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## $\alpha$ -modules and generalized submodules

Rafiquddin, Ayazul Hasan, Mohammad Fareed Ahmad

**Abstract.** A QTAG-module  $M$  is an  $\alpha$ -module, where  $\alpha$  is a limit ordinal, if  $M/H_\beta(M)$  is totally projective for every ordinal  $\beta < \alpha$ . In the present paper  $\alpha$ -modules are studied with the help of  $\alpha$ -pure submodules,  $\alpha$ -basic submodules, and  $\alpha$ -large submodules. It is found that an  $\alpha$ -closed  $\alpha$ -module is an  $\alpha$ -injective. For any ordinal  $\omega \leq \alpha \leq \omega_1$  we prove that an  $\alpha$ -large submodule  $L$  of an  $\omega_1$ -module  $M$  is summable if and only if  $M$  is summable.

### 1 Introduction and Preliminary Terminology

Let  $R$  be any ring. A module  $M$  in which the lattice of its submodule is totally ordered is called a serial module; in addition, if it has finite composition length, it is called a uniserial module. An element  $x \in M$  is uniform, if  $xR$  is a non-zero uniform (hence uniserial) module, and for any  $R$ -module  $M$  with a unique decomposition series,  $d(M)$  denotes its decomposition length.

Modules are the natural generalizations of abelian groups. The results for abelian groups can be generalized for modules after imposing some conditions on modules/rings. In 1976 Singh [15] started the study of TAG-modules satisfying the following two conditions while the rings were associative with unity.

1. Every finitely generated submodule of any homomorphic image of  $M$  is a direct sum of uniserial modules.
2. Given any two uniserial submodules  $U$  and  $V$  of a homomorphic image of  $M$ , for any submodule  $W$  of  $U$ , any non-zero homomorphism  $f: W \rightarrow V$  can be

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extended to a homomorphism  $g: U \rightarrow V$ , provided the composition length  $d(U/W) \leq d(V/f(W))$ .

Later on Benabdallah, Singh, Khan etc. contributed a lot to the study of TAG-modules [2], [17]. In 1987 Singh made an improvement and studied the modules satisfying only the condition 1 and called them QTAG-modules. The study of QTAG-modules and their structure began with work of Singh in [16]. This work, executed by many authors, clearly parallels the earlier work on torsion abelian groups. They studied different notions and structures on QTAG-modules and developed the theory of these modules by introducing different notions and characterizing different submodules of QTAG-modules. Yet there is much to explore.

Throughout this paper, all the rings are associative with unity ( $1 \neq 0$ ) and modules  $M$  are unital QTAG-modules. For a uniform element  $x \in M$ ,  $e(x) = d(xR)$  and

$$H_M(x) = \sup \left\{ d \left( \frac{yR}{xR} \right) \mid y \in M, x \in yR \text{ and } y \text{ uniform} \right\}$$

are the exponent and height of  $x$  in  $M$ , respectively.  $H_k(M)$  denotes the submodule of  $M$  generated by the elements of height at least  $k$  and  $H^k(M)$  is the submodule of  $M$  generated by the elements of exponents at most  $k$ . Let us denote by  $M^1$ , the submodule of  $M$ , containing elements of infinite height. The module  $M$  is  $h$ -divisible if  $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ . The module  $M$  is  $h$ -reduced if it does not contain any  $h$ -divisible submodule. In other words, it is free from the elements of infinite height. The module  $M$  is said to be bounded, if there exists an integer  $n$  such that  $H_M(x) \leq n$  for every uniform element  $x \in M$ .

The sum of all simple submodules of  $M$  is called the socle of  $M$ , denoted by  $\text{Soc}(M)$  and a submodule  $S$  of  $\text{Soc}(M)$  is called a subsocle of  $M$ . The cardinality of the minimal generating set of  $M$  is denoted by  $g(M)$ . For all ordinals  $\alpha$ ,  $f_M(\alpha)$  is the  $\alpha^{\text{th}}$  Ulm invariant of  $M$  and it is equal to  $g(\text{Soc}(H_\alpha(M))/\text{Soc}(H_{\alpha+1}(M)))$ .

A submodule  $N$  of  $M$  is  $h$ -pure in  $M$  if  $N \cap H_k(M) = H_k(N)$ , for every integer  $k \geq 0$ . For an ordinal  $\alpha$ , a submodule  $N \subseteq M$  is an  $\alpha$ -high submodule of  $M$  if  $N$  is maximal among the submodules of  $M$  that intersect  $H_\alpha(M)$  trivially.

For an ordinal  $\alpha$ , a submodule  $N$  of  $M$  is said to be an  $\alpha$ -pure, if  $H_\beta(M) \cap N = H_\beta(N)$  for all  $\beta \leq \alpha$  and a submodule  $N$  of  $M$  is said to be isotype in  $M$ , if it is  $\alpha$ -pure for every ordinal  $\alpha$  [13]. For a submodule  $N \subseteq M$ , the valuation of  $N$  induced by height in  $M$  is defined by  $v(x) = H_M(x)$ , the height of  $x$  in  $M$ , for all  $x \in N$  and  $N = K \oplus L$  is a valued direct sum if  $v(k + \ell) = \min\{v(k), v(\ell)\}$  for all  $k \in K$  and  $\ell \in L$  [5].

