



# An integral transform and its application in the propagation of Lorentz-Gaussian beams

A. Belafhal, E.M. El Halba, T. Usman

## ► To cite this version:

A. Belafhal, E.M. El Halba, T. Usman. An integral transform and its application in the propagation of Lorentz-Gaussian beams. Communications in Mathematics, 2021, Volume 29 (2021), Issue 3 (3), pp.483 - 491. 10.2478/cm-2021-0030 . hal-03665031

**HAL Id: hal-03665031**

**<https://hal.science/hal-03665031v1>**

Submitted on 11 May 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

## An integral transform and its application in the propagation of Lorentz-Gaussian beams

A. Belafhal, E.M. El Halba, T. Usman

**Abstract.** The aim of the present note is to derive an integral transform

$$I = \int_0^\infty x^{s+1} e^{-\beta x^2 + \gamma x} M_{k,\nu}(2\zeta x^2) J_\mu(\chi x) dx,$$

involving the product of the Whittaker function  $M_{k,\nu}$  and the Bessel function of the first kind  $J_\mu$  of order  $\mu$ . As a by-product, we also derive certain new integral transforms as particular cases for some special values of the parameters  $k$  and  $\nu$  of the Whittaker function. Eventually, we show the application of the integral in the propagation of hollow higher-order circular Lorentz-cosh-Gaussian beams through an ABCD optical system (see, for details [13], [3]).

### 1 Introduction

Integral transforms involving Whittaker and Bessel functions have given considerable attention in the literature. In the last decade, many papers studied a considerable number of cases of integral transforms involving the product of Bessel and special functions [6], [2], [5]. The used weight in the integrand is  $e^{\gamma x}$  or  $e^{-\beta x^2}$ . In view of this, it is worth investigating a general integral transform involving Whittaker and Bessel functions with a weight  $e^{-\beta x^2 + \gamma x}$ . This integral is important to evaluate the propagation of some laser beams in the space. A closed form of the considered integral will be derived. To the best of our knowledge, the results of the

---

2020 MSC: 33B15, 33C10, 33C15

**Key words:** Integral transform, Bessel function, Whittaker function, Confluent hypergeometric function, Lorentz-Gaussian beams.

**Affiliation:**

A. Belafhal, E. M. El Halba – LPNAMME, Laser Physics Group, Department of Physics, Faculty of Sciences, Chouaib Doukkali University, P. B 20, 24000 El Jadida, Morocco

*E-mail:* belafhal@gmail.com, elhalbams@gmail.com

Talha Usman (Corresponding author) – Department of Mathematics, School of Basic and Applied Sciences, Lingaya's Vidyapeeth, Faridabad-121002, Haryana, India.

*E-mail:* talhausman.maths@gmail.com

present contribution have not been previously published. The following definitions are essential for the present investigation:

The classical Gauss's hypergeometric function is defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (1)$$

where,  $(a)_n$  ( $a \in \mathbb{C}$ ) is the well known Pochhammer symbol (see [1], [8]). The Kummer confluent hypergeometric function  ${}_1F_1$  (see [1], [8]) is defined by the series representation

$${}_1F_1(a; b; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}, \quad (2)$$

Let us consider the following second-order linear homogenous differential equation given by

$$x^2 \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + (\alpha^2 x^{2r} + \beta^2 y) = 0,$$

whose solution is

$$y = x^{-p} \left[ C_1 J_{q/r} \left( \frac{\alpha}{r} x^r \right) C_2 Y_{q/r} \left( \frac{\alpha}{r} x^r \right) \right] \quad (q = \sqrt{p^2 - \beta^2}),$$

where,  $J_n(x)$  and  $Y_n(x)$  are the Bessel functions of first and second kind, respectively. Indeed we have the following series representation for the Bessel function of the first kind of order  $\mu$  (see [4], [10]):

$$J_{\mu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{\mu+2k}}{k! \Gamma(\mu + k + 1)}; \quad \forall x \in \mathbb{C} \setminus (-\infty, 0). \quad (3)$$

