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Conservative algebras of 2-dimensional algebras, III

Farhodjon Arzikulov, Nodirbek Umrzaqov

Abstract. In the present paper we prove that every local and 2-local derivation on conservative algebras of 2-dimensional algebras are derivations. Also, we prove that every local and 2-local automorphism on conservative algebras of 2-dimensional algebras are automorphisms.

1 Introduction

The present paper is devoted to the study of conservative algebras. In 1972 Kantor [12] introduced conservative algebras as a generalization of Jordan algebras (also, see a good written survey about the study of conservative algebras [25]).

In 1990 Kantor [14] defined the multiplication \cdot on the set of all algebras (i.e. all multiplications) on the n -dimensional vector space V_n over a field \mathbb{F} of characteristic zero as follows: $A \cdot B = [L_e^A, B]$, where A and B are multiplications and $e \in V_n$ is some fixed vector. If $n > 1$, then the algebra $W(n)$ does not belong to any well-known class of algebras (such as associative, Lie, Jordan, or Leibniz algebras). The algebra $W(n)$ is a conservative algebra [12].

In [12] Kantor classified all conservative 2-dimensional algebras and defined the class of terminal algebras as algebras satisfying some certain identity. He proved that every terminal algebra is a conservative algebra and classified all simple finite-dimensional terminal algebras with left quasi-unit over an algebraically closed field of characteristic zero [13]. Terminal algebras were also studied in [18], [19].

In 2017 Kaygorodov and Volkov [16] described automorphisms, one-sided ideals, and idempotents of $W(2)$. Also a similar problem is solved for the algebra W_2 of all commutative algebras on the 2-dimensional vector space and for the algebra

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Affiliation:

Farhodjon Arzikulov – V.I. Romanovskiy Institute of Mathematics Uzbekistan
 Academy of Sciences, Universitet street 9, Tashkent 100174, Uzbekistan.

Andizhan State University, Universitet street 129, Andizhan, 170100, Uzbekistan.

E-mail: arzikulovfn@rambler.ru

Nodirbek Umrzaqov – Andizhan State University, Universitet street 129, Andizhan,
 170100, Uzbekistan.

E-mail: umrzaqov2010@mail.ru

S_2 of all commutative algebras with zero multiplication trace on the 2-dimensional vector space. The papers [15], [17] are also devoted to the study of conservative algebras and superalgebras.

Let \mathcal{A} be an algebra. A linear operator ∇ on \mathcal{A} is called a local derivation if for every $x \in \mathcal{A}$ there exists a derivation ϕ_x of \mathcal{A} , depending on x , such that $\nabla(x) = \phi_x(x)$. The history of local derivations had begun from the paper of Kadison [11]. Kadison introduced the concept of local derivation and proved that each continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation.

A similar notion, which characterizes nonlinear generalizations of derivations, was introduced by Šemrl as 2-local derivations. In his paper [26] was proved that a 2-local derivation of the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H is a derivation. After his works, appear numerous new results related to the description of local and 2-local derivations of associative algebras (see, for example, [1], [3], [4], [20], [21], [23]).

The study of local and 2-local derivations of non-associative algebras was initiated in some papers of Ayupov and Kudaybergenov (for the case of Lie algebras, see [5], [6]). In particular, they proved that there are no non-trivial local and 2-local derivations on semisimple finite-dimensional Lie algebras. In [8] examples of 2-local derivations on nilpotent Lie algebras which are not derivations, were also given. Later, the study of local and 2-local derivations was continued for Leibniz algebras [7], Malcev algebras and Jordan algebras [2]. Local automorphisms and 2-local automorphisms, also were studied in many cases, for example, they were studied on Lie algebras [5], [10].

Now, a linear operator ∇ on \mathcal{A} is called a local automorphism if for every $x \in \mathcal{A}$ there exists an automorphism ϕ_x of \mathcal{A} , depending on x , such that $\nabla(x) = \phi_x(x)$. The concept of local automorphism was introduced by Larson and Sourour [22] in 1990. They proved that, invertible local automorphisms of the algebra of all bounded linear operators on an infinite-dimensional Banach space X are automorphisms.

