Scales

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Abstract

We introduce the notion of *scale* to generalize some aspects of dimension theory. Several versions of scales exist such as Hausdorff, packing, box, local and quantization. We prove general theorems comparing them. They are applied to describe the largeness of ergodic decompositions, Wiener measure and functional spaces. The first application solves a problem set by Berger on the notions of emergence (2020); the second lies in the study of small ball estimates of the Wiener measure and extends an estimation of Dereich-Lifshits (2005); the last refines the Kolmogorov-Tikhomirov theorems (1958) on the ϵ-capacity of functional spaces.

Contents

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1 Introduction and results

Dimension theory was popularized by Mandelbrot in the article *How long is the coast of Britain ?* [Man67] and shed light on the general problem of measuring how large a natural object is. The category of objects considered are metric spaces possibly endowed with a measure.

Dimension theory encompasses not only smooth spaces such as manifolds, but also wild spaces such as fractals, so that the dimension may be any non-negative real number. There are several notions of dimension: for instance Hausdorff [Hau18], packing [Tri82] or box dimensions; also when the space is endowed with a measure, there are moreover the local and the quantization dimensions. These different versions of dimension are bi-Lipschitz invariants. They are in general not equal, so that they reveal different aspects of the underlying space. The seminal works of Hausdorff, Frostman, Tricot, Fan, Tamashiro, Pötzelberger, Graf-Luschgy and Dereich-Lifshits described the relationship between these notions and gave conditions under which they coincide.

Obviously these invariants do not give much information on infinite dimensional spaces. However such spaces are subject to many studies. Kolmogorov-Tikhomirov in [KT93] gave asymptotics of the covering numbers of functional spaces. Dereich-Lifshits gave asymptotics of the mass of the small balls for the Wiener measure and exhibited their relationship with the quantization problem, see [DFMS03, DL05, CM44, Chu47, BR92, KL93]. Also Berger and Bochi [Ber20] gave estimates on the covering number and quantization number of the ergodic decomposition of some smooth dynamical systems. See also [BR92, Klo15, BB21].

We will introduce the notion of *scale* to unify both the different kinds of dimensions and the latter infinite dimensional studies. Scales are families of invariants of different *kinds* (box, Hausdorff, packing, quantization, local) and different *growths* (dim, ord . . .). Scales generalize all the previous kinds of dimensions and allow to describe some aspects of the above examples of infinite dimensional spaces and measures. The different growths of scales are given by *scalings* (see Definition 1.2).

We will generalize the well known comparison theorems between the different kind of dimensions to all the different growths of scales in Theorem A, Theorem B and Theorem C. See Section 1.2. The main difficulty will be to prove Theorem C which enables to compare the quantization scales with both the local and the box scales.

An application of Theorem A, together with new estimates, is to show the coincidence of the box, Hausdorff and packing orders for finitely regular functional spaces; refining [KT93, Thm XV]. Theorem C together with [DL05] immediately implies the coincidence of local, Hausdorff, packing, quantization and box orders of the Wiener measure for the L^p-norm, for any $p \in [1, \infty]$. Moreover, a consequence of Theorem C is that the local order of the ergodic decomposition is at most its quantization order; which solves a problem set by Berger in [BB21]. See Section 1.3.3.

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1.1 From dimension to scale

The Hausdorff, packing and box dimensions of a totally bounded metric space (X, d) are defined by looking at families of subsets of X. For the box dimensions, we consider the set $\mathcal{C}_B(\epsilon)$ of finite coverings of X by balls with radius $\epsilon > 0$. For the Hausdorff dimension, we consider the set $\mathcal{C}_H(\epsilon)$ of countable coverings $(E_i)_{i\geq 1}$ of X by sets with diameter at most $\epsilon > 0$. For $\alpha > 0$, consider the following quantities. For the box dimensions, denote:

$$\mathcal{B}^\alpha_\epsilon(X) := \inf_{C\in\mathcal{C}_B(\epsilon)} \sum_{B\in C} \text{radius}\,(B)^\alpha = \epsilon^\alpha \cdot \inf_{C\in\mathcal{C}_B(\epsilon)} \text{Card}\, C\ .$$

The quantity $\mathcal{N}_\epsilon = \inf_{C\in\mathcal{C}_B(\epsilon)} \text{Card}\, C$ is called the *ϵ-covering number* of (X, d). For the *Hausdorff dimension*, denote:

$$\mathcal{H}^\alpha_\epsilon(X) := \inf_{C\in\mathcal{C}_H(\epsilon)} \sum_{E\in C} \text{diam}(E)^\alpha\ .$$

Then we define the *lower and upper box dimensions* as:

$$\underline{\dim}_B X = \sup\left\{\alpha > 0 : \mathcal{B}^\alpha_\epsilon(X) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} \quad \text{and} \quad \overline{\dim}_B X = \inf\left\{\alpha > 0 : \mathcal{B}^\alpha_\epsilon(X) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}\ .$$

Similarly, we define *Hausdorff dimension* by:

$$\dim_H X = \sup\left\{\alpha > 0 : \mathcal{H}^\alpha_\epsilon(X) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} = \inf\left\{\alpha > 0 : \mathcal{H}^\alpha_\epsilon(X) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}\ .$$

It always holds $\dim_H X \leq \underline{\dim}_B X \leq \overline{\dim}_B X$, in general, none of these inequalities is an equality. For instance the set $(0, 1) \cap \mathbb{Q}$ has null Hausdorff dimension and both box dimensions equal to 1. Moreover, there are non-homogeneous Cantor sets such that $\underline{\dim}_B X < \overline{\dim}_B X$. See e.g. [FLR02]. Hausdorff and box dimensions enjoy different properties, while Hausdorff dimension is countable stable (the dimension of a countable union is the supremum of the dimensions), the box dimensions are stable by closure. One way to define *packing dimension* is to modify upper box dimension to make it countable stable:

$$\dim_P X = \inf \sup_{n\geq 1} \overline{\dim}_B E_n\ ,$$

where the infimum is taken over countable coverings $(E_n)_{n\geq 1}$ of X. These four versions of dimension are bi-Lipschitz invariants; they quantify different aspects of the geometry of the studied metric space. In the next section we will recall how these invariants are related. The idea is to replace the family $(\phi_\alpha)_{\alpha>0} = (\epsilon \in (0, 1) \mapsto \epsilon^\alpha)_{\alpha>0}$ by families of functions, called scalings, defining comparable versions of *scales*, which encompasses the following examples of growth:

Example 1.1. 1. *The family* $\dim = (\epsilon \in (0, 1) \mapsto \epsilon^\alpha)_{\alpha>0}$ *which is used in the definitions of dimensions,*

2. *the family* $\text{ord} = (\epsilon \in (0, 1) \mapsto \exp(-\epsilon^{-\alpha}))_{\alpha>0}$ *which fits with the growth of the covering number of spaces of finitely regular functions studied by Kolmogorov-Tikhomirov [KT93], see Theorem 1.10, or with the one of the space of ergodic measures spaces of chaotic dynamics by Berger-Bochi [Ber20], see Proposition 1.15,*

3. *the family* $\left(\epsilon \in (0,1) \mapsto \exp(-(\log \epsilon^{-1})^{\alpha})\right)_{\alpha>0}$ *which fits with the growth of the covering number of holomorphic functions estimated by Kolmogorov-Tikhomirov [KT93], see Theorem 2.5.*

Yet to keep a comparison between the generalization of box, Hausdorff and packing dimensions, the family $(\phi_\alpha)_{\alpha>0}$ must satisfies some mild assumptions, which leads to the notion of scaling:

Definition 1.2 (Scaling). *A family* $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha\geq 0}$ *of positive non-decreasing functions on* $(0,1)$ *is a* scaling *when for any* $\alpha > \beta > 0$ *and any* $\lambda > 1$ *close enough to* 1*, it holds:*

$$(*) \qquad \mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon^\lambda)\right) \quad \text{and} \quad \mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon)^\lambda\right) \quad \text{when } \epsilon \to 0\ .$$

Remark 1.3. *The left hand side condition is used in all the proof of our theorems represented on Fig. 1. The right hand side condition is only used to prove the equalities between packing and upper local scales in Theorem B and to compare upper local scales with upper box and upper quantization scales in Theorem C inequalities* $(c)\&(g)$ *. It moreover allows to characterize packing scale with packing measure.*

Remark 1.4. *There are scalings that are smaller than dimension, e.g.* $(\epsilon \in (0,1) \mapsto \log(\epsilon^{-1})^{-\alpha})_{\alpha>0}$ *that is adapted to study the set* $\{2^{-n^\beta} : n \geq 1\}$ *where* β *is a positive real number.*

We will show in Proposition 2.4 that $1-2-3$ in Example 1.1 are scalings. Scalings allow to define *scales* which generalize packing dimension, Hausdorff dimension, box dimensions, quantization dimensions and local dimensions that are local counterparts for measures. For each scaling, the different kind of scales do not a priori coincide on a generic space. Nevertheless in Section 1.3, as a direct application of our comparison theorems, we bring examples of metric spaces and measures where all those definitions coincide. This equality between the different scales gives us hints of the regularity of such spaces and measures.

Now for a metric space (X,d), replacing the specific family dim in the definition of box dimensions by any scaling $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha>0}$ gives the following:

Definition 1.5 (Box scales). Lower and upper box scales *of a metric space* (X,d) *are defined by:*

$$\underline{\mathsf{scl}}_B X = \sup\left\{\alpha > 0 : \mathcal{N}_\epsilon(X)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} \quad \text{and} \quad \overline{\mathsf{scl}}_B X = \inf\left\{\alpha > 0 : \mathcal{N}_\epsilon(X)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}\ .$$

Moreover we will generalize the notion Hausdorff and packing dimensions to the *Hausdorff scale* denoted $\mathsf{scl}_H X$ (see Definition 2.13) and *packing scale* denoted $\mathsf{scl}_P X$ (see Definition 2.14).

1.2 Results on comparisons of scales

In this section, we introduce other kind of scales and Theorems A, B and C which state the inequalities between them as illustrated in Fig. 1. First, we bring the following generalization of inequalities comparing dimensions of metric spaces to the frame of scales:

Theorem A. *Let* (X,d) *be a metric space and* scl *a scaling, the following inequalities hold:*

$$\mathsf{scl}_H X \leq \mathsf{scl}_P X \leq \overline{\mathsf{scl}}_B X \quad \text{and} \quad \mathsf{scl}_H X \leq \underline{\mathsf{scl}}_B X \leq \overline{\mathsf{scl}}_B X\ .$$

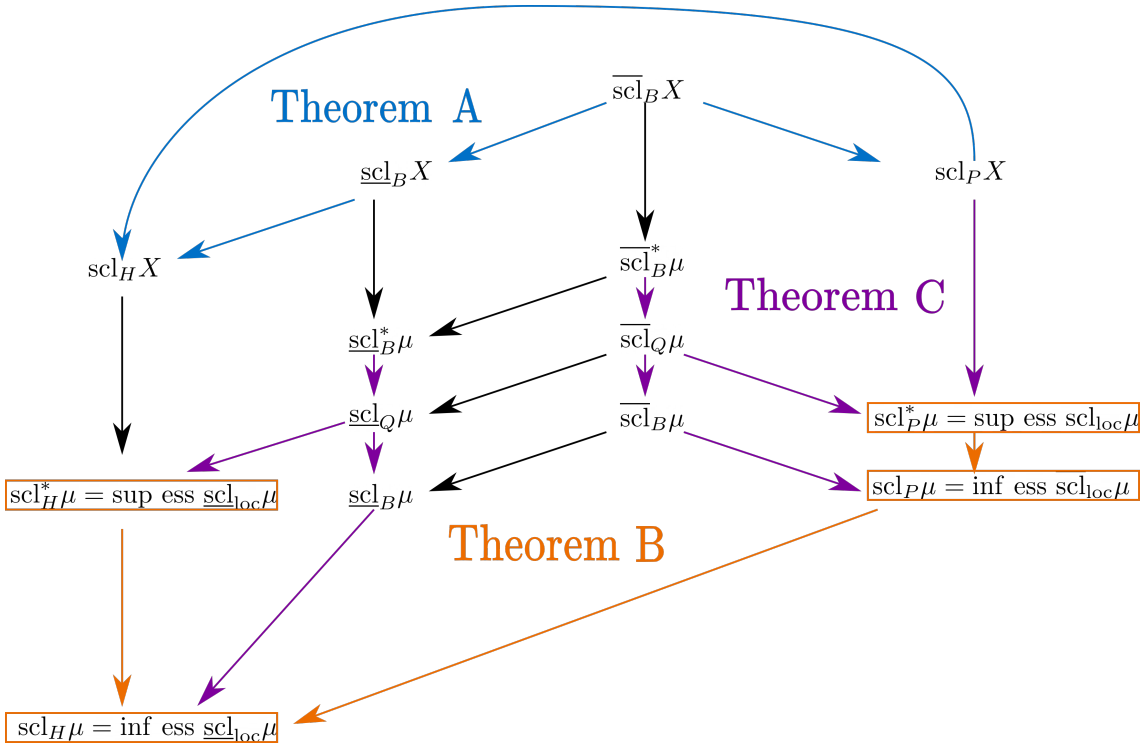

Figure 1: Diagram presenting results of Theorems A, B and C. Each arrow is an inequality, the scale at the starting point of the arrow is at least the one at its ending point : " $\to$ " = " $\geq$ ". None of them is an equality in the general case. If there is no path between two scales scl_1 and scl_2 then there exist examples of spaces endowed with a measure such that both $\mathsf{scl}_1 > \mathsf{scl}_2$ and $\mathsf{scl}_1 < \mathsf{scl}_2$ can happen. See Example 4.6.

In the specific case of dimension these inequalities are well known and redacted for instance by Tricot [Tri82] or Falconer [FF97, Fal04]. The proof of this theorem will be done in Section 2.5.

When the metric space (X, d) is endowed with a measure μ, Frostman first studied the relationship between the Hausdorff dimension and the growth of the mass of the small balls. This has been intensively studied by Fan [FLR02, Fan94], Pötzelberger [Pöt99], Tamashiro [Tam95] as local dimension. Similarly we introduce local scales that extend the notion of local dimensions of a measure:

Definition 1.6 (Local scales). *Let μ be Borel measure on a metric space (X, d) and* scl *a scaling. The* lower and upper scales *of μ are the functions that map a point $x \in X$ to:*

$$\underline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) = \sup\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} 0\right\} \quad \textit{and} \quad \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) = \inf\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}.$$

As in dimension theory, we should not compare the local scales with the scales of X but to the ones of its subsets with positive mass. This observation leads to consider the following:

Definition 1.7 (Hausdorff, packing and box scales of a measure). *Let* scl *be a scaling and* μ *a non-null Borel measure on a metric space* (X,d)*. For any* $\mathsf{scl}_\bullet \in \left\{\mathsf{scl}_H, \mathsf{scl}_P, \underline{\mathsf{scl}}_B, \overline{\mathsf{scl}}_B\right\}$ *we define lower and upper scales of the measure* μ *by:*

$$\mathsf{scl}_\bullet\mu = \inf_{E\in\mathcal{B}}\{\mathsf{scl}_\bullet E : \mu(E) > 0\} \quad \textit{and} \quad \mathsf{scl}^*_\bullet\mu = \inf_{E\in\mathcal{B}}\{\mathsf{scl}_\bullet E : \mu(X\backslash E) = 0\}\ ,$$

where $\mathcal{B}$ *is the set of Borel subsets of* X*.*

In the case of dimension, Frostman [Fro35], Tricot [Tri82], Fan [Fan94, FLR02] and Tamashiro [Tam95] exhibited the relationship between the Hausdorff and packing dimensions of measures and their local dimensions. The following theorem is a straightforward generalization of their results:

Theorem B. *Let* μ *be a Borel measure on a metric space* (X,d)*, then for any scaling* scl*, Hausdorff and packing scales of* μ *are characterized by:*

$$\mathsf{scl}_H\mu = \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu, \quad \mathsf{scl}^*_H\mu = \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu, \quad \mathsf{scl}_P\mu = \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu, \quad \mathsf{scl}^*_P\mu = \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu\ ,$$

where ess sup *and* ess inf *denote the essential suprema and infima of a function.*

The proof of the latter theorem is done in Section 3.1.

