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α -modules and generalized submodules

Rafiquddin, Ayazul Hasan, Mohammad Fareed Ahmad

Abstract. A QTAG-module M is an α -module, where α is a limit ordinal, if $M/H_\beta(M)$ is totally projective for every ordinal $\beta < \alpha$. In the present paper α -modules are studied with the help of α -pure submodules, α -basic submodules, and α -large submodules. It is found that an α -closed α -module is an α -injective. For any ordinal $\omega \leq \alpha \leq \omega_1$ we prove that an α -large submodule L of an ω_1 -module M is summable if and only if M is summable.

1 Introduction and Preliminary Terminology

Let R be any ring. A module M in which the lattice of its submodule is totally ordered is called a serial module; in addition, if it has finite composition length, it is called a uniserial module. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module, and for any R -module M with a unique decomposition series, $d(M)$ denotes its decomposition length.

Modules are the natural generalizations of abelian groups. The results for abelian groups can be generalized for modules after imposing some conditions on modules/rings. In 1976 Singh [15] started the study of TAG-modules satisfying the following two conditions while the rings were associative with unity.

1. Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.
2. Given any two uniserial submodules U and V of a homomorphic image of M , for any submodule W of U , any non-zero homomorphism $f: W \rightarrow V$ can be

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extended to a homomorphism $g: U \rightarrow V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

Later on Benabdallah, Singh, Khan etc. contributed a lot to the study of TAG-modules [2], [17]. In 1987 Singh made an improvement and studied the modules satisfying only the condition 1 and called them QTAG-modules. The study of QTAG-modules and their structure began with work of Singh in [16]. This work, executed by many authors, clearly parallels the earlier work on torsion abelian groups. They studied different notions and structures on QTAG-modules and developed the theory of these modules by introducing different notions and characterizing different submodules of QTAG-modules. Yet there is much to explore.

Throughout this paper, all the rings are associative with unity ($1 \neq 0$) and modules M are unital QTAG-modules. For a uniform element $x \in M$, $e(x) = d(xR)$ and

$$H_M(x) = \sup \left\{ d \left(\frac{yR}{xR} \right) \mid y \in M, x \in yR \text{ and } y \text{ uniform} \right\}$$

are the exponent and height of x in M , respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k . Let us denote by M^1 , the submodule of M , containing elements of infinite height. The module M is h -divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$. The module M is h -reduced if it does not contain any h -divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded, if there exists an integer n such that $H_M(x) \leq n$ for every uniform element $x \in M$.

The sum of all simple submodules of M is called the socle of M , denoted by $\text{Soc}(M)$ and a submodule S of $\text{Soc}(M)$ is called a subsocle of M . The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α , $f_M(\alpha)$ is the α^{th} Ulm invariant of M and it is equal to $g(\text{Soc}(H_\alpha(M))/\text{Soc}(H_{\alpha+1}(M)))$.

A submodule N of M is h -pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. For an ordinal α , a submodule $N \subseteq M$ is an α -high submodule of M if N is maximal among the submodules of M that intersect $H_\alpha(M)$ trivially.

For an ordinal α , a submodule N of M is said to be an α -pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \alpha$ and a submodule N of M is said to be isotype in M , if it is α -pure for every ordinal α [13]. For a submodule $N \subseteq M$, the valuation of N induced by height in M is defined by $v(x) = H_M(x)$, the height of x in M , for all $x \in N$ and $N = K \oplus L$ is a valued direct sum if $v(k + \ell) = \min\{v(k), v(\ell)\}$ for all $k \in K$ and $\ell \in L$ [5].

A submodule $B \subseteq M$ is a basic submodule [9] of M , if B is h -pure in M , $B = \bigoplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h -divisible. A fully invariant submodule $L \subseteq M$ is large [1], if $L + B = M$, for every basic submodule B in M .

Imitating [11], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as h -topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as

$\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h -topology if $\overline{N} = N$.

An h -reduced QTAG-module M is summable [14] if $\text{Soc}(M) = \bigoplus_{\beta < \alpha} S_{\beta}$, where S_{β} is the set of all elements of $H_{\beta}(M)$ which are not in $H_{\beta+1}(M)$, where α is the length of M . Moreover, M is called totally projective [10], if

$$H_{\alpha}(\text{Ext}(M/H_{\alpha}(M), M')) = 0$$

for all ordinal α and QTAG-modules M' .

