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# A $\sigma_3$ condition for arbitrarily partitionable graphs

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## Abstract

A graph  $G$  of order  $n$  is arbitrarily partitionable (AP for short) if, for every partition  $(\lambda_1, \dots, \lambda_p)$  of  $n$ , there is a partition  $(V_1, \dots, V_p)$  of  $V(G)$  such that  $G[V_i]$  is a connected graph of order  $\lambda_i$  for every  $i \in \{1, \dots, p\}$ . Several aspects of AP graphs have been investigated to date, including their connection to Hamiltonian graphs and traceable graphs. Every traceable graph (and, thus, Hamiltonian graph) is indeed known to be AP, and a line of research on AP graphs is thus about weakening, to APness, known sufficient conditions for graphs to be Hamiltonian or traceable.

In this work, we provide a sufficient condition for APness involving the parameter  $\overline{\sigma}_3$ , where, for a given graph  $G$ , the parameter  $\overline{\sigma}_3(G)$  is defined as the maximum value of  $d(u) + d(v) + d(w) - |N(u) \cap N(v) \cap N(w)|$  for a set  $\{u, v, w\}$  of three pairwise independent vertices  $u$ ,  $v$ , and  $w$  of  $G$ . Flandrin, Jung, and Li proved that any graph  $G$  of order  $n$  is Hamiltonian provided  $\overline{\sigma}_3(G) \geq n$ , and traceable provided  $\overline{\sigma}_3(G) \geq n - 1$ . Unfortunately, we exhibit examples showing that having  $\overline{\sigma}_3(G) \geq n - 2$  is not a guarantee for  $G$  to be AP. However, we prove that  $G$  is AP provided  $G$  is 2-connected,  $\overline{\sigma}_3(G) \geq n - 2$ , and  $G$  has a perfect matching or quasi-perfect matching.

*Keywords:* arbitrarily partitionable graph; partition into connected subgraphs;  $\sigma_3$  condition; Hamiltonian graph; traceable graph.

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## 1. Introduction

This paper deals with so-called **arbitrarily partitionable graphs**, which are defined formally as follows. Let  $G$  be an  $n$ -graph, *i.e.*, a graph of order  $n$ . Let also  $\pi = (\lambda_1, \dots, \lambda_p)$  be an  $n$ -partition, *i.e.*, a partition of  $n$  (that is,  $\lambda_1 + \dots + \lambda_p = n$ ). We say that  $\pi$  is *realisable* in  $G$  if  $V(G)$  can be partitioned into  $p$  parts  $V_1, \dots, V_p$  such that  $G[V_i]$  is a connected graph of order  $\lambda_i$  for every  $i \in \{1, \dots, p\}$ , and we call  $(V_1, \dots, V_p)$  a *realisation* (of  $\pi$  in  $G$ ). Finally,  $G$  is said *arbitrarily partitionable* (AP for short) if all  $n$ -partitions are realisable in  $G$ . In other words,  $G$  is AP if we can partition  $G$  into arbitrarily many connected subgraphs, regardless of their requested orders.

AP graphs were introduced independently by Barth, Baudon, and Puech and Horňák and Woźniak [1, 10] in early 2000s, but the problem of partitioning graphs into connected subgraphs has been attracting attention since at least the 1970s, recall for instance the influential Győri-Lovász Theorem [8, 12]. Since then, quite some aspects of AP graphs have been investigated in the literature, including structural aspects, algorithmic questions, numerous variants, and others. For a recent survey on the topic, we refer the interested reader to, *e.g.*, [2].

AP graphs are objects of interest for numerous reasons. A notable one is the fact that they sort of lie in-between two other important types of graph notions, being perfect matchings and Hamiltonian cycles. Note indeed that, if  $n$  is even, then any realisation of the  $n$ -partition  $(2, \dots, 2)$  in an  $n$ -graph forms a perfect matching, while, if  $n$  is odd, then any realisation of the  $n$ -partition  $(2, \dots, 2, 1)$  forms a quasi-perfect matching. Remark also that adding edges to an AP graph cannot make it loose its APness, and, from this simple observation, we get that any graph spanned by an AP graph is AP itself. Since paths are obviously AP, this implies that Hamiltonian graphs and even traceable graphs (graphs having a Hamiltonian path) are AP. Thus, having a perfect matching or a quasi-perfect matching is a necessary condition for a graph to be AP, while being traceable is a sufficient condition for a graph to be AP.

These simple thoughts lead to one of the most interesting lines of research regarding AP graphs, which is on the weakening, to APness, of sufficient conditions for Hamiltonicity and traceability. The general idea is that one can consider any of the numerous sufficient conditions for a graph to

be Hamiltonian or traceable, and investigate whether it can be weakened to a sufficient condition for APness. This line of research was initiated by Marczyk in [13], in which he focused on the parameter  $\sigma_2$ , being defined as

$$\sigma_2(G) = \min \{d(u) + d(v) : u \text{ and } v \text{ are independent vertices of } G\}$$

for any graph  $G$ . By a famous result of Ore [16], recall indeed that any connected  $n$ -graph  $G$  is Hamiltonian whenever  $\sigma_2(G) \geq n$ , while  $G$  is traceable whenever  $\sigma_2(G) \geq n - 1$ . In [13], Marczyk proved that  $G$  is AP provided  $\sigma_2(G) \geq n - 2$  and  $\alpha(G) \leq \lceil n/2 \rceil$  (that is, provided  $G$  has a perfect matching or a quasi-perfect matching). Later on, in [9, 14], Marczyk, together with Horňák, Schiermeyer, and Woźniak, improved this sufficient condition to  $G$  verifying only  $\sigma_2(G) \geq n - 5$  and additional conditions (such as the previous condition on  $\alpha(G)$ , and also conditions on  $n$ ).

Sufficient conditions for Hamiltonicity and traceability being surely one of the most investigated and rich areas of graph theory, the work of Marczyk opened the way to many promising investigations in that line. For instance, in [11], Kalinowski, Piłśniak, Schiermeyer, and Woźniak, motivated by similar conditions for Hamiltonicity and traceability, exhibited sufficient conditions in terms of size (number of edges) guaranteeing a graph is AP. In [3], Bensmail and Li considered several sufficient conditions for Hamiltonicity and traceability (covering squares of graphs and forbidden structures) and proved that some of these weaken to APness, while some do not.

The current work takes place in the line of the previous investigations, and, more particularly, relates to the initiating work of Marczyk on the topic. Precisely, we deal with the parameter  $\sigma_3$  being defined as

$$\sigma_3(G) = \min \{d(u) + d(v) + d(w) : u, v, \text{ and } w \text{ are pairwise independent vertices of } G\}$$

for a given graph  $G$ . As reported in several surveys on the topic (such as, *e.g.*, [7, 18]), this parameter  $\sigma_3$  has indeed been employed, together with other graph properties (such as connectivity and claw-freeness), to express sufficient conditions for graphs to be Hamiltonian, and sometimes more or less than that. As far as we are aware, this was also considered in the context of AP graphs. Indeed, at the occasion of the 18th workshop “Colourings, Independence and Domination” (CID 2013) led in 2013, Brandt was invited to give a lecture, entitled “Finding Vertex Decompositions in Dense Graphs”, during which he announced the following result:

**Theorem 1.1** (Brandt). *If  $G$  is a connected  $n$ -graph with  $\sigma_3(G) \geq n$ , then  $G$  is AP if and only if  $G$  admits a perfect matching or a quasi-perfect matching.*

Unfortunately, it seems that Brandt has never published a proof of Theorem 1.1, and the only remains of his investigations to date are the title of his talk (mentioned in the preface [4]), as well as the corresponding abstract, which can be found online and which we report word for word in concluding Section 4 for the sake of keeping track of it. On the positive side, this abstract provides hints regarding the main lines of Brandt’s proof of Theorem 1.1.

Our investigations in the current work were inspired by Brandt’s result, and our original intent was to provide a result that would be sort of reminiscent of Theorem 1.1. As a result, we deal with a parameter that is very close to the parameter  $\sigma_3$ , as it relies on a slightly different notion of degree sum for triples of pairwise independent vertices. That is, for a graph  $G$  and any three pairwise distinct independent vertices  $u, v$ , and  $w$  of  $G$ , set

$$d^*(u, v, w) = d(u) + d(v) + d(w) - |N(u) \cap N(v) \cap N(w)|.$$

Now, define

$$\overline{\sigma}_3(G) = \min \{d^*(u, v, w) : u, v, \text{ and } w \text{ are pairwise independent vertices of } G\}.$$

Note that  $\overline{\sigma}_3(G) \leq \sigma_3(G)$  for every graph  $G$ . It turns out that this parameter  $\overline{\sigma}_3$  has also been used to express sufficient conditions for Hamiltonicity and traceability. In particular, Flandrin, Jung, and Li proved in [6] that any 2-connected  $n$ -graph  $G$  with  $\overline{\sigma}_3(G) \geq n$  is Hamiltonian, while any connected  $n$ -graph  $G$  with  $\overline{\sigma}_3(G) \geq n - 1$  is traceable. [18] reports that the latter bound for traceability was proved to also hold when  $\overline{\sigma}_3(G) \geq n - 2$  provided  $G$  fulfils additional strong conditions (involving 2-connectivity and claw-freeness).

The current work is dedicated to proving the next result, which provides a new sufficient condition, involving the parameter  $\overline{\sigma}_3$ , for a graph to be AP:

**Theorem 1.2.** *If  $G$  is a 2-connected  $n$ -graph with  $\overline{\sigma}_3(G) \geq n - 2$ , then  $G$  is AP if and only if  $G$  admits a perfect matching or a quasi-perfect matching.*

We searched the literature for a while, and, as far as we can tell, it seems that Theorem 1.2 does not follow immediately from existing results on the parameter  $\overline{\sigma}_3$  (such as conditions implying traceability). We have to add, however, that the literature on the topic is quite vast, rich, and that several old works mentioned, *e.g.*, in surveys seem to be impossible to access easily nowadays. Thus, it is hard to be fully certain we have not missed something. However, even if a previous result implying Theorem 1.2 was to exist, we believe the proof we give would remain of interest, as it relies on understanding how APness and a parameter such as  $\overline{\sigma}_3$  relate in general.

We start by introducing useful material in Section 2, before focusing on proving Theorem 1.2 in Section 3. We discuss our result in Section 4, in which we also discuss Brandt's Theorem 1.1.