Let us introduce a last kind of scale, the quantization scale. It generalizes the quantization dimension which dragged much research interest [GL07, Pöt99, DFMS03, DL05, Ber17, BB21, Ber20].

Definition 1.8 (Quantization scales). *Let* (X,d) *be a metric space and* μ *a Borel measure on* X*. The* quantization number $\mathcal{Q}_\mu$ *of* μ *is the function that maps* $\epsilon > 0$ *to the minimal cardinality of a set of points that is on average* ϵ*-close to any point in* X*:*

$$\mathcal{Q}_\mu(\epsilon) = \inf\left\{N \geq 0 : \exists\,\{c_i\}_{i=1,\dots,N} \subset X, \int_X d(x,\{c_i\}_{1\leq i\leq N})d\mu(x) \leq \epsilon\right\}\ .$$

Then lower and upper quantization scales *of* μ *for a given scaling* scl *are defined by:*

$$\underline{\mathsf{scl}}_Q\mu = \sup\left\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} \quad \textit{and} \quad \overline{\mathsf{scl}}_Q\mu = \inf\left\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}\ .$$

The following gives relationships between the remaining kind of introduced scales of measures:

Theorem C (Main). *Let* (X,d) *be a metric space. Let* μ *be a Borel measure on* X*. For any scaling* scl*, the following inequalities on the scales of* μ *hold:*

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \underbrace{\leq}_{(a)} \underline{\mathsf{scl}}_B\mu \underbrace{\leq}_{(b)} \underline{\mathsf{scl}}_Q\mu \quad ; \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \underbrace{\leq}_{(c)} \overline{\mathsf{scl}}_B\mu \underbrace{\leq}_{(d)} \overline{\mathsf{scl}}_Q\mu$$

and

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \underbrace{\leq}_{(e)} \underline{\mathsf{scl}}_Q\mu \underbrace{\leq}_{(f)} \underline{\mathsf{scl}}^*_B\mu \quad ; \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \underbrace{\leq}_{(g)} \overline{\mathsf{scl}}_Q\mu \underbrace{\leq}_{(h)} \overline{\mathsf{scl}}^*_B\mu\ .$$

The proof of inequalities (b) and (d) is done at Theorem 3.10 and relies mainly on the use of Borel-Cantelli lemma. Even in the specific case of dimension, these inequalities were not shown yet, as far as we know. The proof of inequalities (f) and (h) is straightforward, see Lemma 3.8. Inequalities (e) and

(g) were shown by Pötzelberger in [Pöt99] for dimension and in $[0,1]^d$. A new approach for the general case of scales of inequalities (e) and (g) is brought in Theorem 3.11. We deduce the inequality (a) from (e) and (f) and inequality (c) from (g) and (h). As a direct application, inequality (e) allows to answer to a problem set by Berger in [Ber20] (see Section 1.3.3). We will give in Section 4.1 examples of topological compact groups different versions of orders do not coincide. Moreover in that same section we show that for a metric group where the law is Lipschitz, the Hausdorff scale coincides with the lower box scale and the packing scale coincides with the upper box scale.

1.3 Applications

Let us see how our main theorems imply easily the coincidence of the scales of some natural infinite dimensional spaces.

1.3.1 Wiener measure

First example is the calculus of the orders of the Wiener measure W that describes uni-dimensional standard Brownian motion on $[0,1]$. Recall that W is the law of a continuous process $(B_t)_{t\in[0,1]}$ with independent increments. It is such that for any $t \geq s$ the law of the random variable $B_t - B_s$ is $\mathcal{N}(0, t-s)$. Computation of the local scales of the Wiener measure relies on small ball estimates which received much interest [CM44, Chu47, BR92, KL93]. These results gave asymptotics on the measure of small balls centered at 0 for L^p norms and Hölder norms. Moreover for a random ball the Dereich-Lifshits made the following estimation for L^p-norms:

Theorem 1.9 (Dereich-Lifshits [DL05][Results 3.2, 5.1, 6.1, 6.3]). *For the Wiener measure on $C^0([0,1],\mathbb{R})$ endowed with the L^p-norm, for $p \in [1,\infty]$, there exists [1] a constant $\kappa > 0$ such that for W-almost any $\omega \in C^0([0,1],\mathbb{R})$:*

$$-\epsilon^2 \cdot \log W(B(\omega,\epsilon)) \to \kappa, \text{ when } \epsilon \to 0 \ ,$$

and moreover the quantization number of W verifies:

$$\epsilon^2 \cdot \log \mathcal{Q}_W(\epsilon) \to \kappa, \text{ when } \epsilon \to 0 \ .$$

As a direct consequence of Theorem B and Theorem C we get that the new invariants we introduced for a measure with growth given by ord all coincide:

Theorem D (Orders of the Wiener measure). *For the Wiener measure on $C^0([0,1],\mathbb{R})$ endowed with the L^p-norm, for $p \in [1,\infty]$, verifies for W-almost every $\omega \in C^0([0,1])$:*

$$\underline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = \mathsf{ord}_H W = \mathsf{ord}^*_H W = \overline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = \mathsf{ord}_P W = \mathsf{ord}^*_P W = \underline{\mathsf{ord}}_B W = \underline{\mathsf{ord}}_Q W = \overline{\mathsf{ord}}_B W = \overline{\mathsf{ord}}_Q W = 2 \ .$$

Proof. By Theorem 1.9, for W-almost ω and for any $p \in [1,\infty]$, in the L^p-norm it holds:

$$\underline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = \overline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = 2 = \underline{\mathsf{ord}}_Q W = \overline{\mathsf{ord}}_Q W \ .$$

[1] Note that for $p < \infty$, the constant κ does not depend on the value of p.

Now by Theorem B since local orders of the Wiener measure are almost constant everywhere, it holds:

$$\mathsf{ord}_H W = \underline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = \mathsf{ord}^*_H W \quad \text{and} \quad \mathsf{ord}_P W = \overline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = \mathsf{ord}^*_P W \ .$$

Finally since by Theorem C, it holds:

$$\operatorname{ess\,inf} \underline{\mathsf{ord}}_{\mathrm{loc}} W \leq \underline{\mathsf{ord}}_B W \leq \overline{\mathsf{ord}}_B W \leq \overline{\mathsf{ord}}_Q W \ ,$$

the sought result comes by combining the three above lines of equalities and inequalities. □

1.3.2 Functional spaces endowed with the C^0-norm

Let d be a positive integer. For any integer $k \geq 0$ and for any $\alpha \in [0,1]$ denote:

$$\mathcal{F}^{d,k,\alpha} := \left\{ f \in C^k([0,1]^d, [-1,1]) : \|f\|_{C^k} \leq 1, \text{ and if } \alpha > 0, \text{ the map } D^k f \text{ is } \alpha\text{-Hölder with constant } 1 \right\}.$$

We endow this space with the C^0 norm. See Section 4.2 for the definition of the C^k-norms.Kolmogorov-Tikhomirov gave the following asymptotics:

Theorem 1.10 (Kolmogorov-Tikhomirov, [KT93][Thm XV])**.** *Let d be a positive integer. For any integer k and for any $\alpha \in [0,1]$, there exist two constants $C_1 > C_2 > 0$ such that the covering number $\mathcal{N}_\epsilon(\mathcal{F}^{d,k,\alpha})$ of the space $(\mathcal{F}^{d,k,\alpha}, \|\cdot\|_\infty)$ verifies:*

$$C_1 \cdot \epsilon^{-\frac{d}{k+\alpha}} \geq \log \mathcal{N}_\epsilon(\mathcal{F}^{d,k,\alpha}) \geq C_2 \cdot \epsilon^{-\frac{d}{k+\alpha}} \ .$$

In Section 4.2 by embedding a group whose Hausdorff order is bounded from below into $\mathcal{F}^{k,d,\alpha}$ (see Section 4.2), via an expanding map, we will prove:

Lemma 1.11. *Let d be a positive integer. For any integer k and for any $\alpha \in [0,1]$, it holds:*

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \geq \frac{d}{k+\alpha} \ .$$

The above lemma together with Theorem A gives the following extension of Kolmogorov-Tikhomirov's Theorem:

Theorem E. *Let d be a positive integer. For any integer k and for any $\alpha \in [0,1]$, it holds:*

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} = \mathsf{ord}_P \mathcal{F}^{d,k,\alpha} = \underline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} = \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} = \frac{d}{k+\alpha} \ .$$

Proof of Theorem E. First, by Theorem A, it holds:

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \leq \underline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} \leq \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} \quad \text{and} \quad \mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \leq \mathsf{ord}_P \mathcal{F}^{d,k,\alpha} \leq \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} \ .$$

From there by Theorem 1.10 and Lemma 1.11, it holds:

$$\frac{d}{k+\alpha} \leq \mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \leq \underline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} \leq \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} = \frac{d}{k+\alpha} \ ,$$

and

$$\frac{d}{k+\alpha} \leq \mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \leq \mathsf{ord}_P \mathcal{F}^{d,k,\alpha} \leq \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} = \frac{d}{k+\alpha} \ .$$

From there, all of the above inequalities are indeed equalities, which gives the sought result. □

1.3.3 Local and global emergence

The framework of scales moreover allow to answer to a problem set by Berger in [Ber20] on wild dynamical systems. We now consider a compact metric space (X, d) and a measurable map $f : X \to X$. We denote $\mathcal{M}$ the set of probability Borel measures on X and $\mathcal{M}_f$ the subset of $\mathcal{M}$ of f-invariant measures. The space $\mathcal{M}$ is endowed with the Wasserstein distance W_1 defined by:

$$W_1(\nu_1, \nu_2) = \sup_{\phi \in Lip^1(X)} \int \phi d(\nu_1 - \nu_2) \ ,$$

inducing the weak $*$- topology for which $\mathcal{M}$ is compact. A way to measure the wildness of a dynamical system is to measure how far from being ergodic an invariant measure μ is. Then by Birkhoff's theorem given a measure $\mu \in \mathcal{M}_f$, for μ-almost every $x \in X$ the following measure is well defined:

$$e(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \ ,$$

and moreover the limit measure is ergodic. The definition of emergence, introduced by Berger, describes the size of the set of ergodic measures reachable by limits of empirical measures given an f-invariant probability measure on X.

Definition 1.12 (Emergence, [Ber17, BB21]). *The* emergence *of a measure $\mu \in \mathcal{M}_f$ at $\epsilon > 0$ is defined by:*

$$\mathcal{E}_\mu(\epsilon) = \min\{N \in \mathbb{N} \ : \ \exists \nu_1, \dots, \nu_N \in \mathcal{M}_f, \ \int_X W_1(e_f(x), \{\nu_i\}_{1 \leq i \leq N}) d\mu(x) \leq \epsilon\} \ .$$

The case of high emergence corresponds to dynamics where the considered measure is not ergodic at all. The following result shows us that the order is an adapted scaling in the study of the ergodic decomposition.

Theorem 1.13 ([BGV07, Klo15, BB21]). *Let (X, d) be a metric compact space of finite then:*

$$\underline{\dim}_B X \leq \underline{\mathsf{ord}}_B(\mathcal{M}) \leq \overline{\mathsf{ord}}_B(\mathcal{M}) \leq \overline{\dim}_B X \ .$$

For a given measure $\mu \in \mathcal{M}_f$ we define its *emergence order* by:

$$\overline{\mathsf{ord}}\mathcal{E}_\mu := \limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_\mu(\epsilon)}{-\log \epsilon} = \inf \left\{ \alpha > 0 : \mathcal{E}_\mu(\epsilon) \cdot \exp(-\epsilon^{-\alpha}) \xrightarrow[\epsilon \to 0]{} 0 \right\} \ .$$

We denote $\mu_{e_f} := e_f \star \mu$ the *ergodic decomposition* of μ; it is the probability measure on $\mathcal{M}_f$ equal to the push forward by e of μ. A local analogous local quantity to the emergence order is the local order of the ergodic decomposition of μ, for $\nu \in \mathcal{M}_f$ it is defined by:

$$\overline{\mathsf{ord}}\mathcal{E}^{\mathrm{loc}}_\mu(\nu) := \limsup_{\epsilon \to 0} \frac{\log - \log(\mu_{e_f}(B(\nu, \epsilon))}{-\log \epsilon} \ .$$

Berger asked if the the following comparison between asymptotic behaviour of the mass of the balls of the ergodic decomposition of μ and the asymptotic behaviour of its quantization holds.

Problem 1.14 (Berger, [Ber20, Pbm 4.22]). *Let (X,d) be a compact metric space, $f : X \to X$ a measurable map and μ a Borel f-invariant measure on X. Does the following holds ?*

$$\int_{\mathcal{M}_{\{}} \overline{\mathsf{ord}}\mathcal{E}_\mu^{\mathrm{loc}} d\mu_{e_f} \leq \overline{\mathsf{ord}}\mathcal{E}_\mu \ .$$

We propose here a stronger result that answer to latter problem as a direct application of Theorem C:

Proposition 1.15. *Let (X,d) be a compact metric space, $f : X \to X$ a measurable map and μ a Borel f-invariant measure on X. For μ_{e_f}-almost every $\nu \in \mathcal{M}$, it holds:*

$$\overline{\mathsf{ord}}\mathcal{E}_\mu^{\mathrm{loc}}(\nu) \leq \overline{\mathsf{ord}}\mathcal{E}_\mu \ .$$

Proof. Note that $\overline{\mathsf{ord}}\mathcal{E}_\mu^{\mathrm{loc}} = \overline{\mathsf{ord}}_{\mathrm{loc}}\mu_{e_f}$ and $\overline{\mathsf{ord}}\mathcal{E}_\mu = \overline{\mathsf{ord}}_Q\mu_{e_f}$. Now by Theorem C, it holds μ_{e_f}-almost surely that $\overline{\mathsf{ord}}_{\mathrm{loc}}\mu_{e_f} \leq \overline{\mathsf{ord}}_Q\mu_{e_f}$ which is the sought result. □

2 Metric scales

2.1 Scalings

We first recall that a scaling is a family $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha\geq 0}$ of positive non-decreasing functions on $(0,1)$ is a *scaling* when for any $\alpha > \beta > 0$ and any $\lambda > 1$ close enough to 1, it holds:

$$(*) \qquad \mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon^\lambda)\right) \quad \text{and} \quad \mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon)^\lambda\right) \ \text{ when } \epsilon \to 0 \ .$$

An immediate consequence of the latter definition is the following:

Fact 2.1. *Let scl be a scaling then for any $\alpha > \beta > 0$ and for any constant $C > 0$ it holds for $\epsilon > 0$ small enough:*

$$\mathrm{scl}_\alpha(\epsilon) \leq \mathrm{scl}_\beta(C \cdot \epsilon) \ .$$

A consequence of the latter fact is the following which compares scales of metric spaces and measures:

Lemma 2.2. *Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be two functions defined such that $f \leq g$ near 0, thus for any constant $C > 0$:*

$$\inf\left\{\alpha > 0 : f(C\cdot\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\} \leq \inf\left\{\alpha > 0 : g(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}$$

and

$$\sup\left\{\alpha > 0 : f(C\cdot\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} \leq \sup\left\{\alpha > 0 : g(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} \ .$$

Proof. It suffices to observe that, by Fact 2.1, for any $\alpha > \beta > 0$, and $\epsilon > 0$ small, it holds:

$$f(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \leq g(\epsilon)\cdot \mathrm{scl}_\beta(C^{-1}\cdot\epsilon) = g(C\cdot\tilde{\epsilon})\cdot \mathrm{scl}_\beta(\tilde{\epsilon}) \ ,$$

with $\tilde{\epsilon} = C \cdot \epsilon$. □

Lemma 2.3 gives a sequential characterization of scales.

