
93 3D interfaces with circular conical tips (see [24, Chapter 2]). Under assumption 2.1, one is able to prove the
94 accompanying shift theorem. We refer to [18, 7, 13, 12, 5].

95 **THEOREM 2.2.** *Assume that Σ is smooth (of class \mathcal{C}^2), polygonal (in 2D) or polyhedral (in 3D) and that*
96 *Problem (1.1) is well-posed in the Hadamard sense. Then, there exists $\sigma_D(\varepsilon) \in (0, 1]$ such that*

$$97 \quad \forall f \in L^2(\Omega), \text{ the solution } u \text{ to Problem (1.1) is such that } \begin{cases} u \in \bigcap_{s \in [0, \sigma_D(\varepsilon))} PH^{1+s}(\Omega) & \text{if } \sigma_D(\varepsilon) < 1 \\ u \in PH^2(\Omega) & \text{if } \sigma_D(\varepsilon) = 1 \end{cases},$$

98 *with continuous dependence.*

99 The number $\sigma_D(\varepsilon)$ in the shift theorem is called the *(limit) regularity exponent*. For instance, when the
100 interface is smooth and when it does not intersect with the boundary, then $\sigma_D(\varepsilon) = 1$ (cf. [18]).

101 *Remark 2.3.* In Problem (1.1), we consider homogeneous Dirichlet boundary conditions. Let us mention that
102 the results below extend quite straightforwardly to other situations, for example with Neumann or Robin-
103 Fourier boundary conditions which can be homogeneous or not, as long as the associated operator is an
104 isomorphism.

105 To introduce the method, we start by writing an equivalent version of (1.1) in which the unknown $u \in H_0^1(\Omega)$
106 is split into two unknowns defined in Ω_1 and Ω_2 : $(u_1, u_2) := (u|_{\Omega_1}, u|_{\Omega_2})$. To do so, we observe that since
107 $f \in L^2(\Omega)$, the solution u to (1.1) is such that the vector field $\varepsilon \nabla u$ belongs to the space $\mathbf{H}(\text{div}, \Omega) = \{\mathbf{v} \in$
108 $(L^2(\Omega))^d$ such that $\text{div } \mathbf{v} \in L^2(\Omega)\}$. Consequently, the pair of functions (u_1, u_2) satisfies the problem

$$109 \quad (2.1) \quad \text{Find } (u_1, u_2) \in V_1(\Omega_1) \times V_2(\Omega_2) \text{ such that } \begin{cases} -\text{div}(\varepsilon_1 \nabla u_1) = f_1 =: f|_{\Omega_1} \\ -\text{div}(\varepsilon_2 \nabla u_2) = f_2 =: f|_{\Omega_2} \\ \varepsilon_1 \partial_n u_1 = \varepsilon_2 \partial_n u_2 \text{ in } (H_{00}^{1/2}(\Sigma))^* \\ u_1 = u_2 \text{ in } H_{00}^{1/2}(\Sigma) \end{cases}$$

in which \mathbf{n} stands for the unit normal vector to Σ oriented to the exterior of Ω_2 (see Figure 1), the spaces
 $V_1(\Omega_1), V_2(\Omega_2)$ are given by

$$V_1(\Omega_1) := \{v \in H^1(\Omega_1), v = 0 \text{ on } \partial\Omega_1 \setminus \Sigma\}, \quad V_2(\Omega_2) := \{v_2 \in H^1(\Omega_2), v_2 = 0 \text{ on } \partial\Omega_2 \setminus \Sigma\},$$

and the space $H_{00}^{1/2}(\Sigma)$ is defined as follows

$$H_{00}^{1/2}(\Sigma) = \begin{cases} H^{1/2}(\Sigma) & \text{if } \partial\Sigma \cap \partial\Omega = \emptyset \text{ (inclusion),} \\ \{\lambda \in H^{1/2}(\Sigma), \tilde{\lambda} \in H^{1/2}(\partial\Omega_2)\} & \text{else.} \end{cases}$$

110 Above, in the definition of the space $H_{00}^{1/2}(\Sigma)$, $\tilde{\lambda}$ denotes the continuation of λ by 0 to $\partial\Omega_2$ (one can also
111 consider the continuation by 0 to $\partial\Omega_1$).

112 Since $\text{meas}_{\partial\Omega}(\partial\Omega_2 \setminus \Sigma) > 0$, all elements of $V_2(\Omega_2)$ fulfill a homogeneous boundary condition on a part of
113 the boundary $\partial\Omega_2$. On the other hand, one can check that if (u_1, u_2) is a solution to (2.1), then the function
114 u defined by $u|_{\Omega_j} = u_j$ for $j = 1, 2$ solves (1.1). The equations satisfied by u_1 and u_2 are elliptic but they
115 are coupled by the transmission conditions on Σ . As a consequence, we cannot solve them independently.
116 The purpose of the next paragraph is to explain how to proceed to write an alternative formulation (an
117 optimization-based one), which can be solved via an iterative procedure such that at each step one has to solve
118 a set of elliptic problems.

119 **3 The smooth extension method and optimal control reformulation of the problem** The
120 smooth extension method was proposed in [21] and can be considered as a special case of **fictitious domain**
121 **methods** (see [2]). It has been adapted to study the classical scalar transmission problem, i.e., with constant
122 sign coefficients, in [19]. In this section, we explain how to apply it to our problem.

123 **3.1 Presentation of the smooth extension method** The idea behind the smooth extension method
124 is the following: instead of looking for $(u_1, u_2) \in V_1(\Omega) \times V_2(\Omega_2)$ solution to (2.1), we search for a pair of
125 functions $(\tilde{u}, u_2) \in H_0^1(\Omega) \times V_2(\Omega_2)$ such that $(\tilde{u}|_{\Omega_1}, u_2)$ is a solution to (2.1).¹ The function \tilde{u} is then a
126 particular continuous extension of u_1 to the whole domain Ω . The difficulty is to find a "good" way to define
127 the function \tilde{u} so that it can be approximated by the classical FEM. The function u_2 can then be approximated
128 by solving the elliptic problem satisfied by u_2 in Ω_2 completed by $\tilde{u}|_{\Sigma}$ (resp. $\varepsilon_1 \partial_n \tilde{u}|_{\Sigma}$) as a Dirichlet (resp.
129 Neumann) boundary condition on Σ . Note that at first sight the construction of such \tilde{u} is not straightforward.
130 This will be achieved thanks to an optimal control reformulation of (2.1). This is the main goal of the next
131 paragraph in which we also reformulate the idea presented above in a more rigorous way.

132 **3.2 An optimal control reformulation of the problem** Before getting into details, let us first
133 introduce $\tilde{\varepsilon}_1 \in L^\infty(\Omega)$ such that $\tilde{\varepsilon}_1 \geq \tilde{\varepsilon}^+ > 0$ a. e. in Ω and $\tilde{\varepsilon}_1 = \varepsilon_1$ in Ω_1 . Then, let $E : V_1(\Omega_1) \rightarrow H_0^1(\Omega)$ be
134 an arbitrary continuous extension operator. By making use of (2.1), one can show easily that

$$135 \left\{ \begin{array}{l} \int_{\Omega} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_1} f_1 v \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v \rangle_{\Sigma} \quad \forall v \in H_0^1(\Omega), \\ \int_{\Omega_2} \varepsilon_2 \nabla u_2 \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Omega_2} f_2 v_2 \, d\mathbf{x} + \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma} \quad \forall v_2 \in V_2(\Omega_2). \end{array} \right.$$

136 Here and elsewhere, $\langle \cdot, \cdot \rangle_{\Sigma}$ denotes the duality product between $(H_{00}^{1/2}(\Sigma))^*$ and $H_{00}^{1/2}(\Sigma)$.

137 Now, given that the linear form $v_2 \mapsto \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma}$ is continuous on $V_2(\Omega_2)$ one can
138 define, thanks to the Riesz representation theorem, for each $E(u_1)$ a unique $w_{E(u_1)} \in V_2(\Omega_2)$ such that

$$139 (3.1) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_{E(u_1)} \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2).$$

140 Above we have used the fact that $(u, v) \mapsto (\tilde{\varepsilon}_1 \nabla u, \nabla v)_{L^2(\Omega)^d}$ is an inner product on $V_2(\Omega_2)$. As a consequence,
141 we have

$$142 \left\{ \begin{array}{l} \int_{\Omega} \tilde{\varepsilon}_1 \nabla E(u_1) \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_1} f_1 v \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_{E(u_1)} \cdot \nabla v \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega), \\ \int_{\Omega_2} \varepsilon_2 \nabla u_2 \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Omega_2} f_2 v_2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla (E(u_1) - w_{E(u_1)}) \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2). \end{array} \right.$$

Since the coefficients $\tilde{\varepsilon}_1$ and ε_2 have fixed signs, the forms

$$(u, v) \mapsto \int_{\Omega} \tilde{\varepsilon}_1 \nabla u \cdot \nabla v \, d\mathbf{x} \quad \text{and} \quad (u_2, v_2) \mapsto - \int_{\Omega_2} \varepsilon_2 \nabla u_2 \cdot \nabla v_2 \, d\mathbf{x},$$

¹In the text below, we choose an extension from Ω_1 to $\Omega^* = \Omega_2$. Obviously, one could choose an extension from Ω_2 to $\Omega^* = \Omega_1$ so that $(u_1, \tilde{u}|_{\Omega_2})$ is a solution to (2.1). In this case, the condition $\text{meas}_{\partial\Omega}(\partial\Omega_1 \setminus \Sigma) > 0$ must hold.

143 are coercive, respectively on $H_0^1(\Omega)$ and on $V_2(\Omega_2)$. With this in mind, we define for all $w \in V_2(\Omega_2)$, the
 144 couple of functions $(u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$ that are solution to the well-posed system of equations:

$$145 \quad (3.2) \quad \begin{cases} \int_{\Omega} \tilde{\varepsilon}_1 \nabla u^w \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_1} f_1 v \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla v \, d\mathbf{x} & \forall v \in H_0^1(\Omega), \\ \int_{\Omega_2} \varepsilon_2 \nabla u_2^w \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Omega_2} f_2 v_2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla(u^w - w) \cdot \nabla v_2 \, d\mathbf{x} & \forall v_2 \in V_2(\Omega_2). \end{cases}$$

146 Well-posedness is achieved by solving the elliptic problem in u^w first, and then the elliptic problem in u_2^w .

147 *Remark 3.1.* Observe that if we choose $w = w_{E(u_1)}$, then it follows that $(u^w|_{\Omega_1}, u_2^w)$ is the solution to (2.1).
 148 Indeed, one finds first that $u^w = E(u_1)$, and then that $u_2^w = u_2$.