## 2. Preliminaries

In our proof of Theorem 1.2, we will often deal with particular situations in which the following notations and terminology will be useful. Let  $G$  be a graph, and let  $P = v_1 \dots v_p$  be a path of  $G$ . For a vertex  $v_i \in V(P)$ , we sometimes denote by  $v_i^-$  and  $v_i^+$  the vertices  $v_{i-1}$  and  $v_{i+1}$  (assuming they exist). For a set  $S \subseteq V(P)$  of vertices of  $P$ , we denote by  $S^-$  and  $S^+$  the sets  $\{v_i^- : v_i \in S\}$  and  $\{v_i^+ : v_i \in S\}$ , respectively. These notions will be particularly useful when dealing with vertices of  $G$  having all their neighbours on  $P$ . Namely, for a  $u \in V(G)$  (we might have  $u \in V(P)$ ) such that  $N(u) \subseteq V(P)$ , we will have to deal with the sets  $N(u)^-$  and  $N(u)^+$  in many occasions.

In the proof of Theorem 1.2, we will sometimes obtain a realisation in  $G$  of some partition by first “splitting” (sometimes only part of)  $P$  into parts of certain size containing consecutive vertices. Note that this is indeed a legitimate way to proceed, as any set of consecutive vertices of  $P$  induces a connected graph. In such occasions, we will process the vertices of  $P$  one by one, from one end-vertex to the other (either as going from  $v_1$  to  $v_p$ , or conversely), and *pick* parts as going along. Assuming we want to split  $P$  into connected parts of size  $\lambda_1, \dots, \lambda_p$ , where  $\lambda_1 + \dots + \lambda_p \leq |V(P)|$ , the first  $\lambda_1$  vertices of  $P$  (following the considered ordering) will form one part, the next  $\lambda_2$  vertices will form another part, and so on. When building one of these parts, we will sometimes have to *grab* a vertex  $u$  of  $G$  from some  $v_i$  on the way, meaning that, at the very moment where  $v_i$  is added to some part, assuming  $v_i u$  is an edge and the current part still misses vertices to reach the desired size, then we add  $u$  to the current part before resuming the picking process with the vertex succeeding  $v_i$  in the considered ordering. As will be apparent later on, we will use this picking process in a rather flexible way (sometimes different from what we described above), and, in every such occasion, we will make sure to describe the exact process properly to avoid any ambiguity.

Still about  $P$ , note that the way we denote its consecutive vertices  $v_1, \dots, v_p$  above yield a virtual orientation of  $P$  from the first vertex ( $v_1$ ) to the last ( $v_p$ ), where  $v_i$  is considered to be a *predecessor* of  $v_j$  if  $i < j$ , while  $v_j$  is considered to be a *successor* of  $v_i$  in such situations. Note that these notions also make sense for the parts we pick during the picking process above. One has to be careful, however, that the notions of preceding and succeeding parts are not with respect to the virtual orientation of  $P$ , but rather with the ordering in which the consecutive vertices of  $P$  as considered through the process. In particular, when picking parts as going from  $v_p$  to  $v_1$ , note that these notions are reversed compared to those we get as going from  $v_1$  to  $v_p$ .

We finish off with a useful result on partitioning graphs with long paths into connected subgraphs. Assume  $G$  is an  $n$ -graph, and let  $\pi$  be an  $n$ -partition. We denote by  $\text{sp}(\pi)$  the *spectrum* of  $\pi$ , being the set of distinct element values that appear in  $\pi$ . An important and crucial fact is that if  $P$  is long enough, then all  $n$ -partitions with sufficiently many distinct element values are realisable in  $G$ . This is captured in the following result of Ravaux:

**Theorem 2.1** (Ravaux [17]). *If  $G$  is a connected  $n$ -graph with a path of length  $n - \alpha$ , then every  $n$ -partition  $\pi$  with  $|\text{sp}(\pi)| \geq \alpha$  is realisable in  $G$ .*

## 3. Proof of Theorem 1.2

We prove Theorem 1.2 in the following way:

- We start by extracting a long path  $P$  of  $G$  such that the set  $R = V(G) \setminus V(P)$  is a stable set with cardinality at most 2.
- We then investigate the APness of  $G$  with respect to the number of vertices in  $R$ .

For the sake of legibility, we split the proof into several subsections dedicated to proving these steps formally. These subsections are to be considered sequentially; in particular, properties proved in some subsections are assumed to hold throughout the next ones.

### 1. Some properties of longest paths $P$

Let  $P = v_1 \dots v_p$  be a longest path of  $G$ , and set  $R = V(G) \setminus V(P)$  and  $r = |R|$ . So we have  $n = p + r$ . Because  $G$  is not traceable (as otherwise  $G$  would obviously be AP), and  $G$  is connected, we have  $r \geq 1$  and  $p \geq 3$ .

Let us raise a few general remarks on  $P$  and  $R$ . Note first that  $(N(v_1) \cup N(v_p)) \cap R = \emptyset$ , as otherwise there would be a  $u \in R$  such that  $uv_1 \dots v_p$  or  $v_1 \dots v_p u$  is a path of  $G$  longer than  $P$ , a contradiction. Similarly, for any  $u \in R$ , there cannot be an  $i \in \{2, \dots, p-2\}$  such that  $\{v_i, v_{i+1}\} \subseteq N(u)$ , as otherwise  $v_1 \dots v_i uv_{i+1} \dots v_p$  would be a path of  $G$  longer than  $P$ . Also, if  $uu'$  is an edge of  $G[R]$ , then we cannot have  $v_2 \in N(u)$  or  $v_{p-1} \in N(u)$ , as otherwise  $u'uv_2 \dots v_p$  or  $u'uv_{p-1} \dots v_1$  would again be a path of  $G$  longer than  $P$ . Lastly, we cannot have the edge  $v_1 v_p$ , as otherwise  $v_1 \dots v_p v_1$  would be a cycle with an edge  $uv_i$  for some  $u \in R$  and  $i \in \{1, \dots, p\}$  (since  $G$  is connected), and thus  $uv_i \dots v_p v_1 \dots v_{i-1}$  would be a path longer than  $P$ .

Also,  $P$ , due to its maximality and the fact that  $\overline{\sigma}_3(G) \geq n - 2$ , must fulfil the following:

**Claim 1.** *If  $G[R]$  contains edges, then we must have  $d_R(u) \geq r - 2$  for every  $u \in R$ .*

*Proof of the claim.* Assume this is wrong, that is,  $G[R]$  contains edges, but there are vertices  $u \in R$  with  $d_R(u) \leq r - 3$ . Let  $xy$  be any edge of  $G[R]$  where  $x$  is incident to an edge going to  $\{v_1, \dots, v_p\}$  (note that such an edge must indeed exist since  $G$  is connected). By remarks made earlier, note that, because  $P$  is a longest path of  $G$ , we must have  $p \geq 5$ , since  $xy$  is an edge.

Let us analyse which edges of  $G$  exist. Note first that  $v_1 \in N(v_1)^-$  (due to the edge  $v_1 v_2$ ), and similarly  $v_p \in N(v_p)^+$  (due to the edge  $v_{p-1} v_p$ ). Also, trivially,  $v_p \notin N(v_1)^-$  and  $v_1 \notin N(v_p)^+$ .

Note also that we cannot have  $N(v_1)^- \cap N(v_p)^+ \neq \emptyset$ . Indeed, assume the contrary, and let  $v_i \notin \{v_1, v_p\}$  be a vertex of  $P$  in  $N(v_1)^- \cap N(v_p)^+$ . For this to happen, by definition  $v_1 v_{i+1}$  and  $v_p v_{i-1}$  must be edges. Note then that  $C = v_1 \dots v_{i-1} v_p \dots v_{i+1} v_1$  is a cycle of length  $p - 1$  of  $G$ . Then remark that if  $x$  has a neighbour  $v_j$  in  $V(C)$ , then, from  $C$  and the edges  $v_j x$  and  $xy$ , we would deduce a path of order  $p + 1$  of  $G$ , a contradiction to the maximality of  $P$ . Now, if  $v_i$  is the only neighbour of  $x$  in  $\{v_1, \dots, v_p\}$ , then, from  $C$  and the edges  $v_{i-1} v_i$ ,  $v_i x$ , and  $xy$ , we would deduce a path of  $G$  of order  $p + 2$  contradicting that  $P$  has maximum length. So, in both cases, we would deduce that  $P$  is not a longest path of  $G$ . Thus, we can assume that  $N(v_1)^- \cap N(v_p)^+$  is empty, which means that  $|N(v_1)^- \cup N(v_p)^+| = |N(v_1)^-| + |N(v_p)^+|$ .

Also, there cannot be an edge  $xv_i$  with  $v_i \in N(v_1)^-$ . Indeed, recall that we cannot have  $xv_1 \in E(G)$ . Now, if we had  $xv_i \in E(G)$  for some  $v_i \in N(v_1)^- \setminus \{v_1\}$ , then, by definition, we would have  $v_{i+1} \in N(v_1)$ . Then we would deduce that  $xv_i \dots v_1 v_{i+1} \dots v_p$  is a path of order  $p + 1$  of  $G$ , a contradiction to the choice of  $P$ . Similarly, we cannot have an edge  $xv_i$  with  $v_i \in N(v_p)^+$ .

So, by earlier arguments, we have  $N(v_1)^- \cap N(v_p)^+ = \emptyset$ . Also,  $uv_1$  and  $uv_p$  are not edges, and  $v_1 v_p$  is not an edge. By our assumption on  $G$ , we thus have  $d(v_1) + d(v_p) + d(u) \geq d^*(v_1, v_p, u) \geq \overline{\sigma}_3(G) \geq n - 2$ . Now, by our assumption on  $u$  and previous remarks,

$$\begin{aligned} d(u) &\leq n - 3 - |N(v_1)^- \cup N(v_p)^+| \\ &\leq n - 3 - |N(v_1)^-| - |N(v_p)^+|. \end{aligned}$$

Now, since  $|N(v_1)^-| = d(v_1)$  and  $|N(v_p)^+| = d(v_p)$ , from this we deduce

$$d^*(v_1, v_p, u) \leq d(v_1) + d(v_p) + d(u) \leq n - 3 < n - 2,$$

which is a contradiction to the fact that we have  $\overline{\sigma}_3(G) \geq n - 2$ . Thus, if  $G[R]$  has edges, then all  $u \in R$  must have  $d_R(u) \geq r - 2$ .  $\square$

## 2. Stability and cardinality of $R$

Due to Claim 1, note that either

- $G[R]$  has no edges, *i.e.*,  $R$  is a stable set, or
- $G[R]$  has edges, in which case  $\delta(G[R]) \geq r - 2$ .