Lemma 2.3 (Sequential characterization of scales). *Let* scl *be a scaling and* $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ *a non increasing function. Let* $(r_n)_{n\geq 1}$ *be a positive sequence decreasing to* 0 *such that* $\log r_{n+1} \sim \log r_n$ *when* $n \to +\infty$*, then it holds:*

$$\inf\left\{\alpha > 0 : f(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} 0\right\} = \inf\left\{\alpha > 0 : f(r_n)\cdot \mathrm{scl}_\alpha(r_n) \xrightarrow[n\to+\infty]{} 0\right\}$$

and

$$\sup\left\{\alpha > 0 : f(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} +\infty\right\} = \sup\left\{\alpha > 0 : f(r_n)\cdot \mathrm{scl}_\alpha(r_n) \xrightarrow[n\to+\infty]{} +\infty\right\} .$$

Proof. Consider $\alpha > 0$ and $\epsilon > 0$. If ϵ is small enough, there exists an integer $n > 0$ that verifies $r_{n+1} < \epsilon \leq r_n$, thus, since f is not increasing and scl_α is increasing, it holds:

$$f(r_n)\cdot \mathrm{scl}_\alpha(r_{n+1}) \leq f(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \leq f(r_{n+1})\cdot \mathrm{scl}_\alpha(r_n) .$$

Consider now β, γ such that $0 < \beta < \alpha < \gamma$, thus for λ close to 1 and ϵ is small enough, one has by Definition 1.2 of scaling that:

$$\mathrm{scl}_\gamma(r_n) \leq \mathrm{scl}_\alpha(r_n^\lambda) \quad \text{and} \quad \mathrm{scl}_\alpha(r_n) \leq \mathrm{scl}_\beta(r_n^\lambda) .$$

Observe now that $r_{n+1} = r_n^{\frac{\log r_{n+1}}{\log r_n}}$, and since $\log r_{n+1} \sim \log r_n$. For n great enough, it holds then:

$$r_n^\lambda \leq r_{n+1} .$$

Since the functions of the scaling are increasing, it follows:

$$\mathrm{scl}_\gamma(r_n) \leq \mathrm{scl}_\alpha(r_{n+1}) \quad \text{and} \quad \mathrm{scl}_\alpha(r_n) \leq \mathrm{scl}_\beta(r_{n+1}) .$$

We can now deduce from the latter and the first line of inequalities that:

$$f(r_n)\cdot \mathrm{scl}_\gamma(r_n) \leq f(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \quad \text{and} \quad f(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \leq f(r_{n+1})\cdot \mathrm{scl}_\beta(r_{n+1}) .$$

Since γ and β can be chosen arbitrarily close to α, we deduce the sought result. □

The following Proposition 2.4 provides many scalings and shows in particular that the families brought in Example 1.1 are indeed scalings.

Proposition 2.4. *For any integers* $p, q \geq 1$*, the family* $\mathsf{scl}^{p,q} = (\mathrm{scl}_\alpha^{p,q})_{\alpha>0}$ *defined for any* $\alpha > 0$ *by:*

$$\mathrm{scl}_\alpha^{p,q} : \epsilon \in (0,1) \mapsto \frac{1}{\exp^{\circ p}(\alpha\cdot \log_+^{\circ q}(\epsilon^{-1}))}$$

is a scaling; where $\log_+ : t \in \mathbb{R} \mapsto \log(t)\cdot \mathbb{1}_{t>1}$ *is the positive part of the logarithm.*

We prove this proposition below. Now note in particular that $\mathsf{scl}^{1,1} = \mathsf{dim} = (\epsilon \in (0,1) \mapsto \epsilon^\alpha)_{\alpha>0}$ and $\mathsf{scl}^{2,1} = \mathsf{ord} = (\epsilon \in (0,1) \mapsto \exp(-\epsilon^{-\alpha}))_{\alpha>0}$ are both scalings. Let us give an example of space

which have finite box scales for the scaling $\mathsf{scl}^{2,2}$ as defined in Proposition 2.4. Consider the space A of holomorphic functions on the disk $\mathbb{D}(R) \subset \mathbb{C}$ of radius $R > 1$ which are uniformly bounded by 1:

$$A = \left\{ \phi = \sum_{n \geq 0} a_n z^n \in C^\omega(\mathbb{D}(R), \mathbb{C}) : \sup_{\mathbb{D}(R)} |\phi| \leq 1 \right\} \text{endowed with the norm} \|\phi\|_\infty := \sup_{z \in \mathbb{D}(1)} |\phi(z)| \ .$$

The following implies:

$$\underline{\mathsf{scl}}_B^{2,2} A = \overline{\mathsf{scl}}_B^{2,2} A = 2 \ .$$

Theorem 2.5 (Kolmogorov, Tikhomirov [KT93][Equality 129])**.** *The following estimation on the covering number of* $(A, \| \cdot \|_\infty)$ *holds:*

$$\log \mathcal{N}_\epsilon(A) = (\log R)^{-1} \cdot |\log \epsilon|^2 + O(\log \epsilon^{-1} \cdot \log \log \epsilon^{-1}), \ \textit{when } \epsilon \textit{ tends to } 0. \ .$$

Let us now prove Proposition 2.4:

Proof of Proposition 2.4. First it is clear that $\mathsf{scl}^{p,q}$ is a family of non-decreasing functions. Moreover the family is non-increasing. We prove the following below:

Lemma 2.6. *For any* $\gamma > 0$ *and* $\nu > 1$ *close to* 1*, it holds for* $\epsilon > 0$ *small:*

$$\mathrm{scl}^{p,q}_{\nu \cdot \gamma}(\epsilon) \leq \mathrm{scl}^{p,q}_{\gamma}(\epsilon^\nu) \quad \textit{and} \quad \mathrm{scl}^{p,q}_{\nu \cdot \gamma}(\epsilon) \leq \mathrm{scl}^{p,q}_{\gamma}(\epsilon)^\nu \ ,$$

when $\epsilon \to 0$.

Let us show how this lemma implies condition $(*)$ in Definition 1.2 of scaling and thus the result of the proposition. For $\alpha > \beta > 0$, consider $\lambda > 1$ such that $\alpha > \lambda^2 \cdot \beta$, since the family is non-increasing, it holds:

$$\mathrm{scl}^{p,q}_{\alpha} \leq \mathrm{scl}^{p,q}_{\lambda^2 \cdot \beta} \ .$$

Now by the above lemma, it holds for $\epsilon > 0$ small:

$$\mathrm{scl}^{p,q}_{\lambda^2 \cdot \beta}(\epsilon) \leq \mathrm{scl}^{p,q}_{\lambda \cdot \beta}(\epsilon^\lambda) \leq \left(\mathrm{scl}^{p,q}_{\beta}(\epsilon^\lambda) \right)^\lambda \quad \text{and} \quad \mathrm{scl}^{p,q}_{\lambda^2 \cdot \beta}(\epsilon) \leq \left(\mathrm{scl}^{p,q}_{\beta}(\epsilon) \right)^{\lambda^2} \ .$$

Thus it comes:

$$\frac{\mathrm{scl}^{p,q}_{\alpha}(\epsilon)}{\mathrm{scl}^{p,q}_{\beta}(\epsilon^\lambda)} \leq \left(\mathrm{scl}^{p,q}_{\beta}(\epsilon^\lambda) \right)^{\lambda - 1} \xrightarrow[\epsilon \to 0]{} 0 \quad \text{and} \quad \frac{\mathrm{scl}^{p,q}_{\alpha}(\epsilon)}{\mathrm{scl}^{p,q}_{\beta}(\epsilon)^\lambda} \leq \left(\mathrm{scl}^{p,q}_{\beta}(\epsilon) \right)^{\lambda(\lambda-1)} \xrightarrow[\epsilon \to 0]{} 0 \ ;$$

which allows to conclude the proof of the proposition. It remains to show the above lemma. The following can be proved recursively on $d \geq 1$:

Fact 2.7. *For any* $\nu > 1$*, for any* $d \geq 1$ *and for* $y > 0$ *great enough, it holds:*

$$\log^{\circ d}(y^\nu) \leq \nu \cdot \log^{\circ d}(y) \ .$$

Using Fact 2.7 with $d = q$ and $y = \epsilon^{-1}$ for $\epsilon > 0$ small gives:

$$\gamma \cdot \log^{\circ q}(\epsilon^{-\nu}) \leq \lambda \cdot \gamma \cdot \log^{\circ q}(\epsilon^{-1}) \ .$$

Since $t \mapsto \exp^{\circ p}(t)^{-1}$ is decreasing, it comes:

$$\mathsf{scl}^{p,q}_{\nu\cdot\gamma}(\epsilon) \leq \mathsf{scl}^{p,q}_{\gamma}(\epsilon^{\nu}) \ ,$$

which gives the first inequality in the lemma.

Moreover by Fact 2.7 with $d = p$ and y great enough, it holds:

$$\log^{\circ p}(y^{\nu}) \leq \nu \cdot \log^{\circ p}(y) \ .$$

Applying $\exp^{\circ p}$ to both sides gives:

$$y^{\nu} \leq \exp^{\circ p}(\nu \cdot \log^{\circ p}(y)) \ .$$

Now with $y^{-1} = \mathrm{scl}_{\gamma}(\epsilon)$, we have:

$$\exp^{\circ p}(\nu \cdot \log^{\circ p}(y)) = (\mathrm{scl}_{\nu\cdot\gamma}(\epsilon))^{-1} \ .$$

From there we obtain:

$$\mathrm{scl}_{\nu\cdot\gamma}(\epsilon) \leq \mathrm{scl}_{\gamma}(\epsilon)^{\nu} \ ,$$

which is the remaining inequality in the lemma. □

2.2 Box scales

As we introduced in Definition 1.5, *lower and upper box scales* of a metric space (X, d) are defined by:

$$\underline{\mathsf{scl}}_B X = \sup\left\{\alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} +\infty\right\} \quad \text{and} \quad \overline{\mathsf{scl}}_B X = \inf\left\{\alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} 0\right\} \ ,$$

where the covering number $\mathcal{N}_\epsilon(X)$ is the minimal cardinality of a covering of X by balls with radius $\epsilon > 0$.

In general, the upper and lower box scales must not coincide, we give new examples for order in Example 4.6. Now we give a few properties of box scales that are well known in the specific case of dimension.

Fact 2.8. *Let (X, d) be a metric space. The following properties hold true:*

1. *if $\underline{\mathsf{scl}}_B(X) < +\infty$, then (X, d) is totally bounded,*
2. *for any subset $E \subset X$ it holds $\underline{\mathsf{scl}}_B E \leq \underline{\mathsf{scl}}_B X$ and $\overline{\mathsf{scl}}_B E \leq \overline{\mathsf{scl}}_B X$,*
3. *for any subset E of X it holds $\underline{\mathsf{scl}}_B E = \underline{\mathsf{scl}}_B \mathrm{cl}(E)$ and $\overline{\mathsf{scl}}_B E = \overline{\mathsf{scl}}_B \mathrm{cl}(E)$.*

1. and 2. are straightforward. To see 3. it is enough to observe that $\mathcal{N}_\epsilon(E) \leq \mathcal{N}_\epsilon(\mathrm{cl}(E)) \leq \mathcal{N}_{\epsilon/2}(E)$ for any $\epsilon > 0$.

Box scales are sometimes easier to compare with other scales using packing number:

Definition 2.9 (Packing number). *For $\epsilon > 0$ let $\tilde{\mathcal{N}}_\epsilon(X)$ be the packing number of the metric space (X, d). It is the maximum cardinality of an ϵ-separated set of points in X for the distance d:*

$$\tilde{\mathcal{N}}_\epsilon(X) = \sup\{N \geq 0 : \exists x_1, \dots x_N \in X, d(x_i, x_j) \geq \epsilon \text{ for any } 1 \leq i < j \leq N \} .$$

A well know comparison between packing and covering number is the following:

Lemma 2.10. *Let (X, d) be a metric space. For any $\epsilon > 0$, it holds:*

$$\tilde{\mathcal{N}}_{2\epsilon}(X) \leq \mathcal{N}_\epsilon(X) \leq \tilde{\mathcal{N}}_\epsilon(X) .$$

A direct application of this lemma is that we can replace the covering number by the packing number in the definitions of box scales without changing their values:

Lemma 2.11. *Let (X, d) be a metric space and scl a scaling, then box scales of X can be written as:*

$$\underline{\mathsf{scl}}_B X = \sup\left\{\alpha > 0 : \tilde{\mathcal{N}}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} +\infty\right\} \quad \text{and} \quad \overline{\mathsf{scl}}_B X = \inf\left\{\alpha > 0 : \tilde{\mathcal{N}}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} 0\right\} .$$

Proof. Since for any $\epsilon > 0$ it holds by Lemma 2.10:

$$\tilde{\mathcal{N}}_{2\epsilon}(X) \leq \mathcal{N}_\epsilon(X) \leq \tilde{\mathcal{N}}_\epsilon(X) ,$$

we obtain the sought result by Lemma 2.2. □

Remark 2.12. *Another property for the scaling $\mathsf{scl}^{p,q}$ from Proposition 2.4, with $p, q \geq 1$, is that the upper and lower box scales for a metric space (X, d) can be written as:*

$$\underline{\mathsf{scl}}_B^{p,q}(X) = \liminf_{\epsilon\to 0} \frac{\log^{\circ p}(\mathcal{N}_\epsilon(X))}{\log^{\circ q}(\epsilon^{-1})}$$

and

$$\overline{\mathsf{scl}}_B^{p,q}(X) = \limsup_{\epsilon\to 0} \frac{\log^{\circ p}(\mathcal{N}_\epsilon(X))}{\log^{\circ q}(\epsilon^{-1})} .$$

In particular, for dimension and order:

$$\underline{\dim}_B(X) = \liminf_{\epsilon\to 0} \frac{\log(\mathcal{N}_\epsilon(X))}{\log(\epsilon^{-1})} \quad , \quad \overline{\dim}_B(X) = \limsup_{\epsilon\to 0} \frac{\log(\mathcal{N}_\epsilon(X))}{\log(\epsilon^{-1})} .$$

and

$$\underline{\mathsf{ord}}_B(X) = \liminf_{\epsilon\to 0} \frac{\log\log(\mathcal{N}_\epsilon(X))}{\log(\epsilon^{-1})} \quad , \quad \overline{\mathsf{ord}}_B(X) = \limsup_{\epsilon\to 0} \frac{\log\log(\mathcal{N}_\epsilon(X))}{\log(\epsilon^{-1})} .$$

The above equalities coincide with the usual definitions of box dimensions and orders.

2.3 Hausdorff scales

The definition of Hausdorff scales, generalizing Hausdorff dimension, is introduced here using the definition of Hausdorff outer measure as given by Tricot in [Tri82]. We still consider a metric space (X, d). Given an increasing function $\phi \in C(\mathbb{R}_+^*, \mathbb{R}_+^*)$, such that $\phi(\epsilon) \to 0$ when $\epsilon \to 0$, we define:

$$\mathcal{H}_\epsilon^\phi(X) := \inf_{J\text{countable}} \left\{ \sum_{j\in J} \phi(|B_j|) : X = \bigcup_{j\in J} B_j,\ \forall j \in J : |B_j| \leq \epsilon \right\} ,$$

where $|B|$ is the radius of a ball $B \subset X$. A countable family $(B_j)_{j\in J}$ of balls with radius at most $\epsilon > 0$ such that $X = \bigcup_{j\in J} B_j$ will be called an ϵ-*cover* of X. [2] Since the set of ϵ-cover is not decreasing for inclusion when ϵ decreases to 0, the following limit exists:

$$\mathcal{H}^{\phi}(X) := \lim_{\epsilon\to 0} \mathcal{H}^{\phi}_{\epsilon}(X) \ .$$

Now replacing (X, d) in the previous definitions by any subset E of X endowed with the same metric d, we observe that $\mathcal{H}^{\phi}$ defines an outer-measure on X. We now introduce the following:

Definition 2.13 (Hausdorff scale)**.** *The* Hausdorff scale *of a metric space (X, d) is defined by:*

$$\mathsf{scl}_H X = \sup\left\{\alpha > 0 : \mathcal{H}^{\mathrm{scl}_\alpha}(X) = +\infty\right\} = \inf\left\{\alpha > 0 : \mathcal{H}^{\mathrm{scl}_\alpha}(X) = 0\right\} \ .$$

Note that the above definition gives us two quantities on the right hand side that are a priori not equal. However, the mild assumptions in the definition of scaling allow to verify that they indeed coincide.

