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## Conservative algebras of 2-dimensional algebras, III

*Farhodjon Arzikulov, Nodirbek Umrzaqov*

**Abstract.** In the present paper we prove that every local and 2-local derivation on conservative algebras of 2-dimensional algebras are derivations. Also, we prove that every local and 2-local automorphism on conservative algebras of 2-dimensional algebras are automorphisms.

### 1 Introduction

The present paper is devoted to the study of conservative algebras. In 1972 Kantor [12] introduced conservative algebras as a generalization of Jordan algebras (also, see a good written survey about the study of conservative algebras [25]).

In 1990 Kantor [14] defined the multiplication  $\cdot$  on the set of all algebras (i.e. all multiplications) on the  $n$ -dimensional vector space  $V_n$  over a field  $\mathbb{F}$  of characteristic zero as follows:  $A \cdot B = [L_e^A, B]$ , where  $A$  and  $B$  are multiplications and  $e \in V_n$  is some fixed vector. If  $n > 1$ , then the algebra  $W(n)$  does not belong to any well-known class of algebras (such as associative, Lie, Jordan, or Leibniz algebras). The algebra  $W(n)$  is a conservative algebra [12].

In [12] Kantor classified all conservative 2-dimensional algebras and defined the class of terminal algebras as algebras satisfying some certain identity. He proved that every terminal algebra is a conservative algebra and classified all simple finite-dimensional terminal algebras with left quasi-unit over an algebraically closed field of characteristic zero [13]. Terminal algebras were also studied in [18], [19].

In 2017 Kaygorodov and Volkov [16] described automorphisms, one-sided ideals, and idempotents of  $W(2)$ . Also a similar problem is solved for the algebra  $W_2$  of all commutative algebras on the 2-dimensional vector space and for the algebra

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$S_2$  of all commutative algebras with zero multiplication trace on the 2-dimensional vector space. The papers [15], [17] are also devoted to the study of conservative algebras and superalgebras.

Let  $\mathcal{A}$  be an algebra. A linear operator  $\nabla$  on  $\mathcal{A}$  is called a local derivation if for every  $x \in \mathcal{A}$  there exists a derivation  $\phi_x$  of  $\mathcal{A}$ , depending on  $x$ , such that  $\nabla(x) = \phi_x(x)$ . The history of local derivations had begun from the paper of Kadison [11]. Kadison introduced the concept of local derivation and proved that each continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation.

A similar notion, which characterizes nonlinear generalizations of derivations, was introduced by Šemrl as 2-local derivations. In his paper [26] was proved that a 2-local derivation of the algebra  $B(H)$  of all bounded linear operators on the infinite-dimensional separable Hilbert space  $H$  is a derivation. After his works, appear numerous new results related to the description of local and 2-local derivations of associative algebras (see, for example, [1], [3], [4], [20], [21], [23]).

The study of local and 2-local derivations of non-associative algebras was initiated in some papers of Ayupov and Kudaybergenov (for the case of Lie algebras, see [5], [6]). In particular, they proved that there are no non-trivial local and 2-local derivations on semisimple finite-dimensional Lie algebras. In [8] examples of 2-local derivations on nilpotent Lie algebras which are not derivations, were also given. Later, the study of local and 2-local derivations was continued for Leibniz algebras [7], Malcev algebras and Jordan algebras [2]. Local automorphisms and 2-local automorphisms, also were studied in many cases, for example, they were studied on Lie algebras [5], [10].

Now, a linear operator  $\nabla$  on  $\mathcal{A}$  is called a local automorphism if for every  $x \in \mathcal{A}$  there exists an automorphism  $\phi_x$  of  $\mathcal{A}$ , depending on  $x$ , such that  $\nabla(x) = \phi_x(x)$ . The concept of local automorphism was introduced by Larson and Sourour [22] in 1990. They proved that, invertible local automorphisms of the algebra of all bounded linear operators on an infinite-dimensional Banach space  $X$  are automorphisms.