Regarding the latter case, note in particular that if  $G[R]$  has edges, then  $G[R]$  must be connected. Furthermore,  $G[R]$  can be obtained from a complete graph by removing the edges of a matching.

Through the next two claims, we actually exclude the latter case above.

**Claim 2.** *If  $G[R]$  has edges, then  $G[R]$  must be a complete graph.*

*Proof of the claim.* Assume this is wrong, and let  $u$  be a vertex of  $R$  such that  $d_R(u) = r - 2$  and  $u$  is incident to edges in  $G[R]$  (this  $u$  exists by Claim 1, since  $G$  is not complete and  $G[R]$  has edges). That is,  $u$  neighbours all other vertices of  $R$  but exactly one of them.

By very similar counting arguments as in the proof of Claim 1, we here have

$$\begin{aligned} d(u) &\leq n - 2 - |N(v_1)^- \cup N(v_p)^+| \\ &\leq n - 2 - |N(v_1)^-| - |N(v_p)^+|. \end{aligned}$$

By similar arguments as in the proof of Claim 1, we have  $|N(v_1)^-| = d(v_1)$  and  $|N(v_p)^+| = d(v_p)$ , and we thus have  $d(v_1) + d(v_p) + d(u) \leq n - 2$ . Since  $\overline{\sigma}_3(G) \geq n - 2$ , we deduce that we thus have  $d(v_1) + d(v_p) + d(u) = n - 2$ , and, because  $d^*(v_1, v_p, u) = d(v_1) + d(v_p) + d(u) - |N(v_1) \cap N(v_p) \cap N(u)| \leq d(v_1) + d(v_p) + d(u)$ , vertices  $v_1$ ,  $v_p$ , and  $u$  must not have common neighbours. In particular, for the equality  $d(v_1) + d(v_p) + d(u) = n - 2$  to hold, vertex  $u$  must be adjacent to every vertex in  $\{v_1, \dots, v_p\} \setminus (N(v_1)^- \cup N(v_p)^+)$ . Recall that  $N(v_1)^- \cap N(v_p)^+$  is empty.

Assume there is an edge  $uv_i$ . Recall that  $v_i \notin \{v_1, v_2, v_{p-1}, v_p\}$ , as otherwise, because  $u$  is incident to edges of  $G[R]$ , we could find a path of  $G$  longer than  $P$ . Thus,  $v_{i-2}$  and  $v_{i+2}$  exist. Also,  $uv_{i-1}$  and  $uv_{i+1}$  cannot be edges, as otherwise, here as well, as earlier we would find a path of  $G$  longer than  $P$ . Since  $u$  must be adjacent to every vertex not in  $N(v_1)^- \cup N(v_p)^+$ , we get that  $v_{i-1}$  and  $v_{i+1}$  belong both to  $N(v_1)^- \cup N(v_p)^+$ .

- If  $v_{i-1}$  and  $v_{i+1}$  belong to distinct of  $N(v_1)^-$  and  $N(v_p)^+$ , then either  $v_{i-1} \in N(v_1)^-$  and  $v_{i+1} \in N(v_p)^+$ , or  $v_{i-1} \in N(v_p)^+$  and  $v_{i+1} \in N(v_1)^-$ . In the first case, note that  $v_i \in N(v_1) \cap N(v_p) \cap N(u)$ , which is not allowed. In the second case, we have  $v_{i-2} \in N(v_p)$  and  $v_{i+2} \in N(v_1)$ , in which case we can find a path of  $G$  longer than  $P$ , a contradiction. One such path first follows  $v_1 \dots v_{i-2} v_p \dots v_i$ , then goes to  $u$ , and then traverses edges of  $G[R]$ .
- If  $v_{i-1}$  and  $v_{i+1}$  belong both to  $N(v_1)^-$  or  $N(v_p)^+$ , then, assuming, without loss of generality, they belong both to  $N(v_1)^-$ , we have  $v_{i-1}, v_{i+1} \in N(v_1)^-$ , and thus  $v_i, v_{i+2} \in N(v_1)$ . Again, a path longer than  $P$  can be deduced: start, *e.g.*, by following  $v_p \dots v_{i+2} v_1 \dots v_i$ , then go to  $u$ , and lastly traverse edges of  $G[R]$ .

Thus there must be no edges incident to  $u$  going to a vertex in  $\{v_1, \dots, v_p\}$ , meaning that every  $v_i$  must lie in  $N(v_1)^- \cup N(v_p)^+$ . Note that  $v_1$  lies trivially in  $N(v_1)^-$  since  $v_2 \in N(v_1)$ . Regarding  $v_2$ , note that we cannot have  $v_2 \in N^+(v_p)$ , as this would imply  $v_p v_1$  is an edge, which is not allowed. Thus,  $v_3 \in N(v_1)$ , so that  $v_2 \in N(v_1)^-$ . Regarding  $v_3$ , note that having  $v_3 \in N^+(v_p)$  would mean  $v_p v_2$  is an edge, and, since  $v_1 v_3$  is an edge, we would deduce that  $v_1 v_3 \dots v_p v_2 v_1$  is a cycle, from which any edge joining  $P$  and  $G[R]$  (which exists since  $G$  is connected) would imply the existence of a path longer than  $P$ , a contradiction. Repeating those arguments to  $v_4, v_5$ , and so on, eventually we deduce that  $v_1 v_p$  must be an edge (because  $v_{p-1}$  must lie in  $N(v_1)^-$ ), which is not allowed. This is a final contradiction. Thus,  $G[R]$  must be complete.  $\square$

**Claim 3.**  *$G[R]$  cannot be a complete graph of order at least 2.*

*Proof of the claim.* Assume this is wrong. By similar arguments as in the proofs of previous Claims 1 and 2, for any  $u \in R$ , since  $G[R]$  is complete and has edges, we now have

$$\begin{aligned} d(u) &\leq n - 1 - |N(v_1)^- \cup N(v_p)^+| \\ &\leq n - 1 - |N(v_1)^-| - |N(v_p)^+|, \end{aligned}$$

and thus  $d(v_1) + d(v_p) + d(u) \leq n - 1$ . Since  $\overline{\sigma}_3(G) \geq n - 2$ , this means that  $u$  must neighbour all vertices of  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$  but possibly one. Also,  $v_1$ ,  $v_p$ , and  $u$  have at most one common neighbour, and, in case they do, then  $u$  must neighbour all vertices of  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$ .

Since  $G$  is connected, we can assume there is an edge  $uv_i$  for some  $i \in \{1, \dots, p\}$ . Furthermore, since  $G[R]$  is a complete graph on at least two vertices, note that  $v_{i-2}$ ,  $v_{i-1}$ ,  $v_{i+1}$ , and  $v_{i+2}$  exist, by maximality of  $P$ . Also, still because  $P$  is a longest path of  $G$ , recall that  $u$  cannot neighbour any of  $v_{i-1}$  and  $v_{i+1}$ . Since  $u$  is supposed to neighbour all vertices of  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$  but at most one, it must be that at least one of  $v_{i-1}$  and  $v_{i+1}$  lies in  $N(v_1)^- \cup N(v_p)^+$ . Assume  $v_{i-1}$  does. Actually, we cannot have  $v_{i-1} \in N(v_p)^+$ , that is,  $v_{i-2} \in N(v_p)$ , as otherwise first following  $v_1 \dots v_{i-2} v_p \dots v_i u$  and then traversing edges of  $G[R]$  would yield a path of  $G$  longer than  $P$ , a contradiction. Thus,  $v_{i-1} \in N(v_1)^-$ , that is,  $v_1 v_i$  is an edge.

Now consider  $u' \neq u$ , any other vertex of  $R$ . Recall that  $uu'$  is an edge. Also, since  $G$  is 2-connected, there must be such a  $u'$  that neighbours a vertex  $v_j \neq v_i$  (as otherwise either  $u$  or  $v_i$  would be an articulation vertex). Note that  $u'$  cannot neighbour any of  $v_{i-2}$ ,  $v_{i-1}$ ,  $v_{i+1}$ , and  $v_{i+2}$  as otherwise we would deduce a path of  $G$  longer than  $P$ . Actually, because  $v_{i-1} \in N(v_1)^-$ , we already know that we cannot have  $u'v_{i-1}$ . Regarding  $u'v_{i+1}$ , note that it cannot be forbidden through having  $v_{i+1} \in N(v_1)^-$ , that is, having the edge  $v_1 v_{i+2}$ , as this would yield the path  $v_p \dots v_{i+2} v_1 \dots v_i u u'$  which is longer than  $P$ . So either  $v_{i+1} \in N(v_p)^+$  (that is,  $v_p v_i$  is an edge), or  $u'v_{i+1}$  is the only missing edge incident to  $u'$  and to a vertex of  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$ .

- Assume  $v_p v_i$  is an edge. In this case, note that  $v_i$  is a common neighbour of  $v_1$ ,  $v_p$ , and  $u$ , meaning that  $u$  must be adjacent to all vertices  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$ . In particular,  $uv_j$  is an edge. By earlier arguments,  $v_j \notin \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$ . Also, as earlier  $v_{j-2}$  and  $v_{j+2}$  must exist,  $uv_{j-1}$  and  $uv_{j+1}$  cannot be edges due to the maximality of  $P$ , and since  $u$  must be adjacent to all vertices of  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$ , similarly as before we deduce that we must have  $v_1 v_j$  and  $v_p v_j$ . Thus,  $v_1$ ,  $v_p$ , and  $u$  have at least two common neighbours, meaning that  $d^*(v_1, v_p, u) < n - 2$ , while  $\overline{\sigma}_3(G) \geq n - 2$ , a contradiction.
- Assume  $u'v_{i+1}$  is the only missing edge incident to  $u'$  and to a vertex in  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$  (that is,  $v_{i+1} \notin N(v_1)^- \cup N(v_p)^+$ ). This means that  $v_1$ ,  $v_p$ , and  $u'$  cannot have common neighbours, and that  $u'$  must be adjacent to every vertex of  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$  different from  $v_{i+1}$ . Since  $uv_{i+1}$  also cannot be an edge, the fact that  $v_{i+1} \notin (N(v_1)^- \cup N(v_p)^+)$  means that  $uv_{i+1}$  is also the only missing edge incident to  $u$  and to a vertex in  $V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$ . So, now, regarding  $v_j$ , by similar arguments as earlier we deduce that  $v_1 v_j$  and  $v_p v_j$  must be edges so that  $uv_{j-1}$  and  $uv_{j+1}$  are not. But then  $v_j$  is a common neighbour of  $v_1$ ,  $v_p$ , and  $u$ , while  $u$  also misses  $uv_{i+1}$  while  $v_{i+1} \notin N(v_1)^- \cup N(v_p)^+$ . Thus, we have  $d^*(v_1, v_p, u) < n - 2$  but  $\overline{\sigma}_3(G) \geq n - 2$ , which is a contradiction.