Proof of the equality in Definition 2.13. It is clear from definition that $\alpha \mapsto \mathcal{H}^{\mathrm{scl}_\alpha}(X)$ is non-increasing. It is then enough to check that if there exists $\alpha > 0$ such that $0 < \mathcal{H}^{\mathrm{scl}_\alpha}(X) < +\infty$ then, for any positive $\delta < \alpha$, it holds:

$$\mathcal{H}^{\mathrm{scl}_{\alpha+\delta}}(X) = 0 \quad \text{and} \quad \mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}(X) = +\infty \ .$$

Let us fix $\eta > 0$, by Definition 1.2 , for $\epsilon > 0$ small it holds:

$$\mathrm{scl}_{\alpha+\delta}(\epsilon) \le \eta \cdot \mathrm{scl}_\alpha(\epsilon) \quad \text{and} \quad \mathrm{scl}_\alpha(\epsilon) \le \eta \cdot \mathrm{scl}_{\alpha-\delta}(\epsilon) \ .$$

Thus by the definition of Hausdorff measure, since ϵ is small, it holds:

$$0 < \frac{1}{2}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \le \mathcal{H}^{\mathrm{scl}_\alpha}_{\epsilon}(X) \le \mathcal{H}^{\mathrm{scl}_\alpha}(X) < +\infty \ .$$

Given $(B_j)_{j\in J}$ an ϵ-cover of X, the following holds:

$$\frac{1}{2}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \le \mathcal{H}^{\mathrm{scl}_\alpha}_{\epsilon}(X) \le \sum_{j\in J} \mathrm{scl}_\alpha(|B_j|) \ ,$$

and then:

$$\frac{1}{2\eta}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \le \frac{1}{\eta}\sum_{j\in J} \mathrm{scl}_\alpha(|B_j|) \le \sum_{j\in J} \mathrm{scl}_{\alpha-\delta}(|B_j|) \ .$$

Since this holds for any ϵ-cover, the latter inequality leads to:

$$\frac{1}{2\eta}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \le \mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}_{\epsilon}(X) \ ,$$

and so:

$$\frac{1}{2\eta}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \le \mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}(X) \ .$$

[2] Note that the historical construction of the Hausdorff measures uses subsets of X with diameter at most ϵ instead of the balls with radius at most ϵ. However both these constructions lead to the same definitions of Hausdorff scales.

On the other side, there exists an ϵ-cover $(B_j)_{j\in J}$ of E such that:

$$\sum_{j\in J} \mathrm{scl}_\alpha(|B_j|) \leq 2\mathcal{H}_\epsilon^{\mathrm{scl}_\alpha}(X) \ .$$

Now since $\mathcal{H}_\epsilon^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}^{\mathrm{scl}_\alpha}(X)$, this leads to:

$$\sum_{j\in J} \mathrm{scl}_{\alpha+\delta}(|B_j|) \leq \eta \cdot \sum_j \mathrm{scl}_\alpha(|B_j|) \leq 2\eta \cdot \mathcal{H}^{\mathrm{scl}_\alpha}(X) \ .$$

From there:

$$\mathcal{H}_\epsilon^{\mathrm{scl}_{\alpha+\delta}}(X) \leq 2\eta \cdot \mathcal{H}^{\mathrm{scl}_\alpha}(X) \ ,$$

and this holds for any small ϵ. We have just shown:

$$\frac{1}{2\eta}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}(X) \quad \text{and} \quad \mathcal{H}_\epsilon^{\mathrm{scl}_{\alpha+\delta}}(X) \leq 2\eta \cdot \mathcal{H}^{\mathrm{scl}_\alpha}(X) \ .$$

Since η can be arbitrarily close to 0, it follows that $\mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}(X) = +\infty$ and $\mathcal{H}^{\mathrm{scl}_{\alpha+\delta}}(X) = 0$, which concludes the proof. □

As box scales, Hausdorff scales are increasing for inclusion. We show a stronger property of Hausdorff scales in Lemma 2.20.

2.4 Packing scales

2.4.1 Packing scales through modified box scales

The original construction of packing dimension relies on the packing measure introduced by Tricot in [Tri82]. We first define packing scales by modifying upper box scales and we show then later how it is related to packing measures.

Definition 2.14 (Packing scale)**.** *Let (X,d) be a metric space and* scl *a scaling. The* packing scale *of X is defined by:*

$$\mathsf{scl}_P X = \inf\left\{ \sup_{n\geq 1} \overline{\mathsf{scl}}_B E_n : (E_n)_{n\geq 1} \subset X^{\mathbb{N}} \text{ s.t. } \bigcup_{n\geq 1} E_n = X \right\} \ .$$

The following comes directly from definition of packing scale:

Proposition 2.15. *Let (X,d) be a metric space and* scl *a scaling. It holds:*

$$\mathsf{scl}_P X \leq \overline{\mathsf{scl}}_B X \ .$$

2.4.2 Packing measures

In this paragraph we show the relationship between packing measures and packing scales. Let us first recall a few definitions.

Given $\epsilon > 0$, an *ϵ-pack* of a metric space (X, d) is a countable collection of disjoint balls of X with radii at most ϵ. As for Hausdorff outer measure, consider $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ an increasing function such that $\phi(\epsilon) \to 0$ when $\epsilon \to 0$. For $\epsilon > 0$, put:

$$\mathcal{P}_\epsilon^\phi(X) := \sup \left\{ \sum_{i \in I} \phi(|B_i|) : (B_i)_{i \in I} \text{ is an } \epsilon\text{-pack of } X \right\}.$$

Since $\mathcal{P}_\epsilon^\phi(X)$ is non-increasing when ϵ decreases to 0, the following quantity is well defined:

$$\mathcal{P}_0^\phi(X) := \lim_{\epsilon \to 0} \mathcal{P}_\epsilon^\phi(X).$$

The idea of Tricot is to build an outer measure from this quantity:

Definition 2.16 (Packing measure)**.** *For any subset E of X endowed with the same metric d, the* packing ϕ-measure *of E is defined by:*

$$\mathcal{P}^\phi(E) = \inf \left\{ \sum_{n \geq 1} \mathcal{P}_0^\phi(E_n) : E = \bigcup_{n \geq 1} E_n \right\}.$$

Note that $\mathcal{P}^\phi$ is an outer-measure on X and can eventually be infinite or null. The following shows the equivalence of Tricot's counterpart definition of the packing scale; this will be useful to show the equality between upper local scale and packing scale of a measure given by Theorem C eq. (c&g).

Proposition 2.17. *The packing scale of a metric space (X, d) verifies:*

$$\sup \left\{ \alpha > 0 : \mathcal{P}^{\mathrm{scl}_\alpha}(X) = +\infty \right\} = \mathsf{scl}_P X = \inf \left\{ \alpha > 0 : \mathcal{P}^{\mathrm{scl}_\alpha}(X) = 0 \right\}.$$

Proof. Let $(E_n)_{n \geq 1}$ be a family of subsets of X. Using that each map $\alpha \mapsto \mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n)$ is not increasing, we prove:

$$\inf \left\{ \alpha > 0 : \sum_{n \geq 1} \mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n) = 0 \right\} = \sup_{n \geq 1} \inf \left\{ \alpha > 0 : \mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n) = 0 \right\}. \tag{2.1}$$

We show below the following:

Lemma 2.18. *Given $\alpha > 0$, if $\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)$ is a finite and positive number then for any $\delta \in (0, \alpha)$, it holds:*

$$\mathcal{P}_0^{\mathrm{scl}_{\alpha+\delta}}(E) = 0 \quad \textit{and} \quad \mathcal{P}_0^{\mathrm{scl}_{\alpha-\delta}}(E) = +\infty.$$

The right hand side equality of the latter lemma implies:

$$\sup \left\{ \alpha > 0 : \sum_{n \geq 1} \mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n) = +\infty \right\} = \sup_{n \geq 1} \sup \left\{ \alpha > 0 : \mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n) = +\infty \right\}. \tag{2.2}$$

We now compare the right hand term using the following shown below:

Lemma 2.19. *For any $E \subset X$, it holds:*

$$\sup \left\{ \alpha > 0 : \mathcal{P}_0^{\mathrm{scl}_\alpha}(E) = +\infty \right\} = \overline{\mathsf{scl}}_B E = \inf \left\{ \alpha > 0 : \mathcal{P}_0^{\mathrm{scl}_\alpha}(E) = 0 \right\}.$$

Consequently by Eqs. (2.1) and (2.2) and Lemmas 2.18 and 2.19:

$$\sup\left\{\alpha>0:\sum_{n\geq 1}\mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n)=+\infty\right\}=\sup_{n\geq 1}\overline{\mathsf{scl}}_B E_n=\inf\left\{\alpha>0:\sum_{n\geq 1}\mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n)=0\right\}\ .$$

Taking the infimum over families $(E_n)_{n\geq 1}$ which covers X we obtain the sought result. □

Proof of Lemma 2.18. Given $\eta>0$, by Definition 1.2 of scaling, for $\epsilon>0$ small enough, it holds:

$$\mathrm{scl}_{\alpha+\delta}(\epsilon)\leq\eta\cdot\mathrm{scl}_\alpha(\epsilon)\quad\text{and}\quad\mathrm{scl}_\alpha(\epsilon)\leq\eta\cdot\mathrm{scl}_{\alpha-\delta}(\epsilon)\ .$$

Moreover there exists $(B_j)_{j\geq 1}$ an ϵ-pack of E such that:

$$\frac{1}{2}\mathcal{P}_\epsilon^{\mathrm{scl}_\alpha}(E)\leq\sum_{j\geq 1}\mathrm{scl}_\alpha(|B_j|)\ .$$

Combining the two inequalities above leads to:

$$\frac{1}{2}\mathcal{P}_\epsilon^{\mathrm{scl}_\alpha}(E)\leq\eta^{-1}\cdot\sum_{j\geq 1}\mathrm{scl}_{\alpha-\delta}(|B_j|)\leq\mathcal{P}_\epsilon^{\mathrm{scl}_{\alpha-\delta}}(E)\ .$$

Taking the limit when ϵ tends to 0 gives $\frac{1}{2}\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)\leq\mathcal{P}_0^{\mathrm{scl}_{\alpha-\delta}}(E)$. On the other side consider $(B_j)_{j\geq 1}$ an ϵ-pack of E. It holds:

$$\sum_{j\geq 1}\mathrm{scl}_\alpha(|B_j|)\leq\mathcal{P}_\epsilon^{\mathrm{scl}_\alpha}(E)\leq\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)\ ,$$

moreover it holds:

$$\eta\cdot\sum_{j\geq 1}\mathrm{scl}_{\alpha+\delta}(|B_j|)\leq\sum_{j\geq 1}\mathrm{scl}_\alpha(|B_j|)\ .$$

Since this holds true for any ϵ-cover and $\epsilon>0$ arbitrary small, it follows:

$$\mathcal{P}_0^{\mathrm{scl}_{\alpha+\delta}}(E)\leq\eta\cdot\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)\ .$$

By taking η arbitrarily small, it comes $\mathcal{P}^{\mathrm{scl}_{\alpha-\delta}}(E)=+\infty$ and $\mathcal{P}^{\mathrm{scl}_{\alpha+\delta}}(E)=0$. □

Proof of Lemma 2.19. By Lemma 2.18, it suffices to show that:

$$\sup\left\{\alpha>0:\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)=+\infty\right\}\leq\overline{\mathsf{scl}}_B E\leq\inf\left\{\alpha>0:\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)=0\right\}\ ,\tag{2.3}$$

Consider $\alpha>0$ such that $\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)=0$. Then for $\epsilon>0$ sufficiently small it holds $\mathcal{P}_\epsilon^{\mathrm{scl}_\alpha}(E)\leq 1$. In particular the packing number (see def. 2.9) satisfies $\tilde{\mathcal{N}}_\epsilon(E)\cdot\mathrm{scl}_\alpha(\epsilon)<1$. Taking the limit when ϵ tends to 0 leads to $\overline{\mathsf{scl}}_B E\leq\alpha$ by Lemma 2.11. This proves the right hand side of Eq. (2.3).

To show the left hand side inequality, it suffices to show that $\overline{\mathsf{scl}}_B E$ is at least any α such that $\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)=+\infty$. For such an α, given $\epsilon>0$, there exists an ϵ-pack $(B_j)_{j\geq 1}$ such that:

$$\sum_{j\geq 1}\mathrm{scl}_\alpha(|B_j|)>1\ .$$

For $k \geq 1$ an integer, put:

$$n_k := \mathrm{Card}\left\{ j \geq 1 : 2^{-(k+1)} \leq \mathrm{scl}_\alpha^{-1}(|B_j|) < 2^{-k} \right\} .$$

Thus, since scl_α is not decreasing, it holds:

$$\sum_{k \geq 1} n_k \cdot 2^{-k} > 1 .$$

Since B_j has radius at most δ, we have $n_k = 0$ for any $k < -\log_2 \mathrm{scl}_\alpha(\delta)$. Then for $\delta > 0$ small, there exists an integer $j \geq 2$ such that $n_j > j^{-2} 2^j$. In fact, otherwise we would have:

$$\sum_{k \geq 1} n_k \cdot 2^{-k} \leq \sum_{k \geq 2} \frac{1}{k^2} < 1 ,$$

which contradicts the above inequality. Then E contains the centers of n_j disjoint balls with radii at least $\mathrm{scl}_\alpha^{-1}(2^{-(j+1)})$, in particular:

$$\tilde{\mathcal{N}}_{\mathrm{scl}_\alpha^{-1}(2^{-(j+1)})}(E) \geq n_j > j^{-2} 2^j ,$$

and moreover:

$$j \geq -\log_2 \mathrm{scl}_\alpha(\delta) .$$

Since this inequality holds true for δ arbitrarily small, there exists an increasing sequence of integers $(j_n)_{n \geq 1}$ such that:

$$\tilde{\mathcal{N}}_{\epsilon_n}(E) > j_n^{-2} 2^{j_n} ,$$

with $\epsilon_n = \mathrm{scl}_\alpha^{-1}(2^{-(j_n+1)})$. Let us consider a positive number $\beta < \alpha$, by Definition 1.2 of scaling, for $\lambda > 1$ close to 1, it holds:

$$\mathrm{scl}_\beta(\epsilon) \cdot (\mathrm{scl}_\alpha(\epsilon))^{-\lambda^{-1}} \xrightarrow[\epsilon \to 0]{} +\infty .$$

On the other hand, given a such $\lambda > 1$, for n large enough, it holds:

$$j_n^{-2} 2^{j_n} \geq 2^{\lambda^{-1}(j_n+1)} ,$$

it follows:

$$\tilde{\mathcal{N}}_{\epsilon_n}(E) \geq \left(2^{-(j_n+1)}\right)^{-\lambda^{-1}} = (\mathrm{scl}_\alpha(\epsilon_n))^{-\lambda^{-1}} .$$

Thus we finally have:

$$\mathrm{scl}_\beta(\epsilon_n) \cdot \tilde{\mathcal{N}}_{\epsilon_n}(E) > \mathrm{scl}_\beta(\epsilon) \cdot (\mathrm{scl}_\alpha(\epsilon))^{-\lambda^{-1}} \xrightarrow[\epsilon \to 0]{} +\infty .$$

By Lemma 2.11 we deduce $\overline{\mathrm{scl}}_B E \geq \alpha$. Since this holds true for β arbitrary close to α, it follows $\overline{\mathrm{scl}}_B E \geq \alpha$. □

The following is similar to the proof Definition 3.1 of Hausdorff scales.

2.5 Properties and comparison of scales of metric spaces

We first give a few basic properties of scales that would allow to compare them. Since both packing and Hausdorff scales are defined via measures, they both are countable stable as shown in the following:

Lemma 2.20 (Countable stability)**.** *Let (X,d) be a metric space. Let I be a countable set and $(E_i)_{i\geq 1}$ a covering of X, then for any scaling* scl*:*

$$\mathsf{scl}_H X = \sup_{i\geq 1} \mathsf{scl}_H E_i \quad \textit{and} \quad \mathsf{scl}_P X = \sup_{i\geq 1} \mathsf{scl}_P E_i \ .$$

Proof. The equality on packing scales is obvious by definition with modified upper box scales. Let us prove the equality on Hausdorff scales. By monotonicity of the Hausdorff measure, it holds $\mathsf{scl}_H X \geq \sup_{i\geq 1} \mathsf{scl}_H E_i$. For the reverse inequality, consider $\alpha > \sup_{i\geq 1} \mathsf{scl}_H E_i$, then for any $i \geq 1$ it holds $\mathcal{H}^{\mathrm{scl}_\alpha}(E_i) = 0$. Thus:

$$\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \sum_{i\geq 1} \mathcal{H}^{\mathrm{scl}_\alpha}(E_i) = 0 \ ,$$

and then $\mathsf{scl}_H X \leq \alpha$. Since this is true for any $\alpha > \sup_{i\geq 1} \mathsf{scl}_H E_i$, the sought result comes. □

Note that countable stability is not a property of box scales. To see that, it suffices to consider a countable dense subset of a metric space (X,d) with positive box scales.