A similar notion, which characterizes non-linear generalizations of automorphisms, was introduced by Šemrl in [26] as 2-local automorphisms. Namely, a map  $\Delta: \mathcal{A} \rightarrow \mathcal{A}$  (not necessarily linear) is called a 2-local automorphism, if for every  $x, y \in \mathcal{A}$  there exists an automorphism  $\phi_{x,y}: \mathcal{A} \rightarrow \mathcal{A}$  such that  $\Delta(x) = \phi_{x,y}(x)$  and  $\Delta(y) = \phi_{x,y}(y)$ . After the work of Šemrl, it appeared numerous new results related to the description of local and 2-local automorphisms of algebras (see, for example, [5], [7], [9], [10], [21]).

In the present paper, we continue the study of derivations, local and 2-local derivations of conservative algebras of 2-dimensional algebras. We prove that every local and 2-local derivation of the conservative algebras of 2-dimensional algebras are derivations. In the present paper, we continue the study of automorphisms, local and 2-local automorphisms in the case of conservative algebras of 2-dimensional algebras. We prove that every local and 2-local automorphism of the conservative algebras of 2-dimensional algebras are automorphisms.

## 2 Preliminaries

Throughout this paper  $\mathbb{F}$  is some fixed field of characteristic zero. A multiplication on 2-dimensional vector space is defined by a  $2 \times 2 \times 2$  matrix. Their classification was given in many papers (see, for example, [24]). Let consider the space  $W(2)$  of all multiplications on the 2-dimensional space  $V_2$  with a basis  $v_1, v_2$ . The definition of the multiplication  $\cdot$  on the algebra  $W(2)$  is defined as follows: we fix the vector  $v_1 \in V_2$  and define

$$(A \cdot B)(x, y) = A(v_1, B(x, y)) - B(A(v_1, x), y) - B(x, A(v_1, y))$$

for  $x, y \in V_2$  and  $A, B \in W(2)$ . The algebra  $W(2)$  is conservative [14]. Let consider the multiplications  $\alpha_{i,j}^k$  ( $i, j, k = 1, 2$ ) on  $V_2$  defined by the formula  $\alpha_{i,j}^k(v_t, v_l) = \delta_{it}\delta_{jl}v_k$  for all  $t, l \in \{1, 2\}$ . It is easy to see that  $\{\alpha_{i,j}^k | i, j, k = 1, 2\}$  is a basis of the algebra  $W(2)$ . The multiplication table of  $W(2)$  in this basis is given in [15]. In this work we use another basis for the algebra  $W(2)$  (from [16]). Let introduce the notation

$$e_1 = \alpha_{11}^1 - \alpha_{12}^2 - \alpha_{21}^2, \quad e_2 = \alpha_{11}^2, \quad e_3 = \alpha_{22}^2 - \alpha_{12}^1 - \alpha_{21}^1, \quad e_4 = \alpha_{22}^1, \quad e_5 = 2\alpha_{11}^1 + \alpha_{12}^2 + \alpha_{21}^2,$$

$$e_6 = 2\alpha_{22}^2 + \alpha_{12}^1 + \alpha_{21}^1, \quad e_7 = \alpha_{12}^1 - \alpha_{21}^1, \quad e_8 = \alpha_{12}^2 - \alpha_{21}^2.$$

It is easy to see that the multiplication table of  $W(2)$  in the basis  $e_1, \dots, e_8$  is the following.

	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$	$e_8$
$e_1$	$-e_1$	$-3e_2$	$e_3$	$3e_4$	$-e_5$	$e_6$	$e_7$	$-e_8$
$e_2$	$3e_2$	$0$	$2e_1$	$e_3$	$0$	$-e_5$	$e_8$	$0$
$e_3$	$-2e_3$	$-e_1$	$-3e_4$	$0$	$e_6$	$0$	$0$	$-e_7$
$e_4$	$0$	$0$	$0$	$0$	$0$	$0$	$0$	$0$
$e_5$	$-2e_1$	$-3e_2$	$-e_3$	$0$	$-2e_5$	$-e_6$	$-e_7$	$-2e_8$
$e_6$	$2e_3$	$e_1$	$3e_4$	$0$	$-e_6$	$0$	$0$	$e_7$
$e_7$	$2e_3$	$e_1$	$3e_4$	$0$	$-e_6$	$0$	$0$	$e_7$
$e_8$	$0$	$e_2$	$-e_3$	$-2e_4$	$0$	$-e_6$	$-e_7$	$0$