It is well known that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

and

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

The Whittaker function  $M_{k,\nu}(x)$  and  $W_{k,\nu}(x)$  (see, [11], [12]), in terms of Kummer confluent hypergeometric function defined by (2) are given, respectively by

$$M_{k,\nu}(x) = x^{\nu+\frac{1}{2}} e^{-\frac{x}{2}} {}_1F_1 \left( \frac{1}{2} + \nu - k, 2\nu + 1; x \right), \quad (4)$$

and

$$W_{k,\nu}(x) = x^{\mu+\frac{1}{2}} e^{-\frac{x}{2}} U \left( \frac{1}{2} + \nu - k, 2\nu + 1; x \right). \quad (5)$$

The generalized Laguerre polynomials  $L_n^{(\alpha)}(x)$  can be defined as

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n; 1+\alpha; x). \quad (6)$$

From (6), it follows that

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n x^n}{k!(n-k)!(1+\alpha)_k}. \quad (7)$$

## 2 Main results

In this section, we compute certain integral transforms involving the product of Bessel function and Whittaker function given in (3) and (4), respectively. These integral formulas are expressed in terms of Kummer confluent hypergeometric function as given in Theorem 1.

**Theorem 1.** Let  $\Re(\mu) > -1$ ,  $\Re(s + \nu + \frac{\mu}{2}) > -1$ , and  $\Re(\beta + \zeta) > 0$ . Consider the integral  $I$

$$I = \int_0^\infty x^{s+1} e^{-\beta x^2 + \gamma x} M_{k,\nu}(2\zeta x^2) J_\mu(\chi x) dx = (2\zeta)^{\nu+1/2} \delta^{2\nu+s+2} e^{\frac{\gamma^2}{4\epsilon^2}} \times \\ \times \sum_{r=0}^{+\infty} \frac{(\nu + 1/2 - k)_r (2\zeta \delta^2)^r}{r!(2\nu + 1)_r} \sum_{q=0}^{\vartheta} \frac{C_{\vartheta}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi \delta) C_{mq}, \quad (8)$$

where

$$C_{mq} = \left\{ \begin{aligned} & \frac{\left(\frac{\chi}{2\epsilon}\right)^m}{2m!} \frac{\Gamma\left(\frac{m+q+1}{2}\right)}{\epsilon^{q+1}} {}_1F_1\left(\frac{m+q+1}{2}; m+1; -\frac{\chi^2}{4\epsilon^2}\right) + \\ & + (-1)^{q+m} \left(\frac{\chi \delta^{1/4}}{2}\right)^m \delta^{\frac{q+1}{4}} e^{-\epsilon^2 \sqrt{\delta}} \sum_{p=0}^{+\infty} \frac{\left(\frac{-\chi^2 \sqrt{\delta}}{4}\right)^p}{p!(2p+q+m+1)\Gamma(m+p+1)} \times \\ & \times {}_1F_1\left(1; 1+p+\frac{q+m+1}{2}; \epsilon^2 \sqrt{\delta}\right) \end{aligned} \right\},$$

$$\epsilon^2 = \beta + \zeta, \delta = \frac{\gamma}{2\epsilon^2} \text{ and } \vartheta = 2\nu + 2r + s + 2. \quad (9)$$

*Proof.* To derive (8), we make use of (2) and (4), and consequently, the expression of the integral  $I$  becomes

$$I = (2\zeta)^{\nu+1/2} e^{\frac{\gamma^2}{4(\beta+\zeta)}} \sum_{r=0}^{+\infty} \frac{(\nu + 1/2 - k)_r (2\zeta)^r}{(2\nu + 1)_r r!} I_r, \quad (10)$$

where  $I_r = I'_r + I''_r$ , with

$$I'_r = \int_0^\infty (t + \delta)^{2\nu+2r+s+2} e^{-(\beta+\zeta)t^2} J_\mu(\chi t + \chi \delta) dt, \quad (11)$$

and

$$I''_r = \int_{-\delta}^0 (t + \delta)^{2\nu+2r+s+2} e^{-(\beta+\zeta)t^2} J_\mu(\chi t + \chi \delta) dt. \quad (12)$$