The following lemma shows in particular that the above scales are bi-Lipschitz invariant quantities.

Lemma 2.21. *Let (X,d) and (Y,d) be two metric spaces such that there exists a Lipschitz map $f:(X,d)\to(Y,d)$. Then for any scaling* scl*, the scales of $f(X)$ are at most the ones of X:*

$$\mathsf{scl}_H f(X) \leq \mathsf{scl}_H X; \quad \mathsf{scl}_P f(X) \leq \mathsf{scl}_P X; \quad \underline{\mathsf{scl}}_B f(X) \leq \underline{\mathsf{scl}}_B X; \quad \overline{\mathsf{scl}}_B f(X) \leq \overline{\mathsf{scl}}_B X \ .$$

We prove this lemma below. As a direct application, we obtain the following:

Corollary 2.22. *Let (X,d) and (Y,d) be two metric spaces. Suppose there exist an embedding $g:(Y,\delta)\to(X,d)$ such that g^{-1} is Lipschitz on $g(X)$. Then for any scaling* scl*, the scales of Y are at most the ones of X:*

$$\mathsf{scl}_H Y \leq \mathsf{scl}_H X; \quad \mathsf{scl}_P Y \leq \mathsf{scl}_P Y; \quad \underline{\mathsf{scl}}_B Y \leq \underline{\mathsf{scl}}_B X; \quad \overline{\mathsf{scl}}_B Y \leq \overline{\mathsf{scl}}_B X \ .$$

Proof of Corollary 2.22. By Lemma 2.21 we have $\mathsf{scl}_\bullet Y \leq \mathsf{scl}_\bullet g(Y)$ for any $\mathsf{scl}_\bullet \in \{\mathsf{scl}_H, \mathsf{scl}_P\underline{\mathsf{scl}}_B, \overline{\mathsf{scl}}_B\}$. As $g(Y)\subset X$, we have also $\mathsf{scl}_\bullet g(Y) \leq \mathsf{scl}_\bullet X$. □

Proof of Lemma 2.21. Let us fix $\epsilon > 0$. Suppose that f is K- Lipschitz for a constant $K > 0$. We first show the inequalities on box and packing scales. Consider a finite covering by a collection of balls $(B(x_j,\epsilon))_{1\leq j\leq N}$ where $x_j \in X$ for any $1\leq j\leq N$ and $N = \mathcal{N}_\epsilon(X)$. Since $X = \bigcup_{j=1}^N B(x_j,\epsilon)$, it comes:

$$f(X) \subset f\left(\bigcup_{j=1}^N B(x_j,\epsilon_j)\right) \subset \bigcup_{j=1}^N B(f(x_j), K\cdot\epsilon_j) \ .$$

Then $(B(f(x_j),\epsilon))_{1\leq j\leq N}$ is a covering by $K\cdot\epsilon$-balls of $f(X)$. Then $\mathcal{N}_{K\cdot\epsilon}(f(X))\leq\mathcal{N}_\epsilon(X)$ and all the inequalities on the box and packing scales are immediately deduced. Now for Hausdorff scales, consider a countable set J and $\{B(x_j,\epsilon_j):j\in J\}$ an ϵ-cover of X. Then it comes:

$$f(X)\subset\bigcup_{j\in J}B(f(x_j),K\cdot\epsilon_j)\ .$$

For any $\alpha>\beta>0$ and $\delta>0$ small enough, by Fact 2.1 , it holds:

$$\mathrm{scl}_\alpha(\delta)\leq\mathrm{scl}_\beta(K^{-1}\cdot\delta)\ .$$

Hence for ϵ small, it holds:

$$\mathcal{H}^{\mathrm{scl}_\alpha}_{K\cdot\epsilon}(f(X))\leq\sum_{j\in J}\mathrm{scl}_\alpha(K\cdot\epsilon_j)\leq\sum_{j\in J}\mathrm{scl}_\beta(\epsilon_j)\ .$$

As $\beta>\mathsf{scl}_H X$, the ϵ-cover $(B(x_j,\epsilon_j))_{j\in J}$ can be chosen such that $\sum_{j\in J}\mathrm{scl}_\beta(\epsilon_j)$ is arbitrary small. Thus $\mathcal{H}^{\mathrm{scl}_\alpha}_{K\cdot\epsilon}(f(X))=0$, and so $\mathsf{scl}_H f(X)\leq\alpha$. As α is arbitrary close to $\mathsf{scl}_H X$, it holds:

$$\mathsf{scl}_H f(X)\leq\mathsf{scl}_H X\ .$$

□

The end of this section consists of comparing the different scales introduced and prove Theorem A. We start by comparing the Hausdorff and lower box scale. The following proposition generalizes well known facts on dimension. See e.g. [Fal04][(3.17)].

Proposition 2.23. *Let (X,d) be a metric space and* scl *a scaling, its Hausdorff scale is at most its lower box scale:*

$$\mathsf{scl}_H X\leq\underline{\mathsf{scl}}_B X\ .$$

Proof. We can assume without any loss of generality that (X,d) is totally bounded. If $\mathsf{scl}_H X=0$ the inequality obviously holds, thus consider a positive number $\alpha<\mathsf{scl}_H X$. For $\delta>0$ small enough, $\mathcal{H}^{\mathrm{scl}_\alpha}_\delta(X)>1$. Thus there exist δ -covers and for any δ-cover $(B_j)_{1\leq j\leq\mathcal{N}_\delta(X)}$, it holds:

$$1<\sum_{1\leq j\leq\mathcal{N}_\delta(F)}\mathrm{scl}_\alpha(|B_j|)=\mathcal{N}_\delta(X)\cdot\mathrm{scl}_\alpha(\delta)\ .$$

From there, it holds $\underline{\mathsf{scl}}_B X\geq\alpha$. We conclude by taking α arbitrarily close to $\mathsf{scl}_H X$. □

We have compared Hausdorff and packing scales with their corresponding box scales. It remains to compare each other with the following:

Proposition 2.24. *Let (X,d) be a metric space and* scl *a scaling. It holds:*

$$\mathsf{scl}_H X\leq\mathsf{scl}_P X\ .$$

Proof. By Lemma 2.20, it holds:

$$\mathsf{scl}_H X = \inf_{\bigcup_{n\geq 1} E_n = X} \sup_{n\geq 1} \mathsf{scl}_H E_n \ ,$$

where the infimum is taken over countable coverings of X. Moreover by Proposition 2.23, we have:

$$\mathsf{scl}_H E \leq \underline{\mathsf{scl}}_B E \leq \overline{\mathsf{scl}}_B E \ ,$$

for any subset E of X. It follows then:

$$\mathsf{scl}_H X \leq \inf_{\bigcup_{n\geq 1} E_n = X} \sup_{n\geq 1} \overline{\mathsf{scl}}_B E_n = \mathsf{scl}_P X \ .$$

□

To conclude this section we now shall prove Theorem A:

Proof of Theorem A. Let (X, d) be a metric space and scl a scaling. By Proposition 2.23, Proposition 2.24 and Proposition 2.15, it holds respectively:

$$\mathsf{scl}_H X \leq \underline{\mathsf{scl}}_B X, \quad \mathsf{scl}_H X \leq \mathsf{scl}_P X \quad \text{and} \quad \mathsf{scl}_H X \leq \overline{\mathsf{scl}}_B X \ .$$

Now since $\underline{\mathsf{scl}}_B X \leq \overline{\mathsf{scl}}_B X$ obviously holds, we deduce the sought result:

$$\mathsf{scl}_H X \leq \mathsf{scl}_P X \leq \overline{\mathsf{scl}}_B X \quad \text{and} \quad \mathsf{scl}_H X \leq \underline{\mathsf{scl}}_B X \leq \overline{\mathsf{scl}}_B X \ .$$

□

3 Scales of measures

In this section we recall the different versions of scales of measures we introduced and show the inequalities and equalities comparing them. In particular we provide proofs of Theorem B and Theorem C. They generalize known facts of dimension theory to any scaling and moreover bring new comparisons (see Theorem 3.10) between quantization and box scales that were not shown yet for even for the case of dimension.

3.1 Hausdorff, packing and local scales of measures

Let us recall the definition of local scales. Let μ be a Borel measure on a metric space (X, d) and scl a scaling. The *lower and upper scales* of μ are the functions that map a point $x \in X$ to:

$$\underline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) = \sup\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \xrightarrow[\epsilon\to 0]{} 0\right\} \quad \text{and} \quad \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) = \inf\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \xrightarrow[\epsilon\to 0]{} +\infty\right\} \ .$$

We shall compare local scales with the followings:

Definition 3.1 (Hausdorff scales of a measure)**.** *Let* scl *be a scaling and* μ *a non-null Borel measure on a metric space* (X, d)*. We define* Hausdorff and $*$-Hausdorff scales *of the measure* μ *by:*

$$\mathsf{scl}_H\mu = \inf_{E\in\mathcal{B}} \{\mathsf{scl}_H E : \mu(E) > 0\} \quad \textit{and} \quad \mathsf{scl}^*_H\mu = \inf_{E\in\mathcal{B}} \{\mathsf{scl}_H E : \mu(X\backslash E) = 0\} \ ,$$

where $\mathcal{B}$ *is the set of Borel subsets of* X*.*

Definition 3.2 (Packing scales of a measure)**.** *Let* scl *be a scaling and* μ *a non-null Borel measure on a metric space* (X, d)*. We define* packing and $*$-packing scales *of* μ *by:*

$$\mathsf{scl}_P\mu = \inf_{E\in\mathcal{B}} \{\mathsf{scl}_P E : \mu(E) > 0\} \quad \textit{and} \quad \mathsf{scl}^*_P\mu = \inf_{E\in\mathcal{B}} \{\mathsf{scl}_P E : \mu(X\backslash E) = 0\} \ .$$

Remark 3.3. *In order to avoid excluding the null measure* 0*, we set* $\mathsf{scl}_H 0 = \mathsf{scl}^*_H 0 = \mathsf{scl}_P 0 = \mathsf{scl}^*_P 0 = 0$*.*

The lemma below will allows to compare local scales with the other scales of measures.

Lemma 3.4. *Let* μ *be a Borel measure on* X*. Then for any Borel subset* F *of* X *such that* $\mu(F) > 0$*, the restriction* σ *of* μ *to* F *verifies:*

$$\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \quad \textit{and} \quad \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma.$$

Moreover, if there exists $\alpha > 0$ *such that* $F \subset \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) > \alpha\right\}$*, it holds then:*

$$\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \geq \alpha \ ,$$

and similarly if $F \subset \{x \in X : \underline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) > \alpha\}$*, it holds:*

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \geq \alpha \ .$$

Proof. Consider a point $x \in X$, then for any $\epsilon > 0$, one has $\sigma(B(x,\epsilon)) \leq \mu(B(x,\epsilon))$, thus by definition of local scales:

$$\overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \quad \text{and} \quad \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \ .$$

Now if there exists $\alpha > 0$ such that $F \subset \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) > \alpha\right\}$, as $\overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) \geq \alpha$ for μ-almost every x in F, it comes by the above inequality that $\overline{\mathsf{scl}}_{\mathrm{loc}}\sigma(x) \geq \alpha$ for μ-almost every x in F, and thus for σ-almost every $x \in X$. It follows $\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \geq \alpha$. And the same holds for lower local scales. □

The following is a first step in the proof of Theorem B. We prove this lemma later. We first use it to prove Theorem C.

Lemma 3.5. *Let* (X, d) *be a metric space and* μ *a Borel measure on* X*. Let* scl *be a scaling. The lower and upper local scales of* μ *are respectively not greater than the Hausdorff and packing scales of the space* X*:*

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_H X \quad \textit{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P X \ .$$

Remark 3.6. *Note that in the above we can replace* X *by any subset of* X *with total mass, this leads to:*

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}^*_H\mu \quad \textit{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P\mu^* \ .$$

3.2 Quantization and box scales of measures

Let us first recall the definition of quantization scales. Let (X,d) be a metric space and μ a Borel measure on X. The *quantization number* $\mathcal{Q}_\mu$ of μ is the function that maps $\epsilon>0$ to the minimal cardinality of a set of points that is on average ϵ-close to any point in X:

$$\mathcal{Q}_\mu(\epsilon)=\inf\left\{N\geq 0:\exists\,\{c_i\}_{i=1,\dots,N}\subset X,\int_X d(x,\{c_i\}_{1\leq i\leq N})d\mu(x)\leq\epsilon\right\}\ .$$

Then *lower and upper quantization scales* of μ for a given scaling scl are defined by:

$$\underline{\mathsf{scl}}_Q\mu=\sup\left\{\alpha>0:\mathcal{Q}_\mu(\epsilon)\cdot\mathrm{scl}_\alpha(\epsilon)\underset{\epsilon\to 0}{\longrightarrow}+\infty\right\}\quad\text{and}\quad\overline{\mathsf{scl}}_Q\mu=\inf\left\{\alpha>0:\mathcal{Q}_\mu(\epsilon)\cdot\mathrm{scl}_\alpha(\epsilon)\underset{\epsilon\to 0}{\longrightarrow}0\right\}\ .$$

Quantization scales of a measure are compared in Theorem C with box scales:

Definition 3.7 (Box scales of a measure)**.** *Let* scl *be a scaling and* μ *a positive Borel measure on a metric space* (X,d)*. We define the* lower box scale *and the* $*$-lower box scale *of* μ *by:*

$$\underline{\mathsf{scl}}_B\mu=\inf_{E\in\mathcal{B}}\{\underline{\mathsf{scl}}_BE:\mu(E)>0\}\quad\textit{and}\quad\underline{\mathsf{scl}}^*_B\mu=\inf_{E\in\mathcal{B}}\{\underline{\mathsf{scl}}_BE:\mu(X\backslash E)=0\}\ .$$

Similarly, we define the upper box scale *and the* $*$-upper box scale *of* μ *by:*

$$\overline{\mathsf{scl}}_B\mu=\inf_{E\in\mathcal{B}}\left\{\overline{\mathsf{scl}}_BE:\mu(E)>0\right\}\quad\textit{and}\quad\overline{\mathsf{scl}}^*_B\mu=\inf_{E\in\mathcal{B}}\left\{\overline{\mathsf{scl}}_BE:\mu(X\backslash E)=0\right\}\ ,$$

where $\mathcal{B}$ *is the set of Borel subsets of* X*.*

As for Hausdorff scales of measures we chose that all box scales of the null measure are equal to 0 as a convention. The following is straightforward:

Lemma 3.8. *Let* (X,d) *be a metric space and* μ *a Borel measure on* X*. Given* scl *a scaling, it holds:*

$$\underline{\mathsf{scl}}_Q\mu\leq\underline{\mathsf{scl}}^*_B\mu\quad\textit{and}\quad\overline{\mathsf{scl}}_Q\mu\leq\overline{\mathsf{scl}}^*_B\mu\ .$$

Proof. We can assume without loss of generality that $\overline{\mathsf{scl}}^*_B\mu$ and $\underline{\mathsf{scl}}^*_B\mu$ are finite. Let E be a Borel set with total mass such that $\overline{\mathsf{scl}}_BE$ is finite, then E is totally bounded. Now for $\epsilon>0$, consider a covering by ϵ-balls centered at some points $x_1,\dots,x_N$ in E. Since $\mu(X\backslash E)=0$, it comes:

$$\int_X d(x,\{x_i\}_{1\leq i\leq N})d\mu(x)=\int_E d(x,\{x_i\}_{1\leq i\leq N})d\mu(x)\leq\epsilon\ .$$

Thus $\mathcal{Q}_\mu(\epsilon)\leq\mathcal{N}_\epsilon(E)$, and by Lemma 2.2:

$$\underline{\mathsf{scl}}_Q\mu\leq\underline{\mathsf{scl}}_BE\quad\text{and}\quad\overline{\mathsf{scl}}_Q\mu\leq\overline{\mathsf{scl}}_BE\ .$$

Since this holds true for any Borel set E with total mass, the sought results comes. □

The following lemma will allow to compare quantization scales with box scales.