A similar notion, which characterizes non-linear generalizations of automorphisms, was introduced by Šemrl in [26] as 2-local automorphisms. Namely, a map $\Delta: \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily linear) is called a 2-local automorphism, if for every $x, y \in \mathcal{A}$ there exists an automorphism $\phi_{x,y}: \mathcal{A} \rightarrow \mathcal{A}$ such that $\Delta(x) = \phi_{x,y}(x)$ and $\Delta(y) = \phi_{x,y}(y)$. After the work of Šemrl, it appeared numerous new results related to the description of local and 2-local automorphisms of algebras (see, for example, [5], [7], [9], [10], [21]).

In the present paper, we continue the study of derivations, local and 2-local derivations of conservative algebras of 2-dimensional algebras. We prove that every local and 2-local derivation of the conservative algebras of 2-dimensional algebras are derivations. In the present paper, we continue the study of automorphisms, local and 2-local automorphisms in the case of conservative algebras of 2-dimensional algebras. We prove that every local and 2-local automorphism of the conservative algebras of 2-dimensional algebras are automorphisms.

2 Preliminaries

Throughout this paper \mathbb{F} is some fixed field of characteristic zero. A multiplication on 2-dimensional vector space is defined by a $2 \times 2 \times 2$ matrix. Their classification was given in many papers (see, for example, [24]). Let consider the space $W(2)$ of all multiplications on the 2-dimensional space V_2 with a basis v_1, v_2 . The definition of the multiplication \cdot on the algebra $W(2)$ is defined as follows: we fix the vector $v_1 \in V_2$ and define

$$(A \cdot B)(x, y) = A(v_1, B(x, y)) - B(A(v_1, x), y) - B(x, A(v_1, y))$$

for $x, y \in V_2$ and $A, B \in W(2)$. The algebra $W(2)$ is conservative [14]. Let consider the multiplications $\alpha_{i,j}^k$ ($i, j, k = 1, 2$) on V_2 defined by the formula $\alpha_{i,j}^k(v_t, v_l) = \delta_{it}\delta_{jl}v_k$ for all $t, l \in \{1, 2\}$. It is easy to see that $\{\alpha_{i,j}^k | i, j, k = 1, 2\}$ is a basis of the algebra $W(2)$. The multiplication table of $W(2)$ in this basis is given in [15]. In this work we use another basis for the algebra $W(2)$ (from [16]). Let introduce the notation

$$e_1 = \alpha_{11}^1 - \alpha_{12}^2 - \alpha_{21}^2, \quad e_2 = \alpha_{11}^2, \quad e_3 = \alpha_{22}^2 - \alpha_{12}^1 - \alpha_{21}^1, \quad e_4 = \alpha_{22}^1, \quad e_5 = 2\alpha_{11}^1 + \alpha_{12}^2 + \alpha_{21}^2,$$

$$e_6 = 2\alpha_{22}^2 + \alpha_{12}^1 + \alpha_{21}^1, \quad e_7 = \alpha_{12}^1 - \alpha_{21}^1, \quad e_8 = \alpha_{12}^2 - \alpha_{21}^2.$$

It is easy to see that the multiplication table of $W(2)$ in the basis e_1, \dots, e_8 is the following.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	$-e_1$	$-3e_2$	e_3	$3e_4$	$-e_5$	e_6	e_7	$-e_8$
e_2	$3e_2$	0	$2e_1$	e_3	0	$-e_5$	e_8	0
e_3	$-2e_3$	$-e_1$	$-3e_4$	0	e_6	0	0	$-e_7$
e_4	0	0	0	0	0	0	0	0
e_5	$-2e_1$	$-3e_2$	$-e_3$	0	$-2e_5$	$-e_6$	$-e_7$	$-2e_8$
e_6	$2e_3$	e_1	$3e_4$	0	$-e_6$	0	0	e_7
e_7	$2e_3$	e_1	$3e_4$	0	$-e_6$	0	0	e_7
e_8	0	e_2	$-e_3$	$-2e_4$	0	$-e_6$	$-e_7$	0

The subalgebra generated by the elements e_1, \dots, e_6 is the conservative (and, moreover, terminal) algebra W_2 of commutative 2-dimensional algebras. The subalgebra generated by the elements e_1, \dots, e_4 is the conservative (and, moreover, terminal) algebra S_2 of all commutative 2-dimensional algebras with zero multiplication trace [15].

Let \mathcal{A} be an algebra. A linear map $D: \mathcal{A} \rightarrow \mathcal{A}$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for any two elements $x, y \in \mathcal{A}$.

