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An arbitrary-order fully discrete Stokes complex on general polyhedral meshes.

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Abstract

In this paper we present an arbitrary-order fully discrete Stokes complex on general polyhedral meshes. We enrich the fully discrete de Rham complex with the addition of a full gradient operator defined on vector fields and fitting into the complex. We show a complete set of results on the novelties of this complex: exactness properties, uniform Poincaré inequalities and primal and adjoint consistency. The Stokes complex is especially well suited for problem involving Jacobian, divergence and curl, like the Stokes problem or magnetohydrodynamic systems. The framework developed here eases the design and analysis of scheme for such problems. Schemes built that way are nonconforming and benefit from the exactness of the complex. We illustrate with the design and study of a scheme to solve the Stokes equations and validate the convergence rates with various numerical tests.

Keywords: Discrete Stokes complex, Discrete de Rham complex, compatible discretization, polyhedral methods, arbitrary order

MSC2010 classification: 65N30, 65N99, 76D07

1 Introduction.

The exactness of the divergence free condition plays an important role in the numerical resolution of incompressible fluid equations, [6] provides a detailed review. This kind of conservation requires the discrete spaces to reproduce relevant algebraic properties of the continuous spaces. Let Ω be a domain of \mathbb{R}^3 . This exactness can be expressed by the following differential complex:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\text{curl}, \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\}. \quad (1.1)$$

Many discrete counterparts of the complex (1.1) have been developed. See [7] for a thorough exposition and an extensive bibliography. Although many partial differential equations can be expressed using the de Rham complex, the lack of smoothness causes issues for some equations, in particular for the Stokes equations (see [3]). So a smoother variant more suited to the Stokes equations (hence called Stokes complex) has been considered. In three dimensions the Stokes complex is written:

$$\mathbb{R} \xrightarrow{i_\Omega} H^2(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}^2(\Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}^1(\Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\}. \quad (1.2)$$

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The development of discrete counterparts of this smoother complex is much more complicated. See [7, Chapter 8.7] for a history. Although such constructions exist (for example [5]) they often have drawbacks. Recurrent problems can be a large minimal degree and thus numerous unknowns as well as difficulties to enforce Dirichlet boundary conditions. The subject is very active with many recent advances: [11, 12]. Another issue of these constructions is that they are frequently constrained to conformal simplicial meshes, which is limiting for some geometries as well as for the possibility of refinement or agglomeration. A construction of the Stokes complex in virtual finite elements on general meshes has also been recently developed (see [13]).

Our construction works on general polyhedral meshes and for arbitrary polynomial degrees. The discrete spaces consist of polynomial spaces on the elements of all geometric dimensions: cells, faces, edges and vertices. Compared to the virtual finite element method, the basis functions are explicitly known but do not live in a subspace of continuous functions. The discrete differential operators are therefore necessarily different from the continuous operators. They are constructed according to integration by parts formulae and in a sense converge with the discrete spaces to the continuous operators (see the consistency results of Section 5). A discretization of the de Rham complex (1.1) has been developed in detail by D. A. Di Pietro and J. Droniou [14]. One can find in the introduction a very complete comparison of the different methods leading to discrete de Rham complex on polytopal meshes. Our paper is a continuation of [14]: Our construction is based upon it, and we add the necessary basis functions required for the increased smoothness of the Stokes complex. We define and analyze in detail the Jacobian operator while checking its compatibility with the complex.

More precisely we show the exactness of the complex, the existence of uniform Poincaré inequalities and many consistency results as well as a discrete version of the right inverse for the divergence for the discrete norm \mathbf{H}^1 . Finally, we apply this to the Stokes equations: we show well-posedness, give an error estimate and find an optimal convergence rate of order $\mathcal{O}(h^{k+1})$, h being the size of the mesh and $k \geq 0$ the chosen polynomial degree. We also explore other choices of boundary conditions and validate numerically every result.

The remaining of the paper is organized as follows. In Section 2 we introduce the general setting. We define the discrete spaces and operators (interpolators, differential operators and norms) in Section 3. In Section 4 we show that our construction is indeed a complex which is exact for contractible domains. In Section 5 we establish consistency properties, including primal and dual consistencies. The Stokes equations are defined in Section 6 and other boundary conditions are studied in Section 7. We display our numerical results in Section 8. Finally we prove technical propositions in the appendices: on polynomial spaces in appendix A and on various lifts in appendix B.

2 Setting.

This section is dedicated to the introduction of the setting and various notations that will be used throughout the paper. We follow the conventions of [14].

2.1 Mesh and orientation.

In the following we consider a polyhedral domain $\Omega \subset \mathbb{R}^3$ and keeping the notation of [14], for any set $Y \subset \mathbb{R}^3$, we write $h_Y := \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in Y\}$ and $|Y|$ its Hausdorff measure. We consider on this domain a mesh sequence $\mathcal{M}_h = \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$ parameterized by a positive real parameter $h \in \mathcal{H}$. Here \mathcal{T}_h is a finite collection of open convex polyhedra such that $\overline{\Omega} = \cup_{T \in \mathcal{T}_h} \overline{T}$ and $h = \max_{T \in \mathcal{T}_h} h_T > 0$, \mathcal{F}_h is the collection of open polygonal faces of

the cells, \mathcal{E}_h is the collection of open polygonal edges, and \mathcal{V}_h the collection of vertices. This sequence must be regular in the sense of [10, Definition 1.9] with the regularity constant ρ . For any cell $T \in \mathcal{T}_h$, we write \mathcal{F}_T the set of faces of this cell. Likewise for any face $F \in \mathcal{F}_h$, we write \mathcal{E}_F the set of edges of this face.

We take $k \geq 0$ a fixed polynomial degree. In the following most inequalities hold up to a positive constant. This constant depends only on some parameters, here on the chosen polynomial degree k , on the regularity parameter of the mesh sequence ρ and on the domain Ω .

We denote the inequality up to a positive constant by

$$A \lesssim B$$

meaning there exists $C \in \mathbb{R}_+^*$ depending only on some parameters (here usually only on k , ρ and Ω) such that $A \leq CB$. We also write

$$A \approx B$$

meaning that $A \lesssim B$ and $B \lesssim A$.

For any h , we set the orientation of any face $F \in \mathcal{F}_h$ and any edge $E \in \mathcal{E}_h$ by prescribing a unit normal vector \mathbf{n}_F and unit tangent vector \mathbf{t}_E . For any face $F \in \mathcal{F}_h$ and any $E \in \mathcal{E}_F$ we also define the unit vector \mathbf{n}_{FE} normal to E lying in the plane tangent to F , and such that $(\mathbf{t}_E, \mathbf{n}_{FE}, \mathbf{n}_F)$ is right-handed in \mathbb{R}^3 . To keep track of the relative orientation we define for any $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, $\omega_{TF} \in \{-1, 1\}$ such that $\omega_{TF}\mathbf{n}_F$ points out of T , and for any $F \in \mathcal{F}_h$, $E \in \mathcal{E}_F$ we define $\omega_{FE} \in \{-1, 1\}$ such that $\omega_{FE}\mathbf{n}_{FE}$ points out of F . We also define \mathbf{n}_Ω as the outward pointing unit normal vector on the boundary $\partial\Omega$. We note by $^\perp$ the rotation of angle $\pi/2$ in the oriented plane F .

2.2 Polynomial spaces.

For any entity $X \in \{E, F, T\}$, we denote by $\mathcal{P}^k(X)$ the set of polynomials of total degree at most k on X , by $\mathcal{P}^k(X)$ the set of vector valued polynomials, and by $(\mathcal{P}^k(X)^\top)^3$ the set of triples of polynomials on X forming the rows of a matrix valued polynomial. We use the conventions $\mathcal{P}^{-1}(X) := \{0\}$ and $\mathcal{P}^{0,k}(X) := \{P \in \mathcal{P}^k(X) : \int_X P = 0\}$. We also define the broken polynomial space

$$\mathcal{P}^k(\mathcal{X}_h) := \{P_h \in L^2(\mathcal{X}_h) : \forall X \in \mathcal{X}_h, P_{h|X} \in \mathcal{P}^k(X)\}, \quad (2.1)$$

as well as its continuous counterpart

$$\mathcal{P}_c^k(\mathcal{X}_h) := \{P_h \in C^0(\mathcal{X}_h) : \forall X \in \mathcal{X}_h, P_{h|X} \in \mathcal{P}^k(X)\}. \quad (2.2)$$

Remark 1. Continuous polynomials can be characterized by their values at the interface and their lower order moments on the elements. An explicit construction is deduced from Lemma 50. In the context of edges we can see the isomorphism between $\mathcal{P}_c^{k+2}(\mathcal{E}_h)$ and $\mathcal{P}^k(\mathcal{E}_h) \times \mathbb{R}^{\mathcal{V}_h}$.

For the sake of readability we quote two lemmas on discrete spaces: [10, Lemma 1.28 and Lemma 1.32] (in a slightly more restrictive setting):

Lemma 2 (Discrete inverse Poincaré). *Let X be an element of $\mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h$. Let a positive integer l and a real number $p \in [1, \infty]$ be fixed. Then, the following inequality holds: For all $v \in \mathcal{P}^l(X)$,*

$$\|\nabla v\|_{L^p(X)} \lesssim h_X^{-1} \|v\|_{L^p(X)}, \quad (2.3)$$

with hidden constant depending only on ρ , l and p .

Lemma 3. *Let $p \in [1, \infty]$ be a fixed real number and $l \geq 0$ be a fixed integer. Then for all $h \in \mathcal{H}$, all $T \in \mathcal{T}_h$ (resp. $F \in \mathcal{F}_h$), all $F \in \mathcal{F}_h$ (resp. $E \in \mathcal{E}_h$), all $v \in \mathcal{P}^l(F)$,*

$$\|v\|_{L^p(F)} \lesssim h_T^{-\frac{1}{p}} \|v\|_{L^p(T)} \quad (2.4)$$

with hidden constant depending only on ρ , l and p .

We will also use Koszul complements (see [14, Section 2.4]). We consider for any face $T \in \mathcal{T}_h$ a point \mathbf{x}_T such that $B(\mathbf{x}_T, \rho h_T) \subset T$. Then we define the following subspaces of $\mathcal{P}^k(T)$:

$$\begin{aligned} \mathcal{G}^k(T) &:= \mathbf{grad} \mathcal{P}^{k+1}(T), & \mathcal{G}^{c,k}(T) &:= (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{k-1}(T), \\ \mathcal{R}^k(T) &:= \mathbf{curl} \mathcal{P}^{k+1}(T), & \mathcal{R}^{c,k}(T) &:= (\mathbf{x} - \mathbf{x}_T) \mathcal{P}^{k-1}(T). \end{aligned} \quad (2.5)$$

These spaces are such that:

$$\mathcal{P}^k(T) = \mathcal{G}^k(T) \oplus \mathcal{G}^{c,k}(T) = \mathcal{R}^k(T) \oplus \mathcal{R}^{c,k}(T), \quad (2.6)$$

however the sum is not orthogonal for the L^2 scalar product.

Similarly, in 2 dimensions for any face $F \in \mathcal{F}_h$, we define:

$$\begin{aligned} \mathcal{G}^k(F) &:= \mathbf{grad} \mathcal{P}^{k+1}(F), & \mathcal{G}^{c,k}(F) &:= (\mathbf{x} - \mathbf{x}_F)^\perp \mathcal{P}^{k-1}(F), \\ \mathcal{R}^k(F) &:= \mathbf{rot} \mathcal{P}^{k+1}(F), & \mathcal{R}^{c,k}(F) &:= (\mathbf{x} - \mathbf{x}_F) \mathcal{P}^{k-1}(F). \end{aligned} \quad (2.7)$$

We also have the following isomorphisms:

$$\mathbf{rot} : \mathcal{P}^{0,k}(F) \rightarrow \mathcal{R}^{k-1}(F), \quad (2.8)$$

$$\mathbf{div} : \mathcal{R}^{c,k}(F) \rightarrow \mathcal{P}^{k-1}(F), \quad \mathbf{div} : \mathcal{R}^{c,k}(T) \rightarrow \mathcal{P}^{k-1}(T), \quad (2.9)$$

$$\mathbf{curl} : \mathcal{G}^{c,k}(T) \rightarrow \mathcal{R}^{k-1}(T). \quad (2.10)$$

We can deduce from Lemma 2 that $\|\mathbf{rot}\| \lesssim h^{-1}$, $\|\mathbf{div}\| \lesssim h^{-1}$, $\|\mathbf{curl}\| \lesssim h^{-1}$ and from [14, Lemma A.9] that $\|(\mathbf{rot})^{-1}\| \lesssim h$, $\|(\mathbf{div})^{-1}\| \lesssim h$, $\|(\mathbf{curl})^{-1}\| \lesssim h$.

For $X = \{T, F\}$ we define the local spaces of Nedelec and of Raviart-Thomas respectively by:

$$\mathcal{N}^k(X) := \mathcal{G}^{k-1}(X) \oplus \mathcal{G}^{c,k}(X), \quad \mathcal{RT}^k(X) := \mathcal{R}^{k-1}(X) \oplus \mathcal{R}^{c,k}(X). \quad (2.11)$$

These spaces are strictly contained between $\mathcal{P}^{k-1}(X)$ and $\mathcal{P}^k(X)$. Another important property given in [14, Proposition A.8] is that for any cell $T \in \mathcal{T}_h$ (resp. $F \in \mathcal{F}_h$) and any face of this cell $F \in \mathcal{F}_T$ (resp. $E \in \mathcal{E}_F$):

$$\begin{aligned} \forall \mathbf{v}_F \in \mathcal{N}^k(F), (\mathbf{v}_F)|_E \cdot \mathbf{t}_E &\in \mathcal{P}^{k-1}(E), \\ \forall \mathbf{w}_F \in \mathcal{RT}^k(F), (\mathbf{w}_F)|_E \cdot \mathbf{n}_{FE} &\in \mathcal{P}^{k-1}(E), \\ \forall \mathbf{v}_T \in \mathcal{N}^k(T), (\mathbf{v}_T)|_E \cdot \mathbf{t}_E &\in \mathcal{P}^{k-1}(E), \\ \forall \mathbf{w}_T \in \mathcal{RT}^k(T), (\mathbf{w}_T)|_F \cdot \mathbf{n}_F &\in \mathcal{P}^{k-1}(F), \\ \forall \mathbf{v}_T \in \mathcal{N}^k(T), (\mathbf{v}_T)|_F \times \mathbf{n}_F &\in \mathcal{RT}^k(F). \end{aligned} \quad (2.12)$$

In order to fix the notation we write

$$(\mathcal{R}^{c,k}(F)^\top)^2 = \begin{pmatrix} \mathcal{R}^{c,k}(F)^\top \\ \mathcal{R}^{c,k}(F)^\top \end{pmatrix}, \quad (\mathcal{R}^{c,k}(T)^\top)^3 = \begin{pmatrix} \mathcal{R}^{c,k}(T)^\top \\ \mathcal{R}^{c,k}(T)^\top \\ \mathcal{R}^{c,k}(T)^\top \end{pmatrix}. \quad (2.13)$$

We take differential operators to be acting row-wise on matrix valued functions, and we use the convention

$$\nabla \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ \partial_1 v_2 & \partial_2 v_2 & \partial_3 v_2 \\ \partial_1 v_3 & \partial_2 v_3 & \partial_3 v_3 \end{pmatrix}.$$

We define the space $\overline{\mathcal{R}}^{c,k}(T)$ by

$$\overline{\mathcal{R}}^{c,k}(T) := \{W \in (\mathcal{R}^{c,k}(T)^\top)^3 : \text{Tr } W = 0\}. \quad (2.14)$$

An explicit description of this space is given by Lemma 42. Let us now construct a complement to this space. First noticing that $\text{Tr}((\mathcal{R}^{c,k}(T)^\top)^3) = \mathcal{P}^{0,k}(T)$, we can consider the inverse operator $\mathbf{P}_{\text{Tr}}^k : \mathcal{P}^{0,k}(T) \rightarrow (\mathcal{R}^{c,k}(T)^\top)^3$:

$$\mathbf{P}_{\text{Tr}}^k := \begin{pmatrix} \text{div}^{-1} \\ \text{div}^{-1} \\ \text{div}^{-1} \end{pmatrix} \circ \mathbf{grad}, \quad (2.15)$$

where div is the isomorphism from $\mathcal{R}^{c,k}(T)$ into $\mathcal{P}^{k-1}(T)$ given by (2.9). Then we define the space:

$$\overline{\mathcal{R}}^k(T) := \mathbf{P}_{\text{Tr}}^k \mathcal{P}^{0,k}(T). \quad (2.16)$$

Lemma 46 shows that the spaces $\overline{\mathcal{R}}^{c,k}(T)$ and $\overline{\mathcal{R}}^k(T)$ are complementary. A similar construction of the spaces $\overline{\mathcal{R}}^{c,k}(F)$ and $\overline{\mathcal{R}}^k(F)$ holds in 2 dimensions, and is used in Appendix C.

Remark 4. By construction, we have: $\nabla \cdot \overline{\mathcal{R}}^k(T) = \nabla \cdot \mathbf{P}_{\text{Tr}}^k \mathcal{P}^{0,k}(T) = \mathbf{grad} \mathcal{P}^k(T) = \mathcal{G}^{k-1}(T)$.

Remark 5. These spaces are hierarchical since $\overline{\mathcal{R}}^{c,k} \subset \overline{\mathcal{R}}^{c,k+1}$, $\overline{\mathcal{R}}^k \subset \overline{\mathcal{R}}^{k+1}$.

We define a matrix valued equivalent to the Raviart-Thomas space as follows

$$\overline{\mathcal{RT}}^k(T) := \overline{\mathcal{R}}^{c,k}(T) \oplus \overline{\mathcal{R}}^{k-1}(T) \oplus (\mathcal{R}^{k-1}(T)^\top)^3. \quad (2.17)$$

Remark 6. For $q \in \mathcal{P}^k(T)$, we have $qI \in \overline{\mathcal{R}}^k(T) \oplus (\mathcal{R}^k(T)^\top)^3$. Indeed $\nabla \cdot (\mathbf{P}_{\text{Tr}} q - qI) = 0$ so $\mathbf{P}_{\text{Tr}} q - qI \in (\mathcal{R}^k(T)^\top)^3$ by the isomorphism (2.9) and (2.6).

Lemma 7. For $X \in \{F, T\}$, $\nabla \cdot$ is an isomorphism from $\overline{\mathcal{R}}^{c,k+1}(X)$ to $\mathcal{G}^{c,k}(X)$.

Proof. The proof for $X = F$ is given by Lemma 44 for $X = T$ in the appendix. The case $X = F$ is far easier and is provable with the same arguments. \square

We will often need to view 2-dimensional spaces as subspace of \mathbb{R}^3 . In particular we introduce two spaces related to the normal plane of an edge and to the tangent plane of a face.

Definition 8. For any edge $E \in \mathcal{E}_h$ a natural 3-dimensional vector is \mathbf{t}_E . We can arbitrarily complete it in an orthonormal basis of \mathbb{R}^3 ($\mathbf{t}_E, \mathbf{n}_1, \mathbf{n}_2$). Assume that \mathbf{n}_1 and \mathbf{n}_2 are fixed once and for all on each edge. We define the space

$$\widetilde{\mathcal{P}}_{\mathbf{n},E}^k(E) = \{p_1 \mathbf{n}_1 + p_2 \mathbf{n}_2 : p_1, p_2 \in \mathcal{P}^k(E)\}. \quad (2.18)$$

Likewise for any face $F \in \mathcal{F}_h$ assume there is a F -dependent fixed basis $(\mathbf{n}_F, \mathbf{n}_1, \mathbf{n}_2)$ of \mathbb{R}^3 . We define the space

$$\tilde{\mathcal{P}}^k(F) = \left\{ \sum_{i,j=\{1,2\}} V_{i,j} \mathbf{n}_i \otimes \mathbf{n}_j + \sum_{i=\{1,2\}} w_i \mathbf{n}_F \otimes \mathbf{n}_i : \right. \\ \left. \mathbf{V} = (V)_{i,j} \in \overline{\mathcal{R}}^{c,k}(F) \oplus \overline{\mathcal{R}}^k(F) \oplus (\mathcal{R}^k(F)^\top)^2, \mathbf{w} = (w)_i \in \mathcal{P}^k(F) \right\}. \quad (2.19)$$

We will implicitly write $\tilde{\mathcal{P}}^k(F) = (\overline{\mathcal{R}}^{c,k}(F) \oplus \overline{\mathcal{R}}^k(F) \oplus (\mathcal{R}^k(F)^\top)^2) \oplus (\mathbf{n}_F \otimes \mathcal{P}^k(F))$ to decompose it into its subcomponents. The last direct sum here is L^2 -orthogonal, hence this will not cause any ambiguity in the scalar products. The space $\tilde{\mathcal{P}}^k(F)$ is isomorph to $M_{3,2}(\mathcal{P}^k(F))$. When embedded in the space of 3 by 3 tensor, elements of $\tilde{\mathcal{P}}^k(F)$ are all orthogonal to \mathbf{n}_F on the right.

3 Discrete complex.

We can now define the discrete complex. We start by giving the degree of freedom and the interpolator of the discrete spaces. Then we define the discrete differential operators and give some basic properties on them. To distinguish operators acting on scalar from those acting on vector we use the notation **grad** for the operator acting on scalar fields and giving vector fields, and ∇ for the operator acting on vector fields and giving a tensor field.

3.1 Complex definition.

We define five discrete spaces $\underline{X}_{\mathbf{grad},h}^k$, $\underline{X}_{\mathbf{curl},h}^k$, $\underline{X}_{\nabla,h}^k$, $\underline{X}_{L^2,h}^{k+1}$ and $\underline{X}_{L^2,h}^k$. Diagram (3.1) summarizes their connection with each other and with their continuous counterpart. Throughout the paper we will use the notations introduced in this section to refer to the components of discrete vectors.

$$\begin{array}{ccccccc} & & & & L^2(\Omega) & & \\ & & & & \uparrow \nabla & & \\ H^2(\Omega) & \xrightarrow{\mathbf{grad}} & \mathbf{H}^2 & \xrightarrow{\mathbf{curl}} & \mathbf{H}^1(\Omega) & \xrightarrow{\mathbf{div}} & L^2(\Omega) \\ \downarrow \underline{I}_{\mathbf{grad},h}^k & & \downarrow \underline{I}_{\mathbf{curl},h}^k & & \downarrow \underline{I}_{\nabla,h}^k & & \downarrow \underline{I}_{L^2,h}^k \\ \underline{X}_{\mathbf{grad},h}^k & \xrightarrow{\underline{C}_h^k} & \underline{X}_{\mathbf{curl},h}^k & \xrightarrow{\underline{C}_h^k} & \underline{X}_{\nabla,h}^k & \xrightarrow{\underline{D}_h^k} & \underline{X}_{L^2,h}^k \\ & & & & \downarrow \underline{\nabla}_h^{k+1} & & \\ & & & & \underline{X}_{L^2,h}^{k+1} & & \leftarrow \end{array} \quad (3.1)$$

Notice that the interpolators (defined in Section 3.2) require more smoothness than the spaces shown in (3.1). Discrete spaces are defined by:

$$\underline{X}_{\mathbf{grad},h}^k := \{ \underline{q}_h = ((\mathbf{G}_{q,V})_{V \in \mathcal{V}_h}, (q_E, \mathbf{G}_{q,E})_{E \in \mathcal{E}_h}, (q_F, \mathbf{G}_{q,F})_{F \in \mathcal{F}_h}, (q_T)_{T \in \mathcal{T}_h}) : \\ \mathbf{G}_{q,V} \in \mathbb{R}^3(V), \forall V \in \mathcal{V}_h, \\ q_E \in \mathcal{P}_c^{k+1}(\mathcal{E}_h), \mathbf{G}_{q,E} \in \tilde{\mathcal{P}}_{n,E}^k(E), \forall E \in \mathcal{E}_h, \\ q_F \in \mathcal{P}^{k-1}(F), \mathbf{G}_{q,F} \in \mathcal{P}^{k-1}(F), \forall F \in \mathcal{F}_h, \\ q_T \in \mathcal{P}^{k-1}(T), \forall T \in \mathcal{T}_h \}, \quad (3.2)$$

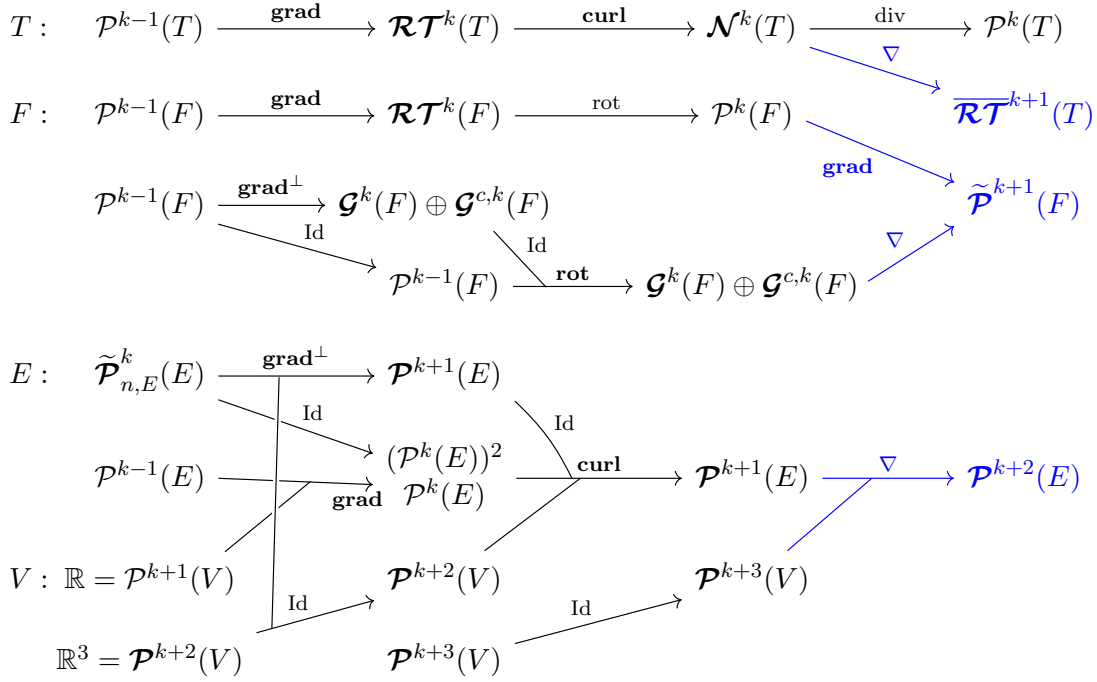


Figure 1: Usage of the local degrees of freedom for the discrete differential operators.

$$\underline{\mathbf{X}}_{\text{curl},h}^k := \{ \underline{\mathbf{v}}_h = ((\mathbf{R}_{v,V})_{V \in \mathcal{V}_h}, (\mathbf{v}_E, \mathbf{R}_{v,E})_{E \in \mathcal{E}_h}, (\mathbf{v}_{\mathcal{R},F}, \mathbf{v}_{\mathcal{R},F}^c, v_F, \mathbf{R}_{v,\mathcal{G},F}, \mathbf{R}_{v,\mathcal{G},F}^c)_{F \in \mathcal{F}_h}, \quad (3.3)$$

$$(\mathbf{v}_{\mathcal{R},T}, \mathbf{v}_{\mathcal{R},T}^c)_{T \in \mathcal{T}_h}) : \mathbf{R}_{v,V} \in \mathbb{R}^3(V), \forall V \in \mathcal{V}_h,$$

$$\mathbf{v}_E \in \mathcal{P}_c^{k+2}(\mathcal{E}_h), \mathbf{R}_{v,E} \in \mathcal{P}^{k+1}(E), \forall E \in \mathcal{E}_h,$$

$$\mathbf{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F), \mathbf{v}_{\mathcal{R},F}^c \in \mathcal{R}^{c,k}(F), v_F \in \mathcal{P}^{k-1}(F),$$

$$\mathbf{R}_{v,\mathcal{G},F} \in \mathcal{G}^k(F), \mathbf{R}_{v,\mathcal{G},F}^c \in \mathcal{G}^{c,k}(F), \forall F \in \mathcal{F}_h,$$

$$\mathbf{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T), \mathbf{v}_{\mathcal{R},T}^c \in \mathcal{R}^{c,k}(T), \forall T \in \mathcal{T}_h \},$$

$$\underline{\mathbf{X}}_{\nabla,h}^k := \{ \underline{\mathbf{w}}_h = ((\mathbf{w}_E)_{E \in \mathcal{E}_h}, (\mathbf{w}_F, \mathbf{w}_{\mathcal{G},F}, \mathbf{w}_{\mathcal{G},F}^c)_{F \in \mathcal{F}_h}, (\mathbf{w}_{\mathcal{G},T}, \mathbf{w}_{\mathcal{G},T}^c)_{T \in \mathcal{T}_h}) : \quad (3.4)$$

$$\mathbf{w}_E \in \mathcal{P}_c^{k+3}(E), \forall E \in \mathcal{E}_h$$

$$\mathbf{w}_F \in \mathcal{P}^k(F), \mathbf{w}_{\mathcal{G},F} \in \mathcal{G}^k(F), \mathbf{w}_{\mathcal{G},F}^c \in \mathcal{G}^{c,k}(F), \forall F \in \mathcal{F}_h,$$

$$\mathbf{w}_{\mathcal{G},T} \in \mathcal{G}^{k-1}(T), \mathbf{w}_{\mathcal{G},T}^c \in \mathcal{G}^{c,k}(T), \forall T \in \mathcal{T}_h \},$$

$$\underline{\mathbf{X}}_{L^2,h}^{k+1} := \{ \underline{\mathbf{W}}_h = ((\mathbf{W}_E)_{E \in \mathcal{E}_h}, (\mathbf{W}_F)_{F \in \mathcal{F}_h}, (\mathbf{W}_T)_{T \in \mathcal{T}_h}) : \quad (3.5)$$

$$\mathbf{W}_E \in \mathcal{P}^{k+2}(E), \forall E \in \mathcal{E}_h,$$

$$\mathbf{W}_F \in \tilde{\mathcal{P}}^{k+1}(F), \forall F \in \mathcal{F}_h, \mathbf{W}_T \in \overline{\mathcal{RT}}^{k+1}(T), \forall T \in \mathcal{T}_h \},$$

$$\underline{\mathbf{X}}_{L^2,h}^k := \{ \underline{\mathbf{q}}_h = ((q_T)_{T \in \mathcal{T}_h}) : q_T \in \mathcal{P}^k(T), \forall T \in \mathcal{T}_h \}. \quad (3.6)$$

Figure 1 summarizes the involvement of the various degrees of freedom with the differential operators.