Lemma 3.9. *Let μ be a Borel measure on (X, d) such that $\mathcal{Q}_\mu(\epsilon) < +\infty$ for any $\epsilon > 0$. Let us fix $\epsilon > 0$ and an integer $N \geq \mathcal{Q}_\mu(\epsilon)$. Thus consider $x_1, \ldots, x_N \in X$ such that:*

$$\int_X d(x, \{x_i\}_{1 \leq i \leq N}) d\mu(x) \leq \epsilon \ .$$

For any $r > 0$, with $E_r := \bigcup_{i=1}^N B(x_i, r)$, it holds:

$$\mu(X \backslash E_r) \leq \frac{\epsilon}{r} \ .$$

Proof. Since $X \backslash E_r$, the complement of E_r in X is the set of points with distance at most r from the set $\{x_1, \ldots, x_n\}$, it holds:

$$r \cdot \mu(X \backslash E_r) \leq \int_{X \backslash E_r} d(x, \{x_i\}_{1 \leq i \leq N}) d\mu(x) \leq \epsilon \ ,$$

which gives the sought result by dividing both sides by r. □

The following result exhibits the relationship between quantization scales and box scales. As far as we know, this result has not yet have been proved even for the specific case of dimension. It is a key element in the answer to Problem 1.14.

Theorem 3.10. *Let μ be a non null Borel measure on a metric space (X, d). For any scaling* scl*, there exists a Borel set F with positive mass such that:*

$$\underline{\mathsf{scl}}_B F \leq \underline{\mathsf{scl}}_Q \mu \quad \textit{and} \quad \overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \mu \ .$$

In particular, it holds:

$$\underline{\mathsf{scl}}_B \mu \leq \underline{\mathsf{scl}}_Q \mu \quad \textit{and} \quad \overline{\mathsf{scl}}_B \mu \leq \overline{\mathsf{scl}}_Q \mu \ .$$

Proof. If $\mathcal{Q}_\mu(\epsilon)$ is not finite for any $\epsilon > 0$, then $F = X$ satisfies the sought properties. Let us suppose now that the quantization number of μ is finite. Given an integer $n \geq 0$, we set $\epsilon_n := \exp(-n)$ and $r_n := n^2 \cdot \exp(-n) = n^2 \cdot \epsilon_n$. We also consider a finite set of points $C_n \subset X$ that contains exactly $\mathcal{Q}_\mu(\epsilon_n)$ points and such that:

$$\int_X d(x, C_n) d\mu(x) \leq \epsilon_n \ .$$

We can then consider the following set:

$$E_n := \bigcup_{c \in C_n} B(c, r_n) \ ,$$

then by Lemma 3.9, it holds:

$$\mu(X \backslash E_n) \leq \frac{\epsilon_n}{r_n} = \frac{1}{n^2} \ .$$

Thus, it holds:

$$\sum_{n \geq 0} \mu(X \backslash E_n) < +\infty \ .$$

By Borell-Cantelli lemma, we obtain:

$$\mu\left(\bigcup_{m\geq 0}\bigcap_{n\geq m} E_n\right) = \mu(X) > 0\ .$$

Thus there exists an integer $m \geq 0$ such that $\mu\left(\bigcap_{n\geq m} E_n\right) > 0$. We fix such an integer m and set $F := \bigcap_{n\geq m} E_n$. It remains to check that $\underline{\mathsf{scl}}_B F \leq \underline{\mathsf{scl}}_Q \mu$ and $\overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \mu$. By definition, one has $F \subset E_n$ for any $n \geq m$. Then since $F \subset E_n = \bigcup_{c\in C_n} B(c, r_n)$, it holds:

$$\mathcal{N}_{r_n}(F) \leq \operatorname{Card} C_n = \mathcal{Q}_\mu(\epsilon_n)$$

Since this holds true for any n greater than m, and since $\log r_n \sim \log \epsilon_n = -n$, we finally have by Lemmas 2.2 and 2.3 that:

$$\underline{\mathsf{scl}}_B F \leq \underline{\mathsf{scl}}_Q \mu \quad \text{and} \quad \overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \mu\ .$$

□

3.3 Comparison between local and global scales of measures and proof of Theorem C

By the latter theorem, to finish the proof of Theorem C, it remains only to show:

Theorem 3.11. *Let (X, d) be a separable metric space and μ a finite Borel measure on X. Let* scl *be a scaling. It holds:*

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}} \mu \leq \underline{\mathsf{scl}}_Q \mu \quad \textit{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}} \mu \leq \overline{\mathsf{scl}}_Q \mu\ .$$

Proof. We can suppose without any loss of generality that there exists $\alpha < \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}} \mu$ and $\beta < \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}} \mu$. We now set $E := \left\{x \in X : \underline{\mathsf{scl}}_{\mathrm{loc}} \mu(x) > \alpha \quad \text{and} \quad \overline{\mathsf{scl}}_{\mathrm{loc}} \mu(x) > \beta\right\}$. By definition of essential suprema, we have $\mu(E) > 0$. Thus the restriction σ of μ to E is a positive measure. Thus by Lemma 3.4 one has $\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}} \sigma \geq \alpha$ and $\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}} \sigma \geq \beta$. Moreover by Theorem 3.10, there is a Borel set $F \subset E$ with $\sigma(F) > 0$ an such that:

$$\underline{\mathsf{scl}}_B F \leq \underline{\mathsf{scl}}_Q \sigma \leq \underline{\mathsf{scl}}_Q \mu \quad \text{and} \quad \overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \sigma \leq \overline{\mathsf{scl}}_Q \mu\ .$$

Yet by Proposition 2.23 and Proposition 2.15, it holds respectively:

$$\mathsf{scl}_H F \leq \underline{\mathsf{scl}}_B F \quad \text{and} \quad \mathsf{scl}_P F \leq \overline{\mathsf{scl}}_B F\ .$$

Now, by setting τ the restriction of μ to F, Lemma 3.4 also gives:

$$\alpha \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}} \sigma \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}} \tau \quad \text{and} \quad \beta \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}} \sigma \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}} \tau\ .$$

By Lemma 3.5, it holds:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}} \tau \leq \mathsf{scl}_H F \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}} \tau \leq \mathsf{scl}_P F\ .$$

Finally, combining all the above inequalities leads to:

$$\alpha \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\tau \leq \mathsf{scl}_H F \leq \underline{\mathsf{scl}}_B F \leq \underline{\mathsf{scl}}_Q \mu$$

and

$$\beta \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\tau \leq \mathsf{scl}_P F \leq \overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \mu \ .$$

Since this holds true for any α and β arbitrarily close to $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$ and $\operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ we have the sought results. □

We shall now prove Theorem C.

Proof of Theorem C. By Theorem 3.10 it holds:

$$\underline{\mathsf{scl}}_B \mu \leq \underline{\mathsf{scl}}_Q \mu \quad \text{and} \quad \overline{\mathsf{scl}}_B \mu \leq \overline{\mathsf{scl}}_Q \mu \ .$$

By Theorem 3.11 and Lemma 3.8 it holds:

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \underline{\mathsf{scl}}_Q \mu \quad \text{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_Q \mu \ .$$

Thus it remains only to show:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \underline{\mathsf{scl}}_B \mu \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_B \mu \ .$$

Given E a subset of X with positive mass, we set σ the restriction of μ to E. By Lemma 3.4, it holds:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \ .$$

By Theorem 3.11 it holds:

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \leq \underline{\mathsf{scl}}_Q \sigma \quad \text{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_Q \sigma \ .$$

Moreover by Lemma 3.8:

$$\underline{\mathsf{scl}}_Q \sigma \leq \underline{\mathsf{scl}}_B E \quad \text{and} \quad \overline{\mathsf{scl}}_Q \sigma \leq \overline{\mathsf{scl}}_B E \ .$$

Combining all of the above leads to:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \underline{\mathsf{scl}}_B E \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_B E \ .$$

Taking the infima over such subsets $E \subset X$ with positive mass leads to the sought result. □

3.4 Proof of Theorem B

This subsection contains the proof of Theorem B, we recall its statement below. We use Vitali's lemma [Vit08] to compare local scales with Hausdorff and packing scales as did Fan or Tamashiro in their proof for the specific case of dimension.

Lemma 3.12 (Vitali). *Let (X, d) be a separable metric space. Given $\delta > 0$, $\mathcal{B}$ a family of open balls in X with radii at most δ and F the union of these balls. There exists a countable set J and a δ-pack $(B(x_j, r_j))_{j\in J} \subset \mathcal{B}$ of F such that:*

$$F \subset \bigcup_j B(x_j, 5r_j) \ .$$

We first prove Lemma 3.5 that we used to prove Theorem C.

Proof of Lemma 3.5. First we can assume that $\mathsf{scl}_H X < +\infty$, $\mathsf{scl}_P X < +\infty$, and that μ is not null, otherwise both inequalities immediately hold true. In particular, we can assume that X is separable. LEFT HAND SIDE INEQUALITY: If $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu = 0$ the inequality is obviously true. Suppose then that this quantity is positive and consider a positive $\alpha < \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$. Thus, there exists $r_0 > 0$ such that the set $A := \{x \in X : \mu(B(x,r)) \leq \mathrm{scl}_\alpha(r),\ \forall r \in (0, r_0)\}$ has positive measure. Consider $\delta \leq r_0$, thus for any δ-cover $(B_j)_{j\in J}$ of A it holds:

$$0 < \mu(A) \leq \sum_{j\in J} \mu(B_j) \leq \sum_{j\in J} \mathrm{scl}_\alpha(|B_j|)\ .$$

Since this holds true for an arbitrary cover, it follows:

$$0 < \mu(A) \leq \mathcal{H}_\delta^{\mathrm{scl}_\alpha}(A)\ .$$

Taking δ arbitrarily close to 0 leads to:

$$0 < \mu(A) < \mathcal{H}^{\mathrm{scl}_\alpha}(A)\ .$$

Finally since Hausdorff scale is non-decreasing for inclusion, it holds:

$$\mathsf{scl}_H X \geq \mathsf{scl}_H A \geq \alpha\ .$$

Note that since this holds true for any $\alpha < \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$, we indeed have $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_H X$.

RIGHT HAND SIDE INEQUALITY: Similarly , we consider without any loss of generality that there exists $0 < \alpha < \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ and put:

$$F = \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) > \alpha\right\}.$$

Let us fix a family of Borel subsets $(F_N)_{N\geq 1}$ of X such that $F = \bigcup_{N\geq 1} F_N$. For $0 < \beta < \alpha$, by Fact 2.1, there exists $\delta_0 > 0$ such that for any $r \leq \delta_0$, it holds:

$$\mathrm{scl}_\alpha(5r) \leq \mathrm{scl}_\beta(r)\ .$$

We fix $\delta \in (0, \delta_0)$ and an integer $N \geq 1$. For any x in F_N by Lemma 2.3 there exists an integer $n(x)$, minimal, such that $r(x) := \exp(-n(x)) \leq \delta$ and:

$$\mu\left(B(x, 5r(x))\right) \leq \mathrm{scl}_\alpha(5r(x))\ .$$

We now set:

$$\mathcal{F} = \{B(x, r(x)) : x \in F_N\}\ .$$

Thus by Vitali Lemma 3.12, since we assumed (X, d) separable, there exists a countable set J and a δ-pack $(B(x_j, r_j))_{j\in J} \subset \mathcal{F}$ of F such that $F_N \subset \bigcup_{j\in J} B(x_j, 5r_j)$. From there:

$$\mu(F_N) \leq \sum_{j\in J} \mu(B(x_j, 5r_j)) \leq \sum_{j\in J} \mathrm{scl}_\alpha(5r_j) \leq \sum_{j\in J} \mathrm{scl}_\beta(r_j)\ .$$

Since this holds true for any δ-pack, it follows:

$$\mathcal{P}_{\delta}^{\mathrm{scl}_\beta}(F_N) \geq \mu(F_N) \ ,$$

and then taking δ arbitrarily close to 0 leads to:

$$\mathcal{P}_{0}^{\mathrm{scl}_\beta}(F_N) \geq \mu(F_N) \ .$$

By taking the sum over $N \geq 1$, it holds:

$$\sum_{N\geq 1} \mathcal{P}_{0}^{\mathrm{scl}_\beta}(F_N) \geq \sum_{N\geq 1} \mu(F_N) \geq \mu(F) > 0 \ .$$

Recall that $(F_N)_{N\geq 1}$ is an arbitrary covering of Borel sets of F, thus:

$$\mathcal{P}^{\mathrm{scl}_\beta}(F) \geq \mu(F) > 0 \ .$$

It holds then $\mathsf{scl}_P F \geq \beta$ for any $\beta < \alpha < \operatorname{ess\,sup}\operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ which allows to conclude the proof. □

We deduce then:

Proposition 3.13. *Let (X, d) be a metric space and μ a Borel measure on X, then:*

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_H\mu \quad \textit{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P\mu \ ,$$

and

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}^*_H\mu \quad \textit{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}^*_P\mu \ .$$

Proof. The second line of inequalities are given by Remark 3.6. It remains to show the first line of inequalities. Let E be a Borel subset of X with μ positive mass. Thus with σ the restriction of μ to E, it holds by Lemma 3.5:

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \leq \mathsf{scl}_H E \quad \text{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \leq \mathsf{scl}_P E \ .$$

By Lemma 3.4, it holds:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_H E \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P E \ .$$

Taking the infima over E with positive mass ends the proof. □

Explicit links between packing scales, Hausdorff scales and local scales of measures can be now established by proving Theorem B. Let us first recall its statement: Let (X, d) be a metric space and μ a Borel measure on X, then:

$$\mathsf{scl}_H\mu = \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P\mu = \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$$

and

$$\mathsf{scl}^*_H\mu = \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}^*_P\mu = \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \ .$$

Proof of Theorem B. By Proposition 3.13 it remains only to show four inequalities.

We first prove $\mathsf{scl}_H\mu \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$. We can assume that $\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}} < +\infty$, otherwise the result immediately comes, and fix $\alpha > \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}$. Consider $\beta > \alpha$, thus by definition of scaling, there exists $\delta > 0$ such that for any $r \in (0, \delta)$ one has $\mathrm{scl}_\beta(5r) \leq \mathrm{scl}_\alpha(r)$. Denote:

$$F := \{x \in X : \underline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) < \alpha\} ,$$

then $\mu(F) > 0$ and by Lemma 2.3 for any x in F there exists an integer $n(x)$, minimal, such that $r(x) := \exp(-n(x)) \leq \delta$ and:

$$\mu\left(B(x, r(x))\right) \geq \mathrm{scl}_\alpha(r(x)) .$$

Now set:

$$\mathcal{F} := \{B(x, r(x)) : x \in F\} .$$

By Vitali Lemma 3.12, there exists a countable set J and a δ-pack $\{B(x_j, r_j)\}_{j\in J} \subset \mathcal{F}$ of F such that $F \subset \bigcup_{j\in J} B(x_j, 5r_j)$. Then, it holds:

$$\sum_{j\in J} \mathrm{scl}_\beta(5r_j) \leq \sum_{j\in J} \mathrm{scl}_\alpha(r_j) \leq \sum_{j\in J} \mu(B(x_j, r_j)) \leq \mu(F) .$$

We then have $\mathcal{H}^{\mathrm{scl}_\beta}_{5\delta}(F) \leq \mu(F)$. Since this holds true δ as small as we want, we deduce $\mathcal{H}^{\mathrm{scl}_\beta}(F) \leq \mu(F)$; and this holds true for any $\beta > \alpha$. We finally get $\mathsf{scl}_H F \leq \alpha$ and then by taking α close to $\operatorname{ess\,inf} \mathsf{scl}_{\mathrm{loc}}\mu$, we indeed have $\mathsf{scl}_H\mu \leq \mathsf{scl}_H F \leq \operatorname{ess\,inf} \mathsf{scl}_{\mathrm{loc}}\mu$.

