



**HAL**  
open science

## A $\sigma_3$ condition for arbitrarily partitionable graphs

Julien Bensmail

► **To cite this version:**

Julien Bensmail. A  $\sigma_3$  condition for arbitrarily partitionable graphs. *Discussiones Mathematicae Graph Theory*, 2024, 44 (2), pp.755-776. 10.7151/dmgt.2471 . hal-03665116v2

**HAL Id: hal-03665116**

**<https://hal.science/hal-03665116v2>**

Submitted on 28 Jul 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A $\sigma_3$ condition for arbitrarily partitionable graphs

Julien Bensmail<sup>a</sup>

<sup>a</sup>Université Côte d’Azur, CNRS, Inria, I3S, France

---

## Abstract

A graph  $G$  of order  $n$  is arbitrarily partitionable (AP for short) if, for every partition  $(\lambda_1, \dots, \lambda_p)$  of  $n$ , there is a partition  $(V_1, \dots, V_p)$  of  $V(G)$  such that  $G[V_i]$  is a connected graph of order  $\lambda_i$  for every  $i \in \{1, \dots, p\}$ . Several aspects of AP graphs have been investigated to date, including their connection to Hamiltonian graphs and traceable graphs. Every traceable graph (and, thus, Hamiltonian graph) is indeed known to be AP, and a line of research on AP graphs is thus about weakening, to APness, known sufficient conditions for graphs to be Hamiltonian or traceable.

In this work, we provide a sufficient condition for APness involving the parameter  $\bar{\sigma}_3$ , where, for a given graph  $G$ , the parameter  $\bar{\sigma}_3(G)$  is defined as the minimum value of  $d(u) + d(v) + d(w) - |N(u) \cap N(v) \cap N(w)|$  for a set  $\{u, v, w\}$  of three pairwise independent vertices  $u, v$ , and  $w$  of  $G$ . Flandrin, Jung, and Li proved that any graph  $G$  of order  $n$  is Hamiltonian provided  $G$  is 2-connected and  $\bar{\sigma}_3(G) \geq n$ , and traceable provided  $\bar{\sigma}_3(G) \geq n - 1$ . Unfortunately, we exhibit examples showing that having  $\bar{\sigma}_3(G) \geq n - 2$  is not a guarantee for  $G$  to be AP. However, we prove that  $G$  is AP provided  $G$  is 2-connected,  $\bar{\sigma}_3(G) \geq n - 2$ , and  $G$  has a perfect matching or quasi-perfect matching.

*Keywords:* arbitrarily partitionable graph; partition into connected subgraphs;  $\sigma_3$  condition; Hamiltonian graph; traceable graph.

---

## 1. Introduction

This paper deals with so-called **arbitrarily partitionable graphs**, which are defined formally as follows. Let  $G$  be an  $n$ -graph, *i.e.*, a graph of order  $n$ . Let also  $\pi = (\lambda_1, \dots, \lambda_p)$  be an  $n$ -partition, *i.e.*, a partition of  $n$  (that is,  $\lambda_1 + \dots + \lambda_p = n$ ). We say that  $\pi$  is *realisable* in  $G$  if  $V(G)$  can be partitioned into  $p$  parts  $V_1, \dots, V_p$  such that  $G[V_i]$  is a connected graph of order  $\lambda_i$  for every  $i \in \{1, \dots, p\}$ , and we call  $(V_1, \dots, V_p)$  a *realisation* (of  $\pi$  in  $G$ ). Finally,  $G$  is called *arbitrarily partitionable* (AP for short) if all  $n$ -partitions are realisable in  $G$ . In other words,  $G$  is AP if we can partition  $G$  into arbitrarily many connected subgraphs, regardless of their requested orders.

AP graphs were introduced independently by Barth, Baudon, and Puech and Horňák and Woźniak [1, 10] in early 2000s, but the problem of partitioning graphs into connected subgraphs has been attracting attention since at least the 1970s, recall for instance the influential Győri-Lovász Theorem [8, 12]. Since then, quite some aspects of AP graphs have been investigated in the literature, including structural aspects, algorithmic questions, numerous variants, and others. For a recent survey on the topic, we refer the interested reader to, *e.g.*, [2].

AP graphs are objects of interest for numerous reasons. A notable one is the fact that they sort of lie in-between two other important types of graph notions, being perfect matchings and Hamiltonian cycles. Note indeed that, if  $n$  is even, then any realisation of the  $n$ -partition  $(2, \dots, 2)$  in an  $n$ -graph forms a perfect matching, while, if  $n$  is odd, then any realisation of the  $n$ -partition  $(2, \dots, 2, 1)$  forms a quasi-perfect matching. Remark also that adding edges to an AP graph cannot make it lose its APness, and, from this simple observation, we get that any graph spanned by an AP graph is AP itself. Since paths are obviously AP, this implies that Hamiltonian graphs and even traceable graphs (graphs having a Hamiltonian path) are AP. Thus, having a perfect matching or a quasi-perfect matching is a necessary condition for a graph to be AP, while being traceable is a sufficient condition for a graph to be AP.

These simple thoughts lead to one of the most interesting lines of research regarding AP graphs, which is on the weakening, to APness, of sufficient conditions for Hamiltonicity and traceability. The general idea is that one can consider any of the numerous sufficient conditions for a graph to

be Hamiltonian or traceable, and investigate whether it can be weakened to a sufficient condition for APness. This line of research was initiated by Marczyk in [13], in which he focused on the parameter  $\sigma_2$ , being defined as

$$\sigma_2(G) = \min \{d(u) + d(v) : u \text{ and } v \text{ are independent vertices of } G\}$$

for any graph  $G$ . By a famous result of Ore [16], recall indeed that any connected  $n$ -graph  $G$  is Hamiltonian whenever  $\sigma_2(G) \geq n$ , while  $G$  is traceable whenever  $\sigma_2(G) \geq n - 1$ . In [13], Marczyk proved that  $G$  is AP provided  $\sigma_2(G) \geq n - 2$  and  $\alpha(G) \leq \lceil n/2 \rceil$  (that is, provided  $G$  has a perfect matching or a quasi-perfect matching). Later on, in [9, 14], Marczyk, together with Horňák, Schiermeyer, and Woźniak, improved this sufficient condition to  $G$  satisfying only  $\sigma_2(G) \geq n - 5$  and additional conditions (such as the previous condition on  $\alpha(G)$ , and also conditions on  $n$ ).

Sufficient conditions for Hamiltonicity and traceability being surely one of the most investigated and rich areas of graph theory, the work of Marczyk opened the way to many promising investigations in that line. For instance, in [11], Kalinowski, Piłśniak, Schiermeyer, and Woźniak, motivated by similar conditions for Hamiltonicity and traceability, exhibited sufficient conditions in terms of size (number of edges) guaranteeing a graph is AP. In [3], Bensmail and Li considered several sufficient conditions for Hamiltonicity and traceability (covering squares of graphs and forbidden structures) and proved that some of these weaken to APness, while some do not.

The current work takes place in the line of the previous investigations, and, more particularly, relates to the initiating work of Marczyk on the topic. Precisely, we deal with the parameter  $\sigma_3$  being defined as

$$\sigma_3(G) = \min \{d(u) + d(v) + d(w) : u, v, \text{ and } w \text{ are pairwise independent vertices of } G\}$$

for a given graph  $G$ . As reported in several surveys on the topic (such as, *e.g.*, [7, 18]), this parameter  $\sigma_3$  has indeed been employed, together with other graph properties (such as connectivity and claw-freeness), to express sufficient conditions for graphs to be Hamiltonian, and sometimes more or less than that. As far as we are aware, this was also considered in the context of AP graphs. Indeed, at the occasion of the 18th workshop ‘‘Colourings, Independence and Domination’’ (CID 2013) led in 2013, Brandt was invited to give a lecture, entitled ‘‘Finding Vertex Decompositions in Dense Graphs’’, during which he announced the following result:

**Theorem 1.1** (Brandt). *If  $G$  is a connected  $n$ -graph with  $\sigma_3(G) \geq n$ , then  $G$  is AP if and only if  $G$  admits a perfect matching or a quasi-perfect matching.*

Unfortunately, it seems that Brandt has never published a proof of Theorem 1.1, and the only remains of his investigations to date are the title of his talk (mentioned in the preface [4]), as well as the corresponding abstract, which can be found online and which we report word for word in concluding Section 4 for the sake of keeping track of it. On the positive side, this abstract provides hints regarding the main lines of Brandt’s proof of Theorem 1.1.

Our investigations in the current work were inspired by Brandt’s result, and our original intent was to provide a result that would be sort of reminiscent of Theorem 1.1. As a result, we deal with a parameter that is very close to the parameter  $\sigma_3$ , as it relies on a slightly different notion of degree sum for triples of pairwise independent vertices. That is, for a graph  $G$  and any three pairwise distinct independent vertices  $u, v$ , and  $w$  of  $G$ , set

$$d^*(u, v, w) = d(u) + d(v) + d(w) - |N(u) \cap N(v) \cap N(w)|.$$

Now, define

$$\bar{\sigma}_3(G) = \min \{d^*(u, v, w) : u, v, \text{ and } w \text{ are pairwise independent vertices of } G\}.$$

Note that  $\bar{\sigma}_3(G) \leq \sigma_3(G)$  for every graph  $G$ . It turns out that this parameter  $\bar{\sigma}_3$  has also been used to express sufficient conditions for Hamiltonicity and traceability. In particular, Flandrin, Jung, and Li proved in [6] that any 2-connected  $n$ -graph  $G$  with  $\bar{\sigma}_3(G) \geq n$  is Hamiltonian, while any connected  $n$ -graph  $G$  with  $\bar{\sigma}_3(G) \geq n - 1$  is traceable. [18] reports that the latter bound for traceability was proved to also hold when  $\bar{\sigma}_3(G) \geq n - 2$  provided  $G$  fulfils additional strong conditions (involving 2-connectivity and claw-freeness).