Our main tool for study of local and 2-local derivations of the algebras S_2, W_2 and $W(2)$ is the following lemma [15, Theorem 6], where the matrix of a derivation is calculated in the new basis e_1, \dots, e_8 .

Lemma 1. *A linear map $D: W(2) \rightarrow W(2)$ is a derivation if and only if the matrix of D has the following matrix form:*

$$\begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha & 2\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

where α, β are elements in \mathbb{F} .

Now, we give a characterization of automorphisms on conservative algebras of 2-dimensional algebras.

Let \mathcal{A} be an algebra. A bijective linear map $\phi: \mathcal{A} \rightarrow \mathcal{A}$ is called an automorphism, if $\phi(xy) = \phi(x)\phi(y)$ for any elements $x, y \in \mathcal{A}$.

Our principal tool for study of local and 2-local automorphisms of the algebras S_2, W_2 and $W(2)$ is the following lemma, which was proved in [16, Theorem 11].

Lemma 2. *A linear map $\phi: W(2) \rightarrow W(2)$ is an automorphism if and only if the matrix of ϕ has the following matrix form:*

$$\begin{pmatrix} 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2ab & a^2b & b & 0 & 0 & 0 & 0 & 0 \\ 3a^2b^2 & a^3b^2 & 3ab^2 & b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -ab & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & ab \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where a, b are elements in \mathbb{F} and $b \neq 0$.

3 Local derivations of conservative algebras of 2-dimensional algebras

In this section we give a characterization of derivations on conservative algebras of 2-dimensional algebras.

Let \mathcal{A} be an algebra. A linear map $\nabla: \mathcal{A} \rightarrow \mathcal{A}$ is called a local derivation, if for any element $x \in \mathcal{A}$ there exists a derivation $D_x: \mathcal{A} \rightarrow \mathcal{A}$ such that $\nabla(x) = D_x(x)$.

Theorem 1. *Every local derivation of the algebra $W(2)$ is a derivation.*

Proof. Let ∇ be an arbitrary local derivation of $W(2)$ and write

$$\nabla(x) = B\bar{x}, x \in W(2),$$

where $B = (b_{i,j})_{i,j=1}^8$, $\bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is the vector corresponding to x . Then for every $x \in W(2)$ there exist elements a_x, b_x in \mathbb{F} such that

$$B\bar{x} = \begin{pmatrix} 0 & a_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_x & 0 & b_x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3a_x & 2b_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_x & b_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_x & a_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}.$$

In other words

$$\begin{cases} b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 + b_{1,4}x_4 + b_{1,5}x_5 + b_{1,6}x_6 + b_{1,7}x_7 + b_{1,8}x_8 = a_x x_2; \\ b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 + b_{2,4}x_4 + b_{2,5}x_5 + b_{2,6}x_6 + b_{2,7}x_7 + b_{2,8}x_8 = -b_x x_2; \\ b_{3,1}x_1 + b_{3,2}x_2 + b_{3,3}x_3 + b_{3,4}x_4 + b_{3,5}x_5 + b_{3,6}x_6 + b_{3,7}x_7 + b_{3,8}x_8 = 2a_x x_1 + b_x x_3; \\ b_{4,1}x_1 + b_{4,2}x_2 + b_{4,3}x_3 + b_{4,4}x_4 + b_{4,5}x_5 + b_{4,6}x_6 + b_{4,7}x_7 + b_{4,8}x_8 = 3a_x x_3 + 2b_x x_4; \\ b_{5,1}x_1 + b_{5,2}x_2 + b_{5,3}x_3 + b_{5,4}x_4 + b_{5,5}x_5 + b_{5,6}x_6 + b_{5,7}x_7 + b_{5,8}x_8 = 0; \\ b_{6,1}x_1 + b_{6,2}x_2 + b_{6,3}x_3 + b_{6,4}x_4 + b_{6,5}x_5 + b_{6,6}x_6 + b_{6,7}x_7 + b_{6,8}x_8 = -a_x x_5 + b_x x_6; \\ b_{7,1}x_1 + b_{7,2}x_2 + b_{7,3}x_3 + b_{7,4}x_4 + b_{7,5}x_5 + b_{7,6}x_6 + b_{7,7}x_7 + b_{7,8}x_8 = b_x x_7 + a_x x_8; \\ b_{8,1}x_1 + b_{8,2}x_2 + b_{8,3}x_3 + b_{8,4}x_4 + b_{8,5}x_5 + b_{8,6}x_6 + b_{8,7}x_7 + b_{8,8}x_8 = 0. \end{cases}$$