The subalgebra generated by the elements  $e_1, \dots, e_6$  is the conservative (and, moreover, terminal) algebra  $W_2$  of commutative 2-dimensional algebras. The subalgebra generated by the elements  $e_1, \dots, e_4$  is the conservative (and, moreover, terminal) algebra  $S_2$  of all commutative 2-dimensional algebras with zero multiplication trace [15].

Let  $\mathcal{A}$  be an algebra. A linear map  $D: \mathcal{A} \rightarrow \mathcal{A}$  is called a derivation, if  $D(xy) = D(x)y + xD(y)$  for any two elements  $x, y \in \mathcal{A}$ .

Our main tool for study of local and 2-local derivations of the algebras  $S_2$ ,  $W_2$  and  $W(2)$  is the following lemma [15, Theorem 6], where the matrix of a derivation is calculated in the new basis  $e_1, \dots, e_8$ .

**Lemma 1.** *A linear map  $D: W(2) \rightarrow W(2)$  is a derivation if and only if the matrix of  $D$  has the following matrix form:*

$$\begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha & 2\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (1)$$

where  $\alpha, \beta$  are elements in  $\mathbb{F}$ .

Now, we give a characterization of automorphisms on conservative algebras of 2-dimensional algebras.

Let  $\mathcal{A}$  be an algebra. A bijective linear map  $\phi: \mathcal{A} \rightarrow \mathcal{A}$  is called an automorphism, if  $\phi(xy) = \phi(x)\phi(y)$  for any elements  $x, y \in \mathcal{A}$ .

Our principal tool for study of local and 2-local automorphisms of the algebras  $S_2$ ,  $W_2$  and  $W(2)$  is the following lemma, which was proved in [16, Theorem 11].

**Lemma 2.** *A linear map  $\phi: W(2) \rightarrow W(2)$  is an automorphism if and only if the matrix of  $\phi$  has the following matrix form:*

$$\begin{pmatrix} 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2ab & a^2b & b & 0 & 0 & 0 & 0 & 0 \\ 3a^2b^2 & a^3b^2 & 3ab^2 & b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -ab & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & ab \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2)$$

where  $a, b$  are elements in  $\mathbb{F}$  and  $b \neq 0$ .

### 3 Local derivations of conservative algebras of 2-dimensional algebras

In this section we give a characterization of derivations on conservative algebras of 2-dimensional algebras.

Let  $\mathcal{A}$  be an algebra. A linear map  $\nabla: \mathcal{A} \rightarrow \mathcal{A}$  is called a local derivation, if for any element  $x \in \mathcal{A}$  there exists a derivation  $D_x: \mathcal{A} \rightarrow \mathcal{A}$  such that  $\nabla(x) = D_x(x)$ .

**Theorem 1.** *Every local derivation of the algebra  $W(2)$  is a derivation.*

*Proof.* Let  $\nabla$  be an arbitrary local derivation of  $W(2)$  and write

$$\nabla(x) = B\bar{x}, x \in W(2),$$

where  $B = (b_{i,j})_{i,j=1}^8$ ,  $\bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$  is the vector corresponding to  $x$ . Then for every  $x \in W(2)$  there exist elements  $a_x, b_x$  in  $\mathbb{F}$  such that

$$B\bar{x} = \begin{pmatrix} 0 & a_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_x & 0 & b_x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3a_x & 2b_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_x & b_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_x & a_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}.$$