Making use of the following identities (see [4])

$$(t + \delta)^n = \sum_{p=0}^n C_n^p \delta^{n-p} t^p, \quad (13)$$

with  $C_n^p = \frac{n!}{p!(n-p)!}$  is the binomial coefficient, and

$$J_\mu(\chi t + \chi \delta) = \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi \delta) J_m(\chi t), \quad (14)$$

and using the integral formulas (see [4])

$$\int_0^\infty t^q e^{-\eta t^2} J_m(\chi t) dt = \frac{\left(\frac{\chi}{2\sqrt{\eta}}\right)^m \Gamma\left(\frac{m+q+1}{2}\right)}{2(\sqrt{\eta})^{q+1} m!} {}_1F_1\left(\frac{m+q+1}{2}; m+1; -\frac{\chi^2}{4\eta}\right), \quad (15)$$

and

$$\int_0^u t^{\nu-1} e^{-\mu t} dt = \frac{1}{\mu^\nu} \gamma(\nu, u\mu), \quad (16)$$

where  $\gamma$  is the incomplete Gamma function given as

$$\gamma(\alpha, x) = \alpha^{-1} x^\alpha {}_1F_1(\alpha; 1 + \alpha; -x),$$

with  $\Re(q+m) > -1$  and  $\Re(\eta) > 0$ , and Eqs. (11) and (12) can be rearranged as

$$\begin{aligned} I'_r &= \sum_{q=0}^{\vartheta} C_{\vartheta}^q \delta^{\vartheta-q} \times \\ &\times \sum_{m=-\infty}^{+\infty} \frac{J_{\mu-m}(\chi \delta) \chi^m \Gamma\left(\frac{m+q+1}{2}\right)}{2^{m+1} m! (\beta + \zeta)^{\frac{m+q+1}{2}}} {}_1F_1\left(\frac{m+q+1}{2}; m+1; -\frac{\chi^2}{4(\zeta + \beta)}\right), \end{aligned} \quad (17)$$

and

$$\begin{aligned} I''_r &= \sum_{q=0}^{\vartheta} C_{\vartheta}^q \delta^{\vartheta-q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi \delta) (-1)^{q+m} \left(\frac{\chi}{2}\right)^m \delta^{\frac{q+m+1}{4}} e^{-\varepsilon^2 \sqrt{\delta}} \times \\ &\times \sum_{p=0}^{+\infty} \frac{\left(\frac{-\chi^2 \sqrt{\delta}}{4}\right)^p}{p! (2p+q+m+1) \Gamma(m+p+1)} {}_1F_1\left(1; 1+p+\frac{q+m+1}{2}; \varepsilon^2 \sqrt{\delta}\right), \end{aligned} \quad (18)$$

with  $\vartheta = 2\nu + 2r + s + 2$ , and the required result easily follows. This completes the proof.  $\square$

**Remark**

For simulations, Eq. (14) can be replaced by

$$J_{\mu}(\chi t + \chi \delta) = \sum_{m=0}^{\mu} J_{\mu-m}(\chi \delta) J_m(\chi t) + \sum_{m=1}^{+\infty} (-1)^m [J_{\mu+m}(\chi \delta) J_m(\chi t) + J_m(\chi \delta) J_{\mu+m}(\chi t)],$$

**3 Special cases**

In this section, we derive certain new formulas by recalling relations of the Whittaker function with some other special functions as the exponential, the Modified Bessel, the sine hyperbolic, the generalized Laguerre polynomial, the Hermite polynomial, the Whittaker and erf functions. By taking some particular values of the index  $k$  and  $\nu$  of this last function, we establish the following corollaries.