Taking $x = (1, 0, 0, 0, 0, 0, 0, 0)$, $x = (0, 0, 1, 0, 0, 0, 0, 0)$, $x = (0, 0, 0, 1, 0, 0, 0, 0)$, etc, from this it follows that

$$\begin{aligned} b_{1,1} &= b_{1,3} = b_{1,4} = b_{1,5} = b_{1,6} = b_{1,7} = b_{1,8} = \\ &= b_{2,1} = b_{2,3} = b_{2,4} = b_{2,5} = b_{2,6} = b_{2,7} = b_{2,8} \\ &= b_{3,2} = b_{3,4} = b_{3,5} = b_{3,6} = b_{3,7} = b_{3,8} \\ &= b_{4,1} = b_{4,2} = b_{4,5} = b_{4,6} = b_{4,7} = b_{4,8} \\ &= b_{5,1} = b_{5,2} = b_{5,3} = b_{5,4} = b_{5,5} = b_{5,6} = b_{5,7} = b_{5,8} \\ &= b_{6,1} = b_{6,2} = b_{6,3} = b_{6,4} = b_{6,7} = b_{6,8} \\ &= b_{7,1} = b_{7,2} = b_{7,3} = b_{7,4} = b_{7,5} = b_{7,6} \\ &= b_{8,1} = b_{8,2} = b_{8,3} = b_{8,4} = b_{8,5} = b_{8,6} = b_{8,7} = b_{8,8} = 0. \end{aligned}$$

Then for every $x \in W(2)$ there exist elements a_x, b_x in \mathbb{F} such that

$$\begin{cases} b_{1,2}x_2 = a_x x_2; \\ b_{2,2}x_2 = -b_x x_2; \\ b_{3,1}x_1 + b_{3,3}x_3 = 2a_x x_1 + b_x x_3; \\ b_{4,3}x_3 + b_{4,4}x_4 = 3a_x x_3 + 2b_x x_4; \\ b_{6,5}x_5 + b_{6,6}x_6 = -a_x x_5 + b_x x_6; \\ b_{7,7}x_7 + b_{7,8}x_8 = b_x x_7 + a_x x_8. \end{cases} \tag{3}$$

Using 1-th and 3-th equalities of system (3) we get

$$\begin{cases} 2b_{1,2}x_1x_2 = 2a_x x_1x_2; \\ b_{3,1}x_1x_2 + b_{3,3}x_2x_3 = 2a_x x_1x_2 + b_x x_2x_3. \end{cases}$$

and

$$(b_{3,1} - 2b_{1,2})x_1x_2 + b_{3,3}x_2x_3 = b_x x_2x_3.$$

Hence, $b_{3,1} = 2b_{1,2}$. Similarly, using equalities of (3) we get

$$b_{4,3} = 3b_{1,2}, b_{2,2} = -b_{3,3}, b_{4,4} = -2b_{2,2}.$$

Using 1-th and 5-th equalities of system (3) we get

$$\begin{cases} b_{1,2}x_2x_5 = a_x x_2x_5; \\ b_{6,5}x_5x_2 + b_{6,6}x_6x_2 = -a_x x_5x_2 + b_x x_6x_2. \end{cases}$$

and

$$(b_{6,5} + b_{1,2})x_2x_5 + b_{6,6}x_6x_2 = b_x x_6x_2.$$

Hence, $b_{6,5} = -b_{1,2}$.

Using 2-th and 5-th equalities of system (3) we get

$$\begin{cases} b_{2,2}x_2x_6 = -b_x x_2x_6; \\ b_{6,5}x_5x_2 + b_{6,6}x_6x_2 = -a_x x_5x_2 + b_x x_6x_2. \end{cases}$$

and

$$b_{6,5}x_5x_2 + (b_{6,6} + b_{2,2})x_6x_2 = -a_x x_5x_2.$$

Hence, $b_{6,6} = -b_{2,2}$.

Using 1-th and 6-th equalities of system (3) we get

$$\begin{cases} b_{1,2}x_2x_8 = a_x x_2x_8; \\ b_{7,7}x_7x_2 + b_{7,8}x_8x_2 = b_x x_7x_2 + a_x x_8x_2. \end{cases}$$

and

$$b_{7,7}x_7x_2 + (b_{7,8} - b_{1,2})x_8x_2 = b_x x_7x_2.$$

Hence, $b_{7,8} = b_{1,2}$.