In other words

$$\begin{cases} b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 + b_{1,4}x_4 + b_{1,5}x_5 + b_{1,6}x_6 + b_{1,7}x_7 + b_{1,8}x_8 = a_x x_2; \\ b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 + b_{2,4}x_4 + b_{2,5}x_5 + b_{2,6}x_6 + b_{2,7}x_7 + b_{2,8}x_8 = -b_x x_2; \\ b_{3,1}x_1 + b_{3,2}x_2 + b_{3,3}x_3 + b_{3,4}x_4 + b_{3,5}x_5 + b_{3,6}x_6 + b_{3,7}x_7 + b_{3,8}x_8 = 2a_x x_1 + b_x x_3; \\ b_{4,1}x_1 + b_{4,2}x_2 + b_{4,3}x_3 + b_{4,4}x_4 + b_{4,5}x_5 + b_{4,6}x_6 + b_{4,7}x_7 + b_{4,8}x_8 = 3a_x x_3 + 2b_x x_4; \\ b_{5,1}x_1 + b_{5,2}x_2 + b_{5,3}x_3 + b_{5,4}x_4 + b_{5,5}x_5 + b_{5,6}x_6 + b_{5,7}x_7 + b_{5,8}x_8 = 0; \\ b_{6,1}x_1 + b_{6,2}x_2 + b_{6,3}x_3 + b_{6,4}x_4 + b_{6,5}x_5 + b_{6,6}x_6 + b_{6,7}x_7 + b_{6,8}x_8 = -a_x x_5 + b_x x_6; \\ b_{7,1}x_1 + b_{7,2}x_2 + b_{7,3}x_3 + b_{7,4}x_4 + b_{7,5}x_5 + b_{7,6}x_6 + b_{7,7}x_7 + b_{7,8}x_8 = b_x x_7 + a_x x_8; \\ b_{8,1}x_1 + b_{8,2}x_2 + b_{8,3}x_3 + b_{8,4}x_4 + b_{8,5}x_5 + b_{8,6}x_6 + b_{8,7}x_7 + b_{8,8}x_8 = 0. \end{cases}$$

Taking  $x = (1, 0, 0, 0, 0, 0, 0, 0)$ ,  $x = (0, 0, 1, 0, 0, 0, 0, 0)$ ,  $x = (0, 0, 0, 1, 0, 0, 0, 0)$ , etc, from this it follows that

$$\begin{aligned} b_{1,1} &= b_{1,3} = b_{1,4} = b_{1,5} = b_{1,6} = b_{1,7} = b_{1,8} = \\ &= b_{2,1} = b_{2,3} = b_{2,4} = b_{2,5} = b_{2,6} = b_{2,7} = b_{2,8} \\ &= b_{3,2} = b_{3,4} = b_{3,5} = b_{3,6} = b_{3,7} = b_{3,8} \\ &= b_{4,1} = b_{4,2} = b_{4,5} = b_{4,6} = b_{4,7} = b_{4,8} \\ &= b_{5,1} = b_{5,2} = b_{5,3} = b_{5,4} = b_{5,5} = b_{5,6} = b_{5,7} = b_{5,8} \\ &= b_{6,1} = b_{6,2} = b_{6,3} = b_{6,4} = b_{6,7} = b_{6,8} \\ &= b_{7,1} = b_{7,2} = b_{7,3} = b_{7,4} = b_{7,5} = b_{7,6} \\ &= b_{8,1} = b_{8,2} = b_{8,3} = b_{8,4} = b_{8,5} = b_{8,6} = b_{8,7} = b_{8,8} = 0. \end{aligned}$$

Then for every  $x \in W(2)$  there exist elements  $a_x, b_x$  in  $\mathbb{F}$  such that

$$\begin{cases} b_{1,2}x_2 = a_x x_2; \\ b_{2,2}x_2 = -b_x x_2; \\ b_{3,1}x_1 + b_{3,3}x_3 = 2a_x x_1 + b_x x_3; \\ b_{4,3}x_3 + b_{4,4}x_4 = 3a_x x_3 + 2b_x x_4; \\ b_{6,5}x_5 + b_{6,6}x_6 = -a_x x_5 + b_x x_6; \\ b_{7,7}x_7 + b_{7,8}x_8 = b_x x_7 + a_x x_8. \end{cases} \quad (3)$$