The current work is dedicated to proving the next result, which provides a new sufficient condition, involving the parameter  $\bar{\sigma}_3$ , for a graph to be AP:

**Theorem 1.2.** *If  $G$  is a 2-connected  $n$ -graph with  $\overline{\sigma}_3(G) \geq n - 2$ , then  $G$  is AP if and only if  $G$  admits a perfect matching or a quasi-perfect matching.*

We searched the literature for a while, and, as far as we can tell, it seems that Theorem 1.2 does not follow immediately from existing results on the parameter  $\overline{\sigma}_3$  (such as conditions implying traceability). We have to add, however, that the literature on the topic is quite vast, rich, and that several old works mentioned, *e.g.*, in surveys seem to be impossible to access easily nowadays. Thus, it is hard to be fully certain we have not missed something. However, even if a previous result implying Theorem 1.2 was to exist, we believe the proof we give would remain of interest, as it relies on understanding how APness and a parameter such as  $\overline{\sigma}_3$  relate in general.

We start by introducing useful material in Section 2, before focusing on proving Theorem 1.2 in Section 3. We discuss our result in Section 4, in which we also discuss Brandt's Theorem 1.1.

## 2. Preliminaries

In our proof of Theorem 1.2, we will often deal with particular situations in which the following notations and terminology will be useful. Let  $G$  be a graph, and let  $P = v_1 \dots v_p$  be a path of  $G$ . For a vertex  $v_i \in V(P)$ , we sometimes denote by  $v_i^-$  and  $v_i^+$  the vertices  $v_{i-1}$  and  $v_{i+1}$  (assuming they exist). For a set  $S \subseteq V(P)$  of vertices of  $P$ , we denote by  $S^-$  and  $S^+$  the sets  $\{v_i^- : v_i \in S\}$  and  $\{v_i^+ : v_i \in S\}$ , respectively. These notions will be particularly useful when dealing with vertices of  $G$  having all their neighbours on  $P$ . Namely, for a  $u \in V(G)$  (we might have  $u \in V(P)$ ) such that  $N(u) \subseteq V(P)$ , we will have to deal with the sets  $N(u)^-$  and  $N(u)^+$  in many occasions.

In the proof of Theorem 1.2, we will sometimes obtain a realisation in  $G$  of some partition by first ‘‘splitting’’ (sometimes only part of)  $P$  into parts of certain size containing consecutive vertices. Note that this is indeed a legitimate way to proceed, as any set of consecutive vertices of  $P$  induces a connected graph. In such occasions, we will process the vertices of  $P$  one by one, from one end-vertex to the other (either as going from  $v_1$  to  $v_p$ , or conversely), and *pick* parts as going along. Assuming we want to split  $P$  into connected parts of size  $\lambda_1, \dots, \lambda_p$ , where  $\lambda_1 + \dots + \lambda_p \leq |V(P)|$ , the first  $\lambda_1$  vertices of  $P$  (following the considered ordering) will form one part, the next  $\lambda_2$  vertices will form another part, and so on. When building one of these parts, we will sometimes have to *grab* a vertex  $u$  of  $G$  from some  $v_i$  on the way, meaning that, at the very moment where  $v_i$  is added to some part, assuming  $v_i u$  is an edge and the current part still misses vertices to reach the desired size, then we add  $u$  to the current part before resuming the picking process with the vertex succeeding  $v_i$  in the considered ordering. As will be apparent later on, we will use this picking process in a rather flexible way (sometimes different from what we described above), and, in every such occasion, we will make sure to describe the exact process properly to avoid any ambiguity.

Still about  $P$ , note that the way we denote its consecutive vertices  $v_1, \dots, v_p$  above yields a virtual orientation of  $P$  from the first vertex ( $v_1$ ) to the last ( $v_p$ ), where  $v_i$  is considered to be a *predecessor* of  $v_j$  if  $i < j$ , while  $v_j$  is considered to be a *successor* of  $v_i$  in such situations. Note that these notions also make sense for the parts we pick during the picking process above. One has to be careful, however, that the notions of preceding and succeeding parts are not with respect to the virtual orientation of  $P$ , but rather with the ordering in which the consecutive vertices of  $P$  as considered through the process. In particular, when picking parts as going from  $v_p$  to  $v_1$ , note that these notions are reversed compared to those we get as going from  $v_1$  to  $v_p$ .

We finish off with a useful result on partitioning graphs with long paths into connected subgraphs. Assume  $G$  is an  $n$ -graph, and let  $\pi$  be an  $n$ -partition. We denote by  $\text{sp}(\pi)$  the *spectrum* of  $\pi$ , being the set of distinct element values that appear in  $\pi$ . An important and crucial fact is that if  $P$  is long enough, then all  $n$ -partitions with sufficiently many distinct element values are realisable in  $G$ . This is captured in the following result of Ravoux:

**Theorem 2.1** (Ravoux [17]). *If  $G$  is a connected  $n$ -graph with a path of length  $n - \alpha$ , then every  $n$ -partition  $\pi$  with  $|\text{sp}(\pi)| \geq \alpha$  is realisable in  $G$ .*

Before we proceed to the proof of Theorem 1.2 in the next section, we first recall some easy properties of longest paths in connected graphs.

**Lemma 2.2.** *Let  $P = v_1 \dots v_p$  be a longest path in a connected graph  $G$ , and set  $R = V(G) \setminus V(P)$ . Then the following items hold for every vertex  $u \in R$ .*

- (1)  $\{u, v_1, v_p\}$  is a stable set.
- (2)  $d(v_1) = |N(v_1)^-|$  and  $d(v_p) = |N(v_p)^+|$ .
- (3)  $N_P(u) \subseteq V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$ .
- (4)  $N_P(u) \cap N_P(u)^- = N_P(u) \cap N_P(u)^+ = \emptyset$ .
- (5) If  $G[R]$  has at least one edge, then:
  - (5a)  $N(v_1)^- \cap N(v_p)^+ = \emptyset$ ;
  - (5b)  $d_P(u) + d(v_1) + d(v_p) \leq p$ .

*Proof.* (1) follows from the fact that having an edge joining two vertices in  $\{u, v_1, v_p\}$  would lead, because  $G$  is connected, to deducing a path of  $G$  longer than  $P$ , a contradiction. (2) follows directly from the fact that  $v_1$  and  $v_p$ , because  $P$  is a longest path of  $G$ , have all their neighbours on  $P$ . (3) is because if  $u$  had a neighbour  $v_i$  with  $v_i \in N(v_1)^-$  (or  $v_i \in N(v_p)^+$ ), then  $uv_i \dots v_1 v_{i+1} \dots v_p$  (or  $uv_i \dots v_p v_{i-1} \dots v_1$ ) would be a path of  $G$  longer than  $P$ , a contradiction. (4) is because if  $u$  had two neighbours  $v_i$  and  $v_{i+1}$  that are consecutive on  $P$ , then  $v_1 \dots v_i u v_{i+1} \dots v_p$  would be a path of  $G$  longer than  $P$ , another contradiction.

Assume now  $G[R]$  has at least one edge. First, note that if there was some  $v_i \in N(v_1)^- \cap N(v_p)^+$ , then  $v_1 \dots v_{i-1} v_p \dots v_{i+1} v_1$  would be a cycle of length  $p-1$ , and, from the facts that  $G$  is connected and that  $G[R]$  contains edges, we would come up with a path of  $G$  longer than  $P$ , a contradiction. Thus, such a  $v_i$  cannot exist, which proves (5a). Now, since (3) and (5a) hold, note that we have  $d_P(u) \leq p - d(v_1) - d(v_p)$ . Thus  $d_P(u) + d(v_1) + d(v_p) \leq p$ , and (5b) also holds.  $\square$

### 3. Proof of Theorem 1.2

We start by establishing facts on the structure of 2-connected  $n$ -graphs  $G$  with  $\overline{\sigma}_3(G) \geq n-2$ .

**Lemma 3.1.** *Let  $P = v_1 \dots v_p$  be a longest path in a 2-connected  $n$ -graph  $G$  with  $\overline{\sigma}_3(G) \geq n-2$ , and set  $R = V(G) \setminus V(P)$ . If  $G$  is not traceable, i.e.,  $p < n$ , then the following items hold.*

- (1)  $R$  is a stable set.
- (2)  $|R| \leq 2$ .
- (3) If  $R = \{u_1, u_2\}$ , then:
  - (3a)  $|N(u_1) \cap N(u_2)| \geq 2$ ;
  - (3b) denoting by  $v_{z_1}$  and  $v_{z_2}$  the first and last vertices of  $N(u_1) \cup N(u_2)$  on  $P$ , respectively, and setting  $Z = \{v_{z_1}, \dots, v_{z_2}\}$ , we have

$$|N_Z(v_1) \cup N(u_1) \cup N(u_2)| > 1 + \frac{z_2 - z_1}{2}.$$

*Proof.* Set  $r = |R|$ . Regarding (1), assume, towards a contradiction, that  $G[R]$  has edges. Let  $u \in R$  be a vertex incident to edges of  $G[R]$ . Then

$$d(u) + d(v_1) + d(v_p) - |N(u) \cap N(v_1) \cap N(v_p)| \geq \overline{\sigma}_3(G) \geq n-2 = p+r-2.$$

However, by Lemma 2.2(5b), we have

$$d_P(u) + d(v_1) + d(v_p) \leq p.$$

Since  $d(u) = d_P(u) + d_R(u)$ , it follows that

$$d_R(u) \geq r-2 + |N(u) \cap N(v_1) \cap N(v_p)|.$$

Now suppose  $d_R(u) = r - 2$ . Then  $N(u) \cap N(v_1) \cap N(v_p) = \emptyset$  and  $d_P(u) + d(v_1) + d(v_p) = p$ . Thus, by Lemma 2.2(2) and (5a), we have

$$d_P(u) = p - (|N(v_1)^-| + |N(v_p)^+|) = p - |N(v_1)^- \cup N(v_p)^+|.$$

From Lemma 2.2(3), it thus follows that every vertex of  $P$  that is not a neighbour of  $u$  lies in  $N(v_1)^- \cup N(v_p)^+$ . Now, because  $G$  is connected, we can further assume that there is a  $v_i$  such that  $uv_i$  is an edge. Recall that  $v_i \notin \{v_1, v_2, v_{p-1}, v_p\}$ , as otherwise, because  $u$  is incident to edges of  $G[R]$ , we could find a path of  $G$  longer than  $P$ . Thus,  $v_{i-2}$  and  $v_{i+2}$  exist. Also,  $uv_{i-1}$  and  $uv_{i+1}$  cannot be edges by Lemma 2.2(4). So, by earlier arguments, we have that  $v_{i-1}$  and  $v_{i+1}$  belong both to  $N(v_1)^- \cup N(v_p)^+$ .