**Corollary 1.** For  $\Re(\mu) > -1$ ,  $\Re(s + \nu + \frac{\mu}{2}) > -1$ ,  $\Re(\beta + \zeta) > 0$ , the undermentioned integral transform holds true:

$$\int_0^{\infty} x^{s+1} e^{-(\beta-\varepsilon)x^2 + \gamma x} J_{\mu}(\chi x) dx = \delta^{s+1-2k} e^{\frac{\gamma^2}{4\varepsilon^2}} \times \sum_{r=0}^{+\infty} \frac{(2\zeta\delta^2)^r}{r!} \sum_{q=0}^{\vartheta_1} \frac{C_{\vartheta_1}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \quad (19)$$

where

$$\vartheta_1 = 2r + s - 2k + 1.$$

For  $\nu = -(k + \frac{1}{2})$ , the Whittaker function in (4) can be expressed in terms of exponential function (see [7], [9]) as

$$M_{k, -k - \frac{1}{2}}(z) = e^{\frac{z}{2}} z^{-k}. \quad (20)$$

Now in view of the (20) and (8), it is easy to establish the result in (19).

**Corollary 2.** With the conditions of (8), the undermentioned integral transform holds true:

$$\int_0^{\infty} x^{s+2} e^{-\beta x^2 + \gamma x} I_{\nu}(\zeta x^2) J_{\mu}(\chi x) dx = \frac{(2\zeta)^{\nu+1/2}}{2^{2\nu+3/2} \varepsilon \sqrt{2\zeta} \nu!} \delta^{2\nu+s+2} e^{\frac{\gamma^2}{4\varepsilon^4}} \times \sum_{r=0}^{+\infty} \frac{(\nu+1/2)_r}{(2\nu+1)_r} \frac{(2\zeta\delta^2)^r}{r!} \sum_{q=0}^{\vartheta_2} \frac{C_{\vartheta_2}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \quad (21)$$

where

$$\vartheta_2 = 2\nu + 2r + s + 2. \quad (22)$$

With the help of (8) and by expressing the Whittaker function in terms of the modified Bessel function of the first kind  $I_\nu$  (see [9], [10]), with  $k = 0$ , as follows

$$M_{0,\nu}(2z) = 2^{2\nu+\frac{1}{2}}\nu!\sqrt{z}I_\nu(z), \quad (23)$$

the required result (21) directly follows.

**Corollary 3.** For  $\Re(\mu) > -1$ ,  $\Re(s + \frac{\mu+1}{2}) > -1$ ,  $\Re(\beta + \zeta) > 0$ , the undermentioned integral transform holds true:

$$\begin{aligned} \int_0^\infty x^{s+1} e^{-\beta x^2 + \gamma x} \sinh(\zeta x^2) J_\mu(\chi x) dx &= \frac{(2\zeta)}{4} \delta^{s+3} e^{\frac{\gamma^2}{4\epsilon^4}} \times \\ &\times \sum_{r=0}^{+\infty} \frac{(2\zeta\delta^2)^r}{(2)_r r!} \sum_{q=0}^{\vartheta_3} \frac{C_{\vartheta_3}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \end{aligned} \quad (24)$$

where

$$\vartheta_3 = 2r + s + 3. \quad (25)$$

If we take  $k = 0$  and  $\nu = \frac{1}{2}$ , the Whittaker function in (4) can be expressed in terms of sine hyperbolic function (see [7], [9]) as

$$M_{0,\frac{1}{2}}(z) = 2 \sinh\left(\frac{z}{2}\right), \quad (26)$$

from which the required result (24) easily follows.