A submodule  $B \subseteq M$  is a basic submodule [9] of  $M$ , if  $B$  is  $h$ -pure in  $M$ ,  $B = \bigoplus B_i$ , where each  $B_i$  is the direct sum of uniserial modules of length  $i$  and  $M/B$  is  $h$ -divisible. A fully invariant submodule  $L \subseteq M$  is large [1], if  $L + B = M$ , for every basic submodule  $B$  in  $M$ .

Imitating [11], the submodules  $H_k(M), k \geq 0$  form a neighborhood system of zero, thus a topology known as  $h$ -topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of  $N \subseteq M$  is defined as

$\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$ . Therefore the submodule  $N \subseteq M$  is closed with respect to  $h$ -topology if  $\overline{N} = N$ .

An  $h$ -reduced QTAG-module  $M$  is summable [14] if  $\text{Soc}(M) = \bigoplus_{\beta < \alpha} S_{\beta}$ , where  $S_{\beta}$  is the set of all elements of  $H_{\beta}(M)$  which are not in  $H_{\beta+1}(M)$ , where  $\alpha$  is the length of  $M$ . Moreover,  $M$  is called totally projective [10], if

$$H_{\alpha}(\text{Ext}(M/H_{\alpha}(M), M')) = 0$$

for all ordinal  $\alpha$  and QTAG-modules  $M'$ .

It is interesting to note that almost all the results which hold for TAG-modules are also valid for QTAG-modules [13]. Our notations and terminology generally agree with those in [3] and [4].

## 2 $\alpha$ -modules and $\alpha$ -basic submodules

Mehdi et al. [12] defined  $\alpha$ -modules and introduced some new concepts for these modules. The same type of study was continued in [6] and a number of results have been obtained in terms of generalized submodules. Here we also continue the similar study of  $\alpha$ -modules that depend on the notions of summability, purity, basic submodules, projectivity and injectivity. For facilitating the exposition and for the convenience of the readers, we recall the definition of  $\alpha$ -modules.

**Definition 1.** Let  $\alpha$  denote the class of all QTAG-modules  $M$  such that  $M/H_{\beta}(M)$  is totally projective for all ordinals  $\beta < \alpha$ , a limit ordinal. These modules are called  $\alpha$ -modules.

To develop the study, we need to prove some results, and we start with the following.

**Proposition 1.** *If  $N$  is an  $\alpha$ -pure submodule of an  $\alpha$ -module  $M$ , then  $N$  is itself an  $\alpha$ -module.*

*Proof.* We actually only need that  $N \cap H_{\gamma}(M) = H_{\gamma}(N)$  for all  $\gamma < \alpha$ . For then it is a simple calculation to show that  $N + H_{\beta}(M)/H_{\beta}(M)$  is isotype in  $M/H_{\beta}(M)$  for each  $\beta < \alpha$ . And therefore,  $N + H_{\beta}(M)/H_{\beta}(M) \cong N/H_{\beta}(N)$  is totally projective for all  $\beta < \alpha$ . □

As generalized the notion of a basic submodule in [12], by defining  $B$  to be an  $\alpha$ -basic submodule of an  $\alpha$ -module  $M$  if  $B$  is totally projective of length at most  $\alpha$ ,  $B$  is  $\alpha$ -pure submodule of  $M$ , and  $M/B$  is  $h$ -divisible.

In order to establish the existence of  $\alpha$ -basic submodules we require the following notion for technical convenience.

**Definition 2.** Let  $\alpha$  be a limit ordinal and  $M$  a QTAG-module. An  $\alpha$ -high tower of  $M$  is a well-ordered ascending chain  $\{M_{\beta}\}_{\beta < \alpha}$  of submodules of  $M$  such that, for each  $\beta$ ,  $M_{\beta}$  is a  $\beta$ -high submodule of  $M$ .

Now we need to prove the following lemma.

**Lemma 1.** *Let  $\alpha$  be a limit ordinal and  $\{M_\beta\}_{\beta < \alpha}$  an  $\alpha$ -high tower of a QTAG-module  $M$ . If each  $M_\beta$  is summable, then  $N = \bigcup_{\beta < \alpha} M_\beta$  is summable.*

*Proof.* As  $\alpha$  is a limit ordinal, we may choose a strictly increasing sequence  $\beta_1 < \beta_2 < \dots < \beta_n < \dots$  of ordinals having  $\alpha$  as its limit. Then  $N = \bigcup_{n < \omega} M_{\beta_n}$ . Set  $T_0 = \text{Soc}(M_{\beta_1})$  and, for  $n > 1$ , let  $T_n$  be such that

$$\text{Soc}(H_{\beta_n}(M)) = T_n \oplus \text{Soc}(H_{\beta_{n+1}}(M))$$

with  $T_n \subseteq M_{\beta_{n+1}}$ . Then we have a direct-sum decomposition  $\text{Soc}(N) = \bigoplus_{n < \omega} T_n$  which is normal in the sense that

$$H_M(t_1 + \dots + t_n) = \min[H_M(t_1), \dots, H_M(t_n)]$$

provided  $t_i \in T_i$  for  $i = 1, \dots, n$ . Now each  $M_\beta$  is isotype, summable, and of countable length. Therefore, each subsocle of  $M_\beta$  is a summable subsocle of  $M$ . In particular, each  $T_n$  is a summable subsocle of  $M$ . Since the decomposition  $\text{Soc}(N) = \bigoplus_{n < \omega} T_n$  is normal, it follows that  $\text{Soc}(N)$  is a summable subsocle of  $M$ . Since each  $M_\beta$  is isotype,  $N$  is itself an isotype submodule of  $M$  and consequently  $N$  is summable.  $\square$

We continue the study with the following corollary.