- If  $v_{i-1}$  and  $v_{i+1}$  belong to distinct of  $N(v_1)^-$  and  $N(v_p)^+$ , then either  $v_{i-1} \in N(v_1)^-$  and  $v_{i+1} \in N(v_p)^+$ , or  $v_{i-1} \in N(v_p)^+$  and  $v_{i+1} \in N(v_1)^-$ . In the first case, note that  $v_i \in N(v_1) \cap N(v_p) \cap N(u)$ , which is not allowed. In the second case, we have  $v_{i-2} \in N(v_p)$  and  $v_{i+2} \in N(v_1)$ , in which case we can find a path of  $G$  longer than  $P$ , a contradiction. One such path first follows  $v_1 \dots v_{i-2} v_p \dots v_i$ , then goes to  $u$ , and then traverses edges of  $G[R]$ .
- If  $v_{i-1}$  and  $v_{i+1}$  belong both to  $N(v_1)^-$  or  $N(v_p)^+$ , then, assuming, without loss of generality, they belong both to  $N(v_1)^-$ , we have  $v_{i-1}, v_{i+1} \in N(v_1)^-$ , and thus  $v_i, v_{i+2} \in N(v_1)$ . Again, a path longer than  $P$  can be deduced: start, *e.g.*, by following  $v_p \dots v_{i+2} v_1 \dots v_i$ , then go to  $u$ , and lastly traverse edges of  $G[R]$ .

In both cases we thus come up with a path of  $G$  longer than  $P$ , which is a contradiction. So  $R$  must be a stable set, which proves (1).

We now prove (2). Set  $R = \{u_1, \dots, u_r\}$ . Since  $R$  is stable due to (1), every vertex  $u \in R$  verifies  $N(u) \subset \{v_1, \dots, v_p\}$ . Remind also that, by Lemma 2.2(1), vertex  $u$  can be adjacent to neither  $v_1$  nor  $v_p$ . Thus the vertices in  $R$  have all their neighbours in  $\{v_2, \dots, v_{p-1}\}$ .

Note that if  $u_i$  and  $u_j$  are two distinct vertices of  $R$ , then  $|N(u_j) \cap N(u_i)^+| \leq 1$ . Indeed, assume there are two distinct vertices  $v_{i_1+1}, v_{i_2+1} \in N(u_j) \cap N(u_i)^+$ , where  $i_1 < i_2$ . Then  $v_{i_1}, v_{i_2} \in N(u_i)$ , and  $v_1 \dots v_{i_1} u_i v_{i_2} \dots v_{i_1+1} u_j v_{i_2+1} \dots v_p$  is a path of  $G$  of order  $p + 2$ , a contradiction to the maximality of  $P$ . Similarly,  $|N(u_j) \cap N(u_i)^-| \leq 1$ .

Towards a contradiction to (2), assume  $r \geq 3$ . So,  $u_1, u_2$ , and  $u_3$  exist. We define the following metric  $\rho$  for the vertices in  $\{v_1, \dots, v_{p-1}\}$ . We set  $\rho(1) = -1$ . Then, for every  $i \in \{2, \dots, p-1\}$  such that  $\rho(i-1)$  is defined, we define  $\rho(i)$  as follows:

- if  $v_i$  is adjacent to no vertex in  $\{u_1, u_2, u_3\}$ , then  $\rho(i) = \rho(i-1) - 1$ ;
- if  $v_i$  is adjacent to exactly one vertex in  $\{u_1, u_2, u_3\}$ , then  $\rho(i) = \rho(i-1)$ ;
- otherwise, if  $v_i$  is adjacent to two or three vertices in  $\{u_1, u_2, u_3\}$ , then  $\rho(i) = \rho(i-1) + 1$ .

In other words,  $\rho(i)$  refers to the difference  $X - Y$ , where  $X$  denotes the number of edges incident to  $v_1, \dots, v_i$  going to vertices in  $\{u_1, u_2, u_3\}$  (with the subtlety that for a common neighbour of  $u_1, u_2$ , and  $u_3$  we count only two edges) while  $Y$  denotes the number of vertices in  $\{v_1, \dots, v_i\}$ , which is precisely  $i$ . Note that, indeed, we have  $\rho(1) = -1$  since  $v_1$  is incident to none of  $u_1, u_2$ , and  $u_3$ .

Note that, for every  $i \geq 3$ , we cannot have  $\rho(i) = \rho(i-1) + 1 = \rho(i-2) + 2$  because two consecutive  $v_i$ 's cannot both be adjacent to the same  $u_i$  by the maximality of  $P$  (recall Lemma 2.2(4)). Thus, for the  $\rho(i)$ 's to grow by 2, some  $v_i$  must be adjacent to two or three vertices in  $\{u_1, u_2, u_3\}$ , some next  $v_i$ 's, say  $v_{i+1}, \dots, v_{j-1}$ , must then be adjacent to exactly one vertex in  $\{u_1, u_2, u_3\}$  each, before the next vertex,  $v_j$ , is adjacent to two vertices in  $\{u_1, u_2, u_3\}$ . Actually, the first  $v_i$  in that sequence cannot be adjacent to the three vertices in  $\{u_1, u_2, u_3\}$ , as, by an earlier remark, this would make it impossible for  $v_{i+1}$  to have neighbours in  $\{u_1, u_2, u_3\}$ . Thus, this first  $v_i$  must be adjacent to exactly two vertices in  $\{u_1, u_2, u_3\}$ . More generally, note that if some  $v_i$  neighbours all three vertices in  $\{u_1, u_2, u_3\}$ , then both  $v_{i-1}$  and  $v_{i+1}$  cannot have neighbours in  $\{u_1, u_2, u_3\}$ .

We claim that  $\rho(i) \leq 1$  for every  $i \in \{1, \dots, p-1\}$ . Assume this is wrong, that is we have  $\rho(i) = 2$  for some  $i \in \{2, \dots, p-1\}$ . Because  $\rho(1) = -1$ , by the remarks above there must be an  $x$  such that  $\rho(x-1) = -1$  and  $\rho(x) = 0$ , a  $y > x + 1$  such that  $\rho(x) = \rho(x+1) = \dots = \rho(y-1) = 0$

and  $\rho(y) = 1$ , and a  $z > y + 1$  such that  $\rho(y) = \rho(y + 1) = \dots = \rho(z - 1) = 1$  and  $\rho(z) = 2$ . Still by the remarks above, we have that  $v_x, v_y$ , and  $v_z$  must each be adjacent to exactly two vertices in  $\{u_1, u_2, u_3\}$ , while all of  $v_{x+1}, \dots, v_{y-1}, v_{y+1}, \dots, v_{z-1}$  must be adjacent to exactly one vertex in  $\{u_1, u_2, u_3\}$  each. Without loss of generality, assume  $v_x$  is adjacent to  $u_1$  and  $u_2$ . Then  $v_{x+1}$  must be adjacent to  $u_3$ . Also, there cannot be another  $i$  such that  $v_i \in N(u_1) \cup N(u_2)$  and  $v_{i+1} \in N(u_3)$ , by an earlier remark. This implies that  $v_y$  cannot be adjacent to  $u_3$ , and  $v_y$  must thus be adjacent to  $u_1$  and  $u_2$ , while  $v_{y-1}$  must be adjacent to  $u_3$ . This implies there cannot be another  $i$  such that  $v_i \in N(u_3)$  and  $v_{i+1} \in N(u_1) \cup N(u_2)$ . Now we get to a contradiction whatever two vertices in  $\{u_1, u_2, u_3\}$  are adjacent to  $v_z$ , regardless of what other vertex in  $\{u_1, u_2, u_3\}$  is adjacent to  $v_{z-1}$ .

Thus, we must have  $\rho(p - 1) \leq 1$ , which means that the number of edges incident to  $v_1, \dots, v_p$  going to vertices in  $\{u_1, u_2, u_3\}$  (counting only two edges for the  $v_i$ 's being common neighbours of  $u_1, u_2$ , and  $u_3$ ) is at most  $p$ . In other words,  $d^*(u_1, u_2, u_3) \leq p$ . Since  $n = p + r$  and  $r \geq 3$ , we have  $p \leq n - 3$ , and thus  $\overline{\sigma}_3(G) \leq d^*(u_1, u_2, u_3) < n - 2$ , a contradiction to the fact that  $\overline{\sigma}_3(G) \geq n - 2$ . Thus,  $r \leq 2$ , and (2) holds.

We now prove (3a), given that (1) and (2) hold. Set  $A = N(u_1) \setminus N(u_2)$ ,  $B = N(u_2) \setminus N(u_1)$ , and  $C = N(u_1) \cap N(u_2)$ . Then,  $N(u_1) \cup N(u_2) = A \cup B \cup C$ . Set also  $|A| = a$ ,  $|B| = b$ , and  $|C| = c$ . Recall that  $v_1$  cannot have neighbours in  $\{u_1, u_2\}$ ,  $A^+$ ,  $B^+$ , or  $C^+$  (by Lemma 2.2(1) and (3)). Therefore, we have

$$d(v_1) \leq p - 1 - a - b - c.$$

Since  $d(u_1) + d(u_2) = a + b + 2c$ , it follows that

$$d^*(v_1, u_1, u_2) \leq p - 1 + c - |N(v_1) \cap C|.$$

However,  $d^*(v_1, u_1, u_2) \geq n - 2 = p$ , and hence

$$|N(v_1) \cap C| \leq c - 1.$$

Thus,  $c \geq 1$ . Now suppose  $c = 1$ . Then  $|N(v_1) \cap C| = 0$ . Thus, assuming  $v_x$  is the vertex in  $C$ , we have  $v_x \notin N(v_1)$ . However,  $v_{x-1}$  lies neither in  $N(u_1)$  nor in  $N(u_2)$  by Lemma 2.2(4). Thus,  $v_x \notin A^+ \cup B^+ \cup C^+$ , and thus

$$d(v_1) \leq p - 2 - a - b - c = p - 3 - a - b,$$

which implies that

$$d^*(v_1, u_1, u_2) \leq (p - 3 - a - b) + (a + b + 2) = p - 1 = n - 3,$$

contradicting that  $\overline{\sigma}_3(G) \geq n - 2$ . Thus, (3a) holds.