For a given cell T we define the local discrete spaces $\underline{\mathbf{X}}_{\text{grad},T}^k$, $\underline{\mathbf{X}}_{\text{curl},T}^k$, $\underline{\mathbf{X}}_{\nabla,T}^k$, $\underline{\mathbf{X}}_{L^2,T}^k$ and $\underline{\mathbf{X}}_{L^2,T}^{k+1}$ as the restriction of the global one to T , i.e. containing only the components attached to T and those attached to the faces, edges and vertices lying on its boundary. We define in the same way the local discrete spaces attached to a face F or an edge E .

3.2 Interpolators.

In this section we define the interpolator linking discrete spaces to their continuous counterpart. Since we project on objects of lower dimension (edges and vertices) we will need a somewhat high smoothness for the continuous functions. For a vertex $V \in \mathcal{V}_h$ we define $\mathbf{x}_V \in \mathbb{R}^3$ to be its coordinate. The interpolator on the space $\underline{X}_{\mathbf{grad},h}^k$ is defined for any $q \in C^1(\bar{\Omega})$ by

$$\begin{aligned} \underline{I}_{\mathbf{grad},h}^k q = & ((\mathbf{grad} q(\mathbf{x}_V))_{V \in \mathcal{V}_h}, (q_E, \boldsymbol{\pi}_{\mathcal{P},E}^k(\mathbf{t}_E \times (\mathbf{grad} q \times \mathbf{t}_E)))_{E \in \mathcal{E}_h}, \\ & (\pi_{\mathcal{P},F}^{k-1}(q), \pi_{\mathcal{P},F}^{k-1}(\mathbf{grad}(q) \cdot \mathbf{n}_F), (\pi_{\mathcal{P},T}^{k-1}(q))_{T \in \mathcal{T}_h}), \end{aligned} \quad (3.7)$$

where for any edge $E \in \mathcal{E}_h$, q_E is such that $\pi_{\mathcal{P},E}^{k-1}(q_E) = \pi_{\mathcal{P},E}^{k-1}(q)$ and for any vertex $V \in \mathcal{V}_E$, $q_E(\mathbf{x}_V) = q(\mathbf{x}_V)$.

The interpolator on the space $\underline{X}_{\mathbf{curl},h}^k$ is defined for any $\mathbf{v} \in C^1(\bar{\Omega})$ by

$$\begin{aligned} \underline{I}_{\mathbf{curl},h}^k \mathbf{v} = & ((\mathbf{curl} \mathbf{v}(\mathbf{x}_V))_{V \in \mathcal{V}_h}, (\mathbf{v}_E, \boldsymbol{\pi}_{\mathcal{P},E}^{k+1}((\mathbf{curl} \mathbf{v} \cdot \mathbf{t}_E)\mathbf{t}_E + \mathbf{grad}(\mathbf{v} \cdot \mathbf{t}_E) \times \mathbf{t}_E))_{E \in \mathcal{E}_h}, \\ & (\boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\mathbf{v}_{t,F}), \boldsymbol{\pi}_{\mathcal{R},F}^{c,k}(\mathbf{v}_{t,F}), \pi_{\mathcal{P},F}^{k-1}(\mathbf{v} \cdot \mathbf{n}_F), \\ & \boldsymbol{\pi}_{\mathcal{G},F}^k(\mathbf{n}_F \times (\nabla \mathbf{v} \cdot \mathbf{n}_F)), \boldsymbol{\pi}_{\mathcal{G},F}^{c,k}(\mathbf{n}_F \times (\nabla \mathbf{v} \cdot \mathbf{n}_F)))_{F \in \mathcal{F}_h}, \\ & (\boldsymbol{\pi}_{\mathcal{R},T}^{k-1}(\mathbf{v}), \boldsymbol{\pi}_{\mathcal{R},T}^{c,k}(\mathbf{v}))_{T \in \mathcal{T}_h}), \end{aligned} \quad (3.8)$$

where $\mathbf{v}_{t,F}$ is the tangential trace of \mathbf{v} on F , and where for any edge $E \in \mathcal{E}_h$, \mathbf{v}_E is such that $\boldsymbol{\pi}_{\mathcal{P},E}^k(\mathbf{v}_E) = \boldsymbol{\pi}_{\mathcal{P},E}^k(\mathbf{v})$ and for any vertex $V \in \mathcal{V}_E$, $\mathbf{v}_E(\mathbf{x}_V) = \mathbf{v}(\mathbf{x}_V)$.

The interpolator on the space $\underline{X}_{\nabla,h}^k$ is defined for any $\mathbf{w} \in C^0(\bar{\Omega})$ by

$$\begin{aligned} \underline{I}_{\nabla,h}^k \mathbf{w} = & ((\mathbf{w}_E)_{E \in \mathcal{E}_h}, (\pi_{\mathcal{P},F}^k(\mathbf{w} \cdot \mathbf{n}_F), \boldsymbol{\pi}_{\mathcal{G},F}^k(\mathbf{w}_{t,F}), \boldsymbol{\pi}_{\mathcal{G},F}^{c,k}(\mathbf{w}_{t,F}))_{F \in \mathcal{F}_h}, \\ & (\boldsymbol{\pi}_{\mathcal{G},T}^{k-1}(\mathbf{w}), \boldsymbol{\pi}_{\mathcal{G},T}^{c,k}(\mathbf{w}))_{T \in \mathcal{T}_h}), \end{aligned} \quad (3.9)$$

where for any edge $E \in \mathcal{E}_h$, $\mathbf{w}_E \in \mathcal{P}^{k+3}(E)$ is such that $\boldsymbol{\pi}_{\mathcal{P},E}^{k+1}(\mathbf{w}_E) = \boldsymbol{\pi}_{\mathcal{P},E}^{k+1}(\mathbf{w})$ and for any vertex $V \in \mathcal{V}_E$, $\mathbf{w}_E(\mathbf{x}_V) = \mathbf{w}(\mathbf{x}_V)$.

The interpolator on the space $\underline{X}_{L^2,h}^{k+1}$ is defined for any $\mathbf{W} \in C^0(\bar{\Omega})^3$ by

$$\underline{I}_{L^2,h}^k \mathbf{W} = ((\boldsymbol{\pi}_{\mathcal{P},E}^{k+2}(\mathbf{W} \mathbf{t}_E))_{E \in \mathcal{E}_h}, (\boldsymbol{\pi}_{\tilde{\mathcal{P}},F}^{k+1}(\mathbf{W}))_{F \in \mathcal{F}_h}, (\boldsymbol{\pi}_{\mathcal{RT},T}^{k+1}(\mathbf{W}))_{T \in \mathcal{T}_h}). \quad (3.10)$$

The interpolator on the space $\underline{X}_{L^2,h}^k$ is defined for any $q \in L^2(\Omega)$ by

$$\underline{I}_{L^2,h}^k q = ((\pi_{\mathcal{P},T}^k(q))_{T \in \mathcal{T}_h}) \quad (3.11)$$

3.3 Discrete operators

3.3.1 Gradient.

In the following sections we define the discrete operators starting with the discrete gradient operator $\underline{\mathbf{G}}_h^k$. The operator $\underline{\mathbf{G}}_h^k$ is the collection of the local discrete operators (3.18) acting on the edges, faces and cells. For any edge $E \in \mathcal{E}_h$ we define the operator $\underline{\mathbf{G}}_E^k : \underline{X}_{\mathbf{grad},E}^k \rightarrow \underline{X}_{\mathbf{curl},E}^k$ such that $\forall q_E \in \underline{X}_{\mathbf{grad},E}^k$

$$\underline{\mathbf{G}}_E^k q_E = ((\mathbf{0})_{V \in \mathcal{V}_E}, \mathbf{v}_E, \mathbf{v}_E' \times \mathbf{t}_E), \quad (3.12)$$

where \mathbf{v}_E is such that $\boldsymbol{\pi}_{\mathcal{P},E}^k(\mathbf{v}_E) = q_E' \mathbf{t}_E + \mathbf{G}_{q,E}$ and $\forall V \in \mathcal{V}_E$, $\mathbf{v}_E(\mathbf{x}_V) = \mathbf{G}_{q,V}$. We write q_E' the derivative of q_E along the edge E (oriented by \mathbf{t}_E).

For any face $F \in \mathcal{F}_h$ we define the operator $\mathbf{G}_F^k : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^k(F)$ such that $\forall \underline{q}_F \in \underline{X}_{\text{grad},F}^k, \forall \mathbf{w}_F \in \mathcal{P}^k(F)$

$$\int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{w}_F = - \int_F q_F \operatorname{div} \mathbf{w}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E \mathbf{w}_F \cdot \mathbf{n}_{FE}. \quad (3.13)$$

We also define $\mathbf{G}_F^{\perp k} : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^k(F)$ such that $\forall \underline{q}_F \in \underline{X}_{\text{grad},F}^k, \forall \mathbf{w}_F \in \mathcal{P}^k(F)$

$$\int_F \mathbf{G}_F^{\perp k} \underline{q}_F \cdot \mathbf{w}_F = - \int_F G_{q,F} \operatorname{rot} \mathbf{w}_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\mathbf{G}_{q,E} \cdot \mathbf{n}_F) (\mathbf{w}_F \cdot \mathbf{t}_E). \quad (3.14)$$

The full operator $\underline{\mathbf{G}}_F^k : \underline{X}_{\text{grad},F}^k \rightarrow \underline{X}_{\text{curl},F}^k$ is defined to be the collection and projection of the local operators. Explicitly for all $\underline{q}_h \in \underline{X}_{\text{grad},F}^k$

$$\underline{\mathbf{G}}_F^k \underline{q}_F = ((\underline{\mathbf{G}}_E^k \underline{q}_E)_{E \in \mathcal{E}_F}, (\pi_{\mathcal{R},F}^{k-1}(\mathbf{G}_F^k \underline{q}_F), \pi_{\mathcal{R},F}^{c,k}(\mathbf{G}_F^k \underline{q}_F), G_{q,F}, \pi_{\mathcal{G},F}^k(\mathbf{G}_F^{\perp k} \underline{q}_F), \pi_{\mathcal{G},F}^{c,k}(\mathbf{G}_F^{\perp k} \underline{q}_F))). \quad (3.15)$$

The scalar trace $\gamma_{\text{grad},F}^{k+1} : \underline{X}_{\text{grad},F}^k \rightarrow \mathcal{P}^{k+1}(F)$ is defined such that $\forall \underline{q}_F \in \underline{X}_{\text{grad},F}^k, \forall \mathbf{v}_F \in \mathcal{R}^{c,k+2}(F)$,

$$\int_F \gamma_{\text{grad},F}^{k+1} \underline{q}_F \operatorname{div} \mathbf{v}_F = - \int_F \mathbf{G}_F^k \underline{q}_F \cdot \mathbf{v}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E (\mathbf{v}_F \cdot \mathbf{n}_{FE}). \quad (3.16)$$

Remark 9. The relation (3.16) holds for all $\mathbf{v}_F \in \mathcal{R}^k(F) \oplus \mathcal{R}^{c,k+2}(F)$. This is the same definition as for the discrete De Rham complex [14].

Remark 10. We can also define a tangential trace in the same manner as [14]. It is not required thanks to the choice of norm (3.54) but should be considered to show consistency results for $\underline{X}_{\text{grad},h}^k$.

For any cell $T \in \mathcal{T}_h$ we define the operator $\mathbf{G}_T^k : \underline{X}_{\text{grad},T}^k \rightarrow \mathcal{P}^k(T)$ such that $\forall \underline{q}_T \in \underline{X}_{\text{grad},T}^k, \forall \mathbf{w}_T \in \mathcal{P}^k(T)$

$$\int_T \mathbf{G}_T^k \underline{q}_T \cdot \mathbf{w}_T = - \int_T q_T \operatorname{div} \mathbf{w}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\text{grad},F}^{k+1} \underline{q}_F \mathbf{w}_T \cdot \mathbf{n}_F. \quad (3.17)$$

Likewise we define the full operator $\underline{\mathbf{G}}_T^k : \underline{X}_{\text{grad},T}^k \rightarrow \underline{X}_{\text{curl},T}^k$ for all $\underline{q}_T \in \underline{X}_{\text{grad},T}^k$ by

$$\underline{\mathbf{G}}_T^k \underline{q}_T = ((\underline{\mathbf{G}}_E^k \underline{q}_E)_{E \in \mathcal{E}_T}, (\underline{\mathbf{G}}_F^k \underline{q}_F)_{F \in \mathcal{F}_T}, (\pi_{\mathcal{R},T}^{k-1}(\mathbf{G}_T^k \underline{q}_T), \pi_{\mathcal{R},T}^{c,k}(\mathbf{G}_T^k \underline{q}_T))). \quad (3.18)$$

The global operator $\underline{\mathbf{G}}_h^k$ is obtained by gathering the local operators $\underline{\mathbf{G}}_T^k, T \in \mathcal{T}_h$. Since the interpolators require taking the full gradient even on edges we must consider functions to be defined in a small neighborhood $E \subset \mathcal{O}_E$ open in \mathbb{R}^3 . We also define \mathbf{X} to be the polynomial $\mathbf{X}(\mathbf{x}) = \mathbf{x}$.

Lemma 11 (Consistency properties). *The discrete gradients and trace satisfy the following consistency properties for all $E \in \mathcal{E}_h, F \in \mathcal{F}_h$ and $T \in \mathcal{T}_h$:*

$$\underline{\mathbf{G}}_E^k(I_{\text{grad},E}^k q) = I_{\text{curl},E}^k(\mathbf{grad} q) \quad \forall q \in C^2(\mathcal{O}_E) \quad (3.19)$$

$$\mathbf{G}_F^k(I_{\text{grad},F}^k q) = \mathbf{grad} q \quad \forall q \in \mathcal{P}^{k+1}(F) \quad (3.20)$$

$$\mathbf{G}_F^{\perp k}(I_{\text{grad},F}^k(q \mathbf{X} \cdot \mathbf{n}_F)) = - \operatorname{rot} q \quad \forall q \in \mathcal{P}^{k+1}(F) \quad (3.21)$$

$$\gamma_{\text{grad},F}^{k+1}(I_{\text{grad},F}^k q) = q \quad \forall q \in \mathcal{P}^{k+1}(F) \quad (3.22)$$

$$\pi_{\mathcal{P},F}^{k-1}(\gamma_{\text{grad},F}^{k+1} \underline{q}_F) = q_F \quad \forall \underline{q}_F \in \underline{X}_{\text{grad},F}^k \quad (3.23)$$

$$\mathbf{G}_T^k(I_{\text{grad},T}^k q) = \mathbf{grad} q \quad \forall q \in \mathcal{P}^{k+1}(T) \quad (3.24)$$

Proof. Properties (3.20), (3.22), (3.23) and (3.24) are proven in [14, Lemma 3.3].

Proof of (3.19). The idea is to use integration by parts together with the continuity on vertices to remove the projection. Let $E \in \mathcal{E}_h$, $q \in C^2(\mathcal{O}_E)$, $\underline{\mathbf{G}}_E^k(\underline{\mathbf{I}}_{\mathbf{grad},E}^k q) := ((\mathbf{0})_{V \in \mathcal{V}_E}, \mathbf{v}_E, \mathbf{v}_{E'} \times \mathbf{t}_E, q_E)$ as in (3.7) and $\mathbf{w}_E \in \mathcal{P}^{k+2}(E)$ such that $\pi_{\mathcal{P},E}^k(\mathbf{w}_E) = \pi_{\mathcal{P},E}^k(\mathbf{grad} q)$ and $\mathbf{w}_E(\mathbf{x}_V) = \mathbf{grad} q(\mathbf{x}_V)$ for $V \in \mathcal{V}_E$. We have $\mathbf{0} = \mathbf{curl} \mathbf{grad} q(\mathbf{x}_V)$, $\forall V \in \mathcal{V}_E$. We must show that $\mathbf{v}_E = \mathbf{w}_E$ and that $\mathbf{v}_{E'} \times \mathbf{t}_E = \pi_{\mathcal{P},E}^{k+1}(((\mathbf{curl} \mathbf{grad} q) \cdot \mathbf{t}_E) \mathbf{t}_E + (\mathbf{grad}(\mathbf{grad} q \cdot \mathbf{t}_E) \times \mathbf{t}_E)) = \pi_{\mathcal{P},E}^{k+1}((\mathbf{grad}(\mathbf{grad} q \cdot \mathbf{t}_E) \times \mathbf{t}_E))$. Take a standard basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ such that $\mathbf{t}_E = (1, 0, 0)$ and $\mathbf{v}_E := (v_0, v_1, v_2)$. For all $\mathbf{r} \in \mathcal{P}^k(E)$,

$$\begin{aligned} \int_E \mathbf{v}_E \cdot \mathbf{r} &= \int_E q_E' r_0 + \int_E \pi_{\mathcal{P},E}^k(\partial_1 q) r_1 + \pi_{\mathcal{P},E}^k(\partial_2 q) r_2 \\ &= - \int_E q_E r_0' + [q_E r_0] + \int_E \pi_{\mathcal{P},E}^k(\partial_1 q) r_1 + \pi_{\mathcal{P},E}^k(\partial_2 q) r_2 \\ &= - \int_E q r_0' + [q r_0] + \int_E \partial_1 q r_1 + \partial_2 q r_2 \\ &= \int_E \mathbf{grad} q \cdot \mathbf{r}. \end{aligned}$$

We use that $q_E(\mathbf{x}_V) = q(\mathbf{x}_V)$, $V \in \mathcal{V}_E$ to get the third line. Hence, $\pi_{\mathcal{P},E}^k(\mathbf{v}_E) = \pi_{\mathcal{P},E}^k(\mathbf{grad} q)$ and since $\mathbf{v}_E(\mathbf{x}_V) = \mathbf{grad} q(\mathbf{x}_V)$ we have $\mathbf{v}_E = \mathbf{w}_E$. We conclude with the same argument for $\mathbf{r} \in \mathcal{P}^{k+1}(E)$ applied to $\mathbf{v}_{E'} \times \mathbf{t}_E$.

Proof of (3.21). Let $q \in \mathcal{P}^{k+1}(F)$. Since q does not depend on the coordinate in the \mathbf{n}_F direction, we have $\mathbf{grad}(q \mathbf{X} \cdot \mathbf{n}_F) \cdot \mathbf{n}_F = q$. The relevant parts of $\underline{\mathbf{I}}_{\mathbf{grad},F}^k(q \mathbf{X} \cdot \mathbf{n}_F)$ are $G_q \mathbf{X} \cdot \mathbf{n}_{F,F} = \pi_{\mathcal{P},F}^{k-1}(\mathbf{grad}(q \mathbf{X} \cdot \mathbf{n}_F) \cdot \mathbf{n}_F) = \pi_{\mathcal{P},F}^{k-1}(q)$ and $\mathbf{G}_q \mathbf{X} \cdot \mathbf{n}_{F,E} = \pi_{\mathcal{P},F}^k(\mathbf{grad}(q \mathbf{X} \cdot \mathbf{n}_F) - (\mathbf{grad}(q \mathbf{X} \cdot \mathbf{n}_F) \cdot \mathbf{t}_E) \mathbf{t}_E)$. Hence we have $\mathbf{G}_q \mathbf{X} \cdot \mathbf{n}_{F,E} \cdot \mathbf{n}_F = \pi_{\mathcal{P},E}^k(q)$ and for all $\mathbf{w} \in \mathcal{P}^k(F)$,

$$\begin{aligned} \int_F \mathbf{G}_F^{\perp k}(\underline{\mathbf{I}}_{\mathbf{grad},F}^k(q \mathbf{X} \cdot \mathbf{n}_F)) \cdot \mathbf{w} &= - \int_F \pi_{\mathcal{P},F}^{k-1}(q) \mathbf{rot} \mathbf{w} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \pi_{\mathcal{P},E}^k(q) \mathbf{w} \cdot \mathbf{t}_E \\ &= - \int_F q \mathbf{rot} \mathbf{w} - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q \mathbf{w} \cdot \mathbf{t}_E \\ &= - \int_F \mathbf{rot} q \cdot \mathbf{w}. \end{aligned}$$

□

3.3.2 Curl.

The operator $\underline{\mathbf{C}}_h^k$ is the collection of the local discrete operators (3.35) acting on the edges and faces. For any edge $E \in \mathcal{E}_h$ we define the operator $\mathbf{C}_E^k : \underline{\mathbf{X}}_{\mathbf{curl},E}^k \rightarrow \mathcal{P}^{k+3}(E)$ such that $\forall \underline{\mathbf{v}}_E = ((\mathbf{R}_{v,V})_{V \in \mathcal{V}_E}, \mathbf{v}_E, \mathbf{R}_{v,E}) \in \underline{\mathbf{X}}_{\mathbf{curl},E}^k$

$$\pi_{\mathcal{P},E}^{k+1}(\mathbf{C}_E^k \underline{\mathbf{v}}_E) = \mathbf{R}_{v,E} - \mathbf{v}_{E'} \times \mathbf{t}_E, \quad \mathbf{C}_E^k \underline{\mathbf{v}}_E(\mathbf{x}_V) = \mathbf{R}_{v,V}. \quad (3.25)$$

For any face $F \in \mathcal{F}_h$ we define the operators $\mathbf{C}_F^k : \underline{\mathbf{X}}_{\mathbf{curl},F}^k \rightarrow \mathcal{P}^k(F)$ and $\mathbf{C}_F^k : \underline{\mathbf{X}}_{\mathbf{curl},F}^k \rightarrow \mathcal{P}^k(F)$ for all $\underline{\mathbf{v}}_F = ((\mathbf{v}_E, \mathbf{R}_{v,E})_{E \in \mathcal{E}_h}, (\mathbf{v}_{\mathcal{R},F}, \mathbf{v}_{\mathcal{R},F}^c, v_F, \mathbf{R}_{v,\mathcal{G},F}, \mathbf{R}_{v,\mathcal{G},F}^c)) \in \underline{\mathbf{X}}_{\mathbf{curl},F}^k$ such that $\forall r_F \in \mathcal{P}^k(F)$

$$\int_F \mathbf{C}_F^k \underline{\mathbf{v}}_F r_F = \int_F \mathbf{v}_{\mathcal{R},F} \cdot \mathbf{rot} r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\mathbf{v}_E \cdot \mathbf{t}_E) r_F, \quad (3.26)$$

and $\forall \mathbf{r}_F \in \mathcal{P}^k(F)$

$$\int_F \mathbf{C}_F^k \underline{\mathbf{v}}_F \cdot \mathbf{r}_F = \int_F v_F \operatorname{rot} \mathbf{r}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\mathbf{v}_E \cdot \mathbf{n}_F) (\mathbf{r}_F \cdot \mathbf{t}_E). \quad (3.27)$$

The full operator $\underline{\mathbf{C}}_F^k : \underline{\mathbf{X}}_{\operatorname{curl},F}^k \rightarrow \underline{\mathbf{X}}_{\nabla,F}^k$ is defined as the collection and projection of the local operators. Explicitly for all $\underline{\mathbf{v}}_F \in \underline{\mathbf{X}}_{\operatorname{curl},F}^k$

$$\underline{\mathbf{C}}_F^k \underline{\mathbf{v}}_F = ((\mathbf{C}_E^k \underline{\mathbf{v}}_E)_{E \in \mathcal{E}_F}, (\mathbf{C}_F^k \underline{\mathbf{v}}_F, \boldsymbol{\pi}_{\mathcal{G},F}^k(\mathbf{C}_F^k \underline{\mathbf{v}}_F) + \mathbf{R}_{\mathbf{v},\mathcal{G},F}, \boldsymbol{\pi}_{\mathcal{G},F}^{c,k}(\mathbf{C}_F^k \underline{\mathbf{v}}_F) + \mathbf{R}_{\mathbf{v},\mathcal{G},F}^c)). \quad (3.28)$$

Lemma 12 (Local complex property). *For all $F \in \mathcal{F}_h$ it holds:*

$$\operatorname{Im} \underline{\mathbf{G}}_F^k \subset \operatorname{Ker} \underline{\mathbf{C}}_F^k. \quad (3.29)$$

Proof. Let $\underline{q}_F \in \underline{\mathbf{X}}_{\operatorname{grad},F}^k$, we have to show that $\underline{\mathbf{C}}_F^k(\underline{\mathbf{G}}_F^k \underline{q}_F) = 0$. We define $\underline{\mathbf{v}}_F = \underline{\mathbf{G}}_F^k \underline{q}_F$. It is immediate to check for the edges since $\boldsymbol{\pi}_{\mathcal{P},E}^{k+1}(\mathbf{C}_E^k \underline{\mathbf{v}}_E) = \mathbf{R}_{\mathbf{v},E} - \mathbf{v}_{E'} \times \mathbf{t}_E$ with $\mathbf{R}_{\mathbf{v},E} = \mathbf{v}_{E'} \times \mathbf{t}_E$ and $\mathbf{C}_E^k \underline{\mathbf{v}}_E(\mathbf{x}_V) = \mathbf{0}$. Next in order to prove that $\mathbf{C}_F^k \underline{\mathbf{v}}_F = 0$ take any $r_F \in \mathcal{P}^k(F)$, then

$$\begin{aligned} \int_F \mathbf{C}_F^k \underline{\mathbf{v}}_F r_F &= \int_F \boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\underline{\mathbf{G}}_F^k \underline{q}_F) \cdot \operatorname{rot} r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{v}_E \cdot \mathbf{t}_E r_F \\ &= \int_F \underline{\mathbf{G}}_F^k \underline{q}_F \cdot \operatorname{rot} r_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_{E'} r_F \\ &= - \int_F q_F \cdot \operatorname{div}(\operatorname{rot} r_F) + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E \operatorname{rot} r_F \cdot \mathbf{n}_{FE} - q_{E'} r_F \\ &= \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E r_{F'} - q_{E'} r_F \\ &= \sum_{E \in \mathcal{E}_F} \omega_{FE} [q_E r_F]. \end{aligned}$$

The last term is null since we integrate over a closed loop. It remains to prove $\mathbf{C}_F^k \underline{\mathbf{v}}_F = 0$. For any \mathbf{r}_F successively in $\mathcal{G}^k(F)$ and $\mathcal{G}^{c,k}(F)$ it is immediate to check that $\int_F \mathbf{C}_F^k(\underline{\mathbf{G}}_F^k \underline{q}_F) \cdot \mathbf{r}_F = 0$ since (3.27) and (3.14) are opposite. \square

We define the tangential trace $\gamma_{t,\operatorname{rot},F}^k : \underline{\mathbf{X}}_{\operatorname{curl},F}^k \rightarrow \mathcal{P}^k(F)$ such that $\forall (r_F, \mathbf{w}_F) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{c,k}(F)$,

$$\int_F \gamma_{t,\operatorname{rot},F}^k \underline{\mathbf{v}}_F \cdot (\operatorname{rot} r_F + \mathbf{w}_F) = \int_F \mathbf{C}_F^k \underline{\mathbf{v}}_F r_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\mathbf{v}_E \cdot \mathbf{t}_E) r_F + \int_F \mathbf{v}_{\mathcal{R},F}^c \cdot \mathbf{w}_F. \quad (3.30)$$

This is almost the same definition as in [14]. As such we have almost the same properties.