**Corollary 1.** *Let  $\alpha$  be a limit ordinal and  $\{M_\beta\}_{\beta < \alpha}$  an  $\alpha$ -high tower of a QTAG-module  $M$ , where each  $M_\beta$  is totally projective, then  $N = \bigcup_{\beta < \alpha} M_\beta$  is totally projective of length at most  $\alpha$ .*

*Proof.* As noted above,  $N$  is an isotype submodule of  $M$  and clearly  $N$  has a length at most  $\alpha$ . Thus  $M_\beta$  is also a  $\beta$ -high submodule of  $N$  for each  $\beta < \alpha$ . Since  $N$  is summable by Lemma 1 implies that  $N$  is totally projective.  $\square$

Now we prove the following.

**Theorem 1.** *Let  $M$  be a QTAG-module. Then  $M$  contains an  $\alpha$ -basic submodule if and only if  $M$  is an  $\alpha$ -module.*

*Proof.* If  $B$  is an  $\alpha$ -pure submodule of  $M$  and if  $M/B$  is  $h$ -divisible, then it follows that  $M/H_\beta(M) \cong B/H_\beta(B)$  for all  $\beta < \alpha$ . Consequently, only  $\alpha$ -modules can have  $\alpha$ -basic submodules (see [12]). Suppose now that  $M$  is an  $\alpha$ -module and select an  $\alpha$ -high tower  $\{M_\beta\}_{\beta < \alpha}$ . Now

$$M_\beta \cong M_\beta + H_\beta(M)/H_\beta(M),$$

and since  $M_\beta$  is isotype in  $M$ ,  $M_\beta + H_\beta(M)/H_\beta(M)$  is isotype in  $M/H_\beta(M)$ . By Corollary 1,  $B = \bigcup_{\beta < \alpha} M_\beta$  is totally projective. It is easily seen that

$$\text{Soc}(M) \subseteq \text{Soc}(B) + H_\beta(M)$$

for each  $\beta < \alpha$ , and therefore  $B$  is  $\alpha$ -pure in  $M$ . Moreover,  $B \cap H_1(M) = H_1(B)$  and

$$\text{Soc}(M) \subseteq \text{Soc}(B) + H_\beta(M)$$

for  $\beta < \omega$  imply that  $M/B$  is  $h$ -divisible. Thus,  $B$  is the required  $\alpha$ -basic submodule of  $M$ .  $\square$

**Lemma 2.** *Suppose  $N$  is an isotype submodule of a QTAG-module  $M$  and that  $\{N_\beta\}_{\beta < \alpha}$  is an  $\alpha$ -high tower of  $N$ , then there exists an  $\alpha$ -high tower  $\{M_\beta\}_{\beta < \alpha}$  of  $M$  such that, for each  $\beta$ ,  $N_\beta \subseteq M_\beta$  and  $N_\beta = N \cap M_\beta$ .*

*Proof.* Let us first note that  $N_\beta = N \cap M_\beta$  is a consequence of  $N_\beta \subseteq M_\beta$ . Indeed,  $N_\beta \subseteq M_\beta$  implies  $N_\beta \subseteq N \cap M_\beta$  and

$$(N \cap M_\beta) \cap H_\beta(N) = (N \cap M_\beta) \cap H_\beta(M) = 0.$$

The maximality of a  $\beta$ -high submodule then yields the equality. Assume now that  $\beta < \alpha$  and that for each  $\gamma < \beta$  we have a  $\gamma$ -high submodule  $M_\gamma$  of  $M$  such that  $N_\gamma \subseteq M_\gamma$  and  $M_\eta \subseteq M_\gamma$  for all  $\eta < \gamma$ . In order to be able to choose the desired  $M_\beta$ , it suffices to show that

$$(N_\beta + \bigcup_{\gamma < \beta} M_\gamma) \cap \text{Soc}(H_\beta(M)) = 0.$$

Suppose  $x + y \in \text{Soc}(H_\beta(M))$  where  $x \in N_\beta$  and  $y \in M_\gamma$  for some  $\gamma < \beta$ . Then

$$H(x') = -H(y') \in H_1(M) \cap N \cap M_\gamma = H_1(M) \cap N_\gamma = H_1(N_\gamma),$$

where  $d(\frac{xR}{x'R}) = d(\frac{yR}{y'R}) = 1$ , and hence there is  $u \in N_\gamma$  such that

$$x - u \in \text{Soc}(N) = \text{Soc}(N_\gamma) \oplus \text{Soc}(H_\gamma(N)).$$

Thus we can write  $x = u + v + z$  where  $v \in \text{Soc}(N_\gamma)$  and  $z \in \text{Soc}(H_\gamma(N))$ . Then

$$u + v + y = x + y - z \in H_\gamma(M) \cap M_\gamma = 0 \quad \text{and} \quad x + y = z \in N.$$

Therefore  $y \in N \cap M_\gamma = N_\gamma \subseteq N_\beta$  and, consequently,

$$x + y \in N_\beta \cap H_\beta(M) = N_\beta \cap H_\beta(N) = 0$$

as desired.  $\square$

**Lemma 3.** *Let  $M$  be a totally projective QTAG-module such that  $M = \bigcup_{\beta < \alpha} M_\beta$  where  $\{M_\beta\}_{\beta < \alpha}$  is an  $\alpha$ -high tower. If  $N$  is an  $\alpha$ -pure submodule of  $M$  such that for each  $\beta$ ,  $N \cap M_\beta$  is a  $\beta$ -high submodule of  $N$ , then  $N$  is a direct summand of  $M$ .*