Using 2-th and 6-th equalities of system (3) we get

$$\begin{cases} b_{2,2}x_2x_7 = -b_x x_2x_7; \\ b_{7,7}x_7x_2 + b_{7,8}x_8x_2 = b_x x_7x_2 + a_x x_8x_2. \end{cases}$$

and

$$(b_{7,7} + b_{2,2})x_7x_2 + b_{7,8}x_8x_2 = a_x x_8x_2.$$

Hence, $b_{7,7} = -b_{2,2}$.

These equalities show that the matrix of the linear map ∇ is of the form (1). Therefore, by lemma 1 ∇ is a derivation. This completes the proof. \square

Since a derivation on $W(2)$ is invariant on the subalgebras S_2 and W_2 , we have the following corollary.

Corollary 1. *Every local derivation of the algebras S_2 and W_2 is a derivation.*

4 2-Local derivations of conservative algebras of 2-dimensional algebras

In this section we give another characterization of derivations on conservative algebras of 2-dimensional algebras.

A (not necessary linear) map $\Delta: \mathcal{A} \rightarrow \mathcal{A}$ is called a 2-local derivation, if for any elements $x, y \in \mathcal{A}$ there exists a derivation $D_{x,y}: \mathcal{A} \rightarrow \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

Theorem 2. *Every 2-local derivation of the algebras S_2, W_2 and $W(2)$ is a derivation.*

Proof. We will prove that every 2-local derivation of $W(2)$ is a derivation.

Let Δ be an arbitrary 2-local derivation of $W(2)$. Then, by the definition, for every element $a \in W(2)$, there exists a derivation D_{a,e_2} of $W(2)$ such that

$$\Delta(a) = D_{a,e_2}(a), \quad \Delta(e_2) = D_{a,e_2}(e_2).$$

By lemma 1, the matrix A^{a,e_2} of the derivation D_{a,e_2} has the following matrix form:

$$A^{a,e_2} = \begin{pmatrix} 0 & \alpha_{a,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_{a,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha_{a,e_2} & 0 & \beta_{a,e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha_{a,e_2} & 2\beta_{a,e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_{a,e_2} & \beta_{a,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_{a,e_2} & \alpha_{a,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let v be an arbitrary element in $W(2)$. Then there exists a derivation D_{v,e_2} of $W(2)$ such that

$$\Delta(v) = D_{v,e_2}(v), \quad \Delta(e_2) = D_{v,e_2}(e_2).$$

By lemma 1, the matrix A^{v,e_2} of the derivation D_{v,e_2} has the following matrix form:

$$A^{v,e_2} = \begin{pmatrix} 0 & \alpha_{v,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_{v,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha_{v,e_2} & 0 & \beta_{v,e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha_{v,e_2} & 2\beta_{v,e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_{v,e_2} & \beta_{v,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_{v,e_2} & \alpha_{v,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\Delta(e_2) = D_{a,e_2}(e_2) = D_{v,e_2}(e_2)$, we have

$$\alpha_{a,e_2} = \alpha_{v,e_2}, \beta_{a,e_2} = \beta_{v,e_2},$$

that it

$$D_{v,e_2} = D_{a,e_2}.$$

Therefore, for any element a of the algebra $W(2)$

$$\Delta(a) = D_{v,e_2}(a),$$

that it D_{v,e_2} does not depend on a . Hence, Δ is a derivation by lemma 1.

The cases of the algebras S_2 and W_2 are also similarly proved. This ends the proof. \square

5 2-Local automorphisms of conservative algebras of 2-dimensional algebras

A (not necessary linear) map $\Delta: \mathcal{A} \rightarrow \mathcal{A}$ is called a 2-local automorphism, if for any elements $x, y \in \mathcal{A}$ there exists an automorphism $\phi_{x,y}: \mathcal{A} \rightarrow \mathcal{A}$ such that $\Delta(x) = \phi_{x,y}(x), \Delta(y) = \phi_{x,y}(y)$.

Theorem 3. *Every 2-local automorphism of the algebras S_2, W_2 and $W(2)$ is an automorphism.*

Proof. We prove that every 2-local automorphism of $W(2)$ is an automorphism.