**Corollary 4.** Let  $\Re(\mu) > -1$ ,  $\Re(s + \frac{\mu+n}{2}) > -1$ ,  $\Re(\beta + \zeta) > 0$ , the undermentioned integral transform holds true:

$$\begin{aligned} \int_0^\infty x^{s+n+2} e^{-(\beta+\zeta)x^2 + \gamma x} L_s^n(2\zeta x^2) J_\mu(\chi x) dx &= \frac{(n+1)_s}{s!} \delta^{s+n+2} e^{\frac{\gamma^2}{4\epsilon^4}} \times \\ &\times \sum_{r=0}^{+\infty} \frac{(-s)_r (2\zeta\delta^2)^r}{(n+1)_r r!} \sum_{q=0}^{\vartheta_4} \frac{C_{\vartheta_4}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \end{aligned} \quad (27)$$

where

$$\vartheta_4 = n + 2r + s + 2. \quad (28)$$

By setting  $k = \frac{n+1}{2} + s$  and  $\nu = \frac{n}{2}$  in (4), and using (6) we have the relation (see [7], [9])

$$M_{\frac{n+1}{2}+s, \frac{n}{2}}(z) = \frac{s!}{(n+1)_s} e^{\frac{-z}{2}} z^{\frac{n+1}{2}} L_s^{(n)}(z), \quad (29)$$

Now in view of (29), the assertion (27) easily follows.

**Corollary 5.**  $\Re(\mu) > -1$ ,  $\Re(s + \frac{2\mu-1}{4}) > -1$ ,  $\Re(\beta + \zeta) > 0$ , the undermentioned integral transform holds true:

$$\begin{aligned} \int_0^\infty x^{s+3/2} e^{-(\beta+\zeta)x^2 + \gamma x} H_{2p}(\sqrt{2\zeta}x) J_\mu(\chi x) dx &= \frac{2p!}{(-1)^p p!} \times \\ &\times \delta^{s+3/2} e^{\frac{\gamma^2}{4\epsilon^4}} \sum_{r=0}^{+\infty} \frac{(-p)_r (2\zeta\delta^2)^r}{(1/2)_r r!} \sum_{q=0}^{\vartheta_5} \frac{C_{\vartheta_5}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \end{aligned} \quad (30)$$

where

$$\vartheta_5 = 2r + s + 3/2. \quad (31)$$

The above corollary can easily be established by setting  $k = p + \frac{1}{4}$  and  $\nu = \frac{-1}{4}$  in (4). This special case corresponds to the Hermite polynomial  $H_{2p}$  (see [7], [9]) with the odd integer order and the relation between  $M$  and  $H_{2p}$  is given by

$$M_{p+\frac{1}{4}, -\frac{1}{4}}(z^2) = (-1)^p \frac{p!}{(2p)!} e^{\frac{-z^2}{2}} \sqrt{z} H_{2p}(z). \quad (32)$$

**Corollary 6.**  $\Re(\mu) > -1$ ,  $\Re(s + \frac{2\mu+1}{4}) > -1$ ,  $\Re(\beta + \zeta) > 0$ , the undermentioned integral transform holds true:

$$\begin{aligned} \int_0^\infty x^{s+3/2} e^{-(\beta+\zeta)x^2 - \gamma x} H_{2p+1}(\sqrt{2\zeta}x) J_\mu(\chi x) dx &= \frac{(2p+1)! \sqrt{2\zeta}}{(-1)^p p!} \times \\ &\times \delta^{s+5/2} e^{\frac{\gamma^2}{4\epsilon^4}} \sum_{r=0}^{+\infty} \frac{(3/4 - k)_r (2\zeta\delta^2)^r}{(3/2)_r r!} \sum_{q=0}^{\vartheta_6} \frac{C_{\vartheta_6}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \end{aligned} \quad (33)$$

where

$$\vartheta_6 = 2r + s + 5/2. \quad (34)$$

This corollary is established by setting  $k = p + \frac{3}{4}$  and  $\nu = \frac{1}{4}$  in (4), which consequently gives the relation between the Whittaker function and the Hermite polynomial  $H_{2p+1}$  (see [7], [9]) of an even order as

$$M_{p+\frac{3}{4}, \frac{1}{4}}(z^2) = (-1)^p \frac{p!}{(2p+1)!} \frac{e^{\frac{-z^2}{2}} \sqrt{z}}{2} H_{2p+1}(z), \quad (35)$$

and by direct application of (8).