*Proof.* We need only show that  $M/N$  is totally projective having length at most  $\alpha$ . Since  $N \cap M_\beta$  is  $(\beta + 1)$ -pure in  $N$  and  $N$  is  $\alpha$ -pure in  $M$ ,  $N \cap M_\beta$  is  $(\beta + 1)$ -pure in  $M$  and, a fortiori,  $(\beta + 1)$ -pure in  $M_\beta$ . Since  $M_\beta$  is totally projective,  $M_\beta$  is

$\beta$ -projective. Therefore, there is direct decomposition  $M_\beta = (N \cap M_\beta) \oplus K_\beta$  for each  $\beta < \alpha$ . Now

$$M/N = \bigcup_{\beta < \alpha} M_\beta + N/N \quad \text{and} \quad M_\beta + N/N \cong M_\beta / (M_\beta \cap N) \cong K_\beta$$

is totally projective for each  $\beta$ . By Corollary 1, it is enough to show that  $M_\beta + N/N$  is a  $\beta$ -high submodule of  $M/N$  whenever  $\omega \leq \beta < \alpha$ . Since  $N$  is  $\alpha$ -pure in  $M$ , we have

$$\text{Soc}(H_\beta(M/N)) = \text{Soc}(H_\beta(M)) + N/N$$

for  $\beta < \alpha$  and it then easily follows that

$$\text{Soc}(M/N) = \text{Soc}(M_\beta + N/N) \oplus \text{Soc}(H_\beta(M/N)).$$

Because of this direct decomposition, it is enough to show that  $M_\beta + N/N$  is an  $h$ -pure submodule of  $M/N$  for  $\beta \geq \omega$ .

Now

$$\begin{aligned} \text{Soc}(M_\beta + N) &= \text{Soc}(K_\beta \oplus N) \\ &= \text{Soc}(K_\beta) \oplus \text{Soc}(N) \\ &= \text{Soc}(K_\beta) \oplus \text{Soc}(N \cap M_\beta) \oplus \text{Soc}(H_\beta(N)) \\ &= \text{Soc}(M_\beta) \oplus \text{Soc}(H_\beta(N)). \end{aligned}$$

If  $\beta \geq \omega$  and if  $x \in \text{Soc}(M_\beta + N)$ , then we can write  $x = y + z$  where  $y \in \text{Soc}(M_\beta)$  and  $z \in \text{Soc}(H_\beta(N)) \subseteq H_\omega(N)$ . If  $x$  has finite height in  $M$ , then this height is just the height of  $y$  in  $M$  ( $=$  height of  $y$  in  $M_\beta$ ) and thus just the height of  $x = y + z$  in  $M_\beta + N$ . On the other hand, if  $x$  has infinite height in  $M$ , then  $y$  has infinite height in  $M_\beta$  and  $x = y + z$  has infinite height in  $M_\beta + N$ , it follows that  $M_\beta + N$  is an  $h$ -pure submodule of  $M$ . Thus  $M_\beta + N/N$  is  $h$ -pure in  $M/N$ .  $\square$

**Proposition 2.** *Let  $N$  be an  $\alpha$ -pure submodule of an  $\alpha$ -module  $M$  such that  $N$  is totally projective of length at most  $\alpha$ . Then there exists a submodule  $K$  of  $M$  such that  $N \oplus K$  is an  $\alpha$ -basic submodule of  $M$ .*

*Proof.* Since  $N$  is totally projective of length  $\leq \alpha$ ,  $N$  is the union of an  $\alpha$ -high tower  $\{N_\beta\}_{\beta < \alpha}$  of itself. By Lemma 2, there exists an  $\alpha$ -high tower  $\{M_\beta\}_{\beta < \alpha}$  of  $M$  such that  $N_\beta = N \cap M_\beta$  for each  $\beta$ . Let  $B = \bigcup_{\beta < \alpha} M_\beta$ . By the proof of Theorem 1,  $B$  is an  $\alpha$ -basic submodule of  $M$ . But  $\{M_\beta\}_{\beta < \alpha}$  is also an  $\alpha$ -high tower of  $B$ , and by Lemma 3 we have the required direct decomposition  $B = N \oplus K$ .  $\square$

Now we prove the following result.

**Theorem 2.** *If  $N$  is an  $\alpha$ -pure submodule of an  $\alpha$ -module  $M$ , then  $M/N$  is an  $\alpha$ -module.*

*Proof.* Let  $B$  be an  $\alpha$ -basic submodule of  $N$  and choose  $K$  such that  $B \oplus K$  is an  $\alpha$ -basic submodule of  $M$ . Now if  $x \in \text{Soc}(N \cap K)$ , we can write for each  $\beta < \alpha$ ,  $x = y_\beta + z_\beta$ , where  $y_\beta \in \text{Soc}(N)$  and  $z_\beta \in H_\beta(N)$ . Thus

$$-y_\beta + x \in H_\beta(B \oplus K) = H_\beta(B) \oplus H_\beta(K)$$

and

$$x \in \bigcap_{\beta < \alpha} H_\beta(K) = H_\alpha(K) = 0.$$

We then have a direct decomposition  $N \oplus K$ . If  $H_1(a') \in N \oplus K$ , then  $H_1(a') = y + H_1(b') + c$ , where  $d(\frac{a'R}{a'R}) = d(\frac{b'R}{b'R}) = 1$ ,  $y \in B$ ,  $b \in N$  and  $c \in K$ . Since