Thus, if  $r \geq 2$ , then  $G[R]$  cannot be complete.  $\square$

Due to Claims 1 to 3, from now on we can thus assume that  $R$  is a stable set. We now prove that, under that assumption,  $r$  must be small, *i.e.*, at most 2.

**Claim 4.**  $r \leq 2$ .

*Proof of the claim.* Set  $R = \{u_1, \dots, u_r\}$ . Since  $R$  is stable, every vertex  $u \in R$  verifies  $N(u) \subset \{v_1, \dots, v_p\}$ . Remind also that, by maximality of  $P$ , vertex  $u$  can be adjacent to neither  $v_1$  nor  $v_p$ . Thus the vertices in  $R$  have all their neighbours in  $\{v_2, \dots, v_{p-1}\}$ .

Note that if  $u_i$  and  $u_j$  are two distinct vertices of  $R$ , then  $|N(u_j) \cap N(u_i)^+| \leq 1$ . Indeed, assume there are two distinct vertices  $v_{i_1+1}, v_{i_2+1} \in N(u_j) \cap N(u_i)^+$ , where  $i_1 < i_2$ . Then  $v_{i_1}, v_{i_2} \in N(u_i)$ , and  $v_1 \dots v_{i_1} u_i v_{i_2} \dots v_{i_1+1} u_j v_{i_2+1} \dots v_p$  is a path of  $G$  of order  $p + 2$ , a contradiction to the maximality of  $P$ . Similarly,  $|N(u_j) \cap N(u_i)^-| \leq 1$ .

Towards a contradiction to the claim, assume  $r \geq 3$ . So,  $u_1$ ,  $u_2$ , and  $u_3$  exist. We define the following metric  $\rho$  for the vertices in  $\{v_1, \dots, v_{p-1}\}$ . We set  $\rho(1) = -1$ . Then, for every  $i \in \{2, \dots, p-1\}$  such that  $\rho(i-1)$  is defined, we define  $\rho(i)$  as follows:

- if  $v_i$  is adjacent to no vertex in  $\{u_1, u_2, u_3\}$ , then  $\rho(i) = \rho(i-1) - 1$ ;
- if  $v_i$  is adjacent to exactly one vertex in  $\{u_1, u_2, u_3\}$ , then  $\rho(i) = \rho(i-1)$ ;
- otherwise, if  $v_i$  is adjacent to two or three vertices in  $\{u_1, u_2, u_3\}$ , then  $\rho(i) = \rho(i-1) + 1$ .

In other words,  $\rho(i)$  refers to the difference  $X - Y$ , where  $X$  denotes the number of edges incident to  $v_1, \dots, v_i$  going to vertices in  $\{u_1, u_2, u_3\}$  (with the subtlety that for a common neighbour of  $u_1$ ,  $u_2$ , and  $u_3$  we count only two edges) while  $Y$  denotes the number of vertices in  $\{v_1, \dots, v_i\}$ , which is precisely  $i$ . Note that, indeed, we have  $\rho(1) = -1$  since  $v_1$  is incident to none of  $u_1$ ,  $u_2$ , and  $u_3$ .

Note that, for every  $i \geq 3$ , we cannot have  $\rho(i) = \rho(i-1) + 1 = \rho(i-2) + 1$  because two consecutive  $v_i$ 's cannot both be adjacent to the same  $u_i$  by the maximality of  $P$ . Thus, for the  $\rho(i)$ 's to grow by 2, some  $v_i$  must be adjacent to two or three vertices in  $\{u_1, u_2, u_3\}$ , some next  $v_i$ 's, say  $v_{i+1}, \dots, v_{j-1}$ , must then be adjacent to exactly one vertex in  $\{u_1, u_2, u_3\}$  each, before the next vertex,  $v_j$ , is adjacent to two vertices in  $\{u_1, u_2, u_3\}$ . Actually, the first  $v_i$  in that sequence cannot be adjacent to the three vertices in  $\{u_1, u_2, u_3\}$ , as, by an earlier remark, this would make it impossible for  $v_{i+1}$  to have neighbours in  $\{u_1, u_2, u_3\}$ . Thus, this first  $v_i$  must be adjacent to exactly two vertices in  $\{u_1, u_2, u_3\}$ . More generally, note that if some  $v_i$  neighbours all three vertices in  $\{u_1, u_2, u_3\}$ , then both  $v_{i-1}$  and  $v_{i+1}$  cannot have neighbours in  $\{u_1, u_2, u_3\}$ .

We claim that  $\rho(i) \leq 1$  for every  $i \in \{1, \dots, p-1\}$ . Assume this is wrong, that is we have  $\rho(i) = 2$  for some  $i \in \{2, \dots, p-1\}$ . Because  $\rho(1) = -1$ , by the remarks above there must be an  $x$  such that  $\rho(x-1) = -1$  and  $\rho(x) = 0$ , a  $y > x+1$  such that  $\rho(x) = \rho(x+1) = \dots = \rho(y-1) = 0$  and  $\rho(y) = 1$ , and a  $z > y+1$  such that  $\rho(y) = \rho(y+1) = \dots = \rho(z-1) = 1$  and  $\rho(z) = 2$ . Still by the remarks above, we have that  $v_x$ ,  $v_y$ , and  $v_z$  must each be adjacent to exactly two vertices in  $\{u_1, u_2, u_3\}$ , while all of  $v_{x+1}, \dots, v_{y-1}, v_{y+1}, \dots, v_{z-1}$  must be adjacent to exactly one vertex in  $\{u_1, u_2, u_3\}$  each. Without loss of generality, assume  $v_x$  is adjacent to  $u_1$  and  $u_2$ . Then  $v_{x+1}$  must be adjacent to  $u_3$ . Also, there cannot be another  $i$  such that  $v_i \in N(u_1) \cup N(u_2)$  and  $v_{i+1} \in N(u_3)$ , by an earlier remark. This implies that  $v_y$  cannot be adjacent to  $u_3$ , and  $v_y$  must thus be adjacent to  $u_1$  and  $u_2$ , while  $v_{y-1}$  must be adjacent to  $u_3$ . This implies there cannot be another  $i$  such that  $v_i \in N(u_3)$  and  $v_{i+1} \in N(u_1) \cup N(u_2)$ . Now we get to a contradiction whatever two vertices in  $\{u_1, u_2, u_3\}$  are adjacent to  $v_z$ , regardless of what other vertex in  $\{u_1, u_2, u_3\}$  is adjacent to  $v_{z-1}$ .

Thus, we must have  $\rho(p-1) \leq 1$ , which means that the number of edges incident to  $v_1, \dots, v_p$  going to vertices in  $\{u_1, u_2, u_3\}$  (counting only two edges for the  $v_i$ 's being common neighbours of  $u_1$ ,  $u_2$ , and  $u_3$ ) is at most  $p$ . In other words,  $d^*(u_1, u_2, u_3) \leq p$ . Since  $n = p + r$  and  $r \geq 3$ , we have  $p \leq n - 3$ , and thus  $\bar{\sigma}_3(G) \leq d^*(u_1, u_2, u_3) < n - 2$ , a contradiction to the fact that  $\bar{\sigma}_3(G) \geq n - 2$ . Thus,  $r \leq 2$ .  $\square$

### 3. APness of $G$

We are now ready to focus on the APness of  $G$ , given that  $R$  is a stable set of cardinality  $r$  at most 2. We consider the two possible values as  $r$  separately.

**Claim 5.** *If  $r = 2$ , then  $G$  is AP if and only if  $\alpha(G) \leq \lceil n/2 \rceil$ .*

*Proof of the claim.* Let us sum up the information we have at this point. In  $G$ , we have the main path  $P = v_1 \dots v_p$  and exactly two vertices  $u_1$  and  $u_2$  in  $R = V(G) \setminus V(P)$ . Then,  $n = p + 2$ . Also,  $R$  is stable, so  $N(u_1) \cup N(u_2) \subset V(P)$ . We set  $A = N(u_1) \setminus N(u_2)$ ,  $B = N(u_2) \setminus N(u_1)$ , and  $C = N(u_1) \cap N(u_2)$ . Then,  $N(u_1) \cup N(u_2) = A \cup B \cup C$ . We also set  $|A| = a$ ,  $|B| = b$ , and  $|C| = c$ .

Recall that  $v_1$  cannot have neighbours in  $\{u_1, u_2\}$ ,  $A^+$ ,  $B^+$ , or  $C^+$ . Therefore, we have

$$d(v_1) \leq p - 1 - a - b - c.$$

Also, because  $\{v_1, u_1, u_2\}$  is independent, we have

$$\bar{\sigma}_3(G) \leq d^*(v_1, u_1, u_2) \leq d(v_1) + d(u_1) + d(u_2) \leq p - 1 - a - b - c + a + b + 2c = n - 3 + c.$$



Now, since  $\overline{\sigma_3}(G) \geq n-2$ , we must have  $c \geq 1$ . Let  $x \in \{2, \dots, p-1\}$  be an index such that  $v_x \in C$ . If  $c = 1$ , then note that, according to the computations above, we have  $d(v_1) + d(u_1) + d(u_2) = n-2$ . In particular,  $v_1 v_x$  cannot be an edge, because this would mean  $v_x \in N(v_1) \cap N(u_1) \cap N(u_2)$ , and this would imply  $\overline{\sigma_3}(G) \leq d^*(v_1, u_1, u_2) \leq n-3 < n-2$ , a contradiction. So  $v_1 v_x$  cannot be an edge, and for  $d(v_1) + d(u_1) + d(u_2) = n-2$  to hold, it must be that  $v_x \in A^+ \cup B^+ \cup C^+$  (so that  $v_1 v_x$  does not contribute to the computations above). Thus  $u_1 v_{x-1}$  or  $u_2 v_{x-1}$  is an edge, and since both  $u_1 v_x$  and  $u_2 v_x$  are edges, we can find a path contradicting the maximality of  $P$ . Hence,  $c \geq 2$ . Let thus  $y \neq x$  be an index such that  $v_y \in C$ . We assume below that  $x < y$ . In particular,  $d(u_1), d(u_2) \geq 2$  (which, actually, also follows from the fact that  $G$  is 2-connected).