Let us now focus on proving (3b). Set  $X = \{v_1, \dots, v_{z_1-1}\}$  and  $Y = \{v_{z_2+1}, \dots, v_p\}$ . In the calculation of  $d^*(v_1, u_1, u_2)$ , every vertex in  $N(v_1) \cup N(u_1) \cup N(u_2)$  is counted at most twice, and hence

$$d^*(v_1, u_1, u_2) \leq |N_X(v_1)| + 2|N_Z(v_1) \cup N(u_1) \cup N(u_2)| + |N_Y(v_1)|.$$

Now, if  $|N_Z(v_1) \cup N(u_1) \cup N(u_2)| \leq 1 + \frac{z_2 - z_1}{2}$ , then

$$d^*(v_1, u_1, u_2) \leq (z_1 - 2) + (2 + z_2 - z_1) + (p - 1 - z_2) = p - 1 = n - 3,$$

which contradicts that  $\overline{\sigma}_3(G) \geq n - 2$ . Thus, (3b) holds.  $\square$

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* As mentioned in Section 1, admitting a perfect matching or a quasi-perfect matching is a necessary condition for a graph to be AP. We can thus focus on proving the other direction of the equivalence. Let  $G$  be a 2-connected  $n$ -graph with  $\overline{\sigma}_3(G) \geq n - 2$  such that  $G$  admits a perfect matching or a quasi-perfect matching. Our goal is to prove that  $G$  is AP, that is, that every  $n$ -partition  $\pi$  is realisable in  $G$ .

Let  $P$  be a longest path of  $G$ , and set  $R = V(G) \setminus V(P)$ . Then, by Lemma 3.1,  $R$  is a stable set with cardinality  $r \leq 2$ . Since  $P$  is a path of length  $n - (r + 1) \geq n - 3$ , it follows from Theorem 2.1 that  $\pi$  admits a realisation in  $G$  if  $|\text{sp}(\pi)| \geq r + 1$ . Thus, we may assume that  $|\text{sp}(\pi)| \leq r$ .

We consider various scenarios depending on  $r$ ,  $\pi$ , and neighbours of vertices in  $R$ . In each possible case, we describe a picking process yielding a realisation of  $\pi$  in  $G$ , or arrive at contradictions to the fact that  $\bar{\sigma}_3(G) \geq n - 2$ .

**Case 1.**  $r = 2$  and  $|\text{sp}(\pi)| = 2$ .

Set  $R = \{u_1, u_2\}$ . By Lemma 3.1(3a),  $u_1$  and  $u_2$  have at least two common neighbours on  $P$ , which we denote by  $v_x$  and  $v_y$ , where  $v_x$  can be assumed to be their common neighbour with the lowest index, while  $v_y$  can be assumed to be that with the largest index. Then,  $x < y$ .

Set  $\pi = (\alpha, \dots, \alpha, \beta, \dots, \beta)$ , where  $\alpha < \beta$ . We start by applying the picking process, described in Section 2, along  $P$ , as going from  $v_1$  to  $v_p$ , picking parts of size  $\alpha$  first (as many such parts as indicated by  $\pi$ ), and, once no more parts of size  $\alpha$  must be picked, then picking parts of size  $\beta$  (because  $|V(P)| = n - 2$ , the process, if led entirely, would actually end up with a part of size  $\beta - 2$ ). Once  $v_x$  is added to a (possibly partial) part, we pause the process, and analyse the situation we have reached. Several scenarios can occur, where throughout what follows, an  $\alpha$ -part (or  $\beta$ -part) refers to a part that is intended to have cardinality  $\alpha$  (or  $\beta$ ) through the picking process.

**Case 1.1.**  $v_x$  is the  $i$ th vertex of an  $\alpha$ -part with  $i \leq \alpha - 2$ , or of a  $\beta$ -part with  $i \leq \beta - 2$ .

In this case, we grab  $u_1$  and  $u_2$  from  $v_x$  before resuming the process (taking into account that the current part has two vertices more). Once the whole process finishes, we end up with a realisation of  $\pi$  in  $G$ . Note in particular that there is a part containing  $v_x$ ,  $u_1$ , and  $u_2$ , which induces a connected graph due to the edges  $v_x u_1$  and  $v_x u_2$ .

**Case 1.2.**  $v_x$  is the  $(\alpha - 1)$ th vertex of an  $\alpha$ -part.

Here, we grab  $u_1$  and  $u_2$  from  $v_x$ , to turn the current part into a part of size  $\alpha + 1 \leq \beta$ . We then resume the process along  $v_{x+1} \dots v_p$  so that this part of size  $\alpha + 1$  is complemented to a part of size  $\beta$ , before picking the remaining parts arbitrarily. Eventually, a realisation of  $\pi$  in  $G$  results.

**Case 1.3.**  $v_x$  is the  $\alpha$ th vertex of an  $\alpha$ -part.

**Case 1.3.1.**  $\beta \geq \alpha + 2$ .

Here, we grab  $u_1$  and  $u_2$  from  $v_x$  to turn the current part into a part of size  $\alpha + 2$ , and then resume the picking process along  $v_{x+1} \dots v_p$ , taking into account that this part must be complemented to a part of size  $\beta$ , and might thus miss vertices (when  $\beta > \alpha + 2$ ). Eventually, the process ends up with a realisation of  $\pi$  in  $G$ .

**Case 1.3.2.**  $\beta = \alpha + 1$ .

Start by grabbing  $u_1$  from  $v_x$  to turn the current part into a part of size  $\beta$ . Note that, if we resume the picking process, then it remains at least one part of size  $\alpha$  to pick. If there also remain parts of size  $\beta$  to be picked, then Theorem 2.1 implies we can pick the remaining parts along  $v_{x+1} \dots v_p$  in such a way that  $u_2$  can be grabbed from  $v_y$ . From here, by then going on arbitrarily along  $v_{y+1} \dots v_p$ , we can obtain a realisation of  $\pi$  in  $G$ . Otherwise, it means that  $\pi = (\alpha, \dots, \alpha, \alpha + 1)$  for some  $\alpha$ , that is  $\beta = \alpha + 1$  and only one part of size  $\beta$  is requested. If  $\alpha = 1$ , then obviously a realisation of  $\pi$  in  $G$  exists. If  $\alpha = 2$ , then note that a realisation of  $\pi = (2, \dots, 2, 3)$  exists if and only if  $(2, \dots, 2, 1)$  is realisable in  $G$ , that is, if and only if  $G$  admits a quasi-perfect matching, an assumption that was made on  $G$ . So, we can further assume  $\alpha \geq 3$ .

Let  $z_1$ ,  $z_2$ , and  $Z$  be as defined in Lemma 3.1(3b). Note that the number of integers in  $\{z_1, \dots, z_2\}$  that are congruent to 0 modulo  $\alpha$  is at most

$$1 + \frac{z_2 - z_1}{\alpha} < 1 + \frac{z_2 - z_1}{2}$$

since  $\alpha \geq 3$ . But, by Lemma 3.1(3b),

$$|N_Z(v_1) \cup N(u_1) \cup N(u_2)| \geq 1 + \frac{z_2 - z_1}{2},$$

and hence there is a vertex  $v_i \in N_Z(v_1) \cup N(u_1) \cup N(u_2)$  such that  $i \not\equiv 0 \pmod{\alpha}$ .



- If  $v_i$  is a neighbour of  $u_1$  or  $u_2$ , then consider running the picking process along  $P$  again, as going from  $v_1$  to  $v_p$ , first picking parts of size  $\alpha$  as much as possible, before eventually picking the part of size  $\alpha + 1$  (if led entirely without grabbing  $u_1$  or  $u_2$ , then note that the last part would actually be of size  $\alpha - 1$ ). When reaching  $v_i$ , because  $i \not\equiv 0 \pmod{\alpha}$ , then  $u_1$  or  $u_2$  can be grabbed from  $v_i$ . Then run the picking process till the end to have a partition of  $G$  into connected subgraphs of order  $\alpha$  missing only one of  $u_1$  and  $u_2$ , and eventually add the missing vertex to any of the parts it is joined to, to turn it into a part of size  $\beta = \alpha + 1$ .
- If  $v_i$  is a neighbour of  $v_1$ , then run the picking process as in the previous case but omitting  $v_1$ , that is, pick parts of size  $\alpha$  as going from  $v_2$  to  $v_p$ . Additionally, during the process, grab a vertex in  $\{u_1, u_2\}$  from  $v_{z_1}$ , and grab  $v_1$  from  $v_i$  (this is possible, by all hypotheses; in particular, note that  $v_{z_1}$  and  $v_i$  necessarily belong to different parts). Once the process finishes, this partitions all vertices of  $G$  but one in  $\{u_1, u_2\}$  into connected subgraphs of order  $\alpha$ . Then add the missing vertex to any adjacent part.

In both situations, we thus end up with a realisation of  $\pi$  in  $G$ , which deals with **Case 1.3.2** and thus with the whole of **Case 1.3**.

We now consider cases where  $v_x$ , through the picking process picking parts of size  $\alpha$  first and then parts of size  $\beta$ , gets added to a  $\beta$ -part. Since **Case 1.1** does not apply, we can assume  $v_x$  is the  $(\beta - 1)$ th or  $\beta$ th vertex of that part. Note that, at this point of the process, after completing the part of size  $\beta$  containing  $v_x$  (by adding  $v_{x+1}$  if that part misses one vertex), we can assume there remain at least two parts of size  $\beta$  to be picked (as otherwise the rest of the vertices of  $G$  would be  $\beta$  vertices only, inducing a connected graph due to  $v_y$ , where  $v_y \neq v_{x+1}$ , and from this a realisation of  $\pi$  would be obtained). Recall also that we have picked all required parts of size  $\alpha$  earlier in the process.