We prove now $\mathsf{scl}_P\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$. Similarly as for Hausdorff scales, we can assume that $\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ is finite. Consider then $\alpha > \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ and set $E = \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) < \alpha\right\}$, thus $\mu(X\backslash E) = 0$. Moreover it holds:

$$E = \bigcup_{i\geq 1} E_i \quad \text{where } E_i = \left\{x \in E : \forall r \leq 2^{-i},\ \mu(B(x, r)) \geq \mathrm{scl}_\alpha(r)\right\} .$$

By Lemma 2.20, it holds $\mathsf{scl}_P E = \sup_{i\geq 1} \mathsf{scl}_P E_i$. It is then enough to show that for any $i \geq 1$, we have $\mathsf{scl}_P E_i \leq \alpha$. Indeed, then taking α arbitrarily close to $\operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ allows to conclude. In that way let us fix $i \geq 1$. Fix $\delta \in (0, 2^{-i})$, and consider J a countable set and $(B_j)_{j\in J}$ a δ-pack of E_i. It follows:

$$\sum_{j\in J} \mathrm{scl}_\alpha(|B_j|) \leq \sum_{j\in J} \mu(B_j) \leq 1 .$$

Since this holds true for any δ-pack, we have:

$$\mathcal{P}^{\mathrm{scl}_\alpha}_\delta(E_i) \leq 1 .$$

Taking δ arbitrarily close to 0 leads to:

$$\mathcal{P}^{\mathrm{scl}_\alpha}(E_i) \leq \mathcal{P}^{\mathrm{scl}_\alpha}_0(E_i) \leq 1 .$$

From there, $\mathsf{scl}_P E_i \leq \alpha$, which concludes the proof of $\mathsf{scl}_P\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$.

Let us now prove $\mathsf{scl}^*_H\mu \leq \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$. We can assume that $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$ is finite and fix a real number $\alpha > \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$. For $\beta > \alpha$, then consider $\delta > 0$ such that for any $r \in (0, \delta)$ we have

$\mathrm{scl}_\beta(5r) \leq \mathrm{scl}_\alpha(r)$. Denote $F := \{x \in X : \underline{\mathsf{scl}}_{loc}\mu(x) < \alpha\}$, thus F has total mass and by Lemma 2.3 for any x in F there exists an integer $n(x)$, minimal, such that $r(x) := \exp(-n(x)) \leq \delta$ and:

$$\mu\left(B(x, r(x))\right) \geq \mathrm{scl}_\alpha(r(x)) \ .$$

Now put:

$$\mathcal{F} := \{B(x, r(x)) : x \in F\} \ .$$

By Vitali's Lemma 3.12, there exists a countable set J and a δ-pack $(B(x_j, r_j))_{j\in J} \subset \mathcal{F}$ of F such that $F \subset \bigcup_{j\in J} B(x_j, 5r_j)$. Thus:

$$\sum_{j\in J} \mathrm{scl}_\beta(5r_j) \leq \sum_{j\in J} \mathrm{scl}_\alpha(r_j) \leq \sum_{j\in J} \mu(B(x_j, r_j)) \leq \mu(F) \ .$$

Since this holds true for δ arbitrarily close to 0, we deduce that $\mathcal{H}^{\mathrm{scl}_\beta}(F) \leq \mu(F)$. Then, taking $\beta > \alpha$ close to α leads to $\mathsf{scl}_H F \leq \alpha$, and thus by taking α arbitrarily close to $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$, we indeed have $\mathsf{scl}^*_H\mu \leq \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$.

To conclude, we prove $\mathsf{scl}_P\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$. Let E be a Borel set with positive measure. Let σ be the restriction of μ to E, thus by Lemma 3.4:

$$\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \ ,$$

and then by Lemma 3.5, it holds:

$$\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \leq \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \leq \mathsf{scl}_P E \ .$$

This holds true for any E such that $\mu(E) > 0$, thus $\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P\mu$.

Finally, let us show $\mathsf{scl}^*_P\mu \leq \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$. Put $\alpha > \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ and set $F := \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu < \alpha\right\}$, then $\mu(F) > 0$, and denote:

$$E := \bigcup_{i\geq 1} E_i \qquad \text{where } E_i = \left\{x \in E : \forall r \leq 2^{-i},\ \mu(B(x, r)) \geq \mathrm{scl}_\alpha(r)\right\} \ .$$

By Lemma 2.20, we have $\mathsf{scl}_P E = \sup_{i\geq 1} \mathsf{scl}_P E_i$, it is then enough to show that for any $i \geq 1$, we have $\mathsf{scl}_P E_i \leq \alpha$. Indeed we can take α arbitrarily close to $\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$. We then fix $i \geq 1$. Fix $\delta \in (0, 2^{-i})$. We consider J a countable set and $(B_j)_{j\in J}$ a δ-pack of E_i. Then:

$$\sum_{j\in J} \mathrm{scl}_\alpha(|B_j|) \leq \sum_{j\in J} \mu(B_j) \leq 1 \ .$$

Since this holds true for any δ-pack, it follows:

$$\mathcal{P}^{\mathrm{scl}_\alpha}_\delta(E_i) \leq 1 \ .$$

When δ tends to 0, the latter inequality leads to:

$$\mathcal{P}^{\mathrm{scl}_\alpha}(E_i) \leq \mathcal{P}^{\mathrm{scl}_\alpha}_0(E_i) \leq 1 \ .$$

From there, we deduce $\mathsf{scl}_P E_i \leq \alpha$, which concludes the proof of the last inequality and thus the one of Theorem B. □

4 Applications

4.1 Scales of infinite products of finite sets

A natural toy model in the study of scales is given by a product $Z = \prod_{n\geq 1} Z_k$ of finite sets. To define the metric δ on this set, we fix a decreasing sequence $(\epsilon_n)_{n\geq 1}$ which verifies $\log \epsilon_{n+1} \sim \log \epsilon_n$ when $n/to + \infty$. We put for $\underline{x} = (x_n)_{n\geq 1} \in Z$ and $\underline{y} = (y_n)_{n\geq 1} \in Z$:

$$\delta(\underline{x}, \underline{y}) := \epsilon_m \ ,$$

where $m := \inf\{n \geq 1 : x_n \neq y_n\}$ is the minimal index such that the sequences $\underline{x}$ and $\underline{y}$ differ. Note that then if each Z_n is endowed with the discrete topology, then δ provides the product topology on Z.

A natural measure on Z is the following product measure:

$$\mu := \otimes_{n\geq 1} \mu_n \ ,$$

where μ_n is the equidistributed measure on Z_n for $n \geq 1$. The scales of Z and μ are given by the following:

Proposition 4.1. *For any scaling* scl*, it holds for any* $\underline{x} \in Z$*:*

$$\underline{\mathsf{scl}}_{\mathrm{loc}}\mu(\underline{x}) = \underline{\mathsf{scl}}_B Z = \sup\left\{\alpha > 0 : \mathrm{scl}_\alpha(\epsilon_n) \prod_{k=1}^{n} \frac{1}{\mathrm{Card}\, Z_k} \xrightarrow[n\to+\infty]{} +\infty\right\}$$

and

$$\overline{\mathsf{scl}}_{\mathrm{loc}}\mu(\underline{x}) = \overline{\mathsf{scl}}_B Z = \inf\left\{\alpha > 0 : \mathrm{scl}_\alpha(\epsilon_n) \prod_{k=1}^{n} \frac{1}{\mathrm{Card}\, Z_k} \xrightarrow[n\to+\infty]{} 0\right\} \ .$$

We shall prove this proposition later. A first corollary can be deduced directly from Theorem A and C:

Corollary 4.2. *For any scaling* scl*, it holds moreover:*

$$\mathsf{scl}_H Z = \underline{\mathsf{scl}}_Q \mu = \underline{\mathsf{scl}}_B Z$$

and

$$\mathsf{scl}_P Z = \overline{\mathsf{scl}}_Q \mu = \overline{\mathsf{scl}}_B Z \ .$$

For some particular choice of the sequence $(\epsilon_n)_{n\geq 1}$ and for $\mathsf{scl} = \mathsf{ord}$ we obtain moreover:

Corollary 4.3. *Suppose that* $\frac{-\log \epsilon_n}{n}$ *converges to* $C > 1$ *when* $n \to +\infty$*. Then for any scaling* scl*, it holds moreover:*

$$\mathsf{ord}_H Z = \underline{\mathsf{ord}}_B Z = \liminf_{n\to+\infty} \frac{1}{n \log C} \log\left(\sum_{k=1}^{n} \log\left(\mathrm{Card}\, Z_k\right)\right)$$

and

$$\mathsf{ord}_P Z = \overline{\mathsf{ord}}_B Z = \limsup_{n\to+\infty} \frac{1}{n \log C} \log\left(\sum_{k=1}^{n} \log\left(\mathrm{Card}\, Z_k\right)\right) \ .$$

Note that $\log \epsilon_n \sim -C \cdot n$ implies $\log \epsilon_n \sim \log \epsilon_{n+1}$ when $n \to +\infty$.

The following lemma allows to prove both Proposition 4.1 and its corollaries:

Lemma 4.4. *For any $n \geq 1$ the ϵ_n-covering number $\mathcal{N}_{\epsilon_n}(Z)$ verifies for any $\underline{z} \in Z$:*

$$\mathcal{N}_{\epsilon_n}(Z) = \mu(B(\underline{z}, \epsilon_n))^{-1} = \prod_{k=1}^{n} \operatorname{Card} Z_k \ .$$

Proof of Proposition 4.1. Since $\log \epsilon_{n+1} \sim \log \epsilon_n$ when $n \to +\infty$, we have by Lemma 2.3:

$$\underline{\mathsf{scl}}_B Z = \sup\left\{\alpha > 0 : \mathrm{scl}_\alpha(\epsilon_n) \cdot \mathcal{N}_{\epsilon_n}(Z) \xrightarrow[n\to\infty]{} +\infty\right\} \quad \text{and} \quad \overline{\mathsf{scl}}_B Z = \inf\left\{\alpha > 0 : \mathrm{scl}_\alpha(\epsilon_n) \cdot \mathcal{N}_{\epsilon_n}(Z) \xrightarrow[n\to\infty]{} 0\right\} \ ,$$

and we have the same form for local scales. Then the sought results follow from Lemma 4.4. □

Before proving the remaining lemma we first prove the second corollary:

Proof of Corollary 4.3. By Lemma 4.4, for any $n \geq 1$ we have $\mathcal{N}_{\epsilon_n}(Z) = \prod_{k=1}^{n} \operatorname{Card} Z_k$, then by Remark 2.12 and Lemma 2.3, it holds:

$$\underline{\mathsf{ord}}_B(X) = \liminf_{n\to+\infty} \frac{\log\log(\mathcal{N}_{\epsilon_n}(Z))}{\log(\epsilon_n^{-1})} = \liminf_{n\to+\infty} \frac{1}{n\log C} \log\left(\sum_{k=1}^{n} \log\left(\operatorname{Card} Z_k\right)\right)$$

and

$$\overline{\mathsf{ord}}_B(X) = \limsup_{\epsilon\to 0} \frac{\log\log(\mathcal{N}_{\epsilon_n}(Z))}{\log(\epsilon_n^{-1})} \limsup_{n\to+\infty} \frac{1}{n\log C} \log\left(\sum_{k=1}^{n} \log\left(\operatorname{Card} Z_k\right)\right) \ .$$

This concludes the proof of the corollary. □

Finally we provide the remaining:

Proof of Lemma 4.4. Note that for any $n \geq 1$ and for any $\underline{z} \in Z$:

$$B(\underline{z}, \epsilon_n) = \{\underline{w} \in Z : w_1 = z_1, \ldots, w_n = z_n\} \ .$$

Thus:

$$\mu(B(\underline{z}, \epsilon_n)) = \prod_{k=1}^{n} n_k^{-1}.$$

This shows the first equality, it remains to show that $\mathcal{N}_{\epsilon_n}(Z) = \prod_{k=1}^{n} n_k$. Let us consider $\left\{\underline{z}^1, \ldots, \underline{z}^N\right\}$ a set of minimal cardinality such that:

$$Z = \bigcup_{j=1}^{N} B(\underline{z}^j, \epsilon_n) \ .$$

For $1 \leq j \leq N$, denote $\underline{z}^j = (z_k^j)_{k\geq 1}$, thus we have the following:

Fact 4.5. *The map:*

$$\phi : i \in \{1, \ldots, N\} \mapsto (z_1^i, \ldots, z_n^i) \in Z_1 \times \cdots \times Z_n \ .$$

is a bijection.

Proof. We first start by showing ϕ injective. Let us assume that there exists $i \neq j$ such that $\phi(i) = \phi(j)$, then it holds $B(\underline{z}^i, \epsilon_n) = B(\underline{z}^j, \epsilon_n)$. It follows that there exists a covering of Z by $N-1$ balls with radius ϵ_n, which contradicts the assumption on minimality of N. Thus ϕ is injective. We now show that ϕ is also surjective. Consider $\alpha \in Z_1 \times \cdots \times Z_N$. Since Z_k is not empty for any $k \geq 1$, there exists $\underline{z} \in Z$ such that for any $1 \leq k \leq n$ it holds $z_k = \alpha_k$. Then there exists $i \in \{1, \ldots, N\}$ such that $\underline{z} \in B(\underline{z}^i, \epsilon_n)$. Thus $\phi(i) = \alpha$ which gives us the surjectivity. □

From there since ϕ is a bijection, we have:

$$\mathcal{N}_{\epsilon_n}(Z) = N = \operatorname{Card} Z_1 \times \cdots \times Z_N = \prod_{k=1}^{N} n_k. \, .$$

□

Such examples of products of groups allow to exhibit compact metric spaces with arbitrary high order:

Example 4.6. *For any $\alpha \geq \beta > 0$, there exists compact metric probability space (Z, δ, μ) such that for any $z \in Z$:*

$$\beta = \underline{\mathsf{ord}}_{\mathrm{loc}}\mu(z) = \mathsf{ord}_H Z = \underline{\mathsf{ord}}_Q\mu = \underline{\mathsf{ord}}_B Z$$

and

$$\alpha = \overline{\mathsf{ord}}_{\mathrm{loc}}\mu(z) = \mathsf{ord}_P Z = \overline{\mathsf{ord}}_Q\mu = \overline{\mathsf{ord}}_B Z \, .$$

In particular with $\alpha > \beta$ we obtain examples of metric spaces with finite order such that the Hausdorff and packing orders do not coincide. Moreover, for a countable dense subset F of X, it holds $\mathsf{ord}_H F = \mathsf{ord}_P F = 0$ and $\underline{\mathsf{ord}}_B F = \beta < \alpha = \overline{\mathsf{ord}}_B F$. It follows that none of the inequalities of Theorem A for the case of order in a equality in the general case. Moreover, using disjoint unions of such spaces allows to produce examples of metric spaces where either of the strict equality can happen between any pair of scales that are not compared in Fig. 1.