Lemma 13 (Properties of the tangential trace). *It holds*

$$\boldsymbol{\pi}_{\mathcal{R},F}^{k-1}(\gamma_{t,\operatorname{rot},F}^k \underline{\mathbf{v}}_F) = \mathbf{v}_{\mathcal{R},F} \text{ and } \boldsymbol{\pi}_{\mathcal{R},F}^{c,k}(\gamma_{t,\operatorname{rot},F}^k \underline{\mathbf{v}}_F) = \mathbf{v}_{\mathcal{R},F}^c \quad \forall \underline{\mathbf{v}}_F \in \underline{\mathbf{X}}_{\operatorname{curl},F}^k, \quad (3.31)$$

$$\gamma_{t,\operatorname{rot},F}^k(\underline{\mathbf{I}}_{\operatorname{curl},F}^k \mathbf{v}) = \boldsymbol{\pi}_{\mathcal{P},F}^k \mathbf{v} \quad \forall \mathbf{v} \in \mathcal{N}^{k+1}(F), \quad (3.32)$$

$$\boldsymbol{\pi}_{\mathcal{R}\mathcal{T},F}^k(\gamma_{t,\operatorname{rot},F}^k(\underline{\mathbf{G}}_F^k \underline{q}_F)) = \boldsymbol{\pi}_{\mathcal{R}\mathcal{T},F}^k(\underline{\mathbf{G}}_F^k \underline{q}_F) \quad \forall \underline{q}_F \in \underline{\mathbf{X}}_{\operatorname{grad},F}^k. \quad (3.33)$$

Proof. The proof of [14, Proposition 3.3] almost works here, the sole difference being the continuity of $\mathbf{v}_E \cdot \mathbf{t}_E$ in the boundary term: we will only have $\pi_{\mathcal{P},E}^k(\mathbf{v}_E \cdot \mathbf{t}_E) = q_{E'}$ instead of $\mathbf{v}_E \cdot \mathbf{t}_E = q_{E'}$. This only affects the proof of (3.33), for which we must restrict ourselves to test functions in $\mathcal{P}^{0,k}(F) \times \mathcal{R}^{c,k}(F)$ instead of $\mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{c,k}(F)$. This explains the addition of $\pi_{\mathcal{RT},F}^k$ since $\mathbf{rot} \mathcal{P}^{0,k}(F) \oplus \mathcal{R}^{c,k}(F) \approx \mathcal{RT}^k(F)$. \square

For any $T \in \mathcal{T}_h$ we define the operator $\mathbf{C}_T^k : \underline{\mathbf{X}}_{\mathbf{curl},T}^k \rightarrow \mathcal{P}^k(T)$ for all $\underline{\mathbf{v}}_T = ((\mathbf{v}_F)_{F \in \mathcal{F}_T}, \mathbf{v}_{\mathcal{R},T}, \mathbf{v}_{\mathcal{R},T}^c) \in \underline{\mathbf{X}}_{\mathbf{curl},T}^k$ such that $\forall \mathbf{r}_T \in \mathcal{P}^k(T)$,

$$\int_T \mathbf{C}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{r}_T = \int_T \mathbf{v}_{\mathcal{R},T} \cdot \mathbf{curl} \mathbf{r}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{t,\mathbf{rot},F}^k \mathbf{v}_F \cdot (\mathbf{r}_T \times \mathbf{n}_F). \quad (3.34)$$

The full operator $\underline{\mathbf{C}}_T^k : \underline{\mathbf{X}}_{\mathbf{curl},T}^k \rightarrow \underline{\mathbf{X}}_{\nabla,T}^k$ is such that, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\mathbf{curl},T}^k$,

$$\underline{\mathbf{C}}_T^k \underline{\mathbf{v}}_T := ((\mathbf{C}_E^k \mathbf{v}_E)_{E \in \mathcal{E}_T}, (\mathbf{C}_F^k \mathbf{v}_F)_{F \in \mathcal{F}_T}, \pi_{\mathcal{G},T}^{k-1}(\mathbf{C}_T^k \underline{\mathbf{v}}_T), \pi_{\mathcal{G},T}^{c,k}(\mathbf{C}_T^k \underline{\mathbf{v}}_T)). \quad (3.35)$$

3.3.3 Jacobian.

The operator $\underline{\nabla}_h^{k+1}$ is the collection of the local discrete operators (3.45). For all edge $E \in \mathcal{F}_h$ we define the operator $\nabla_E^{k+2} : \underline{\mathbf{X}}_{\nabla,E}^k \rightarrow \mathcal{P}^{k+2}(E)$ for all $\mathbf{w}_E \in \underline{\mathbf{X}}_{\nabla,E}^k$ by

$$\nabla_E^{k+2} \mathbf{w}_E = \mathbf{w}_E'. \quad (3.36)$$

For all face $F \in \mathcal{F}_h$ we define the operator $\nabla_F^{k+1} : \underline{\mathbf{X}}_{\nabla,F}^k \rightarrow \tilde{\mathcal{P}}^{k+1}(F)$ for all $\underline{\mathbf{w}}_F = ((\mathbf{w}_E)_{E \in \mathcal{E}_h}, \mathbf{w}_F, \mathbf{w}_{\mathcal{G},F}, \mathbf{w}_{\mathcal{G},F}^c) \in \underline{\mathbf{X}}_{\nabla,F}^k$ such that $\forall \mathbf{V}_F = \mathbf{V}_{\mathcal{P},\mathbf{n}_F} + \mathbf{V}_{\mathcal{R},F}^c + \mathbf{V}_{\mathcal{R},F} + \mathbf{V}_{\mathcal{R},F} \in \tilde{\mathcal{P}}^{k+1}(F)$,

$$\begin{aligned} \int_F \nabla_F^{k+1}(\underline{\mathbf{w}}_F) : \mathbf{V}_F &= - \int_F \mathbf{w}_{\mathcal{G},F}^c \cdot \nabla \cdot (\mathbf{V}_{\mathcal{R},F}^c) - \int_F \mathbf{w}_{\mathcal{G},F} \cdot \nabla \cdot (\mathbf{V}_{\mathcal{R},F}) - \int_F \mathbf{w}_F \operatorname{div}(\mathbf{V}_{\mathcal{P},\mathbf{n}_F}) \\ &\quad + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{w}_E \mathbf{V}_F \mathbf{n}_{FE}. \end{aligned} \quad (3.37)$$

We define the full operator $\underline{\nabla}_F^{k+1} : \underline{\mathbf{X}}_{\nabla,F}^k \rightarrow \underline{\mathbf{X}}_{L^2,F}^k$ by

$$\underline{\nabla}_F^{k+1} \underline{\mathbf{w}}_F = ((\nabla_E^{k+2} \mathbf{w}_E)_{E \in \mathcal{E}_F}, \nabla_F^{k+1} \underline{\mathbf{w}}_F). \quad (3.38)$$

We prove a first commutative property:

Lemma 14. *For all $F \in \mathcal{F}_h$ it holds:*

$$\nabla_F^{k+1}(\underline{\mathbf{I}}_{\nabla,F}^k \mathbf{w}) = \pi_{\tilde{\mathcal{P}},F}^{k+1}(\nabla \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{C}^1(\bar{F}). \quad (3.39)$$

Proof. For all $\mathbf{w} \in \mathbf{C}^1(\bar{F})$ and all $\mathbf{V}_F \in \tilde{\mathcal{P}}^{k+1}(F)$,

$$\begin{aligned}
\int_F \nabla_F^{k+1}(\underline{\mathbf{I}}_{\nabla, F}^k \mathbf{w}) : \mathbf{V}_F &= - \int_F \boldsymbol{\pi}_{\mathcal{G}, F}^{c, k}(\mathbf{w}_{t, F}) \cdot \nabla \cdot (\mathbf{V}_{\bar{\mathcal{R}}, F}^c) - \int_F \boldsymbol{\pi}_{\mathcal{G}, F}^k(\mathbf{w}_{t, F}) \cdot \nabla \cdot (\mathbf{V}_{\bar{\mathcal{R}}, F}) \\
&\quad - \int_F \pi_{\mathcal{P}, F}^k(\mathbf{w} \cdot \mathbf{n}_F) \operatorname{div}(\mathbf{V}_{\mathcal{P}, \mathbf{n}_F}) + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \boldsymbol{\pi}_{\mathcal{P}, E}^{k+1}(\mathbf{w}) \mathbf{V}_F \mathbf{n}_{FE} \\
&= - \int_F \mathbf{w}_{t, F} \cdot \nabla \cdot (\mathbf{V}_{\bar{\mathcal{R}}, F}^c + \mathbf{V}_{\bar{\mathcal{R}}, F}) - \int_F (\mathbf{w} \cdot \mathbf{n}_F) \cdot \operatorname{div}(\mathbf{V}_{\mathcal{P}, \mathbf{n}_F}) \\
&\quad + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{w} \mathbf{V}_F \mathbf{n}_{FE} \\
&= - \int_F \mathbf{w} \cdot \nabla \cdot (\mathbf{V}_F) + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{w} \mathbf{V}_F \mathbf{n}_{FE} \\
&= \int_F \nabla \mathbf{w} : \mathbf{V}_F.
\end{aligned}$$

We used Lemma 7 and the definition (2.16) to remove the first two projections ($\boldsymbol{\pi}_{\mathcal{G}, F}^{c, k}$ and $\boldsymbol{\pi}_{\mathcal{G}, F}^k$) and integration by parts to conclude. \square

We define the trace operator $\gamma_{\nabla, F}^{k+2} : \underline{\mathbf{X}}_{\nabla, F}^k \rightarrow (\mathcal{P}^{k+2}(F))^\mathfrak{T}^3$ by the relation: $\forall \mathbf{V}_F \in (\mathcal{R}^{c, k+3}(F)^\mathfrak{T})^3$, $\forall \underline{\mathbf{w}}_F \in \underline{\mathbf{X}}_{\nabla, F}^k$,

$$\int_F \gamma_{\nabla, F}^{k+2}(\underline{\mathbf{w}}_F) \cdot \nabla \cdot \mathbf{V}_F = - \int_F \nabla_F^{k+1} \underline{\mathbf{w}}_F : \mathbf{V}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{w}_E \mathbf{V}_F \mathbf{n}_{FE}. \quad (3.40)$$

The isomorphism (2.9) ensures the well-posedness.

Remark 15. The relation (3.40) also holds for all $\mathbf{V}_F \in (\mathcal{P}^{k+1}(F)^\mathfrak{T})^3$. Indeed if \mathbf{V}_F belongs to $(\mathcal{R}^{k+1}(F)^\mathfrak{T})^3$ then $\nabla \cdot \mathbf{V}_F = 0$ and the left-hand side of (3.40) is null. And since $(\mathcal{R}^{k+1}(F)^\mathfrak{T})^3 \subset (\mathcal{P}^{k+1}(F)^\mathfrak{T})^3 \approx \tilde{\mathcal{P}}^{k+1}(F)$ we can apply (3.37) to show that the right-hand side is also zero. Hence, the relation holds for all $(\mathcal{R}^{k+1}(F)^\mathfrak{T})^3 \oplus (\mathcal{R}^{c, k+3}(F)^\mathfrak{T})^3 \supset (\mathcal{P}^{k+1}(F)^\mathfrak{T})^3$.

Lemma 16 (Consistency properties). *For all $F \in \mathcal{F}_h$ the following relations hold:*

$$\gamma_{\nabla, F}^{k+2}(\underline{\mathbf{I}}_{\nabla, F}^k \mathbf{w}) = \mathbf{w}, \quad \forall \mathbf{w} \in \mathcal{P}^{k+2}(F), \quad (3.41)$$

$$\begin{aligned}
\pi_{\mathcal{P}, F}^k(\gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F \cdot \mathbf{n}_F) &= w_F, \\
\boldsymbol{\pi}_{\mathcal{G}, F}^{c, k}(\gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F) &= \mathbf{w}_{\mathcal{G}, F}, \\
\boldsymbol{\pi}_{\mathcal{G}, F}^k(\gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F) &= \mathbf{w}_{\mathcal{G}, F},
\end{aligned} \quad \forall \underline{\mathbf{w}}_F \in \underline{\mathbf{X}}_{\nabla, F}^k. \quad (3.42)$$

Proof. Proof of (3.42). Let $\underline{\mathbf{w}}_F \in \underline{\mathbf{X}}_{\nabla, F}^k$ and $\mathbf{v}_F \in \mathcal{R}^{c, k+1}(F)$. Remark 15 allows to write:

$$\begin{aligned}
\int_F \gamma_{\nabla, F}^{k+2}(\underline{\mathbf{w}}_F) \cdot \mathbf{n}_F \operatorname{div}(\mathbf{v}_F) &= \int_F \gamma_{\nabla, F}^{k+2}(\underline{\mathbf{w}}_F) \cdot \nabla \cdot (\mathbf{n}_F \otimes \mathbf{v}_F) \\
&= - \int_F \nabla_F^{k+1} \underline{\mathbf{w}}_F : (\mathbf{n}_F \otimes \mathbf{v}_F) + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\mathbf{w}_E \cdot \mathbf{n}_F) (\mathbf{v}_F \cdot \mathbf{n}_{FE}) \\
&= \int_F w_F \operatorname{div}(\mathbf{v}_F).
\end{aligned}$$

Since this holds for all $\mathbf{v}_F \in \mathcal{R}^{c,k+1}(F)$, inferring isomorphism (2.9) we have:

$$\pi_{\mathcal{P},F}^k(\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \cdot \mathbf{n}_F) = \pi_{\mathcal{P},F}^k(w_F) = w_F.$$

The other two equations are proven in the same fashion.

Proof of (3.41). Let $\mathbf{w} \in \mathcal{P}^{k+2}(F)$ and $\mathbf{V}_F \in (\mathcal{R}^{c,k+3}(F)^\Gamma)^3$, it holds

$$\begin{aligned} \int_F \gamma_{\nabla,F}^{k+2}(\underline{\mathbf{I}}_{\nabla,F}^k \mathbf{w}) \cdot \nabla \cdot \mathbf{V}_F &= - \int_F \nabla_F^{k+1} \underline{\mathbf{I}}_{\nabla,F}^k \mathbf{w} : \mathbf{V}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\mathbf{w}_E) \mathbf{V}_F \mathbf{n}_{FE} \\ &= - \int_F \nabla \mathbf{w} : \mathbf{V}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{w} \mathbf{V}_F \mathbf{n}_{FE} \\ &= \int_F \mathbf{w} \cdot \nabla \cdot \mathbf{V}_F. \end{aligned}$$

We used Lemma 14 to write $\nabla_F^{k+1} \underline{\mathbf{I}}_{\nabla,F}^k \mathbf{w} = \pi_{\mathcal{P},F}^{k+1}(\nabla \mathbf{w}) = \nabla \mathbf{w}$ since $\nabla \mathbf{w} \in (\mathcal{P}^{k+1}(F)^\Gamma)^3$, and $\mathbf{w}_E = \mathbf{w}$ since $\mathbf{w} \in \mathcal{P}^{k+2} \subset \mathcal{P}^{k+3}$ is continuous to remove the projections. \square

For all $T \in \mathcal{T}_h$ we define the operator $\nabla_T^{k+1} : \underline{\mathbf{X}}_{\nabla,T}^k \rightarrow \overline{\mathcal{RT}}^{k+1}(T)$ such that $\forall \underline{\mathbf{w}}_T = ((\mathbf{w}_F)_{F \in \mathcal{F}_T}, \mathbf{w}_{\mathcal{G},T}, \mathbf{w}_{\mathcal{G},T}^c) \in \underline{\mathbf{X}}_{\nabla,T}^k$, $\forall \mathbf{V}_T = \mathbf{V}_{\overline{\mathcal{R}},T}^c + \mathbf{V}_{\overline{\mathcal{R}},T} + \mathbf{V}_{\mathcal{R},T} \in \overline{\mathcal{RT}}^{k+1}(T)$,

$$\int_T \nabla_T^{k+1}(\underline{\mathbf{w}}_T) : \mathbf{V}_T = - \int_T \mathbf{w}_{\mathcal{G},T}^c \cdot \nabla \cdot (\mathbf{V}_{\overline{\mathcal{R}},T}^c) - \int_T \mathbf{w}_{\mathcal{G},T} \cdot \nabla \cdot (\mathbf{V}_{\overline{\mathcal{R}},T}) + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \mathbf{V}_T \mathbf{n}_F. \quad (3.43)$$

We also define the potential reconstruction operator $P_{\nabla,T}^{k+1} : \underline{\mathbf{X}}_{\nabla,T}^k \rightarrow \mathcal{P}^{k+1}(T)$ by the relation: $\forall \mathbf{V}_T \in (\mathcal{R}^{c,k+2}(T)^\Gamma)^3$, $\forall \underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla,T}^k$,

$$\int_T P_{\nabla,T}^{k+1}(\underline{\mathbf{w}}_T) \cdot \nabla \cdot \mathbf{V}_T = - \int_T \nabla_T^{k+1} \underline{\mathbf{w}}_T : \mathbf{V}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \mathbf{V}_T \mathbf{n}_F. \quad (3.44)$$

The global operator $\underline{\nabla}_T^{k+1} : \underline{\mathbf{X}}_{\nabla,T}^k \rightarrow \underline{\mathbf{X}}_{L^2,T}^{k+1}$ is defined for all $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla,T}^k$ by

$$\underline{\nabla}_T^{k+1} \underline{\mathbf{w}}_T = ((\underline{\nabla}_F^{k+1} \underline{\mathbf{w}}_F)_{F \in \mathcal{F}_T}, \nabla_T^{k+1} \underline{\mathbf{w}}_T). \quad (3.45)$$

Remark 17. Since $\nabla \cdot (\mathcal{R}^k(T)^\Gamma)^3 = 0$ by Remark 6 and (3.42) we see that $\forall q \in \mathcal{P}^k(T)$,

$$\int_T \nabla_T^{k+1}(\underline{\mathbf{w}}_T) : (q \mathbf{I}_{3,3}) = \int_T \mathbf{w}_{\mathcal{G},T} \cdot \mathbf{grad} q + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F q w_F.$$

Remark 18. With the same argument as in Remark 15, (3.44) is valid for all $\mathbf{V}_T \in (\mathcal{P}^{k+1}(T)^\Gamma)^3$.

Lemma 19. For all $T \in \mathcal{T}_h$ it holds:

$$\nabla_T^{k+1}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}) = \nabla \mathbf{w}, \quad \forall \mathbf{w} \in \mathcal{P}^{k+1}(T), \quad (3.46)$$

$$P_{\nabla,T}^{k+1}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}) = \mathbf{w}, \quad \forall \mathbf{w} \in \mathcal{P}^{k+1}(T), \quad (3.47)$$

$$\begin{aligned} \pi_{\mathcal{G},T}^{c,k}(P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T) &= \mathbf{w}_{\mathcal{G},T}^c, \\ \pi_{\mathcal{G},T}^{k-1}(P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T) &= \mathbf{w}_{\mathcal{G},T}, \end{aligned} \quad \forall \underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla,T}^k. \quad (3.48)$$

Proof. The proof of (3.46) is similar to the proof of Lemma 14, using (3.41) to remove the projection on the boundary term. The proofs of (3.47) and (3.48) are the same as the proofs of (3.41) and (3.42). \square

3.3.4 Divergence.

Finally, we define the discrete divergence operator, for all $T \in \mathcal{T}_h$ by:

$$D_T^k := \text{Tr} \nabla_T^{k+1} \in \mathcal{P}^k(T).$$

As in the continuous case the divergence is the trace of the gradient, but we can also define it by a formula mimicking the integration by parts. By Remark 17, $\forall \underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla, T}^k$, D_T^k is such that $\forall q_T \in \mathcal{P}^k(T)$,

$$\begin{aligned} \int_T D_T^k \underline{\mathbf{w}}_T q_T &= \int_T \text{Tr}(\nabla_T^{k+1} \underline{\mathbf{w}}_T) q_T \\ &= \int_T \nabla_T^{k+1} \underline{\mathbf{w}}_T : (q_T \mathbf{I}_{3,3}) \\ &= - \int_T \underline{\mathbf{w}}_{\mathcal{G}, T} \cdot \mathbf{grad} q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F (\gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F \cdot \mathbf{n}_F) q_T \\ &= - \int_T \underline{\mathbf{w}}_{\mathcal{G}, T} \cdot \mathbf{grad} q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F w_F q_T. \end{aligned} \quad (3.49)$$

We get the same definition as the one of the de Rham complex of [14].

3.4 Discrete L^2 -product.

We build scalar products on the discrete spaces. They are made of the sum of the L^2 scalar product on each cell and of a stabilization term taking the lower dimensional objects (edges, vertices and faces) into account. Since we will not need their definitions on $\underline{\mathbf{X}}_{\mathbf{grad}, T}^k$ and $\underline{\mathbf{X}}_{\mathbf{curl}, T}^k$ to study the Jacobian operator, we will not write them down explicitly. They are quite similar to the L^2 product of $\underline{\mathbf{X}}_{\nabla, T}^k$ but require the introduction of potential reconstruction operators akin to those of [14]. First we define them locally for all $T \in \mathcal{T}_h$: For all $\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla, T}^k$ we set

$$(\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T)_{\nabla, T} = \int_T P_{\nabla, T}^{k+1} \underline{\mathbf{v}}_T \cdot P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T + s_{\nabla, T}(\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T), \quad (3.50)$$

$$\begin{aligned} s_{\nabla, T}(\underline{\mathbf{v}}_T, \underline{\mathbf{w}}_T) &= \sum_{F \in \mathcal{F}_T} h_F \int_F (P_{\nabla, T}^{k+1} \underline{\mathbf{v}}_T - \gamma_{\nabla, F}^{k+2} \underline{\mathbf{v}}_F) \cdot (P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T - \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F) \\ &\quad + \sum_{E \in \mathcal{E}_F} h_E^2 \int_E (P_{\nabla, T}^{k+1} \underline{\mathbf{v}}_T - \underline{\mathbf{v}}_E) \cdot (P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T - \underline{\mathbf{w}}_E). \end{aligned} \quad (3.51)$$

For all $\underline{\mathbf{V}}_T, \underline{\mathbf{W}}_T \in \underline{\mathbf{X}}_{L^2, T}^{k+1}$ we set

$$(\underline{\mathbf{V}}_T, \underline{\mathbf{W}}_T)_{L^2, T} = \int_T \underline{\mathbf{V}}_T : \underline{\mathbf{W}}_T + s_{L^2, T}(\underline{\mathbf{V}}_T, \underline{\mathbf{W}}_T), \quad (3.52)$$

$$\begin{aligned} s_{L^2, T}(\underline{\mathbf{V}}_T, \underline{\mathbf{W}}_T) &= \sum_{F \in \mathcal{F}_T} h_F \int_F (\underline{\mathbf{V}}_{\otimes t, F} - \underline{\mathbf{V}}_F) : (\underline{\mathbf{W}}_{\otimes t, F} - \underline{\mathbf{W}}_F) \\ &\quad + \sum_{E \in \mathcal{E}_F} h_E^2 \int_E (\underline{\mathbf{V}}_T \underline{\mathbf{t}}_E - \underline{\mathbf{V}}_E) \cdot (\underline{\mathbf{W}}_T \underline{\mathbf{t}}_E - \underline{\mathbf{W}}_E), \end{aligned} \quad (3.53)$$

where $\mathbf{V}_{\otimes t, F} := \mathbf{V}_T - (\mathbf{V}_T \mathbf{n}_F) \otimes \mathbf{n}_F$. Global scalar products are then merely the sum of local scalar product over every face $T \in \mathcal{T}_h$. For all $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\nabla, T}^k$ and $\underline{\mathbf{W}}_T \in \underline{\mathbf{X}}_{\mathbf{L}^2, T}^{k+1}$ the norm induced by this scalar product is denoted by:

$$\|\underline{\mathbf{v}}_T\|_{\nabla, T} = (\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T)_{\nabla, T}^{1/2}, \quad \|\underline{\mathbf{W}}_T\|_{\mathbf{L}^2, T} = (\underline{\mathbf{W}}_T, \underline{\mathbf{W}}_T)_{\mathbf{L}^2, T}^{1/2}.$$

We also define norms built from the sum over the objects of every dimension. For all $\underline{q}_T \in \underline{\mathbf{X}}_{\mathbf{grad}, T}^k$ we define

$$\begin{aligned} \|\underline{q}_T\|_{\mathbf{grad}, T}^2 &= \|q_F\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \left(\|q_f\|_F^2 + h_F \|G_{q, F}\|_F^2 \right. \\ &\quad \left. + \sum_{E \in \mathcal{E}_F} h_E \left[\|q_E\|_E^2 + h_E \|G_{q, E}\|_E^2 + \sum_{V \in \mathcal{V}_E} h_E^2 |G_{q, V}|^2 \right] \right). \end{aligned} \quad (3.54)$$

For all $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\mathbf{curl}, T}^k$ we define

$$\begin{aligned} \|\underline{\mathbf{v}}_T\|_{\mathbf{curl}, T}^2 &= \|\mathbf{v}_{\mathcal{R}, T}\|_T^2 + \|\mathbf{v}_{\mathcal{R}, T}^c\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \left(\|\mathbf{v}_{\mathcal{R}, F}\|_F^2 + \|\mathbf{v}_{\mathcal{R}, F}^c\|_F^2 + \|\mathbf{v}_F\|_F^2 \right. \\ &\quad \left. + h_F \|\mathbf{R}_{v, \mathcal{G}, F}\|_F^2 + h_F \|\mathbf{R}_{v, \mathcal{G}, F}^c\|_F^2 + \sum_{E \in \mathcal{E}_F} h_E \left[\|\mathbf{v}_E\|_E^2 + h_E \|\mathbf{R}_{v, E}\|_E^2 + \sum_{V \in \mathcal{V}_E} h_E^2 |\mathbf{R}_{v, V}|^2 \right] \right). \end{aligned} \quad (3.55)$$

For all $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla, T}^k$ we define

$$\begin{aligned} \|\underline{\mathbf{w}}_T\|_{\nabla, T}^2 &= \|\mathbf{w}_{\mathcal{G}, T}\|_T^2 + \|\mathbf{w}_{\mathcal{G}, T}^c\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \left(\|\mathbf{w}_{\mathcal{G}, F}\|_F^2 + \|\mathbf{w}_{\mathcal{G}, F}^c\|_F^2 + \|\mathbf{w}_F\|_F^2 \right. \\ &\quad \left. + \sum_{E \in \mathcal{E}_F} h_E \|\mathbf{w}_E\|_E^2 \right). \end{aligned} \quad (3.56)$$

For all $\underline{\mathbf{W}}_T \in \underline{\mathbf{X}}_{\mathbf{L}^2, T}^{k+1}$ we define

$$\|\underline{\mathbf{W}}_T\|_{\mathbf{L}^2, T}^2 = \|\mathbf{W}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \left(\|\mathbf{W}_F\|_F^2 + \sum_{E \in \mathcal{E}_F} h_E \|\mathbf{W}_E\|_E^2 \right). \quad (3.57)$$

And for all $q_T \in \underline{\mathbf{X}}_{\mathbf{L}^2, T}^k$ we define

$$\|q_T\|_{\mathbf{L}^2, T}^2 = \|q_T\|_T^2. \quad (3.58)$$

We show the equivalence between the norm induced by (3.50) and (3.56) in Lemma 24 and the equivalence between those induced by (3.52) and (3.57) in Lemma 25.