It is interesting to note that almost all the results which hold for TAG-modules are also valid for QTAG-modules [13]. Our notations and terminology generally agree with those in [3] and [4].

2 α -modules and α -basic submodules

Mehdi et al. [12] defined α -modules and introduced some new concepts for these modules. The same type of study was continued in [6] and a number of results have been obtained in terms of generalized submodules. Here we also continue the similar study of α -modules that depend on the notions of summability, purity, basic submodules, projectivity and injectivity. For facilitating the exposition and for the convenience of the readers, we recall the definition of α -modules.

Definition 1. Let α denote the class of all QTAG-modules M such that $M/H_{\beta}(M)$ is totally projective for all ordinals $\beta < \alpha$, a limit ordinal. These modules are called α -modules.

To develop the study, we need to prove some results, and we start with the following.

Proposition 1. *If N is an α -pure submodule of an α -module M , then N is itself an α -module.*

Proof. We actually only need that $N \cap H_{\gamma}(M) = H_{\gamma}(N)$ for all $\gamma < \alpha$. For then it is a simple calculation to show that $N + H_{\beta}(M)/H_{\beta}(M)$ is isotype in $M/H_{\beta}(M)$ for each $\beta < \alpha$. And therefore, $N + H_{\beta}(M)/H_{\beta}(M) \cong N/H_{\beta}(N)$ is totally projective for all $\beta < \alpha$. \square

As generalized the notion of a basic submodule in [12], by defining B to be an α -basic submodule of an α -module M if B is totally projective of length at most α , B is α -pure submodule of M , and M/B is h -divisible.

In order to establish the existence of α -basic submodules we require the following notion for technical convenience.

Definition 2. Let α be a limit ordinal and M a QTAG-module. An α -high tower of M is a well-ordered ascending chain $\{M_{\beta}\}_{\beta < \alpha}$ of submodules of M such that, for each β , M_{β} is a β -high submodule of M .

Now we need to prove the following lemma.

Lemma 1. *Let α be a limit ordinal and $\{M_\beta\}_{\beta < \alpha}$ an α -high tower of a QTAG-module M . If each M_β is summable, then $N = \bigcup_{\beta < \alpha} M_\beta$ is summable.*

Proof. As α is a limit ordinal, we may choose a strictly increasing sequence $\beta_1 < \beta_2 < \dots < \beta_n < \dots$ of ordinals having α as its limit. Then $N = \bigcup_{n < \omega} M_{\beta_n}$. Set $T_0 = \text{Soc}(M_{\beta_1})$ and, for $n > 1$, let T_n be such that

$$\text{Soc}(H_{\beta_n}(M)) = T_n \oplus \text{Soc}(H_{\beta_{n+1}}(M))$$

with $T_n \subseteq M_{\beta_{n+1}}$. Then we have a direct-sum decomposition $\text{Soc}(N) = \bigoplus_{n < \omega} T_n$ which is normal in the sense that

$$H_M(t_1 + \dots + t_n) = \min[H_M(t_1), \dots, H_M(t_n)]$$

provided $t_i \in T_i$ for $i = 1, \dots, n$. Now each M_β is isotype, summable, and of countable length. Therefore, each subsocle of M_β is a summable subsocle of M . In particular, each T_n is a summable subsocle of M . Since the decomposition $\text{Soc}(N) = \bigoplus_{n < \omega} T_n$ is normal, it follows that $\text{Soc}(N)$ is a summable subsocle of M . Since each M_β is isotype, N is itself an isotype submodule of M and consequently N is summable. \square

We continue the study with the following corollary.

Corollary 1. *Let α be a limit ordinal and $\{M_\beta\}_{\beta < \alpha}$ an α -high tower of a QTAG-module M , where each M_β is totally projective, then $N = \bigcup_{\beta < \alpha} M_\beta$ is totally projective of length at most α .*

Proof. As noted above, N is an isotype submodule of M and clearly N has a length at most α . Thus M_β is also a β -high submodule of N for each $\beta < \alpha$. Since N is summable by Lemma 1 implies that N is totally projective. \square

Now we prove the following.