149 Given any auxiliary "control" function w , the solutions to (3.2) enjoy the properties listed below.

150 PROPOSITION 3.2. For all $w \in V_2(\Omega_2)$, the functions $u_1^w := u^w|_{\Omega_1}$ and u_2^w are such that

$$151 \quad \begin{cases} -\operatorname{div}(\varepsilon_1 \nabla u_1^w) = f_1 & \text{in } \Omega_1, \\ -\operatorname{div}(\varepsilon_2 \nabla u_2^w) = f_2 & \text{in } \Omega_2 \\ \varepsilon_1 \partial_n u_1^w = \varepsilon_2 \partial_n u_2^w & \text{on } \Sigma. \end{cases}$$

152 *Remark 3.3.* In other words, the introduction of an auxiliary "control" function w allows us to construct
 153 pseudo-solutions to the equation (2.1) for which the condition on the normal derivatives is automatically
 154 satisfied. However we do not have in general continuity across the interface.

155 *Proof.* Take $\varphi_1 \in \mathcal{C}_0^\infty(\Omega_1)$ and extend it by 0 to the whole Ω to obtain the function $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Take $v = \varphi$
 156 in the problem satisfied by u^w . One finds that $-\operatorname{div}(\varepsilon_1 \nabla u_1^w) = f_1$ in Ω_1 . Next, take some $\varphi_2 \in \mathcal{C}_0^\infty(\Omega_2)$,
 157 extend it by 0 in Ω_1 and denote by φ the new function. By taking $v = \varphi$ in the problem satisfied by u^w and
 158 $v_2 = \varphi_2$ in the problem satisfied by u_2^w one finds that

$$159 \quad \begin{aligned} \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla u^w \cdot \nabla \varphi_2 \, d\mathbf{x} &= \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla \varphi_2 \, d\mathbf{x}, \\ \int_{\Omega_2} \varepsilon_2 \nabla u_2^w \cdot \nabla \varphi_2 \, d\mathbf{x} &= \int_{\Omega_2} f_2 \varphi_2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla(u^w - w) \cdot \nabla \varphi_2 \, d\mathbf{x}. \end{aligned}$$

160 By considering the sum of the two formulations, we conclude that $-\operatorname{div}(\varepsilon_2 \nabla u_2^w) = f_2$ in Ω_2 . To end the proof,
 161 it remains to show that $\varepsilon_1 \partial_n u_1^w = \varepsilon_2 \partial_n u_2^w$. For this, let $v \in H_0^1(\Omega)$ and define $v_2 = v|_{\Omega_2} \in V_2(\Omega_2)$. By taking
 162 v and v_2 as test functions in (3.2), considering the sum of the two equations, integrating by parts in both
 163 formulations and then, using the equations satisfied by u_1^w and u_2^w , we infer that

$$164 \quad -\langle \varepsilon_1 \partial_n u_1^w, v \rangle_{\Sigma} = -\langle \varepsilon_2 \partial_n u_2^w, v \rangle_{\Sigma}, \quad v \in H_0^1(\Omega).$$

165 According to the surjectivity of the trace mapping on Σ , this gives $\varepsilon_1 \partial_n u_1^w = \varepsilon_2 \partial_n u_2^w$ on Σ . \square

166 It follows that

167 LEMMA 3.4. If there exists $w^* \in V_2(\Omega_2)$ such that the solution to (3.2) satisfies $u^{w^*} = u_2^{w^*}$ on Σ , then
 168 $(u^{w^*}|_{\Omega_1}, u_2^{w^*})$ solves (2.1).

169 Thanks to what we have explained in Remark 3.1, we know that to every continuous extension of u_1 to Ω , one
 170 can define $w^* \in V_2(\Omega_2)$ for which $u^{w^*} = u_2^{w^*}$ on Σ . This leads us to the following result.

171 LEMMA 3.5. Let u_1 be the first part of the solution to (2.1). Then, the set of $w^* \in V_2(\Omega_2)$ such that the
 172 solution to (3.2) satisfies $u^{w^*} = u_2^{w^*}$ on Σ is isomorphic to the set of all possible continuous extensions of u_1
 173 to Ω . Furthermore, w^* and u^{w^*} are linked by relation

$$174 \quad (3.3) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla u^{w^*} \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w^* \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2).$$

175 Now, we have all the tools to introduce the optimal control reformulation of the problem (2.1). As a matter
 176 of fact, in order to find a function $w^* \in V_2(\Omega_2)$ for which $u^{w^*} = u_2^{w^*}$ on Σ , it is enough to solve the following
 177 optimal control problem:

$$178 \quad (3.4) \quad \text{Find } w^* = \operatorname{argmin}_{w \in V_2(\Omega_2)} J(w) \quad \text{with} \quad J(w) = \frac{1}{2} \int_{\Sigma} |u^w - u_2^w|^2 \, d\sigma,$$

179 where $(u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$ is the solution to (3.2). Note that in (3.4), the functional J plays the role
 180 of the cost functional, while (3.2) plays the role of the state equation. Obviously, thanks to Lemma 3.5, the
 181 problem (3.4) has an infinite number of solutions. As a result, one may need to use a regularization technique
 182 in order to be able to construct a convergent discretization of the problem: this will be the subject of §4.3
 183 where we will study the classical Tikhonov regularization method applied to Problem (3.4).

184 **4 Basic properties of the optimization problem and its regularization** In this section, we
 185 present in §4.1 some useful properties of the cost functional J and of the set of its minimizers in §4.2. After
 186 that in §4.3, we study the Tikhonov regularization of the problem. Furthermore, we explain, in §4.4, how to
 187 use the the adjoint approach in order to find an explicit expression of the gradient of J .

188 **4.1 Properties of the cost functional** Since we have used the $L^2(\Sigma)$ norm instead of the $H^{1/2}(\Sigma)$
 189 norm in the definition of J , one has the following results.

190 PROPOSITION 4.1. *The cost functional J satisfies the following properties:*

- 191 1. *Let $(w_n)_n$ be a sequence of elements of $V_2(\Omega_2)$ that converges weakly to $w_0 \in V_2(\Omega_2)$. Then, $(J(w_n))_n$
 192 converges to $J(w_0)$.*
- 193 2. *The functional J is continuous and convex on $V_2(\Omega_2)$.*

194 *Proof.* 1. For all $n \in \mathbb{N}$, denote by $(u^n, u_2^n) \in H_0^1(\Omega) \times V_2(\Omega)$ the solution to (3.2) with $w = w_n$. From the
 195 ellipticity of the problems involved in (3.2), it follows that $(u^n)_n$ (resp. $(u_2^n)_n$) converges weakly in $H_0^1(\Omega)$
 196 (resp. $V_2(\Omega_2)$) to some $u \in H^1(\Omega)$ (resp. $u_2 \in V_2(\Omega_2)$) such that (u, u_2) is the solution to (3.2) with $w = w_0$.
 197 The continuity of the trace operator from $H^1(\Omega)$ to $H^{1/2}(\Sigma)$ ensures that $(u_{|\Sigma}^n - u_{2|\Sigma}^n)_n$ converges weakly to
 198 $u_{|\Sigma} - u_{2|\Sigma}$ in $H^{1/2}(\Sigma)$. Given that the embedding of $H^{1/2}(\Sigma)$ into $L^2(\Sigma)$ is compact, it actually converges
 199 strongly to $u_{|\Sigma} - u_{2|\Sigma}$ in $L^2(\Sigma)$. Thus $(J(w_n))$ converges to $J(w_0)$. The result is proved.

200 2. While the continuity is a direct consequence of the first item, the convexity follows from the fact that $J :$
 201 $V_2(\Omega_2) \rightarrow \mathbb{R}$ is the composition of the affine map $j_1 : V_2(\Omega_2) \rightarrow L^2(\Sigma)$ and of the convex map $j_2 : L^2(\Sigma) \rightarrow \mathbb{R}$
 202 such that for all $w \in V_2(\Omega_2)$, $g \in L^2(\Sigma)$ we have

$$203 \quad (4.1) \quad \begin{cases} j_1(w) = (u^w - u_2^w)_{|\Sigma} \text{ where } (u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2) \text{ is the solution to (3.2),} \\ j_2(g) = \frac{1}{2} \int_{\Sigma} |g|^2 \, d\sigma. \end{cases} \quad \square$$

204 **4.2 The set of minimizers of the functional J** Thanks to Lemma 3.5, we know that J has an
 205 infinite number of minimizers. This (non-empty) set will be denoted by M_J . Without any difficulty, one can
 206 see that M_J coincides with the set of zeros of the functional J . As a result, since J is continuous, convex and
 207 positive, the set M_J is closed and convex in $V_2(\Omega_2)$. This allows us to say that the following minimization
 208 problem:

$$209 \quad \min_{w \in M_J} \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w|^2 \, d\mathbf{x}$$

210 has a unique solution, as a consequence of the strict convexity of $v_2 \mapsto \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla v_2|^2 \, d\mathbf{x}$ in $V_2(\Omega_2)$, and of
 211 the fact that M_J is a closed, convex subset of $V_2(\Omega_2)$. In the following, we shall denote by w_J^* the smallest
 212 minimizer of the functional J :

$$213 \quad (4.2) \quad w_J^* = \operatorname{argmin}_{w \in M_J} \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w|^2 \, d\mathbf{x}.$$

214 The goal of the rest of this paragraph is to find a characterization of $E_{w_J^*}(u_1)$, the continuous extension of u_1
 215 that is associated with w_J^* . Note that the link between $E_{w_J^*}(u_1)$ and w_J^* is given by the following (see relation
 216 (3.3)):

$$217 \quad (4.3) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E_{w_J^*}(u_1) \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_{\Sigma} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_J^* \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2).$$

218 To proceed, we define $E_H(u_1) \in H_0^1(\Omega)$ the continuous extension of u_1 that satisfies

$$219 \quad (4.4) \quad \operatorname{div}(\tilde{\varepsilon}_1 \nabla E_H(u_1)) = 0 \text{ in } \Omega_2.$$

220 In particular, we have

$$221 \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E_H(u_1) \cdot \nabla v_2 \, d\mathbf{x} = 0 \quad \forall v_2 \in H_0^1(\Omega_2).$$