Let  $\pi$  be an  $n$ -partition. We investigate whether  $\pi$  is realisable in  $G$ . If  $|\text{sp}(\pi)| \geq 3$ , then, by Theorem 2.1,  $\pi$  can be realised in  $G$ . Thus, we can assume  $|\text{sp}(\pi)| \leq 2$ . In what follows, we assume first that  $\pi = (\alpha, \dots, \alpha, \beta, \dots, \beta)$ , where  $\alpha < \beta$ , that is,  $|\text{sp}(\pi)| = 2$ .

We start by applying the picking process, described in Section 2, along  $P$ , as going from  $v_1$  to  $v_p$ , picking parts of size  $\alpha$  first (as many such parts as indicated by  $\pi$ ), and, once no more parts of size  $\alpha$  must be picked, then picking parts of size  $\beta$  (because  $|V(P)| = n-2$ , the process, if led entirely, would actually end up with a part of size  $\beta-2$ ). Once  $v_x$  is added to a (possibly partial) part, we pause the process, and analyse the situation we have reached. Several scenarios can occur.

- $v_x$  is added to a part (of size  $\alpha$  or  $\beta$ ) that misses at least two vertices to reach the desired size. In this case, we grab  $u_1$  and  $u_2$  from  $v_x$  before resuming the process (taking into account that the current part has two vertices more). Once the whole process finishes, we end up with a realisation of  $\pi$  in  $G$ . Note in particular that there is a part containing  $v_x$ ,  $u_1$ , and  $u_2$ , which induces a connected graph due to the edges  $v_x u_1$  and  $v_x u_2$ .
  - $v_x$  is added to a part of size  $\alpha$  that misses at most one vertex.
    - If that part misses exactly one vertex, then we grab  $u_1$  and  $u_2$  from  $v_x$ , to turn that part into a part of size  $\alpha+1 \leq \beta$ . We then resume the process along  $v_{x+1} \dots v_p$  so that this part of size  $\alpha+1$  is complemented to a part of size  $\beta$ , before picking the remaining parts arbitrarily. Eventually, a realisation of  $\pi$  in  $G$  results.
    - Assume now that part misses no vertex.
      - \* If  $\beta \geq \alpha+2$ , then grab  $u_1$  and  $u_2$  from  $v_x$  to turn that part into a part of size  $\alpha+2$ , and then resume the picking process along  $v_{x+1} \dots v_p$ , taking into account that the current part must be complemented to a part of size  $\beta$ , and might thus miss vertices (when  $\beta > \alpha+2$ ). Eventually, the process ends up with a realisation of  $\pi$  in  $G$ .
      - \* Now assume  $\beta = \alpha+1$ . Start by grabbing  $u_1$  from  $v_x$  to form a part of size  $\beta$ . Note that, if we resume the picking process, then it remains at least one part of size  $\alpha$  to pick. If there also remain parts of size  $\beta$  to be picked, then Theorem 2.1 implies we can pick the remaining parts along  $v_{x+1} \dots v_p$  in such a way that  $u_2$  can be grabbed from  $v_y$ . From here, by then going on arbitrarily along  $v_{y+1} \dots v_p$ , we can obtain a realisation of  $\pi$  in  $G$ . Otherwise, it means that  $\pi = (\alpha, \dots, \alpha, \alpha+1)$  for some  $\alpha$ , that is  $\beta = \alpha+1$  and only one part of size  $\beta$  is requested. If  $\alpha = 1$ , then obviously a realisation of  $\pi$  in  $G$  is easy to construct from scratch. If  $\alpha = 2$ , then note that a realisation of  $\pi$  exists if and only if  $(2, \dots, 2, 1)$  is realisable in  $G$ , that is, if and only if  $G$  admits a quasi-perfect matching. So, we can further assume  $\alpha \geq 3$ .
- By all hypotheses we made so far, we have  $n-1 \equiv 0 \pmod{\alpha}$ . Consider running the picking process along  $P$  again, as going from  $v_1$  to  $v_p$ , picking parts of size  $\alpha$  as much as possible (in particular, the last part would be of size  $\alpha-1$ ). At any point, if we can grab any of  $u_1$  or  $u_2$ , then we can obtain a realisation of  $\pi$  in  $G$ : just grab one of  $u_1$  and  $u_2$  as soon as possible, run the picking process till the end to have a partition of  $G$  into connected subgraphs of order  $\alpha$  missing only one of  $u_1$  and  $u_2$ , and eventually add the missing vertex to any of the parts it is joined to. So assume we can never grab any of  $u_1$  and  $u_2$  during the picking process. This means that all neighbours of  $u_1$  and  $u_2$  are of the form  $v_i$  for some  $i \equiv 0 \pmod{\alpha}$ . Let  $z_1 \in \{2, \dots, p-1\}$  be the smallest index such that  $v_{z_1} \in N(u_1) \cup N(u_2)$ . Similarly, let  $z_2 \in \{2, \dots, p-1\}$  be the largest index such that  $v_{z_2} \in N(u_1) \cup N(u_2)$ .

Then  $z_1, z_2 \equiv 0 \pmod{\alpha}$ . Also, since  $c \geq 2$ , we have  $z_1 \neq z_2$ . We claim that if  $v_z$  is a neighbour of  $v_1$  with  $z \in \{z_1 + 1, \dots, z_2 - 1\}$ , then  $z \equiv 0 \pmod{\alpha}$ . This is because otherwise we can obtain a realisation of  $\pi$  in the following way. Start the picking process as above but omitting  $v_1$ , that is, pick parts of size  $\alpha$  as going from  $v_2$  to  $v_p$ . Additionally, during the process, grab a vertex in  $\{u_1, u_2\}$  from  $v_{z_1}$ , and grab  $v_1$  from  $v_z$  (this is possible, by all hypotheses – in particular, note that  $v_{z_1}$  and  $v_z$  necessarily belong to different parts). Once the process finishes, this partitions all vertices of  $G$  but one in  $\{u_1, u_2\}$  into connected subgraphs of order  $\alpha$ . Then add the missing vertex to any adjacent part to obtain a realisation of  $\pi$  in  $G$ . So, to sum up,  $u_1$  and  $u_2$  can only neighbour vertices in  $\{v_i : i \equiv 0 \pmod{\alpha}\}$ . Meanwhile,  $v_1$  can only neighbour vertices in

$$\{v_2, \dots, v_{z_1-1}\} \cup \{v_i : i \in \{z_1, \dots, z_2\} \wedge i \equiv 0 \pmod{\alpha}\} \cup \{v_{z_2+2}, \dots, v_{p-1}\}$$

(in particular, recall that  $v_1$  cannot neighbour  $v_{z_2+1}$ , which is not  $v_p$  since, if  $v_{z_2+1}$  was  $v_p$ , then, because  $\alpha \geq 3$ , we would be done through running the picking process along  $P$  as going in reverse order, from  $v_p$  to  $v_1$ ). Since  $\alpha \geq 3$ , we thus have

$$d(v_1) + d(u_1) + d(u_2) \leq (z_1 - 2) + 2|\{i : i \in \{z_1, \dots, z_2\} \wedge i \equiv 0 \pmod{\alpha}\}| + (p - z_2 - 2).$$

Also, because  $\alpha \geq 3$ , note that

$$2|\{i : i \in \{z_1, \dots, z_2\} \wedge i \equiv 0 \pmod{\alpha}\}| \leq 2 \cdot \frac{z_2 - z_1}{3} + 2.$$

In particular, in  $\{v_{z_1}, \dots, v_{z_2}\}$ , only  $v_z$ 's with  $z \equiv 0 \pmod{\alpha}$  can neighbour vertices in  $\{v_1, u_1, u_2\}$ , and, in case one such vertex  $v_z$  contributes up to 2 to  $d^*(v_1, u_1, u_2)$ , then, unless  $z = z_2$ , it is succeeded by  $\alpha - 1 \geq 2$  vertices that do not contribute to  $d^*(v_1, u_1, u_2)$ . So, at worst, the vertices in  $\{v_z, v_{z+1}, v_{z+2}\}$  contribute at most 2 to  $d^*(v_1, u_1, u_2)$ , while they are three vertices. From all these, we deduce that  $d^*(v_1, u_1, u_2) < n - 2$ , which is a contradiction to the assumption on  $\overline{\sigma}_3(G)$ .

- $v_x$  is added to a part of size  $\beta$  that misses at most one vertex.

Note that, at this point of the process, after completing the part of size  $\beta$  containing  $v_x$  (by adding  $v_{x+1}$  if that part misses one vertex), we can assume there remain at least two parts of size  $\beta$  to be picked (as otherwise the rest of the vertices would be  $\beta$  vertices only, inducing a connected graph due to  $v_y$ , where  $v_y \neq v_{x+1}$ , and from this a realisation of  $\pi$  would be obtained). Recall also that we have picked all required parts of size  $\alpha$  earlier in the process.