Theorem 1. *Let M be a QTAG-module. Then M contains an α -basic submodule if and only if M is an α -module.*

Proof. If B is an α -pure submodule of M and if M/B is h -divisible, then it follows that $M/H_\beta(M) \cong B/H_\beta(B)$ for all $\beta < \alpha$. Consequently, only α -modules can have α -basic submodules (see [12]). Suppose now that M is an α -module and select an α -high tower $\{M_\beta\}_{\beta < \alpha}$. Now

$$M_\beta \cong M_\beta + H_\beta(M)/H_\beta(M),$$

and since M_β is isotype in M , $M_\beta + H_\beta(M)/H_\beta(M)$ is isotype in $M/H_\beta(M)$. By Corollary 1, $B = \bigcup_{\beta < \alpha} M_\beta$ is totally projective. It is easily seen that

$$\text{Soc}(M) \subseteq \text{Soc}(B) + H_\beta(M)$$

for each $\beta < \alpha$, and therefore B is α -pure in M . Moreover, $B \cap H_1(M) = H_1(B)$ and

$$\text{Soc}(M) \subseteq \text{Soc}(B) + H_\beta(M)$$

for $\beta < \omega$ imply that M/B is h -divisible. Thus, B is the required α -basic submodule of M . \square

Lemma 2. *Suppose N is an isotype submodule of a QTAG-module M and that $\{N_\beta\}_{\beta < \alpha}$ is an α -high tower of N , then there exists an α -high tower $\{M_\beta\}_{\beta < \alpha}$ of M such that, for each β , $N_\beta \subseteq M_\beta$ and $N_\beta = N \cap M_\beta$.*

Proof. Let us first note that $N_\beta = N \cap M_\beta$ is a consequence of $N_\beta \subseteq M_\beta$. Indeed, $N_\beta \subseteq M_\beta$ implies $N_\beta \subseteq N \cap M_\beta$ and

$$(N \cap M_\beta) \cap H_\beta(N) = (N \cap M_\beta) \cap H_\beta(M) = 0.$$

The maximality of a β -high submodule then yields the equality. Assume now that $\beta < \alpha$ and that for each $\gamma < \beta$ we have a γ -high submodule M_γ of M such that $N_\gamma \subseteq M_\gamma$ and $M_\eta \subseteq M_\gamma$ for all $\eta < \gamma$. In order to be able to choose the desired M_β , it suffices to show that

$$(N_\beta + \bigcup_{\gamma < \beta} M_\gamma) \cap \text{Soc}(H_\beta(M)) = 0.$$

Suppose $x + y \in \text{Soc}(H_\beta(M))$ where $x \in N_\beta$ and $y \in M_\gamma$ for some $\gamma < \beta$. Then

$$H(x') = -H(y') \in H_1(M) \cap N \cap M_\gamma = H_1(M) \cap N_\gamma = H_1(N_\gamma),$$

where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{yR}{y'R}\right) = 1$, and hence there is $u \in N_\gamma$ such that

$$x - u \in \text{Soc}(N) = \text{Soc}(N_\gamma) \oplus \text{Soc}(H_\gamma(N)).$$

Thus we can write $x = u + v + z$ where $v \in \text{Soc}(N_\gamma)$ and $z \in \text{Soc}(H_\gamma(N))$. Then

$$u + v + y = x + y - z \in H_\gamma(M) \cap M_\gamma = 0 \quad \text{and} \quad x + y = z \in N.$$

Therefore $y \in N \cap M_\gamma = N_\gamma \subseteq N_\beta$ and, consequently,

$$x + y \in N_\beta \cap H_\beta(M) = N_\beta \cap H_\beta(N) = 0$$

as desired. \square

Lemma 3. *Let M be a totally projective QTAG-module such that $M = \bigcup_{\beta < \alpha} M_\beta$ where $\{M_\beta\}_{\beta < \alpha}$ is an α -high tower. If N is an α -pure submodule of M such that for each β , $N \cap M_\beta$ is a β -high submodule of N , then N is a direct summand of M .*

Proof. We need only show that M/N is totally projective having length at most α . Since $N \cap M_\beta$ is $(\beta + 1)$ -pure in N and N is α -pure in M , $N \cap M_\beta$ is $(\beta + 1)$ -pure in M and, a fortiori, $(\beta + 1)$ -pure in M_β . Since M_β is totally projective, M_β is