$$H_1(M) \cap (B \oplus K) = H_1(B \oplus K),$$

we conclude that

$$H_1(M) \cap (N \oplus K) = H_1(N \oplus K).$$

Now

$$\text{Soc}(M) \subseteq \text{Soc}(B \oplus K) + H_\beta(M) \subseteq \text{Soc}(N \oplus K) + H_\beta(M)$$

for all  $\beta < \alpha$ , and therefore  $N \oplus K$  is an  $\alpha$ -pure submodule of  $M$ . Consequently,  $N \oplus K/N$  is  $\alpha$ -pure in  $M/N$ . Also

$$N \oplus K/N \cong K$$

and

$$(M/N)/(N \oplus K/N) \cong (M/B \oplus K)/[(N \oplus K)/(B \oplus K)]$$

is  $h$ -divisible. We have constructed an  $\alpha$ -basic submodule of  $M/N$  and we conclude that  $M/N$  is indeed an  $\alpha$ -module.  $\square$

As a consequence of the above theorem, we have the following striking analog of a familiar property of  $h$ -pure submodules.

**Corollary 2.** *Let  $N$  be a submodule of an  $\alpha$ -module  $M$ . Then  $N$  is an  $\alpha$ -pure submodule of  $M$  if and only if  $N + H_\beta(M)/H_\beta(M)$  is a direct summand of  $M/H_\beta(M)$  for all  $\beta < \alpha$ .*

*Proof.*  $N + H_\beta(M)/H_\beta(M)$  being a direct summand of  $M/H_\beta(M)$  implies that  $N + H_\beta(M)/H_\beta(M)$  is  $\beta$ -pure in  $M/H_\beta(M)$ , which is equivalent to  $N$  being  $\beta$ -pure in  $M$ . Since  $\alpha$  is a limit ordinal,  $N$  is  $\alpha$ -pure in  $M$  if and only if  $N$  is  $\beta$ -pure in  $M$  for all  $\beta < \alpha$ .

Conversely, assume that  $N$  is  $\alpha$ -pure in  $M$ . Then  $M/N$  is an  $\alpha$ -module and therefore, for  $\beta < \alpha$ ,

$$\begin{aligned} (M/N)/H_\beta(M/N) &= (M/N)/(H_\beta(M) + N/N) \\ &\cong (M/H_\beta(M))/(N + H_\beta(M)/H_\beta(M)) \end{aligned}$$

is totally projective of length at most  $\beta$ . Since  $N + H_\beta(M)/H_\beta(M)$  is  $\beta$ -pure in  $M/H_\beta(M)$ ,  $N + H_\beta(M)/H_\beta(M)$  is a direct summand of  $M/H_\beta(M)$ .  $\square$

**Proposition 3.** *If  $N$  is an  $\alpha$ -pure submodule of an  $\alpha$ -module  $M$ , and if  $H_\beta(N)$  is a direct summand of  $H_\beta(M)$  for some  $\beta < \alpha$ , then  $N$  is a direct summand of  $M$ .*

*Proof.* Assuming the conditions of the Theorem 2, we have for some  $\beta < \alpha$ :



- (i)  $(M/N)/H_\beta(M/N)$  is totally projective;
- (ii)  $N \cap H_\beta(M) = H_\beta(N)$ ;
- (iii)  $N + H_\beta(M)/H_\beta(M)$  is a direct summand of  $M/H_\beta(M)$ ; and
- (iv)  $H_\beta(M) = H_\beta(N) \oplus K$ .

It follows that  $M = N \oplus L$  where  $L \supseteq K$ . □

As a corollary, we have the following generalization of the well-known fact that bounded  $h$ -pure submodules are direct summands.

**Corollary 3.** *If  $N$  is an  $\alpha$ -pure submodule of an  $\alpha$ -module  $M$  and if  $H_\beta(N) = 0$  for some  $\beta < \alpha$ , then  $N$  is a direct summand of  $M$ .*

As defined in [10], a QTAG-module  $M$  is fully transitive if for every pair of uniform elements  $x, y \in M$ ,  $H_M(x_i) \leq H_M(y_i)$  for all  $i \geq 0$  implies that there exists an endomorphism of  $M$  that maps  $x$  onto  $y$ . Here  $d\left(\frac{xR}{x_iR}\right) = d\left(\frac{yR}{y_iR}\right) = i$ .

The next corollary tells us that  $\alpha$ -modules of length  $\alpha$  are fully transitive (see [6]). This, of course, is merely a reflection of the fact that modules of length  $\leq \alpha$  behave in the  $\alpha$  context exactly as modules without elements of infinite height in the classical situations.

**Corollary 4.** *If  $M$  is an  $\alpha$ -module of length  $\alpha$ , then every finite subset of  $M$  is contained in a countably generated direct summand.*

*Proof.* Let  $S$  be a finite subset of  $M$ . Then  $S \subseteq T$  for some countably generated,  $\alpha$ -pure submodule  $T$  of  $M$ . We may assume that  $T$  has length  $\alpha$ . Then  $T$  is a direct sum of modules of length less than  $\alpha$ . Consequently,  $T$  is contained in a direct summand  $K$  of  $T$  having length less than  $\alpha$ . By the preceding corollary,  $K$  is a direct summand of  $M$ . □

For a limit ordinal  $\alpha$ , an  $\alpha$ -module  $M$  is an  $\alpha$ -projective if

$$H_\alpha(\text{Ext}(M, M')) = 0$$

for all  $\alpha$ -modules  $M'$ , that is, there exists a submodule  $N$  bounded by  $\alpha$  such that  $M/N$  is totally projective, and an  $\alpha$ -module  $M$  is an  $\alpha$ -injective if

$$H_\alpha(\text{Ext}(M', M)) = 0$$

for all  $\alpha$ -modules  $M'$ , that is, it is a direct summand of every  $\alpha$ -module in which it occurs as an  $\alpha$ -pure submodule.