**Case 1.4.**  $v_x$  is the  $\beta$ th vertex of a  $\beta$ -part.

**Case 1.4.1.**  $\beta \geq \alpha + 2$ .

By “replacing” one of the previous parts of size  $\alpha$  constructed earlier through the picking process with a part of size  $\beta$  (or, in other words, by running the picking process so that we pick a first part of size  $\beta$ , then all parts of size  $\alpha$ , and eventually all remaining parts of size  $\beta$ ), we essentially “shift” the succeeding parts towards  $v_p$  by exactly  $\beta - \alpha$  vertices, including the part that contains  $v_x$ , which makes it now possible, after removing the last  $\beta - \alpha$  vertices of the part, to grab  $u_1$  and  $u_2$  from  $v_x$  (note indeed that shifting parts this way cannot make  $v_x$  “change” part, as this would require  $\beta - \alpha \geq \beta$ , which is not possible since  $1 \leq \alpha < \beta$ ). Then by resuming the picking process, we eventually get a realisation of  $\pi$  in  $G$ , once the process achieves.

**Case 1.4.2.**  $\beta = \alpha + 1$ .

Note that if  $\alpha = 1$ , then  $\beta = 2$ , and a realization of  $\pi = (1, \dots, 1, 2, \dots, 2)$  in  $G$  exists since  $G$  admits a perfect matching or a quasi-perfect matching. Thus, we can assume  $\alpha \geq 2$ . Recall that, in the present case, the part containing  $v_x$  misses no vertex. Then, just as in **Case 1.4.1**, replace a previous part of size  $\alpha$  with a part of size  $\alpha + 1$  to shift parts towards  $v_p$  by one vertex, so that, after removing the last vertex of the part containing  $v_x$ , we can now grab  $u_1$  from  $v_x$ . It remains the vertices of  $\{v_{x+1}, \dots, v_p\} \cup \{u_2\}$  to partition, which induce a connected graph due to  $v_y$ . Also, the remaining part sizes form a partition of the form  $(\alpha, \alpha + 1, \dots, \alpha + 1)$ . Theorem 2.1 tells it is possible to pick parts so that, eventually, a realisation of  $\pi$  in  $G$  results.

**Case 1.5.**  $v_x$  is the  $(\beta - 1)$ th vertex of a  $\beta$ -part.

**Case 1.5.1.**  $\beta \geq \alpha + 2$ .

If  $\alpha \neq 1$ , then we are done by proceeding as in previous **Case 1.4**. That is, by replacing a preceding part of size  $\alpha$  with a part of size  $\beta$  (to shift parts towards  $v_p$ ), having  $v_x$  changing part requires  $\beta - \alpha \geq \beta - 1$ , which holds only if  $\alpha = 1$ . Thus, this cannot occur here. By removing the last  $\beta - \alpha + 1$  vertices of  $v_x$ 's part, we can then grab  $u_1$  and  $u_2$  from  $v_x$ , and then resume the picking process to eventually obtain a realisation of  $\pi$  in  $G$ .

Now, if  $\alpha = 1$ , then we can assume that  $\pi = (1, \beta, \dots, \beta)$ , that is,  $\pi$  contains only one 1. Indeed, if  $\pi$  had at least two 1's, then a realisation of  $\pi$  in  $G$  could be obtained by considering  $\{u_1\}$  and

$\{u_2\}$  as parts of size 1, and then applying the picking process along  $P$  as going from  $v_1$  to  $v_p$ , picking parts of size  $\beta$ . So assume  $\pi = (1, \beta, \dots, \beta)$ . We can further assume that  $\beta \geq 3$ , since  $G$  was assumed to admit a quasi-perfect matching. The same arguments as in **Case 1.3.2** can now be employed to prove that, for  $Z = \{v_{z_1}, \dots, v_{z_2}\}$  as defined in Lemma 3.1(3b), there is some  $v_i \in N_Z(v_1) \cup N(u_1) \cup N(u_2)$  with  $i \not\equiv 0 \pmod{\beta}$  and  $x < i < y$ .

- If  $v_i$  neighbours  $u_1$  or  $u_2$ , then a realisation of  $\pi$  in  $G$  can be obtained by picking parts of size  $\beta$  along  $v_1, \dots, v_p$  so that  $u_1$  or  $u_2$  is grabbed from  $v_i$  during the process (which is possible since  $i \not\equiv 0 \pmod{\beta}$ ), and eventually having the remaining of  $u_1$  and  $u_2$  forming the desired part of size  $\alpha = 1$ .
- If  $v_i$  neighbours  $v_1$ , then a realisation of  $\pi$  in  $G$  can be obtained by picking parts of size  $\beta$  along  $v_2, \dots, v_p$  so that, during the process,  $u_1$  is grabbed from  $v_x$  and  $v_1$  is grabbed from  $v_i$  (which is possible since  $i \not\equiv 0 \pmod{\beta}$ ), and eventually defining  $\{u_2\}$  as the desired part of size  $\alpha = 1$ .

**Case 1.5.2.**  $\beta = \alpha + 1$ .

For similar reasons as in **Case 1.5.1**, we can assume  $\alpha \geq 2$ . Now, since the part containing  $v_x$  misses only one vertex, then, as in previous **Case 1.4.2**, we are done through replacing one of the previous parts of size  $\alpha$  with a part of size  $\beta = \alpha + 1$ , as it shifts parts toward  $v_p$  by one vertex and makes it possible now, because  $\beta \geq 3$ , after removing the last vertex of the part containing  $v_x$ , to grab both  $u_1$  and  $u_2$  from  $v_x$ .

**Case 2.**  $r = 2$  and  $|\text{sp}(\pi)| = 1$ .

We again set  $R = \{u_1, u_2\}$ , and denote by  $v_x$  and  $v_y$  two common neighbours (on  $P$ ) of  $u_1$  and  $u_2$ , where  $x$  and  $y$  verify the same conditions as in **Case 1**. Set  $\pi = (\lambda, \dots, \lambda)$ . We can suppose that  $\lambda \geq 3$  since  $G$  admits a perfect matching. Now apply the picking process along  $P$  as going from  $v_1$  to  $v_p$ , picking parts of size  $\lambda$  as long as possible (in particular, if the process was achieved completely, then the part containing  $v_p$  would be of size  $\lambda - 2$ ). We consider a few cases.

**Case 2.1.**  $v_x$  is the  $i$ th vertex of a  $\lambda$ -part with  $i \leq \lambda - 2$ .

We can here grab  $u_1$  and  $u_2$  from  $v_x$  and resume the process to obtain a realisation of  $\pi$  in  $G$ .

**Case 2.2.**  $v_x$  is the  $\lambda$ th vertex of a  $\lambda$ -part.

Repeat the picking process, but as going from  $v_p$  to  $v_1$  instead. It can be checked that because  $n \equiv 0 \pmod{\lambda}$  and  $p = n - 2$ , this time when treating  $v_x$  the part misses at least one vertex. If the part misses at least two vertices, then we fall back into previous **Case 2.1**. The remaining case is when the part misses exactly one vertex, which is precisely next **Case 2.3**.

**Case 2.3.**  $v_x$  is the  $(\lambda - 1)$ th vertex of a  $\lambda$ -part.

At this point of the picking process, note that if we grab  $u_1$  from  $v_x$  and resume the process, then, later on,  $v_y$  can be assumed to be the  $\lambda$ th vertex added to its part, as otherwise we could just grab  $u_2$  from  $v_y$ , and, upon leading the picking process to its end, get a realisation of  $\pi$  in  $G$ .

To make things clear, let us run the picking process from  $v_1$  to  $v_p$  again, picking parts of size  $\lambda$ , with the exception that we make sure  $v_x$  is the last vertex of its part  $X$  (thus of size  $\lambda - 1$ ), and the very last part (containing  $v_p$ ) is of size  $\lambda - 1$ . Regarding the resulting parts preceding  $X$ , note that  $u_1$  and  $u_2$  can be assumed to only neighbour the  $\lambda$ th vertex as otherwise, if they neighboured another of these vertices, say,  $v_z$ , then a realisation of  $\pi$  would be obtained through running the picking process and grabbing  $u_1$  and  $u_2$  from  $v_z$  and  $v_y$  (as, indeed, grabbing one of  $u_1$  and  $u_2$  from  $v_z$ , because  $\lambda \geq 3$ , would have the effect to shift succeeding parts one vertex towards  $v_1$ , and thus make it possible to grab the second vertex from  $v_y$ ). Now, obviously, in all parts succeeding  $X$ , vertices  $u_1$  and  $u_2$  can be assumed to only neighbour the  $\lambda$ th vertex (because  $v_x$  neighbours both  $u_1$  and  $u_2$ , and its part  $X$  has size  $\lambda - 1$ ). Regarding  $v_1$ , we recall it cannot neighbour a vertex in  $N(u_1)^+ \cup N(u_2)^+$  by Lemma 2.2(3). So if  $u_1$  and/or  $u_2$  neighbour the  $\lambda$ th vertex of some part, then  $v_1$  cannot neighbour the first vertex of the next part.

Similar assumptions can be made regarding the neighbourhood of  $v_1$ ,  $u_1$ , and  $u_2$  in  $X$ . By definition, the  $(\lambda - 1)$ th vertex of  $X$ , which is  $v_x$ , is actually the last vertex of  $X$ , and it neighbours

both  $u_1$  and  $u_2$ . By an earlier remark, this means  $v_1$  cannot neighbour the first vertex of the part succeeding  $X$ . Now let  $v_z \in Y \setminus \{v_x\}$  be any vertex of  $X$  different from  $v_x$ . By our choice of  $v_x$ , it cannot be that  $v_z$  neighbours both  $u_1$  and  $u_2$ . If  $v_z$  neighbours  $v_1$  and, say,  $u_1$ , then  $v_{z+1}$  cannot neighbour any of  $v_1$  and  $u_1$  (By Lemma 2.2(3) and (4)), and it can be assumed that  $v_{z+1}$  does not neighbour  $u_2$ . Indeed, because  $v_{z+1}$  does not neighbour  $u_1$ , we have  $v_{z+1} \neq v_x$  since  $v_x$  neighbours both  $u_1$  and  $u_2$ . This means  $v_{z+1}$  would be, at worst, the  $(\lambda - 2)$ th vertex of  $X$ , and thus  $v_z$  would be, at worst, the  $(\lambda - 3)$ th vertex of  $X$ . Here, a realisation of  $\pi$  in  $G$  could be obtained by running the picking process as earlier, grabbing  $u_1$  from  $v_z$  and  $u_2$  from  $v_{z+1}$  during the process, thereby forming a part of size  $\lambda$ , and resuming the picking process. Thus, we can assume each  $v_z$  of the first  $\lambda - 2$  vertices of  $X$  either neighbours at most one vertex in  $\{v_1, u_1, u_2\}$ , or it neighbours two vertices in  $\{v_1, u_1, u_2\}$  in which case the next vertex has no neighbour in that set (in particular, this implies  $z \neq x - 1$ ).