β -projective. Therefore, there is direct decomposition $M_\beta = (N \cap M_\beta) \oplus K_\beta$ for each $\beta < \alpha$. Now

$$M/N = \bigcup_{\beta < \alpha} M_\beta + N/N \quad \text{and} \quad M_\beta + N/N \cong M_\beta / (M_\beta \cap N) \cong K_\beta$$

is totally projective for each β . By Corollary 1, it is enough to show that $M_\beta + N/N$ is a β -high submodule of M/N whenever $\omega \leq \beta < \alpha$. Since N is α -pure in M , we have

$$\text{Soc}(H_\beta(M/N)) = \text{Soc}(H_\beta(M)) + N/N$$

for $\beta < \alpha$ and it then easily follows that

$$\text{Soc}(M/N) = \text{Soc}(M_\beta + N/N) \oplus \text{Soc}(H_\beta(M/N)).$$

Because of this direct decomposition, it is enough to show that $M_\beta + N/N$ is an h -pure submodule of M/N for $\beta \geq \omega$.

Now

$$\begin{aligned} \text{Soc}(M_\beta + N) &= \text{Soc}(K_\beta \oplus N) \\ &= \text{Soc}(K_\beta) \oplus \text{Soc}(N) \\ &= \text{Soc}(K_\beta) \oplus \text{Soc}(N \cap M_\beta) \oplus \text{Soc}(H_\beta(N)) \\ &= \text{Soc}(M_\beta) \oplus \text{Soc}(H_\beta(N)). \end{aligned}$$

If $\beta \geq \omega$ and if $x \in \text{Soc}(M_\beta + N)$, then we can write $x = y + z$ where $y \in \text{Soc}(M_\beta)$ and $z \in \text{Soc}(H_\beta(N)) \subseteq H_\omega(N)$. If x has finite height in M , then this height is just the height of y in M ($=$ height of y in M_β) and thus just the height of $x = y + z$ in $M_\beta + N$. On the other hand, if x has infinite height in M , then y has infinite height in M_β and $x = y + z$ has infinite height in $M_\beta + N$, it follows that $M_\beta + N$ is an h -pure submodule of M . Thus $M_\beta + N/N$ is h -pure in M/N . \square

Proposition 2. *Let N be an α -pure submodule of an α -module M such that N is totally projective of length at most α . Then there exists a submodule K of M such that $N \oplus K$ is an α -basic submodule of M .*

Proof. Since N is totally projective of length $\leq \alpha$, N is the union of an α -high tower $\{N_\beta\}_{\beta < \alpha}$ of itself. By Lemma 2, there exists an α -high tower $\{M_\beta\}_{\beta < \alpha}$ of M such that $N_\beta = N \cap M_\beta$ for each β . Let $B = \bigcup_{\beta < \alpha} M_\beta$. By the proof of Theorem 1, B is an α -basic submodule of M . But $\{M_\beta\}_{\beta < \alpha}$ is also an α -high tower of B , and by Lemma 3 we have the required direct decomposition $B = N \oplus K$. \square

Now we prove the following result.

Theorem 2. *If N is an α -pure submodule of an α -module M , then M/N is an α -module.*

Proof. Let B be an α -basic submodule of N and choose K such that $B \oplus K$ is an α -basic submodule of M . Now if $x \in \text{Soc}(N \cap K)$, we can write for each $\beta < \alpha$, $x = y_\beta + z_\beta$, where $y_\beta \in \text{Soc}(N)$ and $z_\beta \in H_\beta(N)$. Thus

$$-y_\beta + x \in H_\beta(B \oplus K) = H_\beta(B) \oplus H_\beta(K)$$

and

$$x \in \bigcap_{\beta < \alpha} H_\beta(K) = H_\alpha(K) = 0.$$

We then have a direct decomposition $N \oplus K$. If $H_1(a') \in N \oplus K$, then $H_1(a') = y + H_1(b') + c$, where $d(\frac{a'R}{a'R}) = d(\frac{b'R}{b'R}) = 1$, $y \in B$, $b \in N$ and $c \in K$. Since

$$H_1(M) \cap (B \oplus K) = H_1(B \oplus K),$$

we conclude that

$$H_1(M) \cap (N \oplus K) = H_1(N \oplus K).$$

Now

$$\text{Soc}(M) \subseteq \text{Soc}(B \oplus K) + H_\beta(M) \subseteq \text{Soc}(N \oplus K) + H_\beta(M)$$

for all $\beta < \alpha$, and therefore $N \oplus K$ is an α -pure submodule of M . Consequently, $N \oplus K/N$ is α -pure in M/N . Also

$$N \oplus K/N \cong K$$

and

$$(M/N)/(N \oplus K/N) \cong (M/B \oplus K)/[(N \oplus K)/(B \oplus K)]$$

is h -divisible. We have constructed an α -basic submodule of M/N and we conclude that M/N is indeed an α -module. \square