222 Denote by $w_H \in M_J$ the minimizer associated with $E_H(u_1)$. Thanks to (3.3), we know that

$$223 \quad (4.5) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla E_H(u_1) \cdot \nabla v_2 \, d\mathbf{x} - \langle \varepsilon_1 \partial_n u_1, v_2 \rangle_\Sigma = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_H \cdot \nabla v_2 \, d\mathbf{x} \quad \forall v_2 \in V_2(\Omega_2).$$

224 We infer that

$$225 \quad (4.6) \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w_H \cdot \nabla v_2 \, d\mathbf{x} = 0 \quad \forall v_2 \in H_0^1(\Omega_2).$$

226 By taking the difference between (4.3) and (4.5), taking $v_2 = w_H$, using the fact that $E_H(u_1) - E_{w_j^*}(u_1) \in$
227 $H_0^1(\Omega_2)$ and owing to (4.6), we infer that

$$228 \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla (w_H - w_j^*) \cdot \nabla w_H \, d\mathbf{x} = 0,$$

229 so

$$230 \quad \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w_j^*|^2 \, d\mathbf{x} = \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w_j^* - \nabla w_H|^2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w_H|^2 \, d\mathbf{x} \geq \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w_H|^2 \, d\mathbf{x}.$$

231 Hence, from the definition of w_j^* , we then obtain the following

232 **PROPOSITION 4.2.** *The functions w_H and w_j^* coincide.*

233 *Remark 4.3.* It is worth noting that, thanks to (4.5) and using the definition of $E_H(u_1)$, the function w_H
234 satisfies the problem:

$$235 \quad (4.7) \quad \operatorname{div}(\tilde{\varepsilon} \nabla w_H) = 0 \text{ in } \Omega_2 \text{ and } \tilde{\varepsilon}_1 \partial_n w_H|_\Sigma = \tilde{\varepsilon}_1 \partial_n E_H(u_1)|_\Sigma - \varepsilon_1 \partial_n u_1|_\Sigma.$$

236 Recall that \mathbf{n} is the unit normal vector to Σ oriented to the exterior of Ω_2 .

237 **4.3 Tikhonov regularization of the problem** Tikhonov regularization, which was originally intro-
238 duced in [25], is a classical method to regularize a convex optimization problem. Classically, this method is
239 used in the context of regularization of ill-posed inverse problems (see [20] and the references therein). In this
240 paragraph, we study the convergence of such regularization when it is applied to our problem. For $\delta > 0$, we
241 introduce the functional $J^\delta : V_2(\Omega_2) \rightarrow \mathbb{R}$ defined by

$$242 \quad J^\delta(w) = J(w) + \delta \int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w|^2 \, d\mathbf{x} \quad \forall w \in V_2(\Omega_2).$$

243 To simplify notation, we will denote by $\|\cdot\|_{\tilde{\varepsilon}_1} : V_2(\Omega_2) \rightarrow \mathbb{R}_+$ the norm that is defined as follows:

$$244 \quad \|w\|_{\tilde{\varepsilon}_1} := \left(\int_{\Omega_2} \tilde{\varepsilon}_1 |\nabla w|^2 \, d\mathbf{x} \right)^{1/2}, \quad \forall w \in V_2(\Omega_2).$$

245 Endowed with the associated scalar product $(\cdot, \cdot)_{\tilde{\varepsilon}_1}$, the space $V_2(\Omega_2)$ is a Hilbert space. Since J is convex
246 and $\delta > 0$, the functional J^δ is strictly convex and coercive. Therefore the minimization problem

$$247 \quad \min_{w \in V_2(\Omega_2)} J^\delta(w)$$

248 has a unique solution that we denote by w_δ^* . Our goal is to study the behaviour of w_δ^* as δ tends to zero. One
249 expects $(w_\delta^*)_\delta$ to converge to one of the solutions (3.4). If this is the case and because the problem (3.4) has an
250 infinite number of solutions, it will be interesting to characterize the particular solution towards which $(w_\delta^*)_\delta$
251 converges. Our findings are summarized in the following

252 **PROPOSITION 4.4.** *When $\delta \rightarrow 0$, the sequence $(w_\delta^*)_\delta$ converges towards w_j^* , the smallest minimizer of J .*

253 The proof of the previous result is quite classical. However, for the convenience of the reader, we will detail it
254 in Appendix A.

255 In conclusion, we can say that the Tikhonov regularization allows us to obtain a stabilized version of the
256 optimization problem (3.4). This will be used in order to introduce a stabilization of the discretization of the
257 problem (3.4), but in that case the stabilization parameter δ will be chosen as a function of the discretization
258 parameter. This will be detailed in §5.3. Note that the same idea was employed in [1].

259 **4.4 Gradient of the functional J** As indicated in the introduction, the main objective of this work
 260 is to propose a new numerical method for approximating the solution to (1.1). This method will be based
 261 on the numerical approximation of the solution to the optimization problem (3.4). In this section, we will
 262 explain how to obtain an explicit expression of $J'(w)$ the gradient of J at some $w \in V_2(\Omega)$. We recall that the
 263 functional J is differentiable, because it can be written as a composition of the two differentiable maps j_1 and
 264 j_2 , cf. (4.1). Since the functional J is scalar valued, its differential at $w \in V_2(\Omega_2)$ can be represented by its
 265 gradient $J'(w) \in V_2(\Omega_2)$:

$$266 \quad \text{For all } h \in V_2(\Omega_2), \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla J'(w) \cdot \nabla h \, d\mathbf{x} = \lim_{t \rightarrow 0} \frac{J(w + th) - J(w)}{t}.$$

267 To find an explicit expression of $J'(w)$, we use the adjoint approach [11]. Details about the application of this
 268 approach to our problem are given in Appendix B (see also [19]). Here, we present final result. To do so, we start
 269 by introducing the so-called adjoint equations. For all $w \in V_2(\Omega_2)$, recalling that $(u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$
 270 is the solution to (3.2), we introduce $(g^w, g_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$ such that

$$271 \quad (4.8) \quad \begin{cases} \int_{\Omega} \tilde{\varepsilon}_1 \nabla g^w \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla g_2^w \cdot \nabla v \, d\mathbf{x} - \int_{\Sigma} (u^w - u_2^w) v \, d\sigma & \forall v \in H_0^1(\Omega) \\ \int_{\Omega_2} \varepsilon_2 \nabla g_2^w \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Sigma} (u^w - u_2^w) v_2 \, d\sigma & \forall v_2 \in V_2(\Omega_2). \end{cases}$$

272 As observed before, the functions g^w, g_2^w are well-defined. In Appendix B, we prove the

273 LEMMA 4.5. *For all $w \in V_2(\Omega_2)$, there holds $J'(w) = g_2^w - g^w|_{\Omega_2}$, where (g^w, g_2^w) solve (4.8).*

274 We have the following optimality result

275 COROLLARY 4.6. *We have the equivalence*

$$276 \quad [w^* \in V_2(\Omega_2) \text{ is such that } J'(w^*) = 0] \iff w^* \in M_J.$$

277 *Proof.* Let us start with the proof of the direct implication. Suppose that there exists some $w^* \in V_2(\Omega_2)$ such
 278 that $g^{w^*}|_{\Omega_2} = g_2^{w^*}$. By taking the sum of the variational formulations of (4.8), we deduce that

$$279 \quad \int_{\Omega} \varepsilon \nabla g^{w^*} \cdot \nabla v \, d\mathbf{x} = 0 \quad \forall v \in H_0^1(\Omega).$$

280 This means $A_\varepsilon(g^{w^*}) = 0$ and then, thanks to Assumption 2.1, $g^{w^*} = 0$. This implies that $g_2^{w^*} = 0$ and then by
 281 using the second equation of (4.8), that $u^{w^*} = u_2^{w^*}$ on Σ . This shows that w^* is a minimizer of J . The reverse
 282 implication is a consequence of the fact that if $w^* \in M_J$ we have $J(w^*) = 0$ and then $u^{w^*} = u_2^{w^*}$ on Σ . This
 283 implies that $g_2^{w^*} = 0$ and that $g^{w^*} = 0$. \square

284 We end this paragraph with the following result that can be useful to prove the convergence of the classical
 285 gradient descent algorithm.

286 COROLLARY 4.7. *The functional $J' : V_2(\Omega_2) \rightarrow V_2(\Omega_2)$ is Lipschitz continuous.*

287 *Proof.* Starting from (3.2), we deduce that $w \mapsto u^w, w \mapsto u_2^w$ are Lipschitz continuous. Inserting this into
 288 (4.8), we obtain the result. \square

289 **5 Numerical discretization of the problem** In this part, we are concerned with the numerical
 290 approximation of (3.4) by means of the Finite Element Method. To do so, we start by presenting some details
 291 and notations about the family of meshes that will be used. To simplify the presentation, the domain Ω and
 292 the subdomains $(\Omega_i)_{i=1,2}$ are supposed to have polygonal (when $d = 2$) or polyhedral (resp. $d = 3$) boundaries.

293 **5.1 Meshes and discrete spaces** Let $(\mathcal{T}_h)_h$ be a regular family of meshes of $\overline{\Omega}$ (see [15]), composed
 294 of (closed) simplices. The subscript h stands for the meshsize.

295 *Assumption 5.1.* We suppose that for all h , every simplex of \mathcal{T}_h belongs either to $\overline{\Omega_1}$ or to $\overline{\Omega_2}$.

296 According to Assumption 5.1, for $i = 1, 2$, one can consider the family of meshes $(\mathcal{T}_h^i)_h$ made of those simplices
 297 that belong to $\overline{\Omega_i}$.

298 For all $k \in \mathbb{N}^*$, we set

$$299 \quad V_h^k(\Omega) := \{v_h \in H_0^1(\Omega) \mid v_h|_T \in P^k(T), \forall T \in \mathcal{T}_h\}.$$

300 Here $P^k(T)$ stands for the space of polynomials (of d variables) defined on T of degree at most equal to k . In
 301 the same way, we define for $i = 1, 2$,

$$302 \quad V_h^k(\Omega_i) := \{v_{i,h} \in V_i(\Omega_i) \mid v_{i,h}|_T \in P^k(T), \forall T \in \mathcal{T}_h^i\}.$$

303

304 *Remark 5.2.* Note that for all $h > 0$, for $i = 1, 2$ the space $V_h^k(\Omega_i)$ coincides with $\{u|_{\Omega_i} \mid u \in V_h^k(\Omega)\}$.