Using 1-th and 3-th equalities of system (3) we get

$$\begin{cases} 2b_{1,2}x_1x_2 = 2a_x x_1x_2; \\ b_{3,1}x_1x_2 + b_{3,3}x_2x_3 = 2a_x x_1x_2 + b_x x_2x_3. \end{cases}$$

and

$$(b_{3,1} - 2b_{1,2})x_1x_2 + b_{3,3}x_2x_3 = b_x x_2x_3.$$

Hence,  $b_{3,1} = 2b_{1,2}$ . Similarly, using equalities of (3) we get

$$b_{4,3} = 3b_{1,2}, b_{2,2} = -b_{3,3}, b_{4,4} = -2b_{2,2}.$$

Using 1-th and 5-th equalities of system (3) we get

$$\begin{cases} b_{1,2}x_2x_5 = a_x x_2x_5; \\ b_{6,5}x_5x_2 + b_{6,6}x_6x_2 = -a_x x_5x_2 + b_x x_6x_2. \end{cases}$$

and

$$(b_{6,5} + b_{1,2})x_2x_5 + b_{6,6}x_6x_2 = b_x x_6x_2.$$

Hence,  $b_{6,5} = -b_{1,2}$ .

Using 2-th and 5-th equalities of system (3) we get

$$\begin{cases} b_{2,2}x_2x_6 = -b_x x_2x_6; \\ b_{6,5}x_5x_2 + b_{6,6}x_6x_2 = -a_x x_5x_2 + b_x x_6x_2. \end{cases}$$

and

$$b_{6,5}x_5x_2 + (b_{6,6} + b_{2,2})x_6x_2 = -a_x x_5x_2.$$

Hence,  $b_{6,6} = -b_{2,2}$ .

Using 1-th and 6-th equalities of system (3) we get

$$\begin{cases} b_{1,2}x_2x_8 = a_x x_2x_8; \\ b_{7,7}x_7x_2 + b_{7,8}x_8x_2 = b_x x_7x_2 + a_x x_8x_2. \end{cases}$$

and

$$b_{7,7}x_7x_2 + (b_{7,8} - b_{1,2})x_8x_2 = b_x x_7x_2.$$

Hence,  $b_{7,8} = b_{1,2}$ .

Using 2-th and 6-th equalities of system (3) we get

$$\begin{cases} b_{2,2}x_2x_7 = -b_x x_2x_7; \\ b_{7,7}x_7x_2 + b_{7,8}x_8x_2 = b_x x_7x_2 + a_x x_8x_2. \end{cases}$$

and

$$(b_{7,7} + b_{2,2})x_7x_2 + b_{7,8}x_8x_2 = a_x x_8x_2.$$

Hence,  $b_{7,7} = -b_{2,2}$ .

These equalities show that the matrix of the linear map  $\nabla$  is of the form (1). Therefore, by lemma 1  $\nabla$  is a derivation. This completes the proof.  $\square$

Since a derivation on  $W(2)$  is invariant on the subalgebras  $S_2$  and  $W_2$ , we have the following corollary.

**Corollary 1.** *Every local derivation of the algebras  $S_2$  and  $W_2$  is a derivation.*

## 4 2-Local derivations of conservative algebras of 2-dimensional algebras

In this section we give another characterization of derivations on conservative algebras of 2-dimensional algebras.

A (not necessary linear) map  $\Delta: \mathcal{A} \rightarrow \mathcal{A}$  is called a 2-local derivation, if for any elements  $x, y \in \mathcal{A}$  there exists a derivation  $D_{x,y}: \mathcal{A} \rightarrow \mathcal{A}$  such that  $\Delta(x) = D_{x,y}(x)$ ,  $\Delta(y) = D_{x,y}(y)$ .

**Theorem 2.** *Every 2-local derivation of the algebras  $S_2$ ,  $W_2$  and  $W(2)$  is a derivation.*

*Proof.* We will prove that every 2-local derivation of  $W(2)$  is a derivation.