As a consequence of the above theorem, we have the following striking analog of a familiar property of h -pure submodules.

Corollary 2. *Let N be a submodule of an α -module M . Then N is an α -pure submodule of M if and only if $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$ for all $\beta < \alpha$.*

Proof. $N + H_\beta(M)/H_\beta(M)$ being a direct summand of $M/H_\beta(M)$ implies that $N + H_\beta(M)/H_\beta(M)$ is β -pure in $M/H_\beta(M)$, which is equivalent to N being β -pure in M . Since α is a limit ordinal, N is α -pure in M if and only if N is β -pure in M for all $\beta < \alpha$.

Conversely, assume that N is α -pure in M . Then M/N is an α -module and therefore, for $\beta < \alpha$,

$$\begin{aligned} (M/N)/H_\beta(M/N) &= (M/N)/(H_\beta(M) + N/N) \\ &\cong (M/H_\beta(M))/(N + H_\beta(M)/H_\beta(M)) \end{aligned}$$

is totally projective of length at most β . Since $N + H_\beta(M)/H_\beta(M)$ is β -pure in $M/H_\beta(M)$, $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$. \square

Proposition 3. *If N is an α -pure submodule of an α -module M , and if $H_\beta(N)$ is a direct summand of $H_\beta(M)$ for some $\beta < \alpha$, then N is a direct summand of M .*

Proof. Assuming the conditions of the Theorem 2, we have for some $\beta < \alpha$:

- (i) $(M/N)/H_\beta(M/N)$ is totally projective;
- (ii) $N \cap H_\beta(M) = H_\beta(N)$;
- (iii) $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$; and
- (iv) $H_\beta(M) = H_\beta(N) \oplus K$.

It follows that $M = N \oplus L$ where $L \supseteq K$. □

As a corollary, we have the following generalization of the well-known fact that bounded h -pure submodules are direct summands.

Corollary 3. *If N is an α -pure submodule of an α -module M and if $H_\beta(N) = 0$ for some $\beta < \alpha$, then N is a direct summand of M .*

As defined in [10], a QTAG-module M is fully transitive if for every pair of uniform elements $x, y \in M$, $H_M(x_i) \leq H_M(y_i)$ for all $i \geq 0$ implies that there exists an endomorphism of M that maps x onto y . Here $d\left(\frac{xR}{x_iR}\right) = d\left(\frac{yR}{y_iR}\right) = i$.

The next corollary tells us that α -modules of length α are fully transitive (see [6]). This, of course, is merely a reflection of the fact that modules of length $\leq \alpha$ behave in the α context exactly as modules without elements of infinite height in the classical situations.

Corollary 4. *If M is an α -module of length α , then every finite subset of M is contained in a countably generated direct summand.*

Proof. Let S be a finite subset of M . Then $S \subseteq T$ for some countably generated, α -pure submodule T of M . We may assume that T has length α . Then T is a direct sum of modules of length less than α . Consequently, T is contained in a direct summand K of T having length less than α . By the preceding corollary, K is a direct summand of M . □

For a limit ordinal α , an α -module M is an α -projective if

$$H_\alpha(\text{Ext}(M, M')) = 0$$

for all α -modules M' , that is, there exists a submodule N bounded by α such that M/N is totally projective, and an α -module M is an α -injective if

$$H_\alpha(\text{Ext}(M', M)) = 0$$

for all α -modules M' , that is, it is a direct summand of every α -module in which it occurs as an α -pure submodule.