To characterize the  $\alpha$ -injective modules we must generalize the notion of a closed module. Mimicking [12], for any QTAG-module  $M$ , the submodules  $\{H_k(M)\}_k$ ,  $k = 0, 1, 2, \dots, \infty$  from a neighborhood system of zero, giving rise to  $h$ -topology. If  $k$  is replaced by an arbitrary limit ordinal less than or equal to  $\alpha$ , then  $h$ -topology may be extended to  $\alpha$ -topology, and all the definitions and results which hold for  $h$ -topology may be extended for  $\alpha$ -topology. In  $\alpha$ -topology, for any submodule  $N$  of  $M$ , the closure of  $N$  as  $\bigcap_{\beta < \alpha} (N + H_\beta(M))$  denoted by  $\bar{N}$ .

**Definition 3.** We call a QTAG-module an  $\alpha$ -closed module if it is the maximal closed submodule of its closure in the  $\alpha$ -topology.

With the help of the above discussion, we are able to infer the following.

**Proposition 4.** *Let  $M$  be an  $\alpha$ -closed  $\alpha$ -module. Then  $M$  is an  $\alpha$ -injective.*

*Proof.* We first show that  $H_\alpha(\text{Ext}(T, M)) = 0$  for all  $\alpha$ -modules  $T$ . Assume that  $M$  is an  $\alpha$ -pure submodule of  $M'$  with  $M'/M \cong T$  for all  $\alpha$ -modules  $M'$ . Since  $\alpha$  is a limit ordinal, it follows that  $M' = H_\beta(M') + M$  for all  $\beta < \alpha$ . Therefore, if  $y \in M'$ , we can find for each  $\beta < \alpha$  a  $x_\beta \in M$  such that  $y - x_\beta \in H_\beta(M')$ . Moreover, we can assume that the exponent of  $x_\beta$  does not exceed that of  $y$ . Indeed, if  $y$  has exponent  $n$ , then

$$H_n(x'_\beta) \in H_{\beta+n}(M') \cap M = H_{\beta+n}(M),$$

where  $d(\frac{x_\beta R}{x'_\beta R}) = n$  and  $H_n(x'_\beta) = H_n(z'_\beta)$ , where  $d(\frac{x_\beta R}{x'_\beta R}) = d(\frac{z_\beta R}{z'_\beta R}) = n$  for some  $z_\beta \in H_\beta(M)$ . Then  $\bar{x}_\beta = x_\beta - z_\beta$  has an exponent at most  $n$  and  $y - \bar{x}_\beta \in H_\beta(M')$ . But  $\{x_\beta : \beta < \alpha\}$  is a chain in  $M$  with elements uniformly bounded in exponent and, therefore, converges to some  $x \in M$ . Hence

$$y - x \in \bigcap_{\beta < \alpha} H_\beta(M') = H_\alpha(M').$$

We conclude that  $M' = M \oplus H_\alpha(M')$ .

Now let  $M'$  be an arbitrary  $\alpha$ -module and let  $B$  be an  $\alpha$ -basic submodule of  $M'$ . We then have the exact sequence

$$H_\alpha(\text{Ext}(M'/B, M)) \longrightarrow H_\alpha(\text{Ext}(M', M)) \longrightarrow H_\alpha(\text{Ext}(B, M)).$$

The left-hand term of the above sequence vanishes since  $M'/B$  is isomorphic to a direct sum of copies of  $T$  and the right-hand term vanishes since  $B$  is an  $\alpha$ -projective. Thus,  $H_\alpha(\text{Ext}(M', M)) = 0$  and we conclude that  $M$  is an  $\alpha$ -injective.  $\square$

We can now show that there are enough  $\alpha$ -injective modules and that an  $\alpha$ -injective module is the sum of an  $\alpha$ -closed module and an  $h$ -divisible module.

**Theorem 3.** *Let  $M$  be an  $\alpha$ -module. Then  $M$  is an  $\alpha$ -pure submodule of an  $\alpha$ -injective module and  $M$  is an  $\alpha$ -injective module if and only if  $M$  is the direct sum of an  $h$ -divisible module and an  $\alpha$ -closed  $\alpha$ -module.*

*Proof.* It is evident from Proposition 4 that the direct sum of an  $h$ -divisible module and an  $\alpha$ -closed  $\alpha$ -module is necessarily an  $\alpha$ -injective. Next, we need the observation that every  $\alpha$ -module  $M$  of length at most  $\alpha$  can be imbedded as an  $\alpha$ -pure submodule of an  $\alpha$ -closed module  $T_M(\alpha)$  such that  $T_M(\alpha)/M$  is  $h$ -divisible. Indeed,  $T_M(\alpha)$  may be taken as the maximal closed submodule of the closure of  $M$

in the  $\alpha$ -topology. It follows, by the same reasoning as in the proof of Theorem 1, that

$$T_M(\alpha)/H_\beta(T_M(\alpha)) \cong M/H_\beta(M)$$

for all  $\beta < \alpha$ , and therefore that  $T_M(\alpha)$  is an  $\alpha$ -module.