Let  $\Delta$  be an arbitrary 2-local derivation of  $W(2)$ . Then, by the definition, for every element  $a \in W(2)$ , there exists a derivation  $D_{a,e_2}$  of  $W(2)$  such that

$$\Delta(a) = D_{a,e_2}(a), \quad \Delta(e_2) = D_{a,e_2}(e_2).$$

By lemma 1, the matrix  $A^{a,e_2}$  of the derivation  $D_{a,e_2}$  has the following matrix form:

$$A^{a,e_2} = \begin{pmatrix} 0 & \alpha_{a,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_{a,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha_{a,e_2} & 0 & \beta_{a,e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha_{a,e_2} & 2\beta_{a,e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_{a,e_2} & \beta_{a,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_{a,e_2} & \alpha_{a,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let  $v$  be an arbitrary element in  $W(2)$ . Then there exists a derivation  $D_{v,e_2}$  of  $W(2)$  such that

$$\Delta(v) = D_{v,e_2}(v), \quad \Delta(e_2) = D_{v,e_2}(e_2).$$

By lemma 1, the matrix  $A^{v,e_2}$  of the derivation  $D_{v,e_2}$  has the following matrix form:

$$A^{v,e_2} = \begin{pmatrix} 0 & \alpha_{v,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_{v,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha_{v,e_2} & 0 & \beta_{v,e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha_{v,e_2} & 2\beta_{v,e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_{v,e_2} & \beta_{v,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_{v,e_2} & \alpha_{v,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since  $\Delta(e_2) = D_{a,e_2}(e_2) = D_{v,e_2}(e_2)$ , we have

$$\alpha_{a,e_2} = \alpha_{v,e_2}, \quad \beta_{a,e_2} = \beta_{v,e_2},$$

that it

$$D_{v,e_2} = D_{a,e_2}.$$

Therefore, for any element  $a$  of the algebra  $W(2)$

$$\Delta(a) = D_{v,e_2}(a),$$

that it  $D_{v,e_2}$  does not depend on  $a$ . Hence,  $\Delta$  is a derivation by lemma 1.

The cases of the algebras  $S_2$  and  $W_2$  are also similarly proved. This ends the proof.  $\square$

## 5 2-Local automorphisms of conservative algebras of 2-dimensional algebras

A (not necessary linear) map  $\Delta: \mathcal{A} \rightarrow \mathcal{A}$  is called a 2-local automorphism, if for any elements  $x, y \in \mathcal{A}$  there exists an automorphism  $\phi_{x,y}: \mathcal{A} \rightarrow \mathcal{A}$  such that  $\Delta(x) = \phi_{x,y}(x)$ ,  $\Delta(y) = \phi_{x,y}(y)$ .

**Theorem 3.** *Every 2-local automorphism of the algebras  $S_2$ ,  $W_2$  and  $W(2)$  is an automorphism.*

*Proof.* We prove that every 2-local automorphism of  $W(2)$  is an automorphism.

Let  $\Delta$  be an arbitrary 2-local automorphism of  $W(2)$ . Then, by the definition, for every element  $x \in W(2)$ ,

$$x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 + x_8e_8,$$

there exist elements  $a_{x,e_2}, b_{x,e_2}$  such that

$$A_{x,e_2} = \begin{pmatrix} 1 & a_{x,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{x,e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{x,e_2}b_{x,e_2} & a_{x,e_2}^2b_{x,e_2} & b_{x,e_2} & 0 & 0 & 0 & 0 & 0 \\ 3a_{x,e_2}^2b_{x,e_2}^2 & a_{x,e_2}^3b_{x,e_2}^2 & 3a_{x,e_2}b_{x,e_2}^2 & b_{x,e_2}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{x,e_2}b_{x,e_2} & b_{x,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{x,e_2} & a_{x,e_2}b_{x,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$\Delta(x) = A_{x,e_2}\bar{x}$ , where  $\bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$  is the vector corresponding to  $x$ , and

$$\Delta(e_2) = A_{x,e_2}e_2 = (a_{x,e_2}, \frac{1}{b_{x,e_2}}, a_{x,e_2}^2b_{x,e_2}, a_{x,e_2}^3b_{x,e_2}^2, 0, 0, 0, 0).$$