We define the global norms over Ω as the sum of the local norms over every cell $T \in \mathcal{T}_h$, i.e. $\|\underline{\mathbf{v}}_h\|_{\nabla, h}^2 = \sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{v}}_T\|_{\nabla, T}^2$.

3.5 Results on discrete L^2 -products.

We show some results to justify the choice of discrete norms.

Lemma 20. *For all $F \in \mathcal{F}_h$ and all $\underline{\mathbf{w}}_F \in \underline{\mathbf{X}}_{\nabla, F}^k$ it holds:*

$$\left\| \nabla_F^{k+1} \underline{\mathbf{w}}_F \right\|_F \lesssim h_F^{-1} \|\underline{\mathbf{w}}_F\|_{\nabla, F}.$$

Proof. This is a direct consequence of Lemma 2 and 3 to the definition (3.37) with $\mathbf{V}_F = \nabla_F^{k+1} \underline{\mathbf{w}}_F$. \square

Lemma 21 (Boundedness of the local trace). *For all $F \in \mathcal{F}_h$ and all $\underline{\mathbf{w}}_F \in \underline{\mathbf{X}}_{\nabla, F}^k$ it holds:*

$$\left\| \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F \right\|_F \lesssim \|\mathbf{w}_{\mathcal{G}, F}\|_F + \|\mathbf{w}_{\mathcal{G}, F}^c\|_F + \|\mathbf{w}_F\|_F + \sum_{E \in \mathcal{E}_F} h_E^{\frac{1}{2}} \|\mathbf{w}_E\|_E.$$

Proof. For any $\underline{\mathbf{w}}_F \in \underline{\mathbf{X}}_{\nabla, F}^k$, let $\mathbf{V}_F \in (\mathcal{R}^{c, k+3}(F)^\top)^3$ be such that $\nabla \cdot \mathbf{V}_F = \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F$. From the estimate on the isomorphism (2.9) it holds $\|\mathbf{V}_F\|_F \lesssim h_F \left\| \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F \right\|_F$. Then from (3.40) and (3.37) we write:

$$\begin{aligned} \int_F \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F \cdot \nabla \cdot \mathbf{V}_F &= - \int_F \nabla_F^{k+1} \underline{\mathbf{w}}_F : \mathbf{V}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \mathbf{w}_E \mathbf{V}_F \mathbf{n}_{FE} \\ &\lesssim \left\| \nabla_F^{k+1} \underline{\mathbf{w}}_F \right\|_F \|\mathbf{V}_F\|_F + \sum_{E \in \mathcal{E}_F} h_E^{\frac{1}{2}} \|\mathbf{w}_E\|_E h_E^{-1} \|\mathbf{V}_F\|_F. \end{aligned}$$

We used the discrete trace inequality Lemma 3 on the boundary term, and we conclude with Lemma 20. \square

Lemma 22 (Inverse Poincaré inequality). *For all $T \in \mathcal{T}_h$ and all $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla, T}^k$ it holds:*

$$\left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| \lesssim h^{-1} \|\underline{\mathbf{w}}_T\|_{\nabla, T}.$$

Proof. Let $T \in \mathcal{T}_h$ and $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla, T}^k$. The proof hinge on the bound of isomorphism (2.9) $\left\| \nabla \cdot \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| \lesssim h^{-1} \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\|$ to show that:

$$\begin{aligned} \int_T \nabla_F^{k+1} \underline{\mathbf{w}}_T : \nabla_T^{k+1} \underline{\mathbf{w}}_T &\lesssim \|\mathbf{w}_{\mathcal{G}, T}^c\| h^{-1} \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| + \|\mathbf{w}_{\mathcal{G}, T}\| h^{-1} \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| \\ &\quad + \left\| \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F \right\|_F h^{-\frac{1}{2}} \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\|_T \\ &\lesssim h^{-1} \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| \left(\|\mathbf{v}_{\mathcal{G}, T}^c\| + \|\mathbf{v}_{\mathcal{G}, T}\| + \sum_{F \in \mathcal{F}_T} h^{\frac{1}{2}} \left\| \gamma_{\nabla, F}^{k+2} \underline{\mathbf{v}}_F \right\| \right). \end{aligned}$$

We conclude with Lemma 21. \square

Lemma 23 (Boundedness of the local potential). *For all $T \in \mathcal{T}_h$ and all $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla, T}^k$ it holds:*

$$\left\| P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T \right\| \lesssim \|\underline{\mathbf{w}}_T\|_{\nabla, T} \quad (3.59)$$

Proof. Let $T \in \mathcal{T}_h$, $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla,T}^k$ and let $\mathbf{V}_T \in (\mathcal{R}^{c,k+2}(T)\tau)^3$ such that $\nabla \cdot \mathbf{V}_T = P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T$. From the definition (3.44) we have

$$\begin{aligned} \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|^2 &= \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \cdot \nabla \cdot \mathbf{V}_T \\ &\leq \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| \|\mathbf{V}_T\| + \sum_{F \in \mathcal{F}_T} \left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F \|\mathbf{V}_T\|_F \\ &\lesssim \left(h_T \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| + \sum_{F \in \mathcal{F}_T} h_F^{\frac{1}{2}} \left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F \right) \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_T. \end{aligned}$$

We conclude with Lemma 21 and 22. \square

Lemma 24. *It holds, for all $T \in \mathcal{T}_h$*

$$\|\underline{\mathbf{w}}_T\|_{\nabla,T} \approx \|\underline{\mathbf{w}}_T\|_{\nabla,T}, \forall \underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla,T}^k.$$

Proof. We apply the trace inequality to the definitions (3.50) and (3.51) to write:

$$\begin{aligned} \|\underline{\mathbf{w}}_T\|_{\nabla,T}^2 &\leq \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \left(\left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F^2 + \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_F^2 \right) + \sum_{E \in \mathcal{E}_T} h_E^2 \left(\|\mathbf{w}_E\|_E^2 + \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_E^2 \right) \\ &\lesssim \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F^2 + \sum_{E \in \mathcal{E}_T} h_E^2 \|\mathbf{w}_E\|_E^2 \\ &\lesssim \|\underline{\mathbf{w}}_T\|_{\nabla,T}^2. \end{aligned}$$

We conclude with Lemma 23 and 21.

Conversely, to prove $\|\underline{\mathbf{w}}_T\|_{\nabla,T} \lesssim \|\underline{\mathbf{w}}_T\|_{\nabla,T}$ we begin to write

$$\begin{aligned} \|\mathbf{w}_E\|_E^2 &\lesssim \left\| \mathbf{w}_E - P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_E^2 + h_E^{-2} \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_T^2, \\ \left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F^2 &\lesssim \left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F - P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_F^2 + h_F^{-1} \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_T^2. \end{aligned}$$

Then we conclude with consistency properties (3.48) and (3.42) which allow us to bound $\|\mathbf{w}_{\mathcal{G},T}\|_T$, $\|\mathbf{w}_{\mathcal{G},T}^c\|_T$, $\|\mathbf{w}_{\mathcal{G},F}\|_F$, $\|\mathbf{w}_{\mathcal{G},F}^c\|_F$ and $\|\mathbf{w}_F\|_F$. For example $\|\mathbf{w}_{\mathcal{G},F}^c\|_F = \|\pi_{\mathcal{G},F}^{c,k} \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F\|_F \leq \left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F$. \square

Lemma 25. *It holds, for all $T \in \mathcal{T}_h$*

$$\|\underline{\mathbf{W}}_T\|_{\mathbf{L}^2,T} \approx \|\underline{\mathbf{W}}_T\|_{\mathbf{L}^2,T}, \forall \underline{\mathbf{W}}_T \in \underline{\mathbf{X}}_{\mathbf{L}^2,T}^{k+1}.$$

Proof. The same proof as Lemma 24 works. \square

4 Complex property.

In this section we study the following sequence:

$$\underline{\mathbf{X}}_{\mathbf{grad},T}^k \xrightarrow{\mathbf{G}_h^k} \underline{\mathbf{X}}_{\mathbf{curl},h}^k \xrightarrow{\mathbf{C}_h^k} \underline{\mathbf{X}}_{\nabla,h}^k \xrightarrow{\mathbf{D}_h^k} \underline{\mathbf{X}}_{\mathbf{L}^2,h}^k. \quad (4.1)$$

We will show in Theorem 27 that (4.1) is indeed a complex, but first we show that the interpolators form a cochain morphism from a continuous de Rham complex into the sequence (4.1).

Lemma 26 (Local commutation properties). *It holds for all $T \in \mathcal{T}_h$,*

$$\underline{\mathbf{G}}_T^k(\underline{\mathbf{I}}_{\mathbf{grad},T}^k q) = \underline{\mathbf{I}}_{\mathbf{curl},T}^k(\mathbf{grad} q), \quad \forall q \in C^2(\bar{T}), \quad (4.2a)$$

$$\underline{\mathbf{C}}_T^k(\underline{\mathbf{I}}_{\mathbf{curl},T}^k \mathbf{v}) = \underline{\mathbf{I}}_{\nabla,T}^k(\mathbf{curl} \mathbf{v}), \quad \forall \mathbf{v} \in C^1(\bar{T}), \quad (4.2b)$$

$$\nabla_T^{k+1}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}) = \underline{\mathbf{I}}_{L^2(T)}^k(\nabla \mathbf{w}), \quad \forall \mathbf{w} \in C^1(\bar{T}), \quad (4.2c)$$

$$D_T^k(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}) = \pi_{\mathcal{P},T}^k(\operatorname{div} \mathbf{w}), \quad \forall \mathbf{w} \in C^0(\bar{T}) \cap \mathbf{H}^1(T). \quad (4.2d)$$

Proof. Proof of (4.2a). We already have proved the relation on edges (3.19). Let $q \in C^2(\bar{T})$, we will show that the relation holds for $\mathbf{G}_F^{\perp k}$. The proofs for \mathbf{G}_F^k and \mathbf{G}_T^k are almost the same. For all $\mathbf{w}_F \in \mathcal{P}^k(F)$,

$$\begin{aligned} \int_F \mathbf{G}_F^{\perp k}(\underline{\mathbf{I}}_{\mathbf{grad},F}^k q) \cdot \mathbf{w}_F &= - \int_F \pi_{\mathcal{P},F}^{k-1}(\partial_{\mathbf{n}_F} q) \operatorname{rot} \mathbf{w}_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \pi_{\mathcal{P},E}^k(\partial_{\mathbf{n}_F} q) \mathbf{w}_F \cdot \mathbf{t}_E \\ &= - \int_F \partial_{\mathbf{n}_F} q \operatorname{rot} \mathbf{w}_F - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E \partial_{\mathbf{n}_F} q \mathbf{w}_F \cdot \mathbf{t}_E \\ &= - \int_F \mathbf{rot}(\partial_{\mathbf{n}_F} q) \cdot \mathbf{w}_F. \end{aligned} \quad (4.3)$$

We conclude applying (4.3) for $\mathbf{w}_F \in \mathcal{G}^k(F)$ and $\mathbf{w}_F \in \mathcal{G}^{c,k}(F)$ since $\mathbf{n}_F \times (\nabla(\mathbf{grad} q) \cdot \mathbf{n}_F) = -\mathbf{rot}(\partial_{\mathbf{n}_F} q)$.

Proof of (4.2b). Let $\mathbf{v} \in C^1(\bar{T})$ and $\underline{\mathbf{v}}_T = \underline{\mathbf{I}}_{\mathbf{curl},T}^k \mathbf{v}$. We will show the property for \mathbf{C}_E^k and $\pi_{\mathcal{G},F}^{c,k} \mathbf{C}_F^k + \mathbf{R}_{v,\mathcal{G},F}^c$, the other components are easier and similar. The same proof as (3.19) shows that $\mathbf{v}_{E'} = \pi_{\mathcal{P},E}^{k+1}(\partial_{\mathbf{t}_E} \mathbf{v})$. Let us choose an arbitrary basis $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ such that $\mathbf{t}_E = (1, 0, 0)$ and write $\mathbf{v} = (v_0, v_1, v_2)$. In this basis we have

$$\begin{aligned} \pi_{\mathcal{P},E}^{k+1}(\mathbf{C}_E^k \underline{\mathbf{v}}_E) &= \mathbf{R}_{v,E} - \mathbf{v}_{E'} \times \mathbf{t}_E \\ &= \pi_{\mathcal{P},E}^{k+1} [(\mathbf{curl} \mathbf{v} \cdot \mathbf{t}_E) \mathbf{t}_E + \mathbf{grad}(\mathbf{v} \cdot \mathbf{t}_E) \times \mathbf{t}_E] + \pi_{\mathcal{P},E}^{k+1}(\partial_0 \mathbf{v}) \times \mathbf{t}_E \\ &= \pi_{\mathcal{P},E}^{k+1} \left[\begin{pmatrix} \partial_1 v_2 - \partial_2 v_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \partial_2 v_0 - \partial_0 v_2 \\ \partial_0 v_1 - \partial_1 v_0 \end{pmatrix} \right] \\ &= \pi_{\mathcal{P},E}^{k+1}(\mathbf{curl} \mathbf{v}). \end{aligned}$$

Now let us take another basis such that $\mathbf{n}_F = (1, 0, 0)$. By the same argument we have $\mathbf{C}_F^k(\underline{\mathbf{I}}_{\mathbf{curl},F}^k \mathbf{v}) = \pi_{\mathcal{P},F}^k(\mathbf{rot} v_0)$ so

$$\begin{aligned} \pi_{\mathcal{G},F}^{c,k}(\mathbf{C}_F^k \underline{\mathbf{v}}_F) + \mathbf{R}_{v,\mathcal{G},F}^c &= \pi_{\mathcal{G},F}^{c,k} \left[\begin{pmatrix} 0 \\ \partial_2 v_0 \\ -\partial_1 v_0 \end{pmatrix} \right] + \pi_{\mathcal{G},F}^{c,k} \left[\begin{pmatrix} 0 \\ -\partial_0 v_2 \\ \partial_0 v_1 \end{pmatrix} \right] \\ &= \pi_{\mathcal{G},F}^{c,k}(\mathbf{v}_{t,F}). \end{aligned}$$

Proof of (4.2c). This is an immediate consequence of (3.36), (3.39) and (3.46).

Proof of (4.2d). Let $\mathbf{w} \in C^0(\bar{T}) \cap \mathbf{H}^1(T)$. For all $q_T \in \mathcal{P}^k(T)$, since $\mathbf{grad} q_T \in \mathcal{G}^{k-1}(T)$, we

have:

$$\begin{aligned}
\int_T D_T^k(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}) &= - \int_T \pi_{\mathcal{G},T}^{k-1}(\mathbf{w}) \cdot \mathbf{grad} q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \pi_{\mathcal{P},F}^k(\mathbf{w} \cdot \mathbf{n}_F) q_T \\
&= - \int_T \mathbf{w} \cdot \mathbf{grad} q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F (\mathbf{w} \cdot \mathbf{n}_F) q_T \\
&= \int_T \operatorname{div} \mathbf{w} q_T.
\end{aligned}$$

□

Theorem 27 (Complex property). *It holds:*

$$\underline{\mathbf{I}}_{\mathbf{grad},h}^k \mathbb{R} = \operatorname{Ker} \underline{\mathbf{G}}_h^k, \quad (4.4a)$$

$$\operatorname{Im} \underline{\mathbf{G}}_h^k \subset \operatorname{Ker} \underline{\mathbf{C}}_h^k, \quad (4.4b)$$

$$\operatorname{Im} \underline{\mathbf{C}}_h^k \subset \operatorname{Ker} \underline{\mathbf{D}}_h^k, \quad (4.4c)$$

$$\operatorname{Im} \underline{\mathbf{D}}_h^k = \mathcal{P}^k(\mathcal{T}_h). \quad (4.4d)$$

Proof. Proof of (4.4a). The inclusion $\underline{\mathbf{I}}_{\mathbf{grad},h}^k \mathbb{R} \subset \operatorname{Ker} \underline{\mathbf{G}}_h^k$ follows directly from (4.2a). Conversely if $q_h \in \underline{X}_{\mathbf{grad},h}^k$ is such that $\underline{\mathbf{G}}_h^k q_h = 0$ then since $\underline{\mathbf{G}}_E^k q_E = 0$, (3.12) implies $\mathbf{G}_{q,V} = 0$, $\forall V \in \mathcal{V}_h$, $\mathbf{G}_{q,E} = 0$ and $q_{E'} = 0$, $\forall E \in \mathcal{E}_h$. So q_E is constant on every edge, however q_E is also continuous and Ω has a single connected component. Thus, there is $C \in \mathbb{R}$ such that $\forall E \in \mathcal{E}_h$, $q_E \equiv C$. From (3.13) and (3.15) we have $\forall \mathbf{w}_F \in \mathcal{R}^{c,k}(F)$,

$$0 = - \int_F q_F \operatorname{div} \mathbf{w}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E C \mathbf{w}_F \cdot \mathbf{n}_{FE} = \int_F (C - q_F) \operatorname{div} \mathbf{w}_F.$$

Since $\operatorname{div} : \mathcal{R}^{c,k}(F) \rightarrow \mathcal{P}^{k-1}(F)$ is onto we must have $q_F \equiv C$, $\forall F \in \mathcal{F}_h$. Likewise, since $\mathbf{G}_{q,E} = 0$, (3.14) and (3.15) give

$$- \int_F G_{q,F} \operatorname{rot} \mathbf{w}_F = 0, \quad \forall \mathbf{w} \in \mathcal{P}^k(F).$$

Once again we must have $G_{q,F} = 0$, $\forall F \in \mathcal{F}_h$. The same argument gives $\gamma_{\nabla,F}^{k+2} q_F \equiv C$ and $q_T \equiv C$, $\forall T \in \mathcal{T}_h$. Thus $\operatorname{Ker} \underline{\mathbf{G}}_h^k \subset \underline{\mathbf{I}}_{\mathbf{grad},h}^k \mathbb{R}$.

Proof of (4.4b). We already have $\operatorname{Im} \underline{\mathbf{G}}_F^k \subset \operatorname{Ker} \underline{\mathbf{C}}_F^k$ by Lemma 12. Let $q \in \underline{X}_{\mathbf{grad},h}^k$. For any $T \in \mathcal{T}_h$, since we project on $\mathcal{G}^{k-1}(T) \oplus \mathcal{G}^{c,k}(T)$ in (3.35) it is enough to show $\int_T \underline{\mathbf{C}}_T^k(\underline{\mathbf{G}}_T^k q_T) \cdot \mathbf{r}_T =$

0, $\forall \mathbf{r}_T \in \mathcal{N}^k(T)$. Starting from (3.34) we write

$$\begin{aligned}
\int_T \mathbf{C}_T^k(\underline{\mathbf{G}}_T^k \underline{q}_T) \cdot \mathbf{r}_T &= \int_T \boldsymbol{\pi}_{\mathcal{R},T}^{k-1}(\mathbf{G}_T^k \underline{q}_T) \cdot \mathbf{curl} \mathbf{r}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{t,\text{rot},F}^k(\underline{\mathbf{G}}_F^k \underline{q}_F) \cdot (\mathbf{r}_T \times \mathbf{n}_F) \\
&= \int_T (\mathbf{G}_T^k \underline{q}_T) \cdot \mathbf{curl} \mathbf{r}_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F (\mathbf{G}_F^k \underline{q}_F) \cdot (\mathbf{r}_T \times \mathbf{n}_F) \\
&= - \int_T q_T \operatorname{div}(\mathbf{curl} \mathbf{r}_T) + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\text{grad},F}^{k+1}(\underline{q}_F) (\mathbf{curl}(\mathbf{r}_T) \cdot \mathbf{n}_F) \\
&\quad + \sum_{F \in \mathcal{F}_T} \omega_{TF} \left[- \int_F q_F \operatorname{div}(\mathbf{r}_T \times \mathbf{n}_F) + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E (\mathbf{r}_T \times \mathbf{n}_F) \cdot \mathbf{n}_{FE} \right] \\
&= \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F q_F [\mathbf{curl}(\mathbf{r}_T) \cdot \mathbf{n}_F - \operatorname{div}(\mathbf{r}_T \times \mathbf{n}_F)] \\
&\quad - \sum_{F \in \mathcal{F}_T} \omega_{TF} \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E \mathbf{r}_T \cdot \mathbf{t}_E.
\end{aligned}$$

We used $\mathbf{curl} \mathbf{r}_T \in \mathcal{R}^{k-1}$ (2.5) and $\mathbf{r}_T \times \mathbf{n}_F \subset \mathcal{RT}^k(F)$ (2.12) along with (3.33) on the first line. Then we used (3.17) and (3.13) on the second line and (3.23) on the last. We conclude inferring $\mathbf{curl}(\mathbf{r}_T) \cdot \mathbf{n}_F = \operatorname{div}(\mathbf{r}_T \times \mathbf{n}_F)$ and $\sum_{F \in \mathcal{F}_T} \omega_{TF} \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E q_E \mathbf{r}_T \cdot \mathbf{t}_E = 0$. The last sum is zero since each edge shares exactly two faces on $\partial \bar{T}$ with opposite orientation. Hence we are counting each term twice, with a different sign each time.

Proof of (4.4c). The same proof as [14, Theorem 20] works.

Proof of (4.4d). See Lemma 34. □

The complex is exact if and only if the inclusions (4.4b) and (4.4c) are in fact equalities. We can show that this is the same as asking for Ω to be contractible. The proof is a slight adaptation of [9, Section 4.3] and will not be duplicated here.

Theorem 28 (Exactness). *If Ω is contractible then*

$$\operatorname{Im} \underline{\mathbf{G}}_h^k = \operatorname{Ker} \underline{\mathbf{C}}_h^k, \quad (4.5a)$$

$$\operatorname{Im} \underline{\mathbf{C}}_h^k = \operatorname{Ker} \underline{\mathbf{D}}_h^k, \quad (4.5b)$$

Proof. Proof of (4.5a). Let $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k$ be such that $\underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h = 0$. We want to find $\underline{q}_h \in \underline{\mathbf{X}}_{\text{grad},h}^k$ such that $\underline{\mathbf{G}}_h^k \underline{q}_h = \underline{\mathbf{v}}_h$. Starting from the proof of [14, Theorem 3.1], if Ω is contractible and $\underline{\mathbf{C}}_h^k \underline{\mathbf{v}}_h = 0$ then we can find $q \in \mathcal{P}_c^{k+1}(\mathcal{E}_h)$ such that $\forall E \in \mathcal{E}_h$, $q' = \pi_{\mathcal{P},E}^k(\mathbf{v}_E)$. Let $\mathbf{G}_{q,E} := \pi_{\mathcal{P},E}^k(\mathbf{v}_E)$ and $\mathbf{G}_{q,V} = \mathbf{v}_E(\mathbf{x}_V)$. Since $\underline{\mathbf{C}}_E^k \underline{\mathbf{v}}_E = 0 \implies \mathbf{R}_{v,E} = \mathbf{v}_E' \times \mathbf{t}_E$ and $\mathbf{R}_{v,V} = 0$ we have $\underline{\mathbf{v}}_E = \underline{\mathbf{G}}_E^k(\underline{q}_E)$. We must also have $\mathbf{R}_{v,\mathcal{G},F} = -\pi_{\mathcal{G},F}^k(\mathbf{C}_F^k \underline{\mathbf{v}}_E)$ hence $\forall \mathbf{r}_F \in \mathcal{G}^k(F)$

$$\begin{aligned}
\int_F \underline{\mathbf{C}}_F^k \underline{\mathbf{v}}_h \cdot \mathbf{r}_F &= \int_F v_F \operatorname{rot} \mathbf{r}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F (\mathbf{v}_E \cdot \mathbf{n}_F) (\mathbf{r}_F \cdot \mathbf{t}_E), \\
- \int_F \underline{\mathbf{G}}_F^k(\underline{q}) \cdot \mathbf{r}_F &= \int_F G_{q,F} \operatorname{rot} \mathbf{r}_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_F (\mathbf{G}_{q,E} \cdot \mathbf{n}_F) (\mathbf{r}_F \cdot \mathbf{t}_E).
\end{aligned}$$

Thus we can take $G_{q,F} := v_F$. We construct q_F and q_T exactly as in [14, Theorem 3.2].

Proof of (4.5b). Let $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k$ be such that $\underline{\mathbf{D}}_h^k \underline{\mathbf{w}}_h = 0$. We want to find $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\text{curl},h}^k$

such that $\underline{C}_h^k \underline{\mathbf{v}}_h = \underline{\mathbf{w}}_h$. If Ω is contractible then [14, Theorem 3.2] provides $\mathbf{v}_{\mathcal{R},T} \in \mathcal{R}^{k-1}(T)$, $\mathbf{v}_{\mathcal{R},T}^c \in \mathcal{R}^{c,k}(T)$, $\mathbf{v}_{\mathcal{R},F} \in \mathcal{R}^{k-1}(F)$, $\mathbf{v}_{\mathcal{R},F}^c \in \mathcal{R}^{c,k}(F)$ and a normal component (along \mathbf{t}_E) of \mathbf{v}_E such that $C_F^k \underline{\mathbf{v}}_F = \mathbf{w}_F$ and $\pi_{\mathcal{G},T}^{k-1}(C_T^k \underline{\mathbf{v}}_T) = \mathbf{w}_{\mathcal{G},T}$, $\pi_{\mathcal{G},T}^{c,k}(C_T^k \underline{\mathbf{v}}_T) = \mathbf{w}_{\mathcal{G},T}^c$. It remains to show that C_E^k and C_F^k are onto without using the above-mentioned degrees of freedom. Let $\mathbf{R}_{v,E} := \mathbf{v}_{E'} \times \mathbf{t}_E + \mathbf{w}_E$, $v_F := 0$, $\mathbf{R}_{v,\mathcal{G},F} := \mathbf{w}_{\mathcal{G},F} - \pi_{\mathcal{G},F}^k(C_F^k \underline{\mathbf{v}}_F)$ and $\mathbf{R}_{v,\mathcal{G},F}^c := \mathbf{w}_{\mathcal{G},F}^c - \pi_{\mathcal{G},F}^{c,k}(C_F^k \underline{\mathbf{v}}_F)$ (this makes sense since C_F^k does not depend on $\mathbf{R}_{v,\mathcal{G},F}$ nor on $\mathbf{R}_{v,\mathcal{G},F}^c$). It is easily checked that this choice gives $\underline{C}_h^k \underline{\mathbf{v}}_h = \underline{\mathbf{w}}_h$. \square

5 Consistency results.

The last things we need to show in order to efficiently use this complex are consistency results. First we show primal consistency results, controlling the error made when we use the interpolators. Then we show some Poincare type results useful to show stability, including a discrete counterpart to the right inverse for the divergence Lemma 34. Finally we show adjoint consistency results, which control the error made when we perform a discrete integration by parts.