Under all those assumptions, let us now determine the maximum value of  $d^*(v_1, u_1, u_2)$ . To that aim, we consider the parts we have constructed above one by one, as going from  $v_2$  to  $v_p$ . As considering every  $v_i$  one by one this way, we determine a value  $\rho(i)$  deduced from  $\rho(i - 1)$ , the exact same way we did to prove Lemma 3.1(2). Essentially, for every  $i \in \{2, \dots, p\}$ , the value  $\rho(i)$  is the number of edges incident to the vertices in  $\{v_2, \dots, v_i\}$  going to  $\{v_1, u_1, u_2\}$  (counting only two incident edges when the three edges exist) minus  $|\{v_2, \dots, v_i\}|$ . We consider two main cases.

- Assume the first part is  $X$ . That is,  $v_1$  and  $v_x$  belong to the same part,  $X$ , which, recall, is of size  $\lambda - 1$ . As a base case, note that  $\rho(2) = 0$  if  $v_2$  neighbours  $v_1$  only, and that  $\rho(2) = 1$  if  $v_2$  also neighbours one of  $u_1$  and  $u_2$ . In the latter case, by a remark above, we know that  $v_3$  neighbours no vertex in  $\{v_1, u_1, u_2\}$ , and thus  $\rho(3) = 0$ . In the former case,  $v_3$  is either adjacent to no vertex in  $\{v_1, u_1, u_2\}$ , in which case  $\rho(3) = \rho(2) - 1 = -1$ , to only one vertex in  $\{v_1, u_1, u_2\}$ , in which case  $\rho(3) = \rho(2) = 0$ , or to  $v_1$  and exactly one vertex in  $\{u_1, u_2\}$ , in which case  $\rho(3) = \rho(2) + 1 = 1$  and we additionally know that  $\rho(4) = 0$ . By repeating these arguments to all vertices of  $X$  one by one, for every  $i \in \{2, \dots, x - 1\}$  it can be determined by  $\rho(i)$  is at most 1, and for  $\rho(i)$  to be exactly 1 it must be that  $v_i$  neighbours  $v_1$  and exactly one vertex in  $\{u_1, u_2\}$ , in which case  $\rho(i + 1)$  is sure to be 0. Since  $v_x$  is adjacent to both  $u_1$  and  $u_2$ , we get that  $\rho(x - 1)$  must be at most 0, and thus  $\rho(x)$  is at most 1.

We now get to the part succeeding  $X$ . For the first vertex,  $v_{x+1}$ , we actually have  $\rho(x + 1) = \rho(x) - 1 \leq 0$  since  $v_x$  is a common neighbour of  $u_1$  and  $u_2$ . From here, recall that, for the next  $\lambda - 1$  vertices, only the last one of the part, *i.e.*,  $v_{x+\lambda}$ , can neighbour  $u_1$  and/or  $u_2$ . Thus, for every  $i \in \{2, \dots, \lambda\}$ , we have  $\rho(x + i) \leq \rho(x + 1) \leq 0$ . Now, regarding  $v_{x+\lambda}$ , either it neighbours no vertex in  $\{v_1, u_1, u_2\}$  and we have  $\rho(x + \lambda) = \rho(x + \lambda - 1) - 1 \leq -1$ , it neighbours only one vertex in  $\{v_1, u_1, u_2\}$  and we have  $\rho(x + \lambda) = \rho(x + \lambda - 1) \leq 0$ , or it neighbours two vertices in  $\{v_1, u_1, u_2\}$  and we have  $\rho(x + \lambda) = \rho(x + \lambda - 1) + 1 \leq 1$ . In the latter case, we also know that  $v_{x+\lambda+1}$  neighbours no vertex in  $\{v_1, u_1, u_2\}$  and  $\rho(x + \lambda + 1) \leq 0$ .

These arguments then repeat from the vertices of a part to the vertices of the succeeding part. In particular, recall that the last part, that containing  $v_p$ , is of size  $\lambda - 1 \geq 2$ . This means that  $v_{p-1}$  cannot be the  $\lambda$ th vertex of a part, and, in particular,  $\rho(p - 1) \leq 0$ . Now, since  $v_p$  can neighbour no vertex in  $\{v_1, u_1, u_2\}$ , we have  $\rho(p) \leq -1$ . Since  $|\{v_2, \dots, v_p\}| = p - 1$ , by the definition of  $\rho$  we deduce that  $d^*(v_1, u_1, u_2) \leq p - 2 = n - 4$ . This is a contradiction to the fact that  $\overline{\sigma}_3(G) \geq n - 2$ .

- Similar arguments can be employed when the first part is not  $X$ . Recall that this first part is of size  $\lambda$  and that only its  $\lambda$ th vertex can neighbour  $u_1$  and/or  $u_2$ . So we have  $\rho(2) = 0$ , while  $\rho(3), \dots, \rho(\lambda - 1) \leq 0$ . Now, if  $v_\lambda$  neighbours at most one vertex in  $\{v_1, u_1, u_2\}$  then we have  $\rho(\lambda) \leq \rho(\lambda - 1) \leq 0$ , while if  $v_\lambda$  neighbours at least two vertices in  $\{v_1, u_1, u_2\}$  then we have  $\rho(\lambda) = \rho(\lambda - 1) + 1 \leq 1$ . In the latter case, note that, by definition of  $v_x$ , the at least two neighbours of  $v_\lambda$  cannot include  $u_1$  and  $u_2$ , so they must be  $v_1$  and exactly one of  $u_1$  and  $u_2$ , implying that  $v_{\lambda+1}$  cannot be adjacent to  $v_1$ . So, if the next part is not  $X$ , then  $\rho(\lambda + 1) = \rho(\lambda) - 1 \leq 0$ , since  $v_{\lambda+1}$  cannot neighbour  $u_1$  and/or  $u_2$  in this case. From these arguments, we deduce that the same calculations hold for all parts preceding  $X$ .

Assume now the first vertex of  $X$  is  $v_{q\lambda+1}$  and that  $\rho(q\lambda)$  was computed. By the arguments above, we have  $\rho(q\lambda) \leq 1$ . Also, if  $\rho(q\lambda) = 1$ , then  $v_{q\lambda}$  is adjacent to  $v_1$  and one of  $u_1$  and

$u_2$ , implying that  $v_{q\lambda}$  cannot be adjacent to two vertices in  $\{v_1, u_1, u_2\}$ . Thus, we deduce that  $\rho(q\lambda + 1) \leq 1$ . If we now employ the exact same arguments as when we dealt with  $X$  in the previous case, we here deduce that for  $v_x$ , the last vertex of  $X$ , we have  $\rho(x) \leq 2$ . Also, since  $v_x$  is adjacent to both  $u_1$  and  $u_2$ , we have that  $v_{x+1}$  has no neighbour in  $\{v_1, u_1, u_2\}$ , and thus  $\rho(x+1) \leq 1$ .

By now repeating the same arguments as earlier to the consecutive parts succeeding  $X$ , we here deduce that  $\rho(p-1) \leq 1$ , and thus  $\rho(p) \leq 0$ . Since  $|\{v_2, \dots, v_p\}| = p-1$ , then  $d^*(v_1, u_1, u_2) \leq p-1 = n-3$ , which is another contradiction to  $\overline{\sigma}_3(G) \geq n-2$ .

**Case 3.**  $r = 1$ .

Set  $R = \{u\}$  and  $\pi = (\lambda, \dots, \lambda)$ . Since  $G$  is assumed to admit a perfect matching or a quasi-perfect matching, we may assume  $\lambda \geq 3$ .

Set  $I = \{v_i : i \equiv 0 \pmod{\lambda}\}$ . Note that if  $u$  is adjacent to a vertex  $v_i \notin I$ , then a realisation of  $\pi$  in  $G$  can be obtained by running the picking process from  $v_1$  to  $v_p$  picking parts of size  $\lambda$ , and grabbing  $u$  from  $v_i$ . So we can assume that all neighbours of  $u$  lie in  $I$ .

Let us denote by  $i_1, \dots, i_d$  the indexes, where  $i_1 < \dots < i_d$ , such that  $v_{i_1}, \dots, v_{i_d}$  are the  $d = d(u)$  neighbours of  $u$ . Note that  $d \geq 2$  since  $G$  is 2-connected. Also,  $v_{i_1}, \dots, v_{i_d} \in I$ . Regarding the upcoming arguments, we partition the vertices of  $P$  into three sets  $V_1, V_2$ , and  $V_3$  as follows:  $V_1$  is  $\{v_i : i < i_1\}$ ,  $V_2$  is  $\{v_i : i \in \{i_1, \dots, i_d\}\}$ , and  $V_3$  is  $\{v_i : i > i_d\}$ . Note that  $v_1 \in V_1$ ,  $v_p \in V_3$ , and  $v_{i_1}, v_{i_d} \in V_2$ . So, none of the three sets is empty. Since  $d \geq 2$  and  $\lambda \geq 3$ , also  $|V_2| \geq 4$ .

We now analyse the possible neighbours of  $v_1$  (and, similarly, of  $v_p$ ) through the next claim.