Now let  $M$  be an arbitrary  $\alpha$ -module. Let  $D$  be a minimal  $h$ -divisible module containing  $H_\alpha(M)$ . Take  $P$  to be the amalgamated sum of  $M$  and  $D$  over  $H_\alpha(M)$ . Then  $P = M' \oplus D$  where  $M' \cong M/H_\alpha(M)$  and  $M' \cap M$  is an  $\alpha$ -high submodule of  $M$ . Also,  $P/M$  is  $h$ -divisible and

$$\text{Soc}(P) \subseteq \text{Soc}(M) + H_\beta(P)$$

for all  $\beta < \alpha$ . It follows that  $M$  is an  $\alpha$ -pure submodule of  $P$ . By the transitivity of  $\alpha$ -purity,  $M$  is an  $\alpha$ -pure in the  $\alpha$ -injective  $T_{M'}(\alpha) \oplus D$ .

Finally, assume that  $M$  is itself an  $\alpha$ -injective and that we have it imbedded, as above, as an  $\alpha$ -pure submodule of  $\bar{P} = T_{M'}(\alpha) \oplus D$ . Since  $M$  is an  $\alpha$ -injective,  $\bar{P} = M \oplus Q$  where  $Q \cong \bar{P}/M$  is obviously  $h$ -divisible, since both  $P/M$  and  $\bar{P}/P$  are  $h$ -divisible. But then  $Q \subseteq D$ , and since  $\text{Soc}(D) \subseteq H_\alpha(M)$ , we conclude that  $Q = 0$  and  $M = T_{M'}(\alpha) \oplus D$ .  $\square$

Now we are in a position to prove the following result.

**Theorem 4.** *If  $M$  and  $M'$  are  $\alpha$ -closed  $\alpha$ -modules with the same Ulm invariants, then  $M \cong M'$ .*

*Proof.* Take  $B$  and  $B'$  to be  $\alpha$ -basic submodules of  $M$  and  $M'$ , respectively. It is easily seen that  $B$  and  $B'$  have the same Ulm invariants as  $M$  and  $M'$ . Therefore, there is an isomorphism  $f$  of  $B$  onto  $B'$ . Since  $B$  is an  $\alpha$ -pure submodule of  $M$ , we have the exact sequence

$$\text{Hom}(M, M') \rightarrow \text{Hom}(B, M') \rightarrow H_\alpha(\text{Ext}(M/B, M')) = 0$$

Thus, there is a homomorphism  $f': M \rightarrow M'$  that extends  $f$ . Let  $x \in \text{Ker } f'$  and assume that  $x \neq 0$ . Then  $x$  has some height  $\beta < \alpha$  and we can write  $x = y + z$  where  $y \in B$  and  $z \in H_{\beta+1}(M)$ . But then  $x$  has height  $\beta$  and  $f(y) = f'(y) = -f'(z)$  has height at least  $\beta + 1$ . This, however, is a contradiction, since  $f$  is an isomorphism of  $B$  onto  $B'$  and  $B'$  is an isotype submodule of  $M'$ . We conclude that  $\text{Ker } f' = 0$ . Then

$$f'(M)/B' = f'(M)/f'(B) \cong M/B$$

is  $h$ -divisible. Hence  $f'(M)/B'$  is a direct summand of  $M'/B'$ , and since  $B'$  is an  $\alpha$ -pure submodule of  $M'$ , it follows that  $f'(M)$  is an  $\alpha$ -pure submodule of  $M'$ . Since  $f'(M) \cong M$  is an  $\alpha$ -injective, we have a direct decomposition  $M' = f'(M) \oplus L$  where  $L \cong M'/f'(M)$  is  $h$ -divisible. But  $M'$  is  $h$ -reduced and therefore  $L = 0$  and  $f'(M) = M'$ , that is,  $f'$  is an isomorphism of  $M$  onto  $M'$ .  $\square$

### 3 $\alpha$ -large submodules of summable modules

If  $F$  is a fully invariant submodule of the  $\alpha$ -module  $M$ , then  $F$  is called an  $\alpha$ -large submodule of  $M$  if  $M = B + F$  for all  $\alpha$ -basic submodules  $B$  of  $M$ . This generalization of the concept of large submodule is studied in [12]. It is well-known that  $H_\beta(M)$  is always an  $\alpha$ -large submodule of  $M$  provided that  $\beta < \alpha$  as well as  $H_\beta(M) \subseteq L$  whenever  $L$  is an  $\alpha$ -large submodule in  $M$ .

Likewise, It was proved that an  $h$ -reduced QTAG-module  $M$  of length  $\geq \alpha$  contains a proper  $\alpha$ -basic submodule  $B$  if and only if  $M$  is an  $\alpha$ -module where  $\alpha$  is cofinal with  $\omega$ . Since  $\omega_1$ , the first uncountable limit ordinal, is not cofinal with  $\omega$ , some additional clarifications are necessary. In fact,  $B$  is an  $\omega_1$ -basic submodule of  $M$  only when  $B = 1$  or  $B = M$ , and so  $L$  is an  $\omega_1$ -large submodule of  $M$  uniquely when  $L = M$  and either  $L = 1$  or  $L \neq 1$  and it can take different forms; for instance  $L = H_\beta(M)$  where  $\beta < \text{length of } M \leq \omega_1$ .

In [12] it was seen that the properties of  $\alpha$ -large submodules for  $\alpha > \omega$  are not preserved in general by these of the QTAG-module and conversely; for instance the direct sum of countably generated modules. However, this is not the case for totally projective modules.