Proof. Let $(u_k)_{k\geq 0}$ be the sequence defined by:

$$u_k = \begin{cases} \lfloor \exp(\exp(\beta \cdot k)) \rfloor & \text{if} \quad c^{2j} \leq k < c^{2j+1} \\ \lfloor \exp(\exp(\alpha \cdot k)) \rfloor & \text{if} \quad c^{2j+1} \leq k < c^{2j+2} \end{cases}$$

where $c = \lfloor \frac{\alpha}{\beta} \rfloor + 1$. We denote $Z := \prod_{n\geq 1} \mathbb{Z}/u_k\mathbb{Z}$ endowed with the metric δ defined by:

$$\delta(\underline{z}, \underline{w}) := \exp(-\inf\{n \geq 1 : z_n \neq w_n\})$$

for $\underline{z} = (z_n)_{n\geq 1}$ and $\underline{w} = (w_n)_{n\geq 1}$ in Z. Let us denote $\lambda_n = \frac{1}{n}\log \sum_{k=1}^{n} \log u_k$. Thus by Corollary 4.3:

$$\mathsf{ord}_H Z = \underline{\mathsf{ord}}_B Z = \liminf_{n\to+\infty} \lambda_n$$

and

$$\mathsf{ord}_P Z = \overline{\mathsf{ord}}_B Z = \limsup_{n\to+\infty} \lambda_n \, .$$

It remains to show that $\lambda^- := \liminf_{n\to+\infty} \lambda_n = \beta$ and $\lambda^+ := \limsup_{n\to+\infty} \lambda_n = \alpha$ in order to show that (Z,δ) satisfies the sought properties. First notice that $\exp(\exp(\beta\cdot n)) \leq u_n \leq \exp(\exp(\alpha\cdot n))$. Thus $\lambda^- \geq \beta$ and $\lambda^+ \leq \alpha$. Denote $n_j = c^{2j+1}$:

$$\lambda_{n_j} \geq \frac{1}{n}\log\log(u_{n_j}) = \alpha\ .$$

Thus, taking $j \to +\infty$ leads to $\lambda^+ \geq \alpha$. Moreover, denote $m_j = c^{2j+1} - 1$. We have the following:

Lemma 4.7. *For any $j \geq 1$ and for any $1 \leq k \leq m_j$, it holds:*

$$u_k < u_{m_j}\ .$$

Proof. If $c^{2j} \leq k \leq m_j$, then $u_k = \exp(\exp(\beta\cdot k)) \leq \exp(\exp(\beta\cdot m_j)) = u_{m_j}$. Otherwise, we have $k < c^{2j}$, and then $u_k < \exp(\exp(\alpha\cdot c^{2j})) < \exp(\exp(\beta\cdot c^{2j+1})) = u_{m_j}$, since $\alpha < \beta\cdot c$. □

From the above lemma, we have:

$$\lambda_{m_j} \leq \frac{1}{m_j}\log m_j \log(u_{m_j}) \xrightarrow[j\to+\infty]{} \beta\ ,$$

and so $\lambda^- \leq \beta$ which concludes the proof of the proposition. □

4.2 Functional spaces

Metric spaces studied here are sub-spaces of differentiable spaces on compact subset of $\mathbb{R}^d$ for d a positive integer. We denote by $\|\cdot\|_{C^k}$ the C^k-uniform norm on $C^k([0,1]^d,\mathbb{R})$:

$$\|f\|_{C^k} := \sup_{0\leq j\leq k} \|D^j f\|_\infty\ .$$

Definition 4.8. *For $d \geq 1$ and k an integer, $\alpha \in [0,1]$ let us define:*

$$\mathcal{F}^{d,k,\alpha} := \left\{ f \in C^k([0,1]^d,[-1,1]) : \|f\|_{C^k} \leq 1, \ \text{ and if } \alpha > 0, \text{ the map } D^k f \text{ is } \alpha\text{-Hölder with constant } 1 \right\}.$$

In the case $\alpha > 0$, for any $x, y \in [0,1]^d$:

$$\|D^k f(x) - D^k f(y)\|_\infty \leq \|x-y\|^\alpha\ .$$

In particular, $\mathcal{F}^{d,k,0}$ is the unit ball for the C^k-norm in $C^k([0,1]^d,[-1,1])$. Let us recall the asymptotic given by Kolmogorov-Tikhomirov [KT93][Thm XV] on the covering number of $(\mathcal{F}^{d,k,\alpha},\|\cdot\|_\infty)$, see Theorem 1.10:

$$C_1\cdot\epsilon^{-\frac{d}{k+\alpha}} \geq \log \mathcal{N}_\epsilon(\mathcal{F}^{d,k,\alpha}) \geq C_2\cdot\epsilon^{-\frac{d}{k+\alpha}}\ ,$$

where $C_1 > C_2 > 0$ are two constants depending on d, k and α. In order to prove Theorem E which states that box, packing and Hausdorff scales of $\mathcal{F}^{d,k,\alpha}$ are all equal to $\frac{d}{k+\alpha}$, by Theorem B, it remains to prove Lemma 1.11. The latter states:

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \geq \frac{d}{k+\alpha}\ .$$

Proof of Lemma 1.11. We first start with the case $\alpha > 0$. The case $\alpha = 0$ will be deduced from it. We consider the following set:

$$\Lambda = \prod_{n \geq 1} \Lambda_n \ .$$

where $\Lambda_n = \{-1, 0, +1\}^{R^{dn}}$ and with $R = \lfloor 5^{\frac{1}{k+\alpha}} \rfloor + 1$. We endow Λ with the metric δ defined by:

$$\delta(\underline{\lambda}, \underline{\lambda}') = \epsilon_m \ ,$$

with m the minimal index such that the sequences $\underline{\lambda}$ and $\underline{\lambda}'$ differ and with $(\epsilon_n)_{n \geq 0}$ is a decreasing sequence of positive real numbers such that $\frac{-\log \epsilon_n}{n} \to R^{k+\alpha}$ when $n \to +\infty$. We can choose $(\epsilon_n)_{n \geq 1}$ such that the following holds:

Lemma 4.9. *There exists an embedding $I : (\Lambda, \delta) \to (\mathcal{F}^{d,k,\alpha}, \|\cdot\|_\infty)$ such that for any $\lambda, \lambda' \in \Lambda$ it holds:*

$$\|I(\lambda) - I(\lambda')\|_\infty \geq \frac{1}{2}\delta(\lambda, \lambda') \ .$$

The above lemma allows to conclude the proof of Lemma 1.11. Indeed, since Λ is a product of finite sets endowed with a product metric, and since $\log \epsilon_{n+1} \sim \log \epsilon_n$, by Corollary 4.3 it holds:

$$\mathsf{ord}_H \Lambda = \liminf_{n \to +\infty} \frac{1}{n \log R^{k+\alpha}} \log \left(\sum_{j=1}^{n} \log \mathrm{Card}\, \Lambda_j \right) = \liminf_{n \to +\infty} \frac{1}{n \log R^{k+\alpha}} \log \left(\sum_{j=1}^{n} R^{dj} \cdot \log 3 \right) \ .$$

Now, for $n \geq 1$ it holds:

$$\frac{1}{n \log R^{k+\alpha}} \log \left(\sum_{j=1}^{n} R^{dj} \cdot \log 3 \right) = \frac{\log \log 3 + \log \frac{R^{d(n+1)} - R}{R-1}}{\log R^{k+\alpha}} \xrightarrow[n \to +\infty]{} \frac{d}{k+\alpha} \ .$$

It comes $\mathsf{ord}_H \Lambda = \frac{d}{k+\alpha}$. Now since by assumption on I in Lemma 4.9, it holds by Corollary 2.22:

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \geq \mathsf{ord}_H \Lambda = \frac{d}{k+\alpha} \ ,$$

which concludes the proof of Lemma 1.11. It remains to show:

Proof of Lemma 4.9. Let us denote $q := k + \alpha$ and recall that $R := \lfloor 5^{\frac{1}{q}} \rfloor + 1$. We consider the following map on $\mathbb{R}$:

$$\phi : t \in \mathbb{R} \mapsto (2t)^q (2 - 2t)^q \cdot \mathbb{1}_{0 < t < 1} \ .$$

Note that the function ϕ has its support in $[0, 1]$ and takes the value 1 at $\frac{1}{2}$. The k^{th} derivative of ϕ is non-constant. For $f \in \mathcal{F}^{d,k,\alpha}$, let $\|f\|_q$ be the infimum of the constants $C > 0$ such that for any $x, y \in \mathbb{R}$:

$$\|D^k f(x) - D^k f(y)\| \leq C \cdot \|x - y\|^\alpha \ .$$

Note that $\|\cdot\|_q$ is a semi-norm on $\mathcal{F}^{d,k,\alpha}$ and moreover:

$$\mathcal{F}^{d,k,\alpha} = \left\{ f \in \mathcal{F}^{d,k,0} : \|f\|_q \leq 1 \right\} \ .$$

Observe that $\|\phi\|_q > 0$. Let $(x_j)_{1\leq j\leq R^{dn}}$ be an exhaustive sequence of the set:

$$\left\{\left(\frac{i_1}{R^n},\dots,\frac{i_d}{R^n}\right) : i_1,\dots,i_d \in \{0,\dots,R^n-1\}\right\} .$$

For any $\lambda = (\lambda_1,\dots,\lambda_{R^{dn}}) \in \Lambda_n = \{-1,0,+1\}^{R^{dn}}$ we associate the following map:

$$f_\lambda : x = (x_1,\dots,x_d) \in [0,1]^d \mapsto \epsilon_n \cdot \sum_{j=1}^{R^{dn}} \lambda_j \cdot \phi(R^n \cdot \|x - x_j\|) ,$$

with

$$\epsilon_n := \frac{6}{\pi^2 \cdot n^2 \cdot R^{qn} \cdot \|\phi\|_q} .$$

Let us denote $\mathcal{S}_n$ the set of such maps:

$$\mathcal{S}_n = \{f_\lambda : \lambda \in \Lambda_n\} .$$

The sequence $(\epsilon_{n\geq 1})$ is chosen such as the following holds:

Lemma 4.10. *The distance between $f_\lambda, f_{\lambda'} \in \mathcal{S}_n$ is given by:*

$$\|f_\lambda - f_{\lambda'}\|_q = \frac{6}{\pi^2\cdot n^2}\cdot\|\lambda-\lambda'\|_\infty \quad \textit{and} \quad \|f_\lambda - f_{\lambda'}\|_\infty = \epsilon_n \cdot \|\lambda-\lambda'\|_\infty ,$$

where $\|\lambda-\lambda'\|_\infty := \sup_{1\leq i\leq R^{dn}} |\lambda_i - \lambda'_i|$.

Proof. For any $x \in [0,1]^d$, there exists at most one value $j \in \left\{1,\dots,R^{dn}\right\}$ such that $\|x - x_j\| < R^{-n}$, thus the maps $x \mapsto \phi(\|x-x_j\|\cdot R^n)$ for $1 \leq j \leq R^{dn}$ have disjoints supports. It comes then:

$$\|f_\lambda - f_{\lambda'}\|_q = \epsilon_n \left\| \sum_{i=1}^{R^{dn}} |\lambda_i - \lambda'_i| \phi(R^n \|\cdot - x_i\|) \right\|_q = \epsilon_n \cdot \sup_{1\leq i\leq R^{dn}} |\lambda_i - \lambda'_i| \cdot R^{qn} \|\phi\|_q = \frac{6}{\pi\cdot n^2}\cdot\|\lambda-\lambda'\|_\infty .$$

Now, for the C^0-norm, it holds:

$$\|f_\lambda - f_{\lambda'}\|_\infty = \epsilon_n \left\| \sum_{i=1}^{R^{dn}} (\lambda-\lambda')\cdot\phi(R^n\|\cdot - x_i\|) \right\|_\infty = \epsilon_n \cdot \|\lambda-\lambda'\|_\infty .$$

□

Note in particular that since $0 \in \mathcal{S}_n$, for any $f_\lambda \in \mathcal{S}_n \setminus \{0\}$, it holds:

$$\|f_\lambda\|_q = \frac{6}{\pi^2\cdot n^2} \quad \text{and} \quad \|f_\lambda\|_\infty = \epsilon_n .$$

We now embed $\prod_{n\geq 1}\mathcal{S}_n$ into $\mathcal{F}^{d,k,\alpha}$. For $\underline{\lambda} = (\lambda_n)_{n\geq 1} \in \Lambda$ we associate the formal series $\sum_{n\geq 1} f_{\lambda_n}$ where $f_{\lambda_n} \in \mathcal{S}_n$. Then we have the following:

Lemma 4.11. *For any $\underline{\lambda} = (\lambda_n)_{n\geq 1} \in \Lambda$ the function series $\sum_{n\geq 1} f_{\lambda_n}$ converges in $C^0([0,1]^d,[-1,1])$ and moreover its limit lies in $\mathcal{F}^{d,k,\alpha}$.*

Proof. By Lemma 4.10, it holds:

$$\sum_{n\geq 1} \|f_{\lambda_n}\|_\infty \leq \sum_{n\geq 1} \epsilon_n < +\infty \ .$$

It comes that the series $\sum_{n\geq 1} f_{\lambda_n}$ is normally convergent, thus it is also point-wise convergent and moreover the limit g is continuous. Now note that for any $n \geq 1$ and for any $1 \leq l \leq k$ it holds $D^l f_{\lambda_n}(0) = 0$, thus by Taylor integral formula, it holds:

$$\|D^l f_{\lambda_n}\|_\infty \leq \|D^k f_{\lambda_n}\|_\infty \ .$$

Moreover, still by Lemma 4.10, it holds:

$$\sum_{n\geq 1} \|f_{\lambda_n}\|_q \leq \sum_{n\geq 1} \frac{6}{\pi^2 n^2} = 1 \ .$$

Now since $D^k f_{\lambda_n}$ is $\frac{6}{\pi^2 n^2}$- α-Hölder and $D^k f_{\lambda_n}(0) = 0$, for any $n \geq 1$, it follows:

$$\sum_{n\geq 1} \|f_{\lambda_n}\|_{C^k} \leq \sum_{n\geq 1} \frac{6}{\pi^2 n^2} = 1 \ .$$

Thus the partial sums lie in $\mathcal{F}^{d,k,\alpha}$ and so does g as a limit of elements of $\mathcal{F}^{d,k,\alpha}$, which is closed for the C^0-norm. □

By Lemma 4.11, the following map is well defined:

$$I : \underline{\lambda} = (\lambda_n)_{n\geq 1} \in (\Lambda, \delta) \mapsto \lim_{n\to+\infty} \sum_{n\geq 1} f_{\lambda_n} \in (\mathcal{F}^{d,k,\alpha}, \|\cdot\|_\infty) \ .$$

To conclude the proof, it remains to show that for any $\underline{\lambda}, \underline{\lambda}' \in \Lambda$:

$$\|I(\underline{\lambda}) - I(\underline{\lambda}')\|_\infty \geq \tfrac{1}{2}\delta(\underline{\lambda}, \underline{\lambda}') \ ,$$

Consider $\underline{\lambda} = (\lambda_n)_{n\geq 1}, \underline{\lambda}' = (\lambda'_n)_{n\geq 1} \in \Lambda$. We denote $k := \inf\{n \geq 1 : x_n \neq y_n\}$. Then it holds:

$$\|I(\underline{\lambda}) - I(\underline{\lambda}')\|_\infty = \left\| \sum_{n\geq k} f_{\lambda_n} - f_{\lambda'_n} \right\|_\infty \geq \|f_{\lambda_k} - f_{\lambda'_k}\|_\infty - \sum_{n>k} \|f_{\lambda_n} - f_{\lambda'_n}\|_\infty \ .$$

Now by Lemma 4.10, it holds respectively:

$$\|f_{\lambda_k} - f_{\lambda'_k}\|_\infty \geq \epsilon_k \quad \text{and} \quad \sum_{n>k} \|f_{\lambda_n} - f_{\lambda'_n}\|_\infty \leq 2 \sum_{n>k} \epsilon_n \ .$$

Now recall that $\epsilon_n = \frac{6}{\pi^2 n^2 R^{qn}}$ for any $n \geq 1$, then:

$$\sum_{n>k} \epsilon_n \leq \sum_{n>k} \epsilon_k \cdot R^{-q(n-k)} = \epsilon_k \cdot \frac{1}{R^q - 1} \ .$$

Now since $R^q \geq 5$, it holds then $\frac{1}{R^q-1} \leq \frac{1}{4}$ and it follows:

$$\|I(\underline{\lambda}) - I(\underline{\lambda}')\|_\infty \geq \tfrac{1}{2}\epsilon_k \ .$$

Now since $\epsilon_k = \delta(\underline{\lambda}, \underline{\lambda}')$ by definition of δ, the sought result comes. □

This concludes the proof of Lemma 1.11 for the case $\alpha > 0$. It remains to deduce the case $\alpha = 0$ from that previous one. For any $\beta > 0$, it holds $\mathcal{F}^{d,k,\beta} \subset \mathcal{F}^{d,k,0}$. From there since Hausdorff scales are non decreasing for inclusion, it holds then $\mathsf{ord}_H \mathcal{F}^{d,k,0} \geq \mathsf{ord}_H \mathcal{F}^{d,k,\beta} \geq \frac{d}{k+\beta}$. Since we can take $\beta > 0$ arbitrary small, it indeed holds $\mathsf{ord}_H \mathcal{F}^{d,k,0} \geq \frac{d}{k}$. □

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