To characterize the α -injective modules we must generalize the notion of a closed module. Mimicking [12], for any QTAG-module M , the submodules $\{H_k(M)\}_k$, $k = 0, 1, 2, \dots, \infty$ from a neighborhood system of zero, giving rise to h -topology. If k is replaced by an arbitrary limit ordinal less than or equal to α , then h -topology may be extended to α -topology, and all the definitions and results which hold for h -topology may be extended for α -topology. In α -topology, for any submodule N of M , the closure of N as $\bigcap_{\beta < \alpha} (N + H_\beta(M))$ denoted by \bar{N} .

Definition 3. We call a QTAG-module an α -closed module if it is the maximal closed submodule of its closure in the α -topology.

With the help of the above discussion, we are able to infer the following.

Proposition 4. *Let M be an α -closed α -module. Then M is an α -injective.*

Proof. We first show that $H_\alpha(\text{Ext}(T, M)) = 0$ for all α -modules T . Assume that M is an α -pure submodule of M' with $M'/M \cong T$ for all α -modules M' . Since α is a limit ordinal, it follows that $M' = H_\beta(M') + M$ for all $\beta < \alpha$. Therefore, if $y \in M'$, we can find for each $\beta < \alpha$ a $x_\beta \in M$ such that $y - x_\beta \in H_\beta(M')$. Moreover, we can assume that the exponent of x_β does not exceed that of y . Indeed, if y has exponent n , then

$$H_n(x'_\beta) \in H_{\beta+n}(M') \cap M = H_{\beta+n}(M),$$

where $d(\frac{x_\beta R}{x'_\beta R}) = n$ and $H_n(x'_\beta) = H_n(z'_\beta)$, where $d(\frac{x_\beta R}{x'_\beta R}) = d(\frac{z_\beta R}{z'_\beta R}) = n$ for some $z_\beta \in H_\beta(M)$. Then $\bar{x}_\beta = x_\beta - z_\beta$ has an exponent at most n and $y - \bar{x}_\beta \in H_\beta(M')$. But $\{x_\beta : \beta < \alpha\}$ is a chain in M with elements uniformly bounded in exponent and, therefore, converges to some $x \in M$. Hence

$$y - x \in \bigcap_{\beta < \alpha} H_\beta(M') = H_\alpha(M').$$

We conclude that $M' = M \oplus H_\alpha(M')$.

Now let M' be an arbitrary α -module and let B be an α -basic submodule of M' . We then have the exact sequence

$$H_\alpha(\text{Ext}(M'/B, M)) \longrightarrow H_\alpha(\text{Ext}(M', M)) \longrightarrow H_\alpha(\text{Ext}(B, M)).$$

The left-hand term of the above sequence vanishes since M'/B is isomorphic to a direct sum of copies of T and the right-hand term vanishes since B is an α -projective. Thus, $H_\alpha(\text{Ext}(M', M)) = 0$ and we conclude that M is an α -injective. \square

We can now show that there are enough α -injective modules and that an α -injective module is the sum of an α -closed module and an h -divisible module.

Theorem 3. *Let M be an α -module. Then M is an α -pure submodule of an α -injective module and M is an α -injective module if and only if M is the direct sum of an h -divisible module and an α -closed α -module.*

Proof. It is evident from Proposition 4 that the direct sum of an h -divisible module and an α -closed α -module is necessarily an α -injective. Next, we need the observation that every α -module M of length at most α can be imbedded as an α -pure submodule of an α -closed module $T_M(\alpha)$ such that $T_M(\alpha)/M$ is h -divisible. Indeed, $T_M(\alpha)$ may be taken as the maximal closed submodule of the closure of M

in the α -topology. It follows, by the same reasoning as in the proof of Theorem 1, that

$$T_M(\alpha)/H_\beta(T_M(\alpha)) \cong M/H_\beta(M)$$

for all $\beta < \alpha$, and therefore that $T_M(\alpha)$ is an α -module.

Now let M be an arbitrary α -module. Let D be a minimal h -divisible module containing $H_\alpha(M)$. Take P to be the amalgamated sum of M and D over $H_\alpha(M)$. Then $P = M' \oplus D$ where $M' \cong M/H_\alpha(M)$ and $M' \cap M$ is an α -high submodule of M . Also, P/M is h -divisible and

$$\text{Soc}(P) \subseteq \text{Soc}(M) + H_\beta(P)$$

for all $\beta < \alpha$. It follows that M is an α -pure submodule of P . By the transitivity of α -purity, M is an α -pure in the α -injective $T_{M'}(\alpha) \oplus D$.