- If  $\beta \geq \alpha + 2$  and  $v_x$  was the  $\beta$ th vertex added to its part, then by “replacing” one of the previous parts of size  $\alpha$  constructed earlier through the picking process with a part of size  $\beta$ , we essentially “shift” the succeeding parts towards  $v_p$  by exactly  $\beta - \alpha$  vertices, including the part that contains  $v_x$ , which makes it now possible, after removing the last  $\beta - \alpha$  vertices of the part, to grab  $u_1$  and  $u_2$  from  $v_x$  (note indeed that shifting parts this way cannot make  $v_x$  “change” part, as this would require  $\beta - \alpha \geq \beta$ , which is not possible since  $1 \leq \alpha < \beta$ ). Then by resuming the picking process, we eventually get a realisation of  $\pi$  in  $G$ , once the process achieves.
- If  $\beta \geq \alpha + 2$ ,  $v_x$  was the  $(\beta - 1)$ th vertex added to its part, and  $\alpha \neq 1$ , then we are done by proceeding as in the previous case. In particular, by replacing a preceding part of size  $\alpha$  with a part of size  $\beta$  (to shift parts towards  $v_p$ ), having  $v_x$  changing part requires  $\beta - \alpha \geq \beta - 1$ , which holds only if  $\alpha = 1$ . Thus, this cannot occur here. By then removing the last  $\beta - \alpha + 1$  vertices of  $v_x$ 's part, we can then grab  $u_1$  and  $u_2$  from  $v_x$ , and then resume the picking process to eventually obtain a realisation of  $\pi$  in  $G$ .
- If  $\beta \geq \alpha + 2$ ,  $v_x$  was the  $(\beta - 1)$ th vertex added to its part, and  $\alpha = 1$ , then we can assume that  $\pi = (1, \beta, \dots, \beta)$  with  $\beta \geq 3$ , that is,  $\pi$  contains only one 1. Indeed, if  $\pi$  had at least two 1's, then a realisation of  $\pi$  in  $G$  could be obtained by considering  $\{u_1\}$  and  $\{u_2\}$  as parts of size 1, and then applying the picking process along  $P$  as going

from  $v_1$  to  $v_p$ , picking parts of size  $\beta$ . So assume  $\pi = (1, \beta, \dots, \beta)$ . Note that if any of  $u_1$  and  $u_2$  neighbours a  $v_i$  with  $i \not\equiv 0 \pmod{\beta}$ , then we would be done by picking parts of size  $\beta$  along  $P$  as going from  $v_1$  to  $v_p$ , as, at some point, it would be possible to grab  $u_1$  or  $u_2$ , and, once the process achieves, a realisation of  $\pi$  in  $G$  would be obtained upon having the second of  $u_1$  or  $u_2$  forming the unique part of size 1. We can now be done through similar computations as in an earlier case. Indeed, it can be checked that if we denote by  $v_{z_1}$  and  $v_{z_2}$  the vertices in  $N(u_1) \cup N(u_2)$  with the smallest and largest index, respectively, then, again, in  $\{v_{z_1}, \dots, v_{z_2}\}$  all neighbours  $v_z$  of  $v_1$  have their index  $z$  verifying  $z \equiv 0 \pmod{\beta}$ . Also,  $v_1$  cannot have a neighbour in  $N(u_1)^+ \cup N(u_2)^+$ , and the common neighbours of  $v_1$ ,  $u_1$ , and  $u_2$  provide only 2 to  $d^*(v_1, u_1, u_2)$ . Also, as previously, if  $v_z$  with  $z \in \{z_1, \dots, z_2 - 1\}$  has at least two neighbours in  $\{v_1, u_1, u_2\}$ , then  $v_{z+1}, \dots, v_{z+\beta-1}$  cannot have neighbours in  $\{v_1, u_1, u_2\}$ . All these arguments show, as earlier, that  $d^*(v_1, u_1, u_2) < n-2$ , which is a contradiction to the fact that  $\overline{\sigma_3}(G) \geq n-2$ .

- Now, assume  $\beta = \alpha + 1$ . Assume first  $\alpha = 1$ . Obviously, if  $\pi$  contains at least two 1's, then a realisation of  $\pi$  in  $G$  can easily be deduced as in a previous case. Now, if  $\pi$  contains exactly one 1, then  $\pi = (1, 2, \dots, 2)$ , and finding a realisation of  $\pi$  in  $G$  is equivalent to finding a quasi-perfect matching of  $G$ .

Assume now  $\alpha \geq 2$ . If the part containing  $v_x$  misses only one vertex, then, as previously, we are done through replacing one of the previous parts of size  $\alpha$  with a part of size  $\alpha + 1$ , as it shifts parts toward  $v_p$  by one vertex and makes it possible now, because  $\beta \geq 3$ , after removing the last vertex of the part, to grab both  $u_1$  and  $u_2$  from  $v_x$ . If the part containing  $v_x$  misses no vertex, then, again, replace a previous part of size  $\alpha$  with a part of size  $\alpha + 1$  to shift parts towards  $v_p$  by one vertex, so that, after removing the last vertex of that part, we can now grab  $u_1$  from  $v_x$ . It remains the vertices of  $\{v_{x+1}, \dots, v_p\} \cup \{u_2\}$  to partition, which induce a connected graph due to  $v_y$ . Also, the remaining part sizes form a partition of the form  $(\alpha, \alpha + 1, \dots, \alpha + 1)$ . Theorem 2.1 tells it is possible to pick parts so that, eventually, a realisation of  $\pi$  in  $G$  results.

Thus, in any case, we end up with a realisation of  $\pi$  in  $G$ , assuming  $|\text{sp}(\pi)| = 2$  and  $\pi$  is not  $(2, \dots, 2)$  or  $(2, \dots, 2, 1)$ .

Now assume  $\pi = (\lambda, \dots, \lambda)$ , that is,  $|\text{sp}(\pi)| = 1$ . Towards proving the claim, assume  $\lambda \geq 3$ . Start by applying the picking process along  $P$  as going from  $v_1$  to  $v_p$ , picking parts of size  $\lambda$  as long as possible (in particular, if the process was achieved completely, then the part containing  $v_p$  would be of size  $\lambda - 2$ ). Along the process, after treating  $v_x$ , we can assume its part misses exactly one vertex. Indeed, if at least two vertices are missing, then we could just grab  $u_1$  and  $u_2$  from  $v_x$  and resume the process to obtain a realisation of  $\pi$  in  $G$ . Now, if no vertex is missing, then repeat the picking process, but as going from  $v_p$  and  $v_1$  instead. It can be checked that because  $n \equiv 0 \pmod{\lambda}$  and  $p = n - 2$ , this time when treating  $v_x$  the part misses at least one vertex. Thus, we can indeed suppose that, after treating  $v_x$ , the part misses exactly one vertex.

Note then that if we grab  $u_1$  from  $v_x$  and resume the process, then, later on,  $v_y$  must be the  $\lambda$ th vertex added to its part, as otherwise we could just grab  $u_2$  from  $v_y$ , and, upon leading the picking process to its end, obtain a realisation of  $\pi$  in  $G$ .

To make things clear, let us run the picking process from  $v_1$  to  $v_p$  again, picking parts of size  $\lambda$ , with the exception that we make sure  $v_x$  is the last vertex of its part  $X$  (thus of size  $\lambda - 1$ ), and the very last part (containing  $v_p$ ) is of size  $\lambda - 1$ . Regarding the resulting parts preceding  $X$ , note that  $u_1$  and  $u_2$  can only neighbour the  $\lambda$ th vertex as otherwise, if they neighboured another of these vertices, say,  $v_z$ , then a realisation of  $\pi$  would be obtained through running the picking process and grabbing  $u_1$  and  $u_2$  from  $v_z$  and  $v_y$  (as, indeed, grabbing one of  $u_1$  and  $u_2$  from  $v_z$ , because  $\lambda \geq 3$ , would have the effect to shift succeeding parts one vertex towards  $v_1$ , and thus make it possible to grab the second vertex from  $v_y$ ). Now, obviously, in all parts succeeding  $X$ , vertices  $u_1$  and  $u_2$  can only neighbour the  $\lambda$ th vertex (because  $v_x$  neighbours both  $u_1$  and  $u_2$ , and its part  $X$  has size  $\lambda - 1$ ). Regarding  $v_1$ , we recall it cannot neighbour a vertex in  $N(u_1)^+ \cup N(u_2)^+$ , by maximality of  $P$ . So, to sum up:

- Any two vertices in  $\{v_1, v_p, u_1, u_2\}$  are not adjacent.

- For any part  $Y$  of size  $\alpha \in \{\lambda - 1, \lambda\}$  we constructed above:
  - If  $Y = X$ , then, for every vertex  $v_z \in Y \setminus \{v_x\}$ , note that, by our choice of  $v_x$ , it cannot be that  $v_z$  neighbours both  $u_1$  and  $u_2$ . If  $v_z$  neighbours  $v_1$  and, say,  $u_1$ , then  $v_{z+1}$  cannot neighbour any of  $v_1$  and  $u_1$  (as otherwise we would come up with a path of  $G$  longer than  $P$ ), and it cannot neighbour  $u_2$  as well. Indeed, because  $v_{z+1}$  cannot neighbour  $u_1$ , we have  $v_{z+1} \neq v_x$ . Then  $v_{z+1}$  would be, at worst, the  $(\lambda - 2)$ th vertex of  $Y$ , and we could obtain a realisation of  $\pi$  in  $G$  by repeating the picking process, but grabbing  $u_1$  and  $u_2$  from  $v_z$  and  $v_{z+1}$ , thereby forming a single part of size  $\lambda$ . Thus, if a  $v_z \in Y$  with  $v_z \neq v_x$  has two neighbours in  $\{v_1, u_1, u_2\}$ , then  $v_{z+1}$  has no neighbour in  $\{v_1, u_1, u_2\}$ . Thus, each of the first  $\lambda - 2$  vertices of  $Y$  either neighbours at most one vertex in  $\{v_1, u_1, u_2\}$ , or it neighbours two vertices in  $\{v_1, u_1, u_2\}$  in which case the next vertex has no neighbour in that set. Meanwhile,  $v_x$  neighbours  $u_1$  and  $u_2$ . Thus, the vertices of  $Y$  contribute up to at most  $\alpha + 1$  to  $d^*(v_1, u_1, u_2)$ . Recall, however, that  $v_x \notin N(v_1)^-$ . For that reason, and because  $v_x$  neighbours both  $u_1$  and  $u_2$ , note that  $v_{x+1}$ , the first vertex of the next part, cannot neighbour any vertex in  $\{v_1, u_1, u_2\}$ .
  - Otherwise,  $Y \neq X$ . Recall that  $u_1$  and  $u_2$  can only neighbour the  $\alpha$ th vertex of  $Y$ . We consider the following cases:
    - \*  $u_1$  and  $u_2$  neighbour the  $\alpha$ th element of  $Y$ ; in that case,  $v_1$  can neighbour all vertices of  $Y$ , with the property that if  $v_1$  neighbours the  $\alpha$ th element of  $Y$ , then this vertex is a common neighbour of  $v_1$ ,  $u_1$ , and  $u_2$ , in which case only two out of its three incident edges going to  $\{v_1, u_1, u_2\}$  contribute to  $d^*(v_1, u_1, u_2)$ . Thus, the vertices of  $Y$  contribute up to at most  $\alpha + 1$  to  $d^*(v_1, u_1, u_2)$ . Also, recall that  $v_1$  cannot neighbour the first vertex of the next part, succeeding  $Y$ , since that vertex lies in  $N(u_1)^+ \cup N(u_2)^+$ .
    - \* Only one of  $u_1$  and  $u_2$  neighbours the  $\alpha$ th element of  $Y$ ; then  $v_1$  can neighbour all vertices of  $Y$ . In that case, all edges incident to the vertices in  $Y$  and  $\{v_1, u_1, u_2\}$  contribute to  $d^*(v_1, u_1, u_2)$ . Thus, the vertices of  $Y$  contribute up to at most  $\alpha + 1$  to  $d^*(v_1, u_1, u_2)$ . Again,  $v_1$  cannot neighbour the first vertex of the next part.
    - \* None of  $u_1$  and  $u_2$  neighbours a vertex in  $Y$ . Then  $v_1$  can neighbour all vertices in  $Y$ , in which case the vertices of  $Y$  contribute up to at most  $\alpha$  to  $d^*(v_1, u_1, u_2)$ .