**Claim 1.** *A realisation of  $\pi$  in  $G$  can be constructed in the following contexts:*

- (1) *if  $v_1$  has a neighbour  $v_i$  that does not lie in  $V_1 \cup I$ ;*
- (2) *if  $v_i \in V_1 \cap I$  is a vertex such that  $v_p v_i$  is an edge, and there is some  $v_j \in V_1 \setminus I$  with  $j > i$  such that  $v_1 v_j$  is an edge.*

*Proof of the claim.* Regarding (1), assume  $v_1$  has a neighbour  $v_i$  that does not lie in  $V_1 \cup I$ . Thus,  $i > i_1$ . A realisation of  $\pi$  in  $G$  can then be obtained as follows. Start running the picking process along  $P$  as going from  $v_p$  to  $v_2$ , picking parts of size  $\lambda$ . When reaching  $v_i$ , then, because  $v_i \notin I$ , at least one more vertex must be added to the part. Then grab  $v_1$  from  $v_i$ , before resuming the process. Later on, when reaching  $v_{i_1}$ , note that the part must be missing vertices. So, here, just grab  $u$  from  $v_{i_1}$  before resuming the process. Once it achieves, this results in a realisation of  $\pi$ .

Now consider (2). Let  $v_i \in V_1 \cap I$  be a vertex such that  $v_p v_i$  is an edge, and assume there is a  $v_j \in V_1 \setminus I$  with  $j > i$  such that  $v_1 v_j$  is an edge. A realisation of  $\pi$  in  $G$  can here be obtained as follows. Let  $\alpha$  be the smallest index with  $\alpha > j$  such that  $v_\alpha \in I$ . We run the picking process along the path  $v_{\alpha-1} \dots v_2$  as going from  $v_{\alpha-1}$  to  $v_2$ , picking parts so that the first part (containing  $v_{\alpha-1}$  and  $v_j$ ) has size  $\lambda-1$ , the part containing  $v_i$  has size  $\lambda-1$ , and the other parts have size  $\lambda$ . Note that the hypothesis on  $\alpha$  and  $j$  implies this is possible (in particular, the choice of  $\alpha$  implies that  $\alpha-1-i > \lambda-1$ , so  $v_j$  and  $v_i$  cannot belong to the same part). Then, grab  $v_1$  from  $v_j$  to form a part of size  $\lambda$ , and grab  $v_p$  from  $v_i$  to form another part of size  $\lambda$ . Now resume the process, but along  $v_{p-1} \dots v_\alpha$  as going from  $v_{p-1}$  to  $v_\alpha$ , picking parts of size  $\alpha$ . In particular, when considering  $v_{i_d}$ , more vertices must be added to that part, and we can freely grab  $u$  from  $v_{i_d}$  before resuming the process. Once the process achieves, this results in a realisation of  $\pi$  in  $G$ .  $\square$

Back to the proof of **Case 3**, assume now that Claim 1 cannot be applied to deduce a realisation of  $\pi$  in  $G$ . That is, we can assume all neighbours of  $v_1$  lie in  $V_1 \cup I$ , and that if  $v_i \in V_1 \cap I$  is a vertex such that  $v_p v_i$  is an edge, then there does not exist any  $v_j \in V_1 \setminus I$  with  $j > i$  such that  $v_1 v_j$  is an edge. Likewise, we can assume similar things for  $v_p$ , that is, all its neighbours lie in  $V_3 \cup I$ , and if  $v_i \in V_3 \cap I$  is a vertex such that  $v_1 v_i$  is an edge, then there cannot be any  $v_j \in V_3 \setminus I$  with  $j < i$  such that  $v_p v_j$  is an edge.

Let us now analyse the possible neighbours in  $\{v_1, v_p, u\}$  for the other vertices, in  $\{v_2, \dots, v_{p-1}\}$ , with respect to all observations we made so far.

- Any  $v_i \in I$  can be a neighbour of any number of vertices in  $\{v_1, v_p, u\}$ .

- Any  $v_i \in V_2 \setminus I$  cannot have any neighbour in  $\{v_1, v_p, u\}$ .
- Any  $v_i \in V_1 \setminus (I \cup \{v_1\})$  has  $v_1$  as its only potential neighbour in  $\{v_1, v_p, u\}$ , but this can only occur if there is no  $j < i$  such that  $v_j \in I \cap N(v_p)$ .
- Any  $v_i \in V_3 \setminus (I \cup \{v_p\})$  has  $v_p$  as its only potential neighbour in  $\{v_1, v_p, u\}$ , but this can only occur if there is no  $j > i$  such that  $v_j \in I \cap N(v_1)$ .

We now count edges incident to  $v_1$ ,  $v_p$ , and  $u$ , omitting one edge for each of their common neighbours. This is exactly the quantity  $d^*(v_1, v_p, u)$ . In order to make the calculations easier, we split the analysis into three main cases.

**Case 3.1.**  $v_1$  has no neighbour in  $I \cap V_3$ , and  $v_p$  has no neighbour in  $I \cap V_1$ .

In that case, all neighbours of  $v_1$  lie in  $V_1 \cup (I \cap V_2)$ , while all neighbours of  $v_p$  lie in  $V_3 \cup (I \cap V_2)$ . Recall also that all neighbours of  $u$  lie in  $I \cap V_2$ . Then  $d^*(v_1, v_p, u)$  is at most

$$(|V_1| - 1) + (|V_3| - 1) + (2|I \cap V_2|).$$

Now, since  $d \geq 2$  and  $\lambda \geq 3$ , from this we deduce that

$$d^*(v_1, v_p, u) \leq (i_1 - 2) + (p - 1 - i_d) + 2 \left( 1 + \frac{i_d - i_1}{\lambda} \right) < p - 1 = n - 2,$$

which is a contradiction to the fact that  $\overline{\sigma}_3(G) \geq n - 2$ .

**Case 3.2.**  $v_p$  has a neighbour in  $I \cap V_1$ , and  $v_1$  has no neighbour in  $I \cap V_3$ .

In this case, let  $v_x$  be the neighbour of  $v_p$  in  $I \cap V_1$  with the lowest index. Set  $V_1' = \{v_1, \dots, v_{x-1}\}$  and  $V_1'' = \{v_x, \dots, v_{i_1-1}\}$ . Note that  $V_1 = V_1' \cup V_1''$ . Also,  $|V_1''|$  is a multiple of  $\lambda$ .

Note that all neighbours of  $v_1$  lie in  $V_1' \cup (I \cap V_1'') \cup (I \cap V_2)$ , while all neighbours of  $v_p$  lie in  $(I \cap V_1'') \cup (I \cap V_2) \cup V_3$ . Also, we still have that all neighbours of  $u$  lie in  $I \cap V_2$ . Thus,  $d^*(v_1, v_p, u)$  is at most

$$(|V_1'| - 1) + (2|I \cap V_1''|) + (|V_3| - 1) + (2|I \cap V_2|).$$

Since  $d \geq 2$  and  $\lambda \geq 3$ , we thus have

$$d^*(v_1, v_p, u) \leq (x - 2) + 2 \left( \frac{i_1 - x}{\lambda} \right) + (p - 1 - i_d) + 2 \left( 1 + \frac{i_d - i_1}{\lambda} \right) < p - 1 = n - 2,$$

another contradiction to  $\overline{\sigma}_3(G)$  being at least  $n - 2$ .

**Case 3.3.**  $v_p$  has a neighbour in  $I \cap V_1$ , and  $v_1$  has a neighbour in  $I \cap V_3$ .

Here, let  $v_x$  be the neighbour of  $v_p$  in  $I \cap V_1$  with the lowest index, and  $v_y$  be the neighbour of  $v_1$  in  $I \cap V_3$  with the largest index. Set  $V_1' = \{v_1, \dots, v_{x-1}\}$  and  $V_1'' = \{v_x, \dots, v_{i_1-1}\}$ , and  $V_3' = \{v_{i_d+1}, \dots, v_y\}$  and  $V_3'' = \{v_{y+1}, \dots, v_p\}$ . Note that  $V_1 = V_1' \cup V_1''$  and that  $V_3 = V_3' \cup V_3''$ . Also,  $|V_1''|$  and  $|V_3'|$  are multiples of  $\lambda$ .

In the current case, all neighbours of  $v_1$  lie in  $V_1' \cup (I \cap V_1'') \cup (I \cap V_2) \cup (I \cap V_3')$ , while all neighbours of  $v_p$  lie in  $(I \cap V_1'') \cup (I \cap V_2) \cup (I \cap V_3') \cup V_3''$ . Again, all neighbours of  $u$  lie in  $I \cap V_2$ . Thus,  $d^*(v_1, v_p, u)$  is at most

$$(|V_1'| - 1) + (2|I \cap V_1''|) + (2|I \cap V_3'|) + (|V_3''| - 1) + (2|I \cap V_2|).$$

We thus have

$$d^*(v_1, v_p, u) \leq (x - 2) + 2 \left( \frac{i_1 - x}{\lambda} \right) + 2 \left( \frac{y - i_d}{\lambda} \right) + (p - y - 1) + 2 \left( 1 + \frac{i_d - i_1}{\lambda} \right),$$

which, because  $d \geq 2$  and  $\lambda \geq 3$ , is strictly less than  $p - 1 = n - 2$ . This, again, contradicts that  $\overline{\sigma}_3(G) \geq n - 2$ .