Lemma 29 (Primal consistency). *For all $T \in \mathcal{T}_h$ it holds:*

$$\left\| P_{\nabla,T}^{k+1}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}) - \mathbf{w} \right\| \lesssim h^{k+2} |\mathbf{w}|_{\mathbf{H}^{k+2}}, \quad \forall \mathbf{w} \in \mathbf{H}^{k+2}(T). \quad (5.1)$$

Proof. For all $T \in \mathcal{T}_h$, (3.47) shows that $P_{\nabla,T}^{k+1} \underline{\mathbf{I}}_{\nabla,T}^k$ is a projection on $\mathcal{P}^{k+1}(T)$. Thus we just have to show that $\left\| P_{\nabla,T}^{k+1}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}) \right\| \lesssim \|\mathbf{w}\| + h \|\mathbf{w}\|_{\mathbf{H}^1} + h^2 \|\mathbf{w}\|_{\mathbf{H}^2}$ to conclude with the lemma on approximation properties of bounded projector [10, Lemma 1.43]. Starting from (3.59) we have

$$\begin{aligned} \left\| P_{\nabla,T}^{k+1}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}) \right\| &\lesssim \left\| \underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w} \right\|_{\nabla,T} \\ &\lesssim \left\| \pi_{\mathcal{G},T}^{k-1} \mathbf{w} \right\|_T + \left\| \pi_{\mathcal{G},T}^{c,k} \mathbf{w} \right\|_T + \sum_{E \in \mathcal{E}_T} h_E \left\| \pi_{\mathcal{P},E}^{k+1} \mathbf{w} \right\|_E + \sum_{V \in \mathcal{V}_E} h_E^{\frac{3}{2}} |\mathbf{w}(\mathbf{x}_V)| \\ &\quad + \sum_{F \in \mathcal{F}_T} h_F^{\frac{1}{2}} \left(\left\| \pi_{\mathcal{P},F}^k(\mathbf{w} \cdot \mathbf{n}_F) \right\|_F + \left\| \pi_{\mathcal{G},F}^k \mathbf{w}_{\mathbf{t},F} \right\|_F + \left\| \pi_{\mathcal{G},F}^{c,k} \mathbf{w}_{\mathbf{t},F} \right\|_F \right) \\ &\lesssim \|\mathbf{w}\|_T + h_T |\mathbf{w}|_{\mathbf{H}^1(T)} + h_T^2 |\mathbf{w}|_{\mathbf{H}^2(T)}, \end{aligned}$$

where we used the continuous trace inequality [10, Lemma 1.31] and the boundedness of L^2 projectors. \square

Lemma 30 (Stabilization forms consistency). *For all $T \in \mathcal{T}_h$ it holds:*

$$s_{\nabla,T} \left(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}, \underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w} \right)^{1/2} \lesssim h^{k+2} |\mathbf{w}|_{\mathbf{H}^{k+2}(T)}, \quad \forall \mathbf{w} \in \mathbf{H}^{k+2}(T). \quad (5.2)$$

$$s_{L^2,T} \left(\underline{\mathbf{I}}_{L^2(T)}^k \mathbf{W}, \underline{\mathbf{I}}_{L^2(T)}^k \mathbf{W} \right)^{1/2} \lesssim h^{k+1} |\mathbf{W}|_{\mathbf{H}^{k+1}(T)}, \quad \forall \mathbf{W} \in \mathbf{H}^{k+1}(T) \cap C^0(\bar{T}). \quad (5.3)$$

Proof. Proof of (5.2). For all $\mathbf{z}_T \in \mathcal{P}^{k+1}(T)$ we have $P_{\nabla,T}^{k+1}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{z}_T) = \mathbf{z}_T$ by (3.47) and $\gamma_{\nabla,F}^{k+2}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{z}_T) = \mathbf{z}_T$ by (3.41) so for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{X}}_{\nabla,T}^k$,

$$\begin{aligned} s_{\nabla,T}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{z}_T, \underline{\mathbf{v}}_T) &= \sum_{F \in \mathcal{F}_T} h_F \int_F (P_{\nabla,T}^{k+1} \underline{\mathbf{I}}_{\nabla,T}^k \mathbf{z}_T - \gamma_{\nabla,F}^{k+2} \underline{\mathbf{I}}_{\nabla,T}^k \mathbf{z}_T) \cdot (P_{\nabla,T}^{k+1} \underline{\mathbf{v}}_T - \gamma_{\nabla,F}^{k+2} \underline{\mathbf{v}}_T) \\ &= \sum_{E \in \mathcal{E}_F} h_E^2 \int_E (P_{\nabla,T}^{k+1} \underline{\mathbf{I}}_{\nabla,T}^k \mathbf{z}_T - \mathbf{z}_T) \cdot (P_{\nabla,T}^{k+1} \underline{\mathbf{v}}_T - \mathbf{v}_E) = 0. \end{aligned}$$

Hence

$$s_{\nabla,T}(\underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}_T, \underline{\mathbf{I}}_{\nabla,T}^k \mathbf{w}_T) = s_{\nabla,T}(\underline{\mathbf{I}}_{\nabla,T}^k (\mathbf{w}_T - \pi_{\mathcal{P},T}^{k+1}), \underline{\mathbf{I}}_{\nabla,T}^k (\mathbf{w}_T - \pi_{\mathcal{P},T}^{k+1})) \lesssim \|\underline{\mathbf{I}}_{\nabla,T}^k (\mathbf{w}_T - \pi_{\mathcal{P},T}^{k+1})\|_{\nabla,T}^2.$$

We conclude by the norm equivalence Lemma 24 and [10, Theorem 1.45].

Proof of (5.3). Let $\mathbf{W} \in \mathbf{H}^{k+1}(T) \cap \mathbf{C}^0(\overline{T})$, we have:

$$\begin{aligned} \sum_{F \in \mathcal{F}_T} h_F \left\| \pi_{\overline{\mathcal{RT}},T}^{k+1}(\mathbf{W})_{\otimes t,F} - \pi_{\overline{\mathcal{P}},F}^{k+1}(\mathbf{W}) \right\|_F^2 &\leq \sum_{F \in \mathcal{F}_T} h_F \left\| (I - \pi_{\overline{\mathcal{RT}},T}^{k+1}) \mathbf{W} \right\|_F^2 \\ &\leq \sum_{F \in \mathcal{F}_T} h_F \left(\left\| \mathbf{W} - \pi_{\mathcal{P},T}^k \mathbf{W} \right\|_F^2 + \left\| \pi_{\mathcal{P},T}^k \mathbf{W} - \pi_{\overline{\mathcal{RT}},T}^{k+1} \mathbf{W} \right\|_F^2 \right) \\ &\lesssim h^{2(k+1)} |\mathbf{W}|_{\mathbf{H}^{k+1}} + \left\| \mathbf{W} - \pi_{\mathcal{P},T}^k \mathbf{W} \right\|_F^2. \end{aligned}$$

Here the second equality comes from $\overline{\mathcal{RT}}^{k+1}(T)_{\otimes t,F} \subset \widetilde{\mathcal{P}}^{k+1}(F)$ and we used the approximation properties on traces [10, Theorem 1.45 and Equation 1.75] on the first term and the discrete trace inequality [10, Lemma 1.32] on the second term to get the last equality. Likewise, we have:

$$\sum_{F \in \mathcal{F}_T} h_F \sum_{E \in \mathcal{E}_F} h_E \left\| \pi_{\overline{\mathcal{RT}},T}^{k+1}(\mathbf{W}) \mathbf{t}_E - \pi_{\overline{\mathcal{P}},E}^{k+2}(\mathbf{W}) \mathbf{t}_E \right\|_E^2 \lesssim \sum_{F \in \mathcal{F}_T} \sum_{E \in \mathcal{E}_F} h_F \left\| (I - \pi_{\overline{\mathcal{RT}},T}^{k+1}) \mathbf{W} \right\|_F^2.$$

We conclude with [10, Theorem 1.45 and Equation 1.74]. \square

5.1 Poincaré inequality.

We begin by stating two lemmas which will be useful to prove the Poincaré inequality.

Lemma 31. *For all $T \in \mathcal{T}_h$, $F \in \mathcal{F}_T$ and all $\underline{\mathbf{w}}_T \in \underline{\mathbf{X}}_{\nabla,T}^k$ it holds that*

$$\begin{aligned} \left\| \nabla \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F^2 + \sum_{E \in \mathcal{E}_F} h_E^{-1} \left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F - \mathbf{w}_E \right\|_E^2 &\lesssim \left\| \nabla_F^{k+1} \underline{\mathbf{v}}_F \right\|_{\mathbf{L}^2,F}^2, \\ \left\| \nabla P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T - \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F^2 &\lesssim \left\| \nabla_T^{k+1} \underline{\mathbf{v}}_T \right\|_{\mathbf{L}^2,T}^2. \end{aligned} \tag{5.4}$$

Proof. The proof is a simple adaptation of [14, Lemma 5.7]. \square

Lemma 32 (Poincaré inequality). *For all $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k$ such that $\sum_{T \in \mathcal{T}_h} \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T = 0$ it holds that*

$$\|\underline{\mathbf{w}}_h\|_{\nabla,h} \lesssim \left\| \nabla_h^{k+1} \underline{\mathbf{w}}_h \right\|_{\mathbf{L}^2,h}. \tag{5.5}$$

Proof. The proof is a simple adaptation of [14, Theorem 5.3]. \square

Remark 33. When $k \geq 1$ the assumption $\sum_{T \in \mathcal{T}_h} \int_T P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T = 0$ translates to $\sum_{T \in \mathcal{T}_h} \int_T \mathbf{w}_T = 0$ by (3.48). However this does not hold when $k = 0$.

Lastly we show that the fully discrete divergence is onto $D_h^k : \underline{\mathbf{X}}_{\nabla, h}^k \rightarrow \underline{\mathbf{X}}_{L^2, h}^k$. The main difficulty is to show the boundedness of the inverse with the discrete norms.

Lemma 34 (Right-inverse for the divergence). *For all $\underline{p}_h \in \underline{\mathbf{X}}_{L^2, h}^k$ there is $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla, h}^k$ such that $D_h^k \underline{\mathbf{w}}_h = \underline{p}_h$ and $\|\underline{\mathbf{w}}_h\|_{\nabla, h} + \left\| \left\| \nabla_h^{k+1} \underline{\mathbf{w}}_h \right\| \right\|_{L^2, h} \lesssim \|\underline{p}_h\|_{L^2, h}$.*

Proof. Existence. Let $\underline{p}_h = (p_T)_{T \in \mathcal{T}_h} \in \underline{\mathbf{X}}_{L^2, h}^k$. Lemma 51 provides $\tilde{p} \in C^1(\bar{\Omega})$ such that $\forall T \in \mathcal{T}_h$, $\tilde{p}|_T \in \mathcal{P}^{k+\max_{h, T \in \mathcal{T}_h} (2|\mathcal{F}_T|)}(T)$, $\pi_{\tilde{p}, T}^k \tilde{p} = p_T$ and $\|\tilde{p}\|_{L^2(\Omega)} \approx \|\underline{p}_h\|_{L^2, h}$. Under the assumption on the regularity of the mesh sequence we have $\max_{h, T \in \mathcal{T}_h} (|\mathcal{F}_T|) \lesssim 1$ ([10, Lemma 1.12]) so that the maximum degree is bounded independently of h . Since \tilde{p} is a piecewise polynomial, continuous, of continuous derivative and of trace zero on the boundary, $\tilde{p} \in H_0^2(\Omega)$. We apply Theorem 54 to find $\mathbf{u} \in \mathbf{H}^3(\Omega)$ such that $\operatorname{div} \mathbf{u} = \tilde{p}$, $\|\mathbf{u}\|_{\mathbf{H}^3} \lesssim \|\tilde{p}\|_{H^2}$, $\|\mathbf{u}\|_{\mathbf{H}^2} \lesssim \|\tilde{p}\|_{H^1}$ and $\|\mathbf{u}\|_{\mathbf{H}^1} \lesssim \|\tilde{p}\|_{L^2}$. We build $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla, h}^k$ by $\underline{\mathbf{w}}_F = \underline{\mathbf{I}}_{\nabla, F}^k(\mathbf{u})$, $\forall F \in \mathcal{F}_h$, $\mathbf{w}_{\mathcal{G}, T}^c = \pi_{\mathcal{G}, T}^{c, k}(\mathbf{u})$, $\forall T \in \mathcal{T}_h$, and $\forall q_T \in \mathcal{P}^{0, k}(T)$:

$$\int_T \mathbf{w}_{\mathcal{G}, T} \mathbf{grad} q_T := - \int_T p_T q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla, F}^{k+2}(\underline{\mathbf{I}}_{\nabla, F}^k \mathbf{u}) \cdot \mathbf{n}_F q_T.$$

Hence by construction, $\forall T \in \mathcal{T}_h$, $\forall q_T \in \mathcal{P}^{0, k}(T)$, $\int_T D_T^k \underline{\mathbf{w}}_T q_T = \int_T p_T q_T$. It remains to show the equality for $q_T \in \mathcal{P}^0(T)$, and using (3.42) we have

$$\begin{aligned} \int_T D_T^k \underline{\mathbf{w}}_T &= \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla, F}^{k+2}(\underline{\mathbf{I}}_{\nabla, F}^k \mathbf{u}) \cdot \mathbf{n}_F \\ &= \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \pi_{\mathcal{P}, F}^k(\mathbf{u} \cdot \mathbf{n}_F) \\ &= \int_T \operatorname{div} \mathbf{u} = \int_T p_T. \end{aligned}$$

Thus we have $D_h^k \underline{\mathbf{w}}_h = \underline{p}_h$.

Boundedness. Many bounds follow directly from the L^2 -boundedness of the interpolator. Indeed, we make use of the continuous trace inequality [10, Lemma 1.31] to get

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^k(F)} &\lesssim h^{-\frac{1}{2}} \|\mathbf{u}\|_{\mathbf{H}^k(T)} + h^{\frac{1}{2}} \|\mathbf{u}\|_{\mathbf{H}^{k+1}(T)}, \\ \|\mathbf{u}\|_{\mathbf{H}^k(E)} &\lesssim h^{-1} \|\mathbf{u}\|_{\mathbf{H}^k(T)} + \|\mathbf{u}\|_{\mathbf{H}^{k+1}(T)} + h \|\mathbf{u}\|_{\mathbf{H}^{k+2}(T)}. \end{aligned}$$

Inferring the estimate on $\|\mathbf{u}\|_{\mathbf{H}^k(T)}$ and Lemma 2 we get

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^k(F)} &\lesssim h^{\min\{0, \frac{1}{2}-k\}} \|\underline{p}_h\|_{L^2, T}, \\ \|\mathbf{u}\|_{\mathbf{H}^k(E)} &\lesssim h^{-k} \|\underline{p}_h\|_{L^2, T}. \end{aligned} \tag{5.6}$$

This allows to bound all terms of $\|\underline{\mathbf{w}}_h\|_{\nabla,h}$ but $\mathbf{w}_{\mathcal{G},T}$. This one is also easily bounded, for all $T \in \mathcal{T}_h$, let $q_T \in \mathcal{P}^{0,k}(T)$ be such that $\mathbf{grad} q_T = \mathbf{w}_{\mathcal{G},T}$. We have $\|q_T\| \lesssim h_T \|\mathbf{w}_{\mathcal{G},T}\|$ and

$$\begin{aligned} \int_T \mathbf{w}_{\mathcal{G},T} \mathbf{grad} q_T &= - \int_T p_T q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla,F}^{k+2} (\underline{\mathbf{I}}_{\nabla,F}^k \mathbf{u}) \cdot \mathbf{n}_F q_T \\ &\lesssim \|p_T\| \|q_T\| + \sum_{F \in \mathcal{F}_T} h^{\frac{1}{2}} \left\| \gamma_{\nabla,F}^{k+2} (\underline{\mathbf{I}}_{\nabla,F}^k \mathbf{u}) \right\|_F h^{-1} \|q_T\|_T \\ &\lesssim \|p_T\| \|\mathbf{w}_{\mathcal{G},T}\|. \end{aligned}$$

It remains to estimate $\left\| \underline{\nabla}_h^{k+1} \underline{\mathbf{w}}_h \right\|_{L^2,T} \approx \sum_{T \in \mathcal{T}_h} \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\|_T + \sum_{F \in \mathcal{F}_h} h_F \left\| \nabla_F^{k+1} \underline{\mathbf{w}}_F \right\|_F + \sum_{E \in \mathcal{E}_h} h_E^2 \left\| \nabla_E^{k+2} \underline{\mathbf{w}}_E \right\|_E$. Lemma 14 and an easily proved variation of (3.19) and (5.6) give

$$\begin{aligned} h^2 \left\| \nabla_E^{k+2} \underline{\mathbf{I}}_{\nabla,E}^k \mathbf{u} \right\|_E &\lesssim h^2 \|\mathbf{u}\|_{\mathbf{H}^1(E)}^2 \lesssim h^2 (h^{-1})^2 \|\underline{p}_h\|_{L^2,T}, \\ h \left\| \nabla_F^{k+1} \underline{\mathbf{I}}_{\nabla,F}^k \mathbf{u} \right\|_F &\lesssim h \|\mathbf{u}\|_{\mathbf{H}^1(F)}^2 \lesssim h (h^{-\frac{1}{2}})^2 \|\underline{p}_h\|_{L^2,T}. \end{aligned}$$

To estimate $\left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\|_T$ let $\mathbf{V}_T := \nabla_T^{k+1} \underline{\mathbf{w}}_T$ and take $q_T \in \mathcal{P}^{0,k}(T)$ such that $\mathbf{grad} q_T = \nabla \cdot \mathbf{V}_{\overline{\mathcal{R}},T}$ with $\|q_T\| \approx \left\| \mathbf{V}_{\overline{\mathcal{R}},T} \right\|$. Starting from its definition (3.43) we write

$$\begin{aligned} \int_T \nabla_T^{k+1} (\underline{\mathbf{w}}_T) : \mathbf{V}_T &= - \int_T \mathbf{w}_{\mathcal{G},T}^c \cdot \nabla \cdot (\mathbf{V}_{\overline{\mathcal{R}},T}^c) - \int_T \mathbf{w}_{\mathcal{G},T} \cdot \nabla \cdot (\mathbf{V}_{\overline{\mathcal{R}},T}) + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \mathbf{V}_T \mathbf{n}_F \\ &= - \int_T \boldsymbol{\pi}_{\mathcal{G},T}^{c,k}(\mathbf{u}) \cdot \nabla \cdot (\mathbf{V}_{\overline{\mathcal{R}},T}^c) + \int_T p_T q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F (\mathbf{V}_T - q_T \mathbf{I}_{3,3}) \mathbf{n}_F \\ &= - \int_T \mathbf{u} \cdot \nabla \cdot (\mathbf{V}_{\overline{\mathcal{R}},T}^c) + \int_T p_T q_T + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \mathbf{u} (\mathbf{V}_T - q_T \mathbf{I}_{3,3}) \mathbf{n}_F \\ &= - \int_T \mathbf{u} \cdot \nabla \cdot (\mathbf{V}_{\overline{\mathcal{R}},T}^c) + \int_T p_T q_T + \int_T \nabla \mathbf{u} : (\mathbf{V}_T - q_T \mathbf{I}_{3,3}) + \int_T \mathbf{u} \cdot \nabla \cdot (\mathbf{V}_T - q_T \mathbf{I}_{3,3}) \\ &\lesssim \|p_T\| \|q_T\| + \|\nabla \mathbf{u}\| \|\mathbf{V}_T - q_T \mathbf{I}_{3,3}\| \\ &\lesssim \|p_T\| \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\|. \end{aligned}$$

We used that $\overline{\mathcal{R}} \mathbf{n}_F \in \mathcal{P}^k(F)$ (from (2.12)) with (3.42) and Lemma 7 on the second line, an integration by parts on the third and that $\nabla \cdot (\mathbf{V}_T - q_T \mathbf{I}_{3,3}) = \nabla \cdot \mathbf{V}_T - \nabla \cdot \mathbf{V}_{\overline{\mathcal{R}},T} = \nabla \cdot (\mathbf{V}_{\overline{\mathcal{R}},T}^c)$ to cancel the terms in the fourth line. The result follows from the Cauchy-Swartz inequality and the estimates on \mathbf{u} . \square

Remark 35. We can easily adapt Lemma 34 to require $\sum_{T \in \mathcal{T}_h} \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T = 0$. Defining $\underline{\mathbf{w}}'_h = \underline{\mathbf{w}}_h - \underline{\mathbf{I}}_{\nabla,h}^k \left(\frac{1}{\int_{\Omega} 1} \sum_{T \in \mathcal{T}_h} \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right)$, it is clear from (3.47) that $\sum_{T \in \mathcal{T}_h} \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}'_T = 0$, from Lemma 26 that $\underline{D}_h^k \underline{\mathbf{w}}'_h = \underline{p}_h$ and from (3.59) that the estimate of Lemma 34 on the norm of $\underline{\mathbf{w}}'_h$ still holds.

5.2 Adjoint consistency.

We define the adjoint consistency error for all $\mathbf{V} \in C^0(\overline{\Omega}) \cap \mathbf{H}_0^1(\Omega)$ and all $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k$ by:

$$\tilde{\mathcal{E}}_{\nabla,h}(\mathbf{V}, \underline{\mathbf{w}}_h) = \sum_{T \in \mathcal{T}_h} \left(\left(\underline{\mathbf{I}}_{L^2(T)}^k \mathbf{V}, \underline{\nabla}_T^{k+1} \underline{\mathbf{w}}_T \right)_{L^2,T} + \int_T \nabla \cdot \mathbf{V} \cdot P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right). \quad (5.7)$$

Theorem 36 (Adjoint consistency for the gradient). *For all $\mathbf{V} \in \mathbf{C}^0(\bar{\Omega}) \cap \mathbf{H}_0^1(\Omega)$ such that $\mathbf{V} \in \mathbf{H}^{k+2}(\mathcal{T}_h)$ and all $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k$, it holds:*

$$\left| \tilde{\mathcal{E}}_{\nabla,h}(\mathbf{V}, \underline{\mathbf{w}}_h) \right| \lesssim h^{k+1} (|\mathbf{V}|_{\mathbf{H}^{k+1}} + |\mathbf{V}|_{\mathbf{H}^{k+2}}) \left(\|\underline{\mathbf{w}}_h\|_{\nabla,h} + \left\| \nabla_h^{k+1} \underline{\mathbf{w}}_h \right\|_{\mathbf{L}^2,h} \right). \quad (5.8)$$

Proof. Remarks 18 and 5 show that $\forall \mathbf{V}_h \in (\mathcal{RT}^{k+1}(\mathcal{T}_h)^\top)^3, \forall T \in \mathcal{T}_h$,

$$\int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \cdot \nabla \cdot \mathbf{V}_T + \int_T \nabla_T^{k+1} \underline{\mathbf{w}}_T : \mathbf{V}_T - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \mathbf{V}_T \mathbf{n}_F = 0.$$

Let $\widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c$ be the continuous polynomial given by Lemma 51 such that $\pi_{\mathcal{P},F}^{k+2}(\widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c) = \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F$, $(\widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c)|_E = \mathbf{0}$, $(\nabla \widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c)|_E = \mathbf{0}$ and $\left\| \widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c \right\|_F \approx \left\| \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F$. It holds that $\sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \mathbf{V}_T \mathbf{n}_F = \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c \mathbf{V}_T \mathbf{n}_F$. Moreover, since $\mathbf{V} \cdot \mathbf{n}_\Omega = 0$ on $\partial\Omega$ and since the $\widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c$ are single valued we have

$$\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c \mathbf{V}_T \mathbf{n}_F = 0. \quad (5.9)$$

Hence we can write:

$$\begin{aligned} \tilde{\mathcal{E}}_{\nabla,h}(\mathbf{V}, \underline{\mathbf{w}}_h) &= \sum_{T \in \mathcal{T}_h} \left(\int_T (\mathbf{V} - \mathbf{V}_T) : \nabla_T^{k+1} \underline{\mathbf{w}}_T + \int_T \nabla \cdot (\mathbf{V} - \mathbf{V}_T) \cdot P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_E \widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c (\mathbf{V}_T - \mathbf{V}) \mathbf{n}_F + s_{\mathbf{L}^2,T} \left(\underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \mathbf{V}, \nabla_T^{k+1} \underline{\mathbf{w}}_T \right) \right) \\ &\lesssim \sum_{T \in \mathcal{T}_h} (\|\mathbf{V} - \mathbf{V}_T\| + \|\nabla \cdot (\mathbf{V} - \mathbf{V}_T)\|) \left(\left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| + \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\| \right) \\ &\quad + s_{\mathbf{L}^2,T} \left(\underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \mathbf{V}, \underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \mathbf{V} \right)^{1/2} s_{\mathbf{L}^2,T} \left(\nabla_T^{k+1} \underline{\mathbf{w}}_T, \nabla_T^{k+1} \underline{\mathbf{w}}_T \right)^{1/2} \\ &\quad + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \widetilde{\gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F}^c (\mathbf{V}_T - \mathbf{V}) \mathbf{n}_F. \end{aligned}$$

Applying (5.3) and Lemma 25 gives:

$$s_{\mathbf{L}^2,T} \left(\underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \mathbf{V}, \underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \mathbf{V} \right)^{1/2} s_{\mathbf{L}^2,T} \left(\nabla_T^{k+1} \underline{\mathbf{w}}_T, \nabla_T^{k+1} \underline{\mathbf{w}}_T \right)^{1/2} \lesssim h^{k+1} |\mathbf{V}|_{\mathbf{H}^{k+1}(T)} \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\|_{\mathbf{L}^2,T}.$$

Using the approximation properties of the spaces $\mathcal{RT}^{k+1}(T)$ given by a slight adaptation of [14, Lemma 6.8] we can find $\mathbf{V}_T \in (\mathcal{RT}^{k+1}(T)^\top)^3$ such that

$$\|\mathbf{V} - \mathbf{V}_T\| + \|\nabla \cdot (\mathbf{V} - \mathbf{V}_T)\| \lesssim h^{k+1} \left(|\mathbf{V}|_{\mathbf{H}^{k+1}(T)} + |\mathbf{V}|_{\mathbf{H}^{k+2}(T)} \right)$$

By (3.59) we see that

$$\left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\| + \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\| \lesssim \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T \right\|_{\mathbf{L}^2,T} + \|\underline{\mathbf{w}}_T\|_{\nabla,T}.$$

Lastly we use Theorem 52 to find $\mathbf{R}_{\underline{\mathbf{w}}_T} \in \mathbf{H}^1(T)$ such that

$$\begin{aligned} \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \widetilde{\gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F} (\mathbf{V}_T - \mathbf{V}) \mathbf{n}_F &= \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \mathbf{R}_{\underline{\mathbf{w}}_T} (\mathbf{V}_T - \mathbf{V}) \mathbf{n}_F \\ &= \int_T \nabla \mathbf{R}_{\underline{\mathbf{w}}_T} : (\mathbf{V}_T - \mathbf{V}) + \int_T \mathbf{R}_{\underline{\mathbf{w}}_T} \cdot \nabla \cdot (\mathbf{V}_T - \mathbf{V}). \end{aligned}$$

Hence

$$\left| \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F (\mathbf{V}_T - \mathbf{V}) \mathbf{n}_F \right| \lesssim (\|\mathbf{V} - \mathbf{V}_T\| + \|\nabla \cdot (\mathbf{V}_T - \mathbf{V})\|) (\|\nabla \mathbf{R}_{\underline{\mathbf{w}}_T}\| + \|\mathbf{R}_{\underline{\mathbf{w}}_T}\|),$$

and we conclude with Theorem 52 which gives the boundedness of $\mathbf{R}_{\underline{\mathbf{w}}_T}$. \square

We can sharpen the estimate (5.7) when \mathbf{V} is the gradient of some field. Indeed, if were to take $\mathbf{V} = \nabla v$ in Theorem 36, we would see that a norm over \mathbf{H}^{k+3} appears in the estimate, which is not optimal.