Finally, assume that M is itself an α -injective and that we have it imbedded, as above, as an α -pure submodule of $\bar{P} = T_{M'}(\alpha) \oplus D$. Since M is an α -injective, $\bar{P} = M \oplus Q$ where $Q \cong \bar{P}/M$ is obviously h -divisible, since both P/M and \bar{P}/P are h -divisible. But then $Q \subseteq D$, and since $\text{Soc}(D) \subseteq H_\alpha(M)$, we conclude that $Q = 0$ and $M = T_{M'}(\alpha) \oplus D$. \square

Now we are in a position to prove the following result.

Theorem 4. *If M and M' are α -closed α -modules with the same Ulm invariants, then $M \cong M'$.*

Proof. Take B and B' to be α -basic submodules of M and M' , respectively. It is easily seen that B and B' have the same Ulm invariants as M and M' . Therefore, there is an isomorphism f of B onto B' . Since B is an α -pure submodule of M , we have the exact sequence

$$\text{Hom}(M, M') \rightarrow \text{Hom}(B, M') \rightarrow H_\alpha(\text{Ext}(M/B, M')) = 0$$

Thus, there is a homomorphism $f': M \rightarrow M'$ that extends f . Let $x \in \text{Ker } f'$ and assume that $x \neq 0$. Then x has some height $\beta < \alpha$ and we can write $x = y + z$ where $y \in B$ and $z \in H_{\beta+1}(M)$. But then x has height β and $f(y) = f'(y) = -f'(z)$ has height at least $\beta + 1$. This, however, is a contradiction, since f is an isomorphism of B onto B' and B' is an isotype submodule of M' . We conclude that $\text{Ker } f' = 0$. Then

$$f'(M)/B' = f'(M)/f'(B) \cong M/B$$

is h -divisible. Hence $f'(M)/B'$ is a direct summand of M'/B' , and since B' is an α -pure submodule of M' , it follows that $f'(M)$ is an α -pure submodule of M' . Since $f'(M) \cong M$ is an α -injective, we have a direct decomposition $M' = f'(M) \oplus L$ where $L \cong M'/f'(M)$ is h -divisible. But M' is h -reduced and therefore $L = 0$ and $f'(M) = M'$, that is, f' is an isomorphism of M onto M' . \square

3 α -large submodules of summable modules

If F is a fully invariant submodule of the α -module M , then F is called an α -large submodule of M if $M = B + F$ for all α -basic submodules B of M . This generalization of the concept of large submodule is studied in [12]. It is well-known that $H_\beta(M)$ is always an α -large submodule of M provided that $\beta < \alpha$ as well as $H_\beta(M) \subseteq L$ whenever L is an α -large submodule in M .

Likewise, It was proved that an h -reduced QTAG-module M of length $\geq \alpha$ contains a proper α -basic submodule B if and only if M is an α -module where α is cofinal with ω . Since ω_1 , the first uncountable limit ordinal, is not cofinal with ω , some additional clarifications are necessary. In fact, B is an ω_1 -basic submodule of M only when $B = 1$ or $B = M$, and so L is an ω_1 -large submodule of M uniquely when $L = M$ and either $L = 1$ or $L \neq 1$ and it can take different forms; for instance $L = H_\beta(M)$ where $\beta < \text{length of } M \leq \omega_1$.

In [12] it was seen that the properties of α -large submodules for $\alpha > \omega$ are not preserved in general by these of the QTAG-module and conversely; for instance the direct sum of countably generated modules. However, this is not the case for totally projective modules.

Theorem 5 ([6], [12]). *Let L be an α -large submodule of the QTAG-module M . Then L is totally projective if and only if M is totally projective.*

The main goal of this section is to strengthen the above assertion to a class of modules, called summable modules. It is evident that direct sum of countably generated modules are themselves summable. In [8] it was constructed a summable ω_1 -module need not be a direct sum of countably generated modules. So, the investigation of the discussed above theme for α -large submodules of summable ω_1 -modules will be of interest.

Now we have accumulated all the machinery necessary to prove the following.