**Corollary 7.** Let  $\Re(\mu) > -1$ ,  $\Re(s + \frac{\mu+n}{2}) > -1$ ,  $\Re(\beta + \zeta) > 0$ , the undermentioned integral transform holds true:

$$\begin{aligned} \int_0^\infty x^{s+3} e^{-\beta x^2 + \gamma x} W_{n+\frac{1+n}{2}, n/2}(2\zeta x^2) J_\mu(\chi x) dx &= \frac{(n+1)_s}{(-1)^s p! (\sqrt{2\zeta})^{\frac{1-n}{2}}} \times \\ &\times \delta^{s+n+2} e^{\frac{\gamma^2}{4\epsilon^4}} \sum_{r=0}^{+\infty} \frac{(-s)_r (2\zeta\delta^2)^r}{(n+1)_r r!} \sum_{q=0}^{\vartheta_7} \frac{C_{\vartheta_7}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \end{aligned} \quad (36)$$

where

$$\vartheta_7 = n + 2r + s + 2, \quad (37)$$

The assertion in (36) directly follows from the relation between the Whittaker function and the generalized Laguerre polynomial (see [7], [9]) given as

$$L_s^{(n)}(z) = \frac{(-1)^s}{s!} e^{\frac{z}{2}} z^{\frac{1-n}{2}} W_{n+\frac{1+n}{2}, n/2}(z), \quad (38)$$



**Corollary 8.** Let  $\Re(\mu) > -1$ ,  $\Re(s + \frac{2\mu+1}{4}) > -1$ ,  $\Re(\beta + \zeta) > 0$ , the undermentioned integral transform holds true:

$$\int_0^\infty x^{s+3/2} e^{-(\beta-\zeta)x^2+\gamma x} \operatorname{erf}(\sqrt{2\zeta}x) J_\mu(\chi x) dx = \frac{4\sqrt{2\zeta}}{\sqrt{\pi}} \delta^{s+5/2} e^{\frac{\gamma^2}{4\epsilon^4}} \times \\ \times \sum_{r=0}^{+\infty} \frac{(1)_r (2\zeta\delta^2)^r}{(3/2)_r r!} \sum_{q=0}^{\vartheta_8} \frac{C_{\vartheta_8}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_{\mu-m}(\chi\delta) C_{mq}, \quad (39)$$

where

$$\vartheta_8 = 2r + s + 5/2, \quad (40)$$

Using the fact that by taking  $k = \frac{-1}{4}$  and  $\nu = \frac{1}{4}$ , the Whittaker function becomes the error function  $\operatorname{erf}$  (see [7], [9]) as

$$M_{-\frac{1}{4}, \frac{1}{4}}(z^2) = \frac{1}{2} e^{\frac{z^2}{2}} \sqrt{\pi z} \operatorname{erf}(z), \quad (41)$$

and in view of (8), it is straightforward to establish (39).

## 4 Application

In this section we consider the propagation of hollow higher-order circular Lorentz-cosh-Gaussian beams [13] through an ABCD optical system, expressed as [3]

$$E_{p,l}^\alpha(\rho, z) = E_0 \sum_{n=0}^N a_{2n}(\alpha) \sum_{m=0}^p \frac{p!}{m!(p-m)!} e^{(m-p/2)^2\delta} I_n, \quad (42)$$

where

$$I_n = e^{-\frac{a^2}{w_0}} \int_0^\infty x^{l+1} e^{-\chi x^2 + \zeta x} H_{2n}(\sqrt{2\alpha}x) J_0(\beta x) dx. \quad (43)$$