305 Finally, we recall the basic approximability properties

$$306 \quad (5.1) \quad \left\{ \begin{array}{l} \forall v \in H_0^1(\Omega), \quad \lim_{h \rightarrow 0} \left(\inf_{v_h \in V_h^k(\Omega)} \|v - v_h\|_{H_0^1(\Omega)} \right) = 0, \\ \forall v_2 \in V_2(\Omega_2), \quad \lim_{h \rightarrow 0} \left(\inf_{v_{2,h} \in V_h^k(\Omega_2)} \|v_2 - v_{2,h}\|_{\tilde{\varepsilon}_1} \right) = 0. \end{array} \right.$$

307

308 **5.2 Discretization strategy** For $h > 0$ and $w \in V_2(\Omega)$, define the functions $u_h^w \in V_h^k(\Omega)$ and $u_{2,h}^w \in$
 309 $V_h^k(\Omega_2)$ as the solutions to the following well-posed discrete problems:

$$310 \quad (5.2) \quad \left\{ \begin{array}{l} \int_{\Omega} \tilde{\varepsilon}_1 \nabla u_h^w \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega_1} f v_h \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla v_h \, d\mathbf{x}, \quad \forall v_h \in V_h^k(\Omega) \\ \int_{\Omega_2} \varepsilon_2 \nabla u_{2,h}^w \cdot \nabla v_{2,h} \, d\mathbf{x} = \int_{\Omega_2} f_2 v_{2,h} \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla (u_h^w - w) \cdot \nabla v_{2,h} \, d\mathbf{x}, \quad \forall v_{2,h} \in V_h^k(\Omega_2). \end{array} \right.$$

311 Then introduce the projection operator $\pi_h^k : V_2(\Omega_2) \rightarrow V_h^k(\Omega_2)$ such that for all $w \in V_2(\Omega_2)$, $\pi_h^k w$ is defined
 312 as the unique element of $V_h^k(\Omega_2)$ that satisfies the problem

$$313 \quad \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla (\pi_h^k w) \cdot \nabla v_{2,h} \, d\mathbf{x} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla v_{2,h} \, d\mathbf{x} \quad \forall v_{2,h} \in V_h^k(\Omega_2).$$

314 Obviously, one has the estimate

$$315 \quad (5.3) \quad \|\pi_h^k w\|_{\tilde{\varepsilon}_1} \leq \|w\|_{\tilde{\varepsilon}_1}.$$

316 From the definition of $\pi_h^k w$, one can easily see that for all $w \in V_2(\Omega_2)$ we have the identities

$$317 \quad (5.4) \quad u_h^{\pi_h^k w} = u_h^w \quad \text{and} \quad u_{2,h}^{\pi_h^k w} = u_{2,h}^w.$$

318 Now, let us turn our attention to the discretization of the optimization problem (3.4). The natural way to do
 319 that is to replace it by the problem

$$320 \quad (5.5) \quad \inf_{w_h \in V_h^k(\Omega_2)} J_{0,h}(w_h) := \frac{1}{2} \int_{\Sigma} |u_h^{w_h} - u_{2,h}^{w_h}|^2 \, d\sigma.$$

321 One can proceed as in the proof of proposition 4.1 to show that the cost functional $J_{0,h} : V_h^k \rightarrow \mathbb{R}$ (defined in
 322 (5.5)) is convex and continuous. Unfortunately this result is not sufficient to justify that the problem (5.5) is
 323 well-posed. The difficulty comes from the fact that, even under Assumption 2.1, one can not guarantee that
 324 the problem

$$325 \quad \text{Find } u_h \in V_h^k(\Omega) \text{ such that } \int_{\Omega} \varepsilon \nabla u_h \cdot \nabla v_h \, d\mathbf{x} = \int_{\Omega} f v_h \, d\mathbf{x} \quad \forall v_h \in V_h^k(\Omega)$$

326 is well-posed even for h small enough. To cope with this difficulty, an idea is to use again the Tikhonov
 327 regularization (see §4.3), with a regularization parameter that depends now on h . This idea was originally
 328 proposed in [22] for the case of elliptic equations and then, was used by Assyr Abdulle et al. in [1] for the case
 329 of problems with sign-changing coefficients. Here, we explain how to adapt it to our approach. The idea is to
 330 replace the cost functional $J_{0,h}$ in (5.5) by the functional $J_h : V_h^k(\Omega_2) \rightarrow \mathbb{R}_+$ such that for all $w_h \in V_h^k(\Omega_2)$,
 331 we have

$$332 \quad J_h(w_h) := \frac{1}{2} \int_{\Sigma} |u_h^{w_h} - u_{2,h}^{w_h}|^2 \, d\sigma + \lambda_h \|w_h\|_{\tilde{\varepsilon}_1}^2,$$

333 where $\lambda_h > 0$ tends to zero as h goes to 0. Since $\lambda_h > 0$ for all $h > 0$, the functional J_h is strictly convex and
 334 coercive. This guarantees that the optimization problem

$$335 \quad (5.6) \quad \min_{w_h \in V_h^k(\Omega_2)} J_h(w_h)$$

336 has a unique solution that we denote by $w_{k,h}^*$. All the difficulty now is to choose the parameter λ_h in order to
 337 be able to ensure the convergence of $(w_{k,h}^*)_h$ towards a solution to (3.4) as h tends to zero. This is the main
 338 goal of the next paragraph.

339 **5.3 Convergence of the method** The starting point of our discussion is the following

340 LEMMA 5.3. *We have the estimate*

$$341 \quad (5.7) \quad J_h(w_{k,h}^*) \leq \frac{1}{2} \int_{\Sigma} |u_h^{w_J^*} - u_{2,h}^{w_J^*}|^2 \, d\sigma + \lambda_h \|w_J^*\|_{\tilde{\varepsilon}_1}^2$$

342 where w_J^* is defined by (4.2).

343 *Proof.* Starting from the fact that $\pi_h^k w_J^* \in V_h^k(\Omega_2)$ and using that $w_{k,h}^*$ is the unique solution to the optimiza-
 344 tion problem (5.6), we conclude that $J_h(w_{k,h}^*) \leq J_h(\pi_h^k w_J^*)$. On the other hand, the identity (5.4) allows us to
 345 write

$$346 \quad J_h(\pi_h^k w_J^*) = \frac{1}{2} \int_{\Sigma} |u_h^{w_J^*} - u_{2,h}^{w_J^*}|^2 \, d\sigma + \lambda_h \|\pi_h^k w_J^*\|_{\tilde{\varepsilon}_1}^2.$$

347 The Lemma is then proved by recalling the estimate (5.3). □

348 In order to simplify notations, for $h > 0$ and $w \in V_2(\Omega_2)$, we denote by $A_h(w)$ the real number

$$349 \quad A_h(w) = \frac{1}{2} \int_{\Sigma} |u_h^w - u_{2,h}^w|^2 \, d\sigma.$$

350 From (5.4), we know that for all $w \in V_2(\Omega_2)$, we have $A_h(w) = J_0^h(\pi_h^k w)$. The main result of this paragraph
 351 is the following theorem.

352 THEOREM 5.4. *Assume that the parameter λ_h can be chosen such that the sequences $(\lambda_h)_h$ and $(A_h(w_J^*)/\lambda_h)_h$
 353 tend to zero as h tends to zero. Then, as h goes to 0:*

- 354 • *the sequence $(w_{k,h}^*)_h$ converges to w_J^* in $V_2(\Omega_2)$;*
- 355 • *the sequence $(u_h^{w_{k,h}^*})_h$ converges to $E_H(u_1)$ in $H_0^1(\Omega)$, resp. the sequence $(u_{2,h}^{w_{k,h}^*})_h$ converges to u_2 in
 356 $V_2(\Omega_2)$, where (u_1, u_2) is the solution to (2.1) and $E_H(u_1)$ is the extension of u_1 defined in (4.4).*

357 *Proof.* The strategy of proof is similar to the one of proposition 4.4. To simplify notations, we denote by
 358 $u^{k,h} \in V_h^k(\Omega)$ and $u_2^{k,h} \in V_h^k(\Omega_2)$ the functions

$$359 \quad u^{k,h} = u_h^{w_{k,h}^*} \quad \text{and} \quad u_2^{k,h} = u_{2,h}^{w_{k,h}^*}.$$

360 In order to make the proof as clear as possible, we divide it into four steps.

361 **Step 1: weak convergence of $(w_{k,h}^*)_h$, $(u^{k,h})_h$ and $(u_2^{k,h})_h$.** Starting from the estimate

$$362 \quad (5.8) \quad \|w_{k,h}^*\|_{\tilde{\varepsilon}_1}^2 \leq J_h(w_{k,h}^*)/\lambda_h \leq A_h(w_J^*)/\lambda_h + \|w_J^*\|_{\tilde{\varepsilon}_1}^2$$

363 and using the fact that $(A_h(w_J^*)/\lambda_h)_h$ tends to 0 as h goes to 0, we infer that $(w_{k,h}^*)_h$ is bounded in $V_2(\Omega_2)$.
 364 This implies that, up to a sub-sequence, $(w_{k,h}^*)_h$ converges weakly to some $w_0 \in V_2(\Omega)$. For the reader's
 365 convenience, this sub-sequence is still denoted by $(w_{k,h}^*)_h$.

366 Since the problems in (5.2) are uniformly elliptic with respect to h , we know that the sequence $(u^{k,h})_h$ (resp.
 367 $(u_2^{k,h})_h$) converges weakly in $H_0^1(\Omega)$ (resp. in $V_2(\Omega_2)$) to some $u \in H_0^1(\Omega)$ (resp. $u_2 \in V_2(\Omega_2)$). Using the basic
 368 approximability property (5.1), we infer that $u = u^{w_0}$ and $u_2 = u_2^{w_0}$.

369 **Step 2: w_0 is a minimizer of J .** The continuity of the trace operator and the compactness of the embedding
 370 $H^{1/2}(\Sigma) \subset L^2(\Sigma)$ ensure that

$$371 \quad u^{k,h}|_{\Sigma} - u_2^{k,h}|_{\Sigma} \rightarrow u^{w_0}|_{\Sigma} - u_2^{w_0}|_{\Sigma}$$

372 in $L^2(\Sigma)$ as $h \rightarrow 0$. By noticing that

$$373 \quad \frac{1}{2} \int_{\Sigma} |u^{k,h} - u_2^{k,h}|^2 d\sigma = J_0^h(w_{k,h}^*) \leq J_h(w_{k,h}^*) \leq \lambda_h(A_h(w_J^*)/\lambda_h + \|w_J^*\|_{\tilde{\varepsilon}_1}^2)$$

374 and using that $\lambda_h, A_h(w_J^*)/\lambda_h \rightarrow 0$ as h goes to zero, we deduce that $u^{w_0}|_{\Sigma} - u_2^{w_0}|_{\Sigma} = 0$. This shows that w_0
375 is a minimizer of J .