Note that it is possible to have  $v_{p-1} \in N(u_1) \cup N(u_2)$  in the special case  $\lambda = 3$ . Still, from the arguments above, we deduce that, in total,  $d^*(v_1, u_1, u_2) \leq p - 1 = n - 3$ , which is a contradiction to the fact that  $\bar{\sigma}_3(G) \leq n - 2$ . Thus,  $G$  must have other edges incident to  $v_1$ ,  $u_1$ , or  $u_2$ , from which we deduce that a realisation of  $\pi$  in  $G$  must exist.

Thus, when  $r = 2$ , the APness of  $G$  is conditioned only to  $G$  having a perfect matching or a quasi-perfect matching.  $\square$

**Claim 6.** *If  $r = 1$ , then  $G$  is AP if and only if  $\alpha(G) \leq \lceil n/2 \rceil$ .*

*Proof of the claim.* In this case, there is a main path  $P = v_1 \dots v_p$  in  $G$ , and a unique vertex  $u \in V(G) \setminus V(P)$ . By Theorem 2.1, every  $n$ -partition  $\pi$  with  $|\text{sp}(\pi)| \geq 2$  is realisable in  $G$ . Thus, regarding the APness of  $G$ , only  $n$ -sequences  $\pi = (\lambda, \dots, \lambda)$  with  $|\text{sp}(\pi)| = 1$  have to be considered.

Assume  $G$  is not AP. Then there is an  $n$ -sequence  $\pi = (\lambda, \dots, \lambda)$  that is not realisable in  $G$ . We prove below that we must have  $\lambda = 2$ . Towards a contradiction, assume  $\lambda \geq 3$ .

Set  $I = \{v_i : i \equiv 0 \pmod{\lambda}\}$ . First off, we claim that for every edge  $uv_i$ , we must have  $v_i \in I$ . This is for otherwise, if this was not true for some  $v_i$ , then we could just run the picking process (constructing parts of size  $\lambda$ ) from  $v_1$  to  $v_p$ , grab  $u$  from  $v_i$  when treating it, and, upon resuming the process, eventually obtain a realisation of  $\pi$  in  $G$ . In particular,  $uv_1$  and  $uv_p$  are not edges.

Let us denote by  $i_1, \dots, i_d$  the indexes,  $i_1 < \dots < i_d$ , such that  $v_{i_1}, \dots, v_{i_d}$  are the  $d = d(u)$  neighbours of  $u$ . Note that  $d \geq 2$  since  $G$  is 2-connected. Also,  $v_{i_1}, \dots, v_{i_d} \in I$ . Regarding the upcoming arguments, we partition the vertices of  $P$  into three sets  $V_1$ ,  $V_2$ , and  $V_3$  as follows:  $V_1$  is  $\{v_i : i < i_1\}$ ,  $V_2$  is  $\{v_i : i \in \{i_1, \dots, i_d\}\}$ , and  $V_3$  is  $\{v_i : i > i_d\}$ . Note that  $v_1 \in V_1$ ,  $v_p \in V_3$ , and  $v_{i_1}, v_{i_d} \in V_2$ . So, none of the three sets is empty. Since  $d \geq 2$  and  $\lambda \geq 3$ , note that  $|V_2| \geq 4$ .

**Subclaim 1.** *All neighbours of  $v_1$  lie in  $V_1 \cup I$ . Furthermore, if  $v_i \in V_1 \cap I$  is a vertex such that  $v_p v_i$  is an edge, then there cannot be any  $v_j \in V_1 \setminus I$  with  $j > i$  such that  $v_1 v_j$  is an edge.*

*Proof of the subclaim.* Assume the first part of the statement is wrong, that is  $v_1$  has a neighbour  $v_i$  that does not lie in  $V_1 \cup I$ . Thus,  $i > i_1$ . A realisation of  $\pi$  in  $G$  can then be obtained as follows. Start running the picking process along  $P$  as going from  $v_p$  to  $v_2$ , picking parts of size  $\lambda$ . When reaching  $v_i$ , then, because  $v_i \notin I$ , at least one more vertex must be added to the part. Then grab  $v_1$  from  $v_i$ , before resuming the process. Later on, when reaching  $v_{i_1}$ , note that the part must be missing vertices. So, here, just grab  $u$  from  $v_{i_1}$  before resuming the process. Once it achieves, this results in a realisation of  $\pi$  in  $G$ , a contradiction. So, all neighbours of  $v_1$  must lie in  $V_1 \cup I$ .

Assume now the second part of the statement is wrong. That is, let  $v_i \in V_1 \cap I$  be a vertex such that  $v_p v_i$  is an edge, and assume there is a  $v_j \in V_1 \setminus I$  with  $j > i$  such that  $v_1 v_j$  is an edge. A realisation of  $\pi$  in  $G$  can here be obtained as follows. Let  $\alpha$  be the smallest index with  $\alpha > j$  such that  $v_\alpha \in I$ . We run the picking process along the path  $v_{\alpha-1} \dots v_2$  as going from  $v_{\alpha-1}$  to  $v_2$ , picking parts so that the first part (containing  $v_{\alpha-1}$  and  $v_j$ ) has size  $\lambda - 1$ , the part containing  $v_i$  has size  $\lambda - 1$ , and the other parts have size  $\lambda$ . Note that the hypothesis on  $\alpha$  and  $j$  implies this is possible (in particular, the choice of  $\alpha$  implies that  $\alpha - 1 - i > \lambda - 1$ , so  $v_j$  and  $v_i$  cannot belong to the same part). Then, grab  $v_1$  from  $v_j$  to form a part of size  $\lambda$ , and grab  $v_p$  from  $v_i$  to form another part of size  $\lambda$ . Now resume the process, but along  $v_{p-1} \dots v_\alpha$  as going from  $v_{p-1}$  to  $v_\alpha$ , picking parts of size  $\alpha$ . In particular, when considering  $v_{i_d}$ , more vertices must be added to that part, and we can freely grab  $u$  from  $v_{i_d}$  before resuming the process. Once the process achieves, this results in a realisation of  $\pi$  in  $G$ , a contradiction. Thus, the second part of the statement holds.  $\diamond$

Note that Subclaim 1 can be derived to a similar claim regarding  $v_p$ , stating all its neighbours lie in  $V_3 \cup I$ , and if  $v_i \in V_3 \cap I$  is a vertex such that  $v_1 v_i$  is an edge, then there cannot be any  $v_j \in V_3 \setminus I$  with  $j < i$  such that  $v_p v_j$  is an edge.

Let us now analyse the possible neighbours in  $\{v_1, v_p, u\}$  for the other vertices, in  $\{v_2, \dots, v_{p-1}\}$ , with respect to all observations we made so far.

- Any  $v_i \in I$  can be a neighbour of any number of vertices in  $\{v_1, v_p, u\}$ .
- Any  $v_i \in V_2 \setminus I$  cannot have any neighbour in  $\{v_1, v_p, u\}$ .
- Any  $v_i \in V_1 \setminus (I \cup \{v_1\})$  has  $v_1$  as its only potential neighbour in  $\{v_1, v_p, u\}$ , but this can only occur if there is no  $j < i$  such that  $v_j \in I \cap N(v_p)$ .
- Any  $v_i \in V_3 \setminus (I \cup \{v_p\})$  has  $v_p$  as its only potential neighbour in  $\{v_1, v_p, u\}$ , but this can only occur if there is no  $j > i$  such that  $v_j \in I \cap N(v_1)$ .

We now count edges incident to  $v_1$ ,  $v_p$ , and  $u$ , omitting one edge for each of their common neighbours. This is exactly the quantity  $d^*(v_1, v_p, u)$ . We consider two cases:

- $v_1$  has no neighbour in  $I \cap V_3$ , and  $v_p$  has no neighbour in  $I \cap V_1$ .

In that case, all neighbours of  $v_1$  lie in  $V_1 \cup (I \cap V_2)$ , while all neighbours of  $v_p$  lie in  $V_3 \cup (I \cap V_2)$ . Recall also that all neighbours of  $u$  lie in  $I \cap V_2$ . Then  $d^*(v_1, v_p, u)$  is at most

$$|V_1| - 1 + |V_3| - 1 + 2|I \cap V_2|.$$

Now, since  $d \geq 2$  and  $\lambda \geq 3$ , from this we deduce that  $d^*(v_1, v_p, u) \leq p - 2 = n - 3$ , which is a contradiction to the fact that  $\bar{\sigma}_3(G) \geq n - 2$ . Note indeed that these arguments imply that even if some  $v_{i_j}$  ( $1 \leq j < d$ ) has two neighbours in  $\{v_1, v_p, u\}$ , then it is followed by  $\lambda - 1 \geq 2$  vertices that do not have any neighbours in  $\{v_1, v_p, u\}$ ; thus, this is, at worst, three vertices for only two incident edges contributing to  $d^*(v_1, v_p, u)$ . In some sense, the tightest scenario is when  $\lambda = 3$  and  $d = 2$ : here,  $i_d = i_1 + 3$ , vertices  $v_{i_1+1}$  and  $v_{i_1+2}$  do not have neighbours in  $\{v_1, v_p, u\}$ , while  $v_{i_1}$  and  $v_{i_d}$  might have two neighbours in  $\{v_1, v_p, u\}$  each. So, in total, the contribution to  $d^*(v_1, v_p, u)$  is 4, for exactly four vertices in  $V_2$ , and things even out.

- $v_p$  has a neighbour in  $I \cap V_1$ , or, conversely,  $v_1$  has a neighbour in  $I \cap V_3$ .

Let us assume that at least the first situation holds. That is, assume there is  $v_i$ , a neighbour of  $v_p$ , in  $I \cap V_1$ . Among all such  $v_i$ 's, we consider the one with the lowest index  $i$ . Let us now analyse the contribution to  $d^*(v_1, v_p, u)$  of the vertices in  $\{v_2, \dots, v_{p-1}\}$ .