With the help of (30),  $I_n$  can be written as

$$I_n = e^{-\frac{a^2}{w_0}} \frac{2n!}{(-1)^n n! \sqrt{2\alpha}} \frac{(2\alpha)^{1/4}}{2\epsilon} \delta^{l+1/2} e^{\frac{\zeta^2}{4\epsilon^2}} \sum_{r=0}^n \frac{(-n)_r}{(1/2)_r} \frac{(2\alpha\delta^2)^r}{r!} \sum_{q=0}^{\vartheta_6} \frac{C_{\vartheta_6}^q}{\delta^q} \sum_{m=-\infty}^{+\infty} J_m(\chi\delta) \times \\ \times \left\{ \begin{aligned} & \left( \frac{\beta}{2\epsilon} \right)^m \frac{\Gamma\left(\frac{m+q+1}{2}\right)}{\epsilon^{q+1}} {}_1F_1\left(\frac{m+q+1}{2}; m+1; -\frac{\beta^2}{4\epsilon^2}\right) + \\ & + (-1)^q \left(\frac{\beta\delta^{1/4}}{2}\right)^m \delta^{\frac{q+1}{4}} e^{-\epsilon^2\sqrt{\delta}} \sum_{p=0}^{+\infty} \frac{\left(\frac{-\beta^2\sqrt{\delta}}{4}\right)^p}{p!(2p+q+m+1)\Gamma(m+p+1)} \times \\ & \times {}_1F_1\left(1; 1+p+\frac{q+m+1}{2}; \epsilon^2\sqrt{\delta}\right) \end{aligned} \right\} \quad (44)$$

Substituting (44) in (42), one finds the output field of the considered beams family and the behaviour of this field in the space.

## 5 Conclusion

In the present investigation, we have elaborated an integral involving the product of the Whittaker and Bessel functions. The results obtained in this document have a general nature and likely to discover certain applications in the theory of beams propagation. Some corollaries are derived from the main theorem as particular cases as they give multifarious representations involving different special functions. An application of our main result is given to derive and investigate an expression of Hollow higher-order circular Lorentz-cosh-Gaussian beams propagating through an ABCD optical system. Finally, the interesting integral derived in the present investigation may be useful in many fields of physics as plasma, radio and laser physics.

## References

- [1] G.E. Andrews, R. Askey, R. Roy: *Special Functions*. Encyclopedia of Mathematics and its Applications 71. Cambridge University Press, Cambridge (1999).
- [2] R. Chen, C. An: On the evaluation of infinite integrals involving Bessel functions. *App. Math. Comput.* 235 (2014) 212–220.
- [3] S.A. Collins: Lens-system diffraction integral written in terms of matrix optics. *J. Opt. Soc. Am.* 60 (9) (1970) 1168–1177.
- [4] I.S. Gradshteyn, I.M. Ryzhik: *Table of Integrals, Series, and Products (5th edition)*. Academic Press Inc., Boston (1994).
- [5] N.U. Khan, T. Kashmin: On infinite series of three variables involving Whittaker and Bessel functions. *Palest. J. Math.* 5 (1) (2015) 185–190.
- [6] N.U. Khan, T. Usman, M. Ghayasuddin: A note on integral transforms associated with Humbert's confluent hypergeometric function. *Electron. J. Math. Anal. Appl.* 4 (2) (2016) 259–265.
- [7] E.D. Rainville: *Intermediate Differential Equations*. Macmillan (1964).
- [8] E.D. Rainville: *Special Functions*. Macmillan Company, New York. Reprinted by Chelsea Publishing Company, Bronx, New York (1971) (1960).
- [9] H.M. Srivastava, H.L. Manocha: *A Treatise on Generating Functions*. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press, New York (1984).
- [10] G.N. Watson: *A Treatise on the Theory of Bessel Functions (second edition)*. Cambridge University Press, Cambridge (1944).
- [11] E.T. Whittaker: An expression of certain known functions as generalized hypergeometric functions. *Bull. Amer. Math. Soc.* 10 (3) (1903) 125–134.
- [12] E.T. Whittaker, G.N. Watson: *A Course of Modern Analysis (reprint of the fourth (1927) edition)*. Cambridge Mathematical Library, Cambridge University Press, Cambridge (1996).
- [13] Y. Xu, G. Zhou: Circular Lorentz-Gauss beams. *J. Opt. Soc. Am. A.* 36 (2) (2019) 179–185.

Received: 16 September 2019

Accepted for publication: 16 September 2019

Communicated by: Karl Dilcher