376 **Step 3: strong convergence of $(w_{k,h}^*)_h$ to w_J^* .** Thanks to the fact that $A_h(w_J^*)/\lambda_h \rightarrow 0$ as $h \rightarrow 0$ and by
377 means of the estimate (5.8), we can write

$$378 \quad \limsup_{h \rightarrow 0} \|w_{k,h}^*\|_{\tilde{\varepsilon}_1} \leq \|w_J^*\|_{\tilde{\varepsilon}_1}.$$

379 On the other hand, since $(w_{k,h}^*)_h$ converges weakly to w_0 as $h \rightarrow 0$, we infer that

$$380 \quad \|w_0\|_{\tilde{\varepsilon}_1} \leq \liminf_{h \rightarrow 0} \|w_{k,h}^*\|_{\tilde{\varepsilon}_1},$$

381 which is a consequence of the fact that the norm of a Banach space is weakly lower semicontinuous, see [9,
382 Proposition III.5 (iii)]. This implies that $\|w_0\|_{\tilde{\varepsilon}_1} \leq \|w_J^*\|_{\tilde{\varepsilon}_1}$. Since w_0 is a minimizer of J , we conclude that
383 $w_0 = w_J^*$. Furthermore, we also deduce that

$$384 \quad \lim_{h \rightarrow 0} \|w_{k,h}^*\|_{\tilde{\varepsilon}_1} = \|w_0\|_{\tilde{\varepsilon}_1}.$$

385 As a result, by applying [9, Proposition III.32], we infer that $(w_{k,h}^*)_h$ converges, strongly, in $V_2(\Omega_2)$ to $w_0 = w_J^*$.

386 **Step 4: strong convergence of $(u^{k,h})_h$ and $(u_2^{k,h})_h$.** The ellipticity of the problems in (3.2), combined
387 with the strong convergence of $(w_{k,h}^*)_h$ to w_J^* , imply the convergence of $(u^{k,h})_h$ in $H_0^1(\Omega)$ to $u^{w_J^*}$ and of $(u_2^{k,h})_h$
388 in $V_2(\Omega_2)$ to $u_2^{w_J^*}$.

389 The result is then proved by using that $u^{w_J^*} = E_H(u_1)$ and by observing that these limits are independent of
390 the chosen sub-sequences. \square

391 The rest of this paragraph is devoted to explain why it is possible to choose the parameter λ_h in such a way
392 that $(\lambda_h)_h$ and $(A_h(w_J^*)/\lambda_h)_h$ both converge to 0 as h tends to 0. To do so, one needs to study the behaviour
393 of $A_h(w_J^*)$ as h tends to 0. We recall that, according to theorem 2.2, we know that $u \in \cap_{s \in [0, \sigma_D(\varepsilon)]} PH^{1+s}(\Omega)$,
394 where $\sigma_D(\varepsilon) \in (0, 1]$ is the so-called regularity exponent. Let us start with the following

395 **PROPOSITION 5.5.** *Suppose that the coefficients $\tilde{\varepsilon}_1$ and ε_2 are smooth, or piecewise smooth. Assume that the*
396 *solution u to (1.1) belongs to $PH^{1+s}(\Omega)$ for some $s > 0$. Then there exists $s' \in (0, s]$ that depends only on the*
397 *geometry of Ω_2 and on the coefficient ε_2 , and there exists $\sigma \in (0, 1]$ that depends only on the geometry of Ω*
398 *and of Ω_2 such that*

$$399 \quad \begin{aligned} \|u^{w_J^*} - u_h^{w_J^*}\|_{H_0^1(\Omega)} &\leq Ch^{p'} \|u\|_{PH^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_J^*} - u_{2,h}^{w_J^*}\|_{\tilde{\varepsilon}_1} \leq Ch^{p'} \|u_2\|_{H^{1+p'}(\Omega_2)}, \\ \|u^{w_J^*} - u_h^{w_J^*}\|_{L^2(\Omega)} &\leq Ch^{p'+\sigma} \|u\|_{PH^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_J^*} - u_{2,h}^{w_J^*}\|_{L^2(\Omega_2)} \leq Ch^{p'+\sigma} \|u_2\|_{H^{1+p'}(\Omega_2)} \end{aligned}$$

400 with C independent of h and $p' = \min(s', k)$.

401 *Proof.* Along this proof, C denotes a positive constant whose value can change from one line to the next but
402 does not depend on h .

403 Given that $u^{w_J^*} = E_H(u_1)$ solves (4.4), and since $u_1 \in H^{1+s}(\Omega_1)$, it follows that $E_H(u_1)|_{\Omega_2}$ exhibits some
404 extra-regularity because ε_2 is (piecewise) smooth (via a classical shift theorem). In other words, there exists
405 $s' \in (0, s]$ such that $u^{w_J^*} \in PH^{1+s'}(\Omega)$.

406 Given that $u_2^{w_J^*} = u_2 \in PH^{1+s}(\Omega_2) \subset PH^{1+s'}(\Omega_2)$ and since the problems in (3.2) are elliptic with (piecewise)
407 smooth coefficients $\tilde{\varepsilon}_1$ and ε_2 , we obtain the estimates (see [15])

$$408 \quad \|u^{w_J^*} - u_h^{w_J^*}\|_{H_0^1(\Omega)} \leq Ch^{p'} \|u\|_{PH^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_J^*} - u_{2,h}^{w_J^*}\|_{\tilde{\varepsilon}_1} \leq Ch^{p'} \|u_2\|_{H^{1+p'}(\Omega_2)},$$

409 where $p' = \min(s', k)$. By applying the classical Aubin-Nitsche's lemma (see [15, Theorem 3.2.4]), we infer
410 that there exists $\sigma \in (0, 1]$ such that

$$411 \quad \|u^{w_J^*} - u_h^{w_J^*}\|_{L^2(\Omega)} \leq Ch^{p'+\sigma} \|u\|_{PH^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_J^*} - u_{2,h}^{w_J^*}\|_{L^2(\Omega_2)} \leq Ch^{p'+\sigma} \|u_2\|_{H^{1+p'}(\Omega_2)}. \quad \square$$

412 *Remark 5.6.* It is worth to note that the value of s' is prescribed by the regularity of $E_H(u_1)$, the harmonic-like
413 extension u_1 that satisfies (4.4). More precisely, s' depends both on the regularity exponent $\sigma_D(\tilde{\varepsilon}_1) \in (0, 1]$,
414 and on the regularity of the boundary data on $\partial\Omega_2$, because $u_{2H} := E_H(u_1)|_{\Omega_2} \in V_2(\Omega_2)$ solves the Dirichlet
415 problem (cf. (4.4)): $-\operatorname{div}(\tilde{\varepsilon}_1 \nabla u_{2H}) = 0$ in Ω_2 , with $u_{2H} = u_1$ in $H_{00}^{1/2}(\Sigma)$. Assume for instance that Ω and
416 Ω_2 are convex domains, and that the coefficients ε_1 and ε_2 are constant. In this case, one can choose $\tilde{\varepsilon}_1$
417 to be constant over Ω . We recall that the solution u to (1.1) belongs to $\text{PH}^{1+s}(\Omega)$ for all $s \in (0, \sigma_D(\varepsilon))$.
418 Then, because $u_{2H} \in H^1(\Omega_2)$ is governed by: $-\Delta u_{2H} = 0$ in the convex domain Ω_2 with Dirichlet data in
419 $H^{1/2+s}(\partial\Omega_2)$, one has $u_{2H} \in H^{1+s}(\Omega_2)$. In other words, $s' = s$, and $p' = \min(s', k) = s' = s$. Finally, because
420 Ω and Ω_2 are convex, one finds that $\sigma = 1$.

421 Now we have all the tools to study the behavior $A_h(w_J^*)$ as h goes to 0.

422 **COROLLARY 5.7.** *Under the same assumptions as in proposition 5.5, one has*

$$423 \quad A_h(w_J^*) \leq Ch^{2p'+\sigma}$$

424 *with C independent of h and $p' = \min(s', k)$.*

425 *Proof.* Applying the multiplicative trace inequality (recalled in proposition A.1) and using the estimates of
426 proposition 5.5 yield the estimates

$$427 \quad \|u^{w_J^*} - u_h^{w_J^*}\|_{L^2(\Sigma)}^2 \leq Ch^{2p'+\sigma} \|u\|_{\text{PH}^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_J^*} - u_{2,h}^{w_J^*}\|_{L^2(\Sigma)}^2 \leq Ch^{2p'+\sigma} \|u_2\|_{H^{1+p'}(\Omega_2)}.$$

428 By design, one has $u^{w_J^*}|_{\Sigma} = u_2^{w_J^*}|_{\Sigma}$. So, observing that

$$429 \quad \|u_h^{w_J^*} - u_{2,h}^{w_J^*}\|_{L^2(\Sigma)}^2 \leq 2(\|u^{w_J^*} - u_h^{w_J^*}\|_{L^2(\Sigma)}^2 + \|u_2^{w_J^*} - u_{2,h}^{w_J^*}\|_{L^2(\Sigma)}^2),$$

430 we conclude that $A_h(w_J^*) \leq Ch^{2p'+\sigma}$. □

431 The previous result gives us a simple way to choose the parameter λ_h in order to ensure that both $(\lambda_h)_h$ and
432 $(A_h(w_J^*)/\lambda_h)_h$ tend to 0 as h tends to 0.

433 **COROLLARY 5.8.** *Under the same assumptions as in proposition 5.5, any parameter λ_h of the form $\lambda_h = Ch^q$
434 with $C > 0$ independent of h and $q \in (0, 2p' + \sigma)$ satisfies the conditions of theorem 5.4.*

435 *Remark 5.9.* Within the framework of remark 5.6, one may choose $q \in (0, 2\sigma_D(\varepsilon) + 1)$ in the statement of
436 corollary 5.8.

437 Thanks to theorem 5.4, using the conditions of corollary 5.8, one obtains the convergence of the discrete
438 solutions to the exact solution.

439 On the one hand, convergence is guaranteed even on meshes that are not T-conforming. Compared to [1],
440 convergence holds in very general situations, namely as soon as there is a shift theorem for problem (1.1), cf.
441 theorem 2.2, even with a regularity exponent $\sigma_D(\varepsilon) < 1/2$.

442 On the other hand, there is no associated convergence rate. Assuming a Céa lemma-like result, and using the
443 same notations as above, the *expected* convergence rate is $h^{p'}$ in H_0^1 -norm, and $h^{p'+\sigma}$ in L^2 -norm. Whereas,
444 classically, the *optimal* convergence rate is h^k in H_0^1 -norm, and h^{k+1} in L^2 -norm.