Since the element  $x$  was chosen arbitrarily, we have

$$\Delta(e_2) = (a_{x,e_2}, \frac{1}{b_{x,e_2}}, a_{x,e_2}^2b_{x,e_2}, a_{x,e_2}^3b_{x,e_2}^2, 0, 0, 0, 0)$$

$$= (a_{y,e_2}, \frac{1}{b_{y,e_2}}, a_{y,e_2}^2 b_{y,e_2}, a_{y,e_2}^3 b_{y,e_2}^2, 0, 0, 0, 0),$$

for each pair  $x, y$  of elements in  $W(2)$ . Hence,  $a_{x,e_2} = a_{y,e_2}$ ,  $b_{x,e_2} = b_{y,e_2}$ . Therefore

$$\Delta(x) = A_{y,e_2}x$$

for any  $x \in W(2)$  and the matrix  $A_{y,e_2}$  does not depend on  $x$ . Thus, by lemma 2  $\Delta$  is an automorphism.

The cases of the algebras  $S_2$  and  $W_2$  are also similarly proved. The proof is complete.  $\square$

## 6 Local automorphisms of conservative algebras of 2-dimensional algebras

Let  $\mathcal{A}$  be an algebra. A linear map  $\nabla: \mathcal{A} \rightarrow \mathcal{A}$  is called a local automorphism, if for any element  $x \in \mathcal{A}$  there exists an automorphism  $\phi_x: \mathcal{A} \rightarrow \mathcal{A}$  such that  $\nabla(x) = \phi_x(x)$ .

**Theorem 4.** *Every local automorphism of the algebras  $S_2$ ,  $W_2$  and  $W(2)$  is an automorphism.*

*Proof.* We prove that every local automorphism of  $W(2)$  is an automorphism.

Let  $\nabla$  be an arbitrary local automorphism of  $W(2)$  and  $B$  be its matrix, i.e.,

$$\nabla(x) = B\bar{x}, x \in W(2),$$

where  $\bar{x}$  is the vector corresponding to  $x$ . Then, by the definition, for every element  $x \in W(2)$ ,

$$x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 + x_8e_8,$$

there exist elements  $a_x, b_x$  such that

$$A_x = \begin{pmatrix} 1 & a_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_xb_x & a_x^2b_x & b_x & 0 & 0 & 0 & 0 & 0 \\ 3a_x^2b_x^2 & a_x^3b_x^2 & 3a_xb_x^2 & b_x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_xb_x & b_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_x & a_xb_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\nabla(x) = B\bar{x} = A_x\bar{x}.$$

Using these equalities and by choosing subsequently  $x = e_1, x = e_2, \dots, x = e_8$  we get

$$B = \begin{pmatrix} 1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{e_1}b_{e_1} & a_{e_2}^2b_{e_2} & b_{e_3} & 0 & 0 & 0 & 0 & 0 \\ 3a_{e_1}^2b_{e_1}^2 & a_{e_2}^3b_{e_2}^2 & 3a_{e_3}b_{e_3}^2 & b_{e_4}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{e_5}b_{e_5} & b_{e_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{e_7} & a_{e_8}b_{e_8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since  $\nabla(e_6 + e_7) = \nabla(e_6) + \nabla(e_7)$ , we have

$$b_{e_6+e_7} = b_{e_6}, b_{e_6+e_7} = b_{e_7}.$$

Hence,

$$b_{e_6} = b_{e_7}.$$

Similarly to this equality we get  $b_{e_3} = b_{e_6}$  and  $b_{e_6} = b_{e_2} \neq 0$ . Hence,

$$b_{e_2} = b_{e_3} = b_{e_6} = b_{e_7}. \quad (4)$$

Since  $\nabla(e_5 + e_8) = \nabla(e_5) + \nabla(e_8)$ , we have

$$a_{e_5+e_8}b_{e_5+e_8} = a_{e_5}b_{e_5}, \quad a_{e_5+e_8}b_{e_5+e_8} = a_{e_8}b_{e_8}.$$