Theorem 6. *Suppose that M is an ω_1 -module with an α -large submodule L for some ordinal α such that $\omega \leq \alpha \leq \omega_1$. Then M is summable if and only if L is summable.*

Proof. “ \Rightarrow ”. In virtue of [12] there is a countable limit ordinal $\tau \leq \alpha$ such that $H_\tau(M) = H_\omega(L)$. Moreover, $L/H_\omega(L) = L/H_\tau(M)$ is an α -large submodule of $M/H_\tau(M)$, where the latter quotient is totally projective by assumption. Therefore, Theorem 5 applies to deduce that $L/H_\omega(L)$ is totally projective, in fact, a direct sum of uniserial modules. That is why, some high submodule N of L is a direct sum of countably generated modules. Indeed, what suffices to show is that $N/H_\omega(N)$ is a direct sum uniserial modules because $H_\omega(N)$ is bounded. In order to do that, we observe that

$$(N + H_\omega(L))/H_\omega(L) \subseteq L/H_\omega(L)$$

is also a direct sum of uniserial modules as a submodule. But N is isotype in L , whence

$$(N + H_\omega(L))/H_\omega(L) \cong N/(N \cap H_\omega(L)) = N/H_\omega(L)$$

which substantiates our claim. On the other hand, M being summable yields that $H_\omega(L) = H_\tau(M)$ is summable, and we are done.

“ \Leftarrow ”. Same as above, $H_\tau(M) = H_\omega(L)$ for some countable limit ordinal $\tau \leq \alpha$. But L being summable implies that $H_\omega(L) = H_\tau(M)$ is summable. Likewise, $M/H_\tau(M)$ is totally projective of countable length, hence a direct sum of countably generated modules. Let N be a τ -high submodule of M . Since $H_\tau(N)$ is high in $H_\tau(M)$ one may write $H_\tau(M) = H_\tau(N) \oplus T$ for some submodule T , whence $H_\tau(M) = T$. Moreover,

$$\text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(H_\tau(M)) = \text{Soc}(N) \oplus \text{Soc}(T).$$

In fact

$$\begin{aligned} \text{Soc}(N) \cap \text{Soc}(T) &\subseteq N \cap T = N \cap (H_\tau(M) \cap T) \\ &= (N \cap H_\tau(M)) \cap T = H_\tau(N) \cap T = 0 \end{aligned}$$

because N is isotype in M . Consequently, there is a valued direct sum

$$\text{Soc}(H_\tau(M)) = H_\tau(N) \oplus \text{Soc}(T).$$

Even more,

$$\text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T)$$

is a valued direct sum, where T is a valued submodule of $H_\tau(M)$ with $\text{Soc}(T) = \text{Soc}(H_\tau(M))$. Indeed, if $x \in \text{Soc}(M)$ then

$$x \in \text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T).$$

Since $\text{Soc}(T) = \text{Soc}(H_\tau(M))$ and N is h -pure in M . It easily follows that

$$x \in N + H_\tau(M) + M = N \oplus T \oplus +M$$

because $H_\tau(M) = H_\tau(K) \oplus T$, and by induction the desired decomposition now follows.

If $y \in \text{Soc}(N)$ and $z \in \text{Soc}(T)$ then

$$H_M(y + z) = \min\{H_M(y), H_M(z)\}$$

since either

$$H_M(y) < \alpha \leq H_M(z) \quad \text{or} \quad H_\tau(M) = H_\tau(N) \oplus T$$

when $H_M(y) \geq \alpha$. Therefore, $H_\tau(M)$ is summable if and only if T has this property.

Next, observe that

$$N \cong N/\{0\} = N/H_\tau(N) = N/(N \cap H_\tau(M)) \cong (N + H_\tau(M))/H_\tau(M),$$

where the last quotient is obviously isotype in $M/H_\tau(M)$, and thus it is a direct sum of countably generated modules as well. It follows that N is a direct sum of countably generated modules. Furthermore, both T and N are summable. But $\text{Soc}(M) = \text{Soc}(N) \oplus \text{Soc}(T)$ is a valued direct sum and from this, our assertion follows directly by the definition of summability. \square

We close the paper with a problem as follows:

Problem 1. Does it follow that if both $H_\beta(M)$ and $M/H_\beta(M)$ are σ -summable modules (see [14]) for some ordinal β , then M is σ -summable?

For summable modules we refer to [7]. Notice also that it can be obtained some results in this aspect under certain limitations on β which depends on $n < \omega$.

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