445 **6 Numerical experiments** In this section we turn our attention to the validation of the numerical
446 method that we have proposed. We limit ourselves to the case of 2D domains and use P^1 Lagrange finite
447 elements. The numerical results that we present below have been obtained with the help of the library
448 **FreeFem++**². Since the well-posedness of (1.1) depends on the shape of the interface Σ , we test the performance
449 of our method in three different configurations. In the first one, Σ is flat, in the second one, Σ is a circular
450 interface and in the last one, Σ has a "corner", in the sense that the angle at the intersection with the boundary
451 is not a right angle. In all these experiments, we suppose that the coefficients ε_1 and ε_2 are constant with
452 $\varepsilon_1 = 1$. We denote by κ_ε the contrast $\kappa_\varepsilon = \varepsilon_2/\varepsilon_1$.

453 The shape, smoothness and (respective) volumes of Ω_1 and Ω_2 are taken into account to choose the domain
454 $\Omega^* \in \{\Omega_1, \Omega_2\}$ to which the extension is performed (we recall that one must have $\text{meas}_{\partial\Omega}(\partial\Omega^* \setminus \Sigma) > 0$, see
455 footnote¹ on page 4). Indeed, to have a better convergence rate, one should choose Ω^* convex, or with as

²See <https://freefem.org/>.

456 smooth a boundary as possible. Also, in order to speed up the convergence of the optimization algorithm,
457 we must choose Ω^* as small as possible. Once Ω^* is fixed, one has to extend the function ε_1 or ε_2 to all
458 the domain Ω . Because the coefficients are constant, we extend ε_1 (resp. ε_2) by ε_1 in Ω_2 (resp. in Ω_1).
459 In the case where Σ is flat or circular, we take $\Omega^* = \Omega_2$. In the third configuration, we take $\Omega^* = \Omega_1$. To
460 solve the optimization problem, we will use two different algorithms that are available in `FreeFem++`. The
461 first one is the algorithm `BFGS` (Broyden-Fletcher-Goldfarb-Shanno) and the second one is the algorithm `NLCG`
462 (Nonlinear Conjugate Gradient). Compared to the `NLCG` algorithm which uses only the gradient of the cost
463 function to solve the optimization problem, the `BFGS` algorithm, which belongs to the class of quasi-Newton
464 methods, uses a particular approximation of the hessian of the objective function. As already mentioned in
465 the documentation of `FreeFem++` (see page 65 of the third version), when the unknown of the optimization
466 problem is a finite element function with a large size, it is preferable to work with the `NLCG` algorithm because
467 the `BFGS` algorithm can be very memory consuming and its convergence rate can be low. In our numerical
468 experiments, we observed that in general both algorithms work similarly for the case where the interface is flat
469 or circular, however for the case of the interface with corner, the algorithm `NLCG` performs better (the observed
470 convergence rate is better). Below we present the numerical results we obtained using the `BFGS` algorithm
471 for the case where the interface is flat or circular and the results we obtained using the `NLCG` algorithm for
472 the case where the interface has a corner. In our numerical experiments, the `BFGS` function was used with
473 the following parameters: `eps=1.e-11,nbiter=10,nbiterline=1` and the `NLCG` function with the following:
474 `nbiter=10,eps=1.e-11`.

475 **6.1 Flat interface** In this paragraph, we take

$$476 \quad \Omega_1 = \{(x, y) \in (0; 1/2) \times (0; 1)\} \quad \text{and} \quad \Omega_2 = \{(x, y) \in (1/2; 1) \times (0; 1)\}$$

477 (a flat interface and a domain which is symmetric with respect to Σ). We consider a family of meshes of $\bar{\Omega}$
478 satisfying Assumption 5.1 (see Figure 2). In the rest of this paragraph we suppose that $\kappa_\varepsilon \neq -1$. To test the
479 performance of our method, we work with the same example considered in [1, 14]. Define the function u_{κ_ε}
480 such that

$$481 \quad u_{\kappa_\varepsilon}(x, y) = \begin{cases} (x^2 + bx) \sin(\pi y) & \text{if } x < 1/2 \\ a(x - 1) \sin(\pi y) & \text{if } 1/2 < x \end{cases}, \quad \text{where } a = \frac{1}{2(\kappa_\varepsilon + 1)} \quad \text{and} \quad b = -\frac{\kappa_\varepsilon + 2}{2(\kappa_\varepsilon + 1)}.$$

482 and consider it as an exact solution to (1.1). This is possible because $\text{div}(\varepsilon \nabla u_{\kappa_\varepsilon}) \in L^2(\Omega)$. The source term
483 f is computed accordingly. As observed in remark 5.6, by choosing $\lambda_h = Ch^q$ with $q \in (0, 3)$, the method
484 is convergent. In our experiment, we take $\lambda_h = 0.002h^2$. We work with $\kappa_\varepsilon = -1.001$. The behaviors of the
485 relative L^2 -norm error ($\|e_h^r\|_0$) and the relative H_0^1 -norm error ($\|e_h^r\|_1$) between the exact solution and the
numerical one are reported in Figure 2. We observe that both rates of convergence are equal to 2.

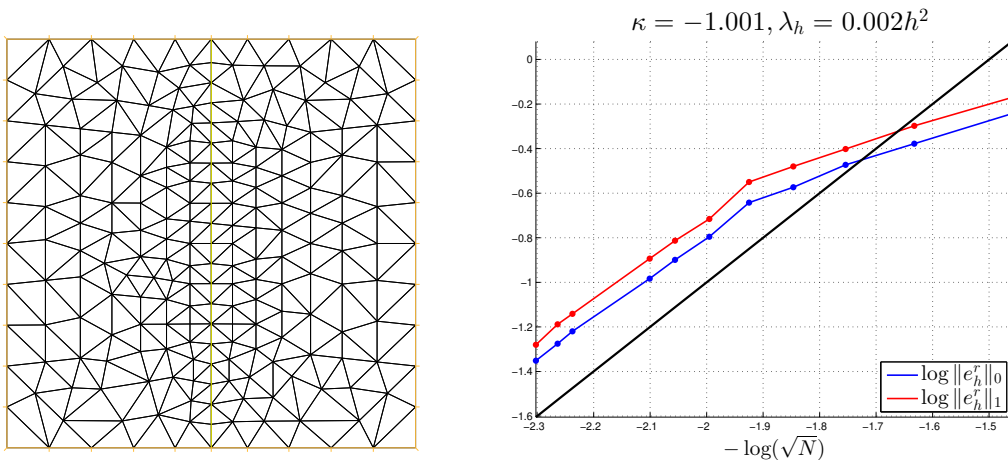


FIG. 2. A given mesh (left). Behavior of the relative L^2 and H_0^1 errors with respect to the meshsize $h \sim \sqrt{N}$, where N is the total number of nodes of the mesh (right).

486

487 *Remark 6.1.* The constant C in $\lambda_h = Ch^q$ must be adjusted by the user according to the contrast κ_ε in order
 488 to obtain a fast convergence of the method. Clearly this depends on $\|w_J^*\|_{\varepsilon_1}$. Using the fact that $w_J^* = w_H$
 489 and owing to (4.7) we see that this depends on the jump of the normal derivative (across Σ) between u_1 and
 490 its harmonic extension. It is also important to note that, once q is fixed and when h is small enough, the
 491 choice of C has little influence on the convergence of the method.

492 **6.2 The case of a circular interface** In this paragraph, we consider the case of a circular inclusion,
 493 precisely the domains Ω_1 and Ω_2 are such that $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\}$ and $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 < |\mathbf{x}| < 2\}$.
 494 In proposition A.2, we prove that A_ε is an isomorphism $\kappa_\varepsilon \notin \{-1\} \cup \mathcal{S}$ with $\mathcal{S} := \{-1 - (1/2)^{2n}/(1 +$
 495 $(1/2)^{2n}) \mid n \in \mathbb{N}^*\}$. For this reason, we consider the case where $\kappa_\varepsilon = -2 \notin \mathcal{S}$. Given that both Ω_2 and Ω have
 496 smooth boundaries, we infer that $\sigma = 1$ and $s' = s$. By taking f as the source term associated to the function

$$497 \quad u_{\kappa_\varepsilon}(x, y) = \begin{cases} r^2 + b & \text{if } r < 1 \\ a(r-2)^2 & \text{if } 1 < r < 2. \end{cases}, \text{ with } r = \sqrt{x^2 + y^2}, a = -1/\kappa_\varepsilon \text{ and } b = a - 1$$

498 and by taking $\lambda_h = 0.002h^2$, we obtain the results displayed in Figure 3. We observe that the method converges
 499 with optimal rate (i.e., the relative L^2 -norm error ($\|e_h^r\|_0$) is of order 2, while the relative H_0^1 -norm error is of
 500 order 1), even though the exterior boundary and the interface are curved.

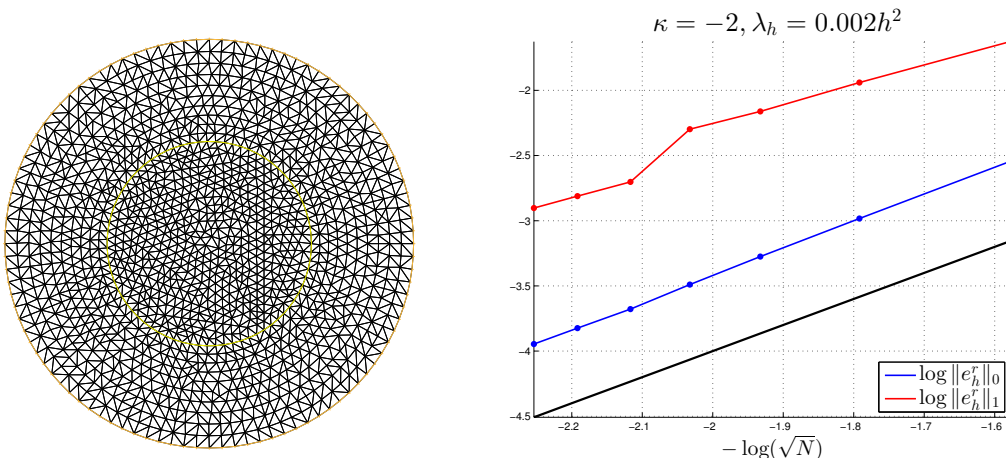


FIG. 3. A given mesh (left). Behavior of the relative L^2 and H_0^1 errors with respect to the meshsize $h \sim \sqrt{N}$, where N is the total number of nodes of the mesh (right)