From this it follows that

$$a_{e_5}b_{e_5} = a_{e_8}b_{e_8}.$$

Similarly to this equality we get  $a_{e_1}b_{e_1} = a_{e_8}b_{e_8}$ . Hence,

$$a_{e_1}b_{e_1} = a_{e_5}b_{e_5} = a_{e_8}b_{e_8}. \quad (5)$$

Since  $\nabla(e_4 + e_6) = \nabla(e_4) + \nabla(e_6)$ , we have

$$b_{e_4+e_6}^2 = b_{e_4}^2, \quad b_{e_4+e_6}^2 = b_{e_6}^2.$$

From this it follows that

$$b_{e_4}^2 = b_{e_6}^2.$$

Hence, by (4), we get

$$b_{e_4}^2 = b_{e_2}^2. \quad (6)$$

Since  $\nabla(e_2 + e_8) = \nabla(e_2) + \nabla(e_8)$ , we have

$$a_{e_2} = a_{e_2+e_8}, \quad a_{e_2+e_8}^2b_{e_2+e_8} = a_{e_2}^2b_{e_2}, \quad a_{e_2+e_8}b_{e_2+e_8} = a_{e_8}b_{e_8}.$$

Hence,

$$b_{e_2+e_8} = b_{e_2}, \quad a_{e_2+e_8}b_{e_2+e_8} = a_{e_2}b_{e_2}$$

and, therefore,

$$a_{e_2}b_{e_2} = a_{e_8}b_{e_8}. \quad (7)$$

Similarly, since  $\nabla(e_2 + e_3) = \nabla(e_2) + \nabla(e_3)$ , we have

$$a_{e_2} = a_{e_2+e_3}, \quad b_{e_2}^{-1} = b_{e_2+e_3}^{-1}, \quad a_{e_2+e_3}^3 b_{e_2+e_3}^2 + 3a_{e_2+e_3}b_{e_2+e_3}^2 = a_{e_2}^3 b_{e_2}^2 + 3a_{e_3}b_{e_3}^2.$$

Hence,

$$b_{e_2} = b_{e_2+e_3}$$

and by (4) and  $a_{e_2} = a_{e_2+e_3}$  we get

$$a_{e_2}^3 + 3a_{e_2} = a_{e_2}^3 + 3a_{e_3}.$$

Therefore,  $a_{e_2} = a_{e_3}$  and

$$a_{e_2}b_{e_2}^2 = a_{e_3}b_{e_3}^2. \quad (8)$$

Finally, since  $\nabla(e_1 + e_8) = \nabla(e_1) + \nabla(e_8)$ , we have

$$a_{e_1+e_8}b_{e_1+e_8} = a_{e_1}b_{e_1}, \quad a_{e_1+e_8}b_{e_1+e_8} = a_{e_8}b_{e_8}.$$

Hence,

$$a_{e_1}b_{e_1} = a_{e_8}b_{e_8}.$$

By (7), from the last equalities it follows that

$$a_{e_1}b_{e_1} = a_{e_2}b_{e_2}, \quad a_{e_1}^2b_{e_1}^2 = (a_{e_1}b_{e_1})^2 = (a_{e_2}b_{e_2})^2 = a_{e_2}^2b_{e_2}^2. \quad (9)$$

By (4), (5), (6), (7), (8), (9) the matrix  $B$  has the following matrix form

$$B = \begin{pmatrix} 1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{e_2}b_{e_2} & a_{e_2}^2b_{e_2} & b_{e_2} & 0 & 0 & 0 & 0 & 0 \\ 3a_{e_2}^2b_{e_2}^2 & a_{e_2}^3b_{e_2}^2 & 3a_{e_2}b_{e_2}^2 & b_{e_2}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{e_2}b_{e_2} & b_{e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{e_2} & a_{e_2}b_{e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, by lemma 2, the local automorphism  $\nabla$  is an automorphism.

The cases of the algebras  $S_2$  and  $W_2$  are also similarly proved. This ends the proof.  $\square$

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