We define the adjoint consistency error for all $\mathbf{v} \in \mathbf{H}^2(\Omega)$ such that $\nabla \mathbf{v} \cdot \mathbf{n}_\Omega = 0$ and all $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla, h}^k$ by:

$$\tilde{\mathcal{E}}_{\Delta, h}(\mathbf{v}, \underline{\mathbf{w}}_h) = \sum_{T \in \mathcal{T}_h} \left(\int_T \Delta \mathbf{v} \cdot P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T + \left(\nabla_T^{k+1} \underline{\mathbf{I}}_{\nabla, T}^k \mathbf{v}, \nabla_T^{k+1} \underline{\mathbf{w}}_T \right)_{\mathbf{L}^2, T} \right). \quad (5.10)$$

Theorem 37 (Adjoint consistency for the Laplacian). *For all $\mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{C}^1(\bar{\Omega})$ such that $\nabla \mathbf{v} \cdot \mathbf{n}_\Omega = 0$ and $\mathbf{v} \in \mathbf{H}^{k+2}(\mathcal{T}_h)$ and for all $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla, h}^k$, it holds:*

$$\left| \tilde{\mathcal{E}}_{\Delta, h}(\mathbf{v}, \underline{\mathbf{w}}_h) \right| \lesssim h^{k+1} |\mathbf{v}|_{\mathbf{H}^{k+2}} \left\| \nabla_h^{k+1} \underline{\mathbf{w}}_h \right\|_{\mathbf{L}^2, h}. \quad (5.11)$$

Proof. For any $T \in \mathcal{T}_h$, (4.2c) gives:

$$\left(\nabla_T^{k+1} \underline{\mathbf{I}}_{\nabla, T}^k \mathbf{v}, \nabla_T^{k+1} \underline{\mathbf{w}}_T \right)_{\mathbf{L}^2, T} = \int_T \pi_{\mathcal{RT}, T}^{k+1} \nabla \mathbf{v} : \nabla_T^{k+1} \underline{\mathbf{w}}_T + s_{\mathbf{L}^2, T} \left(\underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \nabla \mathbf{v}, \nabla_T^{k+1} \underline{\mathbf{w}} \right).$$

With an integration by parts and since $\int_T \pi_{\mathcal{RT}, T}^{k+1} \nabla \mathbf{v} : \nabla_T^{k+1} \underline{\mathbf{w}}_T = \int_T \nabla \mathbf{v} : \nabla_T^{k+1} \underline{\mathbf{w}}_T$ we have:

$$\begin{aligned} \tilde{\mathcal{E}}_{\Delta, h}(\mathbf{v}, \underline{\mathbf{w}}_h) &= \sum_{T \in \mathcal{T}_h} \left(\int_T \nabla \mathbf{v} : (\nabla_T^{k+1} \underline{\mathbf{w}}_T - \nabla P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T) + s_{\mathbf{L}^2, T} \left(\underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \nabla \mathbf{v}, \nabla_T^{k+1} \underline{\mathbf{w}} \right) \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T \nabla \mathbf{v} \mathbf{n}_F \right). \end{aligned}$$

Since we assume $\nabla \mathbf{v} \cdot \mathbf{n}_\Omega = 0$ we have

$$\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F \nabla \mathbf{v} \mathbf{n}_F = 0. \quad (5.12)$$

By Remark 18 it holds $\forall \mathbf{v}_T \in \mathcal{P}^{k+1}(T)$,

$$\int_T \Delta \mathbf{v}_T \cdot P_{\nabla, T}^{k+1} \underline{\mathbf{w}}_T + \int_T \nabla_T^{k+1} \underline{\mathbf{w}}_T : \nabla \mathbf{v}_T - \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F \gamma_{\nabla, F}^{k+2} \underline{\mathbf{w}}_F \nabla \mathbf{v}_T \mathbf{n}_F = 0,$$

so

$$\int_T \nabla \mathbf{v}_T : (\nabla_T^{k+1} \underline{\mathbf{w}}_T - \nabla P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T) + \sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F (P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T - \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F) \nabla \mathbf{v}_T \mathbf{n}_F = 0.$$

This allows us to write for any $\underline{\mathbf{v}}_h = (\mathbf{v}_T)_{T \in \mathcal{T}_h} \in \mathcal{P}^{k+1}(T)$,

$$\begin{aligned} \tilde{\mathcal{E}}_{\Delta,h}(\mathbf{v}, \underline{\mathbf{w}}_h) &= \sum_{T \in \mathcal{T}_h} \left(\int_T \nabla(\mathbf{v} - \mathbf{v}_T) : (\nabla_T^{k+1} \underline{\mathbf{w}}_T - \nabla P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T) + s_{\mathbf{L}^2,T} \left(\underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \nabla \mathbf{v}, \underline{\nabla}_T^{k+1} \underline{\mathbf{w}} \right) \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_T} \omega_{FE} \int_F (P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T - \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F) \nabla(\mathbf{v} - \mathbf{v}_T) \mathbf{n}_F \right), \\ \left| \tilde{\mathcal{E}}_{\Delta,h}(\mathbf{v}, \underline{\mathbf{w}}_h) \right| &\lesssim \sum_{T \in \mathcal{T}_h} \left(\|\nabla(\mathbf{v} - \mathbf{v}_T)\|_T \left\| \nabla_T^{k+1} \underline{\mathbf{w}}_T - \nabla P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \right\|_T \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_T} \left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T - \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F \|\nabla(\mathbf{v} - \mathbf{v}_T)\|_F + \left| s_{\mathbf{L}^2,T} \left(\underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \nabla \mathbf{v}, \underline{\nabla}_T^{k+1} \underline{\mathbf{w}} \right) \right| \right). \end{aligned}$$

Applying Lemma 31 we get

$$\left\| P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T - \gamma_{\nabla,F}^{k+2} \underline{\mathbf{w}}_F \right\|_F \|\nabla(\mathbf{v} - \mathbf{v}_T)\|_F \lesssim \left\| \underline{\nabla}_T^{k+1} \underline{\mathbf{w}}_T \right\|_{\mathbf{L}^2,T} h^{\frac{1}{2}} \|\nabla(\mathbf{v} - \mathbf{v}_T)\|_F.$$

Furthermore (5.3) and Lemma 25 give

$$\left| s_{\mathbf{L}^2,T} \left(\underline{\mathbf{I}}_{\mathbf{L}^2(T)}^k \nabla \mathbf{v}, \underline{\nabla}_T^{k+1} \underline{\mathbf{w}} \right) \right| \lesssim h^{k+1} |\nabla \mathbf{v}|_{\mathbf{H}^{k+1}} \left\| \underline{\nabla}_T^{k+1} \underline{\mathbf{w}}_T \right\|_{\mathbf{L}^2,T}.$$

Hence, applying Lemma 31 we write:

$$\begin{aligned} \left| \tilde{\mathcal{E}}_{\Delta,h}(\mathbf{v}, \underline{\mathbf{w}}_h) \right| &\lesssim \sum_{T \in \mathcal{T}_h} \left(\left\| \underline{\nabla}_T^{k+1} \underline{\mathbf{w}}_T \right\|_{\mathbf{L}^2,T} (\|\nabla(\mathbf{v} - \mathbf{v}_T)\|_T + \sum_{F \in \mathcal{F}_T} h^{\frac{1}{2}} \|\nabla(\mathbf{v} - \mathbf{v}_T)\|_F) \right. \\ &\quad \left. + h^{k+1} |\mathbf{v}|_{\mathbf{H}^{k+2}} \left\| \underline{\nabla}_T^{k+1} \underline{\mathbf{w}}_h \right\|_{\mathbf{L}^2,T} \right). \end{aligned}$$

We conclude by taking $\mathbf{v}_T = \pi_{\mathcal{P},T}^{1,k+1} \mathbf{v}$ the elliptic projection on T (see [10, Definition 1.39]), then [10, Theorem 1.48] gives:

$$\begin{aligned} \left\| \mathbf{v} - \pi_{\mathcal{P},T}^{1,k+1} \mathbf{v} \right\|_{\mathbf{H}^1(T)} &\lesssim h^{k+1} |\mathbf{v}|_{\mathbf{H}^{k+2}}, \\ h^{\frac{1}{2}} \left\| \mathbf{v} - \pi_{\mathcal{P},T}^{1,k+1} \mathbf{v} \right\|_{\mathbf{H}^1(F)} &\lesssim h^{k+1} |\mathbf{v}|_{\mathbf{H}^{k+2}}. \end{aligned}$$

□

6 Stokes equations.

Finally, we illustrate this complex with the design of a scheme for the Stokes equations. For the sake of simplicity we use Neumann boundary conditions over the whole boundary, that it to say with a free outlet condition. More general conditions are not difficult to enforce and

are discussed in Section 7. The solution is determined only up to a constant vector field. This leads to the introduction of a new space:

$$\underline{\mathbf{X}}_{\nabla,h,\star}^k := \{\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k : \sum_{T \in \mathcal{T}_h} \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{v}}_T = 0\}. \quad (6.1)$$

This is the discrete counterpart of $\mathbf{H}^1(\Omega) \cap L_0^2(\Omega)$.

Let μ be a constant viscosity, we define the symmetric bilinear form $\mathfrak{a}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h) \in \underline{\mathbf{X}}_{\nabla,h}^k \times \underline{\mathbf{X}}_{\nabla,h}^k \rightarrow \mathbb{R}$ on all $\underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k$ by

$$\mathfrak{a}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h) := \mu \left(\nabla_h^{k+1} \underline{\mathbf{v}}_h, \nabla_h^{k+1} \underline{\mathbf{w}}_h \right)_{L^2,h}. \quad (6.2)$$

We also define the bilinear form $\mathfrak{b}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{q}}_h) \in \underline{\mathbf{X}}_{\nabla,h}^k \times \underline{\mathbf{X}}_{L^2,h}^k \rightarrow \mathbb{R}$ on all $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k, \underline{\mathbf{q}}_h \in \underline{\mathbf{X}}_{L^2,h}^k$ by

$$\mathfrak{b}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{q}}_h) := \sum_{T \in \mathcal{T}_h} \int_T D_T^k \underline{\mathbf{v}}_T \underline{\mathbf{q}}_T. \quad (6.3)$$

Then we define the bilinear form $\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h), (\underline{\mathbf{w}}_h, \underline{\mathbf{q}}_h)) \in (\underline{\mathbf{X}}_{\nabla,h,\star}^k \times \underline{\mathbf{X}}_{L^2,h}^k) \times (\underline{\mathbf{X}}_{\nabla,h,\star}^k \times \underline{\mathbf{X}}_{L^2,h}^k) \rightarrow \mathbb{R}$ by

$$\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h), (\underline{\mathbf{w}}_h, \underline{\mathbf{q}}_h)) = \mathfrak{a}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h) - \mathfrak{b}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{p}}_h) + \mathfrak{b}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{q}}_h). \quad (6.4)$$

We define a suitable Sobolev-like norm on our discrete spaces such that $\forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k$,

$$\|\underline{\mathbf{v}}_h\|_{\mu,\nabla,1,h} := \left(\|\underline{\mathbf{v}}_h\|_{\nabla,h}^2 + \mathfrak{a}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{v}}_h) \right)^{1/2}. \quad (6.5)$$

And for $f \in L^2(\Omega)$ we set $\mathcal{L}_h : \underline{\mathbf{X}}_{\nabla,h,\star}^k \rightarrow \mathbb{R}$ such that $\forall \underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k$,

$$\mathcal{L}_h(\underline{\mathbf{v}}_h) := \sum_{T \in \mathcal{T}_h} \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{v}}_T \cdot f. \quad (6.6)$$

We define the discrete problem:

Find $(\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h) \in \underline{\mathbf{X}}_{\nabla,h,\star}^k \times \underline{\mathbf{X}}_{L^2,h}^k$ such that for all $(\underline{\mathbf{w}}_h, \underline{\mathbf{q}}_h) \in \underline{\mathbf{X}}_{\nabla,h,\star}^k \times \underline{\mathbf{X}}_{L^2,h}^k$,

$$\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h), (\underline{\mathbf{w}}_h, \underline{\mathbf{q}}_h)) = \mathcal{L}_h(\underline{\mathbf{v}}_h). \quad (6.7)$$

We show well-posedness in Lemma 39.

We consider the following Stokes problem:

Find $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap L_0^2(\Omega)$, $p \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\mu \Delta \mathbf{u} + \mathbf{grad} p &= f, \quad \text{on } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, \quad \text{on } \Omega, \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}_\Omega} &= 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (6.8)$$

Let (\mathbf{u}, p) solves (6.8) and let $(\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h)$ solves (6.7). We assume that the continuous solutions \mathbf{u} and p have the additional smoothness $\mathbf{u} \in \mathbf{H}^{k+2}(\mathcal{T}_h)$ and $p \in H^{k+2}(\mathcal{T}_h)$. We deduce the following error estimate.

Theorem 38 (Error estimate for Stokes). *Under the smoothness assumption on \mathbf{u} and p it holds that*

$$\|\underline{\mathbf{v}}_h - \mathbf{I}_{\nabla,h}^k \mathbf{u}\|_{\mu,\nabla,1,h} + \|\underline{p}_h - \mathbf{I}_{L^2,h}^k p\|_{L^2,T} \lesssim h^{k+1} \left(|\mathbf{u}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)} + |p|_{H^{k+1}(\mathcal{T}_h)} + |p|_{H^{k+2}(\mathcal{T}_h)} \right). \quad (6.9)$$

Proof. The proof is a direct application of the third Strang lemma (see [8]) to the estimates given by Lemma 39 and 40. \square

Lemma 39 (Well-posedness). *For any $(\underline{\mathbf{v}}_h, \underline{p}_h) \in \underline{\mathbf{X}}_{\nabla,h,\star}^k \times \underline{\mathbf{X}}_{L^2,h}^k$ there is $(\underline{\mathbf{w}}_h, \underline{q}_h) \in \underline{\mathbf{X}}_{\nabla,h,\star}^k \times \underline{\mathbf{X}}_{L^2,h}^k$ such that $\|\underline{\mathbf{w}}_h\|_{\mu,\nabla,1,h} + \|\underline{q}_h\|_{L^2,T} \lesssim \|\underline{\mathbf{v}}_h\|_{\mu,\nabla,1,h} + \|\underline{p}_h\|_{L^2,T}$ and*

$$\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{p}_h), (\underline{\mathbf{w}}_h, \underline{q}_h)) \gtrsim \|\underline{\mathbf{v}}_h\|_{\mu,\nabla,1,h}^2 + \|\underline{p}_h\|_{L^2,T}^2.$$

Proof. Let $(\underline{\mathbf{v}}_h, \underline{p}_h) \in \underline{\mathbf{X}}_{\nabla,h,\star}^k \times \underline{\mathbf{X}}_{L^2,h}^k$, we have

$$\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{p}_h), (\underline{\mathbf{v}}_h, \underline{p}_h)) = \mathfrak{a}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{v}}_h) \gtrsim \|\underline{\mathbf{v}}_h\|_{\mu,\nabla,1,h}^2, \quad (6.10)$$

where the last inequality comes from Lemma 32. Moreover by Remark 35 there is $\underline{\mathbf{w}}'_h \in \underline{\mathbf{X}}_{\nabla,h,\star}^k$ such that $D_h^k \underline{\mathbf{w}}'_h = -\underline{p}_h$ and $\|\underline{\mathbf{w}}'_h\|_{\mu,\nabla,1,h} \lesssim \|\underline{p}_h\|_{L^2,T}$. Hence

$$\begin{aligned} \mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{p}_h), (\underline{\mathbf{w}}'_h, 0)) &= \mathfrak{a}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{w}}'_h) + \|\underline{p}_h\|_{L^2,T}^2 \\ &\geq -\frac{1}{2} \|\underline{\mathbf{v}}_h\|_{\mu,\nabla,1,h}^2 - \frac{1}{2} \|\underline{\mathbf{w}}'_h\|_{\mu,\nabla,1,h}^2 + \|\underline{p}_h\|_{L^2,T}^2 \\ &\gtrsim -\frac{1}{2} \|\underline{\mathbf{v}}_h\|_{\mu,\nabla,1,h}^2 + \frac{1}{2} \|\underline{p}_h\|_{L^2,T}^2. \end{aligned} \quad (6.11)$$

We conclude summing (6.10) and (6.11). \square

We define the consistency error $\mathcal{E}_h : \underline{\mathbf{X}}_{\nabla,h,\star}^k \times \underline{\mathbf{X}}_{L^2,h}^k \rightarrow \mathbb{R}$ by

$$\mathcal{E}_h((\underline{\mathbf{w}}_h, \underline{q}_h)) = \mathcal{L}_h(\underline{\mathbf{w}}_h) - \mathcal{A}_h((\mathbf{I}_{\nabla,h}^k \mathbf{u}, \mathbf{I}_{L^2,h}^k p), (\underline{\mathbf{w}}_h, \underline{q}_h)). \quad (6.12)$$

Lemma 40. *For all $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla,h,\star}^k$, $\underline{q}_h \in \underline{\mathbf{X}}_{L^2,h}^k$, it holds*

$$\begin{aligned} \mathcal{E}_h((\underline{\mathbf{w}}_h, \underline{q}_h)) &\lesssim h^{k+1} \left(|\mathbf{u}|_{\mathbf{H}^{k+2}(\mathcal{T}_h)} + |p|_{H^{k+1}(\mathcal{T}_h)} + |p|_{H^{k+2}(\mathcal{T}_h)} \right) \\ &\quad \times \left(\|\underline{\mathbf{w}}_h\|_{\mu,\nabla,1,h} + \|\underline{q}_h\|_{L^2,T} \right). \end{aligned}$$

Proof. Let $\underline{\mathbf{w}}_h \in \underline{\mathbf{X}}_{\nabla,h,\star}^k$, $\underline{q}_h \in \underline{\mathbf{X}}_{L^2,h}^k$, then

$$\begin{aligned} \mathcal{E}_h((\underline{\mathbf{w}}_h, \underline{q}_h)) &= \sum_{T \in \mathcal{T}_h} \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \cdot f - \mu \left(\nabla_T^{k+1} \mathbf{I}_{\nabla,T}^k \mathbf{u}, \nabla_T^{k+1} \underline{\mathbf{w}}_T \right)_{\nabla,T} \\ &\quad + \int_T D_T^k \underline{\mathbf{w}}_T \mathbf{I}_{L^2,h}^k p - \int_T D_T^k (\mathbf{I}_{\nabla,T}^k \mathbf{u}) q_T \\ &= \sum_{T \in \mathcal{T}_h} \int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \cdot \mathbf{grad} p + \int_T D_T^k \underline{\mathbf{w}}_T \mathbf{I}_{L^2,h}^k p \\ &\quad - \mu \left(\int_T P_{\nabla,T}^{k+1} \underline{\mathbf{w}}_T \cdot \Delta \mathbf{u} + \left(\nabla_T^{k+1} \mathbf{I}_{\nabla,T}^k \mathbf{u}, \nabla_T^{k+1} \underline{\mathbf{w}}_T \right)_{\nabla,T} \right) \\ &= \tilde{\mathcal{E}}_{\nabla,h}(p \mathbf{I}_{3,3}, \underline{\mathbf{w}}_h) - s_{L^2,T} \left(\nabla_T^{k+1} \underline{\mathbf{w}}_T, \mathbf{I}_{L^2(T)}^k (p \mathbf{I}_{3,3}) \right) - \mu \tilde{\mathcal{E}}_{\Delta,h}(\mathbf{u}, \underline{\mathbf{w}}_h) \\ &\leq \left| \tilde{\mathcal{E}}_{\nabla,h}(p \mathbf{I}_{3,3}, \underline{\mathbf{w}}_h) \right| + \left| s_{L^2,T} \left(\nabla_T^{k+1} \underline{\mathbf{w}}_T, \nabla_T^{k+1} \underline{\mathbf{w}}_T \right) \right|^{1/2} \\ &\quad + \left| s_{L^2,T} \left(\mathbf{I}_{L^2(T)}^k (p \mathbf{I}_{3,3}), \mathbf{I}_{L^2(T)}^k (p \mathbf{I}_{3,3}) \right) \right|^{1/2} + \mu \left| \tilde{\mathcal{E}}_{\Delta,h}(\mathbf{u}, \underline{\mathbf{w}}_h) \right|. \end{aligned}$$

The second equality comes from $f = -\mu \Delta \mathbf{u} + \mathbf{grad} p$, (4.2c) and $\operatorname{div} \mathbf{u} = 0$, and the third equality comes from (5.7), (5.10) as well as:

$$\int_T D_T^k \underline{\mathbf{w}}_T \pi_{\mathcal{P},T}^k p = \int_T \operatorname{Tr}(\nabla_T^{k+1} \underline{\mathbf{w}}_T) p = \int_T \nabla_T^{k+1} \underline{\mathbf{w}}_T : (p \mathbf{I}_{3,3}) = \int_T \nabla_T^{k+1} \underline{\mathbf{w}}_T : \pi_{\mathcal{RT},T}^{k+1}(p \mathbf{I}_{3,3}).$$

We conclude inferring the estimates of Theorem 36, 37 and the consistency (5.3). \square

7 Alternative boundary conditions.

In this section we show how to extend the results of Section 6 to Dirichlet boundary conditions on $\underline{\mathbf{X}}_{\nabla,h}^k$. This is useful for common condition such as the no slip condition or forced inlet condition and does not require much change.

7.1 Dirichlet boundary conditions.

We introduce the space

$$\underline{\mathbf{X}}_{\nabla,h,0}^k := \{\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\nabla,h}^k : \underline{\mathbf{v}}_F \equiv \mathbf{0}, \forall F \in \mathcal{F}_h, F \subset \partial\Omega\}. \quad (7.1)$$

Since the pressure is only defined up to a constant value, we introduce the natural space:

$$\underline{\mathbf{X}}_{L^2,h,\star}^k := \{\underline{q}_h \in \underline{\mathbf{X}}_{L^2,h}^k : \sum_{T \in \mathcal{T}_h} \int_T q_T = 0\}. \quad (7.2)$$

Then we define the bilinear form: $\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{p}_h), (\underline{\mathbf{w}}_h, \underline{q}_h)) \in (\underline{\mathbf{X}}_{\nabla,h,0}^k \times \underline{\mathbf{X}}_{L^2,h,\star}^k) \times (\underline{\mathbf{X}}_{\nabla,h,0}^k \times \underline{\mathbf{X}}_{L^2,h,\star}^k) \rightarrow \mathbb{R}$ by

$$\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{p}_h), (\underline{\mathbf{w}}_h, \underline{q}_h)) := \mathbf{a}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h) - \mathbf{b}_h(\underline{\mathbf{w}}_h, \underline{p}_h) + \mathbf{b}_h(\underline{\mathbf{v}}_h, \underline{q}_h). \quad (7.3)$$

With \mathbf{a}_h and \mathbf{b}_h defined by (6.2), (6.3), we also keep the same definition (6.6) of the source term \mathcal{L}_h . So the discrete problem is:

Find $(\underline{\mathbf{v}}_h, \underline{p}_h) \in \underline{\mathbf{X}}_{\nabla,h,0}^k \times \underline{\mathbf{X}}_{L^2,h,\star}^k$ such that for all $(\underline{\mathbf{w}}_h, \underline{q}_h) \in \underline{\mathbf{X}}_{\nabla,h,0}^k \times \underline{\mathbf{X}}_{L^2,h,\star}^k$,

$$\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{p}_h), (\underline{\mathbf{w}}_h, \underline{q}_h)) = \mathcal{L}_h(\underline{\mathbf{v}}_h). \quad (7.4)$$

The continuous Stokes problem becomes:

Find $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$, $p \in H^1(\Omega) \cap L_0^2(\Omega)$ such that

$$\begin{aligned} -\mu \Delta \mathbf{u} + \mathbf{grad} p &= f, \text{ on } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, \text{ on } \Omega. \end{aligned} \quad (7.5)$$

Theorem 41. *Under the same assumptions as those of Theorem 38, the problem (7.4) is well-posed and converges toward the continuous solution of problem (7.5) with the same error estimate as in Theorem 38.*

Proof. The proof is similar to the proof of Theorem 38. We need a suitable version of Lemma 34, and we can expect $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\nabla,h,0}^k$ if $\underline{p}_h \in \underline{\mathbf{X}}_{L^2,h,\star}^k$ by substituting the use of Theorem 54 by Theorem 55 in the proof of Lemma 34 (see Remark 56). The consistency errors 36 and 37 require respectively $\mathbf{W} \in \mathbf{H}_0^1(\Omega)$ and $\nabla \mathbf{w} \cdot \mathbf{n}_\Omega = 0$. However we can check that this is only used to get (5.9) and (5.12) both of which also hold if $\underline{\mathbf{v}}_h \in \underline{\mathbf{X}}_{\nabla,h,0}^k$ instead, so that $\underline{\mathbf{v}}_F \equiv \mathbf{0}$, $\forall F \subset \partial\Omega$. Finally, we relied on Lemma 32 to show that \mathcal{A}_h is weakly coercive. This too can be readily adapted if we use [10, Lemma 2.15] instead of [10, Theorem 6.5] in the proof of Lemma 32. With these three results we can proceed exactly in the same manner as we did for Theorem 38. \square

7.2 Mixed boundary conditions.

We can also use Dirichlet conditions on a subset of the boundary and Neumann conditions elsewhere. Let Γ_D be a relatively closed subset of $\partial\Omega$ with a non-zero measure and $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Furthermore assume that each boundary face $\partial\Omega \supset F \in \mathcal{F}_h$ is either contained in Γ_N or in Γ_D but not in both (i.e. either $\overline{F} \subset \Gamma_D = \emptyset$ or $F \cap \Gamma_D = \emptyset$). We also require that Γ_D and Γ_N contain at least one face (else we degenerate to pure Neumann or pure Dirichlet). The continuous Stokes problem is:

Find $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $p \in H^1(\Omega)$ such that

$$\begin{aligned} -\mu \Delta \mathbf{u} + \mathbf{grad} p &= f, & \text{on } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, & \text{on } \Omega, \\ \mathbf{u} &= 0, & \text{on } \Gamma_D, \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}_\Omega} &= 0, & \text{on } \Gamma_N, \\ p &= 0, & \text{on } \Gamma_N. \end{aligned} \tag{7.6}$$

We introduce the discrete space $\underline{\mathbf{X}}_{\nabla,h,D}^k := \{\mathbf{v}_h \in \underline{\mathbf{X}}_{\nabla,h}^k : \mathbf{v}_F \equiv \mathbf{0}, \forall F \in \Gamma_D\}$ and define: $\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h), (\underline{\mathbf{w}}_h, \underline{\mathbf{q}}_h)) \in (\underline{\mathbf{X}}_{\nabla,h,D}^k \times \underline{\mathbf{X}}_{L^2,h}^k) \times (\underline{\mathbf{X}}_{\nabla,h,D}^k \times \underline{\mathbf{X}}_{L^2,h}^k) \rightarrow \mathbb{R}$ by

$$\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h), (\underline{\mathbf{w}}_h, \underline{\mathbf{q}}_h)) = \mathfrak{a}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h) - \mathfrak{b}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{p}}_h) + \mathfrak{b}_h(\underline{\mathbf{v}}_h, \underline{\mathbf{q}}_h). \tag{7.7}$$

The discrete problem becomes:

Find $(\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h) \in \underline{\mathbf{X}}_{\nabla,h,D}^k \times \underline{\mathbf{X}}_{L^2,h}^k$ such that for all $(\underline{\mathbf{w}}_h, \underline{\mathbf{q}}_h) \in \underline{\mathbf{X}}_{\nabla,h,D}^k \times \underline{\mathbf{X}}_{L^2,h}^k$,

$$\mathcal{A}_h((\underline{\mathbf{v}}_h, \underline{\mathbf{p}}_h), (\underline{\mathbf{w}}_h, \underline{\mathbf{q}}_h)) = \mathcal{L}_h(\underline{\mathbf{v}}_h). \tag{7.8}$$

8 Numerical validation.

The Stokes complex was implemented within the HARDCore C++ framework (see <https://github.com/jdroniou/HARDCore>), using the linear algebra facilities from the Eigen3 library (see <https://eigen.tuxfamily.org>) and the linear solver from the Portable, Extensible Toolkit for Scientific Computation PETSc (see <https://petsc.org>). An implementation of the spaces and operators defined in this paper as well as a Stokes solver can be found at <https://github.com/mlhanot/HARDCore3D-Stokes>. The numerical validation is done with a constant viscosity $\mu = 1$. We measure the rate of convergence toward an exact solution for various polynomial degrees $k \in \{0, 1, 2, 3\}$. The error is computed as

$$\frac{\|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}_{\nabla,h}^k \mathbf{u}\|_{\mu,\nabla,1,h} + \|\underline{\mathbf{p}}_h - \underline{\mathbf{I}}_{L^2,h}^k p\|_{L^2,h}}{\|\underline{\mathbf{I}}_{\nabla,h}^k \mathbf{u}\|_{\mu,\nabla,1,h} + \|\underline{\mathbf{I}}_{L^2,h}^k p\|_{L^2,h}}.$$

We expect the error to decrease at a rate $\mathcal{O}(h^{k+1})$ thanks to Theorem 38 and 41. We also validated the 2-dimensional variation detailed in appendix C. These tests are done on various mesh sequences. One mesh of each sequence is shown in Figure 3 for the 2-dimensional cases, and a cross section of the 3-dimensional meshes is shown in Figure 2. The results are given in Figure 4 in 2 dimensions and in Figure 5 in 3 dimensions. We always obtain results consistent with the theory. In 2 dimensions, we can see that the method is robust and the convergence is barely impacted by the various features of the meshes. In 3 dimensions, arbitrary polyhedra can be much wilder than arbitrary polygons. We can see that, while the expected convergence rate

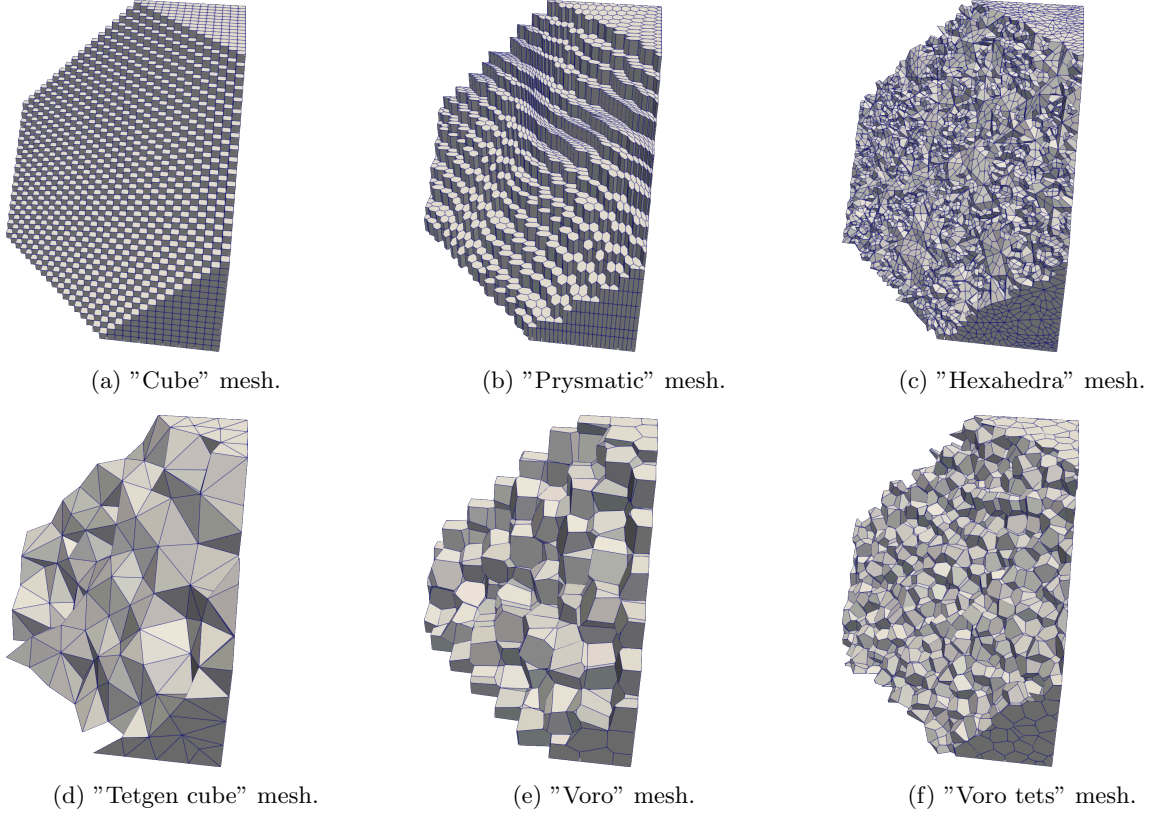


Figure 2: Families of mesh used in 3 dimensions, sliced for visualization.

is asymptotically obtained, some combinations of degree and mesh exhibit better properties. On the coarsest meshes with the lowest polynomial degree we notice a drop in the convergence rate. These meshes are too coarse for the solution sought and the problem disappears when refining or increasing the polynomial degree.

A Results on polynomial spaces.

We begin by showing a few results to complete the introduction of spaces (2.14) and (2.16). Let $T \in \mathcal{T}_h$ be any cell. We assume without loss of generality that $\mathbf{x}_T = 0$, where $\mathbf{x}_T \in T$ is the point given by (2.5). We identify the coordinate with three variables x , y and z , and we define $\mathcal{P}^k[X, Y](T) = \{P \in \mathcal{P}^k(T), \deg_Z(P) = 0\}$. By (2.5) we can write:

$$(\mathcal{R}^{c,k}(T)^\top)^3 = \begin{pmatrix} xP_1 & yP_1 & zP_1 \\ xP_2 & yP_2 & zP_2 \\ xP_3 & yP_3 & zP_3 \end{pmatrix}, P_{i \in \{1,2,3\}} \in \mathcal{P}^{k-1}(T). \quad (\text{A.1})$$

Lemma 42. *Keeping the notations of (A.1), the subset $\overline{\mathcal{R}}^{c,k}(T) = \{W \in (\mathcal{R}^{c,k}(T)^\top)^3 : \text{Tr } W = 0\}$ is characterized by:*

$$\begin{aligned} P_1 &= yzC^1 + y\gamma - z\beta \\ P_2 &= xzC^2 - x\gamma + z\lambda, \\ P_3 &= xyC^3 + x\beta - y\lambda \end{aligned} \quad \begin{aligned} \lambda &\in \mathcal{P}^{k-2}[Y, Z], \beta \in \mathcal{P}^{k-2}[X, Z], \gamma \in \mathcal{P}^{k-2}[X, Y], \\ C^i &\in \mathcal{P}^{k-3}[X, Y, Z], C^1 + C^2 + C^3 = 0. \end{aligned}$$

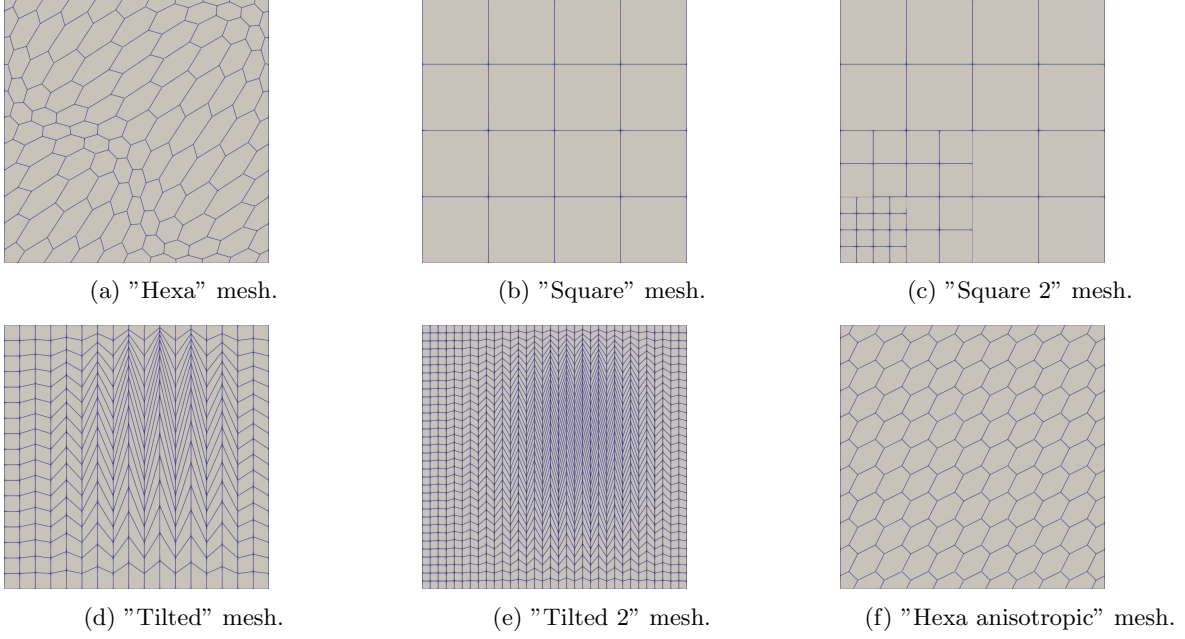


Figure 3: Families of mesh used in 2 dimensions.

Proof. The proof relies on repeated use of Euclidean division. When we append x (resp. y, z) as subscript we mean that the polynomial does not depend on x (resp. y, z). We write:

$$\begin{aligned} P_2 &= xQ_2 + R_x^2, \\ P_3 &= xQ_3 + R_x^3. \end{aligned}$$

Then the condition $\text{Tr } W = 0$ becomes $xP_1 + yP_2 + zP_3 = x(P_1 + yQ_2 + zQ_3) + yR_x^2 + zR_x^3 = 0$. Looking at the degree in X , we must have $P_1 + yQ_2 + zQ_3 = 0$ and $yR_x^2 + zR_x^3 = 0$. Hence there exists $\lambda \in \mathcal{P}^{k-2}[Y, Z]$ such that

$$R_x^2 = z\lambda, \quad R_x^3 = -y\lambda.$$

On the other hand, writing $P_1 = yA^1 + zA_y^1 + A_{y,z}^1$, we have $yA^1 + zA_y^1 + A_{y,z}^1 + yQ_2 + zQ_3 = 0$. Looking at the degree in Y and Z we must have $A_{y,z}^1 = 0$. Let us write $Q_3 = yB^3 + B_y^3$, we have $y(A^1 + Q_2 + zB^3) + z(A_y^1 + B_y^3) = 0$. The degree in Y shows that $A_y^1 + B_y^3 = 0$. Thus there exists $\beta \in \mathcal{P}^{k-2}[X, Z]$ such that

$$A_y^1 = -\beta, \quad B_y^3 = \beta.$$

Let $A^1 = zC^1 + C_z^1$ and $Q_2 = zC^2 + C_z^2$. We are left with $A^1 + Q_2 + zB^3 = z(C^1 + C^2 + B^3) + C_z^1 + C_z^2 = 0$. Once again the degree in Z shows that $C_z^1 + C_z^2 = 0$ and $C^1 + C^2 + B^3 = 0$, so there exists $\gamma \in \mathcal{P}^{k-2}[X, Y]$ such that

$$C_z^1 = \gamma, \quad C_z^2 = -\gamma.$$

We conclude by writing $B^3 := C^3$ and substituting back each term in the development of P_i . □

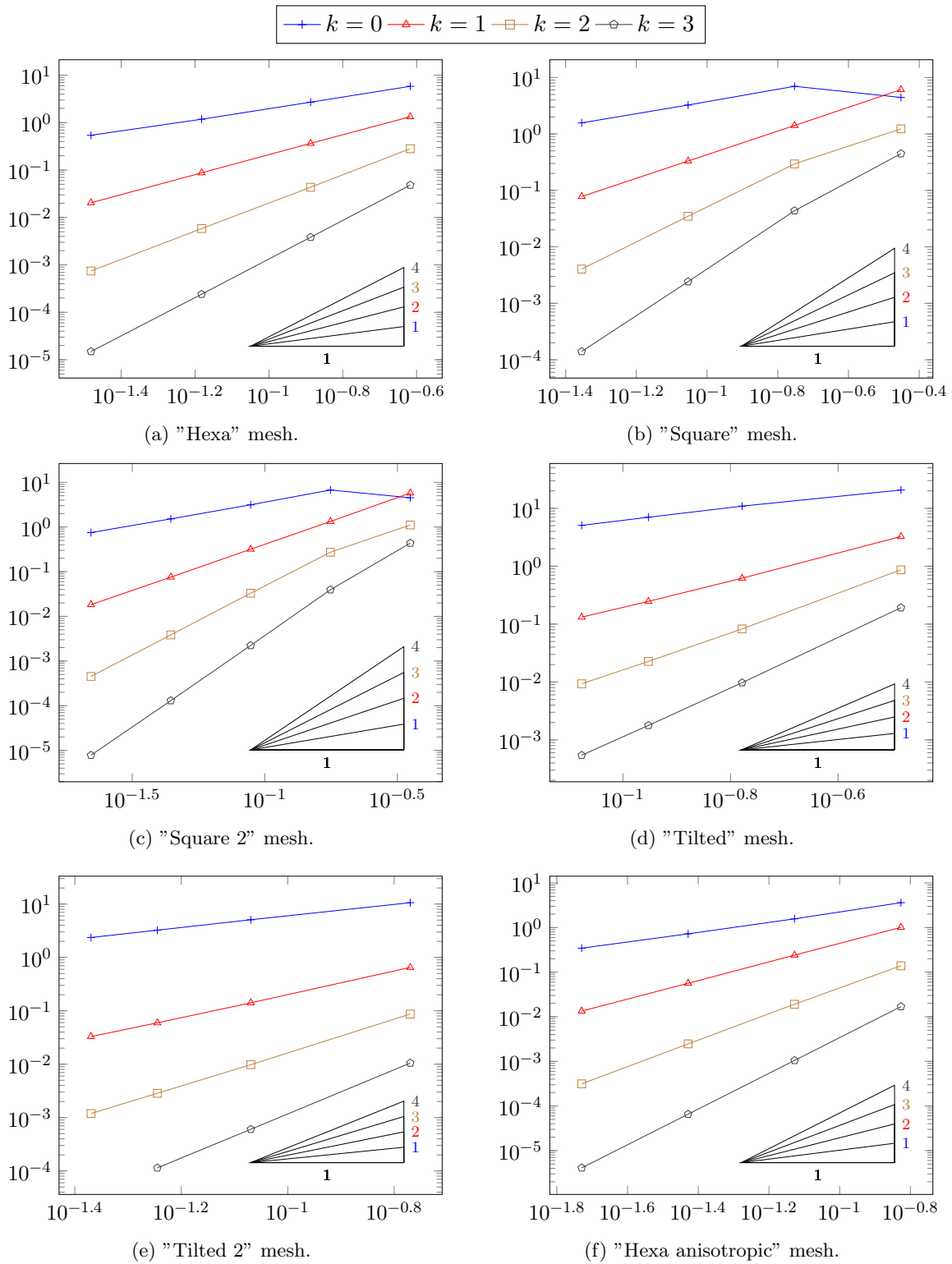


Figure 4: Absolute error estimate in discrete norm $\|\cdot\|_{\mu,\nabla,1,h} + \|\cdot\|_{L^2,h}$ vs. mesh size h in 2 dimensions.

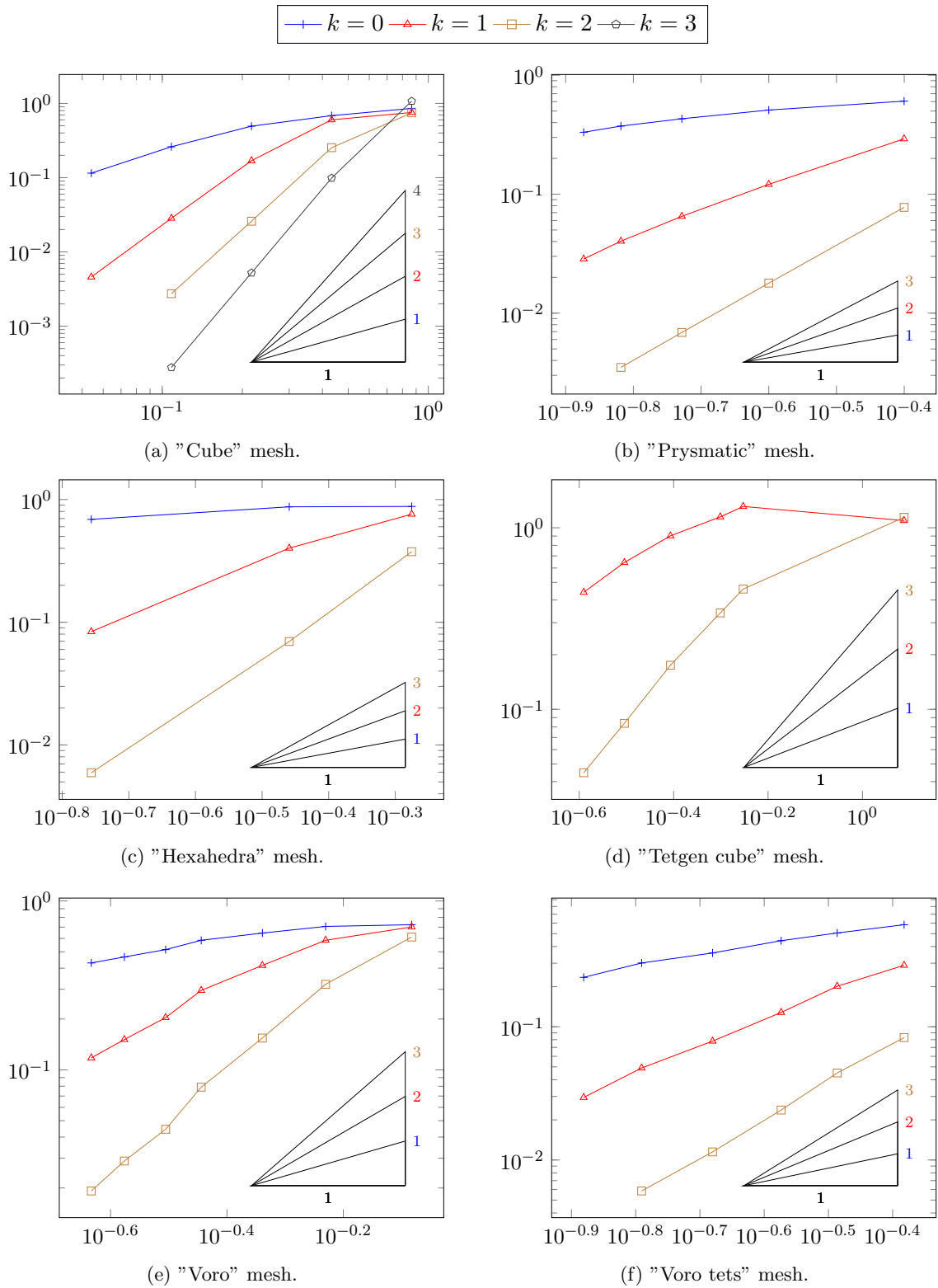


Figure 5: Absolute error estimate in discrete norm $\|\cdot\|_{\mu, \nabla, 1, h} + \|\cdot\|_{L^2, h}$ vs. mesh size h in 3 dimensions.

Remark 43. In 2 dimensions the expression becomes (see Appendix C):

$$\overline{\mathcal{R}}^{c,k+1}(F) = \begin{pmatrix} -(x-x_F)(y-y_F)Q & -(y-y_F)^2Q \\ (x-x_F)^2Q & (x-x_F)(y-y_F)Q \end{pmatrix}, \quad Q \in \mathcal{P}^{k-1}(F).$$

Lemma 44. For any $T \in \mathcal{T}_h$, $\nabla \cdot$ is an isomorphism from $\overline{\mathcal{R}}^{c,k+1}(T)$ to $\mathcal{G}^{c,k}(T)$.

Proof. By the definition (2.5) the space $\mathcal{G}^{c,k}$ is characterized by:

$$\mathcal{G}^{c,k}(T) = \begin{pmatrix} yQ_1 - zQ_2 \\ zQ_3 - xQ_1 \\ xQ_2 - yQ_3 \end{pmatrix}, \quad Q_1, Q_2, Q_3 \in \mathcal{P}^{k-1}(T). \quad (\text{A.2})$$

Let $W = \begin{pmatrix} xP_1 & yP_1 & zP_1 \\ xP_2 & yP_2 & zP_2 \\ xP_3 & yP_3 & zP_3 \end{pmatrix} \in \overline{\mathcal{R}}^{c,k}(T)$, with P_i given by Lemma 42. A direct computation gives

$$\nabla \cdot W = \begin{pmatrix} yzC^{1'} + y\gamma' - z\beta' \\ xzC^{2'} - x\gamma' + z\lambda' \\ xyC^{3'} + x\beta' - y\lambda' \end{pmatrix}, \quad (\text{A.3})$$

with $C^{i'} = (5 + x\partial_x + y\partial_y + z\partial_z)C^i$, $\gamma' = (4 + x\partial_x + y\partial_y)\gamma$, $\beta' = (4 + x\partial_x + z\partial_z)\beta$, $\lambda' = (4 + y\partial_y + z\partial_z)\lambda$. Each of these transformations is a linear automorphism, hence we can drop the $'$ in (A.3) without loss of generality. We must show that there is a one to one correspondence between the Q_i of (A.2) (more precisely their sum, since different Q_i give the same expression) and the C^i , γ , β , λ of (A.3).

Let us write $Q_1 = \gamma + R^1$, $Q_2 = \beta + R^2$ and $Q_3 = \lambda + R^3$ for some R^1, R^2, R^3 . The system becomes:

$$\begin{aligned} yzC^1 + y\gamma - z\beta &= yQ_1 - zQ_2 & yzC^1 &= yR^1 - zR^2 \\ xzC^2 - x\gamma + z\lambda &= zQ_3 - xQ_1 & xzC^2 &= zR^3 - xR^1 \\ xyC^3 + x\beta - y\lambda &= xQ_2 - yQ_3 & xyC^3 &= xR^2 - yR^3 \end{aligned}$$

When can check that $xyz(C^1 + C^2 + C^3) = xyR^1 - xzR^2 + yzR^3 - yxR^1 + zxR^2 - zyR^3 = 0$. Hence given Q_1, Q_2, Q_3 , the Euclidean divisions $Q_1 := \gamma + zS^1$, $Q_2 := \beta + yS^2$ and $Q_3 := \lambda + xS^3$ give suitable $C^1, C^2, C^3, \gamma, \beta, \lambda$ for $C^1 = S^1 - S^2$, $C^2 = S^3 - S^1$ and $C^3 = S^2 - S^3$. Likewise, given $C^1, C^2, C^3, \gamma, \beta, \lambda$, the system

$$\begin{aligned} C^1 &= S^1 - S^2 \\ C^2 &= S^3 - S^1 \\ C^3 &= S^2 - S^3 \end{aligned}$$

is solvable since $C^1 + C^2 + C^3 = 0$ and $Q_1 := \gamma + zS^1$, $Q_2 := \beta + yS^2$, $Q_3 := \lambda + xS^3$ give suitable Q_i . \square

Lemma 45. For any $T \in \mathcal{T}_h$, it holds $\overline{\mathcal{R}}^{c,k}(T) \cap \overline{\mathcal{R}}^k(T) = \{0\}$.

Proof. This is a direct consequence of Lemma 44 and Remark 4. \square

Lemma 46. For any $T \in \mathcal{T}_h$, it holds $(\mathcal{R}^{c,k}(T)^\top)^3 = \overline{\mathcal{R}}^{c,k}(T) \oplus \overline{\mathcal{R}}^k(T)$.

Proof. Lemma 45 already shows that $\overline{\mathcal{R}}^{c,k}(T) \cap \overline{\mathcal{R}}^k(T) = \{0\}$. It is enough to compare the dimension of these spaces:

$$\begin{aligned} \dim \overline{\mathcal{R}}^{c,k}(T) &= 3 \dim \mathcal{P}^{k-2}[X, Y] + 2 \dim \mathcal{P}^{k-3}[X, Y, Z] \\ &= 3 \frac{k!}{2!(k-2)!} + 2 \frac{k!}{3!(k-3)!} = \frac{2k^3 + 3k^2 - 5k}{6}, \\ \dim \overline{\mathcal{R}}^k(T) &= \dim \mathcal{P}^{0,k}(T) = \frac{(k+3)!}{3!k!} - 1 = \frac{k^3 + 6k^2 + 11k}{6}. \end{aligned}$$

The sum of both is

$$3 \frac{(k+2)!}{3!(k-1)!} = 3 \dim \mathcal{P}^{k-1}(T) = \dim (\mathcal{R}^{c,k}(T)^\dagger)^3.$$

□

Next we show some lemmas on convex polytopes.

Lemma 47. *Let $T \in \mathcal{T}_h$, \mathbf{x}_T defined as in (2.5), if $B = B(\mathbf{x}_T, h_B) \subset T$ with $h_B \gtrsim h_T$ and $Q \in \mathcal{P}^k(T)$ then $\|Q\|_{L^\infty(B)} \approx \|Q\|_\infty$.*

Proof. Let $h_o \in \mathbb{R}_+^*$ such that $T \subset B(\mathbf{x}_T, h_o)$ and $h_o \lesssim h_T$ (h_o exists by the regularity assumption on the mesh sequence \mathcal{M}_h). Let \mathbf{v} by any vector such that $\|\mathbf{v}\| = 1$, then $\forall \alpha > 0$ such that $\mathbf{x}_T + \alpha \mathbf{v} \in T$,

$$Q(\mathbf{x}_T + \alpha \mathbf{v}) = \sum_{i=0}^k \frac{\partial_{\mathbf{v}}^i Q(\mathbf{x}_T)}{i!} \alpha^i,$$

so

$$|Q(\mathbf{x}_T + \alpha \mathbf{v})| \leq \sum_{i=0}^k \frac{\|\partial_{\mathbf{v}}^i Q\|_{L^\infty(B)}}{i!} \alpha^i.$$

By the discrete Poincare inequality Lemma 2 we have $\forall i$, $\|\partial_{\mathbf{v}}^i Q\|_{L^\infty(B)} \lesssim h_B^{-i} \|Q\|_{L^\infty(B)}$. Moreover $\mathbf{x}_T + \alpha \mathbf{v} \in T$ so $\alpha < h_o$ and $|Q(\mathbf{x}_T + \alpha \mathbf{v})| \lesssim \|Q\|_{L^\infty(B)} h_o^i h_B^{-i}$. Since $h_o \lesssim h_B$ we can conclude that $\|Q\|_\infty \lesssim \|Q\|_{L^\infty(B)}$. □

Any $T \in \mathcal{T}_h$ is a convex, open polyhedron. Let $(F_i)_{i \leq |\mathcal{F}_T|}$ be the set of its faces, each of normal vector \mathbf{n}_F . For all F_i there exists $P_i \in \mathcal{P}^1(\mathbb{R}^3)$ such that $F_i \subset \text{Ker}(P_i)$, moreover we can normalize P_i such that $\mathbf{x} \in T \implies P_i(\mathbf{x}) > 0$ (since T is convex) and $|\partial_{\mathbf{n}_F} P_i| = 1$.