501 **6.3 The case of an interface with corner** Now, we consider the configuration where the interface
 502 Σ has a corner. More precisely, we assume that $\Omega := \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1 \text{ and } \arg(\mathbf{x}) \in (0; \pi)\}$ and $\Omega_1 := \{\mathbf{x} \in$
 503 $\Omega \mid \arg(\mathbf{x}) \in (0; \pi/4)\}$ (see Figure 4). In such configuration, it can be proved (see [4]) that A_ε is an isomorphism
 504 if and only if $\kappa_\varepsilon \in \mathbb{R}_-^* \setminus [-3, -1]$. Furthermore, contrarily to the two previous cases, in this configuration the
 505 solution to (1.1) can be very singular near the origin. Indeed, it was proved in [12, Chapter 2] that the
 506 regularity of the solution to (1.1) depends in κ_ε and can be very low as κ_ε approaches $[-3, -1]$: more precisely,

$$507 \quad \lim_{\kappa_\varepsilon \rightarrow -3^-} \sigma_D(\varepsilon) = \lim_{\kappa_\varepsilon \rightarrow -1^+} \sigma_D(\varepsilon) = 0.$$

508 As a matter of fact, the value of the regularity exponent $\sigma_D(\varepsilon)$ is $\Re e(\lambda_0)$, where λ_0 is the solution to

$$509 \quad (6.1) \quad \kappa_\varepsilon = -\tan(3\lambda\pi/4)/\tan(\lambda\pi/4)$$

510 that has the smallest positive real part. Note that one can show (see [12, Chapter 3]) that all the solutions
 511 to (6.1) are real-valued. In the particular cases where $\kappa_\varepsilon = -5$ and $\kappa_\varepsilon = -3.1$, one finds, respectively that
 512 $\lambda_0 \approx 0.458$ and $\lambda_0 \approx 0.139$. As mentioned previously this regularity result is optimal. Indeed, one can check
 513 that the function

$$514 \quad u_{\lambda_0}(r, \theta) := (1-r)r^{\lambda_0} \begin{cases} \sin(\lambda_0\theta)/\sin(\lambda_0\pi/4) & \theta \in (0; \pi/4), \\ \sin(\lambda_0(\pi-\theta))/\sin(3\lambda_0\pi/4) & \theta \in (\pi/4; \pi) \end{cases}$$

515 satisfies $\text{div}(\varepsilon \nabla u_{\lambda_0}) \in L^2(\Omega)$. Observe that $u_{\lambda_0} \notin \text{PH}^{\lambda_0}(\Omega)$. This means that $u_{\lambda_0} \notin \text{PH}^{3/2}(\Omega)$. Now, given
516 that Ω and Ω_2 are both convex, owing to proposition 5.4, we can say that by choosing $\lambda_h = Ch^q$ with
517 $q \in (0, 1 + 2\lambda_0)$, the convergence of the method can be guaranteed. In the case $\kappa_\varepsilon = -5$ (resp. $\kappa_\varepsilon = -3.1$),
518 we work with $\lambda_h = 6h^{1.8}$ (resp. $\lambda_h = 1.5h^{1.2}$).
519 The behaviors of the relative L^2 -norm error and the relative H_0^1 -norm error (for the cases $\kappa_\varepsilon = -5$ and
520 $\kappa_\varepsilon = -3.1$) are given in Figure 4. In either case, the expected rate of convergence is equal to λ_0 (≈ 0.458 when
521 $\kappa_\varepsilon = -5$ and ≈ 0.139 when $\kappa_\varepsilon = -3.1$) for the case of the H_0^1 -norm error, while it is equal to $2\lambda_0$ for the case
522 of the L^2 -norm error. We observe that, in both cases, the method converges with **optimal rate of convergence**
523 **for the H^1 -norm and the L^2 one.**

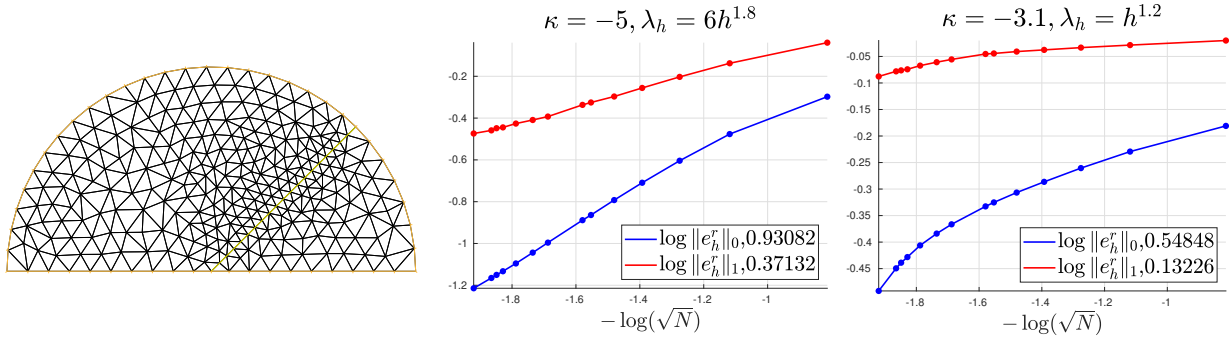


FIG. 4. A given mesh (left). Behavior of the relative L^2 and H_0^1 errors with respect to the meshsize $h \sim \sqrt{N}$, where N is the total number of nodes of the mesh, with the observed convergence rates, when $\kappa_\varepsilon = -5$ (center) and $\kappa_\varepsilon = -3.1$ (right).

524 **7 Concluding remarks** In this work, we have presented a new numerical method to approximate
525 the solution to the scalar transmission problem with sign-changing coefficients. We proved that the method
526 converges without any restriction on the mesh sequence, nor on the regularity of the solution. This result
527 has been illustrated by several 2D numerical experiments. The convergence rate of our method seems to be
528 optimal. In order to improve the performance of the method, several questions can be studied:

- 529 1. Choose the parameter λ_h in order to accelerate convergence. An interesting idea would be to find
530 an adaptive approach to fit its value. Also, one could use adaptive mesh refinement, together with a
531 posteriori estimates. We refer to [17] for estimators that deliver guaranteed error bounds, and that
532 are robust with respect to the sign-changing coefficient ε .
- 533 2. Work with other regularization approaches, i.e., other choices for the coefficient $\tilde{\varepsilon}_1$, and/or an alter-
534 native to the Tikhonov regularization method.
- 535 3. In the case where the interface is not regular, it would be interesting to combine our approach with
536 other existing methods for solving PDE with singular solution such as the Singular Complement
537 Method [16].

538 Besides that, it will be also interesting to extend this approach to other models involving sign-changing
539 coefficients.

540 **Acknowledgments** The authors would like to thank Anne-Sophie Bonnet-Ben Dhia and Lucas Chesnel
541 for their helpful comments on the first draft of this article. We also thank Farah Chaaban, for her constructive
542 remarks during her internship, regarding the implementation of the method.

543 Appendix A. Missing results.

544 *Proof of Proposition 4.4.* From the definition of w_δ^* , we can write that

$$545 \quad \delta \|w_\delta^*\|_{\tilde{\varepsilon}_1}^2 \leq J^\delta(w_\delta^*) \leq J^\delta(w_J^*) = J(w_J^*) + \delta \|w_J^*\|_{\tilde{\varepsilon}_1}^2 = \delta \|w_J^*\|_{\tilde{\varepsilon}_1}^2.$$

546 This means that for all $0 < \delta$, there holds $\|w_\delta^*\|_{\tilde{\varepsilon}_1} \leq \|w_J^*\|_{\tilde{\varepsilon}_1}$. As a result (w_δ^*) is bounded in $V_2(\Omega_2)$. This
547 implies that, up to a sub-sequence, $(w_\delta^*)_\delta$ converges, as δ tends to 0, weakly in $V_2(\Omega_2)$ to some $w_0 \in V_2(\Omega_2)$.
548 For the reader's convenience, this sequence is also denoted by $(w_\delta^*)_\delta$. Now, let us prove that w_0 is a minimizer
549 of J . To do that, we start by observing that for all $\delta > 0$, we have

$$550 \quad 0 \leq J(w_\delta^*) \leq J^\delta(w_\delta^*) \leq J^\delta(w_J^*) = \delta \|w_J^*\|_{\tilde{\varepsilon}_1}^2.$$

551 This shows that $(J(w_\delta^*))_\delta$ converges to zero as δ tends to zero. On the other hand, by using the result of
 552 proposition 4.1, we know that $(J(w_\delta^*))_\delta$ converges to $J(w_0)$. Consequently, $J(w_0) = 0$ and then w_0 is a
 553 minimizer of J .

554 The next step is to show that the convergence of $(w_\delta^*)_\delta$ to w_0 occurs in the strong sense and that $w_0 = w_J^*$.
 555 To do so, we observe that

$$556 \quad \|w_\delta^*\|_{\varepsilon_1} \leq \|w_J^*\|_{\varepsilon_1} \quad \forall \delta \implies \limsup_{\delta \rightarrow 0} \|w_\delta^*\|_{\varepsilon_1} \leq \|w_J^*\|_{\varepsilon_1}, \quad \text{and} \quad w_\delta^* \rightharpoonup w_0 \text{ in } V_2(\Omega_2) \implies \|w_0\|_{\varepsilon_1} \leq \liminf_{\delta \rightarrow 0} \|w_\delta^*\|_{\varepsilon_1},$$

557 where the latter is a consequence of the fact that the norm of a Banach space is weakly lower semicontinuous,
 558 see [9, Proposition III.5 (iii)]. This implies that $\|w_0\|_{\varepsilon_1} \leq \|w_J^*\|_{\varepsilon_1}$. Thanks to the definition of w_J^* , we deduce
 559 that $w_0 = w_J^*$. With this in mind and with the help of the previous inequality, we conclude that

$$560 \quad \lim_{\delta \rightarrow 0} \|w_\delta^*\|_{\varepsilon_1} = \|w_J^*\|_{\varepsilon_1}.$$

561 Since $V_2(\Omega_2)$ is a Hilbert space, it follows (see [9, Proposition III.32]) that $w_\delta \rightarrow w_J^*$ in $V_2(\Omega_2)$. By noticing
 562 that w_J^* is independent of the considered sub-sequence, the result is then proved. \square

563 PROPOSITION A.1. [8, Theorem 1.6.6] Let Ω be an open, bounded, connected subset of \mathbb{R}^d ($d = 2, 3$) with a
 564 Lipschitz boundary. Then the estimate

$$565 \quad \|u\|_{L^2(\partial\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2} \quad \forall u \in H^1(\Omega)$$

566 holds with $0 < C$ independent of u .