Let Δ be an arbitrary 2-local automorphism of $W(2)$. Then, by the definition, for every element $x \in W(2)$,

$$x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 + x_8e_8,$$

there exist elements a_{x,e_2}, b_{x,e_2} such that

$$A_{x,e_2} = \begin{pmatrix} 1 & a_{x,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{x,e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{x,e_2}b_{x,e_2} & a_{x,e_2}^2b_{x,e_2} & b_{x,e_2} & 0 & 0 & 0 & 0 & 0 \\ 3a_{x,e_2}^2b_{x,e_2}^2 & a_{x,e_2}^3b_{x,e_2}^2 & 3a_{x,e_2}b_{x,e_2}^2 & b_{x,e_2}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{x,e_2}b_{x,e_2} & b_{x,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{x,e_2} & a_{x,e_2}b_{x,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$\Delta(x) = A_{x,e_2}\bar{x}$, where $\bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is the vector corresponding to x , and

$$\Delta(e_2) = A_{x,e_2}e_2 = (a_{x,e_2}, \frac{1}{b_{x,e_2}}, a_{x,e_2}^2b_{x,e_2}, a_{x,e_2}^3b_{x,e_2}^2, 0, 0, 0, 0).$$

Since the element x was chosen arbitrarily, we have

$$\Delta(e_2) = (a_{x,e_2}, \frac{1}{b_{x,e_2}}, a_{x,e_2}^2b_{x,e_2}, a_{x,e_2}^3b_{x,e_2}^2, 0, 0, 0, 0)$$

$$= (a_{y,e_2}, \frac{1}{b_{y,e_2}}, a_{y,e_2}^2 b_{y,e_2}, a_{y,e_2}^3 b_{y,e_2}^2, 0, 0, 0, 0),$$

for each pair x, y of elements in $W(2)$. Hence, $a_{x,e_2} = a_{y,e_2}, b_{x,e_2} = b_{y,e_2}$. Therefore

$$\Delta(x) = A_{y,e_2}x$$

for any $x \in W(2)$ and the matrix A_{y,e_2} does not depend on x . Thus, by lemma 2 Δ is an automorphism.

The cases of the algebras S_2 and W_2 are also similarly proved. The proof is complete. □

6 Local automorphisms of conservative algebras of 2-dimensional algebras

Let \mathcal{A} be an algebra. A linear map $\nabla: \mathcal{A} \rightarrow \mathcal{A}$ is called a local automorphism, if for any element $x \in \mathcal{A}$ there exists an automorphism $\phi_x: \mathcal{A} \rightarrow \mathcal{A}$ such that $\nabla(x) = \phi_x(x)$.

Theorem 4. *Every local automorphism of the algebras S_2, W_2 and $W(2)$ is an automorphism.*

Proof. We prove that every local automorphism of $W(2)$ is an automorphism.

Let ∇ be an arbitrary local automorphism of $W(2)$ and B be its matrix, i.e.,

$$\nabla(x) = B\bar{x}, x \in W(2),$$

where \bar{x} is the vector corresponding to x . Then, by the definition, for every element $x \in W(2)$,

$$x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 + x_8e_8,$$

there exist elements a_x, b_x such that

$$A_x = \begin{pmatrix} 1 & a_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_x b_x & a_x^2 b_x & b_x & 0 & 0 & 0 & 0 & 0 \\ 3a_x^2 b_x^2 & a_x^3 b_x^2 & 3a_x b_x^2 & b_x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_x b_x & b_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_x & a_x b_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\nabla(x) = B\bar{x} = A_x\bar{x}.$$

Using these equalities and by choosing subsequently $x = e_1, x = e_2, \dots, x = e_8$ we get

$$B = \begin{pmatrix} 1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{e_1}b_{e_1} & a_{e_2}^2b_{e_2} & b_{e_3} & 0 & 0 & 0 & 0 & 0 \\ 3a_{e_1}^2b_{e_1}^2 & a_{e_2}^3b_{e_2}^2 & 3a_{e_3}b_{e_3}^2 & b_{e_4}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{e_5}b_{e_5} & b_{e_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{e_7} & a_{e_8}b_{e_8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $\nabla(e_6 + e_7) = \nabla(e_6) + \nabla(e_7)$, we have

$$b_{e_6+e_7} = b_{e_6}, b_{e_6+e_7} = b_{e_7}.$$

Hence,

$$b_{e_6} = b_{e_7}.$$

Similarly to this equality we get $b_{e_3} = b_{e_6}$ and $b_{e_6} = b_{e_2} \neq 0$. Hence,

$$b_{e_2} = b_{e_3} = b_{e_6} = b_{e_7}. \quad (4)$$

Since $\nabla(e_5 + e_8) = \nabla(e_5) + \nabla(e_8)$, we have

$$a_{e_5+e_8}b_{e_5+e_8} = a_{e_5}b_{e_5}, a_{e_5+e_8}b_{e_5+e_8} = a_{e_8}b_{e_8}.$$