**Theorem 5 ([6], [12]).** *Let  $L$  be an  $\alpha$ -large submodule of the QTAG-module  $M$ . Then  $L$  is totally projective if and only if  $M$  is totally projective.*

The main goal of this section is to strengthen the above assertion to a class of modules, called summable modules. It is evident that direct sum of countably generated modules are themselves summable. In [8] it was constructed a summable  $\omega_1$ -module need not be a direct sum of countably generated modules. So, the investigation of the discussed above theme for  $\alpha$ -large submodules of summable  $\omega_1$ -modules will be of interest.

Now we have accumulated all the machinery necessary to prove the following.

**Theorem 6.** *Suppose that  $M$  is an  $\omega_1$ -module with an  $\alpha$ -large submodule  $L$  for some ordinal  $\alpha$  such that  $\omega \leq \alpha \leq \omega_1$ . Then  $M$  is summable if and only if  $L$  is summable.*

*Proof.* “ $\Rightarrow$ ”. In virtue of [12] there is a countable limit ordinal  $\tau \leq \alpha$  such that  $H_\tau(M) = H_\omega(L)$ . Moreover,  $L/H_\omega(L) = L/H_\tau(M)$  is an  $\alpha$ -large submodule of  $M/H_\tau(M)$ , where the latter quotient is totally projective by assumption. Therefore, Theorem 5 applies to deduce that  $L/H_\omega(L)$  is totally projective, in fact, a direct sum of uniserial modules. That is why, some high submodule  $N$  of  $L$  is a direct sum of countably generated modules. Indeed, what suffices to show is that  $N/H_\omega(N)$  is a direct sum uniserial modules because  $H_\omega(N)$  is bounded. In order to do that, we observe that

$$(N + H_\omega(L))/H_\omega(L) \subseteq L/H_\omega(L)$$

is also a direct sum of uniserial modules as a submodule. But  $N$  is isotype in  $L$ , whence

$$(N + H_\omega(L))/H_\omega(L) \cong N/(N \cap H_\omega(L)) = N/H_\omega(L)$$

which substantiates our claim. On the other hand,  $M$  being summable yields that  $H_\omega(L) = H_\tau(M)$  is summable, and we are done.

“ $\Leftarrow$ ”. Same as above,  $H_\tau(M) = H_\omega(L)$  for some countable limit ordinal  $\tau \leq \alpha$ . But  $L$  being summable implies that  $H_\omega(L) = H_\tau(M)$  is summable. Likewise,  $M/H_\tau(M)$  is totally projective of countable length, hence a direct sum of countably generated modules. Let  $N$  be a  $\tau$ -high submodule of  $M$ . Since  $H_\tau(N)$  is high in  $H_\tau(M)$  one may write  $H_\tau(M) = H_\tau(N) \oplus T$  for some submodule  $T$ , whence  $H_\tau(M) = T$ . Moreover,

$$\text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(H_\tau(M)) = \text{Soc}(N) \oplus \text{Soc}(T).$$

In fact

$$\begin{aligned} \text{Soc}(N) \cap \text{Soc}(T) &\subseteq N \cap T = N \cap (H_\tau(M) \cap T) \\ &= (N \cap H_\tau(M)) \cap T = H_\tau(N) \cap T = 0 \end{aligned}$$

because  $N$  is isotype in  $M$ . Consequently, there is a valued direct sum

$$\text{Soc}(H_\tau(M)) = H_\tau(N) \oplus \text{Soc}(T).$$

Even more,

$$\text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T)$$

is a valued direct sum, where  $T$  is a valued submodule of  $H_\tau(M)$  with  $\text{Soc}(T) = \text{Soc}(H_\tau(M))$ . Indeed, if  $x \in \text{Soc}(M)$  then

$$x \in \text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T).$$

Since  $\text{Soc}(T) = \text{Soc}(H_\tau(M))$  and  $N$  is  $h$ -pure in  $M$ . It easily follows that

$$x \in N + H_\tau(M) + M = N \oplus T \oplus M$$

because  $H_\tau(M) = H_\tau(K) \oplus T$ , and by induction the desired decomposition now follows.

If  $y \in \text{Soc}(N)$  and  $z \in \text{Soc}(T)$  then

$$H_M(y + z) = \min\{H_M(y), H_M(z)\}$$

since either

$$H_M(y) < \alpha \leq H_M(z) \quad \text{or} \quad H_\tau(M) = H_\tau(N) \oplus T$$

when  $H_M(y) \geq \alpha$ . Therefore,  $H_\tau(M)$  is summable if and only if  $T$  has this property.

Next, observe that

$$N \cong N/\{0\} = N/H_\tau(N) = N/(N \cap H_\tau(M)) \cong (N + H_\tau(M))/H_\tau(M),$$

where the last quotient is obviously isotype in  $M/H_\tau(M)$ , and thus it is a direct sum of countably generated modules as well. It follows that  $N$  is a direct sum of countably generated modules. Furthermore, both  $T$  and  $N$  are summable. But  $\text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T)$  is a valued direct sum and from this, our assertion follows directly by the definition of summability.  $\square$

We close the paper with a problem as follows:

**Problem 1.** Does it follow that if both  $H_\beta(M)$  and  $M/H_\beta(M)$  are  $\sigma$ -summable modules (see [14]) for some ordinal  $\beta$ , then  $M$  is  $\sigma$ -summable?

For summable modules we refer to [7]. Notice also that it can be obtained some results in this aspect under certain limitations on  $\beta$  which depends on  $n < \omega$ .

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