567 PROPOSITION A.2. Let $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\}$ and $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 < |\mathbf{x}| < 2\}$. Assume that $\kappa_\varepsilon := \varepsilon_2/\varepsilon_1 \notin$
 568 $\{-1\} \cup \mathcal{S}$ with

$$569 \quad \mathcal{S} := \left\{ -\frac{1 - (1/2)^{2n}}{1 + (1/2)^{2n}} \mid n \in \mathbb{N}^* \right\}.$$

570 Then the operator $A_\varepsilon : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is an isomorphism.

571 Remark A.3. Note that in accordance with the results concerning the Neumann-Poincaré operator [24, Chapter
 572 1], we observe that -1 is an accumulation point of \mathcal{S} .

573 Proof. [12, Theorem 1.3.3] guarantees that A_ε is Fredholm of index 0 when $\kappa_\varepsilon \neq -1$. Therefore it suffices to
 574 study its kernel. Let $u \in H_0^1(\Omega)$ be such that $A_\varepsilon u = 0$. Then $u_1 := u|_{\Omega_1}$ and $u_2 = u|_{\Omega_2}$ satisfy

$$575 \quad \begin{cases} \Delta u_1 = 0 & \text{in } \Omega_1 \\ \Delta u_2 = 0 & \text{in } \Omega_2 \\ u_1(1, \theta) = u_2(1, \theta) & \text{and} \quad \partial_r u_1(1, \theta) = \kappa_\varepsilon \partial_r u_2(1, \theta) \quad \forall \theta \in [0; 2\pi]. \end{cases}$$

576 Since the problem is invariant with respect to θ , by Fourier decomposition for u_1, u_2 we have the representa-
 577 tions:

$$578 \quad u_1(r, \theta) = \sum_{n \in \mathbb{N}} a_n r^n e^{in\theta} \quad \text{and} \quad u_2(r, \theta) = b_0 \ln(r/2) + \sum_{n \in \mathbb{Z}^*} b_n ((r/2)^n - (r/2)^{-n}) e^{in\theta},$$

579 where $a_n, b_n \in \mathbb{C}$. Using the transmission conditions, we get

$$580 \quad \begin{cases} a_0 = b_0 \ln(1/2), & 0 = b_0 \kappa_\varepsilon \\ a_n = b_n ((1/2)^n - (1/2)^{-n}), & a_n = b_n ((1/2)^n + (1/2)^{-n}) \kappa_\varepsilon, \quad n \in \mathbb{N}^* \\ 0 = b_n ((1/2)^n - (1/2)^{-n}), & 0 = b_n ((1/2)^n + (1/2)^{-n}) \kappa_\varepsilon, \quad -n \in \mathbb{N}^*. \end{cases}$$

581 Therefore we deduce that A_ε is injective when $\kappa_\varepsilon \notin \mathcal{S}$. \square

582 **Appendix B. On the use of the adjoint approach to compute the gradient of the cost functional**
 583 J . The adjoint approach was introduced in [11] as a method for computing the gradient of cost functions
 584 that depend in non-explicit way on the main variable of the problem, namely via the solution of PDEs (the
 585 state equations) in which the main variable plays the role of a parameter. Here, we are going to explain how

586 to apply this method to our case. The idea is to introduce a Lagrangian functional $\mathcal{L} : V_2(\Omega_2) \times H_0^1(\Omega) \times$
 587 $V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2) \rightarrow \mathbb{R}$ such that

$$588 \quad \mathcal{L}(w, u, u_2, g, g_2) = \frac{1}{2} \int_{\Sigma} |u - u_2|^2 \, d\sigma + a_1(w, u, g) + a_2(w, u, u_2, g_2)$$

589 in which $a_1(w, u, g)$ and $a_2(w, u, u_2, g_2)$ are respectively given by

$$590 \quad \begin{cases} a_1(w, u, g) = \int_{\Omega} \tilde{\varepsilon}_1 \nabla u \cdot \nabla g \, d\mathbf{x} - \int_{\Omega_1} f g \, d\mathbf{x} - \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla w \cdot \nabla g \, d\mathbf{x} \\ a_2(w, u, u_2, g_2) = \int_{\Omega_2} \varepsilon_2 \nabla u_2 \cdot \nabla g_2 \, d\mathbf{x} - \int_{\Omega_2} f_2 g_2 \, d\mathbf{x} + \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla(w - u) \cdot \nabla g_2 \, d\mathbf{x}. \end{cases}$$

591 The functions $g \in H_0^1(\Omega)$, $g_2 \in V_2(\Omega_2)$ are the adjoint variables associated to u, u_2 respectively. Let (u^w, u_2^w)
 592 be the solution to (3.2). By design, $a_1(w, u^w, g) = 0$ for all $g \in H_0^1(\Omega)$, and $a_2(w, u^w, u_2^w, g_2) = 0$ for all
 593 $g_2 \in V_2(\Omega_2)$, so one has

$$594 \quad (\text{B.1}) \quad \mathcal{L}(w, u^w, u_2^w, g, g_2) = J(w) \quad \forall g \in H_0^1(\Omega), \quad \forall g_2 \in V_2(\Omega_2).$$

595 Clearly, the functional \mathcal{L} is differentiable with respect to all its variables. For all $\mathbf{v} = (w, u, u_2, g, g_2) \in$
 596 $V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2)$, the partial derivatives of \mathcal{L} at \mathbf{v} belong respectively to
 597 $\partial_w \mathcal{L}(\mathbf{v}) \in (V_2(\Omega_2))^*$, $\partial_u \mathcal{L}(\mathbf{v}) \in (H_0^1(\Omega))^*$, $\partial_{u_2} \mathcal{L}(\mathbf{v}) \in (V_2(\Omega_2))^*$, $\partial_g \mathcal{L}(\mathbf{v}) \in (H_0^1(\Omega))^*$, $\partial_{g_2} \mathcal{L}(\mathbf{v}) \in (V_2(\Omega_2))^*$.
 598 Let $g \in H_0^1(\Omega)$ and $g_2 \in V_2(\Omega_2)$ be given, and $\mathbf{v}^w = (w, u^w, u_2^w, g, g_2)$. By taking the derivative of the relation
 599 (B.1) with respect to w , we find that, by applying the chain rule formula,

$$600 \quad \langle J'(w), h \rangle_{\tilde{\varepsilon}_1} = \langle \partial_w \mathcal{L}(\mathbf{v}^w), h \rangle + \langle \partial_u \mathcal{L}(\mathbf{v}^w), \frac{du^w}{dw}(h) \rangle + \langle \partial_{u_2} \mathcal{L}(\mathbf{v}^w), \frac{du_2^w}{dw}(h) \rangle, \quad \forall h \in V_2(\Omega_2).$$

601 Now, if there exists $(g^w, g_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$ for which the equations

$$602 \quad \partial_u \mathcal{L}(w, u^w, u_2^w, g^w, g_2^w) = 0 \quad \text{and} \quad \partial_{u_2} \mathcal{L}(w, u^w, u_2^w, g^w, g_2^w) = 0$$

603 are satisfied for all $w \in V_2(\Omega_2)$, this yields

$$604 \quad \langle J'(w), h \rangle_{\tilde{\varepsilon}_1} = \langle \partial_w \mathcal{L}(w, u^w, u_2^w, g_1^w, g_2^w), h \rangle \quad \forall w \in V_2(\Omega_2), \quad \forall h \in V_2(\Omega_2).$$

605 To investigate the existence of (g^w, g_2^w) , we need to write down the expression of $\partial_u \mathcal{L}(\mathbf{v}^w)$ and $\partial_{u_2} \mathcal{L}(\mathbf{v}^w)$: By
 606 a direct calculus, one checks that

$$607 \quad \begin{aligned} \langle \partial_u \mathcal{L}(\mathbf{v}^w), v \rangle &= \int_{\Omega} \tilde{\varepsilon}_1 \nabla g \cdot \nabla v \, d\mathbf{x} - \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla g_2 \cdot \nabla v \, d\mathbf{x} + \int_{\Sigma} (u^w - u_2^w) v \, d\sigma \quad \forall v \in H_0^1(\Omega) \\ \langle \partial_{u_2} \mathcal{L}(\mathbf{v}^w), v_2 \rangle &= \int_{\Omega_2} \varepsilon_2 \nabla g_2 \cdot \nabla v_2 \, d\mathbf{x} - \int_{\Sigma} (u^w - u_2^w) v_2 \, d\sigma \quad \forall v_2 \in V_2(\Omega_2). \end{aligned}$$

608 Hence, the functions $(g^w, g_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$ are governed by the following system of equations:

$$609 \quad (\text{B.2}) \quad \begin{cases} \int_{\Omega} \tilde{\varepsilon}_1 \nabla g^w \cdot \nabla v \, d\mathbf{x} = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla g_2^w \cdot \nabla v \, d\mathbf{x} - \int_{\Sigma} (u^w - u_2^w) v \, d\sigma & \forall v \in H_0^1(\Omega) \\ \int_{\Omega_2} \varepsilon_2 \nabla g_2^w \cdot \nabla v_2 \, d\mathbf{x} = \int_{\Sigma} (u^w - u_2^w) v_2 \, d\sigma & \forall v_2 \in V_2(\Omega_2). \end{cases}$$

610 Clearly the previous system of equations is well-posed. Therefore the functions g^w, g_2^w are well-defined. We
 611 then have all the tools to prove the result stated in Lemma 4.5.

612 *Proof of Lemma 4.5.* Take $w \in V_2(\Omega_2)$. From the characterization (B.2) of g^w and g_2^w , we deduce that for all
 613 $h \in V_2(\Omega_2)$, we have

$$614 \quad \langle J'(w), h \rangle_{\tilde{\varepsilon}_1} = \langle \partial_w \mathcal{L}(w, u^w, u_2^w, g_1^w, g_2^w), h \rangle.$$

615 On the other hand, one can compute explicitly the value of $\langle \partial_w \mathcal{L}(w, u, u_2, g, g_2), h \rangle$:

$$616 \quad \langle \partial_w \mathcal{L}(w, u, u_2, g, g_2), h \rangle = \int_{\Omega_2} \tilde{\varepsilon}_1 \nabla h \cdot \nabla (g_2 - g|_{\Omega_2}) \, d\mathbf{x}.$$

617 This shows that $J'(w) = g_2^w - g^w|_{\Omega_2}$ and then the result is proved. \square

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