From this it follows that

$$a_{e_5}b_{e_5} = a_{e_8}b_{e_8}.$$

Similarly to this equality we get $a_{e_1}b_{e_1} = a_{e_8}b_{e_8}$. Hence,

$$a_{e_1}b_{e_1} = a_{e_5}b_{e_5} = a_{e_8}b_{e_8}. \quad (5)$$

Since $\nabla(e_4 + e_6) = \nabla(e_4) + \nabla(e_6)$, we have

$$b_{e_4+e_6}^2 = b_{e_4}^2, b_{e_4+e_6}^2 = b_{e_6}^2.$$

From this it follows that

$$b_{e_4}^2 = b_{e_6}^2.$$

Hence, by (4), we get

$$b_{e_4}^2 = b_{e_2}^2. \quad (6)$$

Since $\nabla(e_2 + e_8) = \nabla(e_2) + \nabla(e_8)$, we have

$$a_{e_2} = a_{e_2+e_8}, a_{e_2+e_8}^2b_{e_2+e_8} = a_{e_2}^2b_{e_2}, a_{e_2+e_8}b_{e_2+e_8} = a_{e_8}b_{e_8}.$$

Hence,

$$b_{e_2+e_8} = b_{e_2}, a_{e_2+e_8}b_{e_2+e_8} = a_{e_2}b_{e_2}$$

and, therefore,

$$a_{e_2}b_{e_2} = a_{e_8}b_{e_8}. \tag{7}$$

Similarly, since $\nabla(e_2 + e_3) = \nabla(e_2) + \nabla(e_3)$, we have

$$a_{e_2} = a_{e_2+e_3}, \quad b_{e_2}^{-1} = b_{e_2+e_3}^{-1}, \quad a_{e_2+e_3}^3 b_{e_2+e_3}^2 + 3a_{e_2+e_3} b_{e_2+e_3}^2 = a_{e_2}^3 b_{e_2}^2 + 3a_{e_3} b_{e_3}^2.$$

Hence,

$$b_{e_2} = b_{e_2+e_3}$$

and by (4) and $a_{e_2} = a_{e_2+e_3}$ we get

$$a_{e_2}^3 + 3a_{e_2} = a_{e_2}^3 + 3a_{e_3}.$$

Therefore, $a_{e_2} = a_{e_3}$ and

$$a_{e_2} b_{e_2}^2 = a_{e_3} b_{e_3}^2. \tag{8}$$

Finally, since $\nabla(e_1 + e_8) = \nabla(e_1) + \nabla(e_8)$, we have

$$a_{e_1+e_8} b_{e_1+e_8} = a_{e_1} b_{e_1}, \quad a_{e_1+e_8} b_{e_1+e_8} = a_{e_8} b_{e_8}.$$

Hence,

$$a_{e_1} b_{e_1} = a_{e_8} b_{e_8}.$$

By (7), from the last equalities it follows that

$$a_{e_1} b_{e_1} = a_{e_2} b_{e_2}, \quad a_{e_1}^2 b_{e_1}^2 = (a_{e_1} b_{e_1})^2 = (a_{e_2} b_{e_2})^2 = a_{e_2}^2 b_{e_2}^2. \tag{9}$$

By (4), (5), (6), (7), (8), (9) the matrix B has the following matrix form

$$B = \begin{pmatrix} 1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{e_2}b_{e_2} & a_{e_2}^2 b_{e_2} & b_{e_2} & 0 & 0 & 0 & 0 & 0 \\ 3a_{e_2}^2 b_{e_2}^2 & a_{e_2}^3 b_{e_2}^2 & 3a_{e_2} b_{e_2}^2 & b_{e_2}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{e_2} b_{e_2} & b_{e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{e_2} & a_{e_2} b_{e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, by lemma 2, the local automorphism ∇ is an automorphism.

The cases of the algebras S_2 and W_2 are also similarly proved. This ends the proof. □

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