In all cases we thus get a contradiction to  $\overline{\sigma}_3(G) \geq n - 2$ ; this concludes the proof.  $\square$

#### 4. Conclusion

Regarding Theorem 1.2, it is worth reporting that there are examples of graphs showing that some of the requirements in the statement are mandatory. In particular, both the condition on  $\alpha(G)$  and on the connectivity of  $G$  cannot be dropped out:

- First, consider, as  $G$ , the complete bipartite graph  $K_{k,k+2}$  for any  $k \geq 2$ . Note that  $G$  has order  $2k + 2$ , which is even, and that  $G$  is 2-connected because  $k \geq 2$ . Furthermore, every three pairwise independent vertices  $u, v$ , and  $w$  of  $G$  must belong to the same partition class, from which we deduce that  $d^*(u, v, w) \geq 2k = n - 2$ , and thus  $\overline{\sigma}_3(G) \geq n - 2$ . Also, the longest paths of  $G$  go through  $n - 1$  vertices, and thus, by Theorem 2.1, all  $n$ -partitions  $\pi$  with  $|\text{sp}(\pi)| \geq 2$  are realisable in  $G$ . Thus, if  $G$  is not AP, then it must be because of  $n$ -partitions  $\pi = (\lambda, \dots, \lambda)$  that are not realisable in  $G$ . Since every path  $P$  on  $n - 1$  vertices of  $G$  has the property that the last vertex of  $G$  neighbours every second vertex of  $P$ , it is easy to see (through, *e.g.*, the picking process) that realisations of  $\pi$  in  $G$  exist whenever  $\lambda \neq 2$ . Meanwhile, it can be noticed that  $G$  does not admit perfect matchings, thus no realisations of  $(2, \dots, 2)$ . Thus, the condition on  $\alpha(G)$  in Theorem 1.2 is an important one.
- Second, consider, as  $G$ , any graph obtained in the following way. Let  $n_1, n_2, n_3 \geq 1$  be integers, and let  $G$  be obtained from the disjoint union of three complete graphs  $C_1, C_2$ , and  $C_3$  of order  $n_1, n_2$ , and  $n_3$ , respectively, by adding a new universal vertex  $x$  (thus joined to all vertices of  $C_1, C_2$ , and  $C_3$ ). Note that  $G$  is not 2-connected, as  $x$  is a cut vertex. Also, we have  $n = n_1 + n_2 + n_3 + 1$ , and every three pairwise independent vertices  $u, v$ , and  $w$  of  $G$  must verify, say,  $u \in V(C_1), v \in V(C_2)$ , and  $w \in V(C_3)$ , from which we deduce that  $\overline{\sigma}_3(G) = d^*(u, v, w) = n_1 + n_2 + n_3 - 1 = n - 2$ .

Now consider any case of  $G$  where  $n_1 \equiv 4 \pmod{6}$ ,  $n_2 \equiv 5 \pmod{6}$ , and  $n_3 \equiv 2 \pmod{6}$ . Under those conditions, note that  $n$  is a multiple of 6, thus a multiple of 2 and 3. On the one hand, note that  $G$  admits perfect matchings (the vertices of  $C_1$  can be matched together, similarly for those of  $C_3$ , while the remaining vertices, of  $C_2$  and  $x$ , can be matched together). On the other hand, note that  $G$  admits no realisation of  $(3, \dots, 3)$  (as, in a realisation, due to the conditions on  $n_1, n_2$ , and  $n_3$ , the vertices of  $C_2$  and  $x$  would have to be covered by parts of size 3, making it impossible for the remaining vertices, of  $C_1$  and  $C_3$ , to be covered by parts of size 3 inducing connected graphs, since  $x$  is a cut vertex). This example shows that the 2-connectivity condition in the statement of Theorem 1.2 cannot be dropped out.

- Third, consider, as  $G$ , the graph obtained by starting from four disjoint edges  $u_1v_1, u_2v_2, u_3v_3$ , and  $u_4v_4$ , and then adding two new independent vertices  $x$  and  $y$  joined to all  $u_i$ 's and  $v_i$ 's. Note that  $G$  has order 10, is 2-connected, and admits perfect matchings (one such is  $\{xu_1, yv_1, u_2v_2, u_3v_3, u_4v_4\}$ ). Also, any three pairwise independent vertices  $u, v$ , and  $w$  of  $G$  do not include  $x$  and  $y$ , and are thus pairwise independent  $u_i$ 's and  $v_i$ 's. From this, we deduce that  $d^*(u, v, w) = 9 - 2 = 7 = n - 3$ , and thus  $\overline{\sigma}_3(G) = n - 3$ . However,  $G$  is not AP, as it admits no realisation of  $(4, 3, 3)$  (to see this is true, note first that, if such a realisation existed, then  $x$  and  $y$  would belong to different parts, while the last part would induce a connected subgraph of order 3 or 4; this is impossible since  $G - \{x, y\}$  has four connected components of order 2). While this example does not generalise well to larger graphs (because of the  $\overline{\sigma}_3$  condition), it shows that the requirement on  $\overline{\sigma}_3(G)$  in the statement of Theorem 1.2 cannot be just lowered to  $n - 3$  right away, even under the other conditions on  $G$ .

Let us add also that our proof of Theorem 1.2 yields a polynomial-time algorithm to decide whether a 2-connected  $n$ -graph  $G$  with  $\overline{\sigma}_3(G) \geq n - 2$  is AP: just determine, in polynomial time, the size of a largest matching of  $G$  (which can be done through Edmonds' Blossom Algorithm [5]).

We end up with a few words on Theorem 1.1, claimed by Brandt. As mentioned in the introductory section, the only remains on this result are the title ("Finding Vertex Decompositions in Dense Graphs") of an invited talk given in 2013, and the corresponding abstract, which can be found online, and which we report now for the sake of keeping track of it.

**Abstract** (Brandt, CID 2013). A graph  $G = (V, E)$  is called *arbitrarily vertex decomposable*, if for any partition  $\pi = (n_1, n_2, \dots, n_k)$  of  $|V| = n = n_1 + n_2 + \dots + n_k$  into positive integers there is a decomposition  $V = V_1 \cup V_2 \cup \dots \cup V_k$  with  $|V_i| = n_i$  such that the graph induced by  $V_i$  is connected for all  $i$ . The decision problem whether  $G$  admits a  $\pi$ -decomposition is known as a notoriously hard problem, while the complexity status of the decision problem whether  $G$  is arbitrarily vertex decomposable is not known in general. The problem is not even known to be in NP.

Let  $\mathcal{G}$  be the class of graphs  $G$  where the degree sum of any set of three independent vertices of  $G$  is at least  $n$ . We show that if  $G \in \mathcal{G}$  satisfies the two necessary conditions of being connected and having a matching on  $\lfloor \frac{n}{2} \rfloor$  edges, then  $G$  is arbitrarily vertex decomposable. This gives a polynomial time algorithm deciding whether  $G \in \mathcal{G}$  is arbitrarily vertex decomposable (just determine the size of a largest matching of  $G$  and determine whether it is connected). The proof is algorithmic and gives an algorithm that finds a decomposition for any partition  $\pi$  in time  $\mathcal{O}(n^3)$ . As a starting point of this algorithm we present another algorithm that finds in a connected graph in  $\mathcal{G}$  either a hamiltonian path or a very long cycle with certain additional properties.

It would be a bit daring to try to retrieve Brandt's proof from this abstract only, but we can at least make some guess. In the very last part of the abstract, Brandt mentions finding long paths or cycles in connected  $n$ -graphs  $G$  with  $\sigma_3(G) \geq n$ , which is indeed the usual way to proceed to establish such results, as illustrated by the results from [9, 13, 14] and our proof of Theorem 1.2. This high-level description actually reminds us of a result of Momège, who proved the following:

**Theorem 4.1** (Momège [15]). *If  $G$  is a connected  $n$ -graph with  $\sigma_3(G) \geq n$ , then either  $G$  is traceable, or any longest cycle of  $G$  is dominating.*

Recall that a cycle  $C$  of a graph  $G$  is *dominating* if every edge of  $G$  is incident to a vertex of  $C$ . Note that we used a sort of similar result in our proof of Theorem 1.2, when we proved that  $R$  must be a stable set. The next step in our proof was then proving that  $|R|$  is small, which seems less immediate to achieve under the condition that  $\sigma_3(G) \geq n$ . So, it might be that Theorem 1.1 can be proved by first making use of Theorem 4.1, but, if it can, we are not sure, however, what the next step would be.

## References

- [1] D. Barth, O. Baudon, J. Puech. Decomposable trees: a polynomial algorithm for tripodes. *Discrete Applied Mathematics*, 119(3):205-216, 2002.
- [2] J. Bensmail. *Partitions and decompositions of graphs*. Ph.D. thesis, Université de Bordeaux, France, 2014.
- [3] J. Bensmail, B. Li. More Aspects of Arbitrarily Partitionable Graphs. *Discussiones Mathematicae Graph Theory*, in press.
- [4] E. Drgas-Burchardt, E. Sidorowicz. Preface. *Discussiones Mathematicae Graph Theory*, 35(2):313-314, 2015.
- [5] J. Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449-467, 1965.
- [6] E. Flandrin, H. Jung, H. Li. Hamiltonism, degree sum and neighborhood intersections. *Discrete mathematics*, 90(1):41-52, 1991.
- [7] R. J. Gould. Recent Advances on the Hamiltonian Problem: Survey III. *Graphs and Combinatorics*, 30:1-46, 2014.
- [8] E. Győri. On division of graphs to connected subgraphs. In *Proceedings 5th Hungarian Combinatorial Colloquium*, 485-494, 1978.
- [9] M. Horňák, A. Marczyk, I. Schiermeyer, M. Woźniak. Dense arbitrarily vertex decomposable graphs. *Graphs and Combinatorics*, 28:807-821, 2012.

- [10] M. Horňák, M. Woźniak. Arbitrarily vertex decomposable trees are of degree at most 6. *Opuscula Mathematica*, 23:49-62, 2003.
- [11] R. Kalinowski, M. Pilśniak, I. Schiermeyer, M. Woźniak. Dense arbitrarily partitionable graphs. *Discussiones Mathematicae Graph Theory*, 36:5-22, 2016.
- [12] L. Lovász. A homology theory for spanning trees of a graph. *Acta Mathematica Academiae Scientiarum Hungaricae*, 30(3-4):241-251, 1977.
- [13] A. Marczyk. A note on arbitrarily vertex decomposable graphs. *Opuscula Mathematica*, 26(1):109-118, 2006.
- [14] A. Marczyk. An Ore-type condition for arbitrarily vertex decomposable graphs. *Discrete Mathematics*, 309:3588-3594, 2009.
- [15] B. Momège. Sufficient Conditions for a Connected Graph to Have a Hamiltonian Path. In *Proceedings of the 2017 International Conference on Current Trends in Theory and Practice of Informatics (SOFSEM 2017)*, 205–216, 2017.
- [16] O. Ore. Note on hamilton circuits. *American Mathematical Monthly*, 67:55, 1960.
- [17] R. Ravaux. Decomposing trees with large diameter. *Theoretical Computer Science*, 411:3068-3072, 2010.
- [18] Z. Tian. *Pancyclicity in hamiltonian graph theory*. Ph.D. thesis, Université Paris-Saclay, France, 2021.