- Along  $V_1$ , note that all vertices in  $\{v_2, \dots, v_{i-1}\}$  neighbour at most one vertex in  $\{v_1, v_p, u\}$ , which is  $v_1$ . Then all vertices  $v_j$  with  $j \geq i$  in  $I \cap V_1$  might neighbour at most two vertices in  $\{v_1, v_p, u\}$ , being  $v_1$  and  $v_p$ . Furthermore, for every such vertex  $v_j \in I \cap V_1$  with  $j \geq i$ , the next  $\lambda - 1 \geq 2$  vertices in  $V_1$  neighbour no vertex in  $\{v_1, v_p, u\}$ . Thus, again, even if some  $v_j$  contributes a lot to  $d^*(v_1, v_p, u)$  (by exactly 2), this is sort of cancelled out by  $v_{j+1}$  and  $v_{j+2}$  not contributing to  $d^*(v_1, v_p, u)$ . Thus, for the vertices in  $\{v_j, v_{j+1}, v_{j+2}\}$ , the contribution to  $d^*(v_1, v_p, u)$  is at most 2, for three vertices.
- Along  $V_3$ , if  $v_1$  has no neighbour in  $V_3 \cap I$ , then all vertices in  $\{v_{i_d+1}, \dots, v_{p-1}\}$  have at most one neighbour in  $\{v_1, v_p, u\}$  being  $v_p$ , while if  $v_1$  has a neighbour in  $I \cap V_3$ , then we have the same description as for  $V_1$ .
- Then, all vertices in  $I \cap V_2$  contribute up to at most 2 to  $d^*(v_1, v_p, u)$ , while the vertices of  $V_2 \setminus I$  do not neighbour any of the vertices in  $\{v_1, v_p, u\}$ . Here, the situation regarding the computation details is thus similar as in the previous case.

From these counting arguments, again we deduce that  $d^*(v_1, v_p, u) < n - 2$ , and thus  $\overline{\sigma}_3(G) < n - 2$ , a contradiction.

Thus, if  $r = 1$ , then we have that  $G$  is AP if and only if  $G$  has a perfect matching or a quasi-perfect matching.  $\square$

The previous two claims end up the proof of Theorem 1.2.

#### 4. Conclusion

Regarding Theorem 1.2, it is worth reporting that there are examples of graphs showing that some of the requirements in the statement are mandatory. In particular, both the condition on  $\alpha(G)$  and on the connectivity of  $G$  cannot be dropped out:

- First, consider, as  $G$ , the complete bipartite graph  $K_{k,k+2}$  for any  $k \geq 2$ . Note that  $G$  has order  $2k + 2$ , which is even, and that  $G$  is 2-connected because  $k \geq 2$ . Furthermore, every three pairwise independent vertices  $u, v$ , and  $w$  of  $G$  must belong to the same partition class, from which we deduce that  $d^*(u, v, w) \geq 2k = n - 2$ , and thus  $\overline{\sigma}_3(G) \geq n - 2$ . Also, the longest paths of  $G$  go through  $n - 1$  vertices, and thus, by Theorem 2.1, all  $n$ -partitions  $\pi$  with  $|\text{sp}(\pi)| \geq 2$  are realisable in  $G$ . Thus, if  $G$  is not AP, then it must be because of  $n$ -partitions  $\pi = (\lambda, \dots, \lambda)$  that are not realisable in  $G$ . Since every path  $P$  on  $n - 1$  vertices of  $G$  has the property that the last vertex of  $G$  neighbours every second vertex of  $P$ , it is easy to see (through, *e.g.*, the picking process) that realisations of  $\pi$  in  $G$  exist whenever  $\lambda \neq 2$ . Meanwhile, it can be noticed that  $G$  does not admit perfect matchings, thus no realisations of  $(2, \dots, 2)$ . Thus, the condition on  $\alpha(G)$  in Theorem 1.2 is an important one.
- Second, consider, as  $G$ , any graph obtained in the following way. Let  $n_1, n_2, n_3 \geq 1$  be integers, and let  $G$  be obtained from the disjoint union of three complete graphs  $C_1, C_2$ , and  $C_3$  of order  $n_1, n_2$ , and  $n_3$ , respectively, by adding a new universal vertex  $x$  (thus joined to all vertices of  $C_1, C_2$ , and  $C_3$ ). Note that  $G$  is not 2-connected, as  $x$  is an articulation vertex. Also, we have  $n = n_1 + n_2 + n_3 + 1$ , and every three pairwise independent vertices  $u, v$ , and  $w$  of  $G$  must verify, say,  $u \in V(C_1)$ ,  $v \in V(C_2)$ , and  $w \in V(C_3)$ , from which we deduce that  $\overline{\sigma}_3(G) = d^*(u, v, w) = n_1 + n_2 + n_3 - 1 = n - 2$ .

Now consider any case of  $G$  where  $n_1 \equiv 4 \pmod 6$ ,  $n_2 \equiv 5 \pmod 6$ , and  $n_3 \equiv 2 \pmod 6$ . Under those conditions, note that  $n$  is a multiple of 6, thus a multiple of 2 and 3. On the one hand, note that  $G$  admits perfect matchings (the vertices of  $C_1$  can be matched together, similarly for those of  $C_3$ , while the remaining vertices, of  $C_2$  and  $x$ , can be matched together). On the other hand, note that  $G$  admits no realisation of  $(3, \dots, 3)$  (as, in a realisation, due to the conditions on  $n_1, n_2$ , and  $n_3$ , the vertices of  $C_2$  and  $x$  would have to be covered by parts of size 3, making it impossible for the remaining vertices, of  $C_1$  and  $C_3$ , to be covered by parts of size 3 inducing connected graphs, since  $x$  is an articulation vertex). This example shows that the 2-connectivity condition in the statement of Theorem 1.2 cannot be dropped out.

- Third, consider, as  $G$ , the graph obtained by starting from four disjoint edges  $u_1v_1$ ,  $u_2v_2$ ,  $u_3v_3$ , and  $u_4v_4$ , and then adding two new independent vertices  $x$  and  $y$  joined to all  $u_i$ 's and  $v_i$ 's. Note that  $G$  has order 10, is 2-connected, and admits perfect matchings (one such is  $\{xu_1, yv_1, u_2v_2, u_3v_3, u_4v_4\}$ ). Also, any three pairwise independent vertices  $u$ ,  $v$ , and  $w$  of  $G$  do not include  $x$  and  $y$ , and are thus pairwise independent  $u_i$ 's and  $v_i$ 's. From this, we deduce that  $d^*(u, v, w) = 9 - 2 = 7 = n - 3$ , and thus  $\bar{\sigma}_3(G) = n - 3$ . However,  $G$  is not AP, as it admits no realisation of  $(4, 3, 3)$  (to see this is true, note first that, if such a realisation existed, then  $x$  and  $y$  would belong to different parts, while the last part would induce a connected subgraph of order 3 or 4 – this is impossible since  $G - \{x, y\}$  has four connected components of order 2). While this example does not generalise well to larger graphs (because of the  $\bar{\sigma}_3$  condition), it shows that the requirement on  $\bar{\sigma}_3(G)$  in the statement of Theorem 1.2 cannot be just lowered to  $n - 3$  right away, even under the other conditions on  $G$ .

Let us add also that our proof of Theorem 1.2 yields a polynomial-time algorithm to decide whether a 2-connected  $n$ -graph  $G$  with  $\bar{\sigma}_3(G) \geq n - 2$  is AP: just determine, in polynomial time, the size of a largest matching of  $G$  (which can be done through Edmonds' Blossom Algorithm [5]).

We end up with a few words on Theorem 1.1, claimed by Brandt. As mentioned in the introductory section, the only remains on this result are the title (“Finding Vertex Decompositions in Dense Graphs”) of an invited talk given in 2013, and the corresponding abstract, which can be found online, and which we report now for the sake of keeping track of it.

**Abstract** (Brandt, CID 2013). *A graph  $G = (V, E)$  is called arbitrarily vertex decomposable, if for any partition  $\pi = (n_1, n_2, \dots, n_k)$  of  $|V| = n = n_1 + n_2 + \dots + n_k$  into positive integers there is a decomposition  $V = V_1 \cup V_2 \cup \dots \cup V_k$  with  $|V_i| = n_i$  such that the graph induced by  $V_i$  is connected for all  $i$ . The decision problem whether  $G$  admits a  $\pi$ -decomposition is known as a notoriously hard problem, while the complexity status of the decision problem whether  $G$  is arbitrarily vertex decomposable is not known in general. The problem is not even known to be in NP.*

*Let  $\mathcal{G}$  be the class of graphs  $G$  where the degree sum of any set of three independent vertices of  $G$  is at least  $n$ . We show that if  $G \in \mathcal{G}$  satisfies the two necessary conditions of being connected and having a matching on  $\lfloor \frac{n}{2} \rfloor$  edges, then  $G$  is arbitrarily vertex decomposable. This gives a polynomial time algorithm deciding whether  $G \in \mathcal{G}$  is arbitrarily vertex decomposable (just determine the size of a largest matching of  $G$  and determine whether it is connected). The proof is algorithmic and gives an algorithm that finds a decomposition for any partition  $\pi$  in time  $\mathcal{O}(n^3)$ . As a starting point of this algorithm we present another algorithm that finds in a connected graph in  $\mathcal{G}$  either a hamiltonian path or a very long cycle with certain additional properties.*

It would be a bit daring to try to retrieve Brandt's proof from this abstract only, but we can at least make some guess. In the very last part of the abstract, Brandt mentions finding long paths or cycles in connected  $n$ -graphs  $G$  with  $\sigma_3(G) \geq n$ , which is indeed the usual way to proceed to establish such results, as illustrated by the results from [9, 13, 14] and our proof of Theorem 1.2. This high-level description actually reminds us of a result of Momège, who proved the following:

**Theorem 4.1** (Momège [15]). *If  $G$  is a connected  $n$ -graph with  $\sigma_3(G) \geq n$ , then either  $G$  is traceable, or any longest cycle of  $G$  is dominating.*

Recall that a cycle  $C$  of a graph  $G$  is *dominating* if every edge of  $G$  intersects  $C$ , or, in other words, if  $G - V(C)$  has no edges. Note that we used a sort of similar result in our proof of Theorem 1.2, when we proved that  $R$  must be a stable set. The next step in our proof was then proving that  $|R|$  is small, which seems less immediate to achieve under the condition that  $\sigma_3(G) \geq n$ . So, it might be that Theorem 1.1 can be proved by first making use of Theorem 4.1, but, if it can, we are not sure, however, what the next step would be.

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