Lemma 48. *Let $P = \prod_{i \leq |\mathcal{F}_T|} P_i$, $\mathbf{x}_0 \in T$ such that $B_0 = B(\mathbf{x}_0, h_T/2) \subset T$ and $B = B(\mathbf{x}_0, h_T/4)$ then $\inf_{\mathbf{x} \in B} P(\mathbf{x}) > \left(\frac{h_T}{4}\right)^{|\mathcal{F}_T|}$ and $\|P\|_{L^\infty(T)} \lesssim h_T^{|\mathcal{F}_T|}$.*

Proof. For any $i \leq |\mathcal{F}_T|$, the value $P_i(\mathbf{x})$ at any point \mathbf{x} is the distance between \mathbf{x} and the plane tangent to F_i and is positive on T . For any $\mathbf{x} \in B$, \mathbf{x} is a least at a distance $h_T/4$ of any face since $B_0 \subset T$. We obtain the lower bound taking the product over all faces. Conversely, using the mesh regularity we can find $h_o > 0$, $h_o \lesssim h_T$ such that T is inscribed in a ball of diameter h_o . Then $\forall i \leq |\mathcal{F}_T|$, $\forall \mathbf{x} \in T$, it holds $0 < P_i(\mathbf{x}) < h_o \lesssim h_T$. Again we conclude by taking the product over all faces. □

Remark 49. Lemma 48 also holds for $P = \prod_{i \leq |\mathcal{F}_T|} P_i^2$, substituting $|\mathcal{F}_T|$ by $2|\mathcal{F}_T|$.

Lemma 50. For any $T \in \mathcal{T}_h$ and $q \in \mathcal{P}^k(T)$ there is $P \in \mathcal{P}^{k+|\mathcal{F}_T|}(\bar{T})$ such that $P|_{\partial T} = 0$, $\pi_{\mathcal{P},T}^k P = q$ and $\|P\| \approx \|q\|$.

Proof. Let $P = \Pi Q$ with Π given by Lemma 48 and $Q \in \mathcal{P}^k(T)$. The application

$$\mathcal{P}^k(T) \ni Q \rightarrow \left(\lambda \rightarrow \int_T P \lambda \right) \in \mathcal{P}^k(T)' \quad (\text{A.4})$$

is linear and between two spaces of same dimension, thus it is enough to check that it is injective. Let $Q \in \mathcal{P}^k(T)$ such that $\forall \lambda \in \mathcal{P}^k(T)$, $\int_T \Pi Q \lambda = 0$; in particular $\int_T \Pi Q^2 = 0$. However since $\Pi > 0$ on T , we can define the function $\sqrt{\Pi} \in L^2(T)$ and have $\int_T (\sqrt{\Pi} Q)^2 = \left\| \sqrt{\Pi} Q \right\|^2 = 0$. This implies $\sqrt{\Pi} Q \equiv 0$, so $Q \equiv 0$ and (A.4) is injective. Thus there is a polynomial $P \in \mathcal{P}^{k+|\mathcal{F}_T|}(\bar{T})$ such that $P|_{\partial T} = 0$ and $\pi_{\mathcal{P},T}^k P = q$. Let us show that P also satisfies the norm equivalence: In particular we must have

$$\begin{aligned} \int_T (P - q) Q &= 0, \\ \int_T \Pi Q^2 &= \int_T q Q, \\ \left\| \sqrt{\Pi} Q \right\|^2 &\leq \|q\| \|Q\| \lesssim \|q\| |T|^{\frac{1}{2}} \|Q\|_\infty. \end{aligned}$$

Therefore the discrete Sobolev inequality [10, Lemma 1.25] gives

$$\left\| \sqrt{\Pi} Q \right\|_\infty^2 \approx |T|^{-1} \left\| \sqrt{\Pi} Q \right\|^2 \lesssim |T|^{-\frac{1}{2}} \|Q\|_\infty \|q\|.$$

On the other consider $B = B(\mathbf{x}_0, h_T/4)$ given in Lemma 48, it holds $\inf_{\mathbf{x} \in B} \sqrt{\Pi}(\mathbf{x}) \gtrsim h_T^{|\mathcal{F}_T|/2}$. Thus by Lemma 47:

$$\left\| \sqrt{\Pi} Q \right\|_\infty \geq \left\| \sqrt{\Pi} Q \right\|_{L^\infty(B)} \gtrsim h_T^{|\mathcal{F}_T|/2} \|Q\|_{L^\infty(B)} \approx h_T^{|\mathcal{F}_T|/2} \|Q\|_\infty.$$

Hence $h_T^{|\mathcal{F}_T|} \|Q\|_\infty^2 \lesssim |T|^{-\frac{1}{2}} \|Q\|_\infty \|q\|$, $\|Q\|_\infty \lesssim |T|^{-\frac{1}{2}} h_T^{-|\mathcal{F}_T|} \|q\|$ and

$$\|\Pi Q\| \approx |T|^{\frac{1}{2}} \|\Pi Q\|_\infty \leq |T|^{\frac{1}{2}} \|\Pi\|_\infty \|Q\|_\infty \lesssim |T|^{\frac{1}{2}} h_T^{|\mathcal{F}_T|} \|Q\|_\infty \lesssim \|q\|.$$

□

We can use the same proof with Remark 49 to enforce the continuity of derivatives.

Lemma 51. For any $T \in \mathcal{T}_h$ and $q \in \mathcal{P}^k(T)$ there is $P \in \mathcal{P}^{k+2|\mathcal{F}_T|}(\bar{T})$ such that $P|_{\partial T} = 0$, $(\mathbf{grad} P)|_{\partial T} = \mathbf{0}$, $\pi_{\mathcal{P},T}^k P = q$ and $\|P\| \approx \|q\|$.

B Trace lifting.

In order to prove consistency results we often need functions of Sobolev spaces with suitable properties. We construct them in this section.

Theorem 52. For all $\mathbf{v}_T \in \underline{\mathbf{X}}_{\nabla,T}^k$ there is $\mathbf{R}_{\mathbf{v}_T} \in \mathbf{H}^1(T)$ such that

$$\begin{aligned} \mathbf{R}_{\mathbf{v}_T} &= \widetilde{\gamma_{\nabla,F}^{k+2}^c} \mathbf{v}_F \text{ on } \partial T, \\ \|\mathbf{R}_{\mathbf{v}_T}\|_T + \|\nabla \mathbf{R}_{\mathbf{v}_T}\|_T &\lesssim \|\mathbf{v}_T\|_{\nabla,T} + \left\| \underline{\nabla}_T^{k+1} \mathbf{v}_T \right\|_{L^2,T}. \end{aligned} \quad (\text{B.1})$$

This lift is built upon [2, Theorem 18.40]: Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be an open set whose boundary $\partial\Omega$ is uniformly Lipschitz continuous of parameters ϵ , L and M (see [2, Definition 13.11]). Then for all $g \in \mathbf{B}^{1/2,2}(\partial\Omega)$, there is $c \in \mathbb{R}$ depending only on N and a function $u \in H^1(\Omega)$ such that $\text{Tr}(u) = g$,

$$\|u\|_{L^2(\Omega)} \leq M^{1/2} \epsilon^{1/2} \|g\|_{L^2(\partial\Omega)} \quad (\text{B.2})$$

and

$$\|\mathbf{grad} u\|_{L^2(\Omega)} \leq cM(1+L)^{3+N/2} \epsilon^{-1/2} \|g\|_{L^2(\partial\Omega)} + cM(1+L)^{2+(N+1)/2} |g|_{\mathbf{B}^{1/2,2}(\partial\Omega)}^\diamond. \quad (\text{B.3})$$

With the Besov seminorm defined by (see [2, Definition 18.36]):

$$|g|_{\mathbf{B}^{1/2,2}(\partial\Omega)}^\diamond := \left(\int_{\partial\Omega} \int_{\partial\Omega \cap B(x,\epsilon)} \frac{|g(x) - g(y)|^2}{\|x - y\|^N} dy dx \right)^{1/2}. \quad (\text{B.4})$$

Proof of theorem 52. We apply the above-mentioned theorem [2, Theorem 18.40] to $\Omega = T$ and g a component of $\widetilde{\gamma_{\nabla,F}^{k+2} \mathbf{v}_F}^c$. Here $N = 3$ and the mesh regularity [10, Definition 1.9] allows us to take an open cover of ∂T making it uniformly Lipschitz continuous in the sense of [2, Definition 13.11] with $L = 1$, $M \approx 1$ and $\epsilon \approx h_T$.

Let $\mathbf{R}_{\mathbf{v}_T}$ be such that $\text{Tr}(\mathbf{R}_{\mathbf{v}_T}) = \widetilde{\gamma_{\nabla,F}^{k+2} \mathbf{v}_F}^c$ and that $\mathbf{R}_{\mathbf{v}_T}$ satisfies (B.2) and (B.3). Let g be a component of $\widetilde{\gamma_{\nabla,F}^{k+2} \mathbf{v}_F}^c$ and u be given by (B.2) and (B.3). Without loss of generality we assume that $\int_{\partial T} g = 0$: Else we take instead $\bar{g} = \int_{\partial T} g$ and $u = \bar{g}$ so that $\mathbf{grad} u = 0$ and $\|u\|_{L^2(T)} \approx h_T^{3/2} |\bar{g}|$, $\|\bar{g}\|_{L^2(\partial T)} \approx h_T |\bar{g}|$ and u, \bar{g} satisfy (B.1). Hence we reduce to the case $\int_{\partial T} g' = 0$ for $g' = g - \bar{g}$.

Equation (B.2) with Lemma 21 gives $\|\mathbf{R}_{\mathbf{v}_T}\|_T \lesssim \|\mathbf{v}_T\|_{\nabla,T}$ since $\epsilon \approx h_T$. Let $A_F := \frac{1}{|F|} \int_F g|_F$ and $A_T := \sum_{F \in \mathcal{F}_T} |F| A_F$. By assumption $A_T = 0$, and for all $F \in \mathcal{F}_T$,

$$\begin{aligned} \sum_{F \in \mathcal{F}_T} \|g|_F\|_F &\leq \|g|_F - A_F\|_F + \|A_F - A_T\|_F \\ &\lesssim h_F \left\| \nabla \widetilde{\gamma_{\nabla,F}^{k+2} \mathbf{v}_F}^c \right\|_F + h_F^{1/2} \|\mathbf{v}_T\|_{\nabla,T}. \end{aligned}$$

We used the Poincaré-Wirtinger inequality on each face on the first term in the right-hand side and the same proof as Lemma 31 (see [14, Equation 5.12]). Hence, Lemma 31 gives $\|g\|_{L^2(\partial T)} \lesssim h_T^{1/2} \|\mathbf{v}_T\|_{\nabla,T}$, and

$$\begin{aligned} \|\mathbf{grad} u\|_{L^2(\Omega)} &\lesssim h_T^{-1/2} \|g\|_{L^2(\partial T)} + |g|_{\mathbf{B}^{1/2,2}(\partial\Omega)}^\diamond \\ &\lesssim \|\mathbf{v}_T\|_{\nabla,T}. \end{aligned}$$

We concluded with the estimate on the Besov seminorm Lemma 53. □

Lemma 53. *Keeping the notations of the proof of Theorem 52, it holds*

$$|g|_{\mathbf{B}^{1/2,2}(\partial\Omega)}^\diamond \lesssim \|\mathbf{v}_T\|_{\nabla,T}. \quad (\text{B.5})$$

Proof. We know that $g \in C^1(\mathcal{F}_T)$, hence by the mean value theorem, $\forall y \in B(x, \epsilon)$, $\exists c \in]0, 1[$ such that

$$g(\mathbf{y}) = g(\mathbf{x}) + \mathbf{grad} g((1-c)\mathbf{x} + c\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x}),$$

thus

$$\frac{|g(\mathbf{x}) - g(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} = \frac{|\mathbf{grad} g((1-c)\mathbf{x} + c\mathbf{y})|^2 \|\mathbf{x} - \mathbf{y}\|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \lesssim \|\mathbf{grad} g\|_{L^\infty(F)}^2 |\mathbf{x} - \mathbf{y}|^{-1}.$$

Switching to polar coordinates gives

$$\int_{B(x,\epsilon)} \frac{|g(x) - g(y)|^2}{\|x - y\|^3} \lesssim \|\mathbf{grad} g\|_{L^\infty(F)}^2 \int_0^\epsilon \frac{1}{r} r \lesssim \|\mathbf{grad} g\|_{L^\infty(F)}^2 \epsilon.$$

Lemma 2 and a Poincaré-Wirtinger inequality show that

$$\begin{aligned} \left\| \nabla \widetilde{\gamma_{\nabla, F}^{k+2} \mathbf{v}_F} \right\|_F &\lesssim h_F^{-1} \left\| \widetilde{\gamma_{\nabla, F}^{k+2} \mathbf{v}_F} - \frac{1}{|F|} \int_F \widetilde{\gamma_{\nabla, F}^{k+2} \mathbf{v}_F} \right\|_F \\ &\approx h_F^{-1} \left\| \gamma_{\nabla, F}^{k+2} \mathbf{v}_F - \frac{1}{|F|} \int_F \gamma_{\nabla, F}^{k+2} \mathbf{v}_F \right\|_F \\ &\lesssim \left\| \nabla \gamma_{\nabla, F}^{k+2} \mathbf{v}_F \right\|_F. \end{aligned}$$

Hence the discrete Lebesgue embedding [10, Lemma 1.25] and Lemma 31 give

$$\|\mathbf{grad} g\|_{L^\infty(F)}^2 \approx h_F^{-2} \|\mathbf{grad} g\|_{L^2(F)}^2 \lesssim h_F^{-3} h_F \left\| \nabla \gamma_{\nabla, F}^{k+2} \mathbf{v}_F \right\|_F^2 \lesssim h_T^{-3} \|\mathbf{v}_T\|_{\nabla, T}^2.$$

Inferring $\epsilon \approx h_T$ and $|\partial T| \approx h_T^2$ we find

$$\begin{aligned} |g|_{\mathbf{B}^{1/2,2}(\partial\Omega)}^\diamond &\lesssim \left(\int_{\partial T} \epsilon \|\mathbf{grad} g\|_{L^\infty(F)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(h_T h_T^{-3} \|\mathbf{v}_T\|_{\nabla, T}^2 \int_{\partial T} 1 \right)^{\frac{1}{2}} \\ &\lesssim \|\mathbf{v}_T\|_{\nabla, T}. \end{aligned}$$

□

Theorem 54. *If $p \in H_0^2(\Omega)$ then there is $\mathbf{u} \in \mathbf{H}^3(\Omega)$ such that $\operatorname{div} \mathbf{u} = p$, $\|\mathbf{u}\|_{\mathbf{H}^3} \lesssim \|p\|_{H^2}$, $\|\mathbf{u}\|_{H^2} \lesssim \|p\|_{H^1}$ and $\|\mathbf{u}\|_{H^1} \lesssim \|p\|_{L^2}$.*

Proof. Let B be a smooth bounded extension (at least $C^{3,1}$) of Ω . For all function $g \in H^{-1}(B)$, following [4, Theorem III.4.1], there is a unique solution $f \in H_0^1(B)$ to the equation $\Delta f = g$ in B . Moreover this solution satisfies $\|f\|_{H^1} \lesssim \|g\|_{H^{-1}}$ and [4, Theorem III.4.2] shows that if B is $C^{k+1,1}$, $k \geq 0$ and $g \in H^k(B)$ then $\|f\|_{H^{k+2}} \lesssim \|g\|_{H^k}$. Since $p \in H_0^2(\Omega)$ we can extend p by zero and define $\tilde{p} \in H_0^2(B)$ with $\|\tilde{p}\|_{H^k(B)} = \|p\|_{H^k(\Omega)}$ ([1, Theorem 3.33]). Hence, since $\tilde{p} \in H^2(B)$, if we take $f \in H_0^1(B)$ such that $\Delta f = \operatorname{div} \mathbf{grad} f = \tilde{p}$ we get $f \in H^4(B)$ with $\|f\|_{H^2(B)} \lesssim \|p\|_{L^2}$, $\|f\|_{H^3(B)} \lesssim \|p\|_{H^1}$, and $\|f\|_{H^4(B)} \lesssim \|p\|_{H^2}$. Let $\mathbf{u} = \mathbf{grad} f|_\Omega$ then we have $\operatorname{div} \mathbf{u} = p$ in Ω and the expected bounds. □

We can adapt the theorem to cover other boundary conditions. However, we still need a smoother domain, and if we want to enforce a condition on the boundary of Ω we can no longer consider a larger domain. Instead, we take a smaller domain, and we will have to do some work to recover the correct values on Ω .

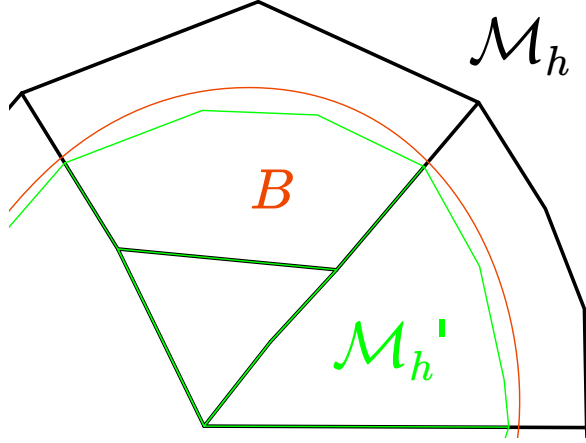


Figure 6: Construction of the smooth subdomain.

Theorem 55. *If $B \subset \Omega$ is $C^{2,1}$, $p \in H^2(B)$ such that $\int_{\Omega} p = 0$ then there is $\mathbf{u} \in \mathbf{H}^3(B) \cap \mathbf{H}_0^1(B)$ such that $\operatorname{div} \mathbf{u} = p$, $\|\mathbf{u}\|_{\mathbf{H}^3} \lesssim \|p\|_{H^2}$, $\|\mathbf{u}\|_{\mathbf{H}^2} \lesssim \|p\|_{H^1}$ and $\|\mathbf{u}\|_{\mathbf{H}^1} \lesssim \|p\|_{L^2}$.*

Proof. Since $\int_B p = 0$ we can use [4, Theorem IV.5.2 and Theorem IV.5.8] to get $\mathbf{u} \in \mathbf{H}^{k+1}(B)$ such that $\operatorname{div} \mathbf{u} = p$ in B , $\mathbf{u} = \mathbf{0}$ on ∂B and $\|\mathbf{u}\|_{\mathbf{H}^{k+1}} \lesssim \|p\|_{H^k}$, $k \in \{0, 1, 2\}$. \square

Remark 56 (Adaptation of Lemma 34). The continuous (for the Sobolev norm) extension of a function in H_0^1 is not necessarily by zero. However, we only need to take L^2 -orthogonal projections, hence we can afford to use a smaller domain B . Let $B \subset \Omega$ be a $C^{2,1}$ domain containing all interior elements and most of the elements next to the boundary. Formally we want B such that $\forall X \in \mathcal{M}_h, \bar{X} \cap \partial\Omega = \emptyset \implies X \subset B$ and $\bar{X} \cap \partial\Omega \subsetneq \bar{X} \implies |X \cap B| \geq |X|/2$.

To adapt Lemma 34 we take another mesh \mathcal{M}'_h with the same interior elements as \mathcal{M}_h and with elements crossing ∂B collinear to those of \mathcal{M}_h but cut before. This way, the domain on which \mathcal{M}'_h is defined lies inside B . Figure 6 illustrates the construction.

Given $p_h \in \underline{\mathbf{X}}_{L^2, h, \star}^k$ we construct p'_h , a piecewise polynomial on each cell of \mathcal{M}'_h with the same degree and moments as p_h has on the corresponding cell of \mathcal{M}_h . We can now proceed as done in the proof of Lemma 34, with p'_h (extended by zero on B) instead of p_h and with a slightly different interpolator. The modified interpolator is the same on interior elements, and on elements $X \in \mathcal{M}_h$ crossing ∂B of corresponding element $X' \in \mathcal{M}_h$ we substitute the L^2 -orthogonal projector by π such that

$$\int_X (\pi_P f) g := \int_{X'} f g, \forall f \in L^2(X'), \forall g \in P.$$

One can check that the proof still works since $\mathbf{u} \in \mathbf{H}_0^1(B)$.

C 2-Dimensional complex.

The Stokes complex also exists in 2 dimensions. Let Ω be a domain of \mathbb{R}^2 instead of \mathbb{R}^3 . The differential complex now reads:

$$\mathbb{R} \xrightarrow{i_{\Omega}} H^1(\Omega) \xrightarrow{\operatorname{rot}} \mathbf{H}(\operatorname{div}, \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega) \xrightarrow{0} \{0\}. \quad (\text{C.1})$$

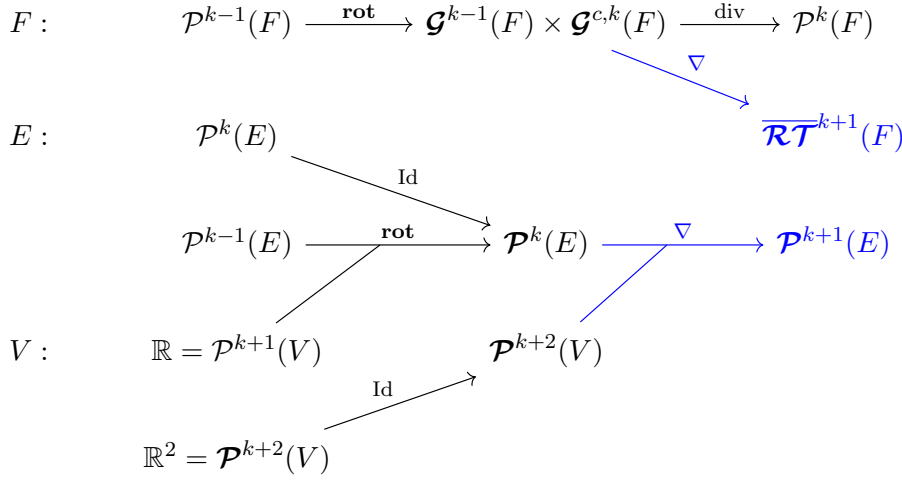


Figure 7: Usage of the local degrees of freedom for the discrete differential operators in 2 dimensions.

We can construct a discrete complex similar to the complex in Figure 1. However the construction is not merely the restriction of the 3-dimensional complex to 2-dimensional objects. It is in fact much simpler.

Discrete spaces.

The 2-dimensional complex only needs four discrete spaces $\underline{X}_{\text{rot},F}^k$, $\underline{X}_{\nabla,T}^k$, $\underline{X}_{L^2,T}^{k+1}$ and $\underline{X}_{L^2,T}^k$. They are defined by:

$$\begin{aligned}
\underline{X}_{\text{rot},h}^k := & \{ \underline{v}_h = ((\mathbf{R}_{v,V})_{V \in \mathcal{V}_h}, (v_E, R_{v,E})_{E \in \mathcal{E}_h}, (v_F)_{F \in \mathcal{F}_h}) : \\
& \mathbf{R}_{v,V} \in \mathbb{R}^2(V), \forall V \in \mathcal{V}_h, v_E \in \mathcal{P}_c^{k+1}(\mathcal{E}_h), R_{v,E} \in \mathcal{P}^k(E), \forall E \in \mathcal{E}_h, \\
& v_F \in \mathcal{P}^{k-1}(F), \forall F \in \mathcal{F}_h \}, \tag{C.2}
\end{aligned}$$

$$\begin{aligned}
\underline{X}_{\nabla,h}^k := & \{ \underline{w}_h = ((\mathbf{w}_E)_{E \in \mathcal{E}_h}, (\mathbf{w}_{\mathcal{G},F}, \mathbf{w}_{\mathcal{G},F}^c)_{F \in \mathcal{F}_h}) : \\
& \mathbf{w}_E \in \mathcal{P}_c^{k+2}(E), \forall E \in \mathcal{E}_h, \mathbf{w}_{\mathcal{G},F} \in \mathcal{G}^k(F), \mathbf{w}_{\mathcal{G},F}^c \in \mathcal{G}^{c,k}(F), \forall F \in \mathcal{F}_h \}, \tag{C.3}
\end{aligned}$$

$$\begin{aligned}
\underline{X}_{L^2,h}^{k+1} := & \{ \underline{W}_h = ((\mathbf{W}_E)_{E \in \mathcal{E}_h}, (\mathbf{W}_F)_{F \in \mathcal{F}_h}) : \\
& \mathbf{W}_E \in \mathcal{P}^{k+1}(E), \forall E \in \mathcal{E}_h, \mathbf{W}_F \in \overline{\mathcal{RT}}^{k+1}(F), \forall F \in \mathcal{F}_h \}, \tag{C.4}
\end{aligned}$$

$$\underline{X}_{L^2,h}^k := \{ \underline{q}_h = ((q_F)_{F \in \mathcal{F}_h}) : q_F \in \mathcal{P}^k(F), \forall F \in \mathcal{F}_h \}. \tag{C.5}$$

Figure 7 is the 2-dimensional equivalent of Figure 1.

The interpolator on the space $\underline{X}_{\text{rot},h}^k$ is defined for any $v \in C^1(\overline{\Omega})$ by

$$\underline{I}_{\text{rot},h}^k v = ((v_E, \pi_{\mathcal{P},E}^k(\text{rot } v \cdot \mathbf{t}_E))_{E \in \mathcal{E}_h}, (\text{rot } v(V))_{V \in \mathcal{V}_h}, (\pi_{\mathcal{P},F}^{k-1}(v))_{F \in \mathcal{F}_h}), \tag{C.6}$$

where for any edge $E \in \mathcal{E}_h$, v_E is such that $\pi_{\mathcal{P},E}^{k-1}(v_E) = \pi_{\mathcal{P},E}^{k-1}(v)$ and for any vertex $V \in \mathcal{V}_E$, $v_E(\mathbf{x}_V) = v(\mathbf{x}_V)$.

The interpolator on the space $\underline{X}_{\nabla,h}^k$ is defined for any $\mathbf{w} \in C^0(\overline{\Omega})$ by

$$\underline{I}_{\nabla,h}^k \mathbf{w} = ((\mathbf{w}_E)_{E \in \mathcal{E}_h}, (\pi_{\mathcal{G},T}^{k-1}(\mathbf{w}), \pi_{\mathcal{G},T}^{c,k}(\mathbf{w}))_{F \in \mathcal{F}_h}), \tag{C.7}$$

where for any edge $E \in \mathcal{E}_h$, \mathbf{w}_E is such that $\pi_{\mathcal{P},E}^k(\mathbf{w}_E) = \pi_{\mathcal{P},E}^k(\mathbf{w})$ and for any vertex $V \in \mathcal{V}_E$, $\mathbf{w}_E(\mathbf{x}_V) = \mathbf{w}(\mathbf{x}_V)$.

The interpolator on the space $\underline{\mathbf{X}}_{L^2,h}^{k+1}$ is defined for any $\mathbf{W} \in (\mathbf{C}^0(\overline{\Omega})^\top)^2$ by

$$\underline{\mathbf{I}}_{L^2,h}^k \mathbf{W} = ((\pi_{\mathcal{P},E}^{k+1}(\mathbf{W} \cdot \mathbf{t}_E))_{E \in \mathcal{E}_h}, (\pi_{\mathcal{RT},T}^{k+1}(\mathbf{W}))_{F \in \mathcal{F}_h}). \quad (\text{C.8})$$

The interpolator on the space $\underline{\mathbf{X}}_{L^2,h}^k$ is $\pi_{\mathcal{P},\mathcal{F}_h}^k$, the piecewise L^2 -orthogonal projection on spaces $\mathcal{P}^k(F)$, $F \in \mathcal{F}_h$.

Operators and properties.

The discrete operators are defined similarly to the faces operators in Section 3.3. Thus all properties of the 3-dimensional complex still applicable hold. Furthermore, the complex property Theorem 27 holds (barring the missing operator $\underline{\mathbf{G}}_h^k$ and equation (4.4b)). The consistency results proven in Section 5 also hold substituting the faces for the edges and the cells for the faces.

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