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Abstract This chapter tackles the output feedback stabilization of a reaction-diffusion PDE in the presence of saturations applying to the command input as well as a finite number of its time derivatives. The control strategy consists of a finite dimensional observer and a finite-dimensional state-feedback. We derive LMI-based sufficient conditions that ensure the local exponential stability of the closed-loop system while providing an estimation of the domain of attraction. These LMI conditions are shown to be feasible provided the order of the observer is selected large enough. The stability analysis is performed by using Lyapunov's direct method while invoking sector conditions commonly used for the analysis of saturated finite-dimensional systems.

1 Introduction

Due to the inherent physical limitations of the actuators, saturation mechanisms are commonly encountered in the practical implementation of control laws. Such saturation mechanisms introduce stringent constraints on the control strategies [1]. It is worth noting that even for finite-dimensional linear time invariant (LTI) systems, such saturation mechanisms are very well-known for introducing strong nonlinear phenomena characterized by multiple equilibrium points and bounded domains of attraction [2]. Due to its practical importance, the topic of feedback stabilization of finite-dimensional LTI systems in the presence of input saturations has been intensively studied [3, 4]. One of the most fruitful approaches in this field relies on

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Lyapunov's direct method combined with the use of a generalized sector condition [3, Lem. 1.6] in order to derive sufficient linear matrix inequality (LMI) conditions ensuring the local stability of the closed-loop plant. A distinguished feature of this approach is that it is also employed to obtain an estimate of the domain of attraction of the origin.

We embrace in this chapter the topic of stabilization of infinite-dimensional systems, and particularly of partial differential equations (PDEs), in the presence of input saturation mechanisms. Among the first contributions in this field of research, such a problem was studied in [5, 6] with saturation mechanisms defined for control input functions evaluated in the norm of an abstract functional space (typically the space of square integrable functions $L^2(0,1)$). Since then, the stabilization of PDEs in the presence of saturation mechanisms has attracted much attention with a particular focus on pointwise saturation mechanisms which are, in general, the most relevant for practical applications. In this context, stabilization of wave and Korteweg-de Vries PDEs under cone-bounded feedback were reported in [7, 8, 9] using Lyapunov's direct method. The case of reaction-diffusion PDEs in the presence of control input constraints was studied in [10] using a model predictive control approach while singular perturbation techniques were reported in [11]. Also in the case of reaction-diffusion PDEs but using spectral reduction methods [12, 13, 14, 15, 16, 17], it was demonstrated in [18] how a state-feedback can be designed to achieve the local stabilization of the plant while deriving an explicit estimation of the domain of attraction using LMIs. In the same context, the possibility to achieve the local output feedback stabilization of a reaction-diffusion PDE using a either distributed or Dirichlet/Neumann boundary measurement, with estimation of the domain of attraction, was demonstrated in [19]. It is worth noting that for the state feedback setting studied in [18], the region of attraction constrains only a finite number of modes of the initial condition. This is essentially because the state-feedback setting allows to achieve the local exponential stabilization of the the first modes of the system while preserving the stability of the residual infinite-dimensional dynamics. See [18, Proposition 1] for a precise statement of this result. In contrast, the output feedback setting does not allow such a strong separation between the to-be-stabilized modes and the residual ones. This is because the measurement is built based on the contribution of all the modes of the system. In this case, the region of attraction imposes constraints on all the modes of the initial condition [19].

We extend in this chapter the results obtained in [19] for the local stabilization of reaction-diffusion PDEs in the presence of an input saturation to the case of saturations applying on the input as well as on a finite number of its time derivatives. Typically, a saturation on the time derivative of the control input can be used to model the limitations in the rate of change of the actuator output. Similarly, a saturation on the second time derivative of the input is employed to model a limit in the acceleration of change of the actuator output, etc. In this context, the output feedback control design strategy adopted in this chapter consists of a finite-dimensional observer-based controller [20, 21, 22, 23, 24, 25] that leverages a control architecture reported first in [21] along with a LMI-based approach introduced in [26] (see also [27]). We particularly adopt the enhanced and general procedures from [28, 29] which allow to design, in a

generic and systematic manner, finite-dimensional observer-based control strategies for general 1-D reaction-diffusion PDEs, with Dirichlet/Neumann/Robin boundary control and Dirichlet/Neumann boundary measurement, in a great variety of settings. These include regulation control [30], nonlinear boundary control [29], and stabilization in the presence of an arbitrarily long input [31], output [32] or state [33] delay. We leverage here similar procedures but in the context of the saturation of the control input u and its n_s first time derivatives $\dot{u}, \ldots, u^{(n_s)}$. Compared to [19] where only a saturation in position was considered, hence the control design was performed directly on the input u, the stabilization problem in the presence of saturations on the n_s first time derivatives of u requires to consider $u^{(n_s)}$ as an auxiliary input for control design. Therefore we perform the control design on the system composed of the original PDE augmented with n_s integral components.

The reminder of this chapter is organized as follows. A general background on the stabilization of finite-dimensional LTI systems in the presence of an input saturation is placed in Section 2. The control design problem addressed in this work is introduced in Section 3. The control design procedure and the resulting stability theorems are reported in Section 4. Some numerical examples and simulations illustrating the main theoretical results are presented in Section 5. Finally, some concluding remarks are formulated in Section 6.

Notation. The real vector spaces $\mathbb{R}^n$ are endowed with the usual Euclidean norm $\|x\| = \sqrt{x^\top x}$. The associated induced norms of matrices are also denoted by $\|\cdot\|$. For any $x, y \in \mathbb{R}^n$, $x \leq y$ means that each component of x is less than or equal to the corresponding component of y. For any $x \in \mathbb{R}^n$ we denote by $|x|$ the vector of $\mathbb{R}^n$ obtained by replacing each component of x by its absolute value. Considering any two vectors X and Y of arbitrary dimensions, we denoted by $\mathrm{col}(X, Y)$ the vector $[X^\top, Y^\top]^\top$. Given a vector ℓ, $\mathrm{diag}(\ell)$ denotes the diagonal matrix with the entries of ℓ on the diagonal. $L^2(0,1)$ stands for the space of square integrable functions on $(0,1)$ and is endowed with the usual inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$. The corresponding norm is denoted by $\|\cdot\|_{L^2}$. For any given integer $m \geq 1$, $H^m(0,1)$ stands for the m-order Sobolev space and is equipped with its usual norm $\|f\|_{H^m} = \left(\sum_{k=0}^m \|f^{(k)}\|_{L^2}^2\right)^{1/2}$. For any given symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \geq 0$ (resp. $P > 0$) means that P is positive semi-definite (resp. positive definite).

Let $(\phi_n)_{n\geq 1}$ be a Hilbert basis of $L^2(0,1)$. For any two integers $1 \leq N < M$, we define the operators of projection:

$$\begin{aligned} \pi_N : L^2(0,1) &\longrightarrow \mathbb{R}^N \\ f &\longmapsto \left[\langle f, \phi_1\rangle \ \ldots \ \langle f, \phi_N\rangle\right]^\top \end{aligned}$$

and

$$\begin{aligned} \pi_{N,M} : L^2(0,1) &\longrightarrow \mathbb{R}^{M-N} \\ f &\longmapsto \left[\langle f, \phi_{N+1}\rangle \ \ldots \ \langle f, \phi_M\rangle\right]^\top. \end{aligned}$$

We finally define

$$\begin{aligned}\mathcal{R}_N : L^2(0,1) &\longrightarrow L^2(0,1)\\ f &\longmapsto f - \sum_{n=1}^{N} \langle f, \phi_n \rangle \, \phi_n = \sum_{n \geq N+1} \langle f, \phi_n \rangle \, \phi_n.\end{aligned}$$

2 Background on the stabilization of finite-dimensional LTI systems in the presence of an input saturation

We start by reviewing in this section some classical results on the stabilization of finite-dimensional LTI systems in the presence of an input saturation. We refer the readers to [3, 4] for a deeper insight into this topic. Let us first introduce the scalar saturation map

$$\begin{aligned}\mathrm{sat}_l : \mathbb{R} &\longrightarrow \mathbb{R}\\ u &\longmapsto \mathrm{sign}(u) \min(|u|, l)\end{aligned}$$

where $l > 0$. Then, for any $\ell = (l_1, \dots, l_m) \in (\mathbb{R}_{>0})^m$, we define the vector saturation map

$$\begin{aligned}\mathrm{sat}_\ell : \mathbb{R}^m &\longrightarrow \mathbb{R}^m\\ (u_1, \dots, u_m) &\longmapsto (\mathrm{sat}_{l_1}(u_1), \dots, \mathrm{sat}_{l_m}(u_m)).\end{aligned}$$

With these notations, consider the system described by

$$\dot{z}(t) = Az(t) + B\mathrm{sat}_\ell(u(t)) \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Here $z(t) \in \mathbb{R}^n$ represents the state at time $t \geq 0$ while $u(t) \in \mathbb{R}^m$ stands for the control input. We assume that the pair (A, B) is stabilizable, i.e., there exists a feedback gain $K \in \mathbb{R}^{m\times n}$ so that $A + BK$ is Hurwitz. Hence, in the absence of saturation, the system described by $\dot{z}(t) = Az(t) + Bu(t)$ can be exponentially stabilized by setting

$$u(t) = Kz(t) \tag{2}$$

because the closed-loop dynamics now reads $\dot{z}(t) = (A + BK)z(t)$. In the case of an input saturation, the situation is more complex since the application of the feedback law (2) to (1) gives the closed-loop dynamics:

$$\dot{z}(t) = Az(t) + B\mathrm{sat}_\ell(Kz(t)). \tag{3}$$

In order to study the stability of (3), a relevant idea is to consider the saturated term $\mathrm{sat}_\ell(Kz(t))$ as a perturbation of the nominal term $Kz(t)$. The objective of this approach is to make explicitly appear the Hurwitz matrix $A + BK$ in the dynamics of the closed-loop system:

$$\begin{aligned}\dot{z}(t) &= (A+BK)z(t) + B\left\{\mathrm{sat}_\ell(Kz(t)) - Kz(t)\right\}\\ &= (A+BK)z(t) + B\phi_\ell(Kz(t))\end{aligned} \tag{4}$$

where we introduce the deadzone nonlinearity defined by

$$\begin{aligned}\phi_\ell : \mathbb{R}^m &\longrightarrow \mathbb{R}^m\\ v &\longmapsto \mathrm{sat}_\ell(v) - v.\end{aligned}$$

When $z(t)$, hence $Kz(t)$, is sufficiently small, the perturbation introduced by the saturation map is such that $\phi_\ell(Kz(t)) = 0$. In that case, the dynamics of the closed-loop system (4) reduced to $\dot{z}(t) = (A+BK)z(t)$, ensuring the local exponential stability of the closed-loop system in a neighborhood of the origin. However, in order to estimate the domain of attraction of the origin of (4), one needs to capture more precisely the impact of the perturbation term $B\phi_\ell(Kz(t))$ on the dynamics of (4). A very efficient approach to achieve this objective is to embed the deadzone nonlinearity ϕ_ℓ into a generalized sector condition. Such a sector condition must be suitable for Lyapunov's direct method in order to derive LMI-based sufficient stability conditions. In this chapter, we leverage the following generalized sector condition that is borrowed from [3, Lem. 1.6].

Lemma 1 *Let $\ell \in (\mathbb{R}_{>0})^m$ be given. For any $v, \omega \in \mathbb{R}^m$ such that $|v_i - \omega_i| \leq \ell_i$ for all $i \in \{1, \dots, m\}$ and for any diagonal positive definite matrix $T \in \mathbb{R}^{m\times m}$ we have $\phi_\ell(v)^\top T(\phi_\ell(v) + \omega) \leq 0$.*

Let us now explain how Lemma 1 can be used to derive an estimation of the domain of attraction of (4). To do so, consider the classical quadratic Lyapunov function candidate

$$V(z) = z^\top P z, \quad \forall z \in \mathbb{R}^n \tag{5}$$

for some matrix $P \succ 0$. The computation of the time-derivative of V along the system trajectories (4) reads

$$\begin{aligned}\dot{V} &= z^\top \left\{(A+BK)^\top P + P(A+BK)\right\} z + 2z^\top PB\phi_\ell(Kz)\\ &= \begin{bmatrix} z \\ \phi_\ell(Kz)\end{bmatrix}^\top \begin{bmatrix} (A+BK)^\top P + P(A+BK) & PB \\ * & 0\end{bmatrix}\begin{bmatrix} z \\ \phi_\ell(Kz)\end{bmatrix}.\end{aligned}$$

Here the symbol $*$ is used to denote a symmetric part. For any matrix $C \in \mathbb{R}^{m\times n}$ and any diagonal positive definite matrix $T \in \mathbb{R}^{m\times m}$, we obtain from Lemma 1 that

$$\forall z \in \mathbb{R}^n, \quad |(K-C)z| \leq \ell \;\Rightarrow\; \phi_\ell(Kz)^\top T(\phi_\ell(Kz) + Cz) \leq 0.$$

Combining the two latter results, we infer that, along the system trajectories (4), $|(K-C)z| \leq \ell$ gives

$$\dot{V} \leq \begin{bmatrix} z \\ \phi_\ell(Kz)\end{bmatrix}^\top \begin{bmatrix} (A+BK)^\top P + P(A+BK) & PB - C^\top T \\ * & -2T\end{bmatrix}\begin{bmatrix} z \\ \phi_\ell(Kz)\end{bmatrix}.$$

Assuming that

$$\Theta_1 = \begin{bmatrix} (A+BK)^\top P + P(A+BK) & PB - C^\top T \\ * & -2T \end{bmatrix} \prec 0,$$

we deduce the existence of $\alpha > 0$ so that $\Theta_1 \preceq -\alpha I$. Hence we have

$$\forall z \in \mathbb{R}^n, \quad |(K-C)z| \leq \ell \implies \dot{V} \leq -\alpha \|z\|^2 \leq -2\kappa V.$$

with $\kappa = \frac{\alpha}{2\lambda_{\max}(P)} > 0$. We now need to derive a set of initial conditions $\mathcal{E}$ containing the origin so that $\mathcal{E}$ is 1) positively invariant; and 2) included into the set $\{z \in \mathbb{R}^n : |(K-C)z| \leq \ell\}$. To do so we define

$$\mathcal{E} = \{z \in \mathbb{R}^n : V(z) = z^\top P z \leq 1\}$$

and we assume that

$$\Theta_2 = \begin{bmatrix} P & (K-C)^\top \\ * & \mathrm{diag}(\ell)^2 \end{bmatrix} \succeq 0.$$

Since $\mathrm{diag}(\ell) > 0$, the Schur complement shows that $P \succeq (K-C)^\top \mathrm{diag}(\ell)^{-2}(K-C)$. Hence $z \in \mathcal{E}$ implies that $|(K-C)z| \leq \ell$. Considering now a non zero initial condition $z_0 \in \mathcal{E} \setminus \{0\}$, we infer that $\dot{V}(z_0) \leq -\alpha\|z_0\| < 0$. From now, a simple contradiction argument shows that $z(t) \in \mathcal{E}$ for all $t \geq 0$ hence $\dot{V}(z(t)) \leq -2\kappa V(z(t))$. Overall, this reasoning shows that if there exist $P \succ 0$, $C \in \mathbb{R}^{m\times n}$, and a diagonal positive definite matrix $T \in \mathbb{R}^{m\times m}$ such that $\Theta_1 \prec 0$ and $\Theta_2 \succeq 0$, then for any initial condition $z_0 \in \mathcal{E}$, the trajectories of (3) exponentially converge to the origin.

Remark 1 It is worth noting that the constraints $\Theta_1 \prec 0$ and $\Theta_2 \succeq 0$ are always feasible for a suitable choice of the decision variables P, C, T. Indeed, setting $C = 0$, $P = \beta P_0$, and $T = \gamma I$ where $\beta, \gamma > 0$ and $P_0 \succ 0$ is the unique solution to the Lyapunov equation $(A+BK)^\top P_0 + P_0(A+BK) = -I$, one can see using Schur complement that $\Theta_2 \succeq 0$ is obtained provided $\beta > 0$ is selected large enough. Using the same argument, $\Theta_1 \prec 0$ can now be obtained provided $\gamma > 0$ is selected large enough.

Remark 2 In their original version, the constraints $\Theta_1 \prec 0$ and $\Theta_2 \succeq 0$ are nonlinear w.r.t. the decision variables. However, they can be equivalently reformulated as LMI constraints for which efficient solvers exist. See, e.g., [18, Sec. III.C].

3 Problem description and properties of Sturm-Liouville operators

We introduce in this section the control design problem for 1-D reaction-diffusion PDEs. These PDEs are strongly related to the class of Sturm-Liouville operators. The main properties of these operators that will be useful to perform the control design are presented in the second part of this section.

3.1 Problem description

We investigate the problem of feedback stabilization of the reaction-diffusion equation described by

$$z_t(t,x) = (p(x)z_x(t,x))_x - \tilde{q}(x)z(t,x) + b(x)\mathrm{sat}_{l_0}(u(t)) \tag{6a}$$
$$\cos(\theta_1)z(t,0) - \sin(\theta_1)z_x(t,0) = 0 \tag{6b}$$
$$\cos(\theta_2)z(t,1) + \sin(\theta_2)z_x(t,1) = 0 \tag{6c}$$
$$z(0,x) = z_0(x). \tag{6d}$$

Here $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in C^1([0,1])$ with $p > 0$, $\tilde{q} \in C^0([0,1])$, and $l_0 > 0$. The shape function $b \in L^2(0,1)$ represents the way the saturated scalar control input $u_{\mathrm{sat}}(t) = \mathrm{sat}_{l_0}(u(t)) \in \mathbb{R}$ acts on the system, $z_0 \in L^2(0,1)$ is the initial condition, and $z(t,\cdot) \in L^2(0,1)$ is the state of the reaction-diffusion PDE. The system output takes the form of the measurement $y(t) \in \mathbb{R}$ described by

$$y(t) = \int_0^1 c(x)z(t,x)\,\mathrm{d}x \tag{7}$$

where $c \in L^2(0,1)$.

Remark 3 The approach presented in this chapter can be extended in a straightforward manner to the case of multiple control inputs for which the PDE (6a) is replaced by

$$z_t(t,x) = (p(x)z_x(t,x))_x - \tilde{q}(x)z(t,x) + \sum_{k=1}^{m} b_k(x)\mathrm{sat}_{l_{k,0}}(u(t))$$

where the shape function $b_k \in L^2(0,1)$, for $k \in \{1,\dots,m\}$, represents the way the saturated scalar control input $\mathrm{sat}_{l_{k,0}}(u(t)) \in \mathbb{R}$ acts on the system. However, we focus the presentation on the single input case ($m = 1$) in order to ease the notations.

Let $n_s \in \mathbb{N}_{>0}$ and $l_1, \dots, l_{n_s} > 0$ be arbitrarily given. The control design objective is to achieve the local exponential stabilization of (6), with explicit estimation of the domain of attraction, while ensuring that

$$|u^{(i)}(t)| \leq l_i, \quad \forall t \geq 0, \quad \forall i \in \{1,\dots,n_s\}. \tag{8}$$

3.2 Properties of Sturm-Liouville operators

Reaction-diffusion PDEs such as described by (6) are strongly related to the so-called Sturm-Liouville operators. We gather in this subsection the key properties of the Sturm-Liouville operators which will be intensively used in the sequel for

both control design and stability analysis. Let $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in C^1([0,1])$, and $q \in C^0([0,1])$ with $p > 0$ and $q \geq 0$. We define the Sturm-Liouville operator by

$$\begin{array}{rccl} \mathcal{A} : & D(\mathcal{A}) & \longrightarrow & L^2(0,1) \\ & f & \longmapsto & -(pf')' + qf \end{array} \tag{9}$$

with the domain of the operator defined by

$$\begin{aligned} D(\mathcal{A}) = \{ f \in H^2(0,1) \, : \, & \cos(\theta_1) f(0) - \sin(\theta_1) f'(0) = 0 \\ & \cos(\theta_2) f(1) + \sin(\theta_2) f'(1) = 0 \}. \end{aligned}$$

Then it holds that the eigenvalues λ_n, $n \geq 1$, of $\mathcal{A}$ are simple, non negative, and form an increasing sequence with $\lambda_n \to +\infty$ as $n \to +\infty$. The corresponding unit eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis. The domain of the operator $\mathcal{A}$ is equivalently characterized by

$$D(\mathcal{A}) = \left\{ f \in L^2(0,1) \, : \, \sum_{n \geq 1} |\lambda_n|^2 |\langle f, \phi_n \rangle|^2 < +\infty \right\}.$$

Moreover we have $\mathcal{A} f = \sum_{n \geq 1} \lambda_n \langle f, \phi_n \rangle \phi_n$ for all $f \in D(\mathcal{A})$.

Let $p_*, p^*, q^* \in \mathbb{R}$ be such that $0 < p_* \leq p(x) \leq p^*$ and $0 \leq q(x) \leq q^*$ for all $x \in [0,1]$, then it holds

$$0 \leq \pi^2 (n-1)^2 p_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^*$$

for all $n \geq 1$ (see, e.g., [34]). Moreover if $p \in C^2([0,1])$, we have (see, e.g., [34]) that $\phi_n(\xi) = O(1)$ and $\phi_n'(\xi) = O(\sqrt{\lambda_n})$ as $n \to +\infty$ for any given $\xi \in [0,1]$.

If we further assume that $q > 0$, an integration by parts and the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$ imply the existence of constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{H^1}^2 \leq \sum_{n \geq 1} \lambda_n \langle f, \phi_n \rangle^2 = \langle \mathcal{A} f, f \rangle \leq C_2 \|f\|_{H^1}^2, \quad \forall f \in D(\mathcal{A}). \tag{10}$$

Inequalities (10) and the Riesz-spectral nature [35] of $\mathcal{A}$ imply that the series expansion $f = \sum_{n \geq 1} \langle f, \phi_n \rangle \phi_n$ holds in $H^2(0,1)$ norm for any $f \in D(\mathcal{A})$. Invoking the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$, we deduce in particular that $f(0) = \sum_{n \geq 1} \langle f, \phi_n \rangle \phi_n(0)$ and $f'(0) = \sum_{n \geq 1} \langle f, \phi_n \rangle \phi_n'(0)$.

4 Control design and main results

4.1 Auxiliary command input and spectral reduction

Recalling that the objective is to achieve the local exponential stabilization of (6) while satisfying the constraints (8) on the n_s first time derivatives of the command input u, we introduce the auxiliary control input $v(t) \in \mathbb{R}$ along with $u_1(t), \dots, u_{n_s-1}(t) \in \mathbb{R}$ as follows:

$$\begin{aligned} \dot{u}(t) &= \mathrm{sat}_{l_1}(u_1(t)), \\ \dot{u}_i(t) &= \mathrm{sat}_{l_{i+1}}(u_{i+1}(t)), \quad 1 \le i \le n_s - 2 \\ \dot{u}_{n_s-1}(t) &= \mathrm{sat}_{l_{n_s}}(v(t)). \end{aligned}$$

We define the short notation $u_{i,\mathrm{sat}}(t) = \mathrm{sat}_{l_i}(u_i(t))$. Hence introducing

$$\mathcal{V} = \begin{bmatrix} u \\ u_1 \\ \vdots \\ u_{n_s-1} \end{bmatrix} \in \mathbb{R}^{n_s}, \qquad J_{n_s} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n_s \times n_s}, \qquad F_{n_s} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n_s}$$

we infer that

$$\begin{aligned} \dot{\mathcal{V}}(t) &= J_{n_s} \mathrm{sat}_{(l_0,\dots,l_{n_s-1})}(\mathcal{V}(t)) + F_{n_s} \mathrm{sat}_{l_{n_s}}(v(t)) \\ &= J_{n_s} \mathcal{V}_{\mathrm{sat}}(t) + F_{n_s} v_{\mathrm{sat}}(t) \end{aligned} \tag{11}$$

where $\mathcal{V}_{\mathrm{sat}}(t) = \mathrm{sat}_{(l_0,\dots,l_{n_s-1})}(\mathcal{V}(t))$ and $v_{\mathrm{sat}}(t) = \mathrm{sat}_{l_{n_s}}(v(t))$.

We now introduce a function $q \in C^0([0,1])$ and a constant $q_c \in \mathbb{R}$ so that

$$\tilde{q} = q - q_c, \quad q \ge 0. \tag{12}$$

Therefore, the reaction-diffusion PDE described by (6) along with (11) can be equivalently represented under the abstract form:

$$z_t(t,\cdot) = \{-\mathcal{A} + q_c \mathrm{Id}_{L^2}\} z(t,\cdot) + b(x) u_{\mathrm{sat}}(t) \tag{13a}$$

$$\dot{\mathcal{V}}(t) = J_{n_s} \mathcal{V}_{\mathrm{sat}}(t) + F_{n_s} v_{\mathrm{sat}}(t) \tag{13b}$$

$$z(0,\cdot) = z_0 \tag{13c}$$

$$\mathcal{V}(0) = \mathcal{V}_0 = \begin{bmatrix} u_0 & u_{1,0} & \dots & u_{n_s-1,0} \end{bmatrix}^\top \tag{13d}$$

with the Sturm-Liouville operator $\mathcal{A}$ defined by (9) and where v is the auxiliary input to perform the control design. Introducing the coefficients of projection

$$z_n(t) = \langle z(t,\cdot), \phi_n \rangle, \quad b_n(t) = \langle b, \phi_n \rangle, \quad c_n(t) = \langle c, \phi_n \rangle,$$

the projection of (13), along with the measurement equation (7), into the Hilbert basis $(\phi_n)_n$ reads

$$\dot{z}_n(t) = (-\lambda_n + q_c) z_n(t) + b_n u_{\text{sat}}(t), \quad n \geq 1 \tag{14a}$$
$$\dot{\mathcal{V}}(t) = J_{n_s} \mathcal{V}_{\text{sat}}(t) + F_{n_s} v_{\text{sat}}(t) \tag{14b}$$
$$y(t) = \sum_{n \geq 1} c_n z_n(t) \tag{14c}$$

4.2 Control architecture

As in [19] dealing with the case of a saturation applying only on the control input but not on its time derivatives (i.e. $n_s = 0$), the control strategy takes the form of a finite-dimensional state-feedback coupled with a finite-dimensional observer. More precisely, let $\delta > 0$ be the desired exponential decay rate for the closed-loop system trajectories and let $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta$ for all $n \geq N_0 + 1$. For any given integer $N \geq N_0 + 1$, which stands for the dimension of the observer and that will be specified later, the control strategy is described by:

$$\begin{aligned} \dot{\hat{z}}_n(t) &= (-\lambda_n + q_c)\hat{z}_n(t) + b_n u_{\text{sat}}(t) \\ &\quad - L_n \left\{ \sum_{k=1}^{N} c_k \hat{z}_k(t) - y(t) \right\}, \quad 1 \leq n \leq N_0 \end{aligned} \tag{15a}$$
$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + b_n u_{\text{sat}}(t), \quad N_0 + 1 \leq n \leq N \tag{15b}$$
$$\dot{u}(t) = u_{1,\text{sat}}(t), \tag{15c}$$
$$\dot{u}_i(t) = u_{i+1,\text{sat}}(t), \quad 1 \leq i \leq n_s - 2 \tag{15d}$$
$$\dot{u}_{n_s-1}(t) = v_{\text{sat}}(t) \tag{15e}$$
$$v(t) = \sum_{k=1}^{N_0} k_{z,k} \hat{z}_k(t) + k_u u(t) + \sum_{l=1}^{n_s-1} k_{u,l} u_l(t) \tag{15f}$$

where $L_n \in \mathbb{R}$ and $k_{z,k}, k_u, k_{u,l} \in \mathbb{R}$ are the observer and feedback gains, respectively.

Remark 4 The idea of splitting the observer architecture, composed of one part with active correction of the error of estimation (15a) and one part without correction (15b), is borrowed from [21].

Introducing the error of observation $e_n = z_n - \hat{z}_n$, the residue of measurement $\zeta = \sum_{n \geq N+1} c_n z_n$, and the vectors

$$\hat{Z}^{N_0} = \begin{bmatrix} \hat{z}_1 \\ \vdots \\ \hat{z}_{N_0} \end{bmatrix}, \quad \hat{Z}^{N-N_0} = \begin{bmatrix} \hat{z}_{N_0+1} \\ \vdots \\ \hat{z}_N \end{bmatrix}, \quad E^{N_0} = \begin{bmatrix} e_1 \\ \vdots \\ e_{N_0} \end{bmatrix}, \quad E^{N-N_0} = \begin{bmatrix} e_{N_0+1} \\ \vdots \\ e_N \end{bmatrix},$$

we infer that

$$\dot{\hat{Z}}^{N_0} = A_0\hat{Z}^{N_0} + B_0E_1\mathcal{V}_{\text{sat}} + LC_0E^{N_0} + LC_1E^{N-N_0} + L\zeta \tag{16a}$$

$$\dot{\mathcal{V}} = J_{n_s}\mathcal{V}_{\text{sat}} + F_{n_s}v_{\text{sat}} \tag{16b}$$

$$\dot{E}^{N_0} = (A_0 - LC_0)E^{N_0} - LC_1E^{N-N_0} - L\zeta \tag{16c}$$

$$\dot{\hat{Z}}^{N-N_0} = A_1\hat{Z}^{N-N_0} + B_1E_1\mathcal{V}_{\text{sat}} \tag{16d}$$

$$\dot{E}^{N-N_0} = A_1E^{N-N_0} \tag{16e}$$

$$v = K_z\hat{Z}^{N_0} + K_u\mathcal{V} \tag{16f}$$

where $A_0 = \text{diag}(-\lambda_1 + q_c, \ldots, -\lambda_{N_0} + q_c)$, $A_1 = \text{diag}(-\lambda_{N_0+1} + q_c, \ldots, -\lambda_N + q_c)$, $B_0 = \begin{bmatrix} b_1 \ldots b_{N_0} \end{bmatrix}^\top$, $B_1 = \begin{bmatrix} b_{N_0+1} \ldots b_N \end{bmatrix}^\top$, $C_0 = \begin{bmatrix} c_1 \ldots c_{N_0} \end{bmatrix}$, $C_1 = \begin{bmatrix} c_{N_0+1} \ldots c_N \end{bmatrix}$, $E_1 = \begin{bmatrix} 1\ 0 \ldots 0 \end{bmatrix}$, $L = \begin{bmatrix} L_1 \ldots L_{N_0} \end{bmatrix}^\top$, $K_z = \begin{bmatrix} k_{z,1} \ldots k_{z,N_0} \end{bmatrix}$, and $K_u = \begin{bmatrix} k_u\ k_{u,1} \ldots k_{u,n_s-1} \end{bmatrix}$. Introducing the augmented vector

$$\hat{Z}_a^{N_0} = \text{col}\left(\hat{Z}^{N_0}, \mathcal{V}\right) \tag{17}$$

we obtain that

$$\dot{\hat{Z}}_a^{N_0} = \tilde{A}_0\hat{Z}_a^{N_0} + \tilde{B}_0\mathcal{V}_{\text{sat}} + \tilde{B}_1v_{\text{sat}} + \tilde{L}C_0E^{N_0} + \tilde{L}C_1E^{N-N_0} + \tilde{L}\zeta \tag{18a}$$

$$v = K\hat{Z}_a^{N_0} \tag{18b}$$

where

$$\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} B_0E_1 \\ J_{n_s} \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 0 \\ F_{n_s} \end{bmatrix}, \quad \tilde{L} = \begin{bmatrix} L \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_z\ K_u \end{bmatrix}.$$

Now, since $\mathcal{V}_{\text{sat}} = \mathcal{V} + \phi_{(l_0,\ldots,l_{n_s-1})}(\mathcal{V})$ and $v_{\text{sat}} = v + \phi_{l_{n_s}}(v)$, we have

$$\begin{aligned} \dot{\hat{Z}}_a^{N_0} &= \tilde{A}_1\hat{Z}_a^{N_0} + \tilde{B}_1v + \tilde{L}C_0E^{N_0} + \tilde{L}C_1E^{N-N_0} + \tilde{L}\zeta \\ &\quad + \tilde{B}_0\phi_{(l_0,\ldots,l_{n_s-1})}(\mathcal{V}) + \tilde{B}_1\phi_{l_{n_s}}(v) \\ &= (\tilde{A}_1 + \tilde{B}_1K)\hat{Z}_a^{N_0} + \tilde{L}C_0E^{N_0} + \tilde{L}C_1E^{N-N_0} + \tilde{L}\zeta + \tilde{B}\phi_\ell(\mathcal{V}_a) \end{aligned}$$

where

$$\mathcal{V}_a = \begin{bmatrix} \mathcal{V} \\ v \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} A_0 & B_0E_1 \\ 0 & J_{n_s} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_0\ \tilde{B}_1 \end{bmatrix}, \quad \ell = (l_0, \ldots, l_{n_s}).$$

Noting finally that $\mathcal{V} = \begin{bmatrix} 0 & I \end{bmatrix} \hat{Z}_a^{N_0}$ and from (16f) we infer that $\mathcal{V}_a = \tilde{K}\hat{Z}_a^{N_0}$ where $\tilde{K} = \begin{bmatrix} 0 & I \\ K_z & K_u \end{bmatrix}$. Therefore, we obtain that

$$\dot{\hat{Z}}_a^{N_0} = (\tilde{A}_1 + \tilde{B}_1 K)\hat{Z}_a^{N_0} + \tilde{L}C_0 E^{N_0} + \tilde{L}C_1 E^{N-N_0} + \tilde{L}\zeta + \tilde{B}\phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \tag{19a}$$

$$\dot{E}^{N_0} = (A_0 - LC_0)E^{N_0} - LC_1 E^{N-N_0} - L\zeta \tag{19b}$$

$$\dot{\hat{Z}}^{N-N_0} = A_1\hat{Z}^{N-N_0} + B_1E_1\begin{bmatrix} 0 & I \end{bmatrix}\hat{Z}_a^{N_0} + B_1E_1\begin{bmatrix} I & 0 \end{bmatrix}\phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \tag{19c}$$

$$\dot{E}^{N-N_0} = A_1 E^{N-N_0}. \tag{19d}$$

Introducing

$$X = \operatorname{col}\left(\hat{Z}_a^{N_0}, E^{N_0}, \hat{Z}^{N-N_0}, E^{N-N_0}\right) \tag{20}$$

we have

$$\dot{X} = FX + \mathcal{L}\zeta + \mathcal{L}_\phi\phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \tag{21}$$

where

$$F = \begin{bmatrix} \tilde{A}_1 + \tilde{B}_1K & \tilde{L}C_0 & 0 & \tilde{L}C_1 \\ 0 & A_0 - LC_0 & 0 & -LC_1 \\ B_1E_1\begin{bmatrix} 0 & I \end{bmatrix} & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} \tilde{L} \\ -L \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{L}_\phi = \begin{bmatrix} \tilde{B} \\ 0 \\ B_1E_1\begin{bmatrix} I & 0 \end{bmatrix} \\ 0 \end{bmatrix}.$$

Remark 5 Due to the structure of the matrix F, it is easy to see that $\operatorname{sp}_{\mathbb{C}}(F) = \operatorname{sp}_{\mathbb{C}}(\tilde{A}_1 + \tilde{B}_1K) \cup \operatorname{sp}_{\mathbb{C}}(A_0 - LC_0) \cup \operatorname{sp}_{\mathbb{C}}(A_1)$. Owing to its definition, the matrix A_1 is Hurwitz, consequently the matrix F is Hurwitz if and only if the matrices $\tilde{A}_1 + \tilde{B}_1K$ and $A_0 - LC_0$ are Hurwitz. Since A_0 is diagonal with simple eigenvalues, the pair (A_0, C_0) satisfies the Kalman condition if and only if $c_n \neq 0$ for all $1 \leq n \leq N_0$. Let us now focus on the pair $(\tilde{A}_1, \tilde{B}_1)$ for which we apply the Hautus test. Let[1] $\lambda \in \mathbb{R}$ and $x = \operatorname{col}(\alpha, \beta)$ with $\alpha \in \mathbb{R}^{N_0}$ and $\beta \in \mathbb{R}^{n_s}$ so that $x^\top\tilde{A}_1 = \lambda x^\top$ and $x^\top\tilde{B}_1 = 0$. This is equivalent to

$$\alpha^\top A_0 = \lambda\alpha^\top, \quad \alpha^\top B_0E_1 + \beta^\top J_{n_s} = \lambda\beta^\top, \quad \beta^\top F_{n_s} = 0. \tag{22}$$

The latter identity is equivalent to $\beta_{n_s} = 0$. Therefore, recalling that $E_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, the second identity now reads $\begin{bmatrix} \alpha^\top B_0 & \beta_1 & \dots & \beta_{n_s-1} \end{bmatrix} = \lambda\begin{bmatrix} \beta_1 & \dots & \beta_{n_s-1} & 0 \end{bmatrix}$. Hence $\beta = 0$ while $\alpha^\top A_0 = \lambda\alpha^\top$ and $\alpha^\top B_0 = 0$. We deduce that $(\tilde{A}_1, \tilde{B}_1)$ satisfies the Kalman condition if and only if (A_0, B_0) satisfies the Kalman condition. This latter condition holds if and only if $b_n \neq 0$ for all $1 \leq n \leq N_0$.

Remark 6 Since the E^{N-N_0} dynamics described by (19c) is stable (recall that the matrix A_1 is Hurwitz) and decoupled from the rest of the modes, see (19), one can perform the stability analysis in two parts. Indeed, one can first complete the stability analysis as performed in the next subsections based on a truncated model similar

[1] We restrict the study to real eigenvalues and eigenvectors owing to the fact that the eigenvalues of the matrix $\tilde{A}_1$ are all real.

to (21) but excluding the E^{N-N_0} dynamics. Then one can deduce the exponential stability of the full closed-loop system by invoking an exponential ISS argument for (19c). Such an approach leads to the derivation of reduced order LMIs compared to the ones presented in the next theorems. For sake of simplicity and conciseness, we focus the presentation on stability conditions derived from the full dynamics (21).

Finally, in preparation for the statement of the main results of this Chapter, we introduce a number of matrices that are useful to relate the different quantities previously introduced. Defining $E = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}$, we have

$$\hat{Z}_a^{N_0} = EX. \tag{23}$$

Introducing $\tilde{E} = \begin{bmatrix} 0_{1\times N_0} & 1 & 0_{1\times(n_s-1)} & 0_{1\times(2N-N_0)} \end{bmatrix}$ and $\tilde{G} = \begin{bmatrix} 1 & 0_{1\times n_s} \end{bmatrix}$, we also have

$$u = \tilde{E}X, \quad \phi_{l_0}(u) = \tilde{G}\phi_\ell(\tilde{K}\hat{Z}_a^{N_0}),$$

hence

$$u_{\text{sat}} = u + \phi_{l_0}(u) = \tilde{E}X + \tilde{G}\phi_\ell(\tilde{K}\hat{Z}_a^{N_0}). \tag{24}$$

4.3 Main stability results

We now study the local exponential stability, while estimating the domain of attraction, of the closed-loop system composed of the plant (6) with measured output (7) and the control law (15). In that case, note that the control design constraints (8) will be structurally achieved due to the introduction of the saturation mechanisms (15c-15e) that are embedded into the control architecture and which constrain the magnitude of the n_s first time derivatives of the control input u.

4.3.1 Stability in L^2 norm

We start by stating a stability result for PDE trajectories evaluated in L^2 norm.

Theorem 1 *Let $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in C^1([0,1])$ with $p > 0$, $\tilde{q} \in C^0([0,1])$, $b, c \in L^2(0,1)$, and $\ell = (l_0, \ldots, l_{n_s}) \in (\mathbb{R}_{>0})^{n_s+1}$. Let $q \in C^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (12) holds. Consider the reaction-diffusion system described by (6) with measured output (7). Let $N_0 \geq 1$ and $\delta > 0$ be given such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0+1$. Assume that $b_n \neq 0$ and $c_n \neq 0$ for all $1 \leq n \leq N_0$. Let $K \in \mathbb{R}^{1\times(N_0+n_s)}$ and $L \in \mathbb{R}^{N_0}$ be such that $\tilde{A}_1 + \tilde{B}_1 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \geq N_0 + 1$, assume that there exist a symmetric positive definite $P \in \mathbb{R}^{(2N+n_s)\times(2N+n_s)}$, $\alpha, \beta, \gamma, \mu, \kappa > 0$, a diagonal positive definite $T \in \mathbb{R}^{(n_s+1)\times(n_s+1)}$, and $C \in \mathbb{R}^{(n_s+1)\times(N_0+n_s)}$ such that*

$$\Theta_1(\kappa) \preceq 0, \quad \Theta_2 \succeq 0, \quad \Theta_3(\kappa) \preceq 0 \tag{25}$$

where

$$\Theta_1(\kappa) = \begin{bmatrix} \Theta_{1,1,1}(\kappa) & P\mathcal{L} & -E^\top C^\top T + P\mathcal{L}_\phi \\ * & -\beta & 0 \\ * & * & \alpha\gamma\|\mathcal{R}_N b\|_{L^2}^2 \tilde{G}^\top \tilde{G} - 2T \end{bmatrix}$$
$$\Theta_2 = \begin{bmatrix} P & E^\top(\tilde{K} - C)^\top \\ * & \mu \operatorname{diag}(\ell)^2 \end{bmatrix},$$
$$\Theta_3(\kappa) = 2\gamma \left\{ -\lambda_{N+1} + q_c + \kappa + \frac{1}{\alpha} \right\} + \beta\|\mathcal{R}_N c\|_{L^2}^2$$

with $\Theta_{1,1,1}(\kappa) = F^\top P + PF + 2\kappa P + \alpha\gamma\|\mathcal{R}_N b\|_{L^2}^2 \tilde{E}^\top \tilde{E}$. *Consider the block representation* $P = (P_{i,j})_{1\le i,j\le 5}$ *with dimensions that are compatible with (17) and (20) and define*

$$\mathcal{E}_1 = \Bigg\{ (z, \mathcal{V}) \in L^2(0,1) \times \mathbb{R}^{n_s} \ : \\ \begin{bmatrix} \mathcal{V} \\ \pi_{N_0} z \\ \pi_{N_0+1,N} z \end{bmatrix}^\top \begin{bmatrix} P_{2,2} & P_{2,3} & P_{2,5} \\ P_{3,2} & P_{3,3} & P_{3,5} \\ P_{5,2} & P_{5,3} & P_{5,5} \end{bmatrix} \begin{bmatrix} \mathcal{V} \\ \pi_{N_0} z \\ \pi_{N_0+1,N} z \end{bmatrix} + \gamma\|\mathcal{R}_N z\|_{L^2}^2 \le \frac{1}{\mu} \Bigg\}. \quad (26)$$

Then, considering the closed-loop system composed of the plant (6) with measured output (7) and the control law (15), there exists $M > 0$ *such that for any initial condition* $(z_0, \mathcal{V}_0) \in \mathcal{E}_1$ *and with a zero initial condition of the observer (i.e.,* $\hat{z}_n(0) = 0$ *for all* $1 \le n \le N$*), the system trajectory satisfies*

$$\|z(t,\cdot)\|_{L^2}^2 + \sum_{n=1}^{N} \hat{z}_n(t)^2 + \|\mathcal{V}(t)\|^2 \le M e^{-2\kappa t} \left(\|z_0\|_{L^2}^2 + \|\mathcal{V}_0\|^2 \right) \quad (27)$$

for all $t \ge 0$. *Moreover, for any fixed* $\kappa \in (0, \delta]$, *the constraints (25) are always feasible for N selected large enough.*

Proof Let the Lyapunov function candidate be defined by

$$V(X, z) = X^\top P X + \gamma \sum_{n \ge N+1} \langle z, \phi_n \rangle^2, \quad \forall X \in \mathbb{R}^{2N+n_s}, \ \forall z \in L^2(0,1).$$

The computation of the time derivative of V along the system trajectories (14) and (21) for classical solutions gives

$$\dot{V} + 2\kappa V = X^\top (F^\top P + PF + 2\kappa P) X + 2X^\top P\mathcal{L}\zeta + 2X^\top P\mathcal{L}_\phi \phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \\ + 2\gamma \sum_{n \ge N+1} (-\lambda_n + q_c + \kappa) z_n^2 + 2\gamma \sum_{n \ge N+1} z_n b_n u_{\mathrm{sat}}.$$

From (24) and using Young's inequality, we infer that

$$
\begin{aligned}
&2\sum_{n\geq N+1} z_n b_n u_{\rm sat}\\
&= 2\sum_{n\geq N+1} z_n b_n \tilde{E}X + 2\sum_{n\geq N+1} z_n b_n \tilde{G}\phi_\ell(\tilde{K}\hat{Z}_a^{N_0})\\
&\leq \frac{2}{\alpha}\sum_{n\geq N+1} z_n^2 + \alpha\|\mathcal{R}_N b\|_{L^2}^2\left\{X^\top\tilde{E}^\top\tilde{E}X + \phi_\ell(\tilde{K}\hat{Z}_a^{N_0})^\top\tilde{G}^\top\tilde{G}\phi_\ell(\tilde{K}\hat{Z}_a^{N_0})\right\}.
\end{aligned}
$$

Moreover, recalling that $\zeta = \sum_{n\geq N+1} c_n z_n$, we infer from Cauchy-Schwarz inequality that

$$
\zeta^2 \leq \|\mathcal{R}_N c\|_{L^2}^2 \sum_{n\geq N+1} z_n^2. \tag{28}
$$

The two latter estimates imply that

$$
\begin{aligned}
&\dot{V} + 2\kappa V\\
&\leq \begin{bmatrix} X\\ \zeta\\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0})\end{bmatrix}^\top \begin{bmatrix} \Theta_{1,1,1}(\kappa) & P\mathcal{L} & P\mathcal{L}_\phi\\ * & -\beta & 0\\ * & * & \alpha\gamma\|\mathcal{R}_N b\|_{L^2}^2\tilde{G}^\top\tilde{G}\end{bmatrix}\begin{bmatrix} X\\ \zeta\\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0})\end{bmatrix} + \sum_{n\geq N+1}\Gamma_n z_n^2
\end{aligned}
$$

where $\Gamma_n = 2\gamma\left(-\lambda_n + q_c + \kappa + \frac{1}{\alpha}\right) + \beta\|\mathcal{R}_N c\|_{L^2}^2$. Invoking now Lemma 1, we know that

$$
\forall \hat{Z}_a^{N_0}\in\mathbb{R}^{N_0+n_s},\quad |(\tilde{K}-C)\hat{Z}_a^{N_0}|\leq \ell \;\Rightarrow\; \phi_\ell(\tilde{K}\hat{Z}_a^{N_0})^\top T(\phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) + C\hat{Z}_a^{N_0}) \leq 0.
$$

Owing to (23), we obtain for any $X\in\mathbb{R}^{2N+n_s}$ with $|(\tilde{K}-C)EX|\leq\ell$ that

$$
\dot{V} + 2\kappa V \leq \begin{bmatrix} X\\ \zeta\\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0})\end{bmatrix}^\top \Theta_1(\kappa)\begin{bmatrix} X\\ \zeta\\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0})\end{bmatrix} + \sum_{n\geq N+1}\Gamma_n z_n^2.
$$

Since $\Gamma_n \leq \Theta_3(\kappa)$ for all $n\geq N+1$, we infer from (25) that

$$
\forall X\in\mathbb{R}^{2N+n_s},\quad |(\tilde{K}-C)EX|\leq\ell \;\Rightarrow\; \dot{V}+2\kappa V\leq 0.
$$

Invoking now the Schur complement, the condition $\Theta_2\geq 0$ from (25) implies that $P\geq\frac{1}{\mu}E^\top(\tilde{K}-C)^\top\mathrm{diag}(\ell)^{-2}(\tilde{K}-C)E$. Hence, for any $X\in\mathbb{R}^{2N+n_s}$ and $z\in L^2(0,1)$ so that $V(X,z)\leq 1/\mu$, we have $\|\mathrm{diag}(\ell)^{-1}(\tilde{K}-C)EX\|\leq 1$ hence $|(\tilde{K}-C)EX|\leq\ell$. Combining this result with the one of the previous paragraph, we deduce that

$$
\forall X\in\mathbb{R}^{2N+n_s},\ \forall z\in L^2(0,1),\quad V(X,z)\leq\frac{1}{\mu}\;\Rightarrow\;\dot{V}+2\kappa V\leq 0.
$$

Consider now an initial condition $(z_0,\mathcal{V}_0)\in\mathcal{E}_1$ with $z_0\in D(\mathcal{A})$. Recalling that the initial condition of the observer is set equal to zero we note that $V(X(0),z_0)\leq 1/\mu$. If $(z_0,\mathcal{V}_0)=0$, the trajectory is identically zero. Otherwise, for $(z_0,\mathcal{V}_0)\neq 0$ we have that $\dot{V}(X(0),z_0)\leq -2\kappa V(X(0),z_0) < 0$. We infer from a classical

contradiction argument that $V(X(t), z(t,\cdot)) \leq 1/\mu$ for all $t \geq 0$. This implies from the above developments that $\dot{V}(X(t), z(t,\cdot)) + 2\kappa V(X(t), z(t,\cdot)) \leq 0$ for all $t \geq 0$ hence $V(X(t), z(t,\cdot)) \leq e^{-2\kappa t} V(X(0), z_0)$. The exponential stability estimate (27) immediately follows for classical solutions. The result for any $(z_0, \mathcal{V}_0) \in \mathcal{E}_1$ is obtained by invoking the concept of mild solutions and by using a classical density argument [36, Thm. 6.1.2].

For any fixed $\kappa \in (0, \delta]$, the feasibility of the constraints (25) for N selected large enough can be obtained using the same arguments as the ones in [19]. □

4.3.2 Stability in H^1 norm

We also state a stability result for PDE trajectories evaluated in H^1 norm.

Theorem 2 *Let $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in C^1([0,1])$ with $p > 0$, $\tilde{q} \in C^0([0,1])$, $b, c \in L^2(0,1)$, and $\ell = (l_0, \dots, l_{n_s}) \in (\mathbb{R}_{>0})^{n_s+1}$. Let $q \in C^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (12) holds with the further constraint $q > 0$. Consider the reaction-diffusion system described by (6) with measured output (7). Let $N_0 \geq 1$ and $\delta > 0$ be given such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. Assume that $b_n \neq 0$ and $c_n \neq 0$ for all $1 \leq n \leq N_0$. Let $K \in \mathbb{R}^{1 \times (N_0 + n_s)}$ and $L \in \mathbb{R}^{N_0}$ be such that $\tilde{A}_1 + \tilde{B}_1 K$ and $A_0 - L C_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \geq N_0 + 1$, assume that there exist a symmetric positive definite $P \in \mathbb{R}^{(2N+n_s)\times(2N+n_s)}$, $\alpha > 1$, $\beta, \gamma, \mu, \kappa > 0$, a diagonal positive definite $T \in \mathbb{R}^{(n_s+1)\times(n_s+1)}$, and $C \in \mathbb{R}^{(n_s+1)\times(N_0+n_s)}$ such that*

$$\Theta_1(\kappa) \preceq 0, \quad \Theta_2 \succeq 0, \quad \Theta_3(\kappa) \leq 0 \tag{29}$$

where $\Theta_1(\kappa)$ and Θ_2 are defined as in Theorem 1 while

$$\Theta_3(\kappa) = 2\gamma \left\{ -\left(1 - \frac{1}{\alpha}\right) \lambda_{N+1} + q_c + \kappa \right\} + \frac{\beta \|\mathcal{R}_N c\|_{L^2}^2}{\lambda_{N+1}}.$$

Consider the block representation $P = (P_{i,j})_{1 \leq i,j \leq 5}$ with dimensions that are compatible with (17) and (20) and define

$$\mathcal{E}_2 = \Bigg\{ (z, \mathcal{V}) \in D(\mathcal{A}) \times \mathbb{R}^{n_s} \ : \\ \begin{bmatrix} \mathcal{V} \\ \pi_{N_0} z \\ \pi_{N_0+1,N} z \end{bmatrix}^\top \begin{bmatrix} P_{2,2} & P_{2,3} & P_{2,5} \\ P_{3,2} & P_{3,3} & P_{3,5} \\ P_{5,2} & P_{5,3} & P_{5,5} \end{bmatrix} \begin{bmatrix} \mathcal{V} \\ \pi_{N_0} z \\ \pi_{N_0+1,N} z \end{bmatrix} + \gamma \|\mathcal{R}_N \mathcal{A}^{1/2} z\|_{L^2}^2 \leq \frac{1}{\mu} \Bigg\}. \tag{30}$$

Then, considering the closed-loop system composed of the plant (6) with measured output (7) and the control law (15), there exists $M > 0$ such that for any initial condition $(z_0, \mathcal{V}_0) \in \mathcal{E}_2$ and with a zero initial condition of the observer (i.e., $\hat{z}_n(0) = 0$ for all $1 \leq n \leq N$), the system trajectory satisfies

$$\|z(t,\cdot)\|_{H^1}^2 + \sum_{n=1}^{N} \hat{z}_n(t)^2 + \|\mathcal{V}(t)\|^2 \leq Me^{-2\kappa t}\left(\|z_0\|_{H^1}^2 + \|\mathcal{V}_0\|^2\right) \tag{31}$$

for all $t \geq 0$. Moreover, for any fixed $\kappa \in (0, \delta]$, the constraints (29) are always feasible for N selected large enough.

Proof Let the Lyapunov function candidate be defined by

$$V(X,z) = X^\top PX + \gamma \sum_{n\geq N+1} \lambda_n \langle z, \phi_n\rangle^2, \quad \forall X \in \mathbb{R}^{2N+n_s},\ \forall z \in D(\mathcal{A}).$$

The series is connected to the H^1 norm of z via (10) because we assumed that $q > 0$. The computation of the time derivative of V along the system trajectories (14) and (21) for classical solutions gives

$$\begin{aligned}\dot{V} + 2\kappa V = {} & X^\top(F^\top P + PF + 2\kappa P)X + 2X^\top P\mathcal{L}\zeta + 2X^\top P\mathcal{L}_\phi \phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \\ & + 2\gamma \sum_{n\geq N+1} \lambda_n(-\lambda_n + q_c + \kappa) z_n^2 + 2\gamma \sum_{n\geq N+1} \lambda_n z_n b_n u_{\text{sat}}.\end{aligned}$$

From (24) and using Young's inequality, we infer that

$$\begin{aligned}& 2\sum_{n\geq N+1} \lambda_n z_n b_n u_{\text{sat}} \\ & = 2\sum_{n\geq N+1} \lambda_n z_n b_n \tilde{E}X + 2\sum_{n\geq N+1} \lambda_n z_n b_n \tilde{G}\phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \\ & \leq \frac{2}{\alpha}\sum_{n\geq N+1} \lambda_n^2 z_n^2 + \alpha\|\mathcal{R}_N b\|_{L^2}^2 \left\{X^\top\tilde{E}^\top\tilde{E}X + \phi_\ell(\tilde{K}\hat{Z}_a^{N_0})^\top\tilde{G}^\top\tilde{G}\phi_\ell(\tilde{K}\hat{Z}_a^{N_0})\right\}.\end{aligned}$$

Combining the above estimate and (28), we obtain that

$$\begin{aligned}& \dot{V} + 2\kappa V \\ & \leq \begin{bmatrix} X \\ \zeta \\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \end{bmatrix}^\top \begin{bmatrix} \Theta_{1,1,1}(\kappa) & P\mathcal{L} & P\mathcal{L}_\phi \\ * & -\beta & 0 \\ * & * & \alpha\gamma\|\mathcal{R}_N b\|_{L^2}^2 \tilde{G}^\top\tilde{G} \end{bmatrix} \begin{bmatrix} X \\ \zeta \\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \end{bmatrix} + \sum_{n\geq N+1} \lambda_n\Gamma_n z_n^2.\end{aligned}$$

where $\Gamma_n = 2\gamma\left\{-\left(1 - \frac{1}{\alpha}\right)\lambda_n + q_c + \kappa\right\} + \frac{\beta\|\mathcal{R}_N c\|_{L^2}^2}{\lambda_n}$. Using again Lemma 1, we have that

$$\forall \hat{Z}_a^{N_0} \in \mathbb{R}^{N_0+n_s}, \quad |(\tilde{K} - C)\hat{Z}_a^{N_0}| \leq \ell \ \Rightarrow\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0})^\top T(\phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) + C\hat{Z}_a^{N_0}) \leq 0.$$

In view of (23), we deduce that, for any $X \in \mathbb{R}^{2N+n_s}$ so that $|(\tilde{K} - C)EX| \leq \ell$,

$$\dot{V} + 2\kappa V \leq \begin{bmatrix} X \\ \zeta \\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \end{bmatrix}^\top \Theta_1(\kappa) \begin{bmatrix} X \\ \zeta \\ \phi_\ell(\tilde{K}\hat{Z}_a^{N_0}) \end{bmatrix} + \sum_{n\geq N+1} \lambda_n\Gamma_n z_n^2.$$

Since $\alpha > 1$, we note that $\Gamma_n \leq \Theta_3(\kappa) \leq 0$ for all $n \geq N+1$. Using in addition the condition $\Theta_1(\kappa) \preceq 0$, we deduce that $\dot{V} + 2\kappa V \leq 0$ for any $X \in \mathbb{R}^{2N+n_s}$ such that $|(K-C)EX| \leq \ell$. The reminder of the proof follows the same arguments that the ones reported in the proof of Theorem 1 while using (10) for the evaluation of the PDE trajectories in H^1 norm. □

4.4 Comments on the main results

4.4.1 Numerical considerations

The design constraints from Theorems 1 and 2 are nonlinear w.r.t. the decision variables. However, following the same approach that the one reported in [19], these constraints can be specialized into tractable LMI conditions. These latter LMI conditions remain feasible provided the order of the observer is selected to be large enough. We refer the reader to [19] for details.

In the context of Theorem 1 (a similar remark applies in the context of Theorem 2), the ellipsoid $\mathcal{E}_1$ defined by (26) is a subset of the region of attraction of the origin. A possible approach for enlarging $\mathcal{E}_1$ is discussed in [19] (see also [18]) and goes as follows. For a given symmetric positive definite matrix $R \in \mathbb{R}^{(N+n_s+1)\times(N+n_s+1)}$, let $r > 0$ be such that

$$\mathcal{P} \triangleq \begin{bmatrix} P_{2,2} & P_{2,3} & P_{2,5} & 0 \\ P_{3,2} & P_{3,3} & P_{3,5} & 0 \\ P_{5,2} & P_{5,3} & P_{5,5} & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \preceq \frac{r}{\mu} R \tag{32}$$

along with the constraints (25) of Theorem 1. Note that since $R > 0$, (32) can always be obtained by selecting $r > 0$ sufficiently large. Under (32) we directly infer that

$$\left\{ (z, \mathcal{V}) \in L^2(0,1) \times \mathbb{R}^{n_s} : \begin{bmatrix} \mathcal{V} \\ \pi_{N_0} z \\ \pi_{N_0+1,N} z \\ \|\mathcal{R}_N z\|_{L^2} \end{bmatrix}^\top R \begin{bmatrix} \mathcal{V} \\ \pi_{N_0} z \\ \pi_{N_0+1,N} z \\ \|\mathcal{R}_N z\|_{L^2} \end{bmatrix} \leq \frac{1}{r} \right\} \subset \mathcal{E}_1 \tag{33}$$

Hence one can try to maximize the size of the ellipsoid $\mathcal{E}_1$ by minimizing the parameter $r > 0$ under the constraints (25).

It is also worth noting that the degrees of freedom available in the choice of the feedback gain K and the observer gain L can be used to enlarge the (estimation of the) domain of attraction. Indeed, fixing K and L as in Theorems 1-2, one can first determine a dimension N of the observer such that the constraints (25) of Theorem 1 or the constraints (29) of Theorem 2 are feasible. Then one can try to maximize the domain of attraction by considering successively: 1) K, L as decision variables and P as fixed; 2) K, L as fixed and P as a decision variable.

4.4.2 Extension to boundary measurements

We have discussed in this chapter the local stabilization of the reaction-diffusion PDE (6) under the saturation constraints (8) for a measurement taking the form of a bounded operator described by (7). The extension to unbounded measurement operators such as Dirichlet boundary measurement $y_D(t) = z(t,0)$ and Neumann boundary measurement $y_N(t) = z_x(t,0)$ in the case $p \in C^2([0,1])$ can be achieved by adopting the procedures reported in [28, 29]. This was achieved in [19] in the case of the reaction-diffusion PDE (6) without constraints on the time derivatives of the control input. The extension to the setting of this chapter can be easily obtained by merging the approach reported in [19] with the arguments of the previous sections. Using arguments from [28, 29], the approach can also be extended to boundary control provided the saturation function is sufficiently smooth for well-posedness issues.

5 Numerical illustration

We propose in this section a numerical illustration of Theorems 1 and 2. We consider the reaction-diffusion PDE (6) with Dirichlet boundary conditions ($\theta_1 = \theta_2 = 0$), $p = 1$, $\tilde{q} = -10$, $b(x) = 1_{[1/10,3/10]}(x)$, and $c(x) = 1_{[7/10,9/10]}(x)$. These parameters yield an open-loop unstable plant.

We consider the case of the saturation of the command input u and its first time derivative ($n_s = 1$) with saturation levels set as $l_0 = l_1 = 1$. The feedback and observer gains are set as $K = \begin{bmatrix} -4.3489 & -1.8304 \end{bmatrix}$ and $L = 31.3729$, respectively.

The constraints (25) of Theorem 1 and (29) of Theorem 2 are found feasible for $N = 3$, ensuring the exponential stability of the system trajectories in L^2 and H^1 norms.

In order to estimate the domain of attraction of the origin, we consider the additional constraint (32) with $R = \mathrm{diag}(10, I, 0.005)$. The results obtained when trying to minimize the value of the parameter $r > 0$ for different dimensions N of the observer are summarized in Tab 1.

Dimension of the observer	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$
L^2 norm	$r = 4.8310$	$r = 3.7142$	$r = 3.5604$	$r = 3.4689$	$r = 3.4238$
H^1 norm	$r = 4.7536$	$r = 3.5864$	$r = 3.5352$	$r = 3.4695$	$r = 3.4237$

Table 1 Results of the estimation of the domain of attraction in L^2 and H^1 norms for different dimensions N of the observer.

For numerical simulations, we consider the dimension of the observer $N = 6$ and we set the initial condition of the reaction-diffusion PDE as $z_0(x) = 2.9x(x-1)$ while $\mathcal{V}_0 = 0$. From the above numerical results, $(z_0, \mathcal{V}_0)$ is checked to be an element of $\mathcal{E}_1 \cap \mathcal{E}_2$, hence is in the domain of attraction of the origin for system trajectories

evaluated in both L^2 and H^1 norms. As shown in Figs. 1-2, the system trajectory converges exponentially to zero in both L^2 and H^1 norms, confirming the theoretical predictions of Theorems 1 and 2. Finally, Fig. 3 shows that this exponential decrease of the system trajectory is obtained despite significant saturations in both position and rate of change of the command input u.

6 Conclusion

This chapter discussed the output feedback stabilization of reaction-diffusion PDEs in the presence of saturations of the command input and its time derivatives. We provide in the process an estimation of the domain of attraction. The presented results extend the approach reported in [19] dealing with the case of a sole input saturation. While the presentation focused on the case of measurements taking the form of a bounded observation operator, the results presented in this chapter can be combined with the approach described in [19] in order to tackle unbounded observation operators. This includes the cases of Dirchlet and Neumann boundary measurements.

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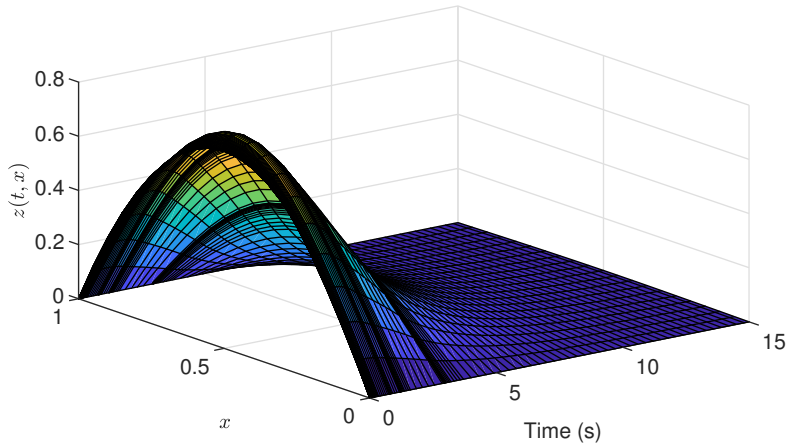
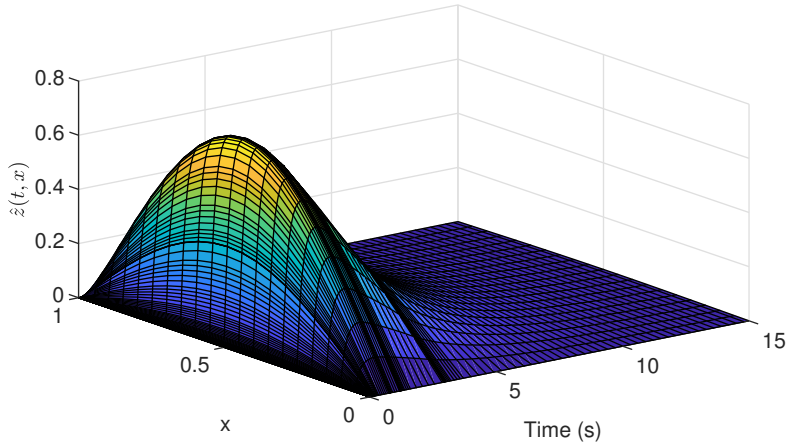
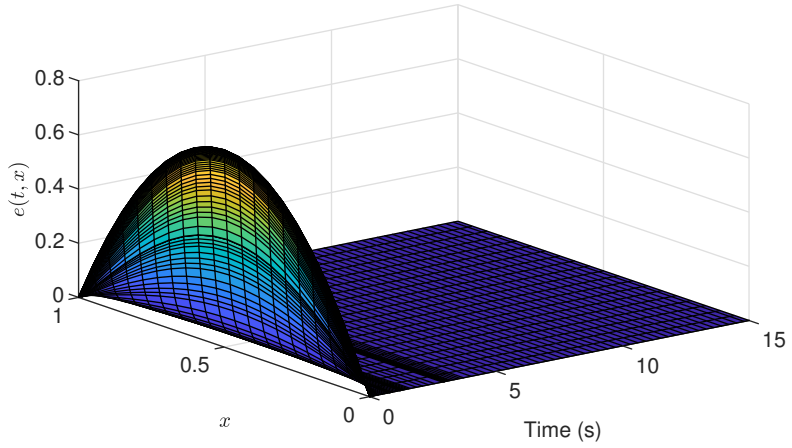


(a) State $z(t,x)$



(b) Estimation $\hat{z}(t,x) = \sum_{n=1}^{N} \hat{z}_n(t)\phi_n(x)$



(c) Observation error $z(t,x) - \hat{z}(t,x)$

Fig. 1 Temporal behavior of the closed-loop system

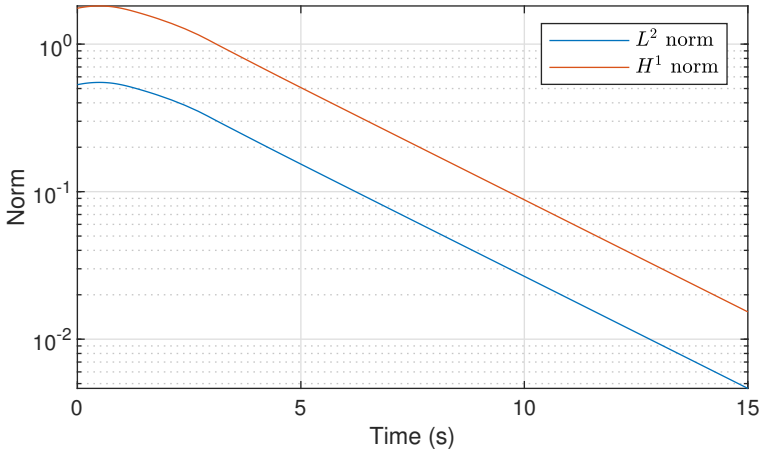


Fig. 2 Temporal behavior of the PDE trajectories in L^2 and H^1 norms

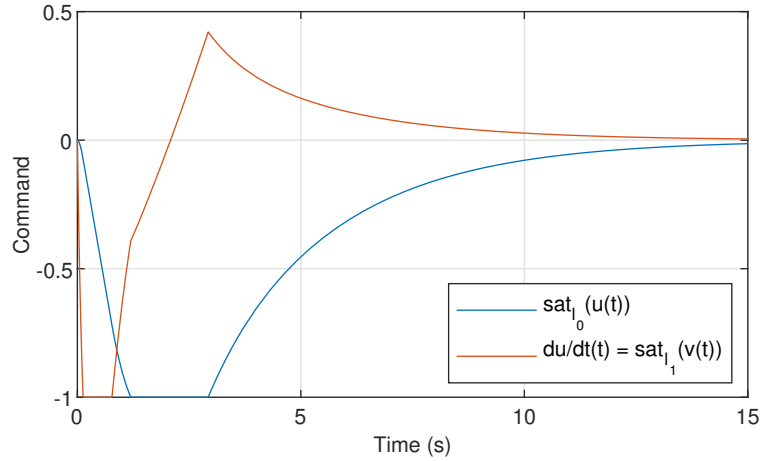


Fig. 3 Temporal